PROBLEM 6.1

(a) $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & 2 \\ 4 & 5 & -3 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}$

$$A\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad B\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 6\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(b) Let $A \in \mathbb{R}^{n\times n}$ be a matrix such that the sum of each of its rows is equal, that is there exists $\mu \in \mathbb{R}$ such that for all $1 \leq i \leq n$ $\sum_j A_{i,j} = \mu$.. Show that A admits at least one eigenvector. Give also the associated eigenvalue.

Consider the vector of $\mathbb{R}^n$ with "1" at each coordinate:

$$A\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^{n} A_{1,j} \\ \vdots \\ \sum_{j=1}^{n} A_{n,j} \end{pmatrix} = \begin{pmatrix} \mu \\ \vdots \\ \mu \end{pmatrix} = \mu\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue μ.

PROBLEM 6.2 :

$\oplus$ Since A is symmetric, there exists P orthogonal and D diagonal $n\times n$ matrices:

$$A = P^T D P$$

$$A^2 = P^T D \underbrace{P P^T}_{=I_d} D P = P^T D^2 P$$

By induction, one can show $A^k = P^T D^k P$.

Rk: $(A^k)^T = A^k$ is also symmetric.

P is a matrix collecting its eigenvectors and the corresponding eigenvalues are $\lambda_1^k \dots \lambda_n^k$.

We can further justify that $\lambda_1^k \dots \lambda_n^k$ are the only eigenvalues of A^k since there multiplicities are the same as the multiplicities of $\lambda_1 \dots \lambda_n$ for A.

$$\sum_{i=1}^{n} m_{\lambda_i^k} = n$$

actually on distinct eigenvalues

$\rightarrow$ yet proposition states that $\sum m_{\lambda_i} \leq n$ for a $n\times n$ matrix. So there cannot be more eigenvalues.

Problem 6-3:

(a) $\operatorname{Tr}(BC) = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} B_{ij} C_{ji} \right)$

$\operatorname{Tr}(CB) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} C_{ji} B_{ij} \right)$

Hence $\operatorname{Tr}(BC) = \operatorname{Tr}(CB)$.

(b) A is symmetric, by the spectral theorem it can be rewritten w.r.t. matrix $P = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$ orthonormal eigenvectors and $D = \begin{pmatrix} \lambda_1 & & (0) \\ & \ddots & \\ (0) & & \lambda_m \end{pmatrix}$ diagonal matrix with associated eigenvalues:

$$A = P^T D P$$

$$\Longrightarrow \operatorname{Tr}(A) = \operatorname{Tr}(P^T D P) \underset{\text{by (a)}}{=} \operatorname{Tr}(D \underbrace{P P^T}_{\substack{= I_d \\ \text{since orthogonal} \\ \text{matrix.}}})$$

$$= \operatorname{Tr}(D) = \sum_{i=1}^{n} \lambda_i \quad \blacksquare.$$

PROBLEM 6.5

(a) $(*)\ x^T M x \geqslant 0 \iff x^T P^T D P x \geqslant 0$ for any $x \in \mathbb{R}^n$

$\iff x'^T D x' \geqslant 0$ for any $x' \in \mathbb{R}^n$ since we are just changing bases.

$\iff \sum_{i=1}^{n} \lambda_i x_i'^2 \geqslant 0 \quad (*)$

Assume $(*) \iff$ positive semi-definite, Take $x' = e_i \Rightarrow \lambda_i \geqslant 0$.

Assume all $\lambda_i \geqslant 0 \Rightarrow \sum_{i=1}^{n} \lambda_i x_i^2 \geqslant 0 \iff$ positive semi-definite

(b) $(\Leftarrow)$ $\lambda_i > 0$ for all $i = 1 \dots n$

$\Rightarrow x^T M x = (xP)^T D P x$

$= \sum_i \lambda_{ii} (Px)_i^2$

since P is invertible, $\mathrm{Ker}(P) = \{0\}$, so there is an index i such that $(Px)_i \neq 0$.

$\Rightarrow (Px)_i^2 > 0$

$\Rightarrow x^T M x > (Px)_i^2 > 0$

$\Rightarrow x^T M x > 0$.

(c) Furthermore if $x^T M x > 0$ for any x,
in particular for v_i eigenvector associated with λ_i:

$$v_i^T M v_i = v_i^T (\lambda_i v_i) = \lambda_i \underbrace{\|v_i\|^2}_{>0} > 0$$

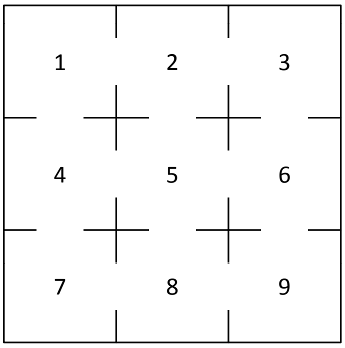
as $v_i \neq 0$.

$\Rightarrow \lambda_i > 0$ for all i in $1 \dots n$.

What this steady-state vector says is that eventually half of the population will be of type Aa and one fourth of the population will be either AA or aa. What this is equivalent to is the following random experiment. Suppose that we have two coins, each one marked with an A on one side and with an a on the other. We flip each coin once, and note what side is showing. There is only one way to get two As - each coin showing an A. Similarly, there is only one way to get two as - each coin shown an a. However, there are two ways to get one A and one a - coin 1 with an A and coin two with an a, and coin 1 with an a and coin two with an A. One out of four possibilities for either AA or aa means a probability of 0.25 and two chances out of four for Aa means a probability of 0.5.

Problem 10.10⋆

A mouse is placed in a box with nine rooms as illustrated in the figure below.



Assume that, at regular intervals of time, it is equally likely that the mouse will decide to go through any door in the room or stay in the room.

(a) Construct the 9×9 transition matrix for this problem and show that it is regular.

(b) Determine the steady-state vector for this matrix.

(c) Use a symmetry argument to show that this problem may be solved using only a 3×3 matrix.

Solution

(a) In this problem, we have nine states, each state corresponding to a room in a 3×3 arrow of rooms and hallways. The mouse may move from one room to another or remain in the same room, each with the same probability of occurrence. Therefore, we can assign transition probabilities in moving from one state (room) to another as illustrated in the following table,

	1	2	3	4	5	6	7	8	9
1	$\frac{1}{3}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0	0	0	0	0
2	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$	0	$\frac{1}{5}$	0	0	0	0
3	0	$\frac{1}{4}$	$\frac{1}{3}$	0	0	$\frac{1}{4}$	0	0	0
4	$\frac{1}{3}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$	0	$\frac{1}{3}$	0	0
5	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{4}$	0	$\frac{1}{4}$	0
6	0	0	$\frac{1}{3}$	0	$\frac{1}{5}$	$\frac{1}{4}$	0	0	$\frac{1}{3}$
7	0	0	0	$\frac{1}{4}$	0	0	$\frac{1}{3}$	$\frac{1}{4}$	0
8	0	0	0	0	$\frac{1}{5}$	0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{3}$
9	0	0	0	0	0	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{3}$

For example, when the mouse is in room 1, he will either stay in room one or move to rooms two or four, all with equal probability of $\frac{1}{3}$. Similarly, if the mouse is in room 2, he will either stay there or move to rooms 1, 3, or 5, all with equal probability of $\frac{1}{4}$. From this table, we immediately have the transition matrix

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{3} & 0 & \frac{1}{5} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{3} & 0 & 0 & \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{4} & \frac{1}{5} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{5} & \frac{1}{4} & 0 & \frac{1}{4} & 0 \\ 0 & 0 & \frac{1}{3} & 0 & \frac{1}{5} & \frac{1}{4} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & \frac{1}{4} & 0 & 0 & \frac{1}{3} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{5} & 0 & \frac{1}{3} & \frac{1}{4} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{4} & 0 & \frac{1}{4} & \frac{1}{3} \end{bmatrix}$$

which is easily seen to be a valid stochastic matrix since all entries are between zero and one and all columns sum to one.

We can see that $\mathbf{P}$ is regular by looking at $\mathbf{P}^n$. Although $\mathbf{P}$ contains many zeros, all of the entries in $\mathbf{P}^4$ are non-zero.

(b) Since $\mathbf{P}$ is regular, $\mathbf{x}(n)$ will converge to a steady-state vector $\mathbf{v}$. We may find $\mathbf{v}$ by either forming the matrix $\mathbf{P}^n$ for a sufficiently large value of n, one for which $\mathbf{P}^{n+1} \approx \mathbf{P}^n$, and then $\mathbf{v}$ will be approximately equal to any one of the columns of $\mathbf{P}^n$. This is true because $\mathbf{v}$ is independent of the starting state, $\mathbf{x}(0)$. Therefore,

$$\mathbf{v} \approx \mathbf{P}^n \mathbf{e}_i$$

where $\mathbf{e}_i$ is a unit vector with all elements equal to zero except the ith element, which is equal to one, and $\mathbf{P}^n\mathbf{e}_i$ is the ith column of $\mathbf{P}^n$.

Alternatively, we can find the eigenvector corresponding to an eigenvalue of one,

$$(\mathbf{P} - \mathbf{I})\mathbf{v} = \mathbf{0}$$

Here we will take this approach. After entering the values of the matrix $\mathbf{P}$ into MATLAB or Octave, we use the `eig` m-file as follows,

```
>> [v,e]=eig(P);
```

We then look at the vector in `v` corresponding to the eigenvalue of one. In my case it is the first column of the matrix `v` (it may be a different one for you), and we find

```
>> v(:,1)
ans =
     -0.26833
     -0.35777
     -0.26833
     -0.35777
     -0.44721
     -0.35777
     -0.26833
     -0.35777
     -0.26833
```

Note that we need to scale $\mathbf{v}$ so that it is a valid probability vector. Specifically, we want the sum of the elements to equal one. Dividing each element by the sum of the terms in the vector we get the desired solution, which is

```
>> v(:,1)/sum(v(:,1))
ans =
      0.090909
      0.121212
      0.090909
      0.121212
```

```
0.151515
0.121212
0.090909
0.121212
0.090909
```

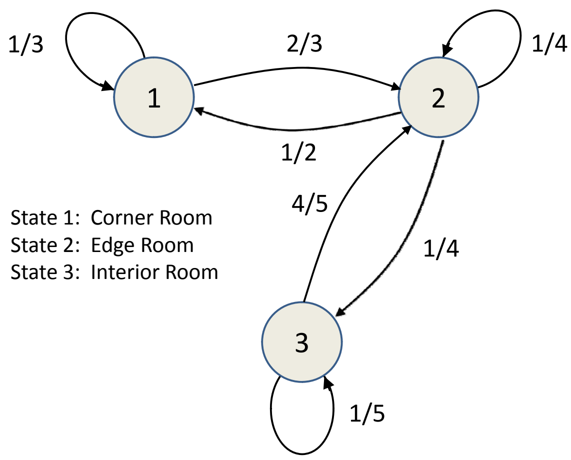
Each of the elements in $\mathbf{v}$ can be converted to a rational number. What we have for our final answer is

$$\mathbf{v} = \begin{bmatrix} 1/11 \\ 4/33 \\ 1/11 \\ 4/33 \\ 5/33 \\ 4/33 \\ 1/11 \\ 4/33 \\ 1/11 \end{bmatrix}$$

(c) If we look at the structure of the box that the mouse is moving around in, there are three types of rooms,

1. A *corner room* within with the mouse has only two options: Stay in the corner room or move to one of two possible *edge rooms.*
2. An *edge room* within which the mouse has three options: Stay in the edge room, move into one of the two neighboring edge rooms, or move into an *interior room.*
3. An *interior room* within which the mouse has two options: Stay in the *interior room* or move into one of the four possible *edge rooms.*

A state diagram of this simplified three-state Markov chain is given in the figure below.



Observe that the probabilities are assigned based on how many options there are in moving from one state to another or remaining in the same state. For example, in state 2, there are two ways to move a corner room, one way to move to an interior room, and one way to stay in the same room. Thus, the probability of moving to state 1 is $2/4$, the probability of moving to state 3 is $1/4$ and the probability of staying in the same state is $1/4$.

The transition matrix for this Markov chain is

$$\mathbf{P} = \begin{bmatrix} \frac{1}{3} & \frac{1}{2} & 0 \\ \frac{2}{3} & \frac{1}{4} & \frac{4}{5} \\ 0 & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$

This is easily seen to be a regular matrix since $\mathbf{P}^2$ has all positive entries. The steady-state solution is

```
>> P=[1/3 1/2 0 ; 2/3 1/4 4/5 ; 0 1/4 1/5]
>> [v,e]=eig(P);
v =
   -0.58209   0.76200   0.50508
   -0.77611  -0.12700  -0.80812
   -0.24254  -0.63500   0.30305

e =

Diagonal Matrix

   1.00000         0         0
   0         0.25000         0
   0               0  -0.46667
```

Therefore, the eigenvector corresponding to an eigenvalue of one is the first column of `v`. Scaling this column vector so that the sum of the elements is equal to one we have

```
>> v(:,1)/sum(v(:,1))
ans =
      0.36364
      0.48485
      0.15152
```

What this says is that

1. The probability of being in a corner room (state 1) is 0.36364,
2. The probability of being in an edge room (state 2) is 0.48485, and
3. The probability of being in an interior room (state 3) is 0.15152.

Since there are four corner rooms, the probability of being in a *specific* corner room is $0.36364/4 = 0.090909 = 1/11$. Since there are four edge rooms, the probability of being in a *specific* edge room is $0.48485/4 = 0.121212 = 4/33$. And since there is only one interior room, then the probability of being in the interior room remains $0.151515 = 5/33$. These results, of course, are consistent with what we found for the full nine-state model for the problem.