Estimation of Possibly Non-Stationary Long Memory Processes via Adaptive Overdifferencing

Maryclare Griffin

University of Massachusetts Amherst
Department of Mathematics and Statistics

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Joint work with David S. Matteson, Gennady Samorodnitsky (Cornell)

Hello and congratulations, Ruey!

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Some Advances in Non-linear and Adaptive Modelling in Time-series

GEORGE C. TIAO AND RUEY S. TSAY
University of Chicago, IL, U.S.A.

Bandwidth selection for kernel regression with long-range dependent errors

By BONNIE K. RAY
Center for Applied Mathemat
nology, Newark, New Jersey (
e-mail: borayx@m.njit.edu
AND RUEY S. TSAY

True or Spurious Long Memory? A New Test

Arek Ohanissian, Jeffrey R. Russell, and Ruey S. Tsay
Graduate School of Business, University of Chicago, Chicago, IL 60637 (*jeffrey.russell@chicagogsb.edu*)

It is well known that long memory characteristics observed in data can be generated by nonstationary structural-break or slow regime switching models. We propose a statistical test to distinguish between true long memory and spurious long memory based on invariance of the long memory parameter for temporal aggregates of the process under the null of true long memory. Geweke Porter-Hudak estimates of the long memory parameter obtained from different temporal aggregates of the underlying time series are shown to be asymptotically jointly normal, leading to a test statistic that is constructed as the quadratic form of a demeaned vector of the estimates. The result is a test statistic that is very simple to implement. Simulations show the test to have good size and power properties for the classic alternatives to true long memory that have been suggested in the literature. The asymptotic distribution of the test statistic is also valid for a stochastic volatility with Gaussian long memory model. The test is applied to foreign exchange rate data. Based on all the models considered in this article, we conclude that the long memory property in exchange rate volatility is generated by a true long memory process.

KEY WORDS: Regime switching; Structural change; Temporal aggregation.

BAYESIAN METHODS FOR CHANGE-POINT DETECTION IN LONG-RANGE DEPENDENT PROCESSES

By Bonnie K. Ray and Ruey S. Tsay

Iterative Bandwidth Estimation for Nonparametric Regression with Long-range Dependent Errors

Bonnie K. Ray
Ruey S. Tsay

ABSTRACT We discuss the problem of bandwidth selection for a kernel
or when the errors are long-range dependent. The
dth selection method is investigated and modified
ory in the errors. We compare the mean average-
nd estimates using the bandwidth obtained from
to that obtained assuming short-range dependent
ve apply the modified method to estimate a trend

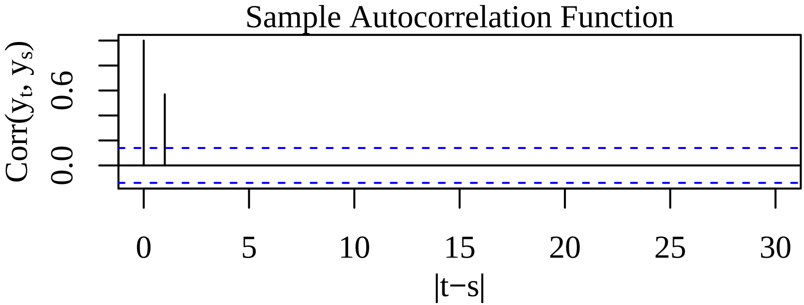
Long-range Dependence in Daily Stock Volatilities

Bonnie K. Ray
Department of Mathematical Sciences and Center for Applied Math and Statistics, New Jersey Institute of Technology, Newark, NJ 07102 (*borayx@m.njit.edu*)

Ruey S. Tsay
Graduate School of Business, University of Chicago, Chicago, IL 60637 (*ruey.tsay@gsb.uchicago.edu*)

Long Memory

Corr (y_t, y_s) decreases very slowly as $|t-s|$ increases



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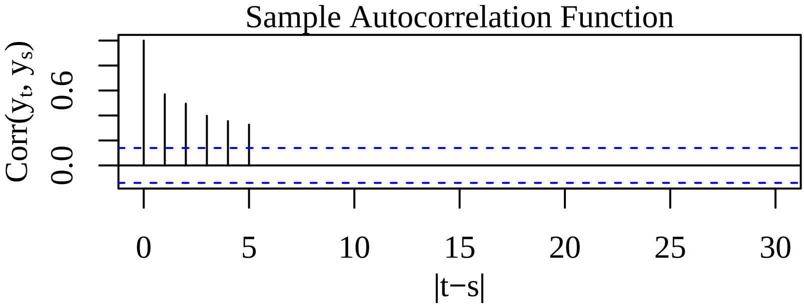

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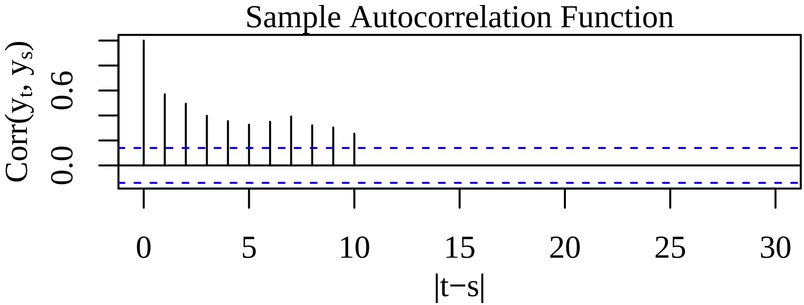

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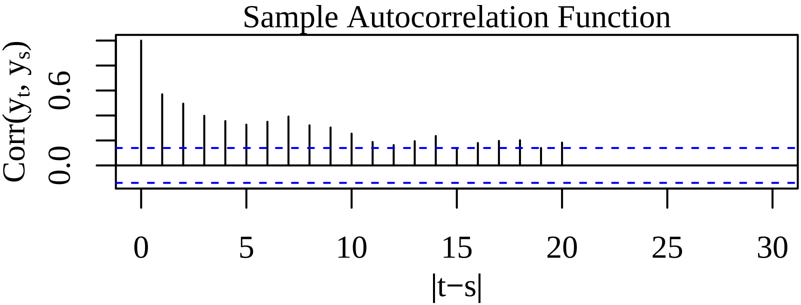

AR and Long Memory

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$$(y_t - \mu_t) = \sum_{j=1}^{p} \phi_j \left(y_{t-j} - \mu_{t-j}\right) + z_t$$

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In practice, long memory requires $p \to \infty$ as $n \to \infty$

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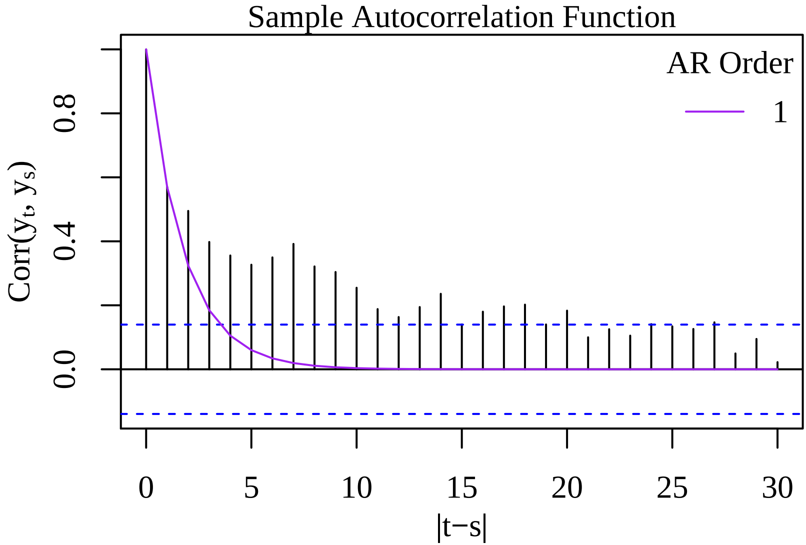

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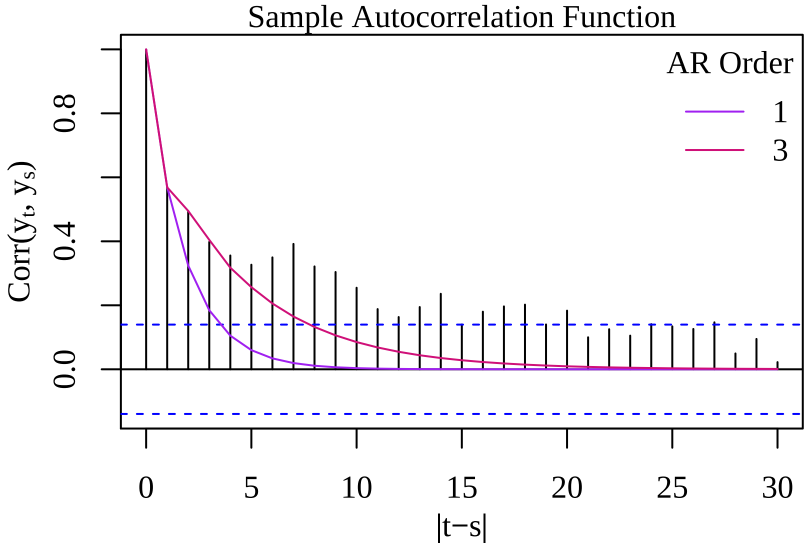

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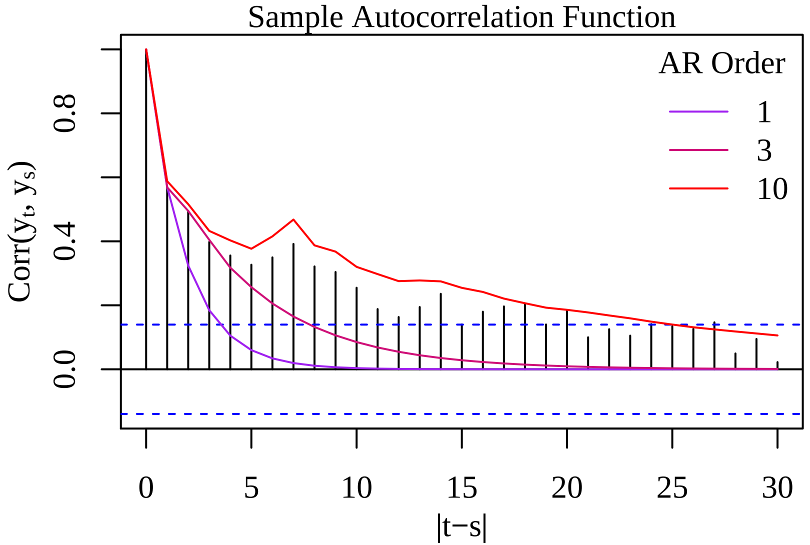

Fractional Integration (FI)

$$\overbrace{(y_t - \mu_t) = \sum_{j=1}^{\infty} \binom{d}{j} (-1)^{j-1} \left(y_{t-j} - \mu_{t-j}\right) + z_t}^{\text{FI}(d)}$$

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Unique values of $d \le -0.5$ give unique likelihood

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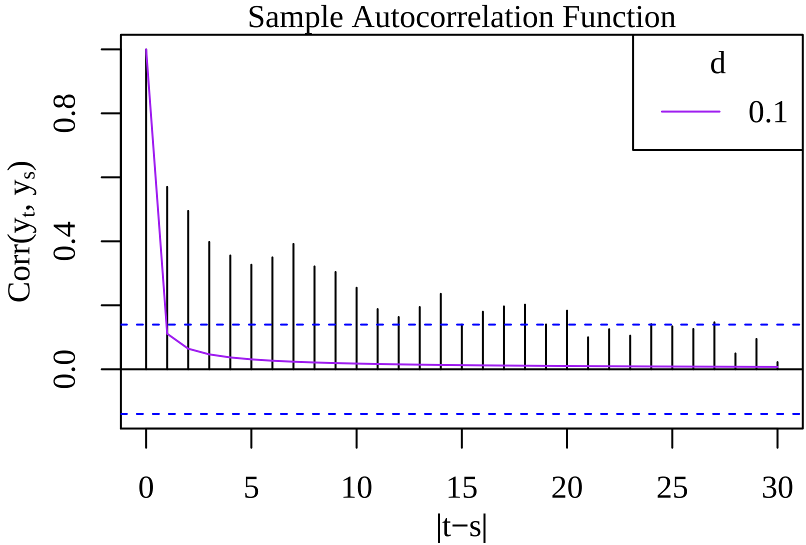

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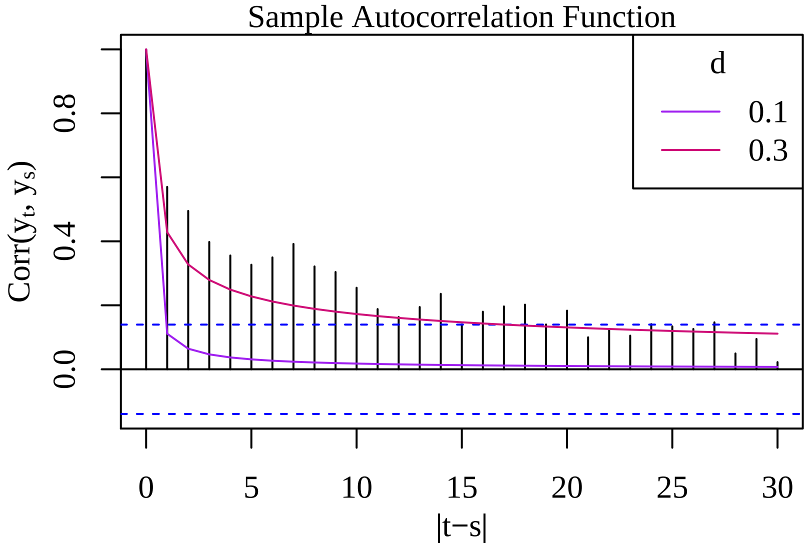

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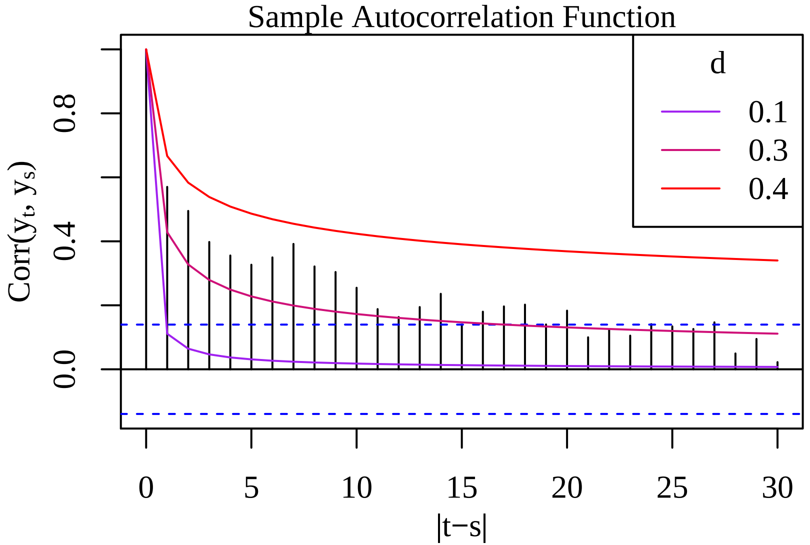

Autoregressive FI Moving Average (ARFIMA) Model

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(Content applies to ARFIMA(p,d,q) as well)

Estimation of ARFIMA(p, d, q) Processes

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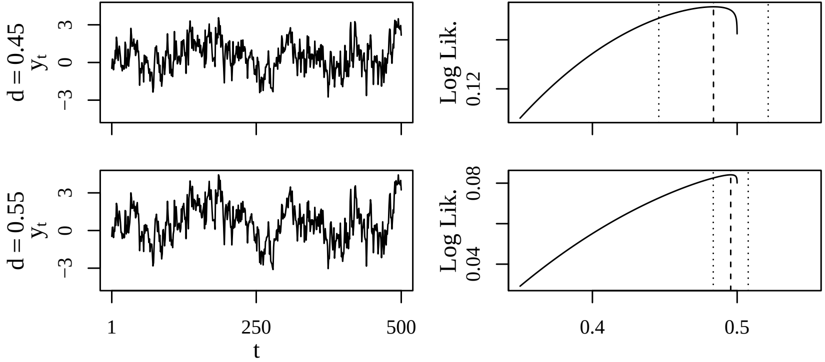
Why do we care about assumptions on d?



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$$(1-B)^{2}(y_t-\mu_t)=y_t-2y_{t-1}+y_{t-2}-(\mu_t-2\mu_{t-1}+\mu_{t-2})$$

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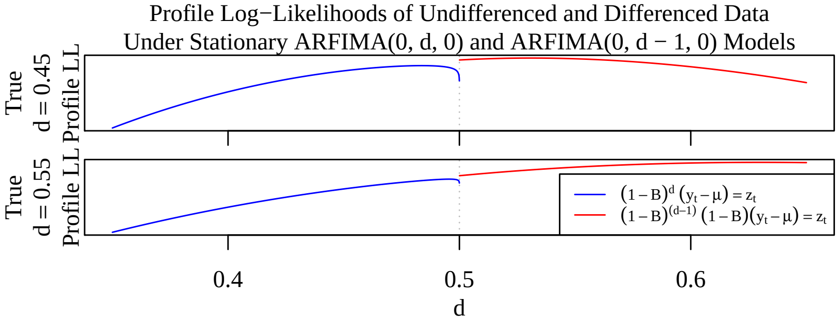
Beran (1995); Velasco and Robinson (2000); Mayoral (2007)

What's wrong with this procedure?



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- **Solution:** Fix data in advance to get stationarity for $d < \bar{d}$

Key Idea

Odaki (1993); Doornik and Ooms (2003)

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$(1-B)^m(y_t - \mu_t)$ stationary, constrained ARFIMA$(0, d-m+r, r)$

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This gives a continuous likelihood in d!

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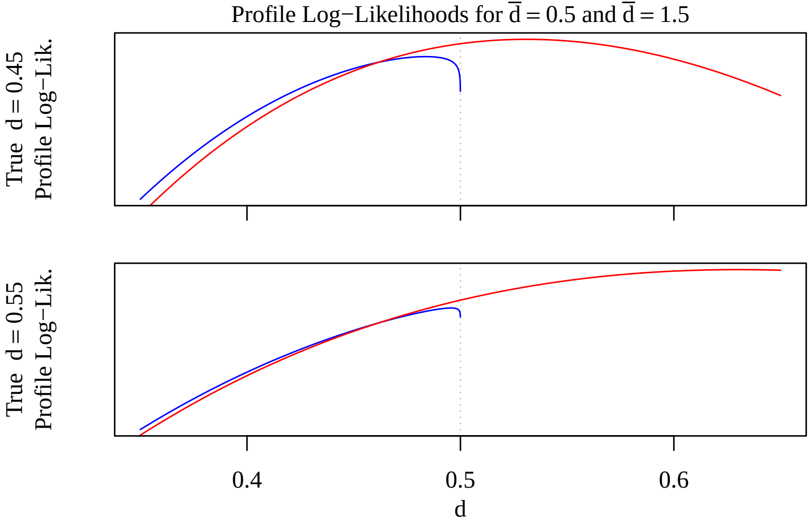

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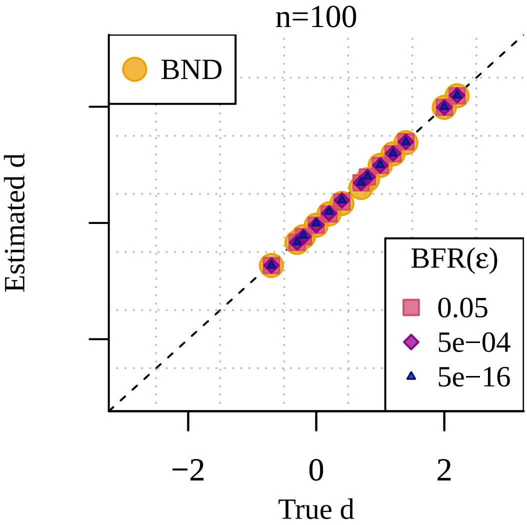
When $\epsilon < 0.5$, we'll call estimator $\mathbf{BFR}(\epsilon)$

Adaptive Procedure Simulations



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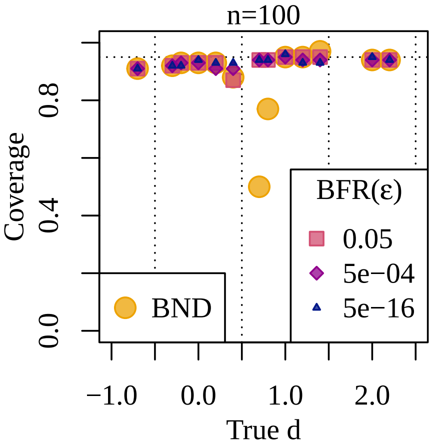

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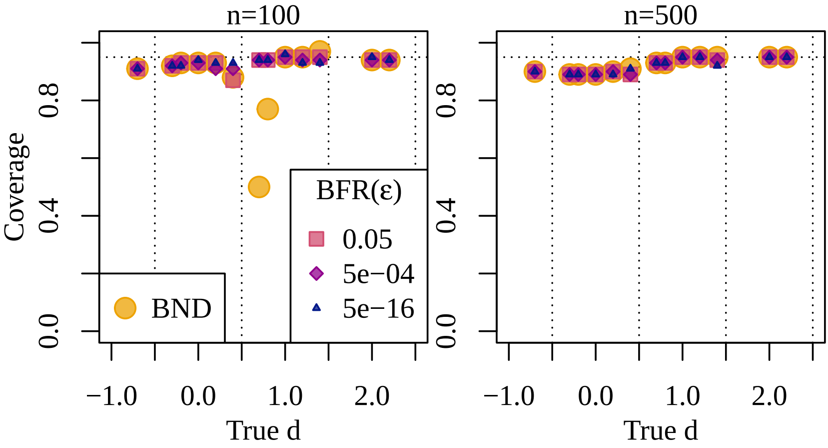

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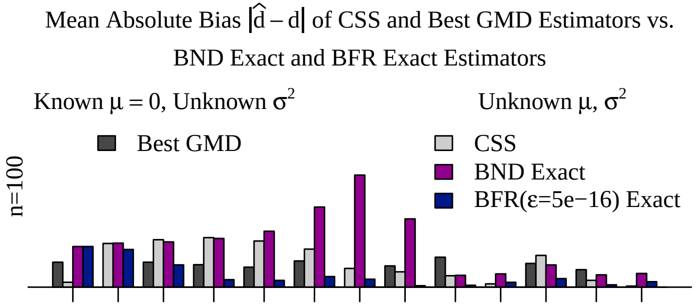

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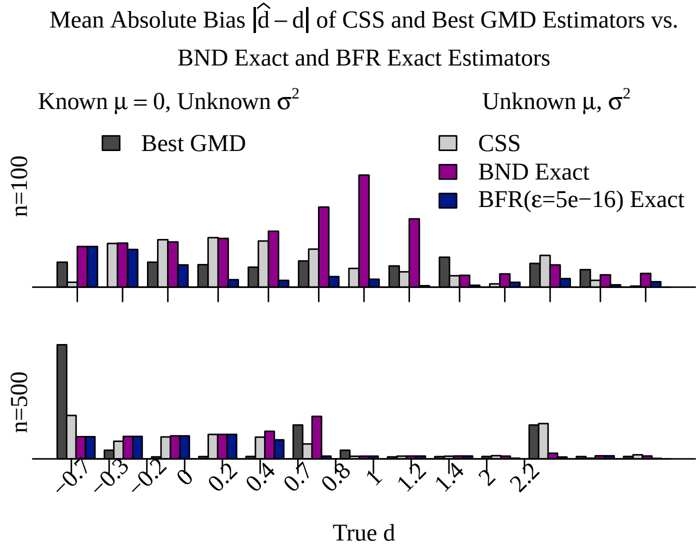
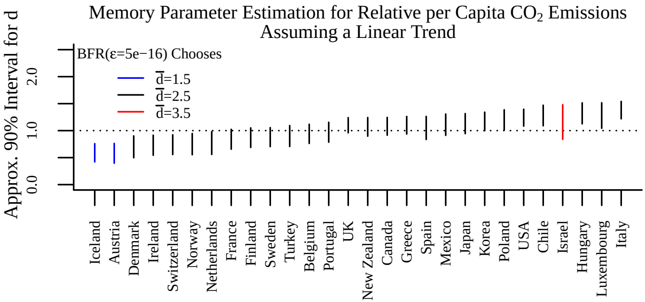


Application to CO^2 Emission Convergence

$$(1-B)^{d_c}\left(y_{tc}-\mu_c-\beta_c t\right)=z_{tc}, \quad z_{tc} \overset{i.i.d.}{\sim} \text{normal}\left(0,\sigma_c^2\right),$$



Barassi et al. (2018)

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