PROBLEM 2.1:

(a) YES.

let $v = (v_1, v_2) \in \mathbb{R}^2$

$w = (w_1, w_2) \in \mathbb{R}^2$

$$
\begin{aligned}
T(v+w) &= T((v_1+w_1,\, v_2+w_2)) \\
&= (v_1+w_1) - (v_2+w_2) \\
&= (v_1-v_2) + (w_1-w_2) \\
&= T(v) + T(w).
\end{aligned}
$$

Let also $\alpha \in \mathbb{R}$,

$$T(\alpha v) = T((\alpha v_1, \alpha v_2)) = \alpha(v_1 - v_2) = \alpha T(v)$$

$\square$

(b) NO (you should have the intuition of the answer seeing the "xy" term coming up, but to prove that T is not linear, you need an example)

take for instance

$$
\begin{cases} v = (1,1) \\ \alpha = 2 \end{cases}
\qquad
\begin{aligned}
T(\alpha v) &= (8, -2) \\
\alpha T(v) &= 2\,(4, 0) = (8, 0)
\end{aligned}
$$

$T(\alpha v) \neq \alpha T(v)$, T cannot be linear.

(c) YES. recall the rules of addition and scalar multiplication of matrices.

let A, B in $\mathbb{R}^{n \times n}$ $\quad \operatorname{diag}(A+B) = \operatorname{diag}(A) + \operatorname{diag}(B)$

let $\alpha \in \mathbb{R}$ $\quad \operatorname{diag}(\alpha A) = \alpha \operatorname{diag}(A)$

d) No Consider $A = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 2 \end{pmatrix}$

A is invertible given that $\begin{pmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/2 \end{pmatrix} A = Id$.

$$T(A+A) = \begin{pmatrix} 1/4 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/4 \end{pmatrix} \neq 2T(A)$$

$$T(A) = \begin{pmatrix} 1/2 & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1/2 \end{pmatrix}$$

PROBLEM 2.2:

Recall that to prove that to sets are equal $(A=B)$ can prove that $A \subseteq B$ and $B \subseteq A$.

- Let $x \in Im(A)$, by definition there exists $v \in \mathbb{R}^n$ such that $x = Av$

$$= \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= v_1 c_1 + v_2 c_2 + \cdots + v_n c_n$$

So that $x \in Span(c_1, \cdots c_n)$.

This shows that $Im(A) \subseteq Span(c_1, \cdots, c_n)$

- Let $x \in Span(c_1, \cdots, c_n)$, so x is a linear combination of $c_1, \cdots c_n$: there exists $\alpha_1, \cdots \alpha_n$ in $\mathbb{R}$ such that

$$x = \alpha_1 c_1 + \dots + \alpha_n c_n$$

$$= \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}$$ → call the vector $v \in \mathbb{R}^m$

$$= Av \implies x \in \operatorname{Im}(A)$$

This proves $\operatorname{Span}(c_1, \dots, c_n) \subseteq \operatorname{Im}(A)$

Overall conclusion: Since $\operatorname{Span}(c_1, \dots c_n) \subseteq \operatorname{Im}(A)$ and $\operatorname{Im}(A) \subseteq \operatorname{Span}(c_1, \dots c_n)$ we can conclude that $\operatorname{Im}(A) = \operatorname{Span}(c_1, \dots c_n)$

PROBLEM 2.3:

(a)- $Ax = 0$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 2x_1 + 4x_2 + 4x_3 = 0 & R2 \\ 3x_1 + 7x_2 + kx_3 = 0 & R3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 0 + 2x_2 + 2x_3 = 0 & R2 - 2R1 \\ 0 + 4x_2 + (k-3)x_3 = 0 & R3 - 3R1 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 0 + 2x_2 + 2x_3 = 0 & R2 \\ 0 + 0 + (k-7)x_3 = 0 & R3 - 2R2 \end{cases}$$

FIRST CASE: $k = 7$ then the linear system is equivalent to $\{ x_1 + x_2 + x_3 = 0$

$$\begin{cases} \ldots \\ x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 0 \\ x_2 = -x_3 \end{cases}$$

$$S = \mathrm{Span}\big((0,-1,1)\big) = \mathrm{Ker}(A)$$

$\dim \mathrm{Ker}(A) = 1$ and $v = (0,-1,1)$ gives the basis (v).

SECOND CASE: $k \neq 7$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = -x_3 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\Rightarrow x_1 = x_2 = x_3 = 0$$

$$S = \{0\} = \mathrm{Ker}(A) = \mathrm{Span}\big((0,0,0)\big).$$

0 vector of $\mathbb{R}^3$
$(0,0,0)$

(b) $$y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathrm{Span}\left(\underset{a_1}{\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}}, \underset{a_2}{\begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}}, \underset{a_3}{\begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix}} \right) = \mathrm{Im}(A)$$

since $y = a_1 \Rightarrow y \in \mathrm{Im}(A)$ there is at least one solution

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

The system has infinitely many solution if the nullspace has infinitely many vectors $\rightarrow k=7$ according to previous question.

(c) $Ax = \begin{pmatrix} 10 \\ 1 \\ 2017 \end{pmatrix}$ one solution $\rightarrow k \neq 7$

$$\begin{cases} x_1 + x_2 + x_3 = 10 \\ 2x_1 + 4x_2 + 4x_3 = 1 \\ 3x_1 + 7x_2 + kx_3 = 2017 \end{cases} \underset{\text{above}}{\overset{\text{Gaussian elimination}}{\Rightarrow}} \begin{cases} x_1 = 10 \\ x_2 = -x_3 \\ (k-7)x_3 = 2017 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 10 \\ x_2 = -x_3 \\ x_3 = 2017/(k-7) \end{cases}$$

$$\boxed{\Rightarrow \text{one solution} \begin{pmatrix} 10 \\ -2017/(k-7) \\ 2017/(k-7) \end{pmatrix}}$$

PROBLEM 2.4:

permuted columns.

(a) $BP = \begin{pmatrix} B_{12} & B_{11} & B_{13} \\ B_{22} & B_{21} & B_{23} \\ B_{32} & B_{31} & B_{33} \end{pmatrix}$

(b) $PB = \begin{pmatrix} B_{2,1} & B_{2,2} & B_{2,3} \\ B_{1,1} & B_{1,2} & B_{1,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}$ two rows were permuted

PROBLEM 2.5:

(a) $AM = \begin{pmatrix} 2M_{11} - M_{21} & 2M_{12} - M_{22} \\ 2M_{11} - M_{21} & 2M_{12} - M_{22} \end{pmatrix}$

(b) • $\mathrm{Im}(T) = \mathrm{Span}\left(\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}\right)$

Which we can see give the expression of AM above, the elements of the first column will always be the same, and same for the others.

We can show the equality by double inclusion.

• $AM = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \longrightarrow \begin{cases} 2M_{11} - M_{21} = 0 \\ 2M_{12} - M_{22} = 0 \end{cases} \longrightarrow \begin{cases} M_{21} = 2M_{11} \\ M_{22} = 2M_{12} \end{cases}$

2 values

choose M_{21} and M_{22} independently and we get a full M matrix

$\dim \ker(T) = 2$

find two such matrices that are linealy independent → base.

$\ker(T) = \mathrm{Span}\left(\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}\right)$