Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 1.1 (3 points)**.** *Are the following sets subspaces of* $\mathbb{R}^2$*? Draw a picture and justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis, add the basis vectors on your drawing.*

(a) $E_1 = \{(x, y) \in \mathbb{R}^2 \,|\, x + 5y = 2\}$.

(b) $E_2 = \{(x, y) \in \mathbb{R}^2 \,|\, x + 5y = 0\}$.

(c) $E_3 = \{(x, y) \in \mathbb{R}^2 \,|\, x^2 + y^2 = 2\}$.

Problem 1.2 (2 points)**.** *Are the following sets subspaces of* $\mathbb{R}^3$*? Justify your answer using the definition of a subspace. If yes, indicate the dimension of the subspace and a basis.*

(a) $E_4 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x + y = 0\}$.

(b) $E_5 = \{(x, y, z) \in \mathbb{R}^3 \,|\, x + y = 0 \text{ and } -y + 3z = 0\}$.

Problem 1.3 (2 points)**.** *Let us define the vectors* $e_1, \ldots, e_n \in \mathbb{R}^n$ *by*

$$\begin{aligned} e_1 &= (1, 0, 0, \ldots, 0) \\ e_2 &= (0, 1, 0, \ldots, 0) \\ &\vdots \\ e_n &= (0, 0, 0, \ldots, 1). \end{aligned}$$

(a) *Verify that the family* $(e_1, \ldots, e_n)$ *is a basis of* $\mathbb{R}^n$*. This basis is called the "canonical basis" of* $\mathbb{R}^n$*. What is the dimension of* $\mathbb{R}^n$*?*

(b) *Give an example of hyperplane and an example of a line of* $\mathbb{R}^n$ *using spans of subsets of* $(e_1, \ldots, e_n)$.

Problem 1.4 (2 points)**.** *Consider* $v_1, \ldots, v_q \in \mathbb{R}^n$ *linearly independent vectors and* $u_1, \ldots, u_p \in \mathbb{R}^n$ *such that* $\mathrm{Span}(u_1, \ldots, u_p) = \mathbb{R}^n$

(a) *Show that* $q \leq n$ *(hint: remember what is the dimension of* $\mathbb{R}^n$*, you can use a lemma from the lecture).*

(b) *Using that dim*$(\text{Span}(u_1, \cdots, u_p)) \leq p$ *(you should convinve yourself that it is true), show that* $p \geq n$*.*

(c) *Assuming that* $q < n$*, one can show that there exists* k $(1 \leq k \leq p)$ *such that* $(v_1, \ldots, v_q, u_k)$ *is linearly independent. From there, allowing a renumbering of the indices of* u_i*-s, there actually exists an interger* m *such that*

(i) $(v_1, \ldots, v_q, u_1, \ldots, u_{m-q})$ *is linearly independent.*

(ii) $\forall i$ $(v_1, \ldots, v_q, u_1, \ldots, u_{m-q}, u_i)$ *is linearly dependent.*

Assuming the above stated properties (i) and (ii), show that $m = n$*.*
That shows that $(v_1, \ldots, v_q, u_1, \ldots, u_{n-q})$ *is a base of* $\mathbb{R}^n$ *and that given a spanning family of a vector space, it is always possible to complement a family of linearly independent vectors to form a base. Let's apply that in* $\mathbb{R}^4$*.*

(d) *We have* $\text{Span}(u_1, \ldots u_5) = \mathbb{R}^4$ *for* $u_1 = (1, 2, 0, 0)$*,* $u_2 = (2, 1, 0, 0)$*,* $u_3 = (1, 0, 0, 1)$*,* $u_4 = (1, 1, 1, 0)$ *and* $u_5 = (0, 1, 1, 0)$ *(again no need to prove it, but you should convince yourself it is true). Using vectors in the sets* $(u_1, \ldots u_5)$ *give a basis of* $\mathbb{R}^4$ *including the canonical basis vectors* $e_1 = (1, 0, 0, 0)$ *and* $e_3 = (0, 0, 1, 0)$*. Justify your answer.*

Problem 1.5 ($\star$)**.** *Consider* V *and* G *two vector spaces of finite dimension. Show the equivalence:*

$$V = G \Longleftrightarrow \begin{cases} \dim(V) = \dim(G) \\ V \subset G. \end{cases}$$