PROBLEM 8.1

$$A = U\Sigma V^T \qquad \text{if } r < \min(n,m)$$

(a) We saw in class that

$$A = \sigma_1 u_1^T v_1 + \cdots + \sigma_r u_r^T v_r$$

$$= \begin{pmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{pmatrix} \begin{pmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{pmatrix} \begin{pmatrix} — v_1^T — \\ \vdots \\ — v_r — \end{pmatrix}$$

just by rewriting.

(b) We have seen in previous homeworks that

$$\mathrm{Ker}(A) = \mathrm{Ker}(A^TA) \subset \mathbb{R}^n \qquad (3.4)$$

a basis of $\mathrm{Ker}(A^TA) = E_0(A^TA)$ are eigenvectors of A^TA associated with 0.

↳ by construction, the right eigenvectors associated with 0 : $\underbrace{v_{r+1}, \dots v_m}_{m-r}$.

$$\begin{cases} \mathrm{Im}(AA^T) \subset \mathrm{Im}(A) \subset \mathbb{R}^n \\ \mathrm{rank}(AA^T) = \mathrm{rank}(A^TA) = m - \dim \mathrm{Ker}(A^TA) = r \end{cases}$$

since same number of non 0 eigenvalue

So $Im(AA^T) = Im(A)$ and a basis of $Im(AA^T)$ are all its eigenvectors with non zero eigen-value: $(u_1 \cdots u_r)$

PROBLEM 8.2

For any $A \in \mathbb{R}^{n \times m}$:

$$\|A\|_F = \sqrt{Tr(A^TA)}$$

$$= \sqrt{Tr(V\Sigma U^T U \Sigma V^T)}$$

→ since orthogonal matrices

$$= \sqrt{Tr(V\Sigma^2 V^T)}$$

and

$Tr(AB) = Tr(AB)$ for all A, B square

$$= \sqrt{Tr(\Sigma^2)}$$

$$= \sqrt{\sum_{i=1}^{\min(n,m)} \sigma_i^2}$$

PROBLEM 8.3

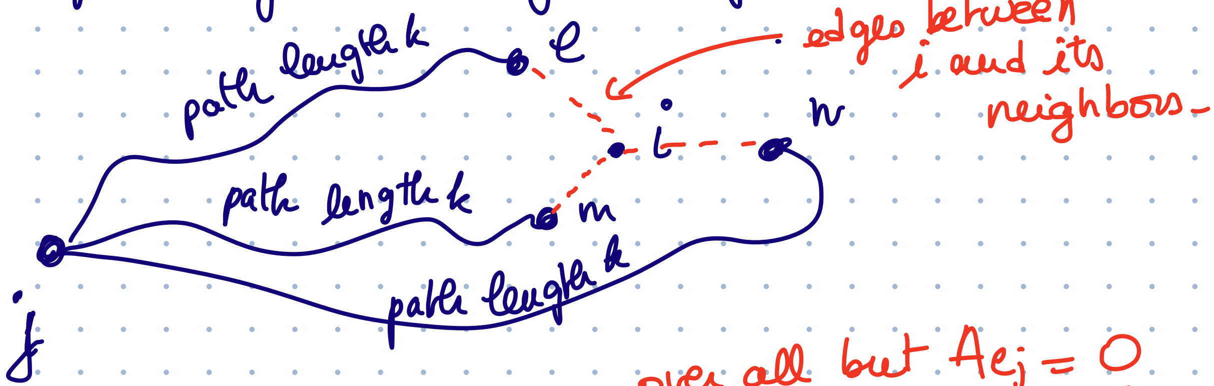
(a) $\mathcal{H}(1)$: Basically the definition of the adjacency matrix.

(b) Assume $\mathcal{H}(k)$ # paths from $i \to j$ of length k is $(A^k)_{ij}$

Number of paths of lengths $k+1$

"number of paths of length k that ended "one edge away"

e.g. from $j \to$ neighbor of i



over all but $A_{\ell j} = 0$ if no edge.

$$\#(k+1) = \sum_{\ell \sim i} (A^k)_{i\ell} = \sum_{\ell=1}^{n} (A^k)_{i\ell} A_{\ell j}$$

over edges

$$= (A^{k+1})_{ij}$$

by definition of matrix multiplicatio

$\mathcal{H}(k+1)$ is true.

Proved by induction. _

PROBLEM 8.5

Let G be a connected graph with n nodes. Define $L \in \mathbb{R}^{n \times n}$ the associate laplacian matrix, with spectrum

$0 = \lambda_1 < \lambda_2 \leq \dots \leq \lambda_n$.

Let G' be a graph constructed from adding an edge to G. Denote by λ_2' its second smallest e.v.

Show that $\lambda_2' \geq \lambda_2$.

say we add an edge between k and l:

$$x^T L' x = x^T L x + (x_k - x_l)^2$$

eigenvector associated with smallest eigenvalue v_1

by definition

$$\lambda_2 = \min_x \frac{x^T L x}{\|x\|} \quad \Big| \quad x \perp \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$$\lambda_2' = \min_x \frac{x^T L' x}{\|x\|} \quad \Big| \quad x \perp \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

$v_1' = v_1$

yet for any x $x^T L x \leq x^T L' x$

so we are minizing over the same set a quantity that is larger or equal:

$\Rightarrow \lambda_2' \geq \lambda_2$.