PROBLEM 3.1

(a) TRUE. Take square matrix $A = \begin{pmatrix} | & | & & 0 \\ c_1 & c_2 & \cdots & 0 \\ | & | & & 0 \end{pmatrix} \in \mathbb{R}^{n \times n}$

$\text{rank}(A) = \dim(\text{Span}(c_1, \ldots, c_{n-1})) \leq n-1$

$\Rightarrow \text{rank}(A) < n \rightarrow$ cannot be invertible

(b) TRUE. Because then $A = \begin{pmatrix} - r_1 - \\ \vdots \\ - r_2 - \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ | \\ 0 \end{pmatrix}$

vector with only ones

which implies that $\begin{pmatrix} 1 \\ | \\ 1 \end{pmatrix} \in \text{Ker}(A)$

$\Rightarrow \dim \text{Ker}(A) \geq 1 \Rightarrow$ A is singular.

(c) TRUE. By what we just proved above $\text{rank}(A^T) < n$

if columns sum to 0 $\Rightarrow$ rows of A^T sum to 0.

yet $\text{rank}(A) = \text{rank}(A^T) \Rightarrow \text{rank}(A) < n$.

(d) FALSE. $A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \rightarrow$ rows and columns sum to zero.

(e) TRUE. A is the inverse of A^{-1} and $A^{-1}A^{-1}$ the inverse of A^2.

$A^{-1}A^{-1}AA = A^{-1}\,\text{Id}\,A = \text{Id}$

$AAA^{-1}A^{-1} = \text{Id}$

PROBLEM 3.2

(a) $A = \begin{pmatrix} 5a-2 & 3a & 3a-3 \\ -4a+2 & -3a+1 & -2a+2 \\ -4a+2 & -3a & -2a+3 \end{pmatrix}$

rank (A) as a function of A:

- $a=0$: $A = \begin{pmatrix} -2 & 0 & -3 \\ +2 & 1 & +2 \\ +2 & 0 & +3 \end{pmatrix}$ rank $(A|_0) = 2$

$\hookrightarrow c_3 \leftarrow c_3 - \frac{3}{2} c_1 + c_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

- $a \neq 0$:

Using operations on rows

$$\text{rank } A = \text{rank} \begin{pmatrix} 5a-2 & 3a & 3a-3 \\ 0 & 1 & -1 \\ -4a+2 & -3a & -2a+3 \end{pmatrix} \begin{matrix} R1 \\ R2-R3 \\ \end{matrix}$$

$$= \text{rank} \begin{pmatrix} a & 0 & a \\ 0 & 1 & -1 \\ -4a+2 & -3a & -2a+3 \end{pmatrix} \begin{matrix} R1+R3 \\ \\ \end{matrix}$$

$\Rightarrow$ R1 and R2 are linearly indep. $\Rightarrow$ rank $A \geqslant 2$.

rank $A = 2$ if R3 is a linear combinaison of R1 and R2
Does there exist α and β such that $\alpha R1 + \beta R2 = R3$?

$$\begin{cases} \alpha a = -4a+2 \\ \beta = -3a \\ \end{cases} \Rightarrow \begin{cases} \alpha = \dfrac{-4a+2}{a} \\ \beta = -3a \\ 5a+3 \end{cases}$$

$$\left|\ \alpha a - \beta = -2a + 3 \right. \qquad \left|\ \alpha = \frac{-3a+3}{a} \right.$$

$$-4a + 2 = -5a + 3$$

$$\underline{a = 1}$$

CONCLUSION:

$$\boxed{\begin{cases} a = 0 \Rightarrow \text{rank } A = 2 \\ a = 1 \Rightarrow \text{rank } A = 2 \\ a \neq 0,\ a \neq 1 \Rightarrow \text{rank } A = 3 \end{cases}}$$

(b) $a = 0$ $\quad A = \begin{pmatrix} -2 & 0 & -3 \\ 2 & 1 & 2 \\ 2 & 0 & 3 \end{pmatrix}$

$$A \begin{pmatrix} -3/2 \\ +1 \\ 1 \end{pmatrix} = \begin{pmatrix} +\frac{3}{2}\,2 + 0 - 3 \\ -3 + 1 + 2 \\ -3 + 0 + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

and since $\text{rank}(A) = 2$, $\dim \text{Ker } A = 1$

(a bases is $\left(\left(-3/2,\ 1,\ 1\right)\right)$)

To build a basis of Im A it is enough to take two of its linearly indep. coulumns.

basis: $\left((-2, 2, 2);\ (0, 1, 0)\right)$ for instance.

PROBLEM 3.3

(a) (*). rank $A = \dim \operatorname{Im}(A) = 2$

show that $((1,2);(0,3))$ is linearly indep.

$$\alpha \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\Leftrightarrow \begin{cases} \alpha + 0 = 0 \\ 2\alpha + 3\beta = 0 \end{cases} \Leftrightarrow \begin{cases} \alpha = 0 \\ \beta = 0 \end{cases}.$$

(*) rank $B \neq 3$ since the last column is zero

$\alpha c_1 + \beta c_2 + \gamma c_3 = 0$ can be verified for $\alpha = 0, \beta = 0$ and $\gamma \neq 0$.

but rank $B = 2$ because two other colums are linearly indep.

(*) Similarly we can show rank $C = 5$, rank $D = 4$

(b) $$\begin{pmatrix} M_{11} & & (0) \\ M_{ij} & M_{22} & \ddots \end{pmatrix} = \begin{pmatrix} | & & | \\ c_1 & \cdots & c_n \\ | & & | \end{pmatrix}$$

Building on the intuition above we will show that M is invertible if all the diagonal entries are $\neq 0$.

<u>Assume</u> $M_{ii} \neq 0$ for all i

Then $\alpha_1 c_1 + \alpha_2 c_2 + \dots + \alpha_n c_n = 0$

$$\Rightarrow \begin{cases} \alpha_1 M_{11} = 0 \\ \alpha_1 M_{21} + \alpha_2 M_{22} = 0 \\ \vdots \\ \alpha_1 M_{n1} + \dots + \alpha_n M_{nn} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 = 0 \\ 0 + \alpha_2 M_{22} = 0 \\ \vdots \end{cases}$$

$$\Rightarrow \begin{cases} \alpha_1 = 0 \\ \alpha_2 = 0 \\ 0 + 0 + \alpha_3 M_{33} = 0 \\ \vdots \end{cases}$$

$\Rightarrow \quad \alpha_1 = \alpha_2 = \dots = \alpha_3 = 0$

$\Rightarrow$ columns are linearly indep $\Rightarrow$ rank $M = n$

$\Leftrightarrow M$ invertible.

(a) Now if there exists one position at which $M_{ii} = 0$

$\hookrightarrow$ one can show that $c_i \in \mathrm{Span}(c_{i+1}, \dots, c_n)$

$\Rightarrow$ columns are not linearly indep. $\Rightarrow M$ not invertible.

(d) $\text{rank}(M) = \text{rank}(M^T)$ $\rightarrow$ so the condition is the same as for lower triangular matrices.

PROBLEM 3.4

(a) $\begin{cases} \text{rank}(A) = \dim \text{Im}(A) \\ \text{rank}(AB) = \dim \text{Im}(AB) \end{cases}$

yet $\text{Im}(AB) \subset \text{Im}(A) \Rightarrow \text{rank}(AB) \leq \text{rank} A$

(b) • trivial to show $\ker(L) \subset \ker(L^T L)$

• now for any $x \in \ker(L^T L)$ $\quad L^T L x = 0$

$$x^T L^T L x = 0$$
$$\Rightarrow \|Lx\| = 0$$
$$\Rightarrow Lx = 0$$
$$\Rightarrow x \in \ker(L)$$

so $\ker(L^T L) \subset \ker L$

Conclusion: $\ker(L^T L) = \ker L$.

(c) $\text{rank}(L^T L) \leq \text{rank}(L^T)$

$\Rightarrow m - \dim\ker(L^T L) \leq \text{rank}(L^T)$ rank nullity theorem

$\Rightarrow m - \dim\ker(L) \leq \text{rank}(L^T)$

$\Rightarrow \text{rank}(L) \leq \text{rank}(L^T)$

+ apply the same inequality to L^T:

$$\text{rank}(L^T) \leq \text{rank}\big(\underbrace{(L^T)^T}_{=L}\big)$$

conclusion $\quad \text{rank}(L) \leq \text{rank}(L^T) \leq \text{rank}(L)$

$\hookrightarrow \boxed{\text{rank}(L) = \text{rank}(L^T)}$.

PROBLEM 3.5

$$A = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix} \qquad A_{ij} = \begin{cases} 1 & \text{if } \; i = n-j+1 \\ 0 & \text{otherwise} \end{cases}$$

$$(A^2)_{ij} = \sum_k A_{ik} A_{kj} = A_{i\,n-i+1}\, A_{n-j+1\,j}\, \delta_{ij}$$

$\neq 0$ $\quad k = n-i+j$

$\neq 0$ $\quad k = n-j+1$

$k = n-j+1 = n-i+1$

$$= \delta_{ij}$$

$\Rightarrow \; A^2 = I$

$$\Rightarrow \begin{cases} A^{2n} = I & \text{even powers.} \\ A^{2n+1} = A & \text{odd powers} \end{cases}$$