Testing and Estimation for Sparsity-Inducing Power Penalties

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Introduction

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Sparsity Inducing Power Penalties

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Objectives

- Test Laplace prior appropriateness
- Adaptively choose a better prior

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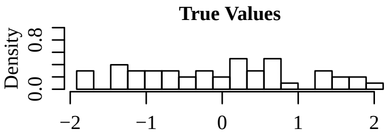
$$\overbrace{\mathbb{E}\left[\boldsymbol{\beta}|\boldsymbol{X},\boldsymbol{y},\sigma^2,\lambda\right]}^{\text{Posterior Mean}} = \text{argmin}_{\boldsymbol{b}} \underbrace{\int ||\boldsymbol{\beta}-\boldsymbol{b}||_2^2\, p\left(\boldsymbol{y}|\boldsymbol{X},\boldsymbol{\beta},\sigma^2\right) p\left(\boldsymbol{\beta}|\lambda\right) d\boldsymbol{\beta}}_{\text{Weighted average squared error loss}}$$

What if $\boldsymbol{\beta}$ is not sparse at all?

Simulated example with $n = 50$, $p = 45$, $\sigma^2 = \tau^2 = 1$



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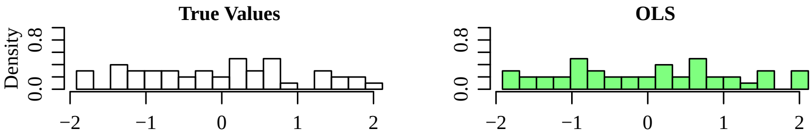

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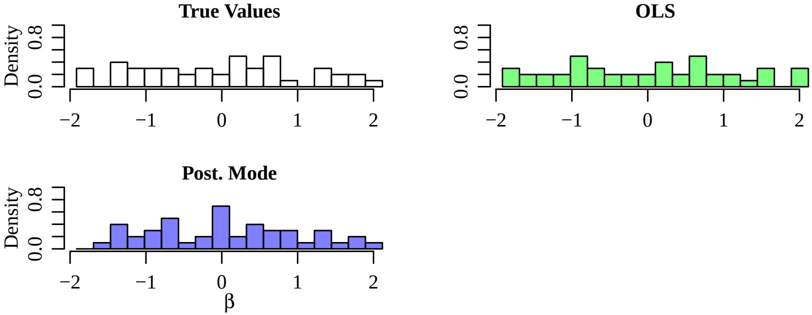

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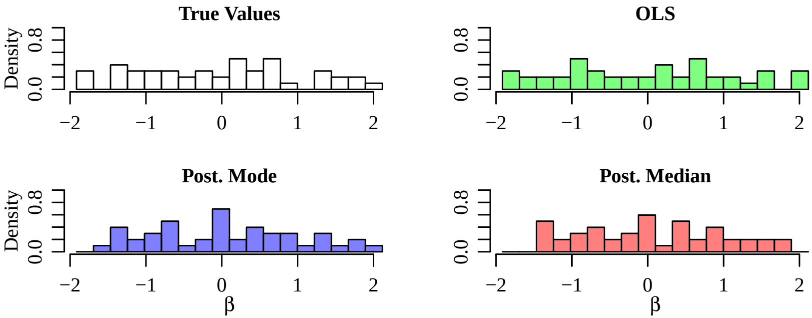

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- Sparsity rate of mode independent of sparsity rate of $\boldsymbol{\beta}$!

Existing Literature on Testing Specification of $p(\boldsymbol{\beta})$

Largely appropriateness of normal random effects models

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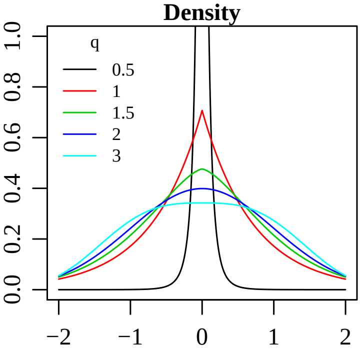
Flexibility of the Exponential Power Distribution



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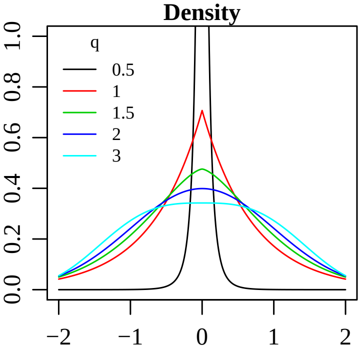

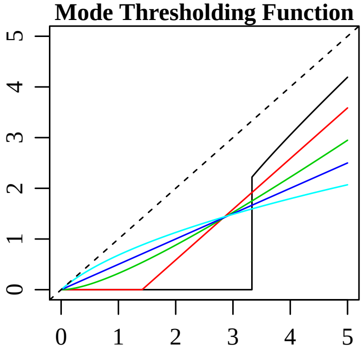

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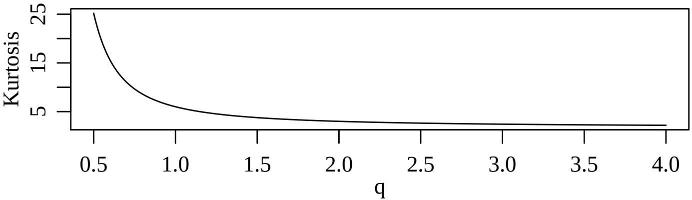

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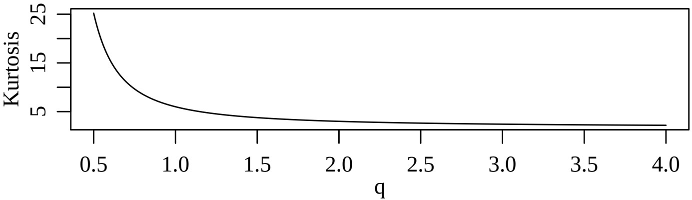

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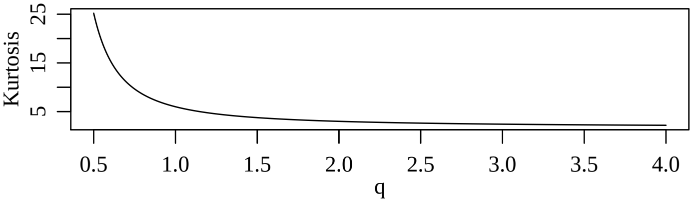

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- Specifying a “better” model if evidence suggests $q \neq 1$

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- Compute $\hat{\psi}_{\delta,\boldsymbol{X}} = \frac{\bar{b}^{(4)}_{\delta,\boldsymbol{X}}}{\left(\bar{b}^{(2)}_{\delta,\boldsymbol{X}}\right)^2}$, compare to quantiles of $\psi_{\delta,\boldsymbol{X}}$

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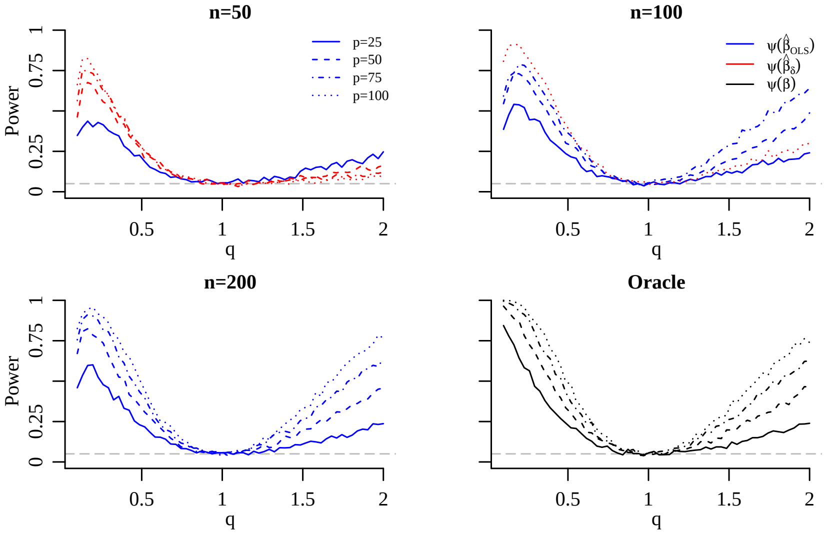
If columns of $\boldsymbol{X}$ standardized to have norm n,

$$\mathrm{tr}\left(\underbrace{\boldsymbol{D}\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}}_{\text{Variance of } \boldsymbol{\beta}_{\delta,\boldsymbol{X}}}\right) \leq \frac{1}{n}\sum_{j=1}^{p}\frac{1}{\delta^2 + \eta_j}$$

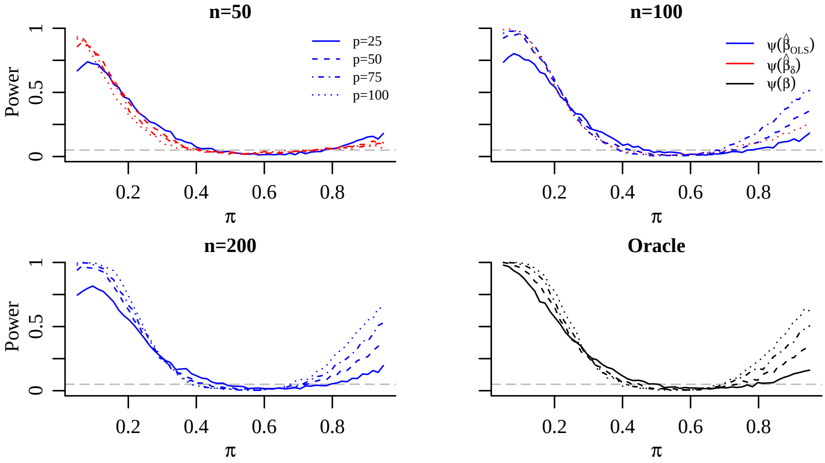
Test Performance: Exponential Power β



Test Performance: Spike-and-Slab $\boldsymbol{\beta}$

Laplace prior often used when $\boldsymbol{\beta}$ may be **sparse**

$$p\left(\beta_j \neq 0\right) = \pi,\ \beta_j | \beta_j \neq 0 \sim \text{normal}\left(0, \tau^2/\pi\right),\ \kappa = 3\left(\frac{1-\pi}{\pi}\right)$$



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Marjanovic and Solo (2014); Polson et al. (2014)

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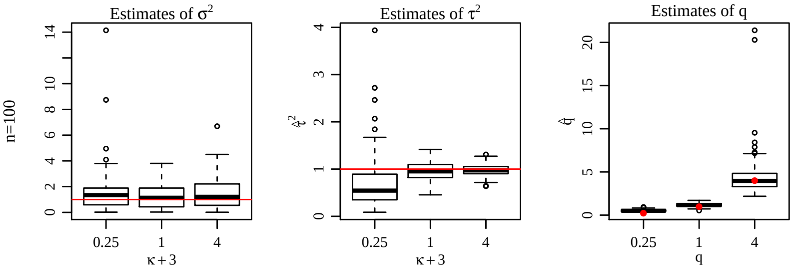
$$\min_{\tau^2,\sigma^2} \log\left(\left|\boldsymbol{X}\boldsymbol{X}^\top\tau^2 + \boldsymbol{I}_n\sigma^2\right|\right) + \mathrm{tr}\left(\boldsymbol{y}\boldsymbol{y}^\top\left(\boldsymbol{X}\boldsymbol{X}^\top\tau^2 + \boldsymbol{I}_n\sigma^2\right)^{-1}\right)$$

Adaptive Simulations: Exponential Power β

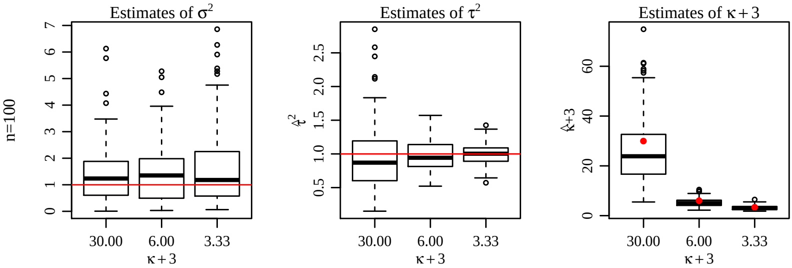


Adaptive Simulations: Exponential Power $\boldsymbol{\beta}$

p	100		
q	1/4	1	4
Null Hypothesis Rejected	83	5	69
MSE APMe $\leq$ MSE LPMe	**97**	**97**	**95**
MSE APMe $\leq$ MSE EPMe	**85**	**85**	50
MSE APMe $\leq$ MSE BB	**100**	**100**	49
MSE APMe $\leq$ MSE CV	**52**	**52**	**63**

Polson et al. (2014); Friedman et al. (2010)

Adaptive Simulations: Spike-and-slab $\boldsymbol{\beta}$



Adaptive Simulations: Spike-and-slab $\boldsymbol{\beta}$

p	100		
π	0.1	0.5	0.9
Pow. ADMo $\geq$ Pow. LPMo	**68**	**98**	**100**
Pow. ADMo $\geq$ Pow. EPMo	**100**	42	31
Pow. ADMo $\geq$ Pow. CV	**84**	**97**	**95**
FDR ADMo $\leq$ FDR LPMo	**100**	**100**	**75**
FDR ADMo $\leq$ FDR EPMo	**93**	**70**	**97**
FDR ADMo $\leq$ FDR CV	**70**	9	24
CS ADMo $\geq$ CS LPMo	**100**	**100**	**98**
CS ADMo $\geq$ CS EPMo	**93**	**70**	31
CS ADMo $\geq$ CS CV	**70**	16	**94**

Polson et al. (2014); Friedman et al. (2010)

Applications: Canonical Examples

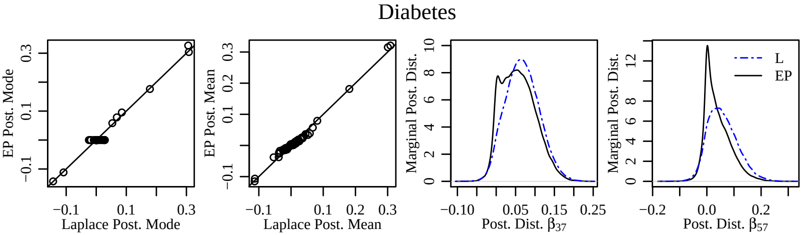
Dataset	n	p	$\psi_{\delta,0.025}$	$\psi_{\delta,0.975}$	$\psi(\hat{\boldsymbol{\beta}}_\delta)$
Diabetes	422	64	2.31	7.68	10.36
Boston Housing	506	104	1.97	7.61	6.57
Motif	287	195	2.87	10.35	5.77
Glucose	68	72	2.31	7.06	7.99

Efron et al. (2004); Park and Casella (2008); Polson et al. (2014); Bühlmann and van de Geer (2011); Priami and Morine (2015)

Application: Diabetes Data with Interactions

- $\boldsymbol{y}$ is a measure of diabetes progression for $n = 442$
- $p = 64$ main effects and interactions for 10 covariates

	Par. Ests.			Mode Sparsity	
Dataset	$\hat{\sigma}^2$	$\hat{\tau}^2$	$\hat{q}$	L	EP
Diabetes	0.4720	0.0071	0.5505	50.0%	87.5%

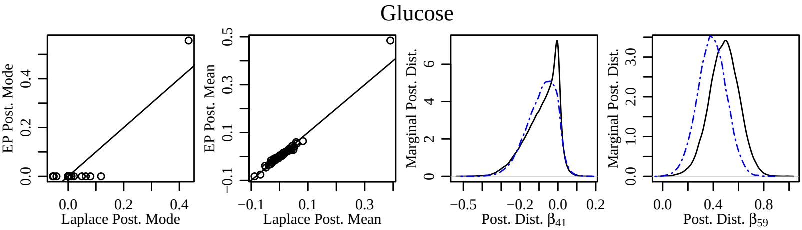


Efron et al. (2004)

Application: Glucose Data

- $\boldsymbol{y}$ is blood glucose concentration $\boldsymbol{y}$ for $n = 68$ subjects
- $p = 72$ health indicators, e.g. metabolite measurements

	Par. Ests.			Mode Sparsity	
Dataset	$\hat{\sigma}^2$	$\hat{\tau}^2$	$\hat{q}$	L	EP
Glucose	0.4754	0.0070	0.5939	83.3%	98.6%



Priami and Morine (2015)

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“Improved Pathwise Coordinate Descent for Power…”

Problem

Mazumder et al. (2011)

“Improved Pathwise Coordinate Descent for Power…”

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Solving $\min_{\boldsymbol{\beta}} \frac{1}{2\sigma^2} \left|\left|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}\right|\right|_2^2 + \lambda \left|\left|\boldsymbol{\beta}\right|\right|_q^q$ is hard

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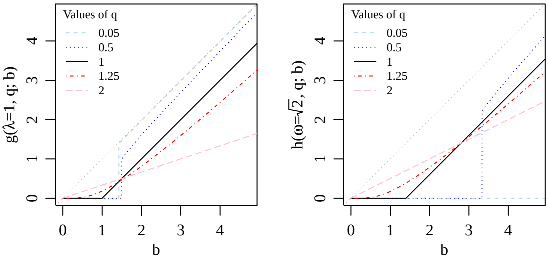

Solution

Mazumder et al. (2011)

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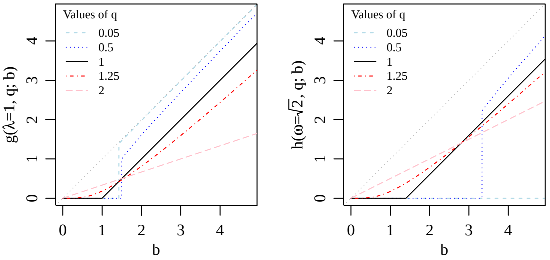


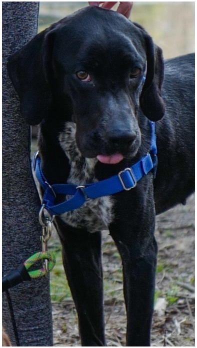
Solution

- ☺ Reparametrization to $\frac{1}{2} ||\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\beta}||_2^2 + \left(\frac{\omega^{2-q}}{q}\right) ||\boldsymbol{\beta}||_q^q$

Mazumder et al. (2011)

Acknowledgements

- My advisor Peter Hoff
- NSF Grants DGE-1256082, DMS-1505136, & DMS-2113079
- My canine assistants



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Details for Estimating $\hat{q}$

$$\left.\begin{aligned} \mathbb{E}[\bar{\beta}^{(2)}_{\delta,\boldsymbol{X}}] &= \alpha_{\delta,\boldsymbol{X}}\tau^2 \\ \mathbb{E}[\bar{\beta}^{(4)}_{\delta,\boldsymbol{X}}] &= \gamma_{\delta,\boldsymbol{X}}\left(\kappa+3\right)\tau^4 + \omega_{\delta,\boldsymbol{X}}\tau^4 \end{aligned}\right\} \to \widehat{\kappa+3} = \left(\frac{\alpha^2_{\delta,\boldsymbol{X}}}{\gamma_{\delta,\boldsymbol{X}}}\right)\left(\hat{\psi}_{\delta,\boldsymbol{X}} - \frac{\omega_{\delta,\boldsymbol{X}}}{\alpha^2_{\delta,\boldsymbol{X}}}\right)$$

$$\begin{aligned} \alpha_{\delta,\boldsymbol{X}} &= \operatorname{tr}\left(\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}\boldsymbol{D}\boldsymbol{X}'\boldsymbol{X}\right)/p \\ \gamma_{\delta,\boldsymbol{X}} &= \sum_{j=1}^{p}\left(\boldsymbol{D}\boldsymbol{X}'\boldsymbol{X}\right)^4_{jj}/p \\ \omega_{\delta,\boldsymbol{X}} &= 3\left(\sum_{j=1}^{p}\left(\boldsymbol{X}'\boldsymbol{X}\boldsymbol{D}\boldsymbol{D}\boldsymbol{X}'\boldsymbol{X}\right)^2_{jj}/p - \gamma_{\delta,\boldsymbol{X}}\right) \end{aligned}$$