PROBLEM 1

① $v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \longrightarrow \boxed{u_1 = \frac{v_1}{\|v_1\|} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

② $\tilde{u}_2 = v_2 - \underbrace{P_{span(u_1)}(v_2)}$

$$\langle v_2, u_1 \rangle u_1 = \left(2/\sqrt{3} + 1/\sqrt{3} + 1/\sqrt{3} \right) u_1$$

$$= 4/\sqrt{3} \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} = 4/3 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \tilde{u}_2 = \begin{pmatrix} 2 - 4/3 \\ 1 - 4/3 \\ 1 - 4/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}$$

and $\|\tilde{u}_2\|_2 = \sqrt{(2/3)^2 + (1/3)^2 + (1/3)^2} = \frac{1}{3}\sqrt{6}$

$$\longrightarrow u_2 = \frac{\tilde{u}_2}{\|\tilde{u}_2\|_2} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$$

③ $\tilde{u}_3 = v_3 - \underbrace{P_{span(u_1, u_2)}(v_3)}_{= \langle v_3, u_1 \rangle u_1 + \langle v_3, u_2 \rangle u_2}$

with $\langle v_3, u_1 \rangle u_1 = \left(2/\sqrt{3} + 0 + 1/\sqrt{3} \right) 1/\sqrt{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$= \frac{3}{3} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and $\langle v_3, u_2\rangle u_2 = \left(2\times 2 + 0 + 1(-1)\right)\frac{1}{\sqrt{6}} \times \frac{1}{\sqrt{6}}\begin{pmatrix}2\\-1\\-1\end{pmatrix}$

$$= 1/2 \begin{pmatrix}2\\-1\\-1\end{pmatrix}$$

so that $\tilde{u}_3 = \begin{pmatrix}1\\0\\2\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix} - 1/2\begin{pmatrix}2\\-1\\-1\end{pmatrix} = \begin{pmatrix}-1\\-1/2\\-1/2\end{pmatrix}$

and $\|\tilde{u}_3\| = \sqrt{1 + 1/4 + 1/4} = \frac{\sqrt{6}}{2}$

so that $\boxed{u_3 = -\frac{1}{\sqrt{6}}\begin{pmatrix}2\\1\\1\end{pmatrix}}$

PROBLEM 2:

(a) $M_V = \begin{pmatrix}1/2\\ \vdots \\ 1/2\end{pmatrix}\left(1/2, \dots, 1/2\right) = 1/4\begin{pmatrix}1&1&1&1\\1&1&1&1\\1&1&1&1\\1&1&1&1\end{pmatrix}$

$\rightarrow$ all columns are the same: rank $M_V = 1$

(b)

$M_V = \begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}\begin{pmatrix}1&0&0&0\\0&1&0&0\end{pmatrix} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$

rank of M_V is 2 -

(c) $M_W x = 1/4 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = 1/4 \begin{pmatrix} 10 \\ 10 \\ 10 \\ 10 \end{pmatrix}$

$M_V x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 0 \\ 0 \end{pmatrix}$

$\neq$

(d)

$M_W M_V =$

$$1/4 \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \frac{1}{4} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

$M_V M_W =$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Tip: You could also use that $(AB)^T = B^T A^T$ noting that M_W and M_V are symmetric matrices -

Intuition: U and V don't overlap except in 0. Projecting last on U will give vectors in U while projecting last on V will give vectors in V. So we expect the order of projection to matter!

(e) $\Pi_V = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$\Pi_U \Pi_V = \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \Pi_U = \Pi_V \Pi_U$$

They commute because $U \subset V$ this time.

PROBLEM 3:

$$\begin{cases} x' = U^T x & \rightarrow \quad x = U x' \\ y' = U^T y \end{cases}$$

let $x \in \mathbb{R}^n$ and $x' = U^T x$ its coordinates in basis U

let $y = \tilde{L} x$, in the canonical basis, in the "U" basis we have $y' = U^T \tilde{L} x$.

And $x' = U^T x \Rightarrow x = Ux'$

(because U is an orthogonal matrix!
$U^{-1} = U^T$
$(U^T)^{-1} = U$)

So that finally $y' = \underline{\underline{U^T \tilde{L} U}} x'$ → transforms in U coordinates.

PROBLEM(4)

(a) as usual

(b) We can use the rank nullity theorem for the linear transformation corresponding to the orthogonal projector on S.

$$\begin{cases} \mathrm{Im}(P_S) = S \\ \ker(P_S) = S^\perp \end{cases}$$

(c) For any $u \in \mathbb{R}^n$ $P_S(u) \in S$ and $u - P_S(u) \in S^\perp$

$$u = P_S(u) + (u - P_S(u))$$
$$= x + y.$$

$\boxtimes$.