Linear Algebra and Optimization DSGA-1014
Fall 2021
CDS at NYU

Lab 1

Zahra Kadkhodaie (based on Brett Bernstein's slides)

Geometric intuition

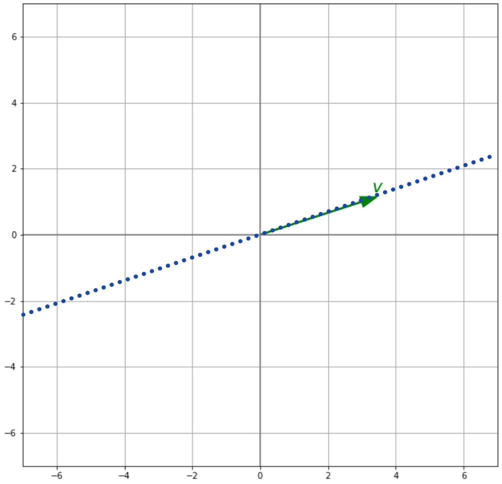
Take these two vectors in $\mathbb{R}^2$:

$v = (3,1)$ and $w = (-2,4)$

Describe the following sets geometrically. Which are subspaces of $\mathbb{R}^2$?

$$Span(v)$$

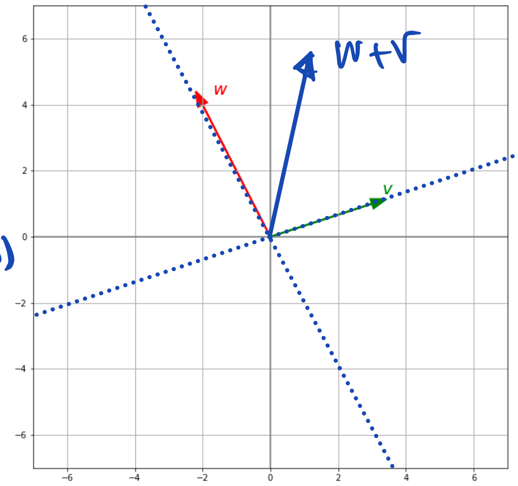
Is a subspace.



$$Span(v) \cup Span(w)$$

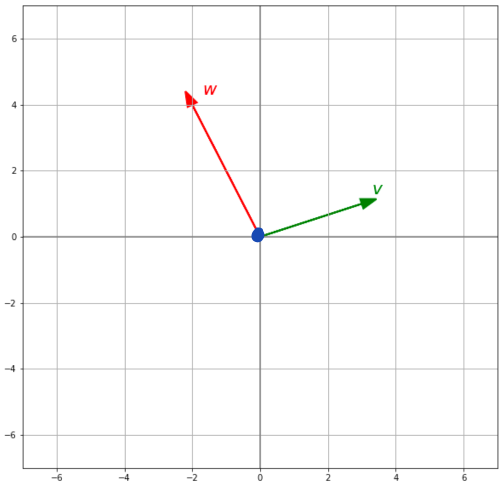
Not a subspace.

$w+v$ is not in $Span(v) \cup Span(w)$



$Span(v) \cap Span(w) = \{0\}$

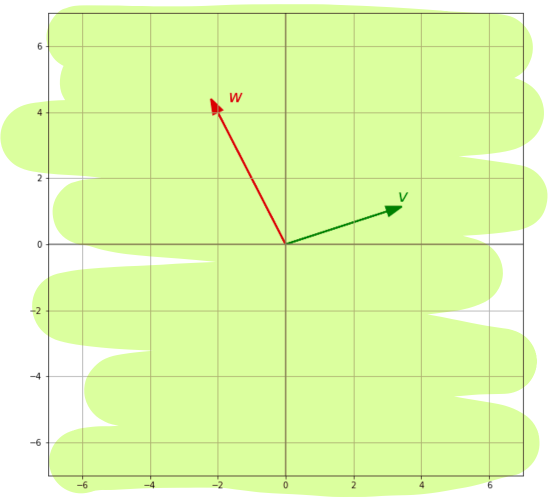
Is a subspace



$$Span(v, w)$$

$$\{\alpha v + \beta w : \alpha, \beta \in \mathbb{R}\ \}$$

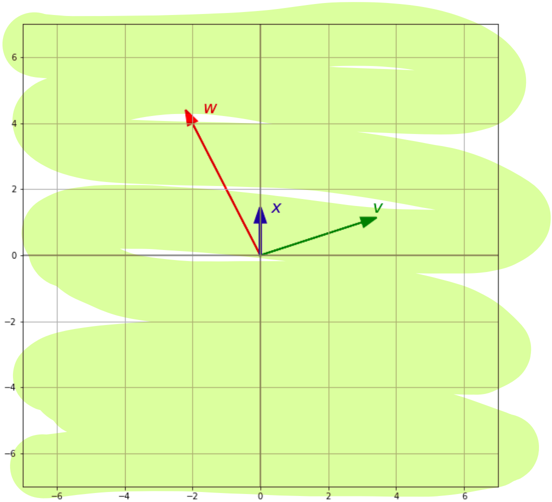
Is a subspace



$$Span(v, w, x)$$

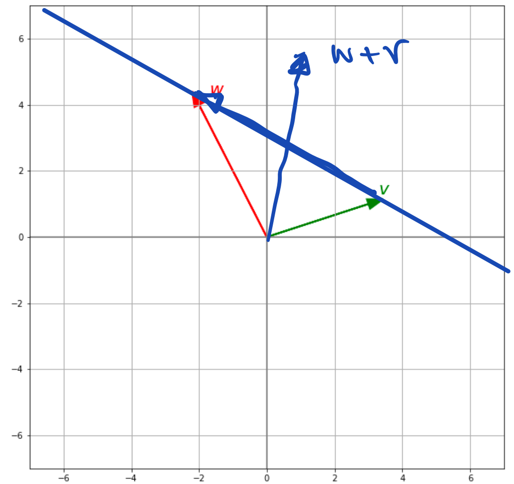
$$x = (0,1)$$

Is a subspace.



$$\{(1-t)v + tw : t \in \mathbb{R}\}$$

not a subspace.



Line equation:

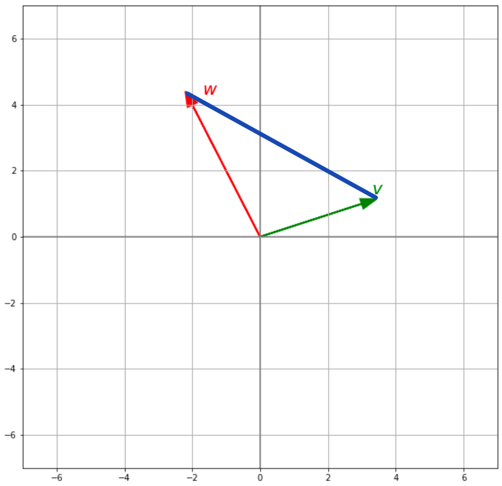
$$t\,(\overbrace{\vec{w}-\vec{v}}^{\text{slope}}) + v$$

$$tw - tv + v$$

$$tw + (1-t)v$$

$$\{(1-t)v + tw : t \in [0,1]\}$$

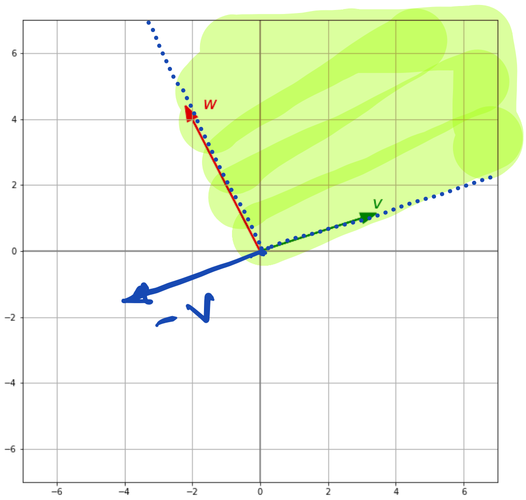
Not a subspace.



$$\{\alpha v + \beta w : \alpha, \beta \geq 0\}$$

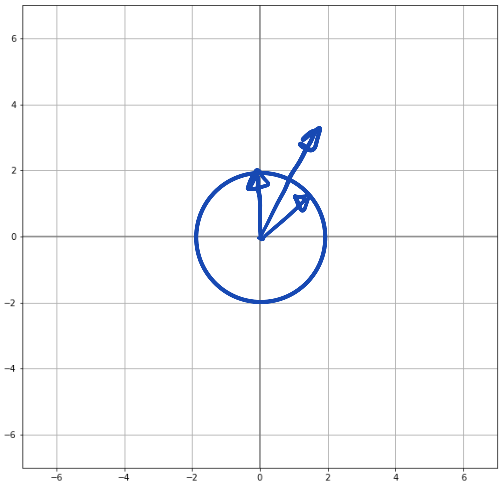
$-v$ not in the set

not a subspace.



$$\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 = 4\}$$

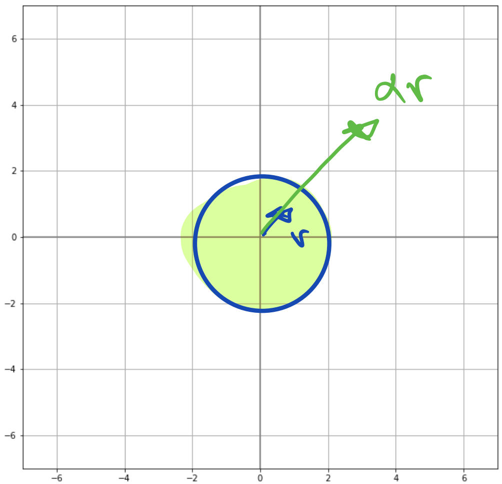
Not a subspace.



$$\{(a, b) \in \mathbb{R}^2 : a^2 + b^2 \leq 4\}$$

Not a subspace

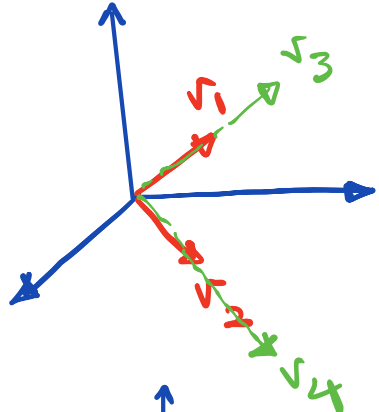
αr not in the set



Let $v_1, v_2, v_3, v_4 \in \mathbb{R}^3$ and define $C_1 = \{v_1, v_2\}$ and $C_2 = \{v_3, v_4\}$. If C_1 and C_2 are both linearly independent, what are the possible values for $dim(Span(v_1, v_2, v_3, v_4))$? No proof necessary.

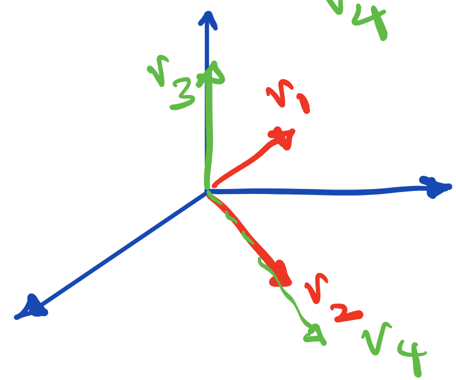
If $v_3 = \alpha v_1 + \beta v_2$ & $v_4 = \alpha' v_1 + \beta' v_2$

$\Rightarrow Dim(Span(v_1, v_2, v_3, v_4)) = 2$

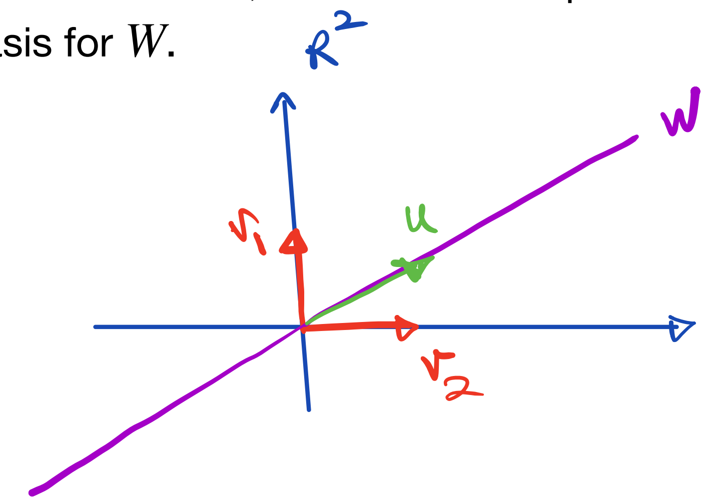


If $v_3 \neq \alpha v_1 + \beta v_2$ or $v_4 \neq \alpha' v_1 + \beta' v_2$

$\Rightarrow Dim(Span(v_1, v_2, v_3, v_4)) = 3$



True or False: If $B = \{v_1, \ldots, v_n\}$ is a basis for $\mathbb{R}^n$, and W is a subspace of $\mathbb{R}^n$, then some subset of B is a basis for W.



False!

Counter example: $n=2$

$v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Any basis for W has to contain αu. But αu is not in $B = \{v_1, v_2\}$

$$\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{bmatrix}$$

Suppose $v_1, \ldots, v_m \in \mathbb{R}^n$ are linearly dependent. Prove that if $x \in Span(v_1, v_2, \ldots, v_m)$ then there are infinitely many $\alpha \in \mathbb{R}^m$ with $x = \alpha_1 v_1 + \ldots + \alpha_m v_m$

Existence : By definition .

By definition of linear dependency:

There exist a non-zero c where:

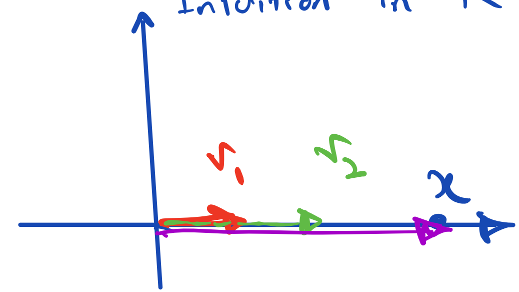
$$c_1 v_1 + c_2 v_2 + \cdots + c_m v_m = 0$$

$$x = \alpha_1 v_1 + \cdots + \alpha_m v_m + r(c_1 v_1 + c_2 v_2 \cdots + c_m v_m)$$

$$= (\alpha_1 + r c_1) v_1 + \cdots + (\alpha_m + r c_m) v_m$$

$$r \in \mathbb{R}$$

Intuition in R^2



$$x = \alpha_1 v_1 + \alpha v_2$$

$g = b_0 + b_1 x + \cdots + b_n x^n$

$\mathbb{R} \to \mathbb{R}$

Let $P_n = \{f(x) = a_0 + a_1 x + \ldots + a_n x^n\}$ and define addition and scalar multiplication as follows

Addition. $(f + g)(x) = (a_0 + b_0) + (a_1 + b_1)x + \ldots + (a_n + b_n)x^n$

Multiplication. $(rf)(x) = (ra_0) + (ra_1)x + \ldots + (ra_n)x^n$

Show that P_n is a vector space. What is a basis for this space and what is the dimension?

[Note: f is a function, but f(x) is a real number. That is, once we plug something into our function, we get a real number not a function. For example, if x is a real number, then x^2 is a real number, not a function. This is subtle but extremely important. For example, we know real numbers commute and thus know that f(x) + g(x) = g(x) + f(x). However, we must prove that f + g = g + f]

To prove P_n is a vector space, we need to show it satisfies all the following conditions

(a)	$\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$	Commutative law
(b)	$(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$	Associative law
(c)	$\mathbf{x}+\boldsymbol{\theta}=\mathbf{x}$	Additive identity
(d)	$\mathbf{x}+(-\mathbf{x})=\boldsymbol{\theta}$	Additive inverse
(e)	$(rs)\mathbf{x}=r(s\mathbf{x})$	Associative law
(f)	$(r+s)\mathbf{x}=r\mathbf{x}+s\mathbf{x}$	Distributive laws
(g)	$r(\mathbf{x}+\mathbf{y})=r\mathbf{x}+r\mathbf{y}$	Distributive laws
(h)	$1\mathbf{x}=\mathbf{x}$	Multiplicative identity

a) $(f+g)x = (a_0+b_0)+(a_1+b_1)x+\cdots+(a_n+b_n)x^n$

$= (b_0+a_0)+(b_1+a_1)x+\cdots+\underbrace{(b_n+a_n)}x^n$ due to commutativity of real numbers

$= (g+f)x$

c) Define: $f(x)=0$ as the zero "vector" of this set.

Is this a polynomial? Yes! It's a polynomial of degree $n=0$ where $a_0=0$.

d) $-f(x)$ is a polynomial:

$$-f(x) = -a_0 + (-a_1)x + (-a_2)x^2 + \dots + (-a_n)x^n$$

$$(f+(-f))(x) = \underbrace{a_0+(-a_0)}_{0} + \underbrace{(a_1+(-a_1))}_{0}x + \dots + \underbrace{(a_n+(-a_n))}_{0}x^n = 0$$

e) $(rsf)(x) = (rs\,a_0) + (rs\,a_1)x + \dots + (rs\,a_n)x^n$

from associative law of real numbers

$$= r(sa_0) + r(sa_1)x + \dots + r(sa_n)x^n$$

$$= r\,[(sa_0) + (sa_1)x + \dots + (sa_n)x^n] = r\,(sf)(x)$$

A basis for this vector space is the set of polynomials

$\{1, x, x^2, \dots, x^n\}$ with $\mathrm{Dim} = n+1$