Rules:

- Unless otherwise stated, all answers must be mathematically justified.
- Partial answers will be graded.
- Your submission has to be uploaded to Gradescope. In Gradescope, indicate the page on which each problem is written.
- You can work in groups but each student must write his/her/their own solution based on his/her/their own understanding of the problem. Please list on your submission the students you work with for the homework (this will not affect your grade).
- Problems with a ($\star$) are extra credit, they will not (directly) contribute to your score of this homework. However, for every 4 extra credit questions successfully answered your lowest homework score get replaced by a perfect score.
- If you have any questions, feel free to contact me (`mgabrie@nyu.edu`) or to stop at the office hours.

Special warning: In this homework, your answers from one problem may help solving another one.

Problem 9.1 (2 points)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a convex function. We assume that the minimum $m \stackrel{\text{def}}{=} \min_{x\in\mathbb{R}^n} f(x)$ of f on $\mathbb{R}^n$ is finite, and that the set of minimizers of f*

$$\mathcal{M} \stackrel{\text{def}}{=} \{v \in \mathbb{R}^n \,|\, f(v) = m\}$$

is non-empty.

(a) *Show that $\mathcal{M}$ is a convex set.*

(b) *Show that if f is strictly convex, then $\mathcal{M}$ has only one element.*

Problem 9.2 (2 points)**.** *Let $M \in \mathbb{R}^{n\times n}$ be a symmetric matrix, $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. For $x \in \mathbb{R}^n$ we define*

$$f(x) = x^{\mathsf{T}} M x + \langle x, b\rangle + c.$$

f is called a quadratic function.

(a) *Compute the gradient $\nabla f(x)$ and the Hessian $H_f(x)$ at all $x \in \mathbb{R}^n$ (show the steps of your computations). Show that f is convex if and only if M is positive semi-definite.*

(b) *In this question, we assume M to be positive semi-definite. Show that f admits a minimizer if and only if $b \in \operatorname{Im}(M)$.*

Problem 9.3 (3 points). *We say that a function $f : \mathbb{R}^n \to \mathbb{R}$ is strongly convex if there exists $\alpha > 0$ such that the function $x \mapsto f(x) - \frac{\alpha}{2}\|x\|^2$ is convex. In other words, f is strongly convex if there exists $\alpha > 0$ and a convex function $g : \mathbb{R}^n \to \mathbb{R}$ such that*

$$f(x) = g(x) + \frac{\alpha}{2}\|x\|^2.$$

(a) *Show that a strongly convex function is strictly convex. (Hint: start by showing that $x \mapsto \|x\|^2$ is strictly convex).*

(b) *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function. Show that φ is strongly convex if and only if there exists $\alpha > 0$ such that for all $x \in \mathbb{R}^n$ the eigenvalues of $H_\varphi(x)$ are greater or equal than α.*

Problem 9.4 (3 points). *Let $A \in \mathbb{R}^{n\times m}$ and $y \in \mathbb{R}^n$. For $x \in \mathbb{R}^m$ we define*

$$f(x) = \|Ax - y\|^2.$$

(a) *Compute the gradient $\nabla f(x)$ and the Hessian $H_f(x)$ at all $x \in \mathbb{R}^m$. Show that f is convex.*

(b) *Show that if* $\operatorname{rank}(A) < m$*, then f is not strictly convex.*

(c) *Show that if* $\operatorname{rank}(A) = m$*, then f is strongly convex.*

Problem 9.5 ($\star$). *Is the following function f convex? Compute its Hessian to conclude.*

$$\begin{array}{rccl} f: & \mathbb{R}^n & \to & \mathbb{R} \\ & x & \mapsto & f(x) = \ln(e^{x_1} + \cdots + e^{x_n}) \end{array}$$