Structured Shrinkage Priors

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ABSTRACT
In many regression settings the unknown coefficients may have some known structure, for instance they may be ordered in space or correspond to a vectorized matrix or tensor. At the same time, the unknown coefficients may be sparse, with many nearly or exactly equal to zero. However, many commonly used priors and corresponding penalties for coefficients do not encourage simultaneously structured and sparse estimates. In this article we develop structured shrinkage priors that generalize multivariate normal, Laplace, exponential power and normal-gamma priors. These priors allow the regression coefficients to be correlated a priori without sacrificing elementwise sparsity or shrinkage. The primary challenges in working with these structured shrinkage priors are computational, as the corresponding penalties are intractable integrals and the full conditional distributions that are needed to approximate the posterior mode or simulate from the posterior distribution may be nonstandard. We overcome these issues using a flexible elliptical slice sampling procedure, and demonstrate that these priors can be used to introduce structure while preserving sparsity. Supplementary materials for this article are available online.

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KEYWORDS
Bayesian lasso; Lasso; Multivariate Laplace; Multivariate normal-scale mixture; Shrinkage; Sparsity

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- Posterior mean, $\mathbb{E}\left[\boldsymbol{\beta}|\boldsymbol{y},\boldsymbol{X},\boldsymbol{\phi},\boldsymbol{\theta}\right]$

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In many settings, $\boldsymbol{\beta}$ may be **structured**

- Treatment effects at different spatial locations or over time
- Autoregressive parameters of different orders
- Main effects and higher order interactions

An Example Motivating Structured Shrinkage

Forney et al. (2013); Bruneel (2018)

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Subject wears cap of **EEG** sensors

- Different spatial locations
- High temporal resolution

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EEG measurements collected over many **trials**

- Grid of randomly selected letters shown each trial
- Subject counts the number of times a target letter appears

Forney et al. (2013); Bruneel (2018)

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Data: For trials $i = 1, \ldots, 240$

- $y_i \in \{0, 1\}$, presence of target
- $\boldsymbol{x}_i$ EEG measurements during trial i

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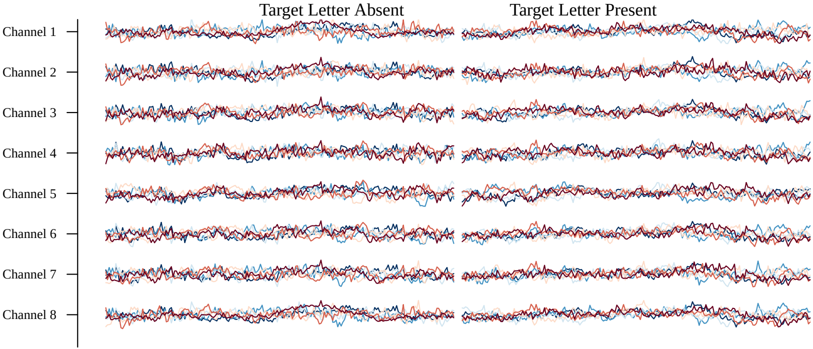

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  - $\rightarrow$ **Structured** $\boldsymbol{\beta}$

Structured shrinkage priors may yield:

- Better estimation of $\boldsymbol{\beta}$
- Improved out-of-sample prediction

Our Approach to Structured Shrinkage

Many **independent** priors are normal scale mixtures

$$\boldsymbol{\beta} \overset{d}{=} \boldsymbol{s} \circ \boldsymbol{z},\ \boldsymbol{z} \sim \text{normal}\left(\boldsymbol{0}, \boldsymbol{I}_p\right),\ \boldsymbol{s} \sim ???$$

Polson and Scott (2010); Finegold and Drton (2011); Griffin and Hoff (2024)

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Dependence introduces challenges

- Marginal prior $p\left(\boldsymbol{\beta}|\boldsymbol{\theta}\right) = \int p\left(\boldsymbol{\beta}|\boldsymbol{s}, \boldsymbol{\theta}\right) p\left(\boldsymbol{s}|\boldsymbol{\theta}\right) d\boldsymbol{s}$ intractable
- $p\left(\boldsymbol{s}|\boldsymbol{\beta}, \boldsymbol{\Omega}, \boldsymbol{\theta}\right)$ nonstandard

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Comparison to Existing Priors

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- $\boldsymbol{\Omega} \propto \boldsymbol{I}_p$ and dependent $\boldsymbol{s}^2 \rightarrow$ Uncorrelated $\boldsymbol{\beta}$ a priori
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Fused Lasso/Structured Penalty Priors:

- Elements of $\boldsymbol{\Omega}$ also random
- Prior parameters hard to interpret, relate to moments of $\boldsymbol{\beta}$

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Our Approach: Three Structured Shrinkage Priors

Structured Product Normal (**SPN**)

- $\boldsymbol{s} \sim \text{normal}\left(\mathbf{0}, \boldsymbol{\Psi}\right)$
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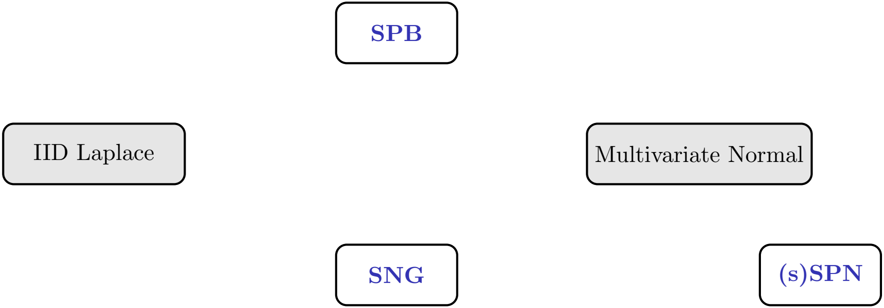
Structured Power/Bridge (SPB)

- $s_j^2 \overset{i.i.d.}{\sim}$ polynomially tilted positive $\frac{q}{2}$-stable distribution
- q treated as fixed and known

Relationships Between SPN, SNG and SPB Priors



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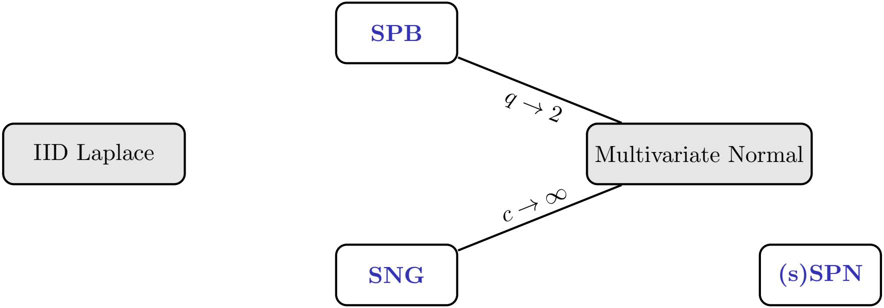

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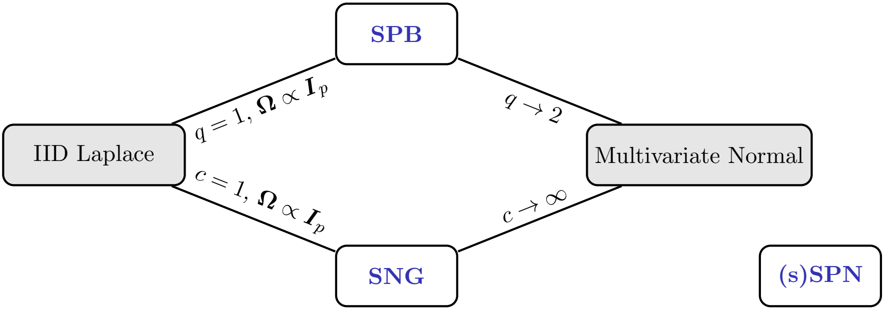

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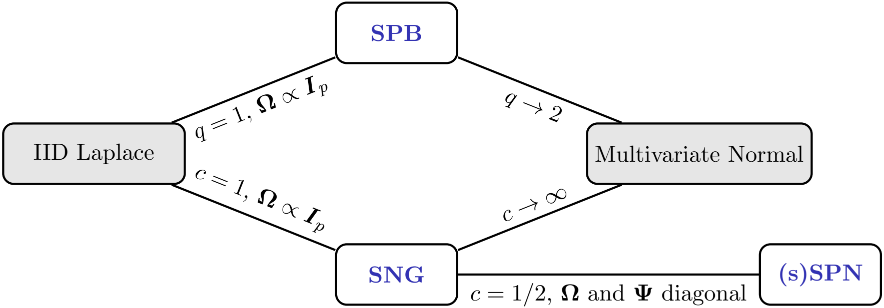

Understanding Properties of SPN, SNG and SPB Priors

No closed form marginal prior, $p\left(\boldsymbol{\beta}|\boldsymbol{\theta}\right) = \int p\left(\boldsymbol{\beta}|\boldsymbol{s}, \boldsymbol{\theta}\right) p\left(\boldsymbol{s}|\boldsymbol{\theta}\right) d\boldsymbol{s}$

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- Prior density contours → **Shrinkage and Structure**

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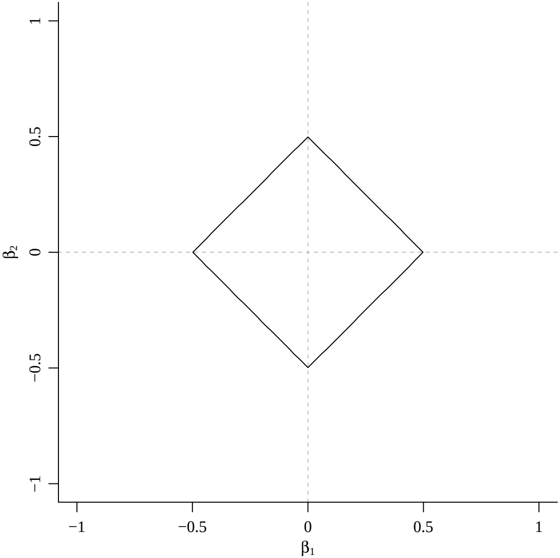
→ Examine behavior of along axes, at origin → **Shrinkage**

→ Study prior moments

- Determine range of $\boldsymbol{\Sigma}$ → Achievable **Structure**

Property	**SNG**	**SPB**	**SPN**
Generalizes an Independent Shrinkage Prior	✓	✓	✓
Univariate Marginal is an Independent Shrinkage Prior	✓	✓	✓
Generalizes a Laplace Prior	$c = 1$	$q = 1$	
Generalizes a Normal Prior	$c \to \infty$	$q \to 2$	
Infinite Spike or Pole at Zero		$c \leq 1/2$	✓
Quadratic Scale Log Full Conditional			✓
Arbitrary Covariance Structure Achievable			✓

Prior Density Contours



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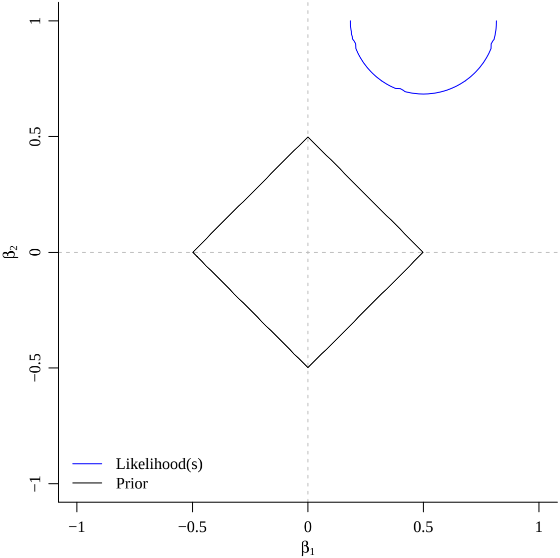

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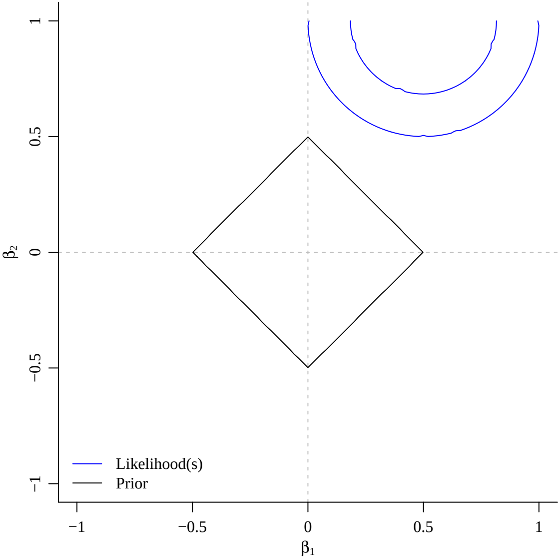

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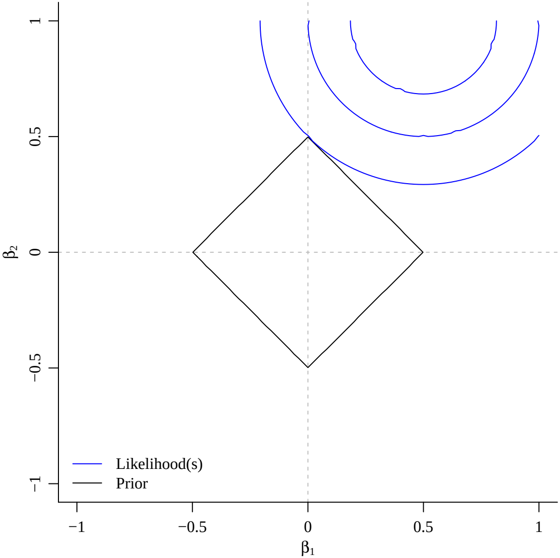

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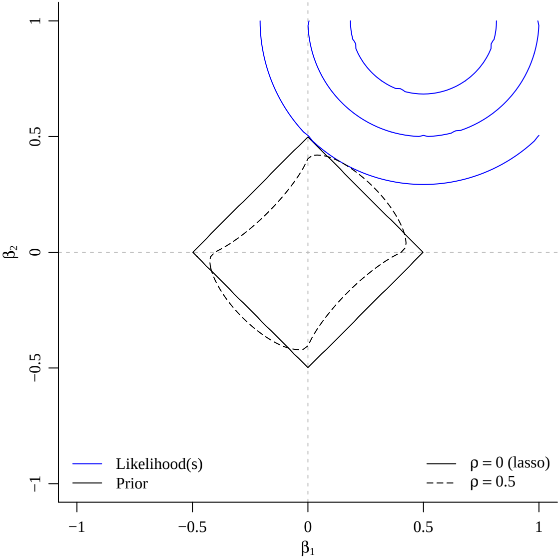

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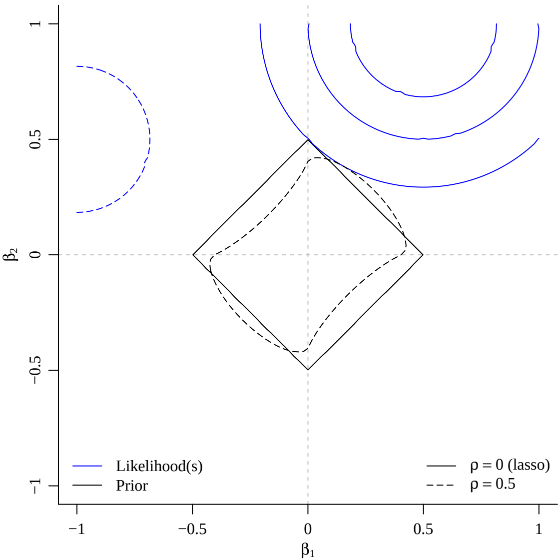

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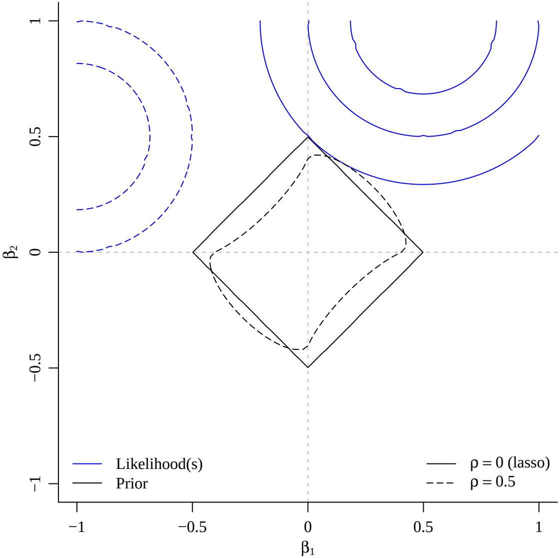

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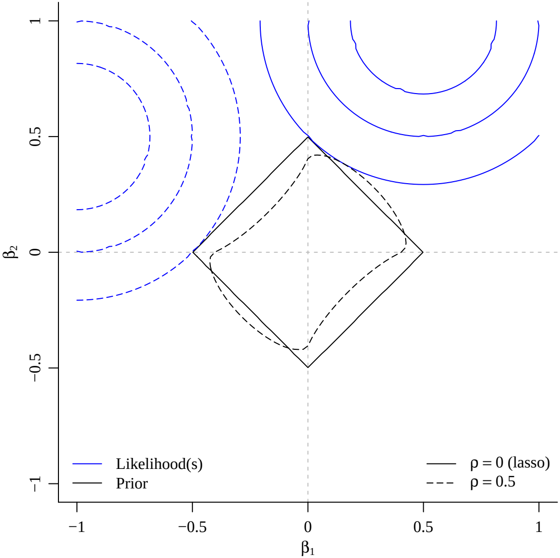

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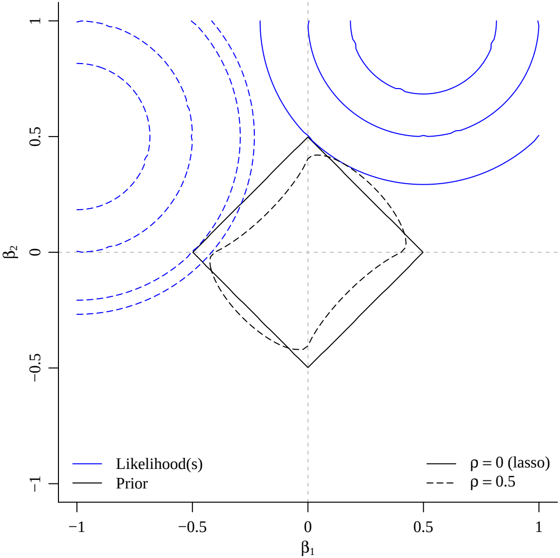

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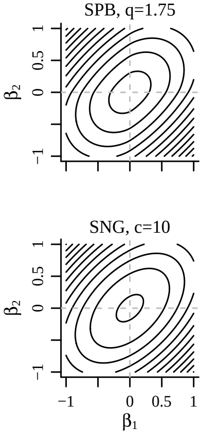

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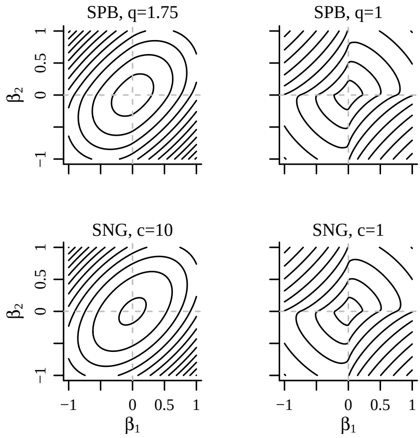

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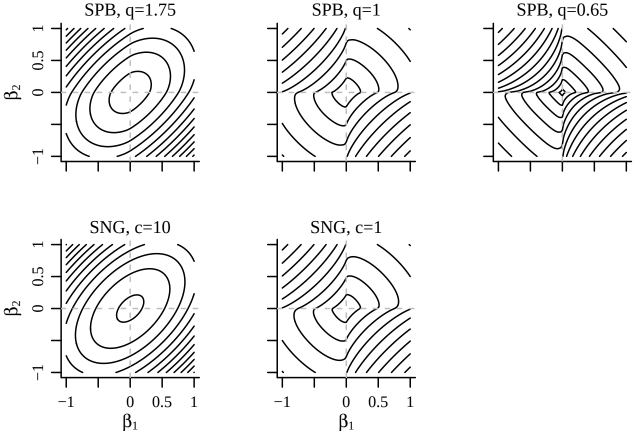

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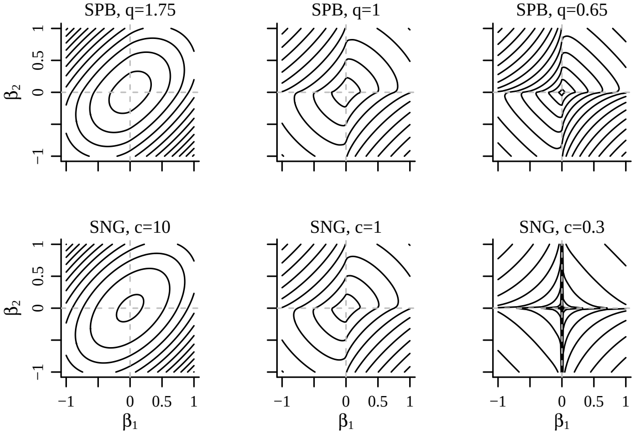

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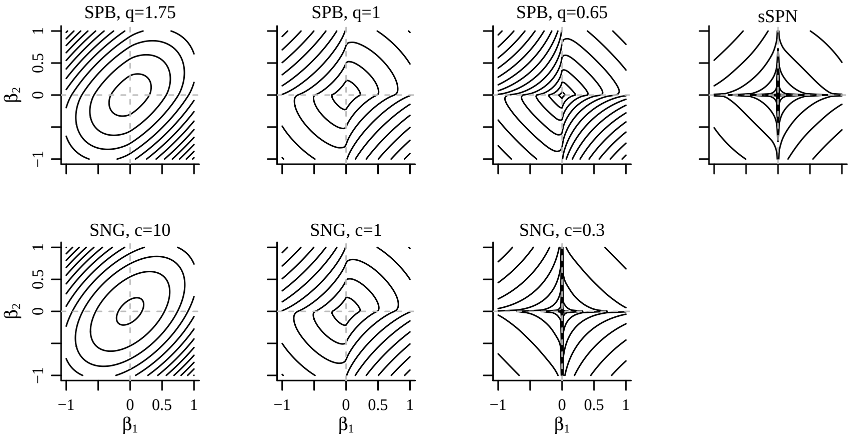

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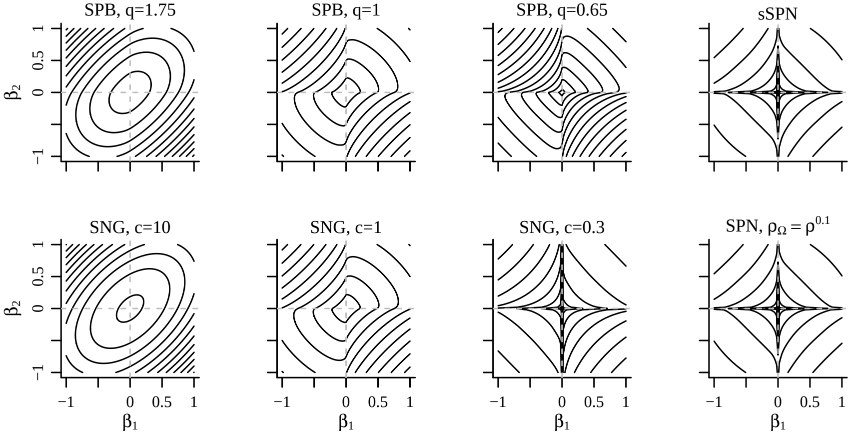
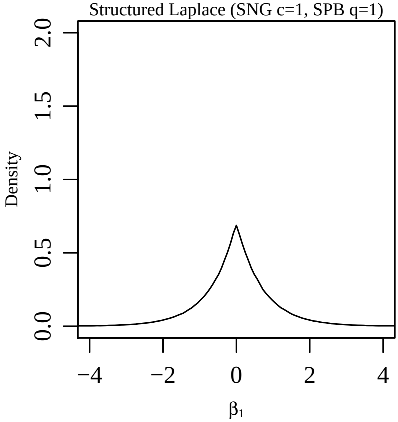


Prior Conditional Distributions



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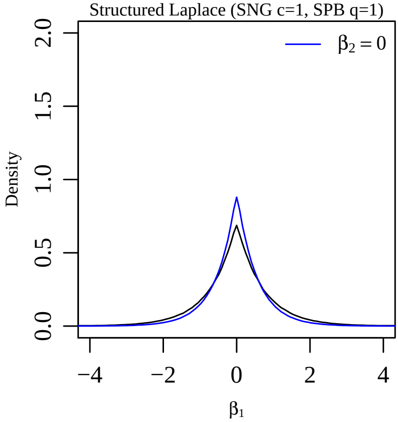

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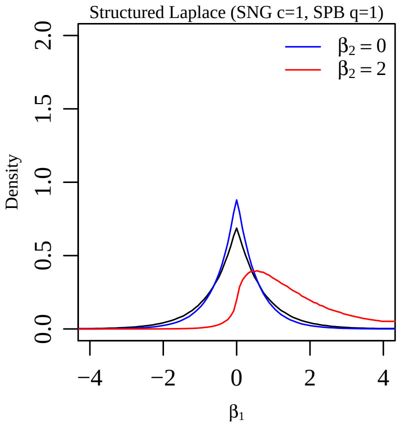

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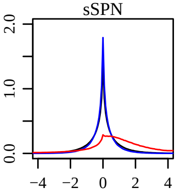

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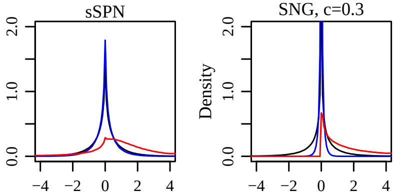

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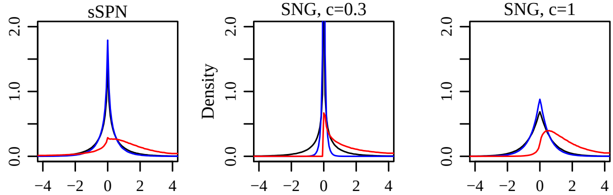

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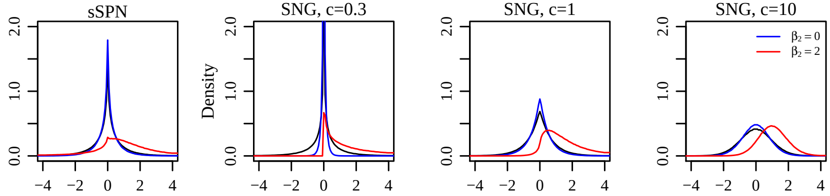

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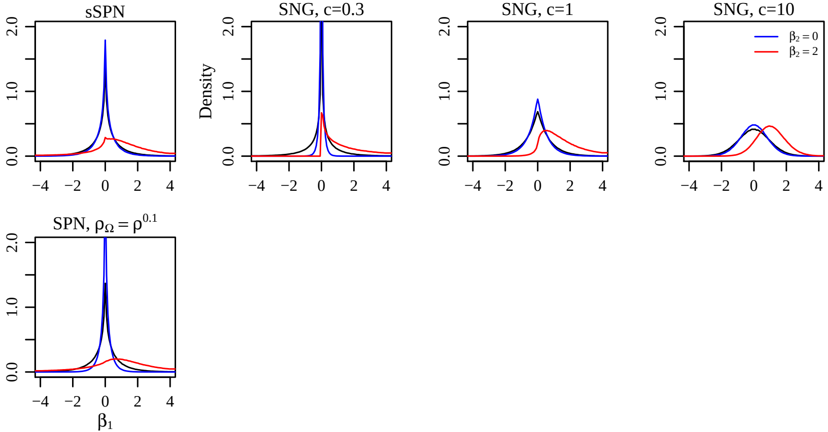

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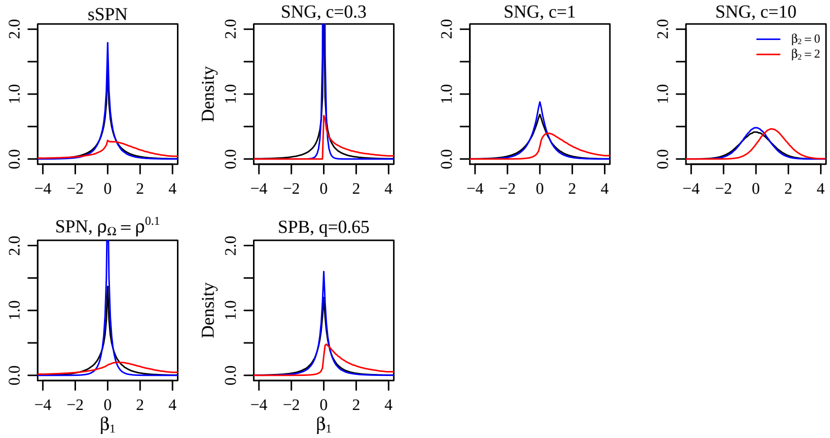

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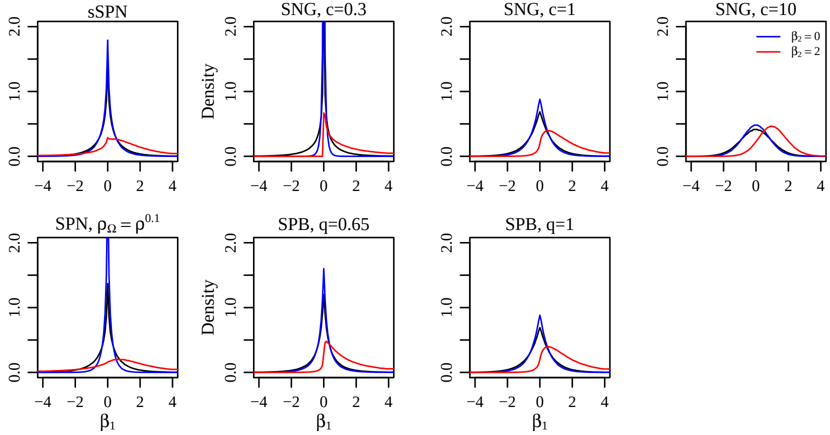

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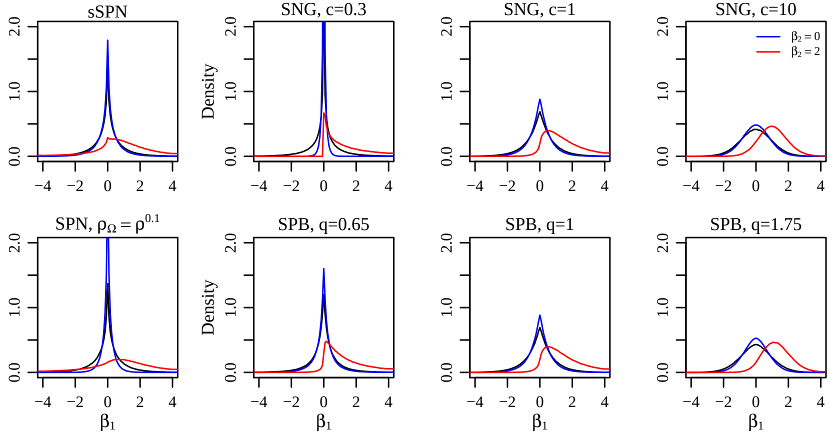

Range of $\boldsymbol{\Sigma} \to$ Achievable Structure

$$\boldsymbol{\Sigma} = \mathbb{E}\left[\boldsymbol{s}\boldsymbol{s}'\right] \circ \boldsymbol{\Omega}$$

Griffin and Hoff (2024)

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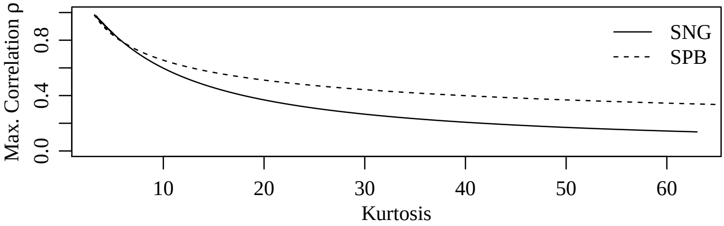
- Clearly restricted under SNG, SPB priors



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Computation under Structured Shrinkage Priors

Posterior Mode:
Approximated by solving

$$\hat{\boldsymbol{\beta}} = \text{maximize}_{\boldsymbol{\beta}} h\left(\boldsymbol{y}|\boldsymbol{X}, \boldsymbol{\beta}, \boldsymbol{\phi}\right) + \log\left(\int p\left(\boldsymbol{\beta}|\boldsymbol{s}, \boldsymbol{\theta}\right) p\left(\boldsymbol{s}|\boldsymbol{\theta}\right) d\boldsymbol{s}\right)$$

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Posterior Mean, Median, Quantiles:

Approximated using samples from posterior

$$p\left(\boldsymbol{\beta}, \boldsymbol{s}|\boldsymbol{y}, \boldsymbol{X}, \boldsymbol{\phi}, \boldsymbol{\theta}\right)$$

$\rightarrow$Requires simulation from $p\left(\boldsymbol{\beta}|\boldsymbol{s}, \boldsymbol{X}, \boldsymbol{y}, \boldsymbol{\theta}, \boldsymbol{\phi}\right)$ and $p\left(\boldsymbol{s}|\boldsymbol{\beta}, \boldsymbol{\theta}\right)$

Simulation from $p\left(\boldsymbol{s}|\boldsymbol{\beta},\boldsymbol{\theta}\right)$

Goal: Simulate $\boldsymbol{s}$ according to

$$\left(\prod_{j=1}^{p}|s|^{-1}\right)\exp\left\{-\frac{1}{2}\left(\mathbf{1}/\boldsymbol{s}\right)'\left(\boldsymbol{\Omega}^{-1}\circ\left(\boldsymbol{\beta}\boldsymbol{\beta}'\right)\right)\left(\mathbf{1}/\boldsymbol{s}\right)\right\}p\left(\boldsymbol{s}|\boldsymbol{\theta}\right)$$

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- Developed for densities with a normal component
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1. Obtain estimate of $\hat{\boldsymbol{\Sigma}}$
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Fully Bayes: Assume prior distributions for $\boldsymbol{\Omega}$, $\boldsymbol{\Psi}$

Perry (2017)

Return to EEG Example

Review of Data:

- $n \times 1$ response $\boldsymbol{y}$, indicators of target letter presence
  - $n = 20$ (220 held out for constructing "ground truth")
- $n \times p$ covariate matrix $\boldsymbol{X}$, contemporaneous EEG
  - $p = p_1 \times p_2 = 26 \times 8 = 208$

Polson et al. (2013)

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$\rightarrow$ Logistic Regression

Structure:

- **Separable** across time and channel, $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_2 \otimes \boldsymbol{\Sigma}_1$
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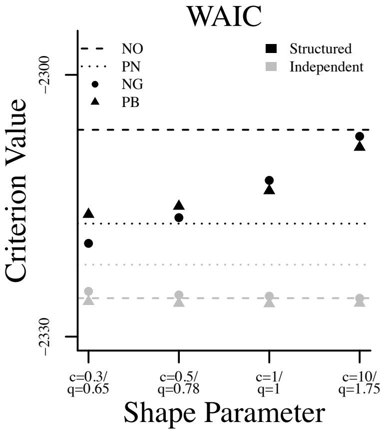
Specification of Hyperparameters:

- $\rho_{\boldsymbol{\Omega}}, \rho_{\boldsymbol{\Psi}} \sim \text{beta}\left(\left(p_1+1\right)/2, \left(p_1+1\right)/2\right)$,
  $\boldsymbol{\Omega}_2^{-1} \sim \text{Wishart}(p_2+2, \kappa^{-1}\boldsymbol{I}_{p_2})$, $\boldsymbol{\Psi}_2^{-1} \sim \text{Wishart}(p_2+2, \boldsymbol{I}_{p_2})$

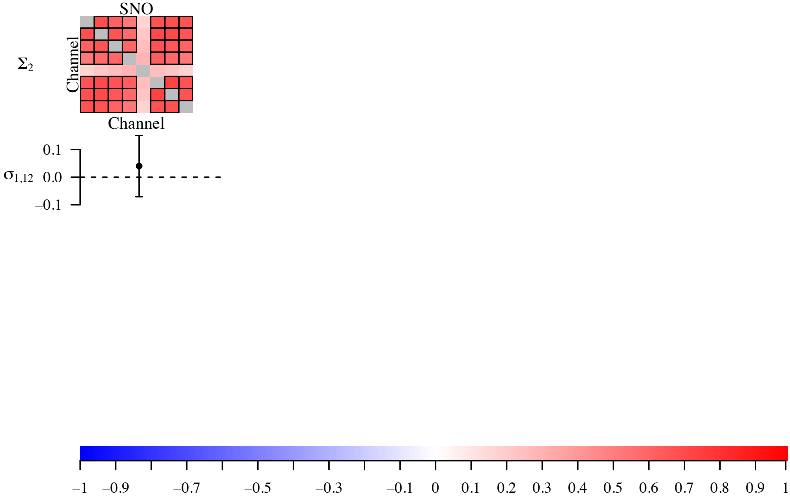
Polson et al. (2013)

Measures of Fit



Approximate Estimates of Covariances

Posterior Means of $\sigma_{1,12}$ and Σ_2



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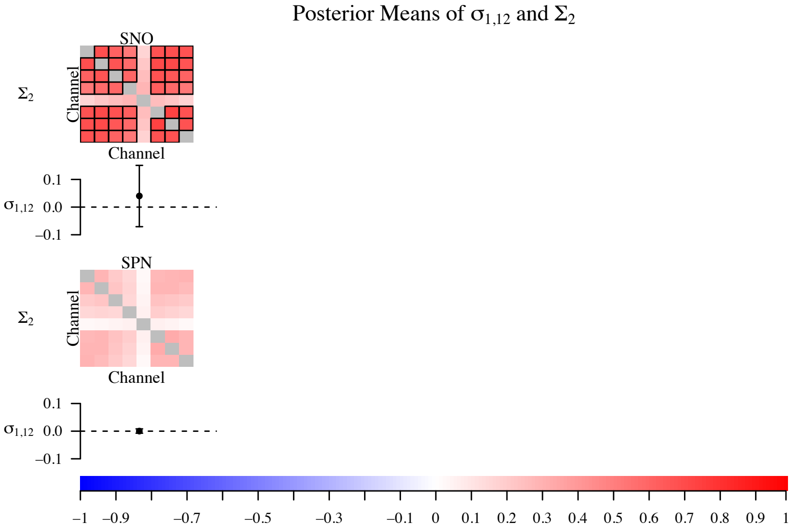

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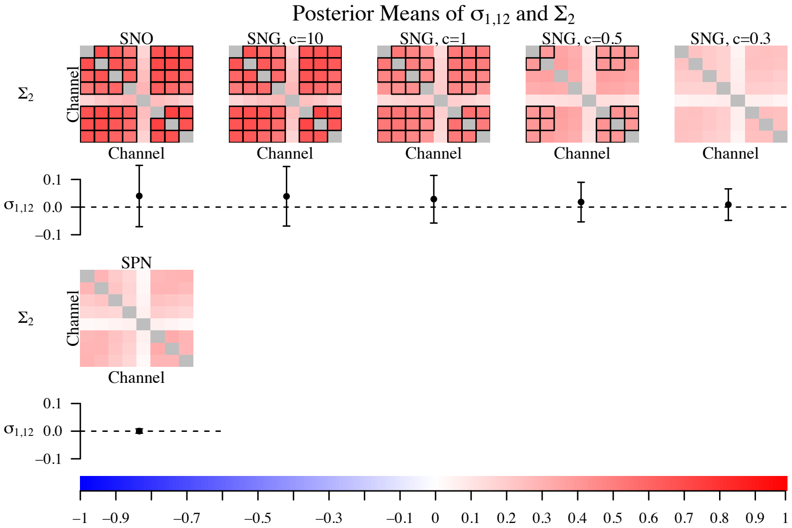

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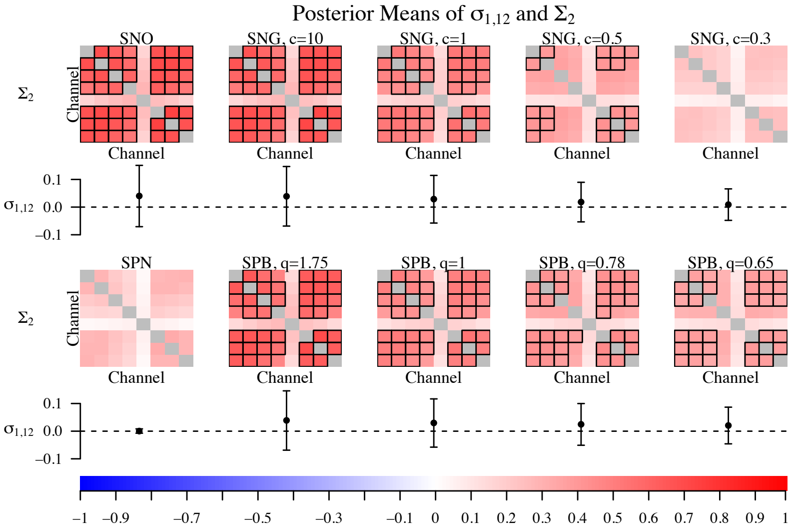
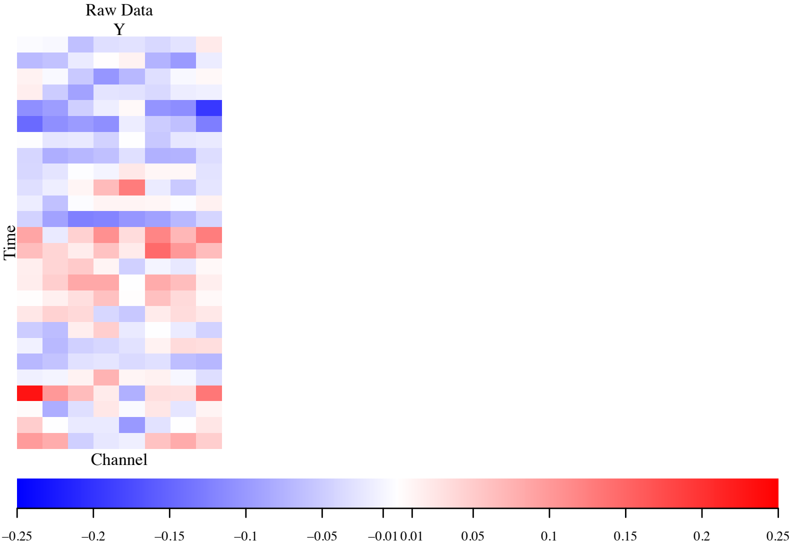


Posterior Mean Estimates



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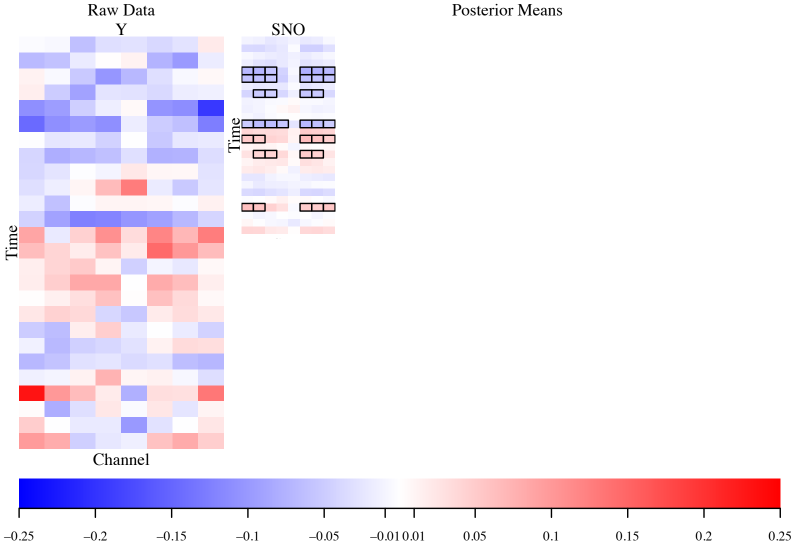

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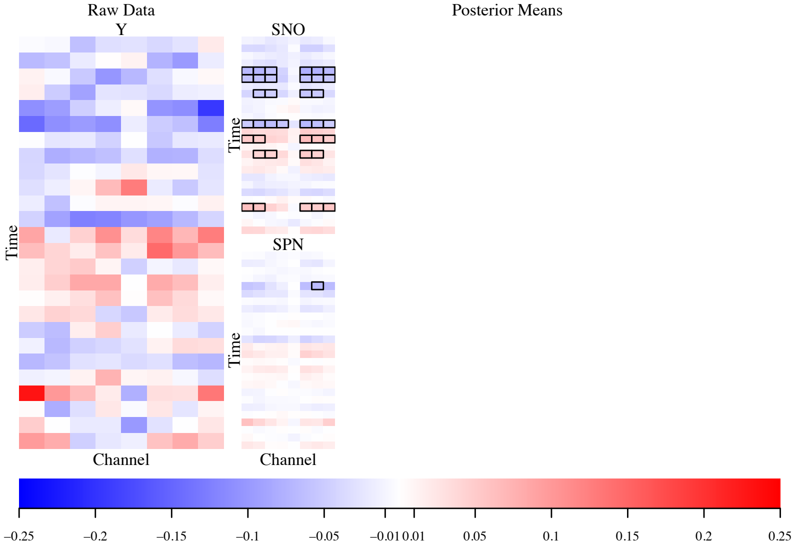

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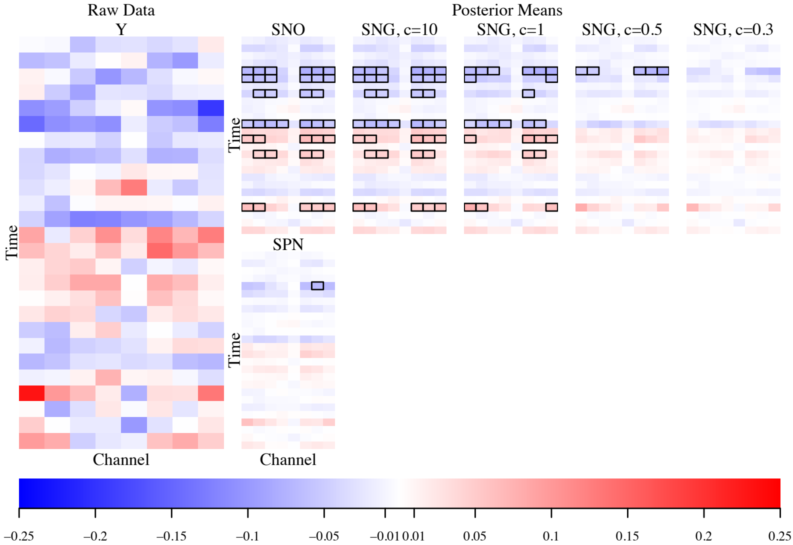

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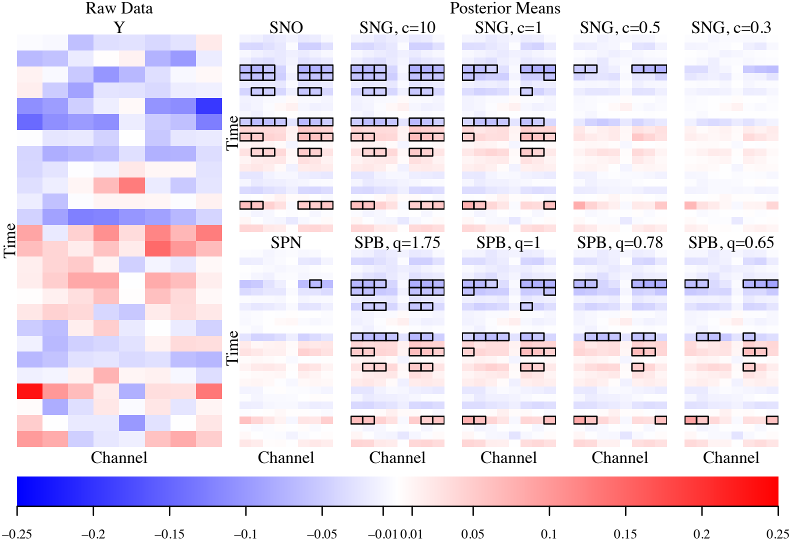

Comparison to “Ground Truth”

How do our estimates from 20 trials compare to estimates from all 240?

	SNO	SNG			
		$c = 10$	$c = 1$	$c = 0.5$	$c = 0.3$
TP%	18.18	18.18	13.64	0.00	0.00
FP%	17.68	17.68	10.98	3.05	0.00
	SPN	SPB			
		$q = 0.65$	$q = 0.78$	$q = 1$	$q = 1.75$
TP%	0.00	18.18	13.64	11.36	9.09
FP%	0.61	15.85	10.98	6.71	6.10

Conclusions and Next Steps

- ☺ Introducing structure to shrinkage priors has benefits
- ☺ Shrinkage priors w/o infinite spikes at zero have benefits

Property	**SNG**	**SPB**	**SPN**
Generalizes an Independent Shrinkage Prior	✓	✓	✓
Univariate Marginal is an Independent Shrinkage Prior	✓	✓	✓
Generalizes a Laplace Prior	$c = 1$	$q = 1$	
Generalizes a Normal Prior	$c \rightarrow \infty$	$q \rightarrow 2$	
Infinite Spike or Pole at Zero		$c \leq 1/2$	✓
Quadratic Scale Log Full Conditional			✓
Arbitrary Covariance Structure Achievable			✓

- How will these priors perform in other applications?
- Can we improve computation more?
- Are there other ways of introducing structure that lead to easier-to-compute modes?

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van Gerven, M., B. Cseke, R. Oostenveld, and T. Heskes (2009). Bayesian source localization with the multivariate laplace prior. *Advances in Neural Information Processing Systems 22*, 1–9. Example of using¡br/¿¡br/¿It's sort of related to the normal product prior¡br/¿¡br/¿Perform approximate inference.

van Gerven, M. A. J., B. Cseke, F. P. de Lange, and T. Heskes (2010). Efficient bayesian multivariate fmri analysis using a sparsifying spatio-temporal prior. *Neuroimage 50*, 150–161. Very relevant!¡br/¿¡br/¿Approximate inference.

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