Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 3.1 (2.5 points)**.** *True or false? Give a reason if true and a counterexample if false.*

(a) *A square matrix with a column of zeros cannot be invertible.*

(b) *If every row of a matrix adds up to zero the matrix must be singular.*

(c) *If every column of a matrix adds up to zero the matrix must be singular.*

(d) *Every matrix with 1's down the main diagonal is invertible.*

(e) *If A is invertible, then A^{-1} and A^2 are invertible.*

Problem 3.2 (2.5 points)**.** *Let* $A = \begin{pmatrix} 5a-2 & 3a & 3a-3 \\ -4a+2 & -3a+1 & -2a+2 \\ -4a+2 & -3a & -2a+3 \end{pmatrix}$ *for a real number a.*

(a) *Determine the rank of A for all values of a, remember that you can use linear operations on rows and columns.*

(b) *Determine a basis of the nullspace $Ker(A)$ and image space $Im(A)$ for $a = 0$.*

Problem 3.3 (2 points)**.** *Let* $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 4 & 5 & 0 \\ 6 & 0 & 7 & 8 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 5 & 6 & 0 \end{pmatrix}$$

(a) *Determine the ranks of A, B, C and D by reasoning on the dimension of the span of the columns of the matrices.*

(b) *Consider a generic lower-triangular matrix $M \in \mathbb{R}^{n \times n}$ (meaning that the entries $M_{ij} = 0$ if $j > i$). Show that if all diagonal entries of M are non-zero, M is invertible.*

(c) *Show that if one of the diagonal entries $M_{ii} = 0$, then the matrix is not invertible.*

(d) *Consider a generic upper-triangular matrix $M \in \mathbb{R}^{n \times n}$ ($M_{ij} = 0$ if $j < i$). At what condition is M invertible?*

Problem 3.4 (2 points). *We will build a formal proof that for all $L \in R^{n\times m}$, $\text{rank}(L) = \text{rank}(L^\top)$.*

(**a**) *Show that for any matrices $A \in \mathbb{R}^{n\times k}$, $B \in \mathbb{R}^{k\times m}$, $\text{rank}(A) \geq \text{rank}(AB)$.*

(**b**) *Given a matrix $L \in R^{n\times m}$, show that $\ker(L) = \ker(L^\top L)$ (hint: you may want to consider $y = Lx$).*

(**c**) *Using* (**a**) *and* (**b**) *Deduce that $\text{rank}(L) = \text{rank}(L^\top)$ (hint: use the rank-nullity theorem).*

Problem 3.5 ($\star$). *Let $A \in \mathbb{R}^{n\times n}$ given by*

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & 1 & 0 \\ \vdots & & & \vdots & \\ 0 & 1 & 0 & \cdots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{pmatrix}.$$

Compute A^k for all $k \in \mathbb{N}$.