Lab 4

DSGA-1014: Linear Algebra and Optimization

CDS at NYU

Fall 2021

Norms and inner products

1. Explain why each of the following functions $f : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ is not an inner product

a) ▶ $\langle x, y \rangle = x_1 y_2 + x_2 y_3 + x_3 y_1$

b) ▶ $\langle x, y \rangle = x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2$

c) ▶ $\langle x, y \rangle = x_1 y_1 + x_2 y_2$

a) Counter example for symmetry:

$\langle x, y \rangle \neq \langle y, x \rangle$

$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad y = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$\langle x, y \rangle = 1 \qquad \langle y, x \rangle = y_1 x_2 + y_2 x_3 + y_3 x_1 = 0$

b) We show a counter example for Linearity:

$$\langle \alpha x, y \rangle = \alpha^2 x_1^2 y_1^2 + \alpha^2 x_2^2 y_2^2 + \alpha^2 x_3^2 y_3^2$$

$$= \alpha^2 \langle x, y \rangle$$

$$\Rightarrow \langle \alpha x, y \rangle \neq \alpha \langle x, y \rangle$$

c) We show a counter example for positive definiteness:

$$x = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \qquad \langle x, x \rangle = 0$$

2 . Let $x = (cos\theta_1, sin\theta_1) \in \mathbb{R}^2$ and $y = (cos\theta_2, sin\theta_2) \in \mathbb{R}^2$ be two vectors on the unit circle (i.e., $||x|| = ||y|| = 1$). Explain the phrase "$x^T y$ gives a measure of the angle between x and y."

$$x^T y = \cos(\theta_1)\cos(\theta_2) + \sin(\theta_1)\sin(\theta_2) = \cos(\theta_1 - \theta_2)$$

For vectors on the unit circle, the Euclidean inner product gives the cosine of the angle.

In general, we have $x^T y = \|x\| \|y\| \cos\theta$ where θ is the angle between x and y measured in the plane $\mathrm{Span}(x, y)$

3. When does $||x+y|| = ||x|| + ||y||$ for $x, y \in \mathbb{R}^n$?

$$\|x+y\|^2 = (\|x\| + \|y\|)^2$$

$$\langle x+y, x+y\rangle = \|x\|^2 + 2\|x\|\|y\| + \|y\|^2$$

$$\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle = \langle x,x\rangle + 2\langle x,x\rangle^{1/2}\langle y,y\rangle^{1/2} + \langle y,y\rangle$$

$$\Rightarrow \langle x,y\rangle = \|x\|\,\|y\|$$

$$\Rightarrow y = \alpha x$$

Orthogonality and orthogonal projection

4. Prove that if $v_1, \dots, v_k \in \mathbb{R}^n$ are orthogonal vectors then they also are linearly independent. (Note: all vectors are non-zero)

Assume linearly dependent, then show Contradiction:

If dependent, the we can write: $v_j = \sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i v_i$

$$\Rightarrow \quad \underbrace{\langle v_j, v_j \rangle}_{\text{Alway greater than zero unless } v_j = 0 \text{ which is not allowed.}} = \left\langle \sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i v_i \,,\, v_j \right\rangle = \sum_{\substack{i=1 \\ i \neq j}}^{k} \alpha_i \underbrace{\langle v_i, v_j \rangle}_{0} = 0$$

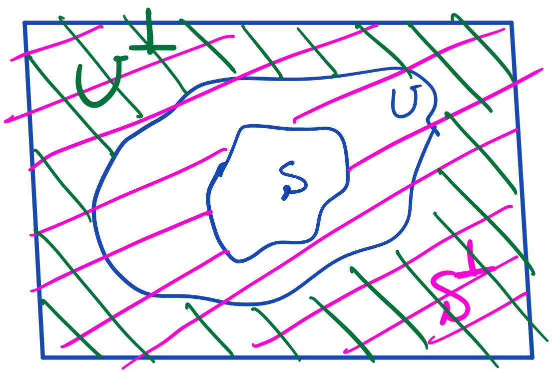
5. Let S and U be subspaces of a vector space V. Prove the following statement: $S \subset U \implies U^{\perp} \subset S^{\perp}$

show $\forall u' \in U^{\perp} \Rightarrow u' \in S^{\perp}$

For $\forall u' \in U^{\perp}, \forall u \in U \underset{\text{by definition}}{\Rightarrow} \langle u', u \rangle = 0$

we have $S \subset U \Rightarrow \forall s \in S \Rightarrow \langle u', s \rangle = 0$

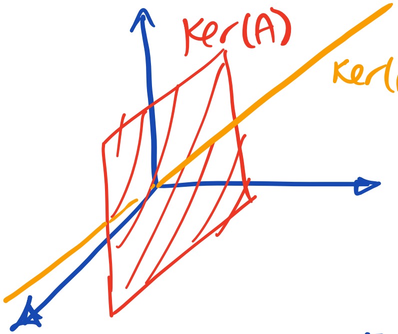
That is, any vector u' in $U^{\perp}$ is orthogonal to all vectors in S. This implies that $u' \in S^{\perp}$



6. Let $A \in \mathbb{R}^{n\times m}$. Assume the Euclidean inner product. Prove that $Ker(A)^{\perp} = \underbrace{Im(A^T)}_{\text{row space}}$.

$A = [\;\equiv\;]$ $\quad A^T = [\;|||\;]$

rows of A



$Ker(A)^{\perp} = Im(A^T)$

Not intuitive!

(i) show $Im(A^T) \subseteq Ker(A)^{\perp} \iff \forall x \in Im(A^T) \Rightarrow x \in Ker(A)^{\perp}$

for $\forall x \in Im(A^T)$ $\exists y$ s.t. $x = A^T y$

$\forall z \in Ker(A)$, $\langle x, z\rangle = x^T z = (A^T y)^T z = y^T \underbrace{Az}_{0} = 0$

That is, x is orthogonal to all vectors in $Ker(A)$

$\Rightarrow x \in Ker(A)^{\perp}$

(ii) show $\ker(A)^{\perp} \subseteq \mathrm{Im}(A^T)$ ← Not easy to show

So use result from problem (5). The above is equivalent to $\mathrm{Im}(A^T)^{\perp} \subseteq \ker(A) \iff \forall x \in \mathrm{Im}(A^T)^{\perp} \Rightarrow x \in \ker(A)$

Show this!

Let $y \in \mathrm{Im}(A^T)$ & $x \in \mathrm{Im}(A^T)^{\perp}$ $\Rightarrow \langle x, y \rangle = x^T y = 0$

and $\exists v$ s.t. $y = A^T v \Rightarrow \langle A^T v, x \rangle = (A^T v)^T x = v^T A x = 0$

not zero

$\Rightarrow Ax = 0$

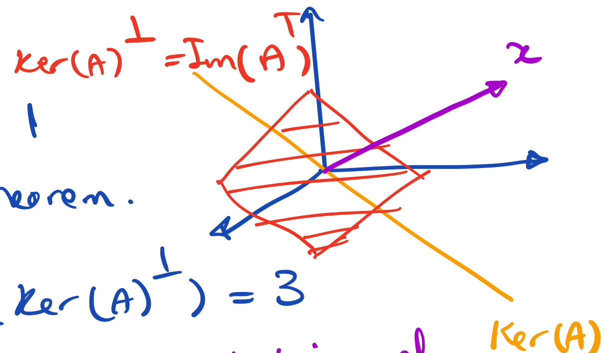
$\Rightarrow x \in \ker(A)$

7. Let $A \in \mathbb{R}^{3\times 3}$ be defined by

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

a) ▶ Find the orthogonal projection of $x \in \mathbb{R}^3$ onto the $Ker(A)$ and $Ker(A)^{\perp}$.

b) ▶ Show that every vector $b \in Im(A)$ comes from one and only one vector in $Im(A^T)$.



a)

$Rank(A) = 2 \Rightarrow Dim(Ker(A)) = 1$

from rank-nullity theorem.

Also, because $Dim(Ker(A)) + Dim(Ker(A)^{\perp}) = 3$

$\Longrightarrow Dim(Ker(A)^{\perp}) = 2$

we want projection of $\vec{x}$ onto these two subspaces.

Find the Ker(A):

$$Ax = 0 \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = 0 \\ \frac{1}{\sqrt{2}} x_2 + \frac{1}{\sqrt{2}} x_3 = 0 \end{cases}$$

$$\operatorname{Ker}(A) = \operatorname{Span}\left(\begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \right)$$

what is projection onto Ker(A)? The matrix whose columns represent a basis for Ker(A) is $v = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}_{3\times 1}$. Luckily this is an orthogonal matrix, so projection matrix will be $P = v v^T$:

$$P_1 = \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix}$$

What's projection onto $\operatorname{Ker}(A)^\perp = \operatorname{Im}(A^T)$?

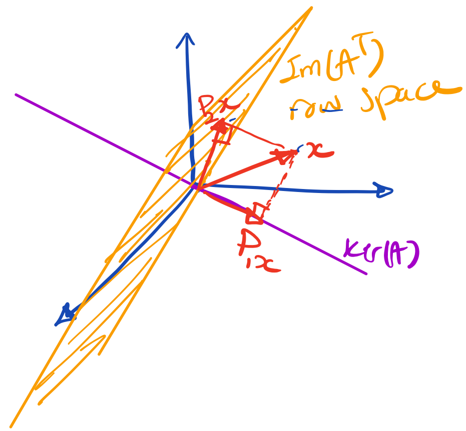
Basis for $\operatorname{Im}(A^T)$

$$A^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$P_2 = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/\sqrt{2} \\ 0 & 1/\sqrt{2} \end{bmatrix}}_{\text{orthogonal matrix}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P_1 x = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix}$$

$$P_2 x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix}$$



Because $\mathrm{Im}(A^T)$ is the orthogonal complement of $\ker(A)$, projection on these two splits x into two orthogonal vectors:

$$P_1 x + P_2 x = x_0 + x_r = \begin{bmatrix} 0 \\ \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix} + \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

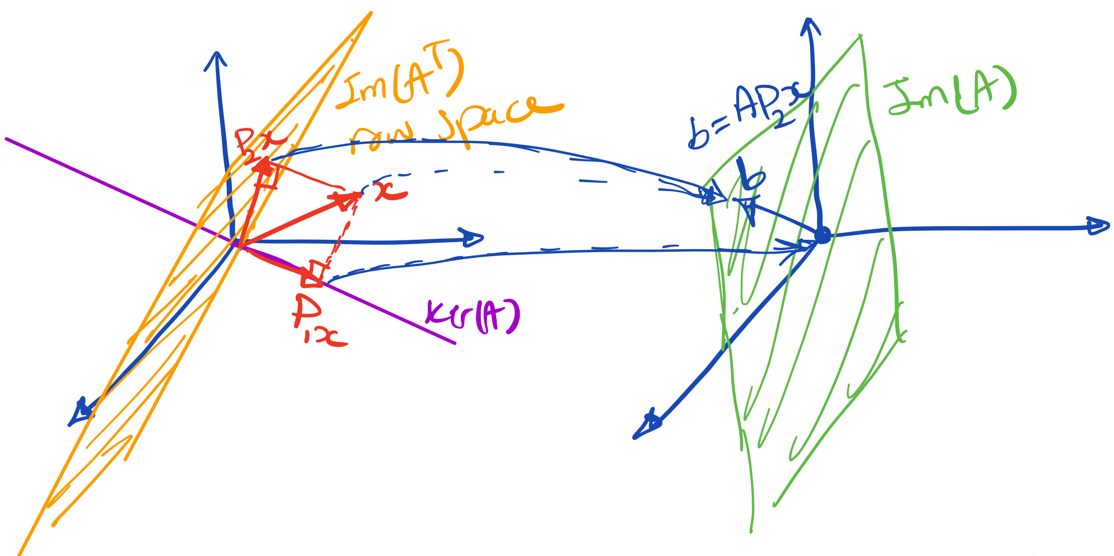
Now, lets see what's the effect of A on x_0 and x_1:

$$A x_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2}x_2 - \frac{1}{2}x_3 \\ -\frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

As we expected!

$$A x_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} x_1 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ \frac{1}{2}x_2 + \frac{1}{2}x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ 0 \\ \frac{1}{\sqrt{2}}x_2 + \frac{1}{\sqrt{2}}x_3 \end{bmatrix} = Ax$$

for $\forall b \in \mathrm{Im}(A)$: $b = Ax = A(x_0 + x_r)$



Any vector x is split into two orthogonal vectors x_0 and x_r. The mapping always takes x_0 to zero and x_r to the $\mathrm{Im}(A)$.

b) Assume x_r & $x'_r \in Im(A^T)$

We assume two vectors in row space are mapped to the same $\vec{b}$ in Column space. Then we show that's contradictory:

If $Ax_r = Ax'_r$

$\Rightarrow A(x_r - x'_r) = 0 \Rightarrow (x_r - x'_r) \in Ker(A)$

Also, we know that $(x_r - x'_r) \in Im(A^T)$

But since $Ker(A) \perp Im(A^T) \Rightarrow$

$x_r - x'_r$ has to be zero vector

$\Rightarrow x = x'$