

TDAB01 Probability and Statistics

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Lecture 9: Hypothesis Testing

Overview

- ▶ **Hypothesis testing**
- ▶ **Z-test**
- ▶ **T-test**
- ▶ **χ^2 -test**

Hypothesis testing

- ▶ Example 1. New medication affects blood pressure

Null hypothesis: $H_0 : \mu = 0$

Alternative hypothesis: $H_A : \mu \neq 0$

→ *two-side* alternative

- ▶ Example 4. Proportion of defective products is higher than 4%

Null hypothesis: $H_0 : p \geq 0.04$

Alternative hypothesis: $H_A : p < 0.04$

→ *one-side, right-tail* alternative

- ▶ Example 3. Average connection speed is worse than the provider promised

Null hypothesis: $H_0 : \mu \geq \mu_0$

Alternative hypothesis: $H_A : \mu < \mu_0$

→ *one-side, left-tail* alternative

Hypothesis testing

- ▶ **Two-sided test:** reject H_0 if μ is greater **or** less than μ_0

Null Hypothesis: $H_0 : \mu = \mu_0$

Alternative hypothesis: $H_A : \mu \neq \mu_0$

- ▶ **One-sided tests**

Null Hypothesis: $H_0 : \mu \leq \mu_0$

Alternative hypothesis: $H_A : \mu > \mu_0$

or

Null hypothesis: $H_0 : \mu \geq \mu_0$

Alternative hypothesis: $H_A : \mu < \mu_0$

- ▶ One-sided tests are often written as

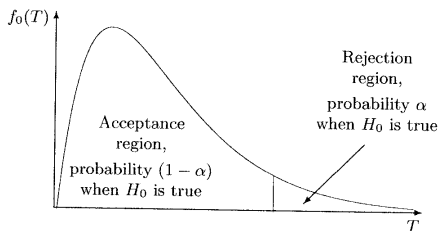
Null hypothesis: $H_0 : \mu = \mu_0$

Alternative hypothesis: $H_A : \mu < \mu_0$ or $H_A : \mu > \mu_0$

→ *the same result*

Steps in hypothesis testing

1. Choose **test statistic** $T = T(X_1, \dots, X_n)$
2. Determine sampling distribution F_0 for T if H_0 is true (null distribution)
3. Determine the **rejection region** \mathcal{R} such that $P(T \in \mathcal{R} | H_0) = \alpha$
4. Reject H_0 at **significance level** α if $T_{obs} \in \mathcal{R}$, T_{obs} - **observed** value of T



On picture: f_0 - pdf of T in dependence of value of T

- ▶ H_0 is about population $\rightarrow H_0$ is true or not with probability 1
Accepting H_0 does **not** mean that H_0 is true with probability $1 - \alpha$
Accepting H_0 means that evidence obtained from data is not sufficient to reject H_0

Type I and Type II errors

- ▶ **Type I error**

$$\alpha = P(\text{Reject } H_0 | H_0 \text{ is true})$$

α - significance level, predetermined, low

- ▶ **Type II error**

$$\beta = P(\text{Accept } H_0 | H_A \text{ is true})$$

β depends on parameter value (set of values in H_A)

	Accept H_0	Reject H_0
H_0 true	Correct decision	Type I error
H_A true	Type II error	Correct decision

- ▶ **Power of test:** $P(\text{Reject } H_0 | H_A \text{ is true}) = 1 - \beta$

- ▶ Formally: Do we reject H_0 or not, we never accept it

Z-test

- ▶ Sampling distribution of test statistic if H_0 is true is normal distribution

- ▶ Case 1. $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$, σ^2 known

$$\Rightarrow Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} \sim N(0, 1)$$

- ▶ Case 2. Central Limit Theorem $\rightarrow Z$ appr. normally distributed

- ▶ One-sided Z-tests: $H_0 : \mu = \mu_0$ and $H_A : \mu > \mu_0$ (right-tail Z-test)

$$\begin{cases} \text{Reject } H_0 & \text{if } Z > z_{\alpha} \\ \text{Accept } H_0 & \text{if } Z \leq z_{\alpha} \end{cases}$$

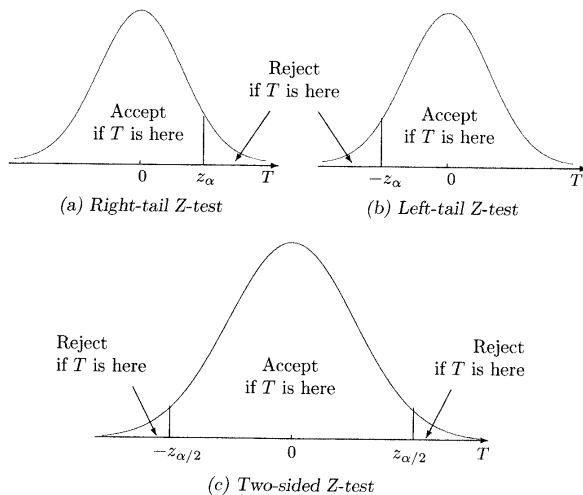
or $H_0 : \mu = \mu_0$ and $H_A : \mu < \mu_0$ (left-tail Z-test)

$$\begin{cases} \text{Reject } H_0 & \text{if } Z < -z_{\alpha} \\ \text{Accept } H_0 & \text{if } Z \geq -z_{\alpha} \end{cases}$$

- ▶ Two-sided Z-test: $H_0 : \mu = \mu_0$ and $H_A : \mu \neq \mu_0$

$$\begin{cases} \text{Reject } H_0 & \text{if } |Z| > z_{\alpha/2} \\ \text{Accept } H_0 & \text{if } |Z| \leq z_{\alpha/2} \end{cases}$$

Z-test



- ▶ Example: See Example 9.25 in textbook

Two-sample Z-test

- ▶ We can also test if **two** populations have the same expected value:

$$H_0 : \mu_X = \mu_Y$$

$$H_A : \mu_X \neq \mu_Y$$

- ▶ Example: Is the average salary the same in Stockholm and Malmö?
- ▶ X_1, \dots, X_n - random sample from $N(\mu_X, \sigma_X^2)$, σ_Y^2 known
- ▶ Y_1, \dots, Y_m - random sample from $N(\mu_Y, \sigma_Y^2)$, σ_Y^2 known
- ▶ **Test Statistics:** $\bar{X} - \bar{Y}$. Sampling distribution under H_0 - ?
 - ▶ Linear combination of normal variables is normally distributed, i.e. $\bar{X} - \bar{Y}$ is normally distributed
 - ▶ $E(\bar{X} - \bar{Y}) = \mu_X - \mu_Y = 0$ under H_0
 - ▶ $Var(\bar{X} - \bar{Y}) = Var(\bar{X}) + Var(\bar{Y}) = \sigma_X^2/n + \sigma_Y^2/m$
 - ▶ Then

$$Z = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0, 1)$$

- ▶ Example: See Example 9.26 in textbook

T-test

- ▶ Z-test is used when sampling distribution of test statistic under H_0 is normal distribution
- ▶ If σ^2 **not** known but estimated with s^2

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}}$$

in **not** normally distributed but **t-distributed** with $n - 1$ degrees of freedom

- ▶ Use t_α instead of z_α
- ▶ $t_\alpha - (1 - \alpha)$ -quantile of t -distribution with $n - 1$ degrees of freedom
- ▶ Values for t_α in Table A5 in textbook

Duality: hypothesis test and confidence interval

A Z-test or T-test of $H_0 : \theta = \theta_0$ against $H_A : \theta \neq \theta_0$ at the significance level α accepts the null hypothesis if and only if θ_0 is included in a symmetric $(1 - \alpha)100\%$ confidence interval for θ .

P-value

- ▶ How do we choose α ?
- ▶ α - Type I error \rightarrow should be low
- ▶ Very low α requires very large / very small observed value of test statistic to reject H_0
- ▶ Idea: Test hypothesis for all α
- ▶ **P-value** = the lowest significance level α for which we can reject H_0
= the highest significance level α for which we can accept H_0
- ▶ Alternative definition: Probability to obtain a test statistic that is as extreme or even more extreme than T_{obs}
- ▶ Example: One-sided, right-tail Z-Test:

$$P\text{-value} = P(Z \geq Z_{obs}) = 1 - \Phi(Z_{obs})$$

- ▶ Reject H_0 at level α if $\alpha > P\text{-value}$
- ▶ Example: See Example 9.38 in textbook

χ^2 -test for population variance

- ▶ Unbiased estimator of σ^2 :

$$s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

- ▶ **Sampling distribution** of s^2 if $X_1, \dots, X_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$:

$$\frac{(n-1)s^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i - \bar{X}}{\sigma} \right)^2 \sim \chi_{n-1}^2$$

- ▶ χ^2 -distribution is used for hypothesis testing and confidence intervals as normal and t -distributions
- ▶ Pdf of $X \sim \chi_\nu^2$:

$$f(x) = \frac{1}{2^{\nu/2} \Gamma(\nu/2)} x^{\nu/2-1} e^{-x/2}, \quad x > 0$$

ν - number of degrees of freedom

- ▶ $\mathbb{E}(X) = \nu$ and $\text{Var}(X) = 2\nu$
- ▶ χ_ν^2 distribution is a special case of Gamma-distribution: $\text{Gamma}(\nu/2, 1/2)$
- ▶ $\chi_{1-\frac{\alpha}{2}}^2$ and $\chi_{\frac{\alpha}{2}}^2$ quantiles of χ^2 -distribution, values in Table A6
- ▶ χ_ν^2 distribution is **not symmetric** \rightarrow both $\chi_{1-\frac{\alpha}{2}}^2$ and $\chi_{\frac{\alpha}{2}}^2$ needed

Thank you for your attention!