DS-GA 1014 Optimization and Computational Linear Algebra

Lab 5: Orthogonal Matrices & Eigenvalues

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Gram-Schmidt

Let $v_1, \ldots, v_m \in \mathbb{R}^n$ be linearly independent. Show there is an orthonormal basis for $\mathrm{Span}(v_1, \ldots, v_m)$.

Previous lab question from Brett Bernstein

Gram-Schmidt

Gram-Schmidt

What is the output of Gram-Schmidt if the input vectors $v_1, \ldots, v_m$ are already orthonormal?

Previous lab question from Brett Bernstein

Gram-Schmidt

Let $V = R^3$ with the Euclidean inner product. We will apply the Gram-Schmidt algorithm to orthogonalize the basis $\{(1, -1, 1), (1, 0, 1), (1, 1, 2)\}$.

Orthonormal basis

We can also use coordinatization for $\mathbb{R}^n$. If we have a basis $B = v_1, \ldots, v_n$ for $\mathbb{R}^n$ then we can define the coordinatization (or change-of-basis) map $\Phi_B : \mathbb{R}^n \to \mathbb{R}^n$ by

$$\Phi_B(\alpha_1 v_1 + \cdots + \alpha_n v_n) = \alpha.$$

(a) Let B denote the basis $(1,0), (-1,1)$ for $\mathbb{R}^2$. Compute

$$\Phi_B\left(\begin{bmatrix}1\\0\end{bmatrix}\right), \quad \Phi_B\left(\begin{bmatrix}-1\\1\end{bmatrix}\right), \quad \text{and} \quad \Phi_B\left(\begin{bmatrix}0\\1\end{bmatrix}\right).$$

(b) Suppose $B = v_1, \ldots, v_n$ is a basis for $\mathbb{R}^n$. Give the matrices corresponding to Φ_B and Φ_B^{-1} (possible since $\Phi_B : \mathbb{R}^n \to \mathbb{R}^n$ is linear and invertible).

(c) For which bases B of $\mathbb{R}^n$ does Φ_B preserve inner products? That is, for which bases B does

$$\langle \Phi_B(x), \Phi_B(y) \rangle = \langle x, y \rangle$$

for all $x, y \in \mathbb{R}^n$?

Previous lab question from Brett Bernstein

Orthonormal basis

Previous lab question from Brett Bernstein

Eigenvalues & Eigenvectors

Consider the matrix

$$A = \begin{pmatrix} 2 & 2 \\ -4 & 8 \end{pmatrix} \quad \text{and vectors} \quad v = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Which are eigenvectors? What are their eigenvalues?

Eigenvalues & Eigenvectors

Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & 6 \end{pmatrix} \qquad v = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Is v an eigenvector of A? If so, what is its eigenvalue?