Recitation 9

https://cims.nyu.edu/ cd2754/

Fall 2021

Convexity and Optimization

- We are temporarily stepping away from *Linearity*
- Two types of convexity
  - Convexity for functions (we care about this)
  - Convexity for sets
  - They are related! (epigraph)
- Convexity implies if a min exists, it must be a global min
  - Optimization is the process to find the minimum
- Convexity is a *global* property
- Contrast to differentiable, which is a *local* property

Convex sets and convex functions

Definition (Convex set)

A set $C \subseteq \mathbb{R}^n$ if for all $x, y \in C$, and all $\alpha \in [0, 1]$,

$$\alpha x + (1 - \alpha)y \in C.$$

Definition (Convex function (and strictly convex function))

A function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if for all $x, y \in \mathbb{R}^n$ and all $\alpha \in [0, 1]$ it holds that

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y). \tag{1}$$

It is strictly convex if moreover $\forall \alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) < \alpha f(x) + (1 - \alpha)f(y). \tag{2}$$

Convex sets

1. Which of the following sets are convex?
   1. $\{x \in \mathbb{R}^2 : \|x\| = 1\}$
   2. $\{x \in \mathbb{R}^2 : \|x\| \leq 1\}$
   3. $\{x \in \mathbb{R}^2 : \|x\| \geq 1\}$
   4. $\{x \in \mathbb{R}^2 : \|x\| < 1\}$
   5. $\{x \in \mathbb{R}^2 : v^\top x \geq a\}$ for fixed $v \in \mathbb{R}^2$ and $a \in \mathbb{R}$.
   6. $\{x \in \mathbb{R}^2 : v^\top x = a\}$ for fixed $v \in \mathbb{R}^2$ and $a \in \mathbb{R}$.
   7. $\{x \in \mathbb{R}^2 : x_2 \geq x_1^2\}$
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Questions: True and False

1. If f has only 1 global min and no local min, then f is convex
2. Linear combination of two convex functions is convex
3. Convex functions are differentiable at all points
4. Norms are convex functions
5. If f is convex, then $g(x) = f(Ax - b)$ is also convex. ($A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$)
6. Sum of a non-convex function w/ another function can never be convex
7. Union of convex sets is convex
8. Intersection of convex sets is convex
9. Maximum of two convex functions is convex
10. Every subspace is a convex set
11. Every convex set is a subspace

Gradients and Hessians

Calculate the gradients and the Hessians of the following functions $f : \mathbb{R}^n \to \mathbb{R}$:

1. $f(x) = \|x\|^2$.
2. $f(x) = \|Ax\|^2$.
3. $f(x) = x^\top Ax$.

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