More Complex State Space Models

observation equation $\quad \underset{1\times 1}{y_t} = \underset{1\times s}{\underset{\sim}{a}'} \underset{\sim}{x}_t + u_t, \quad u_t \overset{iid}{\sim} N(0, \sigma_u^2)$

$\underset{\sim}{x}_t$: $s\times 1$ vector of a multivariate process

state equation $\quad \underset{\sim}{x}_t = \underset{\substack{s\times s \\ \text{matrix}}}{\Gamma} \underset{\sim}{x}_{t-1} + \underset{\sim}{v}_t, \quad v_t \overset{iid}{\sim} N(0, \Sigma_v)$

VAR(1) model for $\underset{\sim}{x}_t$

initial conditions $\quad \underset{\sim}{x}_1$ or $\underset{\sim}{x}_0 = \underset{\sim}{\mu}$

Even though we can write a model like this down, it is tough to say if the parameters are identified, don't usually use in practice

Multivariate state space models have the form

observation equation $\underset{\sim}{y}_{t\,r\times 1} = A_{r\times s}\,\underset{\sim}{x}_{t\,s\times 1} + \underset{\sim}{u}_{t\,r\times 1}, \quad \underset{\sim}{u}_t \overset{iid}{\sim} N(0, \Sigma_u)$

state equation $\underset{\sim}{x}_{t\,s\times 1} = \Gamma_{s\times s}\,\underset{\sim}{x}_{t-1\,s\times 1} + \underset{\sim}{v}_{t\,s\times 1}, \quad \underset{\sim}{v}_t \overset{iid}{\sim} N(0, \Sigma_v)$

Vector autoregressive process of order $\underline{1}$

Parameters:

initial conditions, $\underset{\sim}{x}_1$ or $\underset{\sim}{x}_0 = \underset{\sim}{\mu}$

$r\times s + \frac{r(r-1)}{2} + s + s\times s + \frac{s(s-1)}{2}$

* Could have $s=r$ $\Rightarrow$ our state process is also multivariate with same dim. as observed

* We could have $s<r$, e.g. $s=\underline{1}$

much simpler, lower dimensional processes than in VARMA if $s=1$

Multivariate state space models have the form

observation equation $\underset{\sim}{y}_{t\ r\times 1} = \underset{\sim}{a}_{r\times 1} \times x_t + \underset{\sim}{u}_{t\ r\times 1}, \quad \underset{\sim}{u}_t \overset{iid}{\sim} N(0, \Sigma_u)$

state equation $x_t = \gamma_1 x_{t-1} + \nu_t \qquad \nu_t \overset{iid}{\sim} N(0, \sigma_\nu^2)$

autoregressive process of order 1

initial condition x_1 or $x_0 = \mu$

$p(\underset{\sim}{y} \mid A, \Sigma_u, \Gamma, \Sigma_\nu, \underset{\sim}{\mu})$ may not be convex.

$x_1 \sim N\left(0, \frac{\sigma_v^2}{1-\gamma_{11}^2}\right)$ initial condition

ARMA in state space framework...

First, we'll consider AR(1)

$u_t, u_{t-1} \overset{iid}{\sim} N(0, \sigma_u^2)$

$\sigma_u^2 = 0$

$$\begin{pmatrix} y_t \\ y_{t-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} u_t \\ u_{t-1} \end{pmatrix}$$

$\begin{pmatrix} v_t \\ v_{t-1} \end{pmatrix} \sim N(0, \Sigma_v)$, $\sigma_{v22} = \sigma_{12} = \sigma_{21} = 0$, σ_{v11}^2 freely varying

$$\begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} \gamma_{11} & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_{t-1} \\ x_{t-2} \end{pmatrix} + \begin{pmatrix} v_t \\ v_{t-1} \end{pmatrix}$$

$$\begin{pmatrix} x_t \\ x_{t-1} \end{pmatrix} = \begin{pmatrix} \gamma_{11} x_{t-1}, \\ x_{t-1} \end{pmatrix} + \begin{pmatrix} v_t \\ v_{t-1} \end{pmatrix},$$

Biggest ommission:

GARCH, ARCH, Stochastic Volatility