Why Spectral methods??

Spectral methods let reparametrize stationary models
Instead of a time series model in terms of the autocovariance function $\gamma_Y(h)$, parametrize a time series model in terms of the spectral density, $f(\omega)$

* $f(\omega)$ and $\gamma_Y(h)$ are closely related!

$$\gamma_Y(h) = \int_{-0.5}^{0.5} \exp\{2\pi i \omega h\} \underbrace{f(\omega)\,d\omega}_{dF(\omega)}$$

$dF(\omega)$ ↑ spectral distribution function

$$\gamma_Y(h) \approx \sum_{k=1}^{r} \sigma_k^2 \cos(2\pi \omega_k h)$$

Basically $f(\omega)$ tells us how our time series can be decomposed into arbitrarily many oscillations

WHY ? ?

* <u>computation</u>

$p(\underset{\sim}{y})$ for $\underset{\sim}{y} \sim N(\underset{\sim}{0}, \Sigma)$ ← stationary, $\sigma_{ij} = \gamma_y(|i-j|)$

can be difficult to evaluate

- Σ is difficult to invert
- Methods for avoiding inverting Σ usually require a loop over $n \Rightarrow$ may scale poorly for long time series

periodogram is a transformation of our time series $\underset{\sim}{y}$,

$$y_t = \begin{cases} a_0 + \sum_{k=1}^{(n-1)/2} a_k \cos(2\pi (k/n) t) + b_k \sin(2\pi (k/n) t) & \text{if } n \text{ odd} \\ a_0 + \left(\sum_{k=1}^{n/2-1} a_k \cos(2\pi (k/n) t) + b_k \sin(2\pi (k/n) t)\right) + a_{n/2} \cos(\pi t) & \text{if } n \text{ even} \end{cases}$$

scaled periodogram $P(k/n) = \begin{cases} a_k^2 & \text{if } k=0 \text{ or } k = n/2 \\ a_k^2 + b_k^2 & \text{otherwise} \end{cases}$

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periodogram $I(k/n) = \left(\frac{n}{4}\right) P(k/n)$

we've gone from the observed data with n observations to a transformation of the data with $\frac{n-1}{2} + 1$ obs. if n is odd, $\frac{n}{2}$ observations if n is even

* Periodogram values <u>are</u> approximately independent with a distribution that depends nicely on $f(\omega)$

* Periodogram values $(I(k/n))$ are approximately independent with a distribution that depends nicely on $f(\omega)$

$$\frac{2I(k/n)}{f(k/n)} \xrightarrow{d} \chi^2_2 \text{ random variables}$$

that are independent as $n \to \infty$

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$$p(\text{periodogram}) \approx \prod_{k=1}^{\frac{n-1}{2} \text{ or } \frac{n}{2}} p(I(k/n))$$

↑ depends on parameters of our model through $f(\omega)$

* ARMA(p, q) models have nice spectral densities $f(\omega)$ that are simple, closed form functions of $\sigma^2, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$

WHY ??

* <u>flexibility</u>

Much easier to define a stationary spectral density as opposed to a stationary autocovariance function or stationary ARMA(p,q)

$\Longrightarrow$ Means that estimating $f(\omega)$ is a less constrained problem

If you know the spectral density of two contemporaneously observed time series, $y_1, \dots, y_n$ and $z_1, \dots, z_n$, then it is easy to figure out the spectral density of any linear combination

$$x_t = c_1 y_t + c_2 z_t, \quad c_1, c_2 \text{ are constants}$$

$f(\omega)$: spectral density of y_t

$g(\omega)$: spectral density of z_t

then spectral density of x_t is

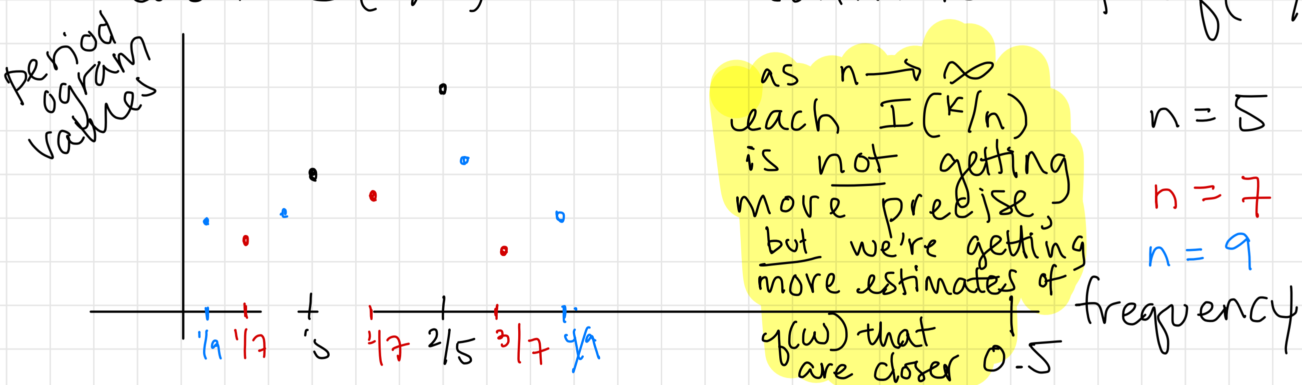
$$c_1^2 f(\omega) + c_2^2 g(\omega)$$

When we use spectral methods, we start with y, and transform to get the periodogram

if n is odd, then periodogram is $\left(I(1/n), I(2/n), \ldots, I\left(\frac{n-1}{2}/n\right)\right)$

if n is even, then periodogram is $\left(I(1/n), I(2/n), \ldots I\left(\frac{n}{2}/n\right)\right)$

each $I(k/n)$ is an estimator of $f(k/n)$



as $n \to \infty$ each $I(k/n)$ is not getting more precise, but we're getting more estimates of $f(\omega)$ that are closer