Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 4.1 (3 points). *For vectors $x = (x_1, x_2)$ in $\mathbb{R}^2$ we define $N(x) = \max(|x_1|, |x_2|)$.*

(a) *Show that $N(\cdot)$ is a norm on $\mathbb{R}^2$.*

(b) *Let $x = (8, 0)$ and $y = (7, 7)$. Using the $N(\cdot)$, which of x and y has bigger norm? Using the Euclidian norm, which of x and y has bigger norm?*

(c) *Imagine that $x = (x_1, x_2)$ represents one student's grade at the midterm (x_1) and at the final (x_2). To rank students in the class, would you rather choose to apply $N(\cdot)$ or the Euclidian norm to their grades? Explain the reason of your choice.*

Problem 4.2 (3 points). *Suppose that we define an inner product $\langle \cdot, \cdot \rangle_A$ by*

$$\langle x, y \rangle_A = x^\top A y = \sum_{i,j=1}^{N} x_i A_{ij} y_j$$

for some $A \in \mathbb{R}^{n \times n}$. Prove that in order for $\langle \cdot, \cdot \rangle_A$ to be an inner product the following conditions are necessary:

(a) *A is symmetric,*

(b) *$A_{ii} > 0$ for $i = 1 \cdots n$,*

(c) *A is invertible.*

Problem 4.3 (1.5 points). *Let S be a subspace of $\mathbb{R}^n$ with $x \mapsto P_S(x)$ the orthogonal projector of all $x \in \mathbb{R}^n$. We will use the Euclidian dot product as our inner product $\langle x, y \rangle = x \cdot y$. Show that for any $x \in \mathbb{R}^n$*

(a) *for any vector y in S, $x \cdot y = P_S(x) \cdot y$,*

(b) *$x - P_S(x)$ is orthogonal to S,*

(c) *$\|P_S(x)\| \leq \|x\|$.*

Problem 4.4 (1.5 point). *Show that for any vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$: $\left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} x_k^2$.*

Problem 4.5 ($\star$). *Recall Cauchy-Schwarz inequality. Let $\|\cdot\|$ be the norm induced by the inner product $\langle\cdot,\cdot\rangle$ on the vector space V. Then for all $x, y \in V$:*

- $|\langle x, y\rangle| \leq \|x\|\,\|y\|$.
- *Moreover, there is equality if and only if x and y are linearly dependent, i.e. $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbb{R}$.*

Prove Cauchy-Schwarz inequality by studying the function $f : t \mapsto \|x - ty\|_2^2$, for t real and for any $x, y \in V$. Be carefull to justify everything precisely.