PROBLEM 4.1

For vectors in $\mathbb{R}^2$, we define $N(x) = \max(|x_1|, |x_2|)$.

(a) norm: (i) non negative values.

(ii) $N(\alpha x) = |\alpha| N(x)$

(iii) $N(x) = 0$ only if $x_1 = x_2 = 0 \Rightarrow x = 0$ in $\mathbb{R}^2$

(iv) triangular inequality: For any x, y in $\mathbb{R}^2$,

$$N(x+y) = \max(|x_1+y_1|, |x_2+y_2|)$$

yet
$$|x_1+y_1| \leq |x_1| + |y_1|$$
$$|x_2+y_2| \leq |x_2| + |y_2|$$

and
$$\begin{cases} N(x) \geq |x_1| \\ N(x) \geq |x_2| \\ N(y) \geq |y_1| \\ N(y) \geq |y_2| \end{cases}$$

$$|x_1+y_1| \leq N(x) + N(y)$$
$$|x_2+y_2| \leq N(x) + N(y)$$

$\Rightarrow N(x+y) \leq N(x) + N(y)$, which proves the triangular inequality.

(b) $x = (8, 0)$

$\|x\|_2 = \sqrt{8^2+0^2} = 8$

$N(x) = 8$

$y = (7, 7)$

$\|y\|_2 = \sqrt{7^2+7^2} = 7\sqrt{2} \approx 9.9$

$N(y) = 7$

$\|x\|_2 < \|y\|_2$ but $N(x) > N(y)$.

(c) Probably the $\|\cdot\|_2$ to give credit to performance on both exams.
(either answer is ok as long as shown that what the norm does).

PROBLEM 4-2:

$$\langle x, y\rangle_A = x^T A y = \sum_{ij} x_i A_{ij} y_j$$

(a) need $\langle x, y\rangle_A = \langle y, x\rangle_A$

↳ in particular for $\begin{cases} x = e_k \\ y = e_\ell \end{cases}$ with ℓ and $k \in \{1, ---, n\}$

$$\langle e_k, e_\ell\rangle_A = \sum_{ij} \underbrace{(e_k)_i}_{\text{only non zero if } k=i} A_{ij} \underbrace{(e_\ell)_j}_{\text{only non-zero if } k=j} = A_{k\ell}$$

$$= \langle e_\ell, e_k\rangle_A = A_{\ell k}$$

So we need $A_{k\ell} = A_{\ell k}$ for all $k, \ell \in \{1, ..., n\}$

$\Rightarrow$ A symmetric.

(b) need $\langle x, x\rangle_A \geq 0$ and equality if and only if $v = 0$

↳ in particular for e_i for any i in $\{1, ..., n\}$

$\langle e_i, e_i\rangle_A = A_{ii} > 0$ (with $>$ and not $\geq$ since the $e_i \neq 0$-vector of $\mathbb{R}^n$)

(c) by contradiction, if A is not invertible then $\dim \mathrm{Im}(A) < n$ and $\dim \mathrm{Ker}(A) > 0$, which implies that there exists a non-zero vector $x \in \mathrm{Ker}(A)$, and $Ax = 0$

$$\Rightarrow x^T A x = 0 = \langle x, x\rangle_A \quad \text{CONTRADICTION}$$

PROBLEM 4.3

(a)

Let y be a vector of S, and $(v_1 ... v_n)$ an orthonormal basis of S.

We have,

$$y = \langle v_1, y\rangle v_1 + \cdots + \langle v_n, y\rangle v_n$$

$$P_S(x) = \langle v_1, x\rangle v_1 + \cdots + \langle v_n, x\rangle v_n$$

So that $\langle y, x\rangle = \langle \langle v_1, y\rangle v_1 + \cdots + , x\rangle$

$$= \langle v_1, y\rangle\langle v_1, x\rangle + \cdots \langle v_n, y\rangle\langle v_n, x\rangle$$

and $\langle y, P_S(x)\rangle = \underbrace{\langle \langle v_1, y\rangle v_1 + \cdots , \langle v_1, x\rangle v_1 + \cdots \rangle}$

would have all crossed terms : $\langle v_i, y\rangle \underbrace{\langle v_i, v_j\rangle}_{\substack{=0 \text{ if } i \neq j \\ =1 \text{ if } = j}} \langle v_j, x\rangle$

because $v_1, ... v_n$ orthonormal

so that only remains:

$$\langle y, P_S(x)\rangle = \langle v_1, y\rangle\langle v_1, x\rangle + \cdots + \langle v_n, y\rangle\langle v_n, x\rangle$$
$$= \langle y, x\rangle$$

(b) Have to show that $x - P_S(x)$ is orthogonal to any vector in S.

For any y in S $\quad \langle y, x - P_S(x)\rangle = \langle y, x\rangle - \langle y, P_S(x)\rangle$

$$= 0$$

according to previous question.

(c) As $P_S(x) - x$ and $P_S(x) \in S$ are orthogonal (previous question) we can use Pythagorean theorem:

$$\|x\|^2 = \|x - P_S(x)\|^2 + \|P_S(x)\|^2$$

$$\Rightarrow \|x\|^2 \geq \|P_S(x)\|^2$$

$$\Rightarrow \|x\| \geq \|P_S(x)\|$$

steps allowed because $\|x\|$ and $\|P_S(x)\|$ are both positive.

PROBLEM 4.4

Apply Cauchy Schwartz to $x \in \mathbb{R}^n$ and $\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \in \mathbb{R}^n$

$$|\langle x, y \rangle| \leq \|x\| \|y\|$$

$$\Rightarrow \langle x, y \rangle^2 \leq \|x\|^2 \|y\|^2$$

$$\Rightarrow \left(\sum_{k=1}^{n} x_k\right)^2 \leq \sum_{k=1}^{n} 1^2 \sum_{k=1}^{n} x_k^2$$

$$\Rightarrow \left(\sum_{k=1}^{n} x_k\right)^2 \leq n \sum_{k=1}^{n} x_k^2$$

$\boxtimes$.

PROBLEM 4.5

↪ see slides 18-20 of lecture 04 slides of 2020.