Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Problem 8.1 (2 points)**.** *Let $A \in \mathbb{R}^{n \times m}$. The Singular Values Decomposition (SVD) tells us that there exists two orthogonal matrices $U \in \mathbb{R}^{n \times n}$ and $V \in \mathbb{R}^{m \times m}$ and a matrix $\Sigma \in \mathbb{R}^{n \times m}$ such that $\Sigma_{1,1} \geq \Sigma_{2,2} \geq \cdots \geq 0$ and $\Sigma_{i,j} = 0$ for $i \neq j$*

$$A = U \Sigma V^\mathsf{T}.$$

The columns $u_1, \ldots, u_n$ of U (respectively the columns $v_1, \ldots, v_m$ of V) are called the left (resp. right) singular vectors of A. The non-negative numbers $\sigma_i \overset{\text{def}}{=} \Sigma_{i,i}$ are the singular values of A. Moreover we also know that $r \overset{\text{def}}{=} \operatorname{rank}(A) = \#\{i \,|\, \Sigma_{i,i} \neq 0\}$.

(a) *Let $\widetilde{U} = \begin{pmatrix} | & & | \\ u_1 & \cdots & u_r \\ | & & | \end{pmatrix} \in \mathbb{R}^{n \times r}$, $\widetilde{V} = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_r \\ | & & | \end{pmatrix} \in \mathbb{R}^{m \times r}$ and $\widetilde{\Sigma} = \operatorname{Diag}(\sigma_1, \ldots, \sigma_r) \in \mathbb{R}^{r \times r}$. Show that $A = \widetilde{U} \widetilde{\Sigma} \widetilde{V}^\mathsf{T}$.*

(b) *Give orthonormal bases of $\operatorname{Ker}(A)$ and $\operatorname{Im}(A)$ in terms of the singular vectors $u_1, \ldots, u_n$ and $v_1, \ldots, v_m$.*

Problem 8.2 (2 points)**.** *For any two matrices $A, B \in \mathbb{R}^{n \times m}$ we define the* Frobenius inner-product *as*

$$\langle A, B \rangle_F = \operatorname{Tr}(A^\mathsf{T} B).$$

We showed in the midterm that it verifies the points of the definition 2.1 of Lecture 4 for the square matrix case (one can also check that it works for rectangular matrices). Show that the induced norm $\|A\|_F = \sqrt{\operatorname{Tr}(A^\mathsf{T} A)}$ can be computed as a function of the singular values $\sigma_1, \ldots, \sigma_{\min(n,m)}$ of A as

$$\|A\|_F = \sqrt{\sum_{i=1}^{\min(n,m)} \sigma_i^2}.$$

Problem 8.3 (2 points)**.** *Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix of a graph G. We define a « path from a node i_1 to a node i_k » as a succession of nodes $i_1, i_2, \ldots, i_k$ such that*

$$i_1 \sim i_2 \sim \cdots \sim i_{k-1} \sim i_k, \quad \text{i.e.} \quad A_{i_1,i_2} = A_{i_2,i_3} = \cdots = A_{i_{k-1},i_k} = 1.$$

The nodes i_j of the path do not need to be distinct. We say that the path $i_1, \ldots, i_k$ has length $k-1$ which is the number of edges in this path. The goal of this exercise is to prove that for all $k \geq 1$

$$\mathcal{H}(k) : \text{« For all } i, j \in \{1, \ldots, n\}, \text{ the number of paths of length } k \text{ from } i \text{ to } j \text{ is } (A^k)_{i,j} \text{ ».}$$

We will prove that $\mathcal{H}(k)$ holds for all k by induction, that is, we will first prove that $\mathcal{H}(1)$ is true. Then we will prove that if $\mathcal{H}(k)$ is true for some k, then $\mathcal{H}(k+1)$ is true. Combining these two things, we get that $\mathcal{H}(2)$ holds, hence $\mathcal{H}(3)$ holds, hence $\mathcal{H}(4)$ holds... and therefore $\mathcal{H}(k)$ will be true for all $k \geq 1$.

(**a**) *Show that $\mathcal{H}(1)$ is true.*

(**b**) *Show that if $\mathcal{H}(k)$ is true for some k, then $\mathcal{H}(k+1)$ is also true.*

Problem 8.4 (4 points)**.** *The goal of this problem is to use spectral clustering techniques on real data. The file* `adjacency.txt` *contains the adjacency matrix of a graph taken from a social network. This graphs has $n = 328$ nodes (that corresponds to users). An edge between user i and user j means that i and j are "friends" in the social network. The notebook* `friends.ipynb` *contains functions to read the adjacency matrix as well as instructions/questions.*

While we focused in the lectures (and in the notes) on the graph Laplacian

$$L = D - A,$$

where A is the adjacency matrix of the graph, and $D = \mathrm{Diag}(\deg(1), \dots, \deg(n))$ is the degree matrix, we will use here the "normalized Laplacian" (instead of L)

$$L_{\mathrm{norm}} = D^{-1/2} L D^{-1/2} = \mathrm{Id}_n - D^{-1/2} A D^{-1/2},$$

where $D^{-1/2} = \mathrm{Diag}(\deg(1)^{-1/2}, \dots, \deg(n)^{-1/2})$. The reason for using a different Laplacian is that then "unnormalized Laplacian" L does not perform well when the degrees in the graph are very broadly distributed, i.e. very heterogeneous. In such situations, the normalized Laplacian L_{norm} is supposed to lead to a more consistent clustering.

It is intended that you code in Python and use the provided Jupyter Notebook. Please only submit a pdf version of your notebook (right-click → 'print' → 'Save as pdf').

Problem 8.5 ($\star$)**.** *Let G be a connected graph with n nodes. Define $L \in \mathbb{R}^{n \times n}$ the associate Laplacian matrix and $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ its eigenvalues. Let G' be a graph constructed from G by simply adding an edge. Similarly denote by λ_2' its second smallest eigenvalue. Show that $\lambda_2' \geq \lambda_2$.*