7.1

(a) M symmetric $\Rightarrow$ There exists $v_1 \dots v_n$ in $\mathbb{R}^n$ orthonormal basis of $\mathbb{R}^n$ with $v_1 \dots v_n$ eigenvectors of M with $\lambda_1 \dots \lambda_n$.

$\Rightarrow$ Assume M positive semi-definite.

$$x^T M x \geqslant 0 \text{ for any } x \Rightarrow v_i^T M v_i = \lambda_i \underbrace{\|v_i\|^2}_{=1} \geqslant 0$$

$$\Rightarrow \lambda_i \geqslant 0 \quad \text{for any } i.$$

$\Leftarrow$ Assume $\lambda_i \geqslant 0$:

For any x, there exists $\alpha_1 \dots \alpha_n$ in $\mathbb{R}$, $x = \sum_{i=1}^n \alpha_i v_i$

$$\Rightarrow x^T M x = \sum_{i,j} \alpha_i \alpha_j v_i^T M v_j$$

$$= \sum_{i,j} \alpha_i \alpha_j \lambda_j \underbrace{v_i^T v_j}_{\begin{cases} =1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}}$$

$$= \sum_i \alpha_i^2 \lambda_i \ , \geqslant 0 \quad \text{if } \lambda_i \geqslant 0$$

(b) $J_n = \begin{pmatrix} 1 & \dots & 1 \\ 1 & & \vdots \\ \vdots & & \vdots \\ 1 & \dots & 1 \end{pmatrix}$ Rank$(J_n) = 1$ since all colums are equal

$*$ Sum of values on one row $= n$ $\Rightarrow$ eigenvalue associated with $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$

$$J_n \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} n \\ n \\ \vdots \\ n \end{pmatrix} = n \begin{pmatrix} 1 \\ \vdots \\ \vdots \\ 1 \end{pmatrix}$$

(b) Since rank$(J_n) = 1 \rightarrow \dim \operatorname{Ker} J_n = n-1$

$$\Rightarrow \dim E_0(J_n) = n-1 = m_0$$

multiplicity eigenvalue 0.

So that the entire spectrum of J_n is $\{0, 1\}$ with has only non-negative elements –

↳ J_n is positive semi-definite.

(c) $$M = PDP^T = P \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_m \end{pmatrix} P^T$$

+ assume $\lambda_1 \geq \dots \geq \lambda_m$.

If $\lambda_m > 0 \rightarrow M$ is already positive definite

$$M + \alpha I d = P \begin{pmatrix} \lambda_1 + \alpha & & (0) \\ & \ddots & \\ (0) & & \lambda_m + \alpha \end{pmatrix} P^T$$ is also positive definite for any $\alpha > 0$.

If $\lambda_m \leq 0 \rightarrow -\lambda_m \geq 0 \rightarrow -\lambda_m + 1 > 0$

for instance ...

$$\Rightarrow \left(M + (1-\lambda_m) I d\right)$$ is positive definite.

7.2

(a) $b = \begin{pmatrix} \langle a, v_1 \rangle \\ \vdots \\ \langle a, v_k \rangle \end{pmatrix}$

$\sum_{i=1}^{n} a_i = 0$

$$\sum_{i=1}^{m} b_i = \sum_{i=1}^{n} \begin{pmatrix} \langle a_i, v_1 \rangle \\ \vdots \\ \langle a_i, v_k \rangle \end{pmatrix} = \begin{pmatrix} \langle \sum a_i, v_1 \rangle \\ \vdots \\ \langle \sum a_i, v_k \rangle \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

(b) $b_i - b_j = \begin{pmatrix} \langle a_i - a_j, v_1 \rangle \\ \vdots \\ \langle a_i - a_j, v_k \rangle \end{pmatrix}$

$$\| b_i - b_j \|^2 = \sum_{m=1}^{k} \langle a_i - a_j, v_m \rangle^2$$

$$\| P_S (a_i - a_j) \| = \| \sum_{m=1}^{k} \langle a_i - a_j, v_m \rangle v_m \| = \| b_i - b_j \|^2$$

orthogonal

orthogonal project onto $S = \mathrm{span}(v_1 \dots v_k)$

Yet by Pythagorean theorem $\rightarrow$ cf HW 0

$\| P_S (a_i - a_j) \| \leq \| a_i - a_j \|$. $\Rightarrow$ $\| b_i - b_j \| \leq \| a_i - a_j \|$.

(c) Consider $A = \begin{pmatrix} - a_1^T - \\ - a_m^T - \end{pmatrix} \in \mathbb{R}^{n \times d}$

$\hookrightarrow$ Orthonormal basis of A^TA: $(v_1 \dots v_d)$.

$f^{(i)} = \begin{pmatrix} \langle a_1, v_i \rangle \\ \vdots \end{pmatrix} = A v_i$

$$\begin{pmatrix} \vdots \\ \langle a_n, v_i \rangle \end{pmatrix}$$

Hence $\langle f^{(j)}, f^{(i)} \rangle = v_j^T A^T A v_i = \lambda_i \langle v_j, v_i \rangle = \begin{cases} 0 & \text{if } i \neq j \\ \lambda_i & \text{otherwize.} \end{cases}$

7-4) $A = PDP^T$ define $D^{1/2}$ $B = PD^{1/2}P^T$.