Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 6.1 (2 points). **(a)** *Show that 6 is an eigenvalue for the matrices A and B in $\mathbb{R}^3$ defined below. In each case give one associate eigenvector.*

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 4 & 5 & -3 \end{pmatrix}$$

(b) *Let $A \in \mathbb{R}^{n\times n}$ be a square matrix such that the sum of its rows is equal. In other words, there exists $\mu \in \mathbb{R}$ such that for any integer i $(1 < i \leq n)$, $\sum_{j=1}^{n} A_{i,j} = \mu$. Show that A admits one pair of eigenvector-eigenvalue and give their values.*

Problem 6.2 (2 points). *For a square symmetric matrix, we call eigen decomposition the collection of all its eigenvector-eigenvalue pairs. Let $A \in \mathbb{R}^{n\times n}$ be a square symmetric matrix. Give the eigen decomposition of A^k as a function of the eigen demcomposition of A for any integer $k > 0$.*

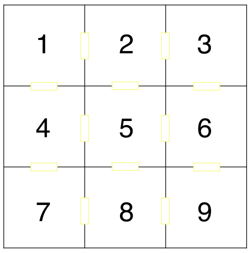
Problem 6.3 (2 points). *The trace of a square $A \in \mathbb{R}^{n\times n}$ matrix is defined as the sum of its diagonal elements*

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{i,i}.$$

(a) *Show that for any $B \in R^{n\times m}$ and $C \in \mathbb{R}^{m\times n}$, $\mathrm{Tr}(BC) = \mathrm{Tr}(CB)$.*

(b) *Use the previous result to show that for a symmetric matrix $A \in \mathbb{R}^{n\times n}$, its trace is equal to the sum of its eigenvalues.*

Problem 6.4 (3 points). *Consider a Washington square squirell trapped in a box divided in 9 rooms. At any point of time, the squirell decides to go through any of the available doors or stay in the room, all actions with equal probability. Use* `numpy` *or any other programming language to help you solve this problem.*



(**a**) *Construct the transition stochastic matrix P for this problem. Can you find an integer $k \leq 1$ such that P^k has only strictly positive entries?*

(**b**) *Find the invariant measure for this problem.*

(**c**) *Using a symmetry argument, show that you can also solve this problem using a 3×3 matrix.*

Problem 6.5 ($\star$). *A symmetric matrix $M \in \mathbb{R}^n$ is positive semi-definite if, for any $x \in \mathbb{R}^n$,*

$$x^\top M x \geq 0.$$

We say furthermore that M is postive definite if $x^\top M x > 0$ for any non-zero vector.

(**a**) *Show that M is positive semi-definite if and only if its eigenvalues are non-negative.*

(**b**) *Give a necessary and sufficient condition on the spectrum of M for the matrix to be definite positive.*