Estimation of Possibly Non-Stationary Long Memory Processes via Adaptive Overdifferencing

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June 4, 2025

Joint work with David S. Matteson, Gennady Samorodnitsky (Cornell)

TECHNOMETRICS
2025, VOL. 00, NO. 0, 1–14
https://doi.org/10.1080/00401706.2025.2453207

Likelihood Inference for Possibly Nonstationary Processes via Adaptive Overdifferencing

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ABSTRACT
We make an observation that facilitates exact likelihood-based inference for the parameters of the popular ARFIMA model without requiring stationarity by allowing the upper bound $\bar{d}$ for the memory parameter d to exceed 0.5: estimating the parameters of a single nonstationary ARFIMA model is equivalent to estimating the parameters of a sequence of stationary ARFIMA models. This allows for the use of existing methods for evaluating the likelihood for an invertible and stationary ARFIMA model. This enables improved inference because many standard methods perform poorly when estimates are close to the boundary of the parameter space. It also allows us to leverage the wealth of likelihood approximations that have been introduced for estimating the parameters of a stationary process. We explore how estimation of the memory parameter d depends on the upper bound $\bar{d}$ and introduce adaptive procedures for choosing $\bar{d}$. We show via simulation how our adaptive procedures estimate the memory parameter well, relative to existing alternatives, when the true value is as large as 2.5.

ARTICLE HISTORY
Received April 2024
Accepted January 2025

KEYWORDS
ARFIMA; FARIMA; long memory

Outline

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- Proposal of a solution
- Demonstration of success

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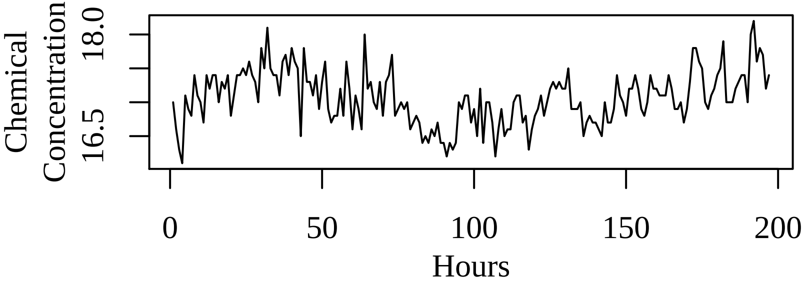

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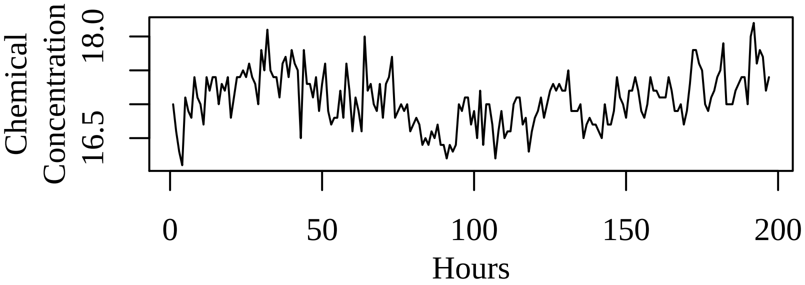

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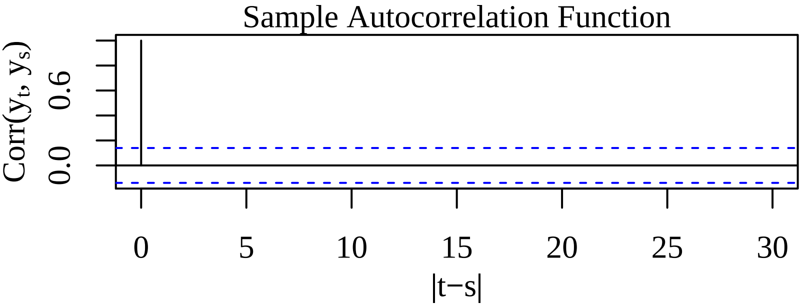

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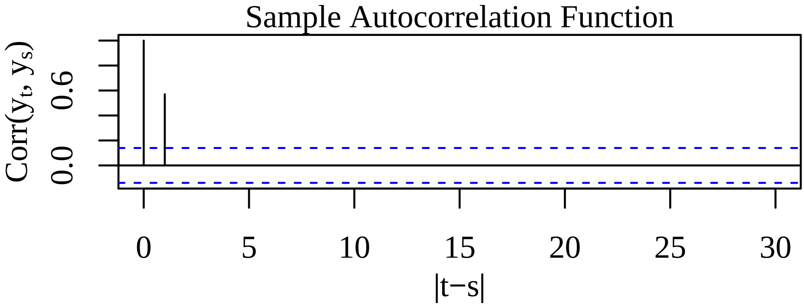

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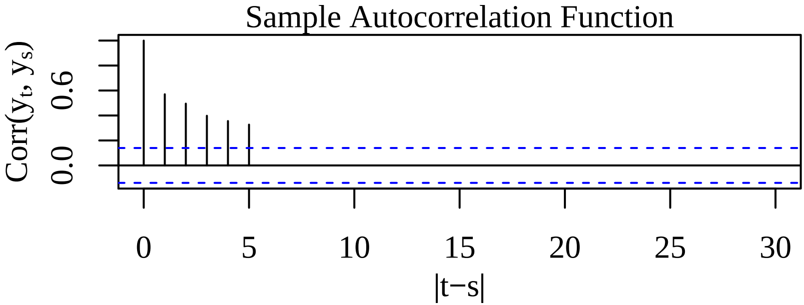

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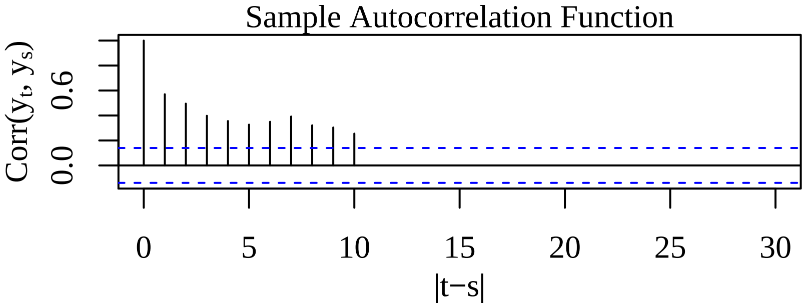

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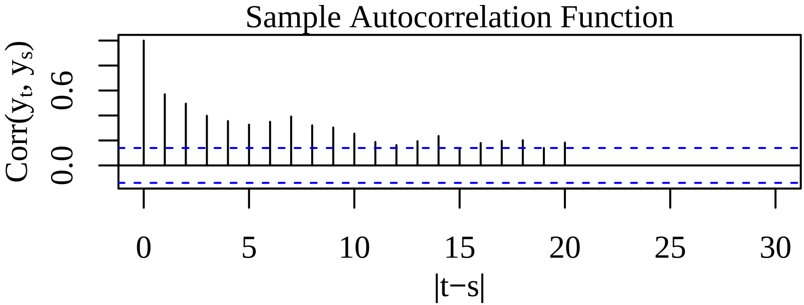

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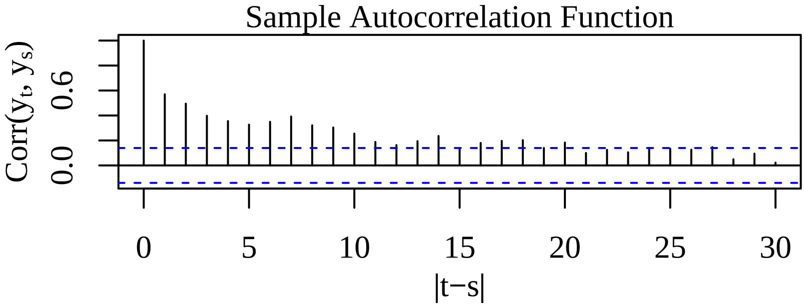

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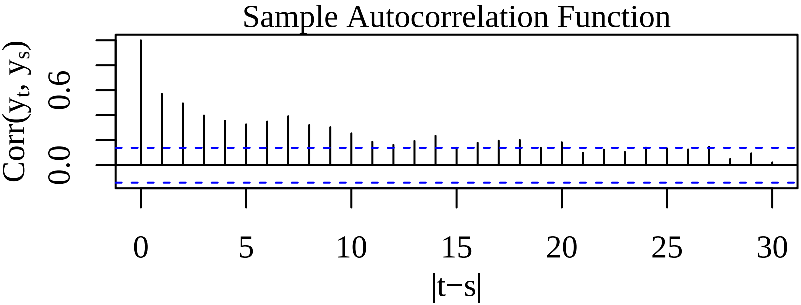

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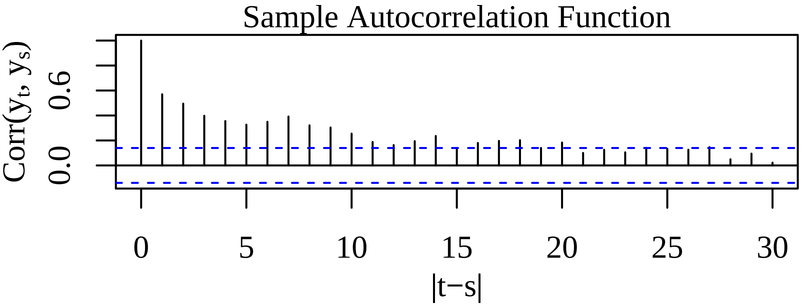

Long memory is observed in many types of data

- Stream flows, earth surface temperature, CO_2 emissions
- EEG, MRI, measurements of movement
- Behavior of cells in culture

Laboissiere et al. (2015); Barassi et al. (2018); Lo et al. (1993)

Autoregressive Moving Average (ARMA) Model

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$$(y_t - \mu_t) = \sum_{j=1}^{p} \phi_j \left(y_{t-j} - \mu_{t-j}\right) + \sum_{l=1}^{q} \theta_l z_{t-l} + z_t$$

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- Mean constant $\mu_t = \mu$ or a linear in covariates $\mu_t = \boldsymbol{x}_t' \boldsymbol{\beta}$

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$$\implies \phi\left(B\right)\left(y_t - \mu_t\right) = \theta\left(B\right) z_t$$

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- **Invertible:** All roots of $\theta(x)$ are outside the unit circle
  $\rightarrow$ Unique values of $\theta_1, \ldots, \theta_q$ give unique likelihood

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MA parameters θ_l contribute up to l-th autocorrelation

$\rightarrow$ Can't help us model long memory

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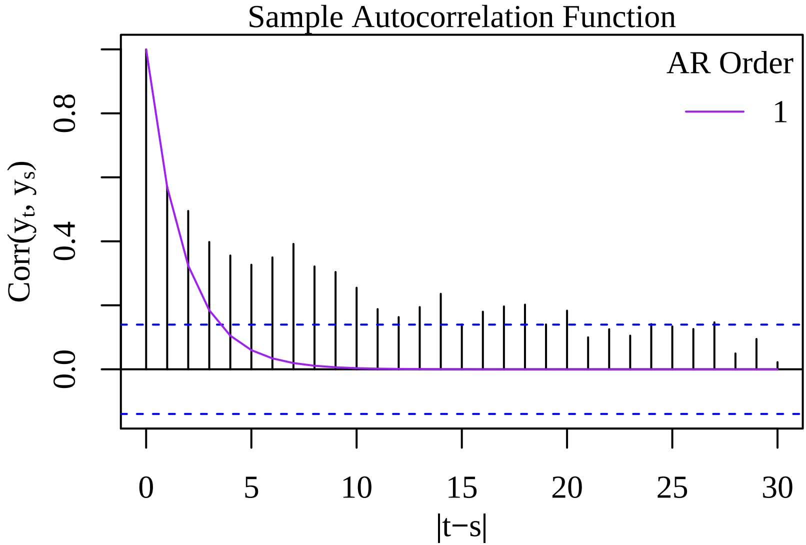
In practice, long memory requires $p \to \infty$ as $n \to \infty$

AR and Long Memory



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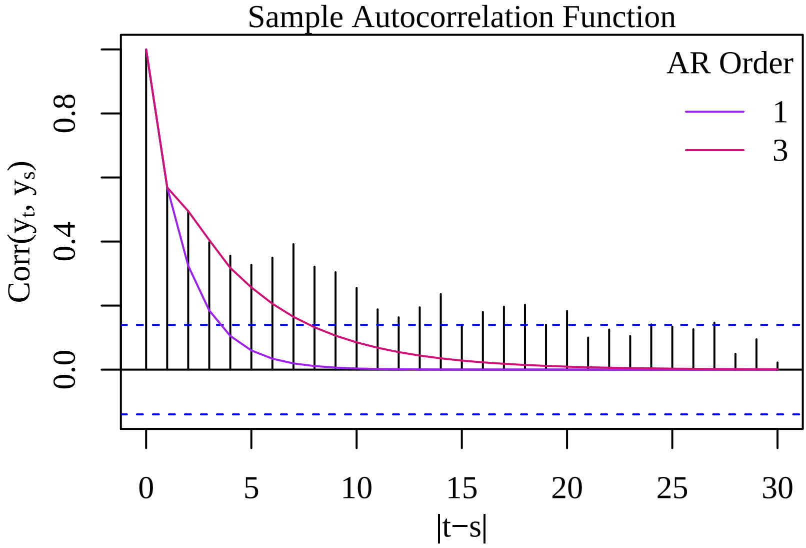

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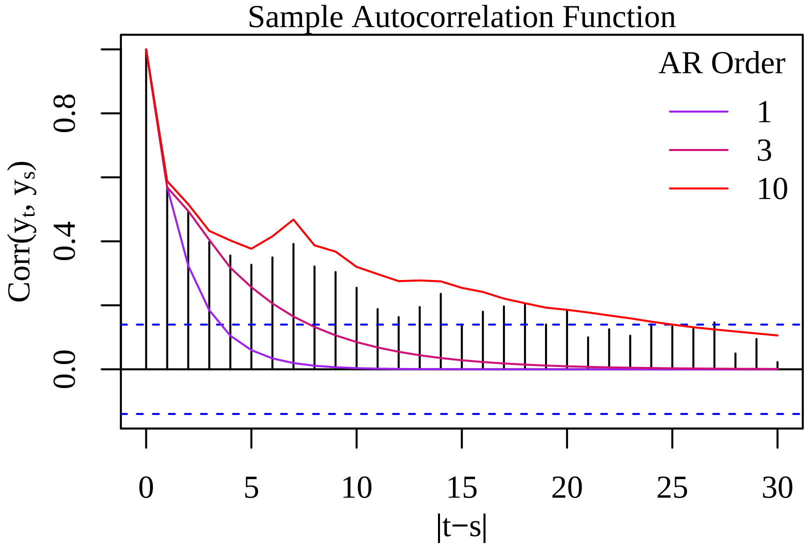

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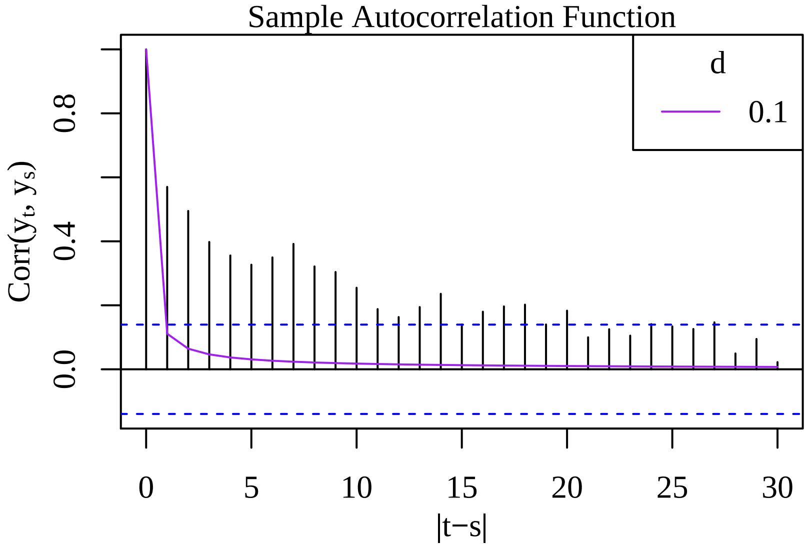

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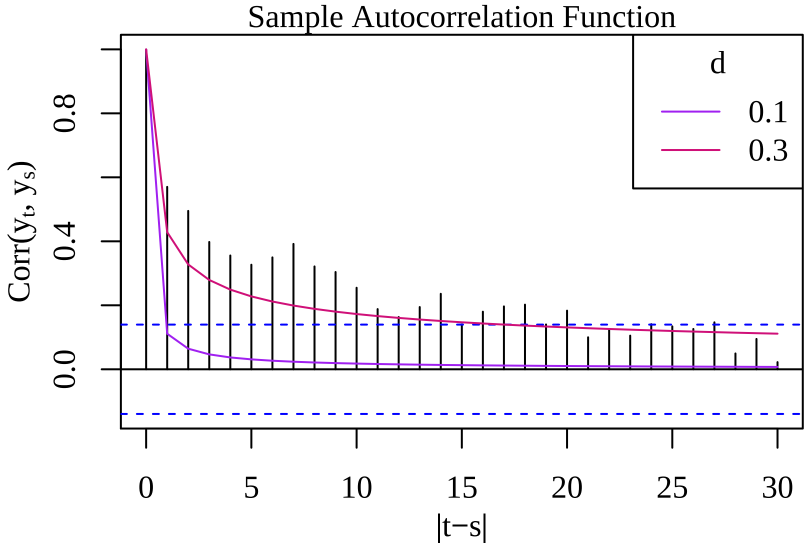

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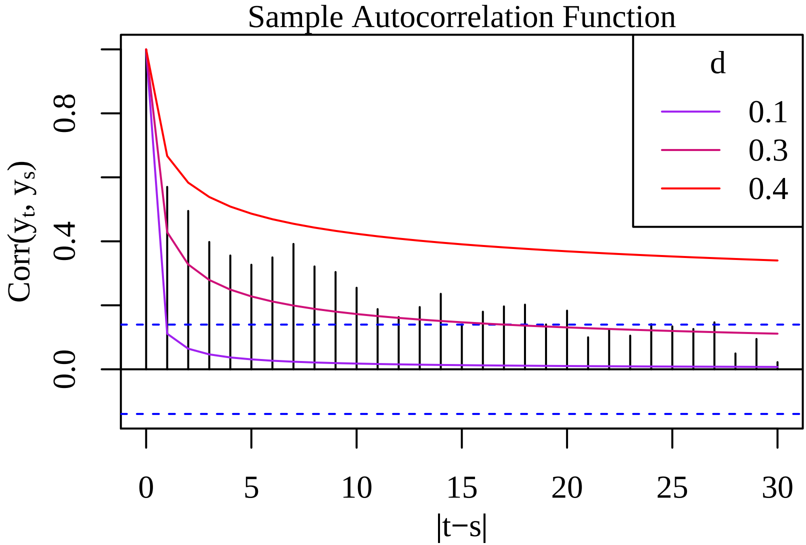

Autoregressive FI Moving Average (ARFIMA) Model

$$\overbrace{\phi\left(B\right)\left(1-B\right)^{d}\left(y_t-\mu_t\right)=\theta\left(B\right)z_t, \quad z_t \stackrel{i.i.d.}{\sim} \text{normal}\left(0,\sigma^2\right)}^{\text{ARFIMA}(p,d,q)}$$

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(Content applies to ARFIMA(p, d, q) as well)

Estimation of ARFIMA(p, d, q) Processes

Baillie (1996)

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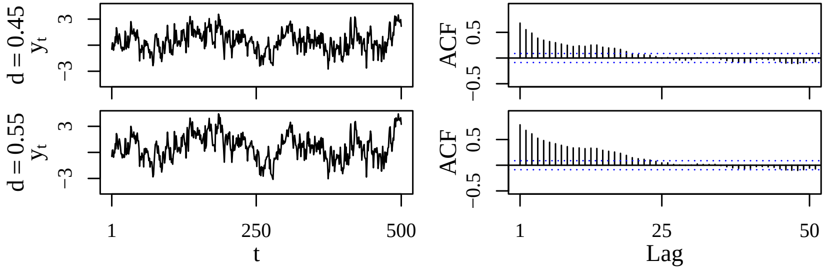
Baillie (1996)

Why do we care about assumptions on d?



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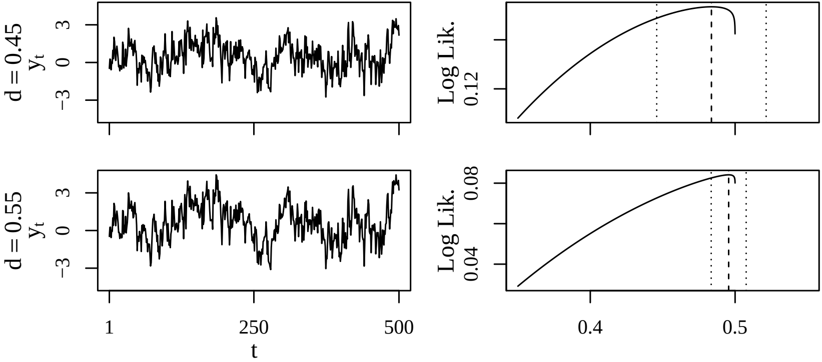

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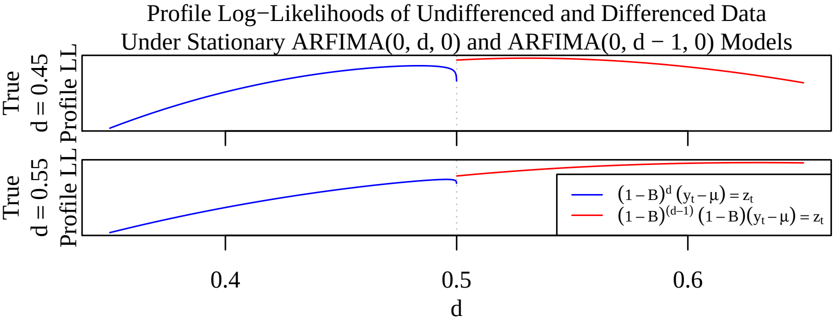
Beran (1995); Velasco and Robinson (2000); Mayoral (2007)

What's wrong with this procedure?



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- **Solution:** Fix data in advance to get stationarity for $d < \bar{d}$

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This gives a continuous likelihood in d!

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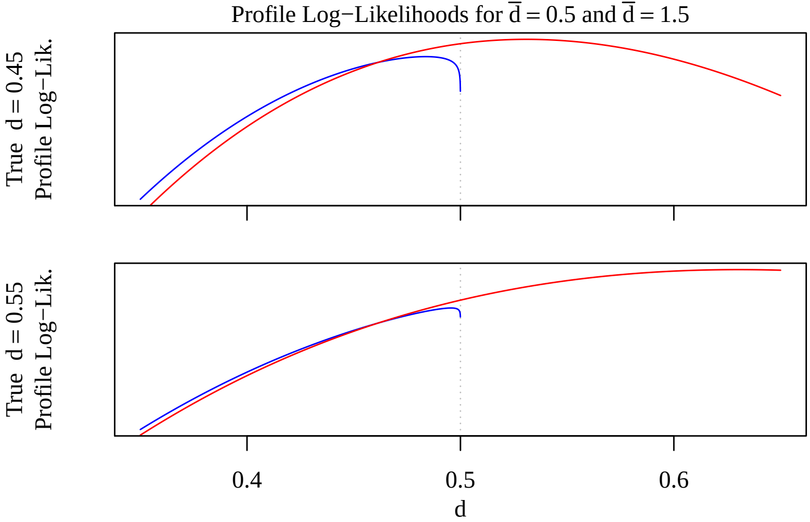

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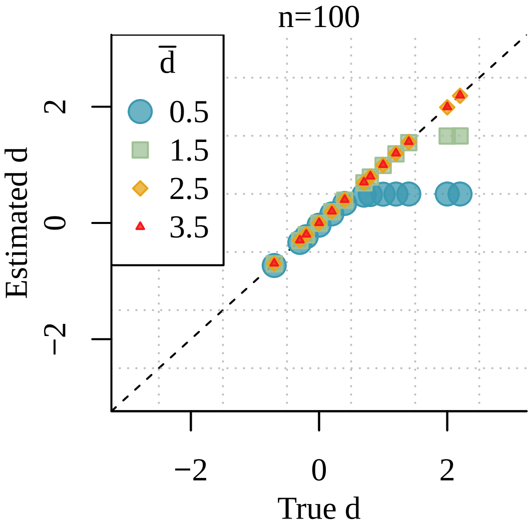
Simulate $10,000$ time series $\boldsymbol{y}$ for each:

- $d \in \{0, 0.1, \ldots, 2.2\}$
- $n \in \{100, 500\}$

Simulations



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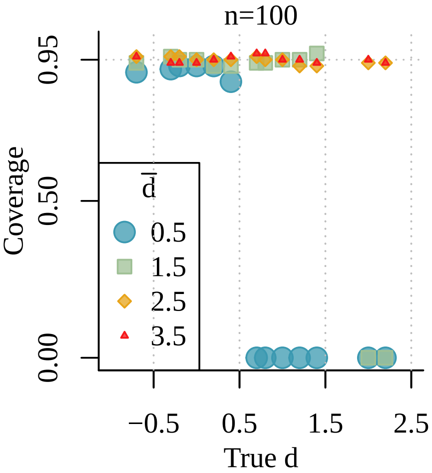

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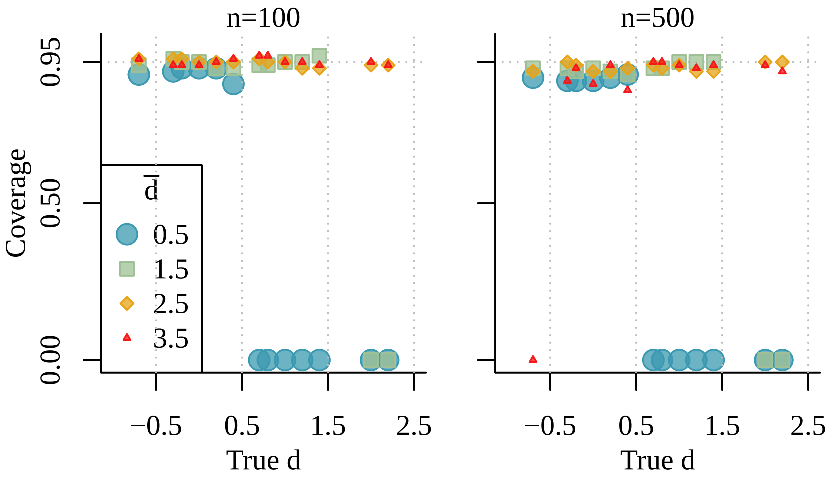

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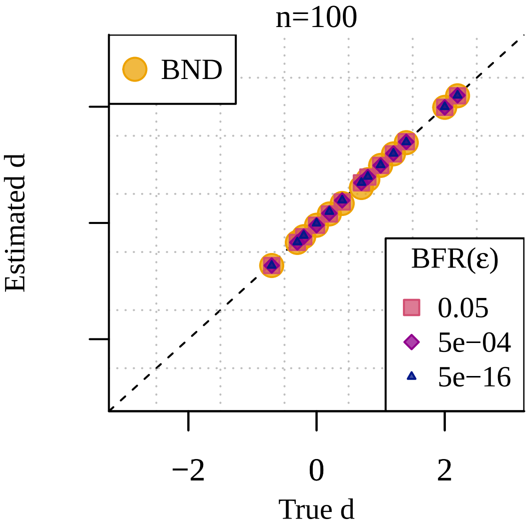
When $\epsilon < 0.5$, we'll call estimator $\mathbf{BFR}(\epsilon)$

Adaptive Procedure Simulations



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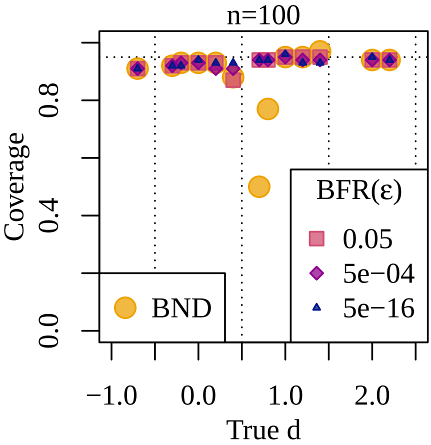

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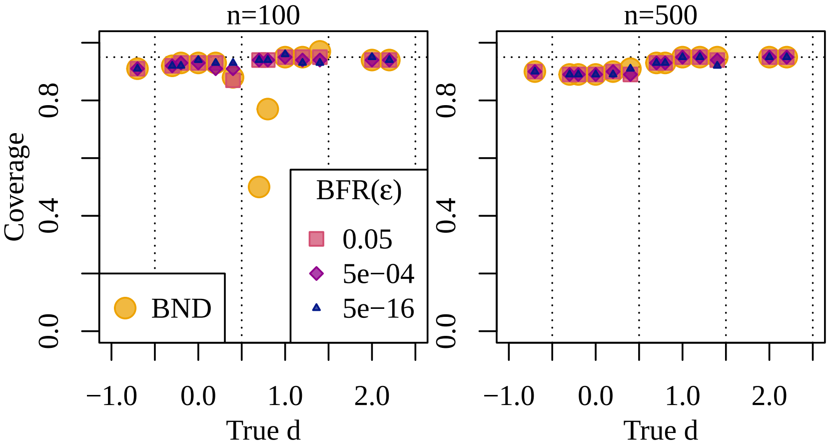
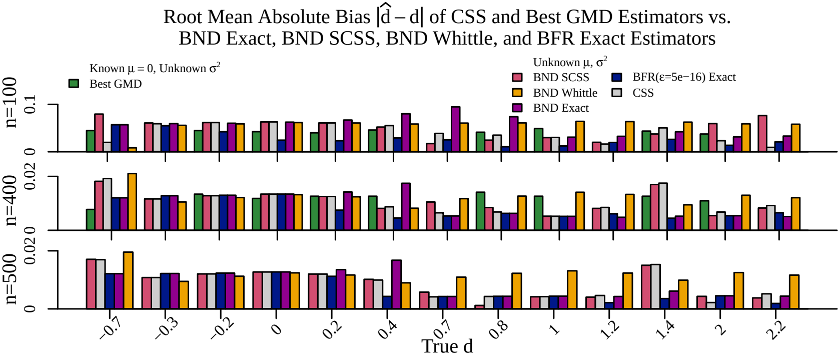


Adaptive Procedure vs. Competition



Application to CO^2 Emission Convergence

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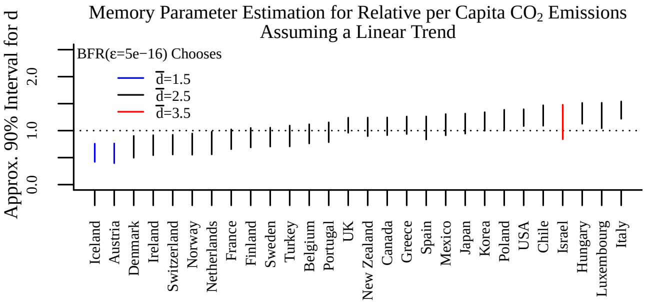

Barassi et al. (2018)

Conclusions and Future Work

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  - ⋆ Optimization is harder
  - ⋆ Likelihood may be multimodal

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Maximizing over possibly non-stationary FI models
$\approx$ Maximizing a sequence of stationary ARFIMA models

Everything we have done:

- Extends to ARFIMA(p, d, q)

$$\phi(B)(1-B)^d(y_t - \mu_t) = \theta(B)z_t,\; z_t \overset{i.i.d.}{\sim} \text{normal}\left(0, \sigma^2\right)$$

  - ⋆ Optimization is harder
  - ⋆ Likelihood may be multimodal
- Can be made approximate

Baillie, R. T. (1996). Long memory processes and fractional integration in econometrics. Journal of Econometrics 73, 5–59.

Barassi, M. R., N. Spagnolo, and Y. Zhao (2018). Fractional Integration Versus Structural Change: Testing the Convergence of CO_2 Emissions. Environmental and Resource Economics 71(4), 923–968.

Beran, J. (1995). Maximum Likelihood Estimation of the Differencing Parameter for Invertible Short and Long Memory Autoregressive Integrated Moving Average Models. Journal of the Royal Statistical Society. Series B (Methodological) 57(4), 659–672.

Box, G. E. P. and G. M. Jenkins (1970). Time Series Analysis: Forecasting and Control. San Francisco: Holden Day.

Doornik, J. A. and M. Ooms (2003). Computational aspects of maximum likelihood estimation of autoregressive fractionally integrated moving average models. Computational Statistics and Data Analysis 42(3), 333–348.

Granger, C. W. and R. Joyeux (1980). An introduction to long-memory time series models and fractional differencing. Journal of Time Series Analysis 1(1), 15–29.

Hosking, J. R. M. (1981). Fractional Differencing. Biometrika 68(1), 165–176.

Laboissiere, R., J. C. Letievant, E. Ionescu, P. A. Barraud, M. Mazzuca, and C. Cian (2015). Relationship between spectral characteristics of spontaneous postural sway and motion sickness susceptibility. PLoS ONE 10(12), 1–15.

Lo, C.-M., C. R. Keese, and I. Giaever (1993). Monitoring Motion of Confluent Cells in Tissue Culture. Experimental Cell Research 204, 102–109.

Mayoral, L. (2007). Minimum distance estimation of stationary and non-stationary ARFIMA processes. Econometrics Journal 10(1), 124–148.

Odaki, M. (1993). On the Invertibility of Fractionally Differenced ARIMA Processes. Biometrika 80(3), 703–709.

Velasco, C. and P. M. Robinson (2000). Whittle Pseudo-Maximum Likelihood Estimation for Nonstationary Time Series. Journal of the American Statistical Association 95(452), 1229–1243.

Acknowledgements

Thank you to:

- Robert Gramacy for inviting me!

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Conclusions and Future Work

We took a difficult problem and turned it into an easy problem!

Maximizing over possibly non-stationary FI models
$\approx$ Maximizing a sequence of stationary ARFIMA models

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- Extends to ARFIMA(p, d, q)
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