Generalizing the linear state-space model

Stochastic regression model

$$y_t = \gamma + t \times x_t + v_t \qquad v_t \overset{iid}{\sim} N(0, \sigma_v^2)$$

$$x_t = \phi x_{t-1} + w_t \qquad w_t \overset{iid}{\sim} N(0, \sigma_w^2)$$

$$x_1 = \mu$$

Spectral Models

Consider a stationary time series y_t

Could think of y_t as a noisy realization of several periodic trends

$$(*) \quad y_t = \sum_{k=1}^{r} v_k \cos(2\pi \omega_k t) + u_k \sin(2\pi \omega_k t)$$

$$u_k, v_k \overset{iid}{\sim} N(0, \sigma_k^2)$$

spectral representation theorem tells us that any stationary process y_t can be represented using a spectral representation $(*)$

There's some number r, variances $\sigma_1^2, \dots, \sigma_r^2$ and frequencies $\omega_1, \dots, \omega_r$ such that $(*)$ represents y_t

Any stationary ARMA(p,q) model with parameters

$\mu, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q, \sigma_w^2$

can be written as a spectral model for some r,

$\sigma_1^2, \dots, \sigma_r^2$, and $\omega_1, \dots, \omega_r$

Analagous to our statements about how any ARMA(p,q) model has a possibly infinite AR representation or MA representation

$$(*) \quad y_t = \sum_{k=1}^{r} v_k \cos(2\pi \omega_k t) + u_k \sin(2\pi \omega_k t)$$

$$u_k, v_k \overset{iid}{\sim} N(0, \sigma_k^2)$$

Popular/useful in part because $\gamma_y(h)$ and the parameters $\sigma_1^2, \ldots, \sigma_r^2$, $\omega_1, \ldots, \omega_r$ have a "nice" relationship

$$\mathbb{E}[y_t] = \sum_{k=1}^{r} \mathbb{E}[v_k] \cos(2\pi \omega_k t) + \mathbb{E}[u_k] \sin(2\pi \omega_k t) = 0$$

$$\gamma_y(0) = \mathbb{E}[y_t^2] = \mathbb{E}\left[\left(\sum_{k=1}^{r} v_k \cos(2\pi \omega_k t) + u_k \sin(2\pi \omega_k t)\right)^2\right]$$

$$= \sum_{k=1}^{r} \mathbb{E}[v_k^2] \cos(2\pi \omega_k t)^2 + \mathbb{E}[u_k^2] \sin(2\pi \omega_k t)^2$$

$$= \sum_{k=1}^{r} \sigma_k^2 \left[\cos^2(2\pi \omega_k t) + \sin^2(2\pi \omega_k t)\right]$$

$$= \sum_{k=1}^{r} \sigma_k^2$$

For any $h > 0$,

$$\gamma_Y(h) = \sum_{k=1}^{r} \sigma_k^2 \cos(2\pi \omega_k h)$$

Basic idea behind the spectral representation theorem..

If you had the values of an autocovariance function $\gamma_Y(0), \ldots, \gamma_Y(h), \ldots$

then because we know how to relate the autocovariance fn. to $\sigma_1^2, \ldots, \sigma_r^2, \omega_1, \ldots, \omega_r$, we can always find a value of r and values of $\omega_1, \ldots, \omega_r, \sigma_1^2, \ldots, \sigma_r^2$ to reconstruct it

Some More Notation

$\sigma_1^2, \ldots, \sigma_r^2$ as values of the spectral density function of y_t at frequencies $\omega_1, \ldots, \omega_r$

$$\gamma_y(h) = \int_{-0.5}^{0.5} \exp\{2\pi \omega h i\} f(\omega)\, d\omega$$

$$\gamma_y(h) = \sum_{k=1}^{r} \sigma_k^2 \cos(2\pi \omega_k h)$$