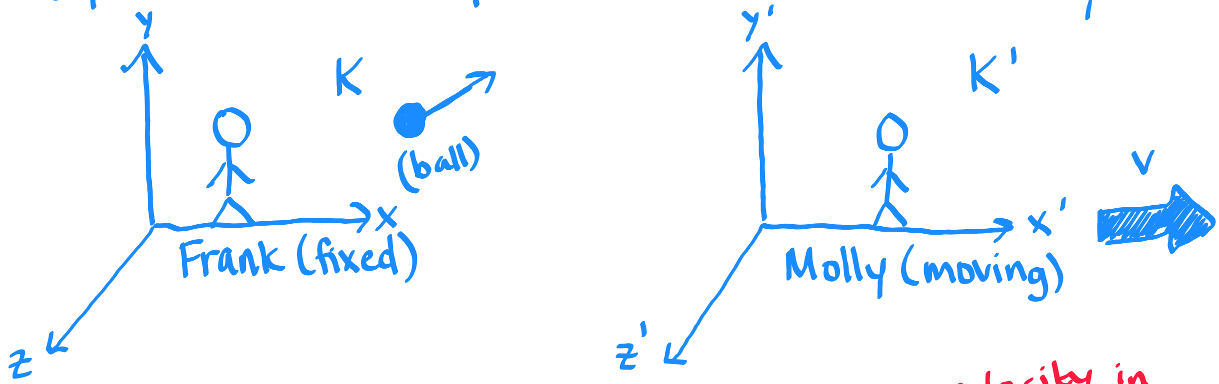
LibreTexts

HONORS PROJECT → Velocity Addition!

Class Notes

2 observers. One is at rest. The other moves according to first observer and fires/throws something. The first observer sees the object going faster than c in their reference frame. This comes from the Galilean transformation, which, as we will see, doesn't hold up in the frame of relativity.



velocity in frame K

velocity in frame K'

Galilean Transformation: how do $\vec{u}$ and $\vec{u}'$ relate to each other?

$$t = t' \quad \Big| \quad \frac{dt'}{dt} = 1$$

$$x' = x - vt \quad \Big| \quad u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \left(\frac{dx}{dt} - v\right)(1) = u_x - v$$

$$y' = y \quad \Big| \quad u_y' = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \left(\frac{dy}{dt}\right)(1) = u_y$$

$$z' = z \quad \Big| \quad u_z' = \frac{dz'}{dt'} = \frac{dz'}{dt}\frac{dt}{dt'} = \left(\frac{dz}{dt}\right)(1) = u_z$$

This gives the existing, non-relativistic result we would expect. However, now, we have to worry about the <u>relativistic</u> aspect of motion between the two frames, leading us to the Lorentz transformations.

γ is related to the relativity b/t frames, not the velocity of the ball! → This γ will be treated as a constant throughout the transformations.

Lorentz Transformation

$$ct' = \gamma\left[ct - \left(\frac{v}{c}\right)x\right] \quad ; \quad \frac{dt'}{dt} = \gamma\left[1 - \left(\frac{v}{c^2}\right)\frac{dx}{dt}\right] = \gamma\left(1 - \frac{u_x v}{c^2}\right)$$

$$x' = \gamma(x - vt) \quad ; \quad u'_x = \frac{dx'}{dt'} = \frac{dx'}{dt}\frac{dt}{dt'} = \gamma\left(\frac{dx}{dt} - v\right)\left[\gamma\left(1 - \frac{u_x v}{c^2}\right)\right]^{-1} = \frac{u_x - v}{1 - \frac{u_x v}{c^2}}$$

$$y' = y \quad ; \quad u'_y = \frac{dy'}{dt'} = \frac{dy'}{dt}\frac{dt}{dt'} = \left(\frac{dy}{dt}\right)\left[\gamma\left(1 - \frac{u_x v}{c^2}\right)\right]^{-1} = \frac{u_y}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

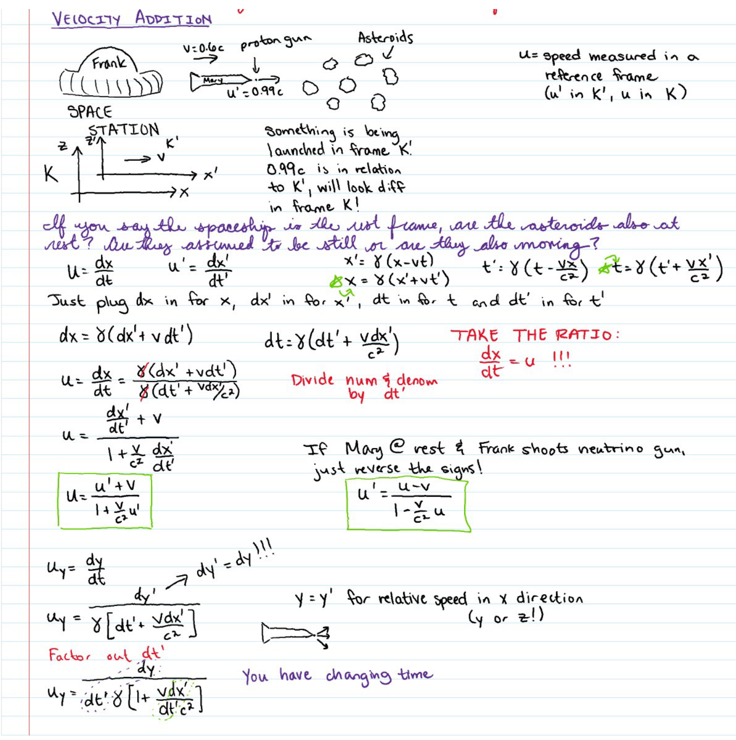
$$z' = z \quad ; \quad u'_z = \frac{dz'}{dt'} = \frac{dz'}{dt}\frac{dt}{dt'} = \left(\frac{dz}{dt}\right)\left[\gamma\left(1 - \frac{u_x v}{c^2}\right)\right]^{-1} = \frac{u_z}{\gamma\left(1 - \frac{u_x v}{c^2}\right)}$$

Ann = Frank
Bob = Molly

SIGNS are super important! When $u_x > 0$, the x-component of the ball's velocity measured by Frank is in the same direction as the relative motion of Molly's frame. Molly will also see the ball going slightly faster than predicted by the Galilean transformation. (denominator < 1 for velocity all! transformation)

$u_x < 0$, the x-comp of the ball's velocity measured by Frank is in the opposite dir. as the relative motion of Molly's frame. The velocities add to less than the Galilean transformation (denominator > 1 for velocity all! transformation)

Plug in in-class example!


VELOCITY ADDITION

$v = 0.6c$ proton gun Asteroids

Frank

Mary $u' = 0.99c$

SPACE STATION

z, z', K', v, x', K, x

u = speed measured in a reference frame (u' in K', u in K)

Something is being launched in frame K'. 0.99c is in relation to K', will look diff in frame K!

If you say the spaceship is the rest frame, are the asteroids also at rest? Are they assumed to be still or are they also moving?

$u = \frac{dx}{dt}$ $\quad u' = \frac{dx'}{dt'}$ $\quad x' = \gamma(x - vt)$ $\quad x = \gamma(x' + vt')$ $\quad t' = \gamma\left(t - \frac{vx}{c^2}\right)$ $\quad t = \gamma\left(t' + \frac{vx'}{c^2}\right)$

Just plug dx in for x, dx' in for x', dt in for t and dt' in for t'

$dx = \gamma(dx' + v\,dt')$ $\quad dt = \gamma\left(dt' + \frac{v\,dx'}{c^2}\right)$

TAKE THE RATIO: $\frac{dx}{dt} = u$!!!

$u = \frac{dx}{dt} = \frac{\cancel{\gamma}(dx' + v\,dt')}{\cancel{\gamma}(dt' + v\,dx'/c^2)}$

Divide num & denom by dt'

$u = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}}$

$\boxed{u = \frac{u' + v}{1 + \frac{v}{c^2}u'}}$

If Mary @ rest & Frank shoots neutrino gun, just reverse the signs!

$\boxed{u' = \frac{u - v}{1 - \frac{v}{c^2}u}}$

$u_y = \frac{dy}{dt}$ $\quad dy' = dy$!!!

$u_y = \frac{dy'}{\gamma\left[dt' + \frac{v\,dx'}{c^2}\right]}$

$y = y'$ for relative speed in x direction (y or z!)

Factor out dt'

$u_y = \frac{dy}{dt'\,\gamma\left[1 + \frac{v\,dx'}{dt'\,c^2}\right]}$

You have changing time

* Plug in the #s. Match this format as well for the equations! *

Effects on Perceived Perpendicular Motion

We already talked about velocity addition along the direction of motion, but relative x-direction motion affects velocity measurements $\perp$ to this motion as well. This occurs due to time dilation. If you already perceive a frame as slower than your own, motion in these perpendicular directions moves slower.

For example, Frank is fixed and fires a laser in the +y dir as Molly passes by. Through the velocity transformation, Molly still sees the laser move at the speed of light, however, the two observers see two different directions.

Frank sees +y-dir. Molly sees an angle w/ y-axis:

$$\sin\theta = \frac{v}{c} \qquad \theta = \sin^{-1}\left(\frac{v}{c}\right)$$

Velocity Transformation:

$$u_x = 0 \Rightarrow u'_x = \frac{0-v}{1-0} = -v$$

$$u_y = 0 \Rightarrow u'_y = \frac{c}{\gamma(1-0)} = \sqrt{c^2 - v^2}$$

$$|\vec{u}'| = \sqrt{u_x'^2 + u_y'^2} = c$$

* Maybe find another example w/ a change in the speed in the $\perp$ dir → cone add on to first problem? *