Basic Time Series Concepts

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The material in this set of notes was initially based on Sections 1.1-1.6 of Robert Shumway and David Stouffer's Time Series Analysis and Its Applications: With R Examples

Suppose we observe an $n \times 1$ vector $\boldsymbol{y} = (y_1, \dots, y_n) = \boldsymbol{\mu}_y + \boldsymbol{\epsilon}$, where $\boldsymbol{\mu}_y$ is a fixed but unknown mean, $\boldsymbol{\epsilon}$ are random errors and elements of $\boldsymbol{y}$ are ordered in time. We will refer to $\boldsymbol{y}$ as a **time series**, although the sequence of elements can also be called a **stochastic process**.

The joint distribution function of $\boldsymbol{y}$ is

$$F\left(c_1, \dots, c_n\right) = P\left(y_1 \leq c_1, \dots, y_n \leq c_n\right).$$

Often, this will be difficult to write out and work with, so it does not provide a useful means of characterizing a time series $\boldsymbol{y}$. Instead, we often characterize a time series $\boldsymbol{y}$ via its:

- **Mean Function:** $\mu_{y,t} = \mathbb{E}[y_t] = \int_{-\infty}^{\infty} y f_t(y) dy$, where $f_t(y)$ is the marginal density of y_t having integrated out all other elements of $\boldsymbol{y}$.
- **Autocovariance Function:** $\gamma_y(s,t) = \mathbb{E}[(y_s - \mu_{y,s})(y_t - \mu_{y,t})]$ for all s and t.
  - When $s = t$, gives the variance $\gamma_y(s,s) = \mathbb{V}[y_s]$.
- **Autocorrelation Function:** $\rho_y(s,t) = \gamma_y(s,t)/\sqrt{\gamma_y(s,s)\gamma_y(t,t)}$ for all s and t.

Without further assumptions, this is still an unwieldy way to characterize a time series because the mean function depends on t and the autocovariance and autocorrelation functions depend on both s and t. To simplify things further, we often assume that the time series is either:

- **Strongly Stationary:** The distribution of any subset of k elements of $(y_{t_1}, \ldots, y_{t_k})$ is exactly the same as the distribution of the shifted set of k elements $(y_{t_1+h}, \ldots, y_{t_k+h})$.
  - The mean function $\mu_{y,t}$ does not depend on t: $\mu_{y,t} = \mathbb{E}[y_t] = \mathbb{E}[y_{t+h}] = \mu_{y,t+h}$.
  - The autocovariance function $\gamma_y(s,t)$ depends on s and t only through their absolute difference $h = |s-t|$:

$$\begin{aligned}\gamma(s+h,s) =& \mathbb{E}[(y_{s+h}-\mu_y)(y_s-\mu_y)] \\ =& \mathbb{E}[(y_h-\mu_y)(y_0-\mu_y)] \\ =& \gamma(h,0).\end{aligned}$$

- **Weakly Stationary:** The second moments of y_t are finite, i.e. $\mathbb{E}\left[y_t^2\right] < \infty$ for all t, the mean function is constant and does not depend on time, $\mu_{y,t} = \mu_y$, and the autocovariance function $\gamma_y(s,t)$ depends on s and t only through their absolute difference $h = |s-t|$.

Note that although strong stationarity with finite second moments $\mathbb{E}\left[y_t^2\right] < \infty$ implies weak stationarity, the reverse does not hold. Strong stationarity is usually too strict to be a reasonable assumption, so from here on out we will call a time series **stationary** if it is **weakly stationary**.

When a time series is stationary, its autocovariance and autocorrelation functions can be written as functions of a single variable h. For this reason, we will drop the second arguments of the autocovariance and autocorrelation functions when a time series is stationary, writing $\gamma_y\left(h\right) = \gamma_y\left(h,0\right)$ and $\rho_y\left(h\right) = \rho_y\left(h,0\right)$.

When we observe a time series $\boldsymbol{y}$, we do not know the mean, autocovariance, or autocorrelation functions a priori - we need to estimate them. When $\boldsymbol{y}$ is stationary we can compute:

- The **sample mean** function:

$$\hat{\mu}_y = \bar{y} = \sum_{t=1}^{n} y_t / n. \tag{1}$$

- The **sample autocovariance function**:

$$\hat{\gamma}_y(h) = \frac{1}{n} \sum_{t=1}^{n-h} (y_{t+h} - \hat{\mu}_y)(y_t - \hat{\mu}_y), \tag{2}$$

  with $\hat{\gamma}_y(-h) = \hat{\gamma}_y(h)$ for $h = 0, 1, \ldots, n-1$.

  - We divide by n and not $n-h$ to ensure that the sample variance of a sum of elements of $\boldsymbol{y}$ computed from the $n \times n$ sample autocovariance matrix with entries $\hat{\gamma}(i-j)$ will always be nonnegative.
  - This is a biased estimate of $\gamma_y(h)$.

- The **sample autocorrelation function**:

$$\hat{\rho}_y(h) = \frac{\hat{\gamma}_y(h)}{\hat{\gamma}_y(0)}. \tag{3}$$

When we examine a sample autocorrelation function, it is natural to ask how different our estimates of the sample autocorrelation are from what we would might expect if $\boldsymbol{y}$ were a **white noise** time series with no autocorrelation at all, i.e. if $\rho_y(h) = 0$ for all $h \neq 0$. We can get a handle on this using the following result:

> If $\boldsymbol{y} = \boldsymbol{\mu}_y + \boldsymbol{\epsilon}$ where $\boldsymbol{\mu}_y = \boldsymbol{0}$ and $\epsilon_i \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2_\epsilon)$ for $i = 1, \ldots n$, then $\hat{\rho}_y(h) \approx v/\sqrt{n}$, for $h = 1, \ldots H$, where $v \sim \mathcal{N}(0,1)$ and H is fixed but arbitrary.

This result allows us to perform an approximate test of the null hypothesis that $\rho_y(h) = 0$ for any $h > 1$.