Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 5.1 (2.5 points). *Give an othonormal basis of $\mathbb{R}^3$ using the Gram-Schmidt algorithm starting from the linearly independent family (v_1, v_2, v_3) where $v_1 = (1,1,1)$, $v_2 = (2,1,1)$ and $v_3 = (2,0,1)$.*

Problem 5.2 (2.5 points). *Consider $U = \text{Span}\left((\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})\right)$ and $V = \text{Span}\left((1,0,0,0),(0,1,0,0)\right)$, two subspaces of $\mathbb{R}^4$.*

(a) *Compute the canonical matrix $M_U \in \mathbb{R}^{4\times 4}$ of orthogonal projection $P_U(\cdot)$ onto subspace U. What is the rank of M_U?*

(b) *Compute the canonical matrix $M_V \in \mathbb{R}^{4\times 4}$ of orthogonal projection $P_V(\cdot)$ onto subspace V. What is the rank of M_V?*

(c) *Let $x = (1,2,3,4)$ in $\mathbb{R}^4$, compute $y = P_U \circ P_V(x)$ and $z = P_V \circ P_U(x)$. Do we have $y = z$?*

(d) *Compute the matrix products $M_U M_V$ and $M_V M_U$. Do M_U and M_V "commute", meaning do we have $M_U M_V = M_V M_U$. Can you give an intuition of why it is the case looking the definitions of U and V?*

(e) *Considering now $U' = \text{Span}\left((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0, 0)\right)$. Compute $M_{U'}$. Do we have $M_{U'} M_V = M_V M_{U'}$? Can you give an intuition why?*

Problem 5.3 (1 points). *Consider L a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ and denote by $\tilde{L} \in \mathbb{R}^{n\times n}$ its canonical matrix. Let $(u_1, \cdot, u_n)$ be any orthonormal basis of $\mathbb{R}^n$ and*

$$U = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \in \mathbb{R}^{n\times n}$$

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Show that $\tilde{L}' = U^\top \tilde{L} U$ computes the transformation of vectors in $\mathbb{R}^n$ using coordinates in the basis $(u_1, \cdots, u_n)$.

Problem 5.4 (3 points). *In this problem, we will see how to compress, by using a particular orthonormal basis called a "discrete cosine basis".*
All the questions are in the jupyter notebook `DCT.ipynb` *and have to be answered directly in the notebook. (Submit only a pdf export of your notebook: Print → Save as pdf)*
You have to use `Python` *and its library* `numpy`*. A useful command:* `A @ B` *: performs the matrix product of the matrix* `A` *with the matrix* `B`*.*

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Problem 5.5 ($\star$). *Let S be a subspace of $\mathbb{R}^n$. We define the orthogonal complement of S by*

$$S^{\perp} \stackrel{\text{def}}{=} \{x \in \mathbb{R}^n \,|\, x \perp S\} = \{x \in \mathbb{R}^n \,|\, \forall y \in S,\, \langle x, y \rangle = 0\}.$$

(**a**) *Show that $S^{\perp}$ is a subspace of $\mathbb{R}^n$.*

(**b**) *Show that* $\dim(S^{\perp}) = n - \dim(S)$.

(**c**) *Show that for any $u \in \mathbb{R}^n$, we can find $x \in S$ and $y \in S^{\perp}$ such that $u = x + y$.*