Recitation 6

https://cims.nyu.edu/ cd2754/

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Warmup: Eigenvector

Let $A \in \mathbb{R}^{n \times n}$ have eigenvalue λ associated to eigenvector v. Show that:

1. $\forall \alpha \in \mathbb{R}$, $\lambda + \alpha$ is an eigenvalue of $A + \alpha I$ w/ eigenvector v.
2. $\forall k \in \mathbb{N}$, λ^k is an eigenvalue of A^k w/ eigenvector v.

1. Since $Av = \lambda v$, then $(A+\alpha I)v = Av + \alpha I v = \lambda v + \alpha v = (\lambda + \alpha) v$

2. $Av = \lambda v$. $A \cdot Av = A \lambda v = \lambda A v = \lambda^2 v$.

Then $A^k v = \lambda^k v$.

Questions 2: Properties of Orthogonal Matrices

Let $Q \in \mathbb{R}^{n \times n}$ be orthogonal.

1. Does Q necessarily have eigenvalues and eigenvectors?

Assume that Q has eigenvalues $\lambda_1, ..., \lambda_k$.

2. Describe the eigenvalues of Q.

1. No. $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ rotation matrix

2. If Q has eigenvalues and eigenvectors $u_1 \cdots u_k$, then

for any u_i.

$$\|Qu_i\| = u_i^T Q^T Q u_i = u_i^T u_i = \|u_i\|$$

Then $\|Qu_i\| = \|\lambda_i u_i\| = |\lambda_i| \|u_i\| = \|u_i\|$

So $\lambda_i = \pm 1$

Questions: Stochastic Matrices

Let $A, B \in \mathbb{R}^{n \times n}$ be stochastic matrices. True or False for 1,2,3.

1. A is always invertible
2. The eigenvector corresponding to the largest eigenvalue of A is unique
3. A cannot have zero has its eigenvalue
4. Prove that AB is a stochastic matrix.

1. F $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

3. F $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

2. F $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

4. True. Let A, B be stocastic, i.e. columns of A, B sum to 1.

Then let $c_1 \dots c_n$, $r_1 \dots r_n$ be columns and rows of A.

$\sum (Ab_1)_i$

$= \sum r_i^T \cdot b_1$

$= (\sum r_i^T)_i b_1$

$= (\sum c_{1j}, \sum c_{2j} \dots \sum c_{nj})^T \cdot b_1$

$= (1, \dots 1)^T \cdot b_1$

$= \sum b_{1j}$

$= 1$

each column of AB has sum 1, it is easy to check all entries are positive. So AB is stocastic.

Questions: Spectral Theorem

1. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. Give a vector v with $\|v\| = 1$ such that $\|Av\|$ is maximized.

let $A = PDP^T$ where P is orthogonal D is diagonal, then

let $u_1 \dots u_n$ be eigenvectors of A, and $v = \sum \alpha_i u_i$. Note u_i forms P.

$$
\begin{aligned}
Av &= PDP^T \sum \alpha_i u_i \\
&= PD \sum \alpha_i P^T u_i \\
&= PD \sum \alpha_i e_i \\
&= P \sum \alpha_i D e_i \\
&= P \sum \alpha_i \lambda_i e_i \\
&= \sum \alpha_i \lambda_i P e_i \\
&= \sum \alpha_i \lambda_i u_i
\end{aligned}
$$

Then $\|Av\|$

$$
\begin{aligned}
&= \| \sum \alpha_i \lambda_i u_i \| \\
&= \sum \alpha_i^2 \lambda_i^2 \|u_i\| \\
&= \sum \alpha_i^2 \lambda_i^2
\end{aligned}
$$

Since $\|v\| = 1$, then $\| \sum \alpha_i u_i \| = 1$

$$\sum \alpha_i^2 = 1$$

Therefore $\|Av\|$ is maximized when $\alpha_i^2 = 1$.

and $\|Av\| = \lambda_i^2$