FIIAN. Exam, H00345091, 10/05/2021

Q1) Consider
$$\frac{\delta u}{\delta t} = D_1 \frac{\delta^2 u}{\delta x^2} + f(u,v)$$
$$\frac{\delta v}{\delta t} = D_2 \frac{\delta^2 v}{\delta x^2} + g(u,v)$$

a) Conditions for diffusion driven instability are

$$f_u + g_v < 0$$
$$f_u g_v - f_v g_u > 0$$
$$D_1 g_v + D_2 f_u > 0$$
$$(D_1 g_v + D_2 f_u)^2 > 4 D_1 D_2 (f_u g_v - f_v g_u)$$

where $f_u = \frac{\delta f}{\delta u}\Big|_{(u_s, v_s)}$ and $g_u = \frac{\delta g}{\delta u}\Big|_{(u_s, v_s)}$

$f_v = \frac{\delta f}{\delta v}\Big|_{(u_s, v_s)}$ and $g_v = \frac{\delta g}{\delta v}\Big|_{(u_s, v_s)}$

b) Consider $f(u,v) = a^2 u^2 v^2 - uv^5$, $g(u,v) = uv - v^2$
$D_1 = 1$, $D_2 = \delta$

Steady states are given by $f(u,v) = 0$ and $g(u,v) = 0$

$a^2u^2v^2 - uv^5 = 0$ if $u = 0$, $v = 0$ OR $a^2 u - v^3 = 0$

$(u-v)$

$uv - v^2 = 0$ if $v = 0$ OR $u - v = 0 \Rightarrow u = v$

$a^2 - u^2$

thus $a^2 u - u^3 = 0 \Rightarrow u = 0$ OR $u = a$

thus $(u_s, v_s) = (a, a)$

Also
$f_u = 2a^2 u_s v_s^2 - v_s^5 = 2a^5 - a^5 = a^5$
$f_v = 2a^2 u_s^2 v_s - 5u_s v_s^4 = 2a^5 - 5a^5 = -3a^5$
$g_u = v_s = a$
$g_v = u_s - 2u_s = a - 2a = -a$

Then conditions:

DDI 1: $f_u + g_v = a^5 - a < 0 \Rightarrow a^4 < 1$

DDI 2: $f_u g_v - f_v g_u = -a^6 + 3a^6 = 2a^6 > 0$, which is true

DDI 3: $D_1 g_v + D_2 f_u = -a + \delta a^5 > 0 \Rightarrow a > \sqrt[4]{1/\delta}$ OR equivalently $a^4 \delta > 1$

$\cdot > 1$

$> \frac{1}{\delta}$

DDI 4: $(\delta a^5 - a)^2 > 4\delta(2a^6)$
$(\delta a^4 - 1)^2 > 8\delta a^4$
$\delta^2 a^8 - 2\delta a^4 + 1 > 8\delta a^4$
$\delta^2 a^8 - 10\delta a^4 + 1 > 0$

setting $A = a^4$, then $\delta^2 A^2 - 10\delta A + 1 = 0$ gives
$$A = \frac{10\delta \pm \sqrt{100\delta^2 - 4\delta^2}}{2\delta^2}$$

$$a^4 = 5/\delta \pm \frac{\sqrt{96}}{2\delta} = \frac{5 \pm 2\sqrt{6}}{\delta}$$

thus $a^4\delta < 5 - 2\sqrt{6}$ OR $a^4\delta > 5 + 2\sqrt{6}$

from DDI 3 we know $a^4\delta > 1$, thus $a^4\delta < 5 - 2\sqrt{6}$ is not possible

thus conditions on a and δ for diffusion driven instability are

$a < 1$ and $a^4\delta > 5 + 2\sqrt{6}$

where $a, \delta > 0$ constants

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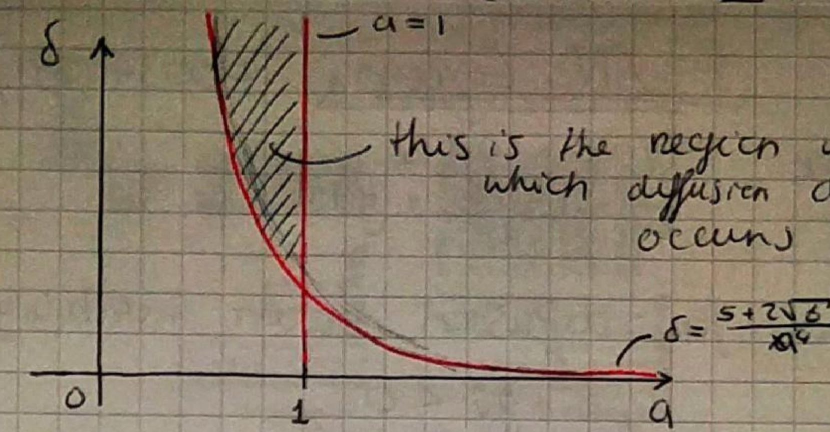
c)



d) $f_u = a^5 > 0$ and $g_v = -a < 0$
thus u is the activator and v the inhibitor

$$\begin{pmatrix} f_u & f_v \\ g_u & g_v \end{pmatrix} = \begin{pmatrix} a^5 & -3a^5 \\ a & -a \end{pmatrix} = \begin{pmatrix} + & - \\ + & - \end{pmatrix}$$
, thus it is a pure activation-inhibitor system

e) wavenumbers k are defined by

$$Q(k^2) = D_1 D_2 k^4 - (D_2 f_u + D_1 g_v)k^2 + f_u g_v - f_v g_u$$
$$= \delta k^4 - (\delta a^5 - a)k^2 + 2a^6$$

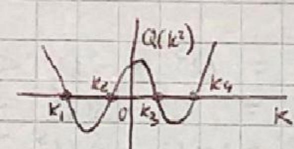
$Q(k^2) < 0$ ~~are on~~ give unstable wavenumbers, thus at critical wavenumber when DDI occurs $Q(k^2) = 0$

$$\delta k^4 - a(\delta a^4 - 1)k^2 + 2a^6 = 0$$

$$k^2 = \frac{a(\delta a^4 - 1) \pm \sqrt{a^2(\delta a^4 - 1)^2 - 4\delta 2a^6}}{2\delta}$$

$$= a\left(\frac{\delta a^4 - 1}{2\delta} \pm \frac{\sqrt{(\delta a^4 - 1)^2 - 8\delta a^4}}{2\delta}\right)$$

$Q(k^2)$ has shape



as $Q(k^2) < 0$ for $k_1 < k < k_2$ and $k_3 < k < k_4$

where k_1, k_2, k_3, k_4 are critical wave numbers at which diffusion driven instability could occur defined by

$$k_{1,2,3,4} = \pm\sqrt{a\left(\frac{\delta a^4 - 1}{2\delta} \pm \frac{\sqrt{(\delta a^4 - 1)^2 - 8\delta a^4}}{2\delta}\right)}$$

2) Consider $\frac{dH}{dt} = 1 - H(t) - cH(t - T)$

a) at steady state $H(t) = H(t-T) = $ constant, ~~thus~~
thus $1 - H - cH = 0 \Rightarrow H_s = 1/(c+1)$

then we consider perturbations about steady state
$H(t) = H_s + h(t)$ and $H(t-T) = H_s + h(t-T)$
where $h(t)$ is a small perturbations
substitution in the ~~ots~~ DDE ~~gives~~ and Taylor's expansion about (H_s, H_s) gives

$$\frac{dh(t)}{dt} = -h(t) - ch(t-T)$$

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To find stability look for solutions of the form $n(t) = ke^{\lambda t}$ where k is a constant and λ are eigenvalues
substitution gives

$$k\lambda e^{\lambda t} = -ke^{\lambda t} - cke^{\lambda(t-T)}$$

$$\lambda = -1 - ce^{-\lambda T}$$

$$\lambda + 1 = -ce^{-\lambda T}$$

where the steady state is stable if all solution for λ have $Re(\lambda) < 0$

b) for $T=0$, $\lambda = -1-c < 0$, thus the steady state is stable

c) Let $c = \sqrt{2}$, then $\lambda + 1 = -\sqrt{2}e^{-\lambda T}$

If we assume for ~~fo~~ some $T > T_c$ the steady state is unstable, then at critical point T_c, we must have $Re(\lambda) = 0 \Rightarrow \lambda = i\omega$
substitution gives

$$i\omega + 1 = -\sqrt{2}e^{-i\omega T_c}$$

real part : $1 = -\sqrt{2}\cos(\omega T_c) \Rightarrow \omega T_c = \cos^{-1}(-1/\sqrt{2})$

imaginary part : $\omega = \sqrt{2}\sin(\omega T_c)$
$= \sqrt{2}\sqrt{1-\cos^2(\omega T_c)}$

$\frac{\sqrt{2}}{2}$

as $\cos(\omega T_c) = -1/\sqrt{2} \rightarrow \omega = \sqrt{2}\sqrt{1 - 1/2} = 1$

thus $T_c = \cos^{-1}(-1/\sqrt{2})$

~~Substitution of T_c and ω into $\lambda + 1 = \sqrt{2}e^{-\lambda T}$ gives~~
thus there is a critical value T_c for which $Re(\lambda) = 0$. As the steady state is stable at $T=0$ ~~there is thus~~ the steady state must be unstable for $T > T_c$

Q3) Consider $E + S \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C \xrightarrow{k_2} 2P + E$

a) using the law of mass action, we can deduce

$$\frac{de}{dt} = -k_1 es + k_{-1}c + k_2 c$$

$$\frac{ds}{dt} = -k_1 es + k_{-1}c$$

$$\frac{dc}{dt} = k_1 es - k_{-1}c - k_2 c$$

$$\frac{dp}{dt} = 2k_2 c$$

Consider $s(0) = s_0$, $e(0) = c(0) = e_0$, $p(0) = 0$

b) Because concentrations e, s, c are all independent of p, we can leave dp/dt out of the system of equations without consequences.
Secondly $\frac{de}{dt} + \frac{dc}{dt} = 0$, thus $e(t) + c(t) =$ ~~co~~ constant
as $e(t) + c(t)$ is a constant $e(t) + c(t) = e(0) + c(0) = 2e_0$
thus we can replace e with $2e_0 - c$ and also leave de/dt out of the system

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this gives $\frac{ds}{dt} = \cancel{k_1 e s + k_{-1} c} - k_1 s(2e_0 - c) + k_{-1} c$

$$\frac{dc}{dt} = k_1 s(2e_0 - c) - (k_{-1} + k_2) c$$

c) Given $s^* = s/s_0$, $c^* = c/2e_0$, $t^* = k_1 e_0 t$, $\varepsilon = e_0/s_0$
$\alpha = k_{-1}/(k_1 s_0)$ $\beta = (k_{-1} + k_2)/(k_1 s_0)$

substitution of $s = s_0 s^*$, $c = 2e_0 c^*$ and $t = t^*/(k_1 e_0)$ gives

$$\frac{ds^*}{dt^*} s_0 k_1 e_0 = -k_1 s_0 s^*(2e_0 - 2e_0 c^*) + k_{-1} 2e_0 c^*$$

$$\frac{ds^*}{dt^*} = -2s^*(1-c^*) + 2 \underbrace{\frac{k_{-1}}{k_1 s_0}}_{=\alpha} c^*$$

thus $\frac{ds^*}{dt^*} = -2s^*(1-c^*) + 2\alpha c^*$

and $\frac{dc^*}{dt^*} 2e_0 k_1 e_0 = k_1 s_0 s^*(2e_0 - 2e_0 c^*) + (k_{-1} + k_2) 2e_0 c^*$

$$\frac{dc^*}{dt^*} \underbrace{\frac{e_0}{s_0}}_{=\varepsilon} = s^*(1-c^*) - \underbrace{\frac{k_{-1}+k_2}{k_1 s_0}}_{=\beta} c^*$$

thus $\varepsilon \frac{dc^*}{dt^*} = s^*(1-c^*) - \beta c^*$

~~after~~ dropping stars gives $\frac{ds}{dt} = -2s(1-c) + 2\alpha c$
$\varepsilon \frac{dc}{dt} = s(1-c) - \beta c$ as required

for initial conditions $s^*(0) = s(0)/s_0 = 1$
and $c^*(0) = c(0)/2e_0 = 1/2$

d) We can assume that ~~$e_0 >> s$~~ $s_0 >> e_0$, meaning that the initial amount of substrate greatly exceeds the ~~init~~ amount of enzyme, which is true for the majority of reactions. as $s_0 >> e_0 \Rightarrow e_0/s_0 << 1$ and thus $\varepsilon = e_0/s_0 << 1$ is a reasonable assumption

neglecting ε-terms gives $\varepsilon \frac{dc}{dt} = 0$ and thus

$$s(1-c) - \beta c = 0 \Rightarrow c(t) = \frac{s(t)}{\beta + s(t)}$$

substitution gives $\frac{ds}{dt} = -2s\left(1 - \frac{s}{\beta+s}\right) + 2\alpha \frac{s}{\beta+s}$

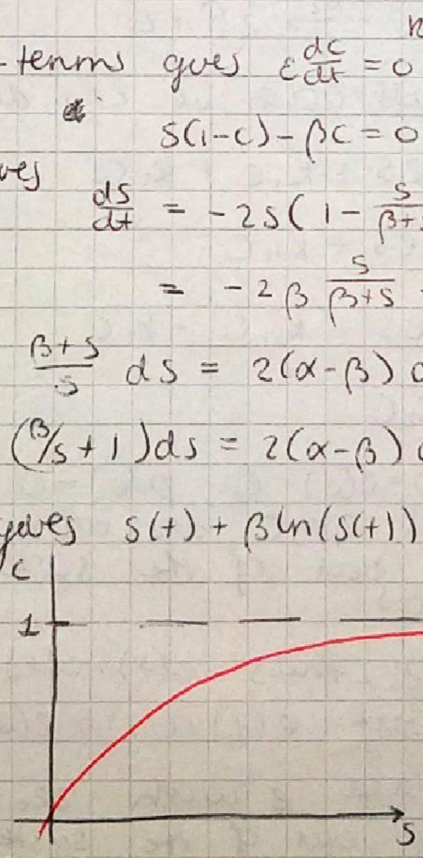
$$= -2\beta \frac{s}{\beta+s} + 2\alpha \frac{s}{\beta+s} = \frac{2s(\alpha-\beta)}{\beta+s}$$

$$\frac{\beta+s}{s} ds = 2(\alpha-\beta) dt$$

$$(\beta/s + 1) ds = 2(\alpha-\beta) dt$$

integration gives $s(t) + \beta \ln(s(t)) = A_1 + 2(\alpha-\beta)t$

$c(t) = \frac{s(t)}{\beta+s(t)}$ gives



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initial conditions ~~gate~~ are $S(0)=1$, $C(0)=1/2$

$$\cancel{C(0) = S(0)+\beta\ln(S(0)) = A+2(\alpha-\beta)t=0}$$

$$C(0)=\frac{S(0)}{\beta+S(0)} = \frac{1}{\beta+1} = 1/2$$

$$\beta = 1$$

thus initial conditions can only both be satisfied if $\beta=1$

e) assume $\beta>1$. Let $\tau = t/\varepsilon$, then substitution gives

$$\frac{dS}{d\tau} = \varepsilon(-2S(1-C)+2\alpha C)$$

$$\frac{dC}{d\tau} = S(1-C)-\beta C$$

again neglecting ε-terms gives $\frac{dS}{d\tau}=0 \Rightarrow S(\tau)=S(0)=1$

thus $\frac{dC}{d\tau} = 1-C-\beta C \Rightarrow C(\tau)=\frac{A}{1+\beta}e^{-(1+\beta)\tau}+1/(1+\beta)$

with initial condition $C(0)=\frac{A_2+1}{1+\beta}=1/2 \Rightarrow A_2 = \frac{\beta-1}{2}$

thus inner solutions $S(t)=1$ and $C(t)=\frac{\beta-1}{2(1+\beta)}e^{-\frac{(1+\beta)}{\varepsilon}t}+1/(1+\beta)$

f) matching inner and outer solutions gives

$$\lim_{t\to0} \text{outer solutions} = \lim_{t\to\infty} \text{inner solutions}$$

for inner solution $S(t)=1$, thus $\lim_{t\to\infty} S(t)=1$

using this as initial condition for outer solution gives

$$A_1 = 1$$

also we note that $\lim_{t\to\infty} A_2 e^{-(1+\beta)t/\varepsilon} + 1/(1+\beta) = \frac{1}{\beta+1}$ and

$$\lim_{t\to0}\frac{S(t)}{\beta+S(t)} = \frac{S(0)}{\beta+S(0)}=\frac{1}{\beta+1}$$

thus solutions are already matched for C

then composite solutions: $S_{composite} = S_{inner}+S_{outer}-S_{match}$
$= 1 + S_{outer} - 1$
$= S_{outer}$

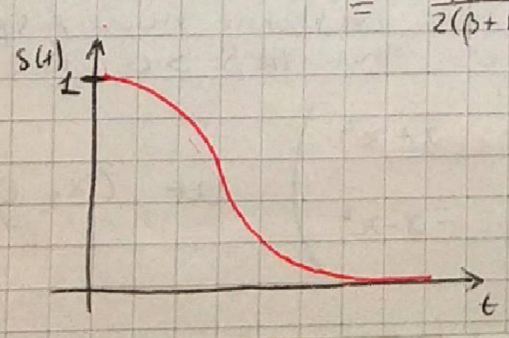
thus composite solution $S(t)$ is determined by

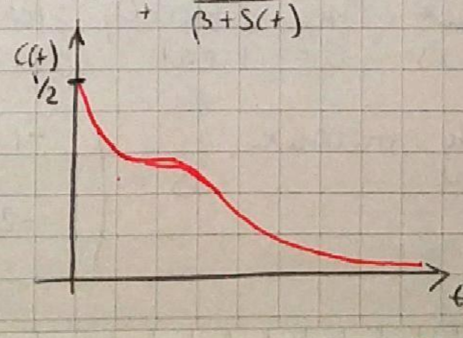
$$S(t)+\beta\ln(S(t)) = 1+2(\alpha-\beta)t$$

and $C(t) = \frac{\beta-1}{2(\beta+1)}e^{-\frac{\beta+1}{\varepsilon}t} + \frac{1}{\beta+1}+\frac{S(t)}{\beta+S(t)} - \frac{1}{\beta+1}$

$$= \frac{\beta-1}{2(\beta+1)}e^{-(\beta+1)t/\varepsilon}+\frac{S(t)}{\beta+S(t)}$$

g)





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Q4) Consider $\frac{dx}{dt} = -x + \alpha y + x^2 y$
$\frac{dy}{dt} = \beta - \alpha y - x^2 y$

First, we find steady states: $-x + \alpha y + x^2 y = 0$ and
$\beta - \alpha y - x^2 y = 0$

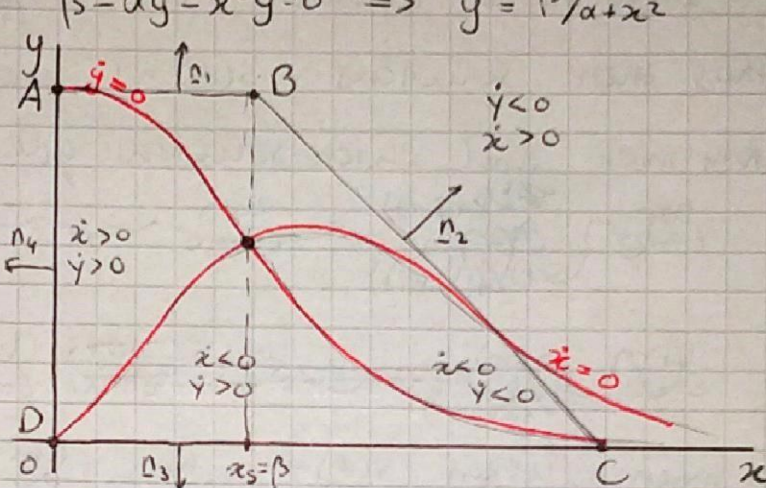
$$y(\alpha + x^2) = \beta \Rightarrow y = \beta / \alpha + x^2$$

$$-x + \alpha \frac{\beta}{\alpha + x^2} + x^2 \frac{\beta}{\alpha + x^2} = 0 \Rightarrow x = \beta$$
$$y = \beta/(\alpha + \beta^2)$$

Thus steady state $(x_s, y_s) = \left(\beta, \frac{\beta}{\alpha + \beta^2}\right)$

then we need to find a ~~confined set~~ closed curve C for which
$(\dot{x}, \dot{y}) \cdot \underline{n} < 0$ where $\underline{n}$ is the outer unit norm

to do this, we draw phase plane:
nullclines $-x + \alpha y + x^2 y = 0 \Rightarrow y = x/\alpha + x^2$
$\beta - \alpha y - x^2 y = 0 \Rightarrow y = \beta/\alpha + x^2$



Let $A = (0, \beta/\alpha)$, $B = (\beta, \beta/_{\alpha+\beta^2})$, $C = (\ , 0)$, and $D = (0,0)$
then ABCD defines a closed curve
with $\underline{n}_1 = (0,1)$, $\underline{n}_2 = \frac{1}{\sqrt{2}}(1,1)$, $\underline{n}_3 = (0,-1)$, $\underline{n}_4 = (-1,0)$
~~then along AB: $\underline{n}_1 \cdot (\dot{x}, \dot{y})$~~

then along AB: $\underline{n}_1 \cdot (\dot{x}, \dot{y}) = \dot{y} < 0$
along BC: $\underline{n}_2 \cdot (\dot{x}, \dot{y}) = \frac{1}{\sqrt{2}}(\dot{x} + \dot{y})$
$= \frac{1}{\sqrt{2}}(\beta - x) < 0$ as $x > \beta$ along BC

along CD: $\underline{n}_3 \cdot (\dot{x}, \dot{y}) = -\dot{y} < 0$
along DA: $\underline{n}_4 \cdot (\dot{x}, \dot{y}) = -\dot{x} < 0$

Thus we have that along ABCD $(\dot{x}, \dot{y}) \cdot \underline{n} < 0$ and
thus ABCD is a confined set ~~and the stea~~
that encloses the steady state (x_s, y_s)

Then the other condition for Poincare–Bendixson theorem is
that the steady state is an unstable node / spiral,
thus we need $\det S > 0$ and $\operatorname{tr} S > 0$ for

stability matrix
$$S = \begin{pmatrix} -1 + 2xy & \alpha + x^2 \\ -2xy & -\alpha - x^2 \end{pmatrix} \text{ at } (x_s, y_s)$$

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$$S(x_s, y_s) = \begin{pmatrix} -1 + 2\frac{\beta^2}{\alpha+\beta^2} & \alpha+\beta^2 \\ -2\frac{\beta^2}{\alpha+\beta^2} & -\alpha-\beta^2 \end{pmatrix}$$

$$\operatorname{tr} S = -1 + 2\frac{\beta^2}{\alpha+\beta^2} - \alpha - \beta^2$$

$$= \frac{-\alpha-\beta^2+2\beta^2}{\alpha+\beta^2} - (\alpha+\beta^2) = \frac{\beta^2-\alpha}{\alpha+\beta^2} - (\alpha+\beta^2) > 0$$

$$\beta^2 - \alpha > (\alpha+\beta^2)^2$$

$$\det S = \left(-1 + \frac{2\beta^2}{\alpha+\beta^2}\right)\left(-(\alpha+\beta^2)\right) - (\alpha+\beta^2)\left(-\frac{2\beta^2}{\alpha+\beta^2}\right)$$

$$= \alpha+\beta^2 - 2\beta^2 + 2\beta^2 = \alpha+\beta^2$$

as $\alpha, \beta > 0$, ~~so~~ $\det S > 0$

thus the condition ~~for~~ on α and β for Poincaré-Bendixson theorem to hold is

$$\beta^2 - \alpha > (\alpha+\beta^2)^2$$

$$\beta^4 + 2\alpha\beta^2 + \alpha - \beta^2 + \alpha^2 < 0$$

$$\beta^4 + (2\alpha-1)\beta^2 + \alpha(\alpha+1) < 0$$

$$\beta^2 = \frac{1-2\alpha \pm \sqrt{(2\alpha-1)^2 - 4\alpha(\alpha+1)}}{2}$$

$$= \frac{1}{2} - \alpha \pm \frac{1}{2}\sqrt{4\alpha^2-4\alpha+1-4\alpha^2-4\alpha}$$

$$= \frac{1}{2} - \alpha \pm \frac{1}{2}\sqrt{1-8\alpha}$$

$$< \frac{1}{2} - \alpha + \frac{1}{2} < 1$$

thus $\beta < 1$ for the condition to hold

Q5] Consider $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} + \frac{k_1 u - k_2 u^2}{k_3 + u^2}$

where x in mm and t in hours
and $u(x,t)$ the density of epidermal cells

a) Let $x = \tilde{x}X$, $u = \tilde{u}U$, $t = \tilde{t}/T$, then substitution gives

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} UT = D\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} UX^2 + U\frac{k_1\tilde{u} - k_2\tilde{u}^2 U}{k_3 + \tilde{u}^2U^2}$$

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = D\frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2}\frac{X^2}{T} + \frac{k_1}{k_3 T}\frac{\tilde{u} - \frac{k_2 U}{k_1}\tilde{u}^2}{1 + U^2/k_3\,\tilde{u}^2}$$

then if $T = k_1/k_3$, $X^2 = T/D = k_1/(k_3 D)$, $U = k_1/k_2$, $\delta = \frac{U^2}{k_3} = \frac{k_1^2}{k_2^2 k_3}$

then

$$\frac{\partial \tilde{u}}{\partial \tilde{t}} = \frac{\partial^2 \tilde{u}}{\partial \tilde{x}^2} + \frac{\tilde{u} - \tilde{u}^2}{1 + \delta\tilde{u}^2}$$

thus scalings are $\tilde{x} = x\sqrt{\frac{k_1}{k_3 D}}$, $\tilde{u} = \frac{k_2}{k_1}u$, $\tilde{t} = \frac{k_1}{k_3}t$

with $\delta = \frac{k_1^2}{k_2^2 k_3}$

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b) travelling wave solution $u(x,t) = U(z)$ where $z = x - ct$ has

$$\frac{\delta u}{\delta t} = \frac{\delta U}{\delta z}\frac{\delta z}{\delta t} = -cU', \quad \frac{\delta u}{\delta x} = U', \quad \frac{\delta^2 u}{\delta x^2} = U''$$

thus then
$$-cU' = DU'' + \frac{k_1 U - k_2 U^2}{k_3 + U^2}$$

$$DU'' + cU' + \frac{k_1 U - k_2 U^2}{k_3 + U^2} = 0$$

Define $V = U'$, then we get set of equations

$$U' = V$$
$$V' = -\frac{c}{D}V - \frac{k_1 U - k_2 U^2}{k_3 + U^2}\frac{1}{D}$$

the system has steady state $(0,0)$ and $(k_1/k_2, 0)$

and has stability matrix
$$S = \begin{pmatrix} 0 & 1 \\ \frac{k_1 U^2 + 2k_2 k_3 U - k_1 k_3}{(k_3+U^2)^2}\frac{1}{D} & -c/D \end{pmatrix}$$

thus $S(0,0) = \begin{pmatrix} 0 & 1 \\ -k_1/k_3 \frac{1}{D} & -c/D \end{pmatrix}$ gives $\operatorname{tr} S = -c/D < 0$
$\det S = k_1/(k_3 D) > 0$
thus $(0,0)$ is stable

$S(k_1/k_2, 0) = \begin{pmatrix} 0 & 1 \\ \frac{k_1}{D(k_3 + k_1^2/k_2^2)} & -c/D \end{pmatrix}$ gives $\operatorname{tr} S = -c/D < 0$
$\det S = \cancel{} - k_1/D(k_3 + k_1^2/k_2^2) < 0$
thus $(k_1/k_2, 0)$ is unstable

Let $f(U) = \frac{k_1 U - k_2 U^2}{D(k_3 + U^2)}$ with $f'(U) = \frac{k_1 U^2 + 2k_2 k_3 U - k_1 k_3}{D(k_3+U^2)^2}$ and $f'(0) = \frac{k_1(k_1^2/k_2^2 + k_3)}{D(k_3 + k_1^2/k_2^2)^2} = \frac{k_1}{D(k_3 + k_1^2/k_2^2)}$

then eigenvalues at $(0,0)$ are given by

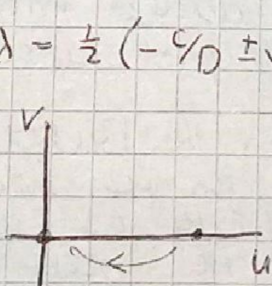
$$-\lambda(-c/D - \lambda) + f'(0) = 0 \Rightarrow \lambda^2 + c/D\,\lambda + f'(0) = 0$$

which gives
$$\lambda = \tfrac{1}{2}\left(-c/D \pm \sqrt{(c/D)^2 - 4f'(0)}\right)$$

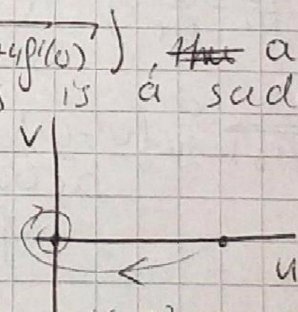
as $f'(0) > 0$:
if $(c/D)^2 \geq 4f'(0)$, eigenvalues are real and negative
thus $(0,0)$ is stable node
if $(c/D)^2 < 4f'(0)$, eigenvalues are complex with negative real part thus $(0,0)$ a stable spiral

at $(k_1/k_2, 0) \Rightarrow \lambda = \tfrac{1}{2}\left(-c/D \pm \sqrt{(c/D)^2 + 4f'(0)}\right)$, ~~thus~~ as $f'(0) > 0$ this is a saddle always

phase planes then give



$(c/D)^2 \geq 4f'(0)$



$(c/D)^2 < 4f'(0)$

thus if $(0,0)$ is a spiral, there would be negative value of U, which is not biologically realistic as u is a density, thus we must have

$$(c/D)^2 \geq 4f'(0) \Rightarrow c^2 \geq 4D\frac{k_1}{k_3 + k_1^2/k_2^2} \Rightarrow c \geq 2\sqrt{\frac{k_1 D}{k_3 + k_1^2/k_2^2}}$$

c) thus $c \geq 2\sqrt{\frac{k_1 D}{k_3 + k_1^2/k_2^2}}$

~~for if you decrease~~ k_1

therapy 1: decreasing k_1 would mean

$$\lim_{k_1 \to 0} c = \lim_{k_1 \to 0} 2\sqrt{\frac{k_1 D}{k_3 + k_1^2/k_2^2}} = 2\sqrt{\frac{0 \cdot D}{k_3}} = 0$$

thus $c \to 0$

therapy 2: decreasing k_2 would mean

$$\lim_{k_2 \to 0} c = \lim_{k_2 \to 0} 2\sqrt{\frac{k_1 D}{k_3 + k_1^2/k_2^2}} = 2\sqrt{\frac{k_1 D}{\infty}} = 0$$

thus $c \to 0$

therapy 3: decreasing k_3 would mean

$$\lim_{k_3 \to 0} c = \lim_{k_3 \to 0} 2\sqrt{\frac{k_1 D}{k_3 + k_1^2/k_2^2}} = 2\sqrt{\frac{k_1 D}{k_1^2/k_2^2}} = 2k_2\sqrt{D/k_1}$$

thus $c \to 2k_2\sqrt{D/k_1}$

thus only with therapy 3 the ~~to~~ wound healing rate would ~~not~~ be longer than 0 as $c \to 2k_2\sqrt{D/k_1}$, thus decreasing k_3 is the most effective

Q6) ~~a)~~ Consider Maxey-Riley equation:

$$m'_{drop}\frac{dv'}{dt'} = m'_{air}\frac{Du'}{Dt'} + (m'_{drop} - m'_{air})g' - 6\pi a'\mu'(v' - u')$$
$$- \frac{m'_{air}}{2}\left[\frac{dv'}{dt'} - \frac{Du'}{Dt'}\right] - 6\pi a'^2\mu'\int_0^{t'} \frac{1}{\sqrt{\pi v'(t'-s')}}\frac{d}{ds'}[v' - u']ds'$$

a) Maxey-Riley equations is simply put ~~in~~ Newton's second law of motion $ma = F$ where $m = m'_{drop}$, $a = \frac{dv'}{dt}$

and F is combined at the right hand side:

$m'_{air}\frac{Du'}{dt'}$ is the force due to the pressure gradient within undisturbed fluid flow

$(m'_{drop} - m'_{air})g'$ is the buoyancy force felt by the droplet immers in fluid

$6\pi a'\mu'(v' - u')$ is the stokes drag force on the droplet

$-\frac{m'_{air}}{2}\left[\frac{dv'}{dt'} - \frac{Du'}{Dt'}\right]$ is the added mass force which results as the droplet accelerates surrounding the fluid

$6\pi a'^2\mu'\int_0^{t'}\frac{1}{\sqrt{\pi v'(t'-s')}}\frac{d}{ds'}[v' - u']ds'$ is the Basset-Boussinesq term that acounts for the drag due to the production of vorticity as the particle accelerates from the rest

the Faxen corrections are zero if $\Delta u = 0$

for example if $u = \frac{x}{|x|^3}$ ~~where n is the number of dimensions~~ for the example in 3 dimensions

~~for 1D $u = \frac{x}{|x|} = 1$, then $\frac{\delta^2 u}{\delta x^2} = 0$~~

~~for 2D $u = \frac{x}{|x|^2}$~~ if $u = \frac{x}{|x|^3}$, then $\Delta u = 0$

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b)

for this part need $m'_{part} = m'_{drop}$, $m'_{fluid} = m'_{air}$, $\rho'_{fluid} = \rho'_{air}$, $\rho'_{part} = \rho'_{drop}$

$$(1) \quad m'_{part}\frac{dv'(t)}{dt'} = m'_{fluid}\frac{Du'}{Dt'} + (m'_{part} - m'_{fluid})g' - 6\pi a'\mu'\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]$$
$$- \frac{m'_{fluid}}{2}\left[\frac{dv'}{dt'} - \frac{D}{Dt'}\left(u' + \cancel{\frac{a'^2}{10}\nabla'^2 u'}\right)\right]$$
$$- 6\pi a'^2\mu'\int_0^{t'}\frac{1}{\sqrt{\pi\nu'(t'-s')}}\frac{d}{ds'}\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]ds'$$

divide by volume of particle $V'_{part} = \frac{4}{3}\pi a'^3$

$$\frac{m'_{part}}{V'_{part}}\frac{dv'(t)}{dt'} = \frac{m'_{fluid}}{V'_{part}}\frac{Du'}{Dt'} + \frac{m'_{part} - m'_{fluid}}{V'_{part}}g' - \frac{6\pi a'\mu'}{4/3\pi a'^3}\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]$$
$$- \frac{m'_{fluid}}{2V'_{part}}\left[\frac{dv'}{dt'} - \frac{D}{Dt'}\left(u' + \cancel{\frac{a'^2}{10}\nabla'^2 u'}\right)\right]$$
$$- \frac{6\pi a'^2\mu'}{4/3\pi a'^3}\int_0^{t'}\frac{1}{\sqrt{\pi\nu'(t'-s')}}\frac{d}{ds'}\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]ds'$$

we know $\rho'_{part} = \frac{m'_{part}}{V'_{part}}$ and $\rho'_{fluid} = \frac{m'_{fluid}}{V'_{part}}$, thus

$$\rho'_{part}\frac{dv'(t)}{dt'} = \rho'_{fluid}\frac{Du'}{Dt'} + (\rho'_{part} - \rho'_{fluid})g' - \frac{18\mu'}{4a'^2}\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]$$
$$(2) \quad - \frac{\rho'_{fluid}}{2}\left[\frac{dv'}{dt'} - \frac{D}{Dt'}\left(u' + \cancel{\frac{a'^2}{10}\nabla'^2 u'}\right)\right]$$
$$- \frac{18\mu'}{4a'^2\sqrt{\pi\nu'}}\int_0^{t'}\frac{1}{\sqrt{t'-s'}}\frac{d}{ds'}\left[v' - u' \cancel{\frac{a'^2}{6}\nabla'^2 u'}\right]ds'$$

To nondimensionalise, we use $U'v^* = v'$, $U'u^* = u'$, $L'x^* = x'$, $(L'/U')t^* = t'$, $(U'^2/L')g^* = g'$ which gives

$$\rho'_{part}\frac{U'dv^*}{L'/U'\,dt^*} = \rho'_{fluid}\frac{U'Du^*}{(L'/U')Dt^*} + (\rho'_{part} - \rho'_{fluid})\frac{U'^2}{L'}g^* - \frac{18\mu'}{4a'^2}\left[U'v^* - U'u^* \cancel{\frac{a'^2}{6}\nabla^2 u^*}\right]$$
$$- \frac{\rho'_{fluid}}{2}\left[\frac{U'dv^*}{L'/U'\,dt^*} - \frac{D}{L'/U'\,Dt^*}\left(U'u^* + \cancel{\frac{a'^2 U'}{10 L'^2}\nabla^2 u^*}\right)\right]$$
$$- \frac{18\mu'}{4a'^2\sqrt{\pi\underbrace{U'v^*}_{=\nu'}}}\int_0^{L'/U't^*}\frac{1}{\sqrt{L'/U'(t^*-s^*)}}\frac{d}{L'/U'\,ds^*}\left[U'v^* - U'u^* \cancel{\frac{a'^2}{6}\nabla^2 u^*}\right]L'/U'\,ds^*$$

thus

$$\frac{\rho'_{part}U'}{L'/U'}\frac{dv^*}{dt^*} = \frac{\rho'_{fluid}U'}{L'/U'}\frac{Du^*}{Dt^*} + (\rho'_{part} - \rho'_{fluid})\frac{U'^2}{L'}g^* - \frac{18\mu'U'}{4a'^2}\left[v^* - u^* \cancel{\frac{a'^2}{6}\nabla^2 u^*}\right]$$
$$(3) \quad - \frac{\rho'_{fluid}}{2}\left[\frac{U'}{L'/U'}\frac{dv^*}{dt^*} - \frac{U'}{L'/U'}\frac{D}{Dt^*}\left(u^* + \cancel{\frac{a'^2}{10}\nabla^2 u^*}\right)\right]$$
$$- \frac{18\mu'}{4a'\sqrt{\pi\nu'}}\int_0^{t^*}\frac{1}{\sqrt{L'/U'(t^*-s^*)}}\frac{U'}{L'/U'}\frac{d}{ds^*}\left[v^* - u^* \cancel{\frac{a'^2}{6}\nabla^2 u^*}\right]\frac{L'}{U'}ds^*$$

Rearrange: ~~and set $\Delta u = 0$~~

$$\left(\rho'_{part} + \frac{\rho'_{fluid}}{2}\right)\frac{U'}{L'/U'}\frac{dv^*}{dt^*} = \left(\rho'_{fluid} + \frac{\rho'_{fluid}}{2}\right)\frac{U'}{L'/U'}\frac{Du^*}{Dt^*} + (\rho'_{part} - \rho'_{fluid})\frac{U'}{L'/U'}g^*$$
$$- \frac{18\mu'U'}{4a'^2}(v^* - u^*) - \frac{18\mu'}{4a'^2\nu'}\frac{U'L'}{L'\sqrt{L'/U'}}\int_0^{t^*}\frac{1}{\sqrt{t^*-s^*}}\frac{d}{ds^*}(v^* - u^*)ds^*$$

Divide by $\frac{U'^2}{L'/U'}$ an by ρ'_{part} and use ~~$v' = \frac{2}{9}\frac{\rho'_{part} - \rho'_{fluid}}{\mu'}a'^2$ (Stokes Law)~~

$$4) \quad \frac{1}{2\rho'_{part}}(2\rho'_{part} + \rho'_{fluid})\frac{dv^*}{dt^*} = \frac{3\rho'_{fluid}}{2\rho'_{part}}\frac{Du^*}{Dt^*} + \frac{\rho'_{part} - \rho'_{fluid}}{\rho'_{part}}g^* - \frac{9}{2}\frac{\mu'L'}{\rho'_{part}U'a'^2}(v^* - u^*)$$

use $\boxed{\nu' = \frac{\mu'}{\rho'_{fluid}}}$

$$- \sqrt{\frac{9}{2\pi}}\frac{\rho'_{fluid}}{\rho'_{part}\sqrt{\frac{2}{9}\left(\frac{a'^2}{L'}\right)\frac{L'U'}{\nu'}}}\int_0^{t^*}\frac{1}{\sqrt{t^*-s^*}}\frac{d}{ds^*}(v^* - u^*)ds'$$

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Now divide by $\frac{2\rho'_{part}+\rho'_{fluid}}{2\rho'_{part}}$ to find

$$\frac{dv^*}{dt^*} = \frac{3\rho'_{fluid}}{2\rho'_{part}+\rho'_{fluid}}\frac{Du^*}{Dt^*} + \frac{2(\rho'_{part}-\rho'_{fluid})}{2\rho'_{part}+\rho'_{fluid}} g^*$$

$$- \frac{9\mu' L' \times 2}{2u' a'^2(2\rho'_{part}+\rho'_{fluid})}(v^*-u^*) - \sqrt{\frac{9}{2\pi}}\frac{2\rho'_{fluid}}{(2\rho'_{part}+\rho'_{fluid})\sqrt{\frac{2}{9}\left(\frac{a'}{L'}\right)^2\frac{u'L'}{\nu'}}}\int_0^{t^*}\frac{1}{\sqrt{t^*-s^*}}\frac{d}{ds^*}(v^*-u^*)\,ds^*$$

$$= \frac{3}{2}\frac{2\rho'_{fluid}}{\rho'_{fluid}+2\rho'_{part}}\frac{Du^*}{Dt^*} + \left(\frac{\rho'_{fluid}+2\rho'_{part}}{\rho'_{fluid}+2\rho'_{part}} - \frac{3}{2}\frac{2\rho'_{fluid}}{\rho'_{fluid}+2\rho'_{part}}\right)g^*$$

$$- \underbrace{\frac{2}{\rho'_{fluid}+2\rho'_{part}}\frac{9}{2}\frac{L'^2}{a'^2}\frac{\mu'}{u'L'}}(v^*-u^*) - \sqrt{\frac{9}{2\pi}}\frac{2\rho'_{fluid}}{2\rho'_{part}+\rho'_{fluid}}\sqrt{\frac{9}{2}\frac{L'^2}{a'^2}\frac{\nu'}{u'L'}}\int_0^{t^*}\frac{1}{\sqrt{t^*-s^*}}\frac{d}{ds^*}(v^*-u^*)\,ds^*$$

used $\boxed{\rho'_{fluid}\nu'=\mu'}$ → true because of definition of kinematic viscosity

Defining $R=\frac{2\rho'_{fluid}}{\rho'_{fluid}+2\rho'_{part}}$, $Re=\frac{u'L'}{\nu'}$, $St=\frac{2}{9}\left(\frac{a'}{L'}\right)^2 Re$, then

$$\frac{dv^*}{dt^*} = \frac{3}{2}R\frac{Du^*}{Dt^*} + \left(1-\frac{3}{2}R\right)g^* - \frac{R}{St}(v^*-u^*) - \sqrt{\frac{9}{2\pi}}\frac{R}{\sqrt{St}}\int_0^{t^*}\frac{1}{\sqrt{t^*-s^*}}\frac{d}{ds^*}(v^*-u^*)\,ds^*$$

and then drop the stars to get

$$\frac{dv}{dt} = \frac{3}{2}R\frac{Du}{Dt}\left(1-\frac{3}{2}R\right)g - \frac{R}{St}(v-u) - \sqrt{\frac{9}{2\pi}}\frac{R}{\sqrt{St}}\frac{1}{(t-s)}\frac{d}{ds}(v-u)\,ds$$

as required

c) By assuming $St \ll 1$, we get $\frac{dv}{dt} = \frac{3}{2}R\frac{Du}{Dt} + \left(1-\frac{3}{2}R\right)g - \frac{k}{St}(v-u)$

(i) if the surrounding airflow is still, then $u=0$ and $\frac{Du}{Dt}=0$ thus

$$\frac{dv}{dt} = \left(1-\frac{3}{2}R\right)g - \frac{R}{St}v$$

solving this gives $\underline{v}(t) = Ce^{-R/St\,t} + \frac{St}{R}\left(1-\frac{3}{2}R\right)g$

with initial condition $\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}C_x\\ C_y - \frac{St}{R}\left(1-\frac{3}{2}R\right)g\\ C_z\end{pmatrix}$ where $g=-g\begin{pmatrix}0\\1\\0\end{pmatrix}$

thus $C_x=1$, $C_z=0$ and

$C_y = \frac{St}{R}\left(1-\frac{3}{2}R\right)g$, thus $C=\left(1, \frac{St}{R}\left(1-\frac{3}{2}R\right)g, 0\right)$

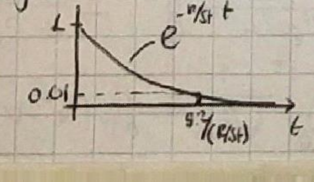
thus $v(t) = \left(e^{-R/St\,t}, \frac{St}{R}\left(1-\frac{3}{2}R\right)g\left(e^{-R/St\,t}-1\right), 0\right)$

(ii) terminal velocity of droplet is ~~when~~ reach as $t\to\infty$

if $t\to\infty$ $v(t)\longrightarrow\left(0, -\frac{St}{R}\left(1-\frac{3}{2}R\right)g, 0\right)$

~~if we~~ the timescale depends on when we can assume $e^{-R/St\,t}=0$

as



we know that after $t=\frac{9.2\,St}{R}$, $e^{-R/St\,t} < 0.01\ \%$.

thus if $t \geq 10\frac{St}{R}$, droplet has reached terminal velocity can be assumed

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(iii) the position of the droplet can be defined as $\underline{x}(t) = \int \underline{v}(t)\, dt$

thus integration of $\underline{v}(t)$ gives

$$\underline{x}(t) = \left(A_x - \frac{St}{R} e^{-R/St\, t},\ \frac{St}{R}\left(1-\frac{3}{2}R\right) g \left(A_y - \frac{St}{R} e^{-R/St\, t} - t \right),\ A_z \right)$$

assuming the initial position of the droplet is $\underline{x}(0) = (0,0,0)$,
then $A_x = St/R$, $A_y = St/R$, & $A_z = 0$, thus

$$\underline{x}(t) = \frac{St}{R}\left(1 - e^{-R/St\, t},\ \left(1-\frac{3}{2}R\right) g \left(\frac{St}{R} - \frac{St}{R} e^{-R/St\, t} - t \right), 0 \right)$$

then as $t \to \infty$, $\underline{x} \to (St/R, -\infty, 0)$

thus the horizontal distance covered is St/R
and time it takes them to reach this is, as computed before, roughly at $t = 10\, St/R$

(iv) horizontal range is given as $\frac{St}{R} = x_{max}$

in dimensional units this becomes

$$x_{max} = \frac{2}{9}\left(\frac{a'}{L'}\right)^2 \frac{U'L'}{\nu'} \frac{\rho'_{air} + 2\rho'_{drop}}{2\rho'_{air}}$$

now suppose $\rho'_{air} = 1.149\ kg/m^{-3}$, $\rho'_{drop} = 1000\ kg/m^{-3}$
$\nu' = 16.36 \times 10^{-6}\ m^2/s$, $a' = d'/2 =$ ~~5-500~~ $5/2$ to $250\ \mu m$

$$x_{max} = \frac{2}{9}\left(\frac{5}{2} \cdot 10^{-6}\right)^2 \frac{1}{16.36 \cdot 10^{-6}} \frac{1.149 + 2000}{2.298} \frac{U'}{L'} \approx \cancel{1.1 \cdot 10^{-4}}\ \cancel{4.435}\ 7.39 \cdot 10^{-4}\ \frac{U'}{L'}\ m$$

$$\text{to} \quad \frac{2}{9}\left(250 \cdot 10^{-6}\right)^2 \frac{1}{16.36 \cdot 10^{-6}} = 0.739\ \frac{U'}{L'}\ m$$