# DS-GA 1014 Optimization and Computational Linear Algebra

## Lab 8: Graphs and linear algebra (& SVD)

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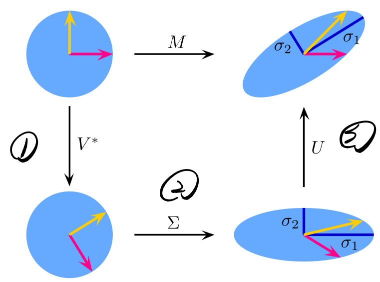
# Singular Value decomposition

**Theorem 3.1** (*Singular value decomposition (SVD)*)

*Let $A \in \mathbb{R}^{n \times m}$. Then there exists two orthogonal matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ and a matrix $\Sigma \in \mathbb{R}^{n \times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$*

$$A = U \Sigma V^{\mathsf{T}}.$$

*The columns $u_1, \ldots, u_n$ of $U$ (respectively the columns $v_1, \ldots, v_m$ of $V$) are called the left (resp. right) singular vectors of $A$. The non-negative numbers $\Sigma_{i,i}$ are the singular values of $A$. Moreover* $\operatorname{rank}(A) = \#\{i \mid \Sigma_{i,i} \neq 0\}$.




$M = U \cdot \Sigma \cdot V^*$

Explain the following statement: For any $A \in \mathbb{R}^{m \times n}$, the set $\{Ax : \|x\| = 1\}$ is an ellipsoid. In other words, the image of the sphere under a linear transformation is always an ellipsoid.

$A = U \Sigma V^T$

① $V$ orthogonal $\Rightarrow$ preserve norm $\Rightarrow$ map the sphere to itself

② $\Sigma$ diagonal $\Rightarrow$ stretch the sphere along each axis $\Rightarrow$ ellipsoid

③ $U$ orthogonal $\Rightarrow$ preserve norm $\Rightarrow$ still ellipsoid

## Singular Value decomposition

Let $A \in \mathbb{R}^{m \times n}$. Give a method for computing $\operatorname{rank}(A)$ using the SVD of $A$.

$$A = U \Sigma V^T$$

$$\operatorname{rank}(A) \overset{*}{=} \operatorname{rank}(\Sigma) = \#\text{ non-zero singular values}$$

* We know $\operatorname{rank}(AB) \leq \min(\operatorname{rank}(A), \operatorname{rank}(B))$.

$$\operatorname{rank}(\Sigma) = \operatorname{rank}(U^{-1} U \Sigma) \leq \operatorname{rank}(U\Sigma) \leq \operatorname{rank}(\Sigma)$$

$$\Rightarrow \operatorname{rank}(\Sigma) = \operatorname{rank}(U\Sigma)$$

left multiplication by an invertible matrix preserves rank

$$\operatorname{rank}(\Sigma) = \operatorname{rank}(\Sigma^T) = \operatorname{rank}((V^T)^{-1} V^T \Sigma^T)$$

$$\leq \operatorname{rank}(V^T \Sigma^T) = \operatorname{rank}(\Sigma V) = \operatorname{rank}(\Sigma)$$

right multiplication by an invertible matrix preserves rank

## Singular Value decomposition

Midterm 2019 Q6: Let $M \in \mathbb{R}^{n \times m}$. Let $n \geq m$, and $M$ have full rank. Let $M$ have SVD $M = U\Sigma V^T$.

1. Show that $M^T M$ is invertible.
2. Which vectors span the $Im(M)$? Write the matrix of orthogonal projection onto $Im(M)$ and give a basis transformation for that matrix.
3. Let $w \in \mathbb{R}^n$, and $u$ be the orthogonal projection of $w$ onto $Im(M)$. Show that $M^T u = M^T w$.
4. Show that $M(M^T M)^{-1} M^T$ is the matrix of an orthogonal projection onto $Im(M)$.

① $M^T M = (U \Sigma V^T)^T (U \Sigma V) = V \Sigma^T \underbrace{U^T U}_{= I \text{ (U orthagnonal)}} \Sigma V = V \Sigma^T \Sigma V$

$\text{rank}(M^T M) = \text{rank}(\Sigma^T \Sigma) = m$

$\Rightarrow M^T M$ invertible

② $Im(M) = \{ y \in R^n : \exists x \text{ s.t. } Mx = y \}$

$y = Mx = U \Sigma V^T x$

Let $x = \sum_{i=1}^{m} \alpha_i v_i$

$V^T x = (\alpha_1 \ \alpha_2 \ \cdots \ \alpha_m)^T$

$\Sigma V^T x = (\sigma_1 \alpha_1 \ \sigma_2 \alpha_2 \ \cdots \ \sigma_m \alpha_m \ 0 \ \cdots \ 0)^T$

$y = U \Sigma V^T x = \sum_{i=1}^{m} \alpha_i \sigma_i u_i$

Note $m$ is also the # non-zero elements in $\Sigma$ because $M$ is full rank

$\Rightarrow Im(M) = \text{span}(u_1, u_2 \cdots u_m)$

Let $U_* = \begin{pmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_m \\ | & | & & | \end{pmatrix}$

$P_{Im(M)} = U_* U_*^T$

③ $M^T u = V \Sigma^T U^T u = V \Sigma^T U^T U_* U_*^T w$

$$= V \Sigma^T U_0^T w \qquad U_0 = \underbrace{\begin{pmatrix} - & u_1 & - \\ & \cdots & \\ - & u_m & - \\ & 0 & \end{pmatrix}}_{n} \begin{array}{l} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} m \\ \left.\vphantom{a}\right\} n-m \end{array}$$

$$= V \Sigma^T U^T w$$

$$= M^T w$$

④ $(MM^T)^{-1} = V(\Sigma^T \Sigma)^{-1} V^T$

$$M(MM^T)^{-1}M^T = U \Sigma V^T V (\Sigma^T \Sigma)^{-1} V^T (U \Sigma V^T)^T$$

$$= U \Sigma (\Sigma^T \Sigma)^{-1} \Sigma^T U^T$$

$$n \quad \underbrace{\begin{pmatrix} \overset{m}{\boxed{\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix}}}\, m & 0 \\ 0 & \end{pmatrix}}_{n}$$

$$= U_* U_*^T$$

# Graphs

- Adjacency matrix $A_{ij} = \begin{cases} 1 & i \sim j \\ 0 & \text{o.w.} \end{cases}$ $\Rightarrow$ A symmetric
- degree matrix $D = \text{diag}(\deg(1), \ldots \deg(n))$
- Laplacian matrix $L = D - A$

**Proposition 2.1**

*The matrix $L$ satisfies the following properties:*

1. *$L$ is symmetric and positive semi-definite.*
2. *The smallest eigenvalue of $L$ is 0 and a corresponding eigenvector is the constant one vector $\mathbb{1} \stackrel{\text{def}}{=} (1, 1, \ldots, 1)$.*
3. *$L$ has $n$ non-negative eigenvalues $0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$.*

The proposition above follows from the following key identity: for all $x \in \mathbb{R}^n$,

$$x^\top L x = \sum_{i \sim j} (x_i - x_j)^2, \tag{1}$$

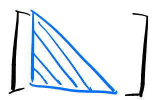
# Graphs

Handshaking lemma: let $G$ be a graph with $n$ nodes and $m$ edges. Show that

$$\sum_{i=1}^{n} deg(\text{node}_i) = 2m$$

(if there is a party with $n$ attendees then an even number of people shakes an odd number of other people's hands)

$$\sum_{i=1}^{n} deg(i) = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} = 2 \overbrace{\sum_{i=1}^{n} \sum_{j=1}^{i} A_{ij}}^{m} = 2m$$



$A$

Previous lab question from Irina Espejo

# Graphs

Let $G$ be a connected graph with $n$ nodes and let $A$ be its adjacency matrix. Show that the highest valued eigenvalue $\lambda_1$ is bounded by the maximum degree, that is

$$\lambda_1 \leq \max_{i\in\{1..n\}} deg(i)$$

Let $v$ be an eigenvector associated with $\lambda_1$

Let $i$ be the vertex on which it takes maximum value $v_i \geq v_k \ \forall k$

$$Av = \lambda_1 v \Rightarrow \sum_j A_{ij} v_j = \lambda_1 v_i$$

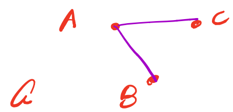
$$\lambda_1 = \frac{\sum_j A_{ij} v_j}{v_i} = \frac{\sum_{i\sim j} v_j}{v_i} = \sum_{i\sim j} \frac{v_j}{v_i}$$

$$\leq \sum_{i\sim j} 1 = deg(i)$$

$$\leq \max_i deg(i)$$

Previous lab question from Irina Espejo

Note that $v(G)$ and $e(G)$ are not necessarily # vertices and # edges




remove A will disconnect G

$\Rightarrow$ $v(G) = 1$

$e(G) = 2$

## Graphs

The vertex connectivity v(G) of G as the minimum number of nodes whose removal would result in losing connectivity of the graph.
The edge connectivity e(G) of G as the minimum number of edges whose removal would result in losing connectivity of the graph.
Show that

definition of graph

$$\lambda_2 \le v(G) \le e(G).$$

Lemma: Let $G^-$ be a graph obtained from $G$ by removing one vertex and all associated edges. Then $\lambda_2(G^-) \ge \lambda_2(G) - 1$

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Remove $v(G)$ vertices to disconnect the graph $\Rightarrow G'$

$$\lambda_2(G) \le \lambda_2(G') + v(G) = v(G) \quad \checkmark$$

Note: The adjacency matrix for a disconnect matrix is $0 = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ & & \vdots & \\ 0 & 0 & \cdots & 0 \end{pmatrix}$

$\Rightarrow \lambda = 0$

**proof** WOLG, assume the last vertex is removed.

Connect the last node to the rest of nodes $\Rightarrow G^+$

$$L_{G^+} = \begin{pmatrix} L_{G^-} + I & -\mathbb{1} \\ -\mathbb{1}^T & n-1 \end{pmatrix}$$

Let $v$ be the eigenvector of $L_{G^-}$ associated with $\lambda_2(G^-)$

(i.e. $L_{G^-} v = \lambda_2(G^-) v$) and $\mathbb{1}^T v = 0$

$$L_{G^+} \begin{pmatrix} v \\ 0 \end{pmatrix} = \begin{pmatrix} (L_{G^-} + I) v \\ -\mathbb{1}^T v \end{pmatrix} = \begin{pmatrix} \lambda_2(G^-) v + v \\ 0 \end{pmatrix} = (\lambda_2(G^-) + 1) \begin{pmatrix} v \\ 0 \end{pmatrix}$$

$\Rightarrow (\lambda_2(G^-) + 1)$ is an eigenvalue of $L_{G^+}$

$\lambda_2(G^+) \leq \lambda_2(G^-) + 1$

Since $G^+$ is obtained from $G$ by adding edges,

$\lambda_2(G) \leq \lambda_2(G^+)$ from the previous question

$\Rightarrow \lambda_2(G) \leq \lambda_2(G^-) + 1$