Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 2.1 (2 points)**.** *Which of the following are linear transformations? Justify.*

(a) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R} \\ (x,y) & \mapsto & x-y \end{array} \right.$

(b) $T : \left| \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (x,y) & \mapsto & (3x+y, x-xy) \end{array} \right.$

(c) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R}^n \\ A & \mapsto & \mathrm{diag}(A) \end{array} \right.$ *where* $\mathrm{diag}(A)$ *is the diagonal of the matrix* A*, defined by*

$$\mathrm{diag}(A) = (A_{1,1}, \dots A_{n,n}) .$$

(d) $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R}^{n\times n} \\ A & \mapsto & A^{-1} \end{array} \right.$ *defined on the set of homothety matrices* $S = \{\lambda \mathrm{Id}_n \,|\, \lambda \in \mathbb{R}^*\}$ *(see slide 19 of lec 02), recall that* $\mathbb{R}^*$ *is the set of real without* 0.

Problem 2.2. *Consider a matrix* $A \in R^{m\times n}$ *and denote by* $(c_1, \dots c_n)$ *its columns (vectors in* $\mathbb{R}^m$*). Prove the equality of the sets*

$$\mathrm{Im}(A) = \mathrm{Span}(c_1, \dots, c_n).$$

Recall that $\mathrm{Im}(A) \overset{def}{=} \{Ax \,|\, x \in \mathbb{R}^n\}$.

Problem 2.3 (3 points)**.** *(Wait for after next lab to complete this problem! or have a look at the last few slides we did not cover during class that are now completed on the website to find out what Gaussian elimination does.) Let*

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 7 & k \end{pmatrix}$$

.

(a) *Using Gaussian elimination solve* $Ax = 0$ *and give a basis of* $\mathrm{Ker}A$ *(show all your steps). You will have to differentiate cases according to values of* k.

(**b**) *Why does the system $Ax = (1,2,3)$ has at least one solution (for any value of k)? Do not solve the system, use previous results of this HW to justify your answer. Find all values of k for which the system $Ax = (1,2,3)$ has infinitely many solutions.*

(**c**) *Find all values of k for which the system $Ax = (10,1,2017)$ has exactly one solution. Give this solution as a function of k.*

Problem 2.4 (2 points)**.** *Let B and P be the matrices of in $\mathbb{R}^{3\times 3}$:*

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} B_{1,1} & B_{1,2} & B_{1,3} \\ B_{2,1} & B_{2,2} & B_{2,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}$$

with arbitrary entries for B.

(**a**) *Compute the matrix product BP. Why is P called a permutation matrix?*

(**b**) *Compute PB. What can you notice?*

Problem 2.5 ($\star$)**.** *Let* $A = \begin{pmatrix} 2 & -1 \\ 2 & -1 \end{pmatrix} \in \mathbb{R}^{2\times 2}$ *and* $T : \left| \begin{array}{ccc} \mathbb{R}^{n\times n} & \to & \mathbb{R}^{n\times n} \\ M & \mapsto & AM \end{array} \right.$.

(**a**) *Show that T is linear.*

(**b**) *Give a basis of* $\mathrm{Ker}(T)$ *and a basis of* $\mathrm{Im}(T)$.