Lab 7

DSGA-1014: Linear Algebra and Optimization

CDS at NYU
Zahra Kadkhodaie

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Diagonalization

Let A be a square $n \times n$ matrix with n linearly independent eigenvectors. Then A is called diagonalizable and we can write

$$A = Q\Lambda Q^{-1}$$

where Q consists of eigenvectors of A and Λ is a diagonal matrix consists of eigenvalues of A. Note that Q^{-1} exits, because eigenvectors are linearly independent, so Q is invertible.

Diagonalization

Not all square matrices are diagonalizable:

1. If A has n distinct eigenvalues, then all the n eigenvectors are independent and A is diagonalizable. That is, multiplicity of all eigenvalues is one, and $\text{rank}(E_{\lambda_i}(A)) = 1$ for $\forall i$
2. If A has repeated eigenvalues, then it *might* be the case that A does not have n linearly independent eigenvectors. That is for some λ_i, $\text{rank}(E_{\lambda_i}(A)) <$ multiplicity of λ_i. In this case, A is not diagonalizable, because Q does not have an inverse.
3. If A is symmetric, then it is guaranteed that A has n linearly independent eigenvectors and is diagonalizable (even if some eigenvalues are repeated!). Moreover, not only eigenvectors are independent, but they are orthogonal too. So

$$A = Q\Lambda Q^T$$

   This is referred to as Spectral Theorem.

1. Let $P \in R^{n \times n}$ be a projection matrix.

(a) Show that P is always diagonalizable.

(b) What are the eigen values?

(c) Is P orthogonal?

(d) Define $P = \begin{bmatrix} .2 & .4 & 0 \\ .4 & .8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. For each eigen value, give the rank of corresponding eigen space, $E_\lambda(P)$.

2. What matrix A has eigenvalues $\lambda = 1, -1$ and eigenvectors $v_1 = (\cos\theta, \sin\theta)$ and $v_2 = (-\sin\theta, \cos\theta)$? Which of these properties hold? $A = A^T$, $A^2 = I$, $A^{-1} = A$.

3. Define $x_1 = (4, 1), x_2 = (-3, 1)$, and $x_3 = (1, 1)$

(a) Give a one-dimensional affine subspace of R^2 that best approximates these three points.

(b) Use this to represent each point using a single number (i.e., reduce the dimension from 2 to 1).

(c) Describe the eigen decomposition of the covariance matrix without computing it directly.

4. Suppose $A \in R^{n \times n}$ has a linearly independent list of n eigenvectors $v_1, ..., v_n$ with real eigenvalues $\lambda_1, ..., \lambda_n$. Can we factor A in a way similar to the spectral decomposition? Show that if $v_1, ..., v_n$ are orthonormal, then A has to be symmetric.

5. Let A and B be diagonalizable matrices. Also assume that α is an eigenvalue of A and β is an eigenvalue of B. Under what condition $\alpha\beta$ is an eigenvalue of AB?