

### PROBLEM 2.1:

(a) **YES**

$$\text{let } v = (v_1, v_2) \in \mathbb{R}^2$$

$$w = (w_1, w_2) \in \mathbb{R}^2$$

$$\begin{aligned} T(v+w) &= T((v_1+w_1, v_2+w_2)) \\ &= (v_1+w_1) - (v_2+w_2) \\ &= (v_1-v_2) + (w_1-w_2) \\ &= T(v) + T(w) \end{aligned}$$

let also  $\alpha \in \mathbb{R}$ ,

$$T(\alpha v) = T(\alpha v_1, \alpha v_2) = \alpha(v_1 - v_2) = \alpha T(v)$$

□

(b) **NO** (you should have the intuition of the answer seeing the "xy" term coming up, but to prove that  $T$  is not linear, you need an example)

take for instance

$$\begin{cases} v = (1, 1) & T(v) = (8, -2) \\ \alpha = 2 & \alpha T(v) = 2(8, -2) = (16, -4) \end{cases}$$

$$T(\alpha v) \neq \alpha T(v), \quad T \text{ cannot be linear.}$$

(c) **YES** recall the rules of addition and scalar multiplication of matrices -

$$\text{let } A, B \text{ in } \mathbb{R}^{n \times n} \quad \text{diag}(A+B) = \text{diag}(A) + \text{diag}(B)$$

$$\text{let } \alpha \in \mathbb{R} \quad \text{diag}(\alpha A) = \alpha \text{diag}(A)$$

d) No Consider  $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

$A$  is invertible given that  $\begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} A = I_d$ .

$T(A + A) = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/4 \end{pmatrix} \neq 2T(A)$

$T(A) = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/2 \end{pmatrix}$

### PROBLEM 2.2:

Recall that to prove that two sets are equal ( $A=B$ ) can prove that  $A \subseteq B$  and  $B \subseteq A$ .

① Let  $x \in \text{Im}(A)$ , by definition there exists  $v \in \mathbb{R}^n$  such that  $x = Av$

$$= \begin{pmatrix} | & & | \\ c_1 & \dots & c_n \\ | & & | \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$= v_1 c_1 + v_2 c_2 + \dots + v_n c_n$$

So that  $x \in \text{Span}(c_1, \dots, c_n)$ .

This shows that  $\text{Im}(A) \subseteq \text{Span}(c_1, \dots, c_n)$

② Let  $x \in \text{Span}(c_1, \dots, c_n)$ , so  $x$  is a linear combination of  $c_1, \dots, c_n$ : there exists  $\alpha_1, \dots, \alpha_n$  in  $\mathbb{R}$  such that

$$x = \alpha_1 c_1 + \dots + \alpha_m c_n$$

$$= \begin{pmatrix} | & & | \\ c_1 & \dots & c_n \\ | & & | \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix} \quad \text{call the vector } v \in \mathbb{R}^m$$

$$= Av \Rightarrow x \in \text{Im}(A)$$

This proves  $\text{Span}(c_1, \dots, c_n) \subseteq \text{Im}(A)$

⊙ Overall conclusion: Since  $\text{Span}(c_1, \dots, c_n) \subseteq \text{Im}(A)$   
and  $\text{Im}(A) \subseteq \text{Span}(c_1, \dots, c_n)$   
we can conclude that  $\text{Im}(A) = \text{Span}(c_1, \dots, c_n)$

PROBLEM 2.3:

(a) -  $Ax = 0$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 2x_1 + 4x_2 + 4x_3 = 0 & R2 \\ 3x_1 + 7x_2 + kx_3 = 0 & R3 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 0 + 2x_2 + 2x_3 = 0 & R2 - 2R1 \\ 0 + 4x_2 + (k-3)x_3 = 0 & R3 - 3R1 \end{cases}$$

$$\begin{cases} x_1 + x_2 + x_3 = 0 & R1 \\ 0 + 2x_2 + 2x_3 = 0 & R2 \\ 0 + 0 + (k-7)x_3 = 0 & R3 - 2R2 \end{cases}$$

FIRST CASE:  $k = 7$  then the linear system is equivalent to  $x_1 + x_2 + x_3 = 0$

$$\begin{cases} x_2 + x_3 = 0 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = -2x_3 \\ x_2 = -x_3 \end{cases}$$

$$S = \text{Span}((-2, -1, 1)) = \text{Ker}(A)$$

$\dim \text{Ker}(A) = 1$  and  $v = (-2, -1, 1)$  gives the basis  $(v)$ .

SECOND CASE:  $k \neq 7$

$$\begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = -x_3 \\ x_3 = 0 \end{cases} \Rightarrow \begin{cases} x_1 + x_2 + x_3 = 0 \\ x_2 = 0 \\ x_3 = 0 \end{cases}$$

$$\Rightarrow x_1 = x_2 = x_3 = 0$$

$$S = \{0\} = \text{Ker}(A) = \text{Span}((0, 0, 0))$$

$\uparrow$   
0 vector of  $\mathbb{R}^3$   
 $(0, 0, 0)$

$$(b) \quad y = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \text{Span} \left( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}, \begin{pmatrix} 1 \\ 4 \\ k \end{pmatrix} \right) = \text{Im}(A)$$

since  $y = a_1 \Rightarrow y \in \text{Im}(A)$  there is at least one solution

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

The system has infinitely many solutions if the nullspace

has infinitely many vectors  $\rightarrow k=7$  according to previous question -

(c)  $Ax = \begin{pmatrix} 10 \\ 1 \\ 2017 \end{pmatrix}$  one solution  $\rightarrow k \neq 7$

$$\begin{cases} x_1 + x_2 + x_3 = 10 \\ 2x_1 + 4x_2 + 4x_3 = 1 \\ 3x_1 + 7x_2 + kx_3 = 2017 \end{cases} \quad \begin{array}{l} \text{Gaussian} \\ \text{elimination} \\ \Rightarrow \\ \text{above} \end{array} \quad \begin{cases} x_1 = 10 \\ x_2 = -x_3 \\ (k-7)x_3 = 2017 \end{cases}$$

$$\Rightarrow \begin{cases} x_1 = 10 \\ x_2 = -x_3 \\ x_3 = 2017/(k-7) \end{cases}$$

$$\Rightarrow \text{one solution} \quad \begin{pmatrix} 10 \\ -2017/(k-7) \\ 2017/(k-7) \end{pmatrix}$$

PROBLEM 2.4:

permuted columns -

(a)  $BP = \begin{pmatrix} B_{12} & B_{11} & B_{13} \\ B_{22} & B_{21} & B_{23} \\ B_{32} & B_{31} & B_{33} \end{pmatrix}$

(b)  $PB = \begin{pmatrix} B_{2,1} & B_{2,2} & B_{2,3} \\ B_{1,1} & B_{1,2} & B_{1,3} \\ B_{3,1} & B_{3,2} & B_{3,3} \end{pmatrix}$  two rows were permuted

PROBLEM 2.5:

$$(a) \quad A\pi = \begin{pmatrix} 2\pi_{11} - \pi_{21} & 2\pi_{12} - \pi_{22} \\ 2\pi_{11} - \pi_{21} & 2\pi_{12} - \pi_{22} \end{pmatrix}$$

$$(b) \quad \text{Im}(T) = \text{Span} \left( \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

Which we can see give the expression of  $A\pi$  above, the elements of the first column will always be the same, and same for the others -

We can show the equality by double inclusion -

$$\text{Im}(T) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \rightarrow \begin{cases} 2\pi_{11} - \pi_{21} = 0 \\ 2\pi_{12} - \pi_{22} = 0 \end{cases} \rightarrow \begin{cases} \pi_{21} = 2\pi_{11} \\ \pi_{22} = 2\pi_{12} \end{cases}$$

choose  $\pi_{21}$  and  $\pi_{22}$  independently and we get a full  $\pi$  matrix

$$\dim \text{Ker}(T) = 2$$

find two such matrices that are linearly independent  $\rightarrow$  base -

$$\text{Ker}(T) = \text{Span} \left( \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} \right)$$

