Multivariate Time Series Concepts

Again, suppose we observe $Y = [y_1, \dots, y_r]$

We can always decompose $Y = M_Y + W$

random errors

fixed mean

Each column of Y comprised of n equally spaced observations ordered in time

Characterized by mean function $E[y_{ij}] = m_{ij}$ and covariance function $\gamma_{ij}(s,t) = \text{Cov}[y_{si}, y_{tj}]$

when $i = j$, $\gamma_{ii}(s,t)$ is the autocovariance function of y_i

when $i \neq j$, we call $\gamma_{ij}(s,t)$ the cross-covariance function of y_i and y_j

Assume joint stationarity to simplify things in the multivariate setting

* Second moments of y_{ti} are finite for all t and i, $\mathbb{E}[y_{ti}^2] < \infty$

* The mean function is constant for each time series and doesn't depend on time

$$m_{ti} = m_i$$

* The autocovariance function $\gamma_{ii}(s,t)$ depends on s and t only through their absolute difference $\gamma_{ii}(s,t)$ can be written as $\gamma_{ii}(|s-t|)$

* The crosscovariance function $\gamma_{ij}(s,t)$ depends on s and t only through their absolute difference $\gamma_{ij}(s,t)$ can be written as $\gamma_{ij}(s-t)$

Suppose $r = 2$, so we observed two processes
Then we have:

* m_1, mean of first process
* m_2, mean of second process
* $\gamma_{11}(h)$ autocovariance function of the first process
* $\gamma_{22}(h)$ autocovariance function of the second process
* $\gamma_{12}(h)$
* $\gamma_{21}(h)$

$\gamma_{12}(h)$, $\gamma_{21}(h)$ } cross-covariance functions

$$\text{cov}(Y_{ti}, Y_{(t+h)j}) = \gamma_{ij}(h)$$

Sample Autocovariances & Cross-Covariances

sample mean function: $\hat{m}_i = \bar{y}_i = \frac{1}{n}\sum_{t=1}^{n} y_{ti}$

sample autocovariance function:

$$\hat{\gamma}_{ii}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(y_{t+h} - \hat{m}_i)(y_t - \hat{m}_i)$$

$$\hat{\gamma}_{ii}(-h) = \hat{\gamma}_{ii}(h)$$

sample autocorrelation function is $\hat{\rho}_{ii}(h) = \frac{\hat{\gamma}_{ii}(h)}{\hat{\gamma}_{ii}(0)}$

sample cross-covariance function

future values of time series i and past vals of ts j

$$\hat{\gamma}_{ij}(h) = \frac{1}{n}\sum_{t=1}^{n-h}(y_{t+h,i} - \hat{m}_i)(y_{tj} - \hat{m}_j)$$

$$\hat{\gamma}_{ij}(-h) = \hat{\gamma}_{ji}(h)$$

sample cross-correlation function is $\hat{\rho}_{ij}(h) = \frac{\hat{\gamma}_{ij}(h)}{\sqrt{\hat{\gamma}_{ii}(0)\hat{\gamma}_{jj}(0)}}$

If $y_{ti} = w_{ti}$, where $w_{ti} \overset{iid}{\sim} N(0, \sigma_{ii})$ **or**

$y_{tj} = w_{tj}$, where $w_{tj} \overset{iid}{\sim} N(0, \sigma_{jj})$ for

$t = 1, \dots, n$, <u>then</u>

$\hat{\rho}_{ij}(h) \overset{d}{\approx} v/\sqrt{n}$, $v \sim N(0,1)$ for $h = 1, \dots, H$

where H is fixed and arbitrary

This allows us to test nulls of the form $\rho_{ij}(h) = 0$, and also $\gamma_{ij}(h) = 0$ for any pair of stationary time series

AR(1) models for multivariate time series

If $r=1$ ...

$$(y_t - \mu) = \phi_1 (y_{t-1} - \mu) + w_t \,, \quad w_t \overset{iid}{\sim} N(0, \sigma_w^2)$$

$$\underset{r\times 1}{(\underset{\sim}{y}_t} - \underset{r\times 1}{\underset{\sim}{m}}) = \underset{r\times r}{\underset{\sim}{\Phi}_1} (\underset{r\times 1}{\underset{\sim}{y}_{t-1}} - \underset{\sim}{m}) + \underset{\sim}{w}_t \,, \quad \underset{\sim}{w}_t \overset{iid}{\sim} N(0, \Sigma_w)$$

If $r=2$

$$\underset{\sim}{\Phi} = \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \qquad \Sigma_w = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

AR(p) Model for multivariate time series

$$(\underset{\sim}{y}_t - \underset{\sim}{m}) = \sum_{j=1}^{p} \underset{\sim}{\Phi}_j \underset{\sim}{y}_{t-j} + \underset{\sim}{w}_t \,, \quad \underset{\sim}{w}_t \overset{iid}{\sim} N(0, \Sigma_w)$$

AR(p) Model for multivariate time series

$$(\underset{\sim}{y}_t - \underset{\sim}{\mu}) = \sum_{j=1}^{p} \underline{\Phi}_j \underset{\sim}{y}_{t-j} + \underset{\sim}{w}_t, \quad \underset{\sim}{w}_t \overset{iid}{\sim} N(0, \Sigma_w)$$

stationary when the roots of the matrix polynomial $\underline{\Phi}(B) = \left(I - \sum_{j=1}^{p} \underline{\Phi}_j B^j\right)$ roots outside the unit circle, i.e.

define companion matrix to be

$$\underline{\Phi}^* = \begin{bmatrix} 0 & I & 0 & 0 & \cdots & 0 \\ 0 & 0 & I & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I \\ \underline{\Phi}_p & \underline{\Phi}_{p-1} & \underline{\Phi}_{p-2} & \underline{\Phi}_{p-3} & \cdots & \underline{\Phi}_1 \end{bmatrix}_{rp \times rp}$$

stationary when eigenvalues of $\underline{\Phi}^*$ are all less than one in absolute value

$$(\underset{\sim}{y}_t - \underset{\sim}{m}) = \sum_{j=1}^{p} \underset{\sim}{\Phi}_j \underset{\sim}{y}_{t-j} + \underset{\sim}{w}_t, \quad \underset{\sim}{w}_t \overset{iid}{\sim} N(0, \Sigma_w)$$

Estimation in practice . . .

* ML is not practical most of the time
* method-of-moments estimates using Yule-Walker equations
  - will produce estimates of $\Phi_1, \ldots, \Phi_p$ that correspond to a jointly stationary process
* use regression, condition on first p observations

estimate $\Phi_1, \ldots, \Phi_p$ by minimizing

$$\sum_{i=p+1}^{n} \| y_t - \underset{\sim}{m} - \sum_{j=1}^{p} \Phi_j x_{t-j} \|_2^2$$

(this is just a general linear model)

Multivariate (vector) ARMA Model

$$(\underset{\sim}{y}_t - \underset{\sim}{m}) = \sum_{j=1}^{p} \underset{\sim}{\Phi}_j (\underset{\sim}{y}_{t-j} - \underset{\sim}{m}) + \sum_{k=1}^{q} \Theta_k \underset{\sim}{w}_{t-k} + \underset{\sim}{w}_t$$

- Θ_k will be $r \times r$ $\quad \underset{\sim}{w}_t \overset{iid}{\sim} N(0, \Sigma_w)$

If $\underset{\sim}{\Phi}_1 = \ldots = \underset{\sim}{\Phi}_p = 0$, we get a multivariate MA process

* Really computationally challenging to estimate
* weird identifiability issues come up

moral of VARMA(p,q) models... don't use them

$$y_t = \underset{\sim}{a}' \underset{\sim}{x}_t + u_t, \quad u_t \overset{iid}{\sim} N(0, \sigma_u^2)$$

$$\underset{\sim}{x}_t = \Gamma \underset{\sim}{x}_{t-1} + \underset{\sim}{v}_t, \quad \underset{\sim}{v}_t \overset{iid}{\sim} N(0, \underset{\sim}{\sigma}_v^2)$$