DS-GA 1014 Optimization and Computational Linear Algebra

Lab 2: Linear Transformations & Matrices

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Linear Transformations

A function $L : \mathbb{R}^m \to \mathbb{R}^n$ *is linear if*

(i) *for all* $v \in \mathbb{R}^m$ *and all* $\alpha \in \mathbb{R}$ *we have* $L(\alpha v) = \alpha L(v)$ *and*

(ii) *for all* $v, w \in \mathbb{R}^m$ *we have* $L(v + w) = L(v) + L(w)$.

Remark

- $L(0) = L(0.0) = 0.L(0) = 0.$
- $L(\sum_{i=1}^k \alpha_i v_i) = \sum_{i=1}^k \alpha_i L(v_i).$

Q

Which of the following functions are linear? If the function is linear, what is the kernel?

(a) $f_1 : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f_1(a, b) = (2a, a + b)$

(b) $f_2 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_2(a, b) = (a + b, 2a + 2b, 0)$

(c) $f_3 : \mathbb{R}^2 \to \mathbb{R}^3$ such that $f_3(a, b) = (2a, a + b, 1)$

(d) $f_4 : \mathbb{R}^2 \to \mathbb{R}$ such that $f_4(a, b) = \sqrt{a^2 + b^2}$

(e) $f_5 : \mathbb{R} \to \mathbb{R}$ such that $f_5(x) = 5x + 3$

Previous lab question from Brett Bernstein

GEOMETRY OF LINEAR TRANSFORMATIONS

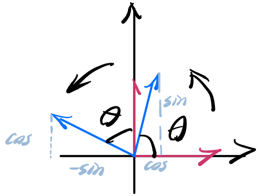
Homothety $H\lambda$

$$\tilde{H}\lambda = \begin{pmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{pmatrix}$$

$\lambda = 1 \Rightarrow$ identity

Rotation $R\theta$

$$\tilde{R}\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$



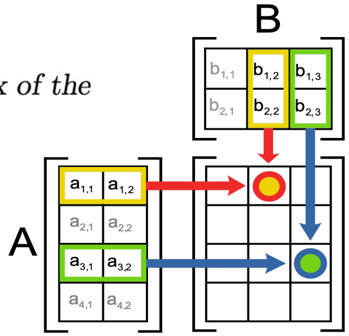
Q

Find a 2 x 2 matrix A (not identity matrix) such that $A^6 = I$.

Matrix Product

Let $L \in \mathbb{R}^{n\times m}$ and $M \in \mathbb{R}^{m\times k}$. We define the matrix product LM as the $n \times k$ matrix of the linear transformation $L \circ M$. His coefficients are given by the formula:

$$(LM)_{i,j} = \sum_{\ell=1}^{m} L_{i,\ell} M_{\ell,j} \quad \textit{for all} \quad 1 \leq i \leq n, \quad 1 \leq j \leq k.$$



Q Calculate the product AB and verify the following properties for matrix multiplication:

- Non-commutativity: AB ≠ BA
- Associativity: A(BC) = (AB)C

$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$

Matrix Inverse

In linear algebra, an n-by-n square matrix $\mathbf{A}$ is called **invertible** (also **nonsingular** or **nondegenerate**), if there exists an n-by-n square matrix $\mathbf{B}$ such that

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$$

Q

Find the inverse of the matrix below and verify. What do you observe?

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Kernels and Images

The kernel $\mathrm{Ker}(L)$ *(or nullspace) of a linear transformation* $L : \mathbb{R}^m \to \mathbb{R}^n$ *is defined as the set of all vectors* $v \in \mathbb{R}^m$ *such that* $L(v) = 0$*, i.e.*

$$\mathrm{Ker}(L) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^m \,|\, L(v) = 0\}.$$

The image $\mathrm{Im}(L)$ *(or column space) of a linear transformation* $L : \mathbb{R}^m \to \mathbb{R}^n$ *is defined as the set of all vectors* $u \in \mathbb{R}^n$ *such that there exists* $v \in \mathbb{R}^m$ *such that* $L(v) = u$*.* $\mathrm{Im}(L)$ *is also the Span of the columns of the matrix representation of* L*.*

Q

Let $T : \mathbb{M}_{22} \mapsto \mathbb{R}^2$ be defined by

$$T \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a - b \\ c + d \end{bmatrix}$$

Then T is a linear transformation. Find a basis for $\ker(T)$ and $\mathrm{im}(T)$.

Solving Linear Systems

1. $y \notin \mathrm{Im}(A)$: there is no solution to $Ax = y$.

2. $y \in \mathrm{Im}(A)$, then there exists $x_0 \in \mathbb{R}^m$ such that $Ax_0 = y$. The set of solutions in then

$$S = \left\{ x_0 + v \,\middle|\, v \in \mathrm{Ker}(A) \right\}.$$

- If $\mathrm{Ker}(A) = \{0\}$, then $S = \{x_0\}$: x_0 is the unique solution.
- If $\mathrm{Ker}(A) \neq \{0\}$, then $\mathrm{Ker}(A)$ contains infinitely many vectors: there are infinitely many solutions.

Slide from the Lecture 2

Gaussian Elimination

Elementary Row Operations

- Swapping two rows, $R_i^* \leftarrow R_j$
- Multiplying a row by a nonzero number, $R_i^* \leftarrow \alpha R_i^*$
- Adding a multiple of one row to another row. $R_i^* \leftarrow R_i + \alpha R_j^*$

Q

Find all values of the coefficients k and c such that the following system has: (i) no solution. (ii) infinitely many solutions. (iii) only one solution.

$$\begin{cases} 2x_2 + 2x_3 = c \\ 3x_1 + 2x_2 + x_3 = 5 \\ x_2 + kx_3 = 3 \end{cases}$$