Lab 4

DSGA-1014: Linear Algebra and Optimization

CDS at NYU

Fall 2021

Norms and inner products

1. Explain why each of the following functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is not an inner product

- $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$
- $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$
- $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

2. Let $x = (cos\theta_1, sin\theta_1) \in \mathbb{R}^2$ and $y = (cos\theta_2, sin\theta_2) \in \mathbb{R}^2$ be two vectors on the unit circle (i.e., $||x|| = ||y|| = 1$). Explain the phrase "$x^T y$ gives a measure of the angle between x and y."

3. When does $||x + y|| = ||x|| + ||y||$ for $x, y \in \mathbb{R}^n$?

Orthogonality and orthogonal projection

4. Prove that if $v_1, ..., v_k \in \mathbb{R}^n$ are orthogonal vectors then they also are linearly independent.

5. Let S and U be subspaces of a vector space V. Prove the following statement: $S \subset U \implies U^{\perp} \subset S^{\perp}$

6. Let $A \in \mathbb{R}^{n \times m}$. Assume the Euclidean inner product. Prove that $Ker(A)^{\perp} = Im(A^T)$.

7. Let $A \in \mathbb{R}^{3\times 3}$ be defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}$$

- Find the orthogonal projection of $x \in \mathbb{R}^3$ onto the $Ker(A)$ and $Ker(A)^\perp$.
- Show that every vector $b \in Im(A)$ comes from one and only one vector in $Im(A^T)$.