# recall: unconstrained optimization

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function. We say that $x \in \mathbb{R}^n$ is

- a **critical point** of $f$ if $\nabla f = 0$
- a **global minimizer** of $f$ if for all $x' \in \mathbb{R}^n$ it holds that $f(x) \leq f(x')$
- a **local minimizer** of $f$ if there exists $\delta > 0$ such that for all $x' \in B(x, \delta)$ it holds that $f(x) \leq f(x')$.
  Note that: $B(x', \delta) = \{x' \mid \|x' - x\| \leq \delta\}$ are the all the elements inside the ball centered at $x$ with radius $\delta$

**Theorem: First order necessary conditions**

Let $x \in \mathbb{R}^n$ be a point at which $f$ is differentiable. Then,

$$\boxed{x \text{ is a local minimizer of } f \implies \nabla f(x) = 0}$$

# Unconstrained optimization

**Theorem: Second order sufficient conditions**

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a differentiable function and let $x \in \mathbb{R}^n$ be a critical point of $f$ (that is $\nabla f(x) = 0$). Then,

- If $H_{f(x)}$ is **positive definite** (*all its eigenvalues are strictly positive*), then $x$ is a local **minimizer** of $f$
- If $H_{f(x)}$ is **negative definite** (*all its eigenvalues are strictly negative*), then $x$ is a local **maximizer** of $f$
- If $H_{f(x)}$ **admits positive and negative eigenvalues**, then $x$ is neither a local minimizer nor a local maximizer of $f$. We call $x$ a **saddle point**

# practice: unconstrained optimization

**Exercise 1**

What happens when $H_{f(x)}$ is positive semidefinite (or negative semidefinite)?

1. Give an example of a twice-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with a local minimizer $x$ such that $H_{f(x)}$ is positive semidefinite.
2. Give an example of a twice-differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$ with critical point $x$ such that $x$ is not a local minimizer and $H_{f(x)}$ is positive semidefinite.

① $f(x,y) = x^2$

$\nabla f = \begin{pmatrix} 2x \\ 0 \end{pmatrix}$ $\Rightarrow$ critical points $(0,y)$ $\forall y \in \mathbb{R}$

local minimizer bc $f(x,y) \geq 0$

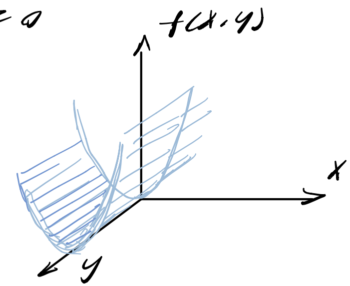
(global)

$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$(v_1\ v_2)\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$= (2v_1\ 0)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 2v_1^2 \geq 0 \ \forall \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$

$\Rightarrow$ $Hf$ positive semidefinite. $(tr(A) = 2 = \underset{\lambda_1}{2} + \underset{\lambda_2}{0})$



**Exercise 1**

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(2) $f(x,y) = x^2 + y^3$

$\nabla f = \begin{pmatrix} 2x \\ 3y^2 \end{pmatrix} \Rightarrow$ critical point $(0,0)$

NOT local minimizer

$\forall \delta > 0,\ f(0,-\delta) = -\delta^3 < 0$

$Hf = \begin{pmatrix} 2 & 0 \\ 0 & 6y \end{pmatrix}$

$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$

$\Rightarrow$ positive semidefinite

# recall: constrained optimization

The problem with constrained optimization is

$$\begin{aligned} \text{minimize} \;\; & f(x) & \\ \text{maximize} \;\; & g_i(x) \leq 0 & i = 1, .., m \\ & h_i(x) = 0 & i = 1, .., p \end{aligned}$$

# recall: constrained optimization

**Theorem KKT: necessary conditions**

Assume that the functions $f, g_1, ..., g_m, h_1, ..., h_p$ in the above setting are differentiable.
Assume that $x$ is a solution of the problem above with $\{\nabla g_i(x) \,|\, g_i(x) = 0\} \cup \{\nabla h_i(x) \,|\; i \in \{1..p\}\}$ are linearly independent vectors.
**Then,** there exists scalars $\lambda_1, .., \lambda_m \geq 0$ *and* $\nu_1, .., \nu_p \in \mathbb{R}$ such that:

$$\boxed{\nabla f(x) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x) + \sum_{i=1}^{m} \nu_i \nabla h_i(x) = 0}$$

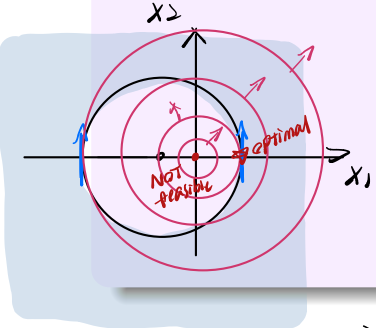
for all $i \in \{1...m\}, \lambda_i = 0$ if $g_i(x) < 0$

# practice: constrained optimization

## Exercise 2

g is not convex here
use First order Optimality conditions instead

Using the ~~KKT~~ necessary conditions, find the minimum and the minimizers of the following constrained optimization problem




$$\text{minimize} \quad {x_1}^2 + {x_2}^2 \qquad f$$

$$\text{subject to} \quad 4 - (x_1 + 1)^2 - {x_2}^2 \leq 0 \qquad g$$

$$\nabla f = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix}$$

$$\nabla g = \begin{pmatrix} -2(x_1+1) \\ -2x_2 \end{pmatrix}$$

$$\nabla f + \lambda \nabla g = 0$$

$$\Rightarrow \begin{pmatrix} 2x_1 - 2\lambda(x_1+1) \\ 2x_2 - 2\lambda x_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} x_1 - \lambda x_1 - \lambda = 0 \\ x_2 - \lambda x_2 = 0 \end{cases}$$

If $x_2 \neq 0$, $\lambda = 1 \Rightarrow x_1 - x_1 - 1 = -1 \neq 0$
contradiction!

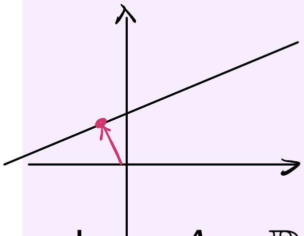
Thus, $x_2 = 0$

(i) $\lambda = 0$
$\Rightarrow x_1 = 0$ but $g(0,0) = 3 > 0$
NOT feasible

(ii) $4 - (x_1+1)^2 - 0 = 0$
$\Rightarrow x_1 = -3$ or $1$
$f(-3,0) = 9 \quad f(1,0) = 1$
The solution is $(1,0)$

# Exercise 9, 2018 review

Consider the optimization problem



$$\text{minimize}_x \ \|x\|^2 \qquad f = x^T x$$

$$\text{subject to } Ax = b \qquad h = Ax - b$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ are fixed and $b \in Im(A)$.

a) Prove that any minimizer $x^*$ must belong to $Im(A^T)$ $\quad A^+ = A^T(AA^T)^{-1}$

b) Give a formula for the minimizer $x^*$ and show it is unique

① $\nabla f = 2x \qquad \nabla h = A^T$

By KKT, any minimizer $x^*$ should satisfy

$\nabla f(x^*) + v \nabla h(x^*) = 0$

$\Rightarrow 2x^* + A^T v = 0$

$\Rightarrow x^* = -\frac{1}{2} A^T v = A^T \underbrace{(-\tfrac{v}{2})}_{\omega}$

Thus, $x \in Im(A^T)$

$Ax^* = AA^T\omega = b \Rightarrow \omega = (AA^T)^{-1} b$

② The solution to $Ax=b$ is

$x = A^+ b + ker(A)$

$\forall\, v \in ker(A)$, $\|A^+b + v\|^2$

$= \|A^+ b\|^2 + \|v\|^2 + 0$ bc. $A^+ b \perp v$

$> \|A^+ b\|$ if $v \neq 0$

Therefore, $x^* = A^+ b$ is the only minimizer

# Extra KKT question

$$\boxed{\begin{array}{ll} \min & x_1^2+x_2^2-4x_1-4x_2 \\ \text{subject to} & x_1^2 \le x_2 \\ & x_1+x_2 \le 2 \end{array}}$$

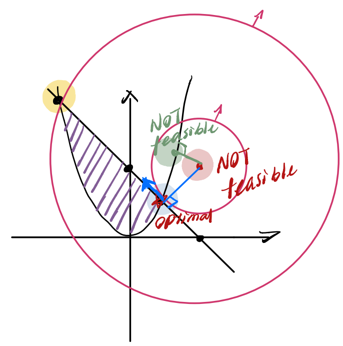
$f = (x_1-2)^2+(x_2-2)^2-8$

$g_1 = x_1^2 - x_2$

$g_2 = x_1+x_2-2$

$$\nabla f = \begin{pmatrix} 2x_1-4 \\ 2x_2-4 \end{pmatrix}$$

$$\nabla g_1 = \begin{pmatrix} 2x_1 \\ -1 \end{pmatrix} \qquad \nabla g_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$




$$\nabla f + \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 = 0$$

$$\Rightarrow \begin{pmatrix} 2x_1-4+2\lambda_1x_1+\lambda_2 \\ 2x_2-4-\lambda_1+\lambda_2 \end{pmatrix} = 0$$

$$\Rightarrow \begin{cases} 2x_1+2\lambda_1x_1+\lambda_2 = 4 & ① \\ 2x_2-\lambda_1+\lambda_2 = 4 & ② \end{cases}$$

① $\lambda_1 = \lambda_2 = 0$

$\Rightarrow x_1 = x_2 = 2$ but $x_1+x_2 = 2+2 = 4 > 2$, $x_1^2 = 4 < 2 = x_2$

NOT feasible

Both constraints need to be active

④ $x_1^2 = x_2$, $x_1+x_2 = 2$
③ ④

Use ③ to eliminate $x_2$ in ④

$\Rightarrow x_1^2+x_1-2 = (x_1+2)(x_1-1) = 0$

$\Rightarrow x_1 = -2$ or $x_1 = 1$

- When $x_1 = -2$, $x_2 = 4$
  $f(-2,4) = 12$
- When $x_1 = 1$, $x_2 = 1$
  $f(1,1) = -6 \Rightarrow$ optimal solution

(Note $\lambda_1 = 0$, $\lambda_2 = 2$ here)

② $\lambda_1 = 0$, $x_1+x_2 = 2$

$$\Rightarrow \begin{cases} 2x_1+\lambda_2 = 4 \\ 2x_2+\lambda_2 = 4 \\ x_1+x_2 = 2 \end{cases}$$

$\Rightarrow x_1 = x_2 = 1$, $\lambda_2 = 2$

$x_1^2 = 1 \le 1 = x_2$ ✓

③ $\lambda_2 = 0$, $x_1^2 = x_2$

$$\Rightarrow \begin{cases} 2x_1+2\lambda_1x_1 = 4 \\ 2x_2-\lambda_1 = 4 \\ x_1^2 = x_2 \end{cases}$$

$\Rightarrow 2x_1+2(2x_1^2-4)x_1 = 4$

$2x_1^3-3x_1-2 = 0$

$x_1 \approx 1.47$

$x_1+x_2 > 2$ NOT feasible