MULTIPLE INTERPHASES FOR FRACTIONAL ALLEN–CAHN EQUATIONS

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Abstract. We consider a nonlocal reaction-diffusion equation in $\mathbb{R}^n$, $n \geq 2$, that physically describes atomic dislocations in crystalline structures. In particular, we study the evolutionary version of the classical Peierls–Nabarro model with initial conditions corresponding to multiple loop dislocations. After suitably rescaling the equation with a small phase parameter ε, the rescaled solution solves a fractional Allen–Cahn equation. We show that, as $\varepsilon \to 0$, the limiting solution exhibits multiple interfaces evolving independently and according to their mean curvature.

1. Introduction

In this paper, we study a nonlocal, reaction-diffusion equation that arrises naturally in the Peierls–Nabarro model for atomic dislocations in crystalline structures. Our initial configuration corresponds to a collection of *loop dislocations* with the same orientation. After suitably rescaling the problem from the microscopic scale to the mesoscopic scale, we show that the dislocation loops move independently, according to their mean curvature.

At the atomic level, one can view a crystal as an infinite cubic lattice. Dislocations are defects from a perfect lattice which evolve when subject to forces, see [13]. The evolution of *edge dislocations* has been recently studied in the literature, that is, when the dislocations are straight, parallel lines. In this special setting, the Peierls–Nabarro model reduces to a *one-dimensional* PDE and the dislocation lines can be associated to single points in $\mathbb{R}$. At the mesoscopic scale, González–Monneau in [12] showed that the dislocation points evolve according to a discrete system of ODEs. See [10] for an excellent overview of the theory. Nevertheless, when the dislocations are not straight edge dislocations, the physical model can only be reduced to a *two-dimensional* PDE and the dislocations are indeed curves in $\mathbb{R}^2$. Unlike the one-dimensional setting in which dislocation points move left or right, curves in higher dimensions can move in infinitely many directions. To the best of our knowledge, we are the first to study the dynamics of dislocation curves in $\mathbb{R}^n$, $n \geq 2$.

Before further reviewing the Peierls–Nabarro model for loop dislocations, let us formalize our problem mathematically.

1.1. Setting of the problem.

We are interested in the fractional Allen–Cahn equation

$$\varepsilon \partial_t u^\varepsilon = \frac{1}{\varepsilon\,|\ln \varepsilon|}\left(\varepsilon \mathcal{I}_n[u^\varepsilon] - W'(u^\varepsilon)\right) \quad \text{in } \mathbb{R}^n,\ n \geq 2, \tag{1.1}$$

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where $\varepsilon > 0$ is a small parameter, $\mathcal{I}_n$ denotes the fractional Laplacian of order 1 in $\mathbb{R}^n$, and W is a multi-well potential. The nonlocal operator $\mathcal{I}_n$ is given by

$$\begin{aligned}\mathcal{I}_n u(x) &= \mathrm{P.V.} \int_{\mathbb{R}^n} (u(x+y) - u(x)) \frac{dy}{|y|^{n+1}}\, dy \\ &= \int_{|y|<1} (u(x+y) - u(x) - \nabla u(x) \cdot y) \frac{dy}{|y|^{n+1}}\, dy + \int_{|y|>1} (u(x+y) - u(x)) \frac{dy}{|y|^{n+1}}\, dy,\end{aligned}$$

where P. V. indicates that the integral is taken in the principal value sense. Up to a multiplicative constant, it can be shown that $\mathcal{I}_n$ satisfies the Fourier transform identity $\widehat{\mathcal{I}_n u}(\xi) = |\xi|\, \widehat{u}(\xi)$, $\xi \in \mathbb{R}^n$. For further background on fractional Laplacians, see for example [8, 25]. Regarding the potential W, we assume that

$$\begin{cases} W \in C^{2,\beta}(\mathbb{R}) & \text{for some } 0 < \beta < 1 \\ W(u+1) = W(u) & \text{for any } u \in \mathbb{R} \\ W = 0 & \text{on } \mathbb{Z} \\ W > 0 & \text{on } \mathbb{R} \setminus \mathbb{Z} \\ W''(0) > 0. \end{cases} \tag{1.2}$$

We let u^ε be the solution to (1.1) when the initial condition u_0^ε is a superposition of layer solutions. The layer solution (also called the phase transition) $\phi : \mathbb{R} \to [0, 1]$ is the unique solution to

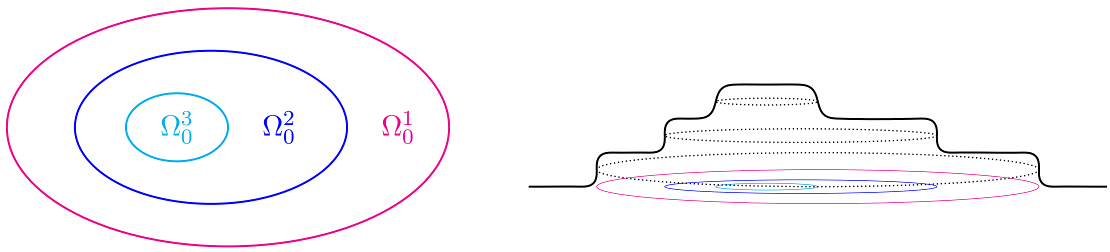
$$\begin{cases} C_n \mathcal{I}_1[\phi] = W'(\phi) & \text{in } \mathbb{R} \\ \dot{\phi} > 0 & \text{in } \mathbb{R} \\ \phi(-\infty) = 0, \quad \phi(+\infty) = 1, \quad \phi(0) = \frac{1}{2}, \end{cases} \tag{1.3}$$

where $\mathcal{I}_1$ denotes the 1/2-Laplacian in $\mathbb{R}$ and the constant $C_n > 0$ (given explicitly in (3.4)) depends only on $n \geq 2$. Further discussion on ϕ will be presented in Section 3.

For a fixed $N \in \mathbb{N}$, let $(\Omega_0^i)_{i=1}^N$ be a finite sequence of open subsets of $\mathbb{R}^n$ that are smooth, bounded, and satisfy $\Omega_0^{i+1} \subset\subset \Omega_0^i$. The corresponding boundaries $\Gamma_0^i = \partial\Omega_0^i$ can be understood as the initial dislocation loops in the crystal. Let $d_i(t, x)$ be the signed distance function associated to Ω_0^i, $i = 1, \ldots, N$, given by

$$d_i(x) = \begin{cases} d(x, \Gamma_0^i) & \text{if } x \in \Omega_0^i \\ -d(x, \Gamma_0^i) & \text{otherwise.} \end{cases} \tag{1.4}$$

For our initial condition to be well-prepared, we let u_0^ε be the N-fold sum of the layer solutions $\phi(d_i(x)/\varepsilon)$, see Figure 1.



(A) Dislocations in the slip plane (B) Initial condition u_0^ε for small $\varepsilon > 0$

FIGURE 1. Initial configuration for $N = 3$

We show that the evolution of the boundaries, denoted by $(\Gamma_t^i)_{t\geq 0}$, corresponds to interfaces moving independently by mean curvature. To be more precise, Γ_t^i is the zero level set at time $t>0$ of a solution u^i to the mean curvature equation whose zero level set at $t=0$ is exactly Γ_0^i. In this case, we say that $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$ where ${}^+\Omega_t^i$ and ${}^-\Omega_t^i$ are the positivity and negativity sets of u^i respectively. See Section 2 for definitions and details.

We now present the main result of our paper.

Theorem 1.1. *Let $u^\varepsilon = u^\varepsilon(t,x)$ be the unique solution of the reaction diffusion equation (1.1) with the initial datum $u_0^\varepsilon : \mathbb{R}^n \to [0,N]$ defined by*

$$u_0^\varepsilon(x) = \sum_{i=1}^N \phi\left(\frac{d_i(x)}{\varepsilon}\right). \tag{1.5}$$

Then, as $\varepsilon \to 0$, the solutions u^ε satisfy

$$\begin{cases} u^\varepsilon \to N & \text{in } {}^+\Omega_t^N, \\ u^\varepsilon \to i & \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}, \quad i=1,\dots,N-1, \\ u^\varepsilon \to 0 & \text{in } {}^-\Omega_t^1, \end{cases}$$

where $({}^+\Omega_t^i, \Gamma_t^i, {}^-\Omega_t^i)$ denotes the level-set evolution of $(\Omega_0^i, \Gamma_0^i, (\overline{\Omega_0^i})^c)$.

Remark 1.2. The interfaces $(\Gamma_t^i)_{t\geq 0}$, $i=1,\dots,N$, in Theorem 1.1 shrink as time increases. Changing the orientation of the initial condition in (1.5) to

$$u_0^\varepsilon(x) = \sum_{i=1}^N \phi\left(-\frac{d_i(x)}{\varepsilon}\right)$$

and taking the limit of u^ε as $\varepsilon \to 0$ instead exhibits interfaces $(\Gamma_t^i)_{t\geq 0}$ moving by mean curvature that expand as time increases. The proof is similar, so we omit the details.

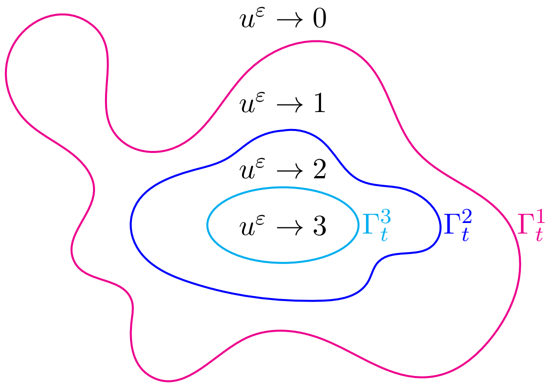


FIGURE 2. Convergence result for $N=3$

As illustrated in Figure 2, the solutions u^ε converge to integers between the interfaces Γ_t^i, but we do not say anything about the limiting solution on the curves themselves. To understand this, we say that the set Γ_t^i does not develop interior if and only if $\Gamma_t^i = \partial({}^+\Omega_t^i) =$

$\partial({}^{-}\Omega_t^i)$. In this special setting, the limiting function in Theorem 1.1 makes integer jumps on the curves Γ_t^i and satisfies

$$\lim_{\varepsilon\to 0} u^\varepsilon = \frac{N}{2} + \frac{1}{2}\sum_{i=1}^{N}\left(\mathbb{1}_{{}^{+}\Omega_t^i} - \mathbb{1}_{\overline{({}^{+}\Omega_t^i)^c}}\right) \quad \text{in } (0,\infty)\times\mathbb{R}^n \setminus \bigcup_{i=1}^{N}\Gamma_i$$

where $\mathbb{1}_\Omega$ denotes the characteristic function of the set Ω. However, due to the degeneracy of the mean curvature equation, Γ_t^i may develop interior, and we cannot say exactly where the jump occurs within these sets. For example, the level set evolution of a dumbbell will develop singularities on its neck in finite time.

The proof of Theorem 1.1 relies on the abstract method introduced in [2] for the study of front propagation. One of the key tools needed for the abstract method is the construction of strict sup/super solutions to (1.1). Since we are working with multiple evolving fronts, our barriers take the form

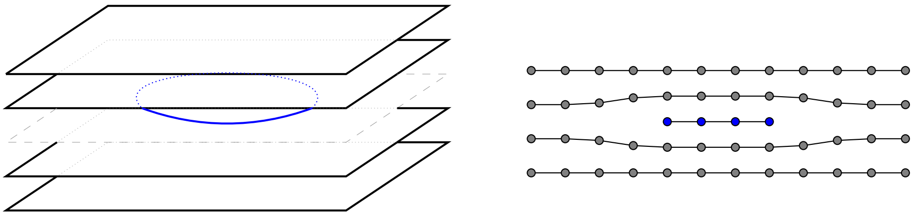
$$v^\varepsilon(t,x) \simeq \sum_{i=1}^{N}\phi\left(\frac{d_i(t,x)}{\varepsilon}\right),$$

where d_i is the signed distance function associated to Γ_t^i. A formal argument for this choice of barrier is presented in Section 4. The difficulty arrises in understanding $v^\varepsilon(t,x)$ when (t,x) is far from the front Γ_t^i since the signed distance function is not smooth at such points. To overcome this, we replace d_i with a smooth extension of the signed distance function away from the curve, see Definition 3.4.

Theorem 1.1 for $N=1$ has been addressed in the literature when W is instead a double-well potential. The local setting for which (1.1) is driven by the usual Laplacian Δ was studied famously by Modica–Mortola [16] for the stationary case and by Chen [5] for the corresponding evolutionary problem. In the fractional setting, the stationary case was studied by Savin–Valdinoci in [24] and the evolution problem was considered by Imbert–Souganidis in the preprint [15]. We are the first to consider when $N>1$ and W is a multi-well potential. Moreover, we found that the proof in [15] does not extend to our setting. Indeed, they show that v^ε (for $N=1$) is a subsolution near Γ_t, then interpolate v^ε between 0 (outside the curve) and 1 (inside the curve). However, when $N>1$, we cannot control the errors associated with the nonlinearity of the potential W using their approach. By extending each signed distance function d_i instead of extending the barrier $\phi(d_i/\varepsilon)$, we are able to use the asymptotic behavior of ϕ to show that v^ε is indeed a sub/super solution (see Section 5). Moreover, in the actual construction of the barrier v^ε, we need to add a corrector ψ to control the error as $\varepsilon\to 0$. In [15], the phase transition and corrector solve a *traveling* wave and linearized traveling wave equation, respectively. Our phase transition and corrector solve a *standing* wave and linearized standing wave equation, respectively, which allows us to avoid dependence on the speed of the wave.

1.2. **The Peierls–Nabarro model for loop dislocations.** The Peierls–Nabarro model is a phase field model for dislocation dynamics which incorporates the atomic features of a crystalline structure into continuum framework [19, 20, 23]. In the phase field approach, dislocations are interfaces represented by a transition of a continuous field. We briefly review the model for loop dislocations (see [13, 14] for more details).

Recall that a perfect crystal can be understood as a simple cubic lattice. A loop dislocation occurs when there is an extra lattice plane of finite area. The picture is similar to that of a round coaster in the middle of a stack of papers, see Figure 3. In Cartesian coordinates $x_1x_2x_3$, we assume that the dislocation loop is located in the x_1x_2-plane, known as the slip plane. The movement of the dislocation is determined by the so-called Burgers' vector b.



(A) Viewed as stacked planes (B) Vertical cross section

FIGURE 3. Loop dislocation at the microscopic level

For a *slip loop*, the vector b lies within the slip plane (also called the glide plane) causing the loop to either expand or shrink. Assume that the Burgers' vector is in the direction of the x_1-axis, say $b = e_1$. Recall that an edge dislocation occurs when the Burgers' vector is perpendicular to the dislocation curve, whereas a screw dislocation occurs when the Burgers' vector is parallel to the dislocation curve. A loop dislocation is a mixture of both edge and screw dislocations that cause the loop to shrink, see Figure 4. Note that taking instead $b = -e_1$ changes the orientation and causes the loop to expand. This corresponds to a set finite area being removed from a lattice plane.

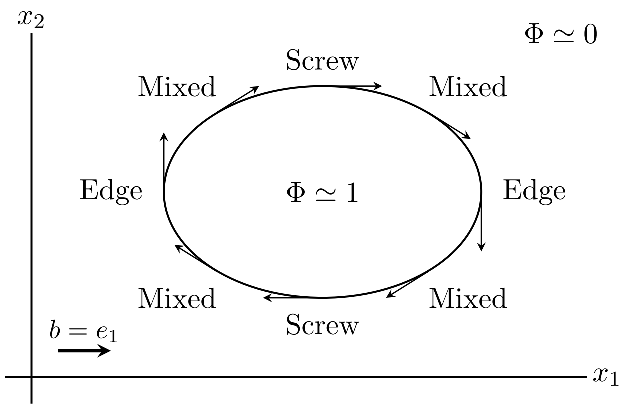


FIGURE 4. Loop dislocation in the slip plane $x_1 x_2$

In order to describe the loop, a phase parameter $\Phi(x_1, x_2)$ between 0 and 1 is used to capture the disregistry of the upper half crystal $\{x_3 > 0\}$ from the lower half crystal $\{x_3 < 0\}$. In particular, the dislocation loop is the set $\Gamma_0 := \{\Phi = 1/2\} \subset \mathbb{R}^2$. We also have that $\Phi \simeq 1$ inside the loop and $\Phi \simeq 0$ outside the loop.

Next, let $U = U(x_1, x_2, x_3)$ be the distance between an atom at location (x_1, x_2, x_3) in the upper half crystal and its rest position. Let $\Phi(x_1, x_2) = U(x_1, x_2, 0)$ be the displacement in the slip plane. Most of the mismatch of atoms from their perfect lattice structure occurs within the slip plane. To quantify this, we use a multi-well potential W satisfying (1.2). Here, the periodicity of W captures the periodicity of the crystal. In the Peierls–Nabarro model, the total energy is the elastic energy for bonds between atoms plus the energy for atomic displacement:

$$\mathcal{E}(U) = \frac{1}{2} \int_{\mathbb{R}^2 \times \mathbb{R}^+} |U(x_1, x_2, x_3)|^2 \, dx_1 \, dx_2 \, dx_3 + \int_{\mathbb{R}^2} W(\Phi(x_1, x_2)) \, dx_1 \, dx_2.$$

The equilibrium configuration is obtained by minimizing the energy $\mathcal{E}$ with respect to U under the constraint that $\Phi \simeq 1$ in the loop and $\Phi \simeq 0$ outside the loop. The corresponding Euler-Lagrange equation is

$$\mathcal{I}_2[\Phi] = W'(\Phi) \quad \text{in } \mathbb{R}^2$$

where $\mathcal{I}_2$ is the fractional Laplacian of order 1 in $\mathbb{R}^2$. It turns out that it is enough to consider $\Phi(x_1, x_2) = \phi(d(x_1, x_2))$ where ϕ solves (1.3) in $\mathbb{R}$ and d is the signed distance function associated to the loop Γ_0. (See, for example, Lemma 3.2 with $n = 2$.)

We are interested in the evolution of multiple loop dislocations in the same slip plane, with the same Burgers' vector. For this, we use a single parameter $u(t, x)$ defined for x in the slip plane $\mathbb{R}^n$, the physical dimension being $n = 2$. The dislocation dynamics are then captured by the evolutionary Peierls–Nabarro model:

$$\partial_t u = \mathcal{I}_n[u] - W'(u) \quad \text{in } (0, \infty) \times \mathbb{R}^n,\ n \geq 2,$$

where $\mathcal{I}_n$ is the fractional Laplacian of order 1 in $\mathbb{R}^n$. At the microscopic scale, we assume that the dislocations curves are at a distance of order 1 from each other. This can be represented by the initial condition

$$u(x, 0) = \sum_{i=1}^{N} \phi\left(\frac{d_i(\varepsilon x)}{\varepsilon}\right) \quad \text{for } x \in \mathbb{R}^n,\ \varepsilon > 0,$$

where ϕ solves (1.3) in $\mathbb{R}$ and d_i is the signed distance function associated to the loop Γ_0^i.

In order to understand the movement of the dislocation curves at a larger (mesoscopic) scale, we consider the rescaling

$$u^\varepsilon(t, x) = u\left(\frac{t}{\varepsilon^2 |\ln \varepsilon|}, \frac{x}{\varepsilon}\right), \quad (t, x) \in [0, \infty) \times \mathbb{R}^n,\ \varepsilon > 0.$$

Consequently, u^ε solves (1.1) with the initial condition (1.5). Here, $\varepsilon > 0$ represents the scaling between the microscopic scale ($\sim 10^{-10}\mu m$) and the mesoscopic scale ($\sim 0.1 - 10\mu m$),. The presence of the factor $|\ln \varepsilon|$ is well-known in physics, see [3, 7, 13]. Roughly speaking, it arrises from an integrability condition for the kernel of the 1/2-fractional Laplacian in $\mathbb{R}^n$. See also [24].

The alternative rescaling

$$u^\delta(t, x) = \frac{1}{\delta} u\left(\frac{t}{\delta}, \frac{x}{\delta}\right)$$

was considered by Monneau–Patrizi in [18] when $u(0, x) = u_0(x)$. In this setting, the parameter $\delta > 0$ is the ratio between the length scale for dislocations (microscopic) and the macroscopic length scale. Taking $\delta \to 0$, the limiting solution u solves an equation describing the plastic flow rule at the macroscopic scale with a density of dislocations.

1.3. **Organization of the paper.** The rest of the paper is organized as follows. First, in Section 2, we provide the necessary background pertaining to motion by mean curvature. Section 3 contains preliminary results on the phase transition ϕ and other auxiliary results needed for the rest of the paper. Then, in Section 4, we provide heuristics for the proof of Theorem 1.1 and for the choice of barrier. The construction of barriers is presented in Section 5. Section 6 contains the proof of Theorem 1.1. Lastly, the proofs of some auxiliary results are given in Section 7.

2. Motion by mean curvature

In this section, we introduce the geometric motions of the fronts. For a smooth function $u = u(t,x)$, consider the sets

$$\begin{aligned}{}^{+}\Omega &= \{(t,x) : u(x,t) > 0\}\\ \Gamma &= \{(t,x) : u(x,t) = 0\}\\ {}^{-}\Omega &= \{(t,x) : u(x,t) < 0\}.\end{aligned}$$

Denote the slices in t of ${}^{+}\Omega$ by

$${}^{+}\Omega_t = {}^{+}\Omega \cap (\{t\} \times \mathbb{R}^n)$$

and similarly for Γ_t and ${}^{-}\Omega_t$. Together, these form a set of triples $({}^{+}\Omega_t, \Gamma_t, {}^{-}\Omega_t)_{t\geq 0}$.

Let $d(t,x)$ denote the signed distance function associated to Γ_t:

$$d(t,x) = \begin{cases} d(x,\Gamma_t) & \text{for } x \in \Gamma_t \cup {}^{+}\Omega_t\\ -d(x,\Gamma_t) & \text{for } x \in {}^{-}\Omega_t.\end{cases}$$

Note that $n(t,x) = \nabla d(t,x)$ is unit normal to the curve Γ_t. Then, as theorized by Osher–Sethian [21] and justified by Evans–Spruck in [11] for viscosity solutions, the zero level sets $(\Gamma_t)_{t>0}$ of u move with normal velocity

$$\begin{aligned} v(t,x,d(x,t)) &= -\frac{\mu}{n-1}\operatorname{div}(n(t,x))n(t,x)\\ &= -\frac{\mu}{n-1}\Delta d(x,t)\nabla d(x,t), \quad \mu > 0,\end{aligned}$$

if and only if u is a solution to the following nonlinear, degenerate equation

$$\partial_t u = \mu \operatorname{tr}\left((I - \widehat{\nabla u} \otimes \widehat{\nabla u})D^2 u\right), \tag{2.1}$$

where $\hat{p} = p/|p|$ for $p \in \mathbb{R}^n$ and $\otimes$ denotes the usual tensor product. That is, the zero level sets of u move according to their mean curvature if and only if u is a solution to the *mean curvature equation* given in (2.1). In fact, the mean curvature equation is a *geometric* equation, so if u solves (2.1), then so does $\Phi(u)$ for any smooth function $\Phi : \mathbb{R} \to \mathbb{R}$. Consequently, u is a solution to the mean curvature equation if and only if every level set of u moves by mean curvature.

For a bounded, open set $\Omega_0 \subset \mathbb{R}^n$, consider the triplet $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$ where $\Gamma_0 = \partial\Omega_0$. Let $u_0(x)$ be such that

$$\Omega_0 = \{x : u_0(x) > 0\} \quad \text{and} \quad \Gamma_0 = \{x : u_0(x) = 0\}.$$

If u is a solution to (2.1) with initial data $u(0,x) = u_0(x)$, then the zero level sets of u move according to their mean curvature and we say that $({}^{+}\Omega_t, \Gamma_t, {}^{-}\Omega_t)_{t\geq 0}$ denotes the *level set evolution* of $(\Omega_0, \Gamma_0, (\overline{\Omega_0})^c)$. Under certain conditions on Ω_0 (such as smooth and convex), the sets $(\Gamma_t)_{t\geq 0}$ do not develop interior, that is, $\Gamma_t = \partial({}^{+}\Omega_t) = \partial({}^{-}\Omega_t)$.

Consider the special case in which the curves Γ_t are smooth and do not develop interior, at least for some time. Then, the signed distance function d is smooth, satisfies $|\nabla d| = 1$ in a neighborhood of Γ_t, and is a solution to (2.1) on Γ_t. Moreover, as a consequence of the strong maximum principle, if two level sets $\{u(x,t) = c_1\}$ and $\{u(x,t) = c_1\}$ start separated, then they remain separated for some time.

2.1. **Weak solutions.** Due to the underlying geometry of Theorem 1.1, it is helpful to pass the notion of viscosity solutions of the PDE (2.1) to weak solutions of the level sets of the solution u. We use the notion of generalized flows for the mean curvature equation presented in [15]. Let $F(p, X)$ be given by

$$F(p, X) = -\mu \operatorname{tr}((I - \widehat{p} \otimes \widehat{p})X)$$

and the lower and upper semi-continuous envelopes of F be denoted by F_* and F^* respectively.

Definition 2.1. A family $(\Omega_t)_{t>0}$ of open (closed) subsets of $\mathbb{R}^n$ is a generalized super-flow (sub-flow) of the mean curvature equation (2.1) if for all $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, $h > 0$, and for all smooth functions $\varphi : (0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ such that

(i) (Boundedness) There exists $r > 0$ such that

$$\{(t, x) \in [t_0, t_0 + h] \times \mathbb{R}^n : \varphi(t, x) \geq 0\} \subset [t_0, t_0 + h] \times B(x_0, r),$$

(ii) (Strict subsolution) There exists $\delta = \delta(\varphi) > 0$ such that

$$\begin{aligned} &\partial_t \varphi + F^*(\nabla \varphi, D^2 \varphi) \leq -\delta \quad \text{in } [t_0, t_0 + h] \times \overline{B}(x_0, r), \\ &(\partial_t \varphi + F_*(\nabla \varphi, D^2 \varphi) \geq \delta) \end{aligned}$$

(iii) (Non-degeneracy)

$$D\varphi \neq 0 \quad \text{in } \{(t, x) \in [t_0, t_0 + h] \times \overline{B}(x_0, r) : \varphi(t, x) = 0\},$$

(iv) (Initial condition)

$$\begin{aligned} &\{x \in \overline{B}(x_0, r) : \varphi(t_0, x) \geq 0\} \subset \Omega_{t_0} \\ &(\{x \in \overline{B}(x_0, r) : \varphi(t_0, x) \leq 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0}), \end{aligned}$$

then

$$\begin{aligned} &\{x \in \bar{B}(x_0, r) : \varphi(t_0 + h, x) > 0\} \subset \Omega_{t_0 + h} \\ &(\{x \in \bar{B}(x_0, r) : \varphi(t_0 + h, x) < 0\} \subset \mathbb{R}^n \setminus \Omega_{t_0 + h}). \end{aligned}$$

For the interested reader, we remark that $(\Omega_t)_{t \geq 0}$ is a generalized super-flow (sub-flow) of (2.1) if and only if $\mathbb{1}_{\Omega_t} - \mathbb{1}_{(\overline{\Omega}_t)^c}$ is a viscosity super (sub) solution of (2.1), see [2, Theorem 2.4]. For an introduction and background on viscosity solutions, see for example [6].

3. The phase transition, the corrector, and the auxiliary functions

In this section, we will introduce the phase transition ϕ and the corrector ψ. Along the way, we will also define the auxiliary functions a_ε and $\bar{a}_\varepsilon$ and exhibit their relationship with fractional Laplacians and the mean curvature equation, respectively.

3.1. **The phase transition ϕ.** Let ϕ be the solution to (1.3). In [4], they proved existence and uniqueness of the solution ϕ. Asymptotics on the decay of ϕ were established in [22] with finer estimates in [9, 12, 17]. We summarize their results in the next lemma. For convenience in the notation, let c_0 and α be given respectively by

$$c_0^{-1} = \int_{\mathbb{R}} [\dot{\phi}(\xi)]^2 \, d\xi \quad \text{and} \quad \alpha = \frac{W''(0)}{C_n} \tag{3.1}$$

and let $H(\xi)$ be the heavyside function.

Lemma 3.1. *There is a unique solution $\phi \in C^{2,\beta}(\mathbb{R})$ of (1.3). Moreover, there exists a constant $C = C(\phi) > 0$ such that*

$$\left|\phi(\xi) - H(\xi) + \frac{1}{\alpha\xi}\right| \le \frac{C}{|\xi|^2}, \quad |\xi| \ge 1 \tag{3.2}$$

and

$$|\dot{\phi}(\xi)| \le \frac{C}{|\xi|^2}, \quad |\ddot{\phi}(\xi)| \le \frac{C}{|\xi|^2}, \quad |\xi| \ge 1. \tag{3.3}$$

The following is an auxiliary lemma that allows us to view one-dimensional fractional Laplacians of $\phi : \mathbb{R} \to \mathbb{R}$ equivalently as n-dimensional fractional Laplacians.

Lemma 3.2. *For a unit vector $e \in \mathbb{S}^n$, let $\phi_e(x) = \phi(e \cdot x) : \mathbb{R}^n \to \mathbb{R}$. Then,*

$$\mathcal{I}_n[\phi_e](x) = C_n \mathcal{I}_1[\phi](e \cdot x)$$

where

$$C_n = \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}}\, dy. \tag{3.4}$$

Consequently,

$$C_n \mathcal{I}_1[\phi](\xi) = \mathrm{P.\,V.} \int_{\mathbb{R}^n} (\phi(\xi + e \cdot z) - \phi(\xi)) \frac{dz}{|z|^{n+1}}, \quad \xi \in \mathbb{R}. \tag{3.5}$$

Proof. Begin by writing

$$\mathcal{I}_n[\phi_e](x) = \mathrm{P.\,V.} \int_{\mathbb{R}^n} (\phi(e \cdot x + e \cdot z) - \phi(e \cdot x)) \frac{dz}{|z|^{n+1}}.$$

We claim that it is enough to prove the result for $e = e_1$. Indeed, let T be a rotation matrix such that $Te = e_1$ and apply the change of variables $Tz = y$ to obtain

$$\begin{aligned}
\mathcal{I}_n[\phi_e](x) &= \mathrm{P.\,V.} \int_{\mathbb{R}^n} \big(\phi(e \cdot x + e \cdot T^{-1}y) - \phi(e \cdot x)\big) \frac{dy}{|T^{-1}y|^{n+1}} \\
&= \mathrm{P.\,V.} \int_{\mathbb{R}^n} (\phi(e_1 \cdot Tx + e_1 \cdot y) - \phi(e_1 \cdot Tx)) \frac{dy}{|y|^{n+1}} = \mathcal{I}_n[\phi_{e_1}](Tx).
\end{aligned}$$

If $\mathcal{I}_n[\phi_{e_1}](x_0) = C_n \mathcal{I}_1[\phi](e_1 \cdot x_0)$ for any $x_0 \in \mathbb{R}$, then we take $x_0 = Tx$ and notice that

$$\mathcal{I}_n[\phi_e](x) = \mathcal{I}_n[\phi_{e_1}](Tx) = C_n \mathcal{I}_1[\phi](e_1 \cdot Tx) = C_n \mathcal{I}_1[\phi](e \cdot x).$$

Hence, the result holds.

It remains to prove the lemma for $e = e_1$. Observe for $x = (x_1, x') \in \mathbb{R} \times \mathbb{R}^{n-1}$ that

$$\begin{aligned}
\mathcal{I}_n[\phi_{e_1}](x) &= \mathrm{P.\,V.} \int_{\mathbb{R}^n} (\phi(x_1 + z_1) - \phi(z_1)) \frac{dz}{|z|^{n+1}} \\
&= \mathrm{P.\,V.} \int_{\mathbb{R}} (\phi(x_1 + z_1) - \phi(z_1)) \left(\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+1}}\, dz' \right) dz_1.
\end{aligned}$$

Since

$$\begin{aligned}
\int_{\mathbb{R}^{n-1}} \frac{1}{|(z_1, z')|^{n+1}}\, dz' &= \int_{\mathbb{R}^{n-1}} \frac{1}{(|z'|^2 + z_1^2)^{\frac{n+1}{2}}}\, dz' \\
&= \frac{1}{|z_1|^{n+1}} \int_{\mathbb{R}^{n-1}} \frac{1}{(|y|^2 + 1)^{\frac{n+1}{2}}}\, |z_1|^{n-1}\, dy = \frac{C_n}{|z_1|^2},
\end{aligned}$$

we have

$$\mathcal{I}_n[\phi_{e_1}](\xi) = C_n \int_{\mathbb{R}} (\phi(x_1+z_1) - \phi(z_1)) \frac{dz_1}{|z_1|^2} = C_n \mathcal{I}_1[\phi](e_1 \cdot x).$$

To prove (3.5), fix $\xi \in \mathbb{R}$. Let $x \in \mathbb{R}^n$ be such that $\xi = e \cdot x$ and simply observe that

$$\begin{aligned} C_n \mathcal{I}_1[\phi](\xi) = C_n \mathcal{I}_1[\phi](e \cdot x) = \mathcal{I}_n[\phi_e](x) \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} (\phi(e\cdot x + e \cdot z) - \phi(e \cdot x)) \frac{dz}{|z|^{n+1}} \\ &= \mathrm{P.V.} \int_{\mathbb{R}^n} (\phi(\xi + e\cdot z) - \phi(\xi)) \frac{dz}{|z|^{n+1}}. \end{aligned}$$

□

3.2. **The auxiliary functions a_ε and $\bar{a}_\varepsilon$.** Here, we will introduce two auxiliary functions that are necessary for our analysis. Let $d = d(t,x)$ be a given smooth function. Define the function $a_\varepsilon = a_\varepsilon(\xi; t, x)$ by

$$a_\varepsilon = \mathrm{P.V.} \int_{\mathbb{R}^n} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x) \cdot z\right) \right) \frac{dz}{|z|^{n+1}},$$

where $(\xi, t, x) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}^n$. The corresponding function $\bar{a}_\varepsilon = \bar{a}_\varepsilon(t,x)$ is

$$\bar{a}_\varepsilon(t,x) = \frac{1}{\varepsilon\, |\ln \varepsilon|} \int_{\mathbb{R}^n} a_\varepsilon(\xi; t, x)\, \dot{\phi}(\xi)\, d\xi. \tag{3.6}$$

We have the following general estimate on $a_\varepsilon, \bar{a}_\varepsilon$. The proof is delayed until Section 7.

Lemma 3.3. *There is a constant $C = C(n, \phi, d) > 0$ such that*

$$|a_\varepsilon(\xi; t, x)| \le C\varepsilon^{1/2}.$$

for all $(\xi, t, x) \in \mathbb{R} \times [0,\infty) \times \mathbb{R}^n$. Consequently,

$$|\bar{a}_\varepsilon(t,x)| \le \frac{C}{\varepsilon^{1/2}\, |\ln \varepsilon|}.$$

We will be interested in a_ε and $\overline{a}_\varepsilon$ when d is the signed distance function to a front Γ_t. In this case, one of the main results in [15] is that $\bar{a}_\varepsilon$ converges to the mean curvature of d in a neighborhood of Γ_t, see Lemma 3.5. However, we must take care because the signed distance function itself is not smooth everywhere. Throughout the paper, we will use the following smooth extension of the distance function away from Γ_t.

Definition 3.4 (Extension of the signed distance function)**.** Let $\rho > 0$ be such that the signed distance function $\tilde{d}$ associated to a curve Γ_t is smooth in

$$Q_{2\rho} = \{(t,x) : |\tilde{d}(t,x)| \le 2\rho\}.$$

Consequently, $|\nabla \tilde{d}| = 1$ in $Q_{2\rho}$. Let $\eta(t,x)$ be a smooth, bounded function such that

$$\eta = 1 \text{ in } \{\tilde{d} < \rho\}, \quad \eta = 0 \text{ in } \{\tilde{d} > 2\rho\}, \quad 0 \le \eta \le 1.$$

We extend $\tilde{d}(t,x)$ in the set $\{\tilde{d} > \rho\}$ with the smooth bounded function $d(t,x)$ given by

$$d(t,x) = \begin{cases} \tilde{d}(t,x) & \text{in } \{\tilde{d}(t,x) \le \rho\} \\ \tilde{d}(t,x)\eta(t,x) + 2\rho(1-\eta(t,x)) & \text{in } \{\rho < \tilde{d}(t,x) < 2\rho\} \\ 2\rho & \text{in } \{\tilde{d}(t,x) \ge 2\rho\}. \end{cases}$$

In $\{\rho < \tilde{d} < 2\rho\}$, notice that d satisfies

$$d = 2\rho + (\tilde{d} - 2\rho)\eta > 2\rho - \rho\eta \geq \rho.$$

We similarly extend outside of Q_ρ, so that d is smooth, bounded and satisfies

$$d = \tilde{d} \text{ in } Q_\rho, \quad \rho < d \leq 2\rho \text{ in } \{\tilde{d} > \rho\}, \quad -2\rho \leq d < -\rho \text{ in } \{\tilde{d} < -\rho\}.$$

Lemma 3.5 (Lemma 4 in [15]). *Let d be as in Definition 3.4. Then,*

$$\lim_{\varepsilon \to 0} c_0 \bar{a}_\varepsilon(t,x) = \mu \Delta d(t,x) = \mu \operatorname{tr}\left((I - \widehat{\nabla d} \otimes \widehat{\nabla d}) D^2 d\right)$$

uniformly in $(t,x) \in Q_\rho$.

It is also important to notice that, morally, a_ε is the difference between an n-dimensional and a 1-dimensional fractional Laplacian of $\phi(d/\varepsilon)$. This is seen in the following two lemmas.

Lemma 3.6 (Near the front). *Let d be as in Definition 3.4. If $|d(t,x)| \leq \rho$, then*

$$a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}; t, x\right) = \varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right).$$

Proof. First, we write $a_\varepsilon = a_\varepsilon\left(d(t,x)/\varepsilon; t, x\right)$ as

$$\begin{aligned}
a_\varepsilon &= \mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon} + \nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\
&= \mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} \\
&\quad - \mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x)}{\varepsilon} + \nabla d(t,x)\cdot z\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}.
\end{aligned}$$

Since $e = \nabla d(t,x)$ is a unit vector when $|d(t,x)| \leq \rho$, we may apply Lemma 3.2 to the second integral. With this and a change of variables in the first integral, we obtain

$$\begin{aligned}
a_\varepsilon &= \varepsilon\,\mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(t,x+z)}{\varepsilon}\right) - \phi\left(\frac{d(t,x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right) \\
&= \varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right).
\end{aligned}$$

□

Lemma 3.7 (Far from the front). *Let d be as in Defintion 3.4. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}; t, x\right) - \left[\varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right]\right| \leq \frac{C\varepsilon}{\rho}.$$

Proof. In Section 7, we will show that all three terms are bounded by $C\varepsilon/\rho$. In particular,

$$\left|a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}; t, x\right)\right| + \left|\varepsilon \mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)\right| + \left|\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right| \leq \frac{C\varepsilon}{\rho}$$

follows from Lemmas 7.1, 7.2, and 7.3. □

3.3. **The corrector ψ.** The linearized operator $\mathcal{L}$ associated to (1.3) is given by

$$\mathcal{L}[\psi] = -C_n\mathcal{I}_1[\psi] + W''(\phi)\psi. \tag{3.7}$$

Let $\psi = \psi(\xi; t, x)$ be the solution to the linearized equation

$$\begin{cases} \mathcal{L}[\psi] = \dfrac{a_\varepsilon(\xi; t, x)}{\varepsilon\,|\ln\varepsilon|} + c_0\dot{\phi}(\xi)\left(\sigma - \bar{a}_\varepsilon(t,x)\right) + \tilde{\sigma}\left(W''(\phi(\xi)) - W''(0)\right) & \xi \in \mathbb{R} \\ \psi(\pm\infty; t, x) = 0, \end{cases} \tag{3.8}$$

where $\sigma > 0$ is a small positive constant and $\tilde{\sigma} > 0$ is such that $\sigma = W''(0)\tilde{\sigma}$. See Section 4 for a formal derivation of (3.8).

Lemma 3.8. *There is a unique solution $\psi = \psi(\xi; t, x)$ to (3.8) such that*

$$|\psi| + |\dot{\psi}| + |\psi_t| + \left|D_x^2\psi\right| \le \frac{C}{\varepsilon^{1/2}\,|\ln\varepsilon|}.$$

for some $C = C(n, \phi, d) > 0$.

We conclude this section by stating the following estimate for the n- and 1-dimensional fractional Laplacians of ψ. The proof will be in Section 7.

Lemma 3.9. *Let d be as in Definition 3.4. There is a constant $C = C(n, \psi, d) > 0$ such that*

$$\left|\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d(t,\cdot)}{\varepsilon}; t, \cdot\right)\right](x) - C_n\mathcal{I}_1[\psi(\cdot; t, x)]\left(\frac{d(t,x)}{\varepsilon}\right)\right| \le C\varepsilon^{1/2}.$$

for any $(t, x) \in [0, \infty) \times \mathbb{R}^n$.

4. Heuristics

4.1. **Ansatz for motion by mean curvature.** We believe it is helpful to view the heuristical derivation of the evolution of the fronts Γ_t^i by mean curvature in Theorem 1.1. For simplicity, we consider the case $N = 2$.

For the following formal computations, assume that the signed distance function $d_i(t, x)$ associated to Γ_t^i is smooth and that $|\nabla d_i| = 1$. Moreover, we assume that there is a positive, uniform distance ρ between Γ_t^1 and Γ_t^2.

The ansatz for the solution to the reaction-diffusion equation (1.1) is given by

$$u^\varepsilon(t, x) \simeq \phi\left(\frac{d_1(t,x)}{\varepsilon}\right) + \phi\left(\frac{d_2(t,x)}{\varepsilon}\right). \tag{4.1}$$

Plugging the ansatz into (1.1), the left-hand side gives

$$\varepsilon\partial_t u^\varepsilon \simeq \dot{\phi}\left(\frac{d_1}{\varepsilon}\right)\partial_t d_1 + \dot{\phi}\left(\frac{d_2}{\varepsilon}\right)\partial_t d_2. \tag{4.2}$$

Up to dividing by $\varepsilon\,|\ln\varepsilon|$, we use the equation for ϕ (see (1.3)) and estimates on a_ε (see Lemma 3.6 and Lemma 3.7) to write the right-hand side of (1.1) for the ansatz as

$$\begin{aligned}
&\varepsilon\mathcal{I}_n[u^\varepsilon]-W'(u^\varepsilon)\\
&\quad\simeq \varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_1}{\varepsilon}\right)\right]+\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_2}{\varepsilon}\right)\right]-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\quad=\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_1}{\varepsilon}\right)\right]-C_n\mathcal{I}_1[\phi]\left(\frac{d_1}{\varepsilon}\right)\right)+\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_2}{\varepsilon}\right)\right]-C_n\mathcal{I}_1[\phi]\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\qquad+C_n\mathcal{I}_1[\phi]\left(\frac{d_1}{\varepsilon}\right)+C_n\mathcal{I}_1[\phi]\left(\frac{d_2}{\varepsilon}\right)-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right)\\
&\quad\simeq a_\varepsilon\left(\frac{d_1}{\varepsilon}\right)+a_\varepsilon\left(\frac{d_2}{\varepsilon}\right)+W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)\right)+W'\left(\phi\left(\frac{d_2}{\varepsilon}\right)\right)-W'\left(\phi\left(\frac{d_1}{\varepsilon}\right)+\phi\left(\frac{d_2}{\varepsilon}\right)\right).
\end{aligned}\tag{4.3}$$

Freeze a point (t,x) near the front Γ_t^1. Let $\xi=d_1(t,x)/\varepsilon$ and assume separation of scales. That is, assume that ξ and (t,x) are unrelated. In this regard, let $\eta=|d_2(t,x)|\geq\rho$, so that η^{-1} is bounded. Since the ansatz u^ε is a solution to (1.1), we can multiply the equation by by $\dot\phi(\xi)$ and integrate over $\xi\in\mathbb{R}$ to write

$$\int_{\mathbb{R}}\varepsilon\partial_t u^\varepsilon\dot\phi\,d\xi\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\left(\varepsilon\mathcal{I}_n u^\varepsilon-W'(u^\varepsilon)\right)\dot\phi(\xi)\,d\xi.\tag{4.4}$$

For convenience, we will consider the left and right-hand sides separately again.

First, the left-hand side of (4.4) with (4.2) gives

$$\begin{aligned}
\int_{\mathbb{R}}\varepsilon\partial_t u^\varepsilon\,\dot\phi(\xi)\,d\xi&\simeq\partial_t d_1(t,x)\int_{\mathbb{R}}[\dot\phi\,(\xi)]^2\,d\xi+\dot\phi\left(\frac{\eta}{\varepsilon}\right)\partial_t d_2(t,x)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi\\
&\simeq c_0^{-1}\partial_t d_1(t,x)+\frac{C\varepsilon^2}{\eta^2}\partial_t d_2(t,x)\\
&\simeq c_0^{-1}\partial_t d_1(t,x),
\end{aligned}$$

where we used (3.1), (1.3), and the asymptotics on $\dot\phi$ (see (3.3)).

Then, we look at the right-hand side of (4.4) with (4.3). First, using that (1.3) and that W is periodic, we have

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}W'\left(\phi(\xi)\right)\dot\phi(\xi)\,d\xi=\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\frac{d}{d\xi}[W\left(\phi\left(\xi\right)\right)]\,d\xi=0.$$

Next, we use the asymptotics and properties of ϕ (see (1.3), (3.2)) and Taylor expand W' around the origin to estimate

$$\begin{aligned}
\frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi\left(\frac{\eta}{\varepsilon}\right)\right)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi&=\frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\\
&\simeq\frac{1}{\varepsilon\,|\ln\varepsilon|}\left[W'(0)+W''(0)\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\right]\\
&\simeq 0+\frac{1}{\varepsilon\,|\ln\varepsilon|}\frac{C\varepsilon}{\eta}\simeq 0.
\end{aligned}$$

For the remaining W' term, we Taylor expand around $\phi(\xi)$ and use similar estimates to obtain

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}W'\left(\phi\left(\xi\right)+\phi\left(\frac{\eta}{\varepsilon}\right)\right)\dot\phi(\xi)\,d\xi$$

$$
\begin{aligned}
&= \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} W'\left(\phi\left(\xi\right)+\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\dot\phi(\xi)\,d\xi\\
&\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}}\left[W'\left(\phi\left(\xi\right)\right)+W''\left(\phi\left(\xi\right)\right)\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\right]\dot\phi(\xi)\,d\xi\\
&= \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} W'\left(\phi\left(\xi\right)\right)\dot\phi(\xi)\,d\xi+\frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\phi\left(\frac{\eta}{\varepsilon}\right)-H\left(\frac{\eta}{\varepsilon}\right)\right)\int_{\mathbb{R}} W''\left(\phi\left(\xi\right)\right)\dot\phi(\xi)\,d\xi\\
&\simeq 0+\frac{1}{\varepsilon\,|\ln\varepsilon|}\frac{C\varepsilon}{\eta}\int_{\mathbb{R}}\frac{d}{d\xi}[W'\left(\phi\left(\xi\right)\right)]\,d\xi=0.
\end{aligned}
$$

Lastly, for the nonlocal terms, we first use Lemma 7.1 to justify that

$$
\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{\eta}{\varepsilon}\right)\int_{\mathbb{R}}\dot\phi(\xi)\,d\xi\simeq 0
$$

and then Lemma 3.5 to obtain

$$
\begin{aligned}
\frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} a_\varepsilon\left(\xi\right)\dot\phi(\xi)\,d\xi &= \bar a_\varepsilon(t,x)\\
&\simeq c_0^{-1}\mu\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2 d_1(t,x)\right).
\end{aligned}
$$

Combing all these pieces, the (4.4) for the ansatz gives

$$
c_0^{-1}\partial_t d_1(t,x)\simeq \mu c_0^{-1}\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2 d_1(t,x)\right).
$$

The computation for (t,x) frozen near Γ_t^2 is similar. We conclude that the fronts move according to their mean curvature:

$$
\begin{cases}
\partial_t d_1(t,x)\simeq \mu\,\mathrm{tr}\left((I-\widehat{\nabla d_1(t,x)}\otimes\widehat{\nabla d_1(t,x)})D^2 d_1(t,x)\right) & \text{near } \Gamma_t^1\\
\partial_t d_2(t,x)\simeq \mu\,\mathrm{tr}\left((I-\widehat{\nabla d_2(t,x)}\otimes\widehat{\nabla d_2(t,x)})D^2 d_2(t,x)\right) & \text{near } \Gamma_t^2.
\end{cases}
$$

4.2. **Ansatz for corrector.** One of the key ingredients in proving Theorem 1.1 is the construction of strict subsolutions (supersolutions), denoted by $v^\varepsilon = v^\varepsilon(t,x)$. For this, it is necessary to add a small corrector ψ to the ansatz in (4.1). In order to showcase the equation for ψ, we will consider the simplest case in which $N=1$ and assume that $d(t,x)=d_1(t,x)$ is smooth with $|\nabla d|=1$ and satisfies

$$
\partial_t d=\mu\Delta d-c_0\sigma\simeq c_0\bar a_\varepsilon(t,x)-c_0\sigma. \tag{4.5}
$$

To find the corrector ψ for the barrier, we consider the ansatz

$$
v^\varepsilon(t,x)\simeq\phi\left(\frac{d(t,x)}{\varepsilon}\right)+\varepsilon\,|\ln\varepsilon|\,\psi\left(\frac{d(t,x)}{\varepsilon}\right)-\varepsilon\,|\ln\varepsilon|\,\tilde\sigma,
$$

where the function ψ is to be determined and $\tilde\sigma>0$ is a small, given constant. Assume for now that ψ is smooth and bounded with bounded derivative.

Since v^ε is a supersolution to (1.1), then heuristically, there is a $\sigma>0$ such that

$$
\varepsilon\partial_t v^\varepsilon=\frac{1}{\varepsilon\,|\ln\varepsilon|}(\varepsilon\mathcal{I}_n v^\varepsilon-W'(v^\varepsilon))-\sigma. \tag{4.6}
$$

Plugging the ansatz into (4.6), the left-hand side gives

$$
\varepsilon\partial_t v^\varepsilon\simeq\dot\phi\left(\frac{d}{\varepsilon}\right)\partial_t d+\varepsilon\,|\ln\varepsilon|\,\dot\psi\left(\frac{d}{\varepsilon}\right)\partial_t d\simeq\dot\phi\left(\frac{d}{\varepsilon}\right)\partial_t d, \tag{4.7}
$$

where we use that $\dot{\psi}$ and $\partial_t d$ are bounded. Next, we look at the right-hand side of (4.6) for the ansatz. First, we use the equation for ϕ (see (1.3)) and estimates on a_ε (see Lemmas 3.6, 3.7, 3.9) to find that

$$
\begin{aligned}
\frac{\varepsilon}{\varepsilon\,|\ln\varepsilon|}\mathcal{I}_n[v^\varepsilon] &\simeq \frac{\varepsilon}{\varepsilon\,|\ln\varepsilon|}\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] + \varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right] \\
&= \frac{1}{\varepsilon\,|\ln\varepsilon|}\left(\varepsilon\mathcal{I}_n\left[\phi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right)\right) + \frac{1}{\varepsilon\,|\ln\varepsilon|}C_n\mathcal{I}_1[\phi]\left(\frac{d}{\varepsilon}\right) \\
&\quad + \left(\varepsilon\mathcal{I}_n\left[\psi\left(\frac{d}{\varepsilon}\right)\right] - C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right) \\
&\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) + \frac{1}{\varepsilon\,|\ln\varepsilon|}W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right).
\end{aligned}
\tag{4.8}
$$

On the other hand, we do a Taylor expansion for W' around $\phi(d/\varepsilon)$ to estimate

$$
\begin{aligned}
\frac{1}{\varepsilon\,|\ln\varepsilon|}W'(v^\varepsilon) &\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d)}{\varepsilon}\right)\right)\left(v^\varepsilon - \phi\left(\frac{d}{\varepsilon}\right)\right)\right] \\
&\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}\left[W'\left(\phi\left(\frac{d}{\varepsilon}\right)\right) + W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\left(\varepsilon\,|\ln\varepsilon|\,\psi\left(\frac{d}{\varepsilon}\right) - \varepsilon\,|\ln\varepsilon|\,\tilde{\sigma}\right)\right].
\end{aligned}
\tag{4.9}
$$

Equating (4.7) with (4.8) and (4.9), the equation for the ansatz gives

$$
\begin{aligned}
\dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d \simeq{}& \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) + C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right) \\
&- W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma.
\end{aligned}
\tag{4.10}
$$

Rearranging and using (4.5), we have

$$
\begin{aligned}
&-C_n\mathcal{I}_1[\psi]\left(\frac{d}{\varepsilon}\right) + W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right)\psi\left(\frac{d}{\varepsilon}\right) \\
&\quad\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) - \dot{\phi}\left(\frac{d}{\varepsilon}\right)\partial_t d + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma \\
&\quad\simeq \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d}{\varepsilon}\right) - \dot{\phi}\left(\frac{d}{\varepsilon}\right)c_0\bar{a}_\varepsilon + c_0\sigma\dot{\phi}\left(\frac{d}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d}{\varepsilon}\right)\right) - \sigma.
\end{aligned}
$$

We let ψ be the solution to this equation. In particular, let $\mathcal{L}$ be the linearized operator in (3.7). Then, that corrector ψ satisfies the equation

$$
\begin{aligned}
\mathcal{L}[\psi]\left(\frac{d(t,x)}{\varepsilon}\right) ={}& \frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\left(\frac{d(t,x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(t,x)}{\varepsilon}\right)c_0\bar{a}_\varepsilon(t,x) \\
&+ c_0\sigma\dot{\phi}\left(\frac{d(t,x)}{\varepsilon}\right) + \tilde{\sigma}W''\left(\phi\left(\frac{d(t,x)}{\varepsilon}\right)\right) - \sigma,
\end{aligned}
\tag{4.11}
$$

as desired. See (3.8) with $\sigma = W''(0)\tilde{\sigma}$.

In order to check the validity equation (4.11), at least formally, we freeze a point (t,x) near Γ_t^1. Let $\xi = d(t,x)/\varepsilon$ and assume separation of scales. We multiply both sides of (4.11) by $\dot{\phi}(\xi)$ and integrate over $\mathbb{R}$ to write

$$
\int_{\mathbb{R}}\mathcal{L}[\psi]\,(\xi)\,\dot{\phi}(\xi)\,d\xi = \int_{\mathbb{R}}\left(\frac{1}{\varepsilon\,|\ln\varepsilon|}a_\varepsilon\,(\xi) - \dot{\phi}\,(\xi)\,c_0\bar{a}_\varepsilon(t,x)\right)\dot{\phi}(\xi)\,d\xi
$$

$$+\int_{\mathbb{R}}\Big(c_0\sigma\dot{\phi}\left(\xi\right)+\tilde{\sigma}W''\left(\phi\left(\xi\right)\right)-\sigma\Big)\,\dot{\phi}(\xi)\,d\xi.$$

Since $\mathcal{I}_1$ is self-adjoint and ϕ satisfies (1.3), the left-hand side of the equation gives

$$\begin{aligned}\int_{\mathbb{R}}\mathcal{L}[\psi]\dot{\phi}\,d\xi&=\int_{\mathbb{R}}\Big(-C_n\mathcal{I}_1[\dot{\phi}]+W''\left(\phi\right)\dot{\phi}\Big)\,\psi\,d\xi\\&=\int_{\mathbb{R}}\frac{d}{d\xi}\left(-C_n\mathcal{I}_1[\phi]+W'\left(\phi\right)\right)\psi\,d\xi=0.\end{aligned}$$

To show that the right-hand side is also zero, we first use the definitions of $\bar{a}_\varepsilon$ and c_0 to find

$$\int_{\mathbb{R}}\left(\frac{1}{\varepsilon\left|\ln\varepsilon\right|}a_\varepsilon\left(\xi\right)-\dot{\phi}\left(\xi\right)c_0\bar{a}_\varepsilon(t,x)\right)\dot{\phi}(\xi)\,d\xi=\frac{1}{\varepsilon\left|\ln\varepsilon\right|}\int_{\mathbb{R}}a_\varepsilon\left(\xi\right)\dot{\phi}(\xi)\,d\xi-\bar{a}_\varepsilon(t,x)=0.$$

Then, we use that W' is periodic to find that

$$\int_{\mathbb{R}}\tilde{\sigma}W''\left(\phi(\xi)\right)\dot{\phi}(\xi)\,d\xi=\tilde{\sigma}\int_{\mathbb{R}}\frac{d}{d\xi}[W'\left(\phi(\xi)\right)]\,d\xi=\tilde{\sigma}[W'(1)-W'(0)]=0$$

and the definition of c_0 to see that

$$\int_{\mathbb{R}}\Big(c_0\sigma\dot{\phi}(\xi)-\sigma\Big)\,\dot{\phi}(\xi)\,d\xi=c_0\sigma\int_{\mathbb{R}}[\dot{\phi}(\xi)]^2\,d\xi-\sigma=0,$$

as desired.

Remark 4.1. Notice that ψ depends on the signed distance function $d(t,x)$. Hence, when $N>1$, we have a finite sequence of correctors, denoted by $\psi_1,\dots,\psi_N$, depending on the signed distance function $d_i(t,x)$ to the front Γ_t^i, $i=1,\dots,N$.

Remark 4.2. To see that $\sigma=W''(0)\tilde{\sigma}$, assume that $d(t,x)<<-1$ and $\psi\equiv0$. Then, $(t,x)\in{}^-\Omega_t^1$ is far from the front Γ_t^1 which implies $\phi(d(t,x)/\varepsilon)\approx0$ and, by Lemma 7.1, $a_\varepsilon((t,x)/\varepsilon)\simeq0$. Therefore, in (4.10), we have

$$0\simeq0+\tilde{\sigma}W''\left(0\right)-\sigma.$$

5. Construction of barriers

The main challenge in proving Theorem 1.1 is the construction of strict subsolutions (supersolutions) to (1.1). In particular, we will use barriers to prove that a sequence of sets are generalized super(sub)-flows. We will focus on the construction of subsolutions as the construction of supersolutions is similar.

Let $\varphi_i(t,x)$, $i=1,\dots,N$, be smooth functions satisfying *(i),(ii),(iii)* in Definition 2.1. Moreover, assume that

$$\{(t,x):\varphi_{i+1}(t,x)>0\}\subset\subset\{(t,x):\varphi_i(t,x)>0\}\quad\text{for } i=1,\dots,N-1. \tag{5.1}$$

Let $\tilde{d}_i(t,x)$ be the signed distance function associated to $\{x:\varphi_i(t,x)>0\}$. Then, we can denote the zero level set of φ_i by $\Gamma_t^i=\{x:\tilde{d}_i(t,x)=0\}$. As a consequence of *(iii)* in Definition 2.1, there is a $\rho>0$ such that $\tilde{d}_i(t,x)$ is smooth in the set

$$Q_{2\rho}^i=\{(t,x):|\tilde{d}_i(t,x)|\le2\rho\}$$

and $|\nabla\tilde{d}_i|=1$ in $Q_{2\rho}^i$. Moreover, by (5.1), and perhaps making ρ smaller, we can assume that $Q_\rho^i\cap Q_\rho^j=\emptyset$ for $i\neq j$. Let d_i be the smooth, bounded extension of $\tilde{d}_i$ outside of Q_ρ^i as defined in Definition 3.4.

As a consequence of *(ii)* in Definition 2.1, for $\sigma > 0$ sufficiently small,

$$\partial_t d_i \leq \mu \,\mathrm{tr}\left((I - \widehat{\nabla d_i} \otimes \widehat{\nabla d_i}) D^2 d_i\right) - c_0\sigma = \mu \Delta d_i - c_0 \sigma \quad \text{in } Q^i_\rho. \tag{5.2}$$

Let $\tilde{\sigma} > 0$ be such that $\sigma = W''(0)\tilde{\sigma}$. Then, we define the smooth barrier $v^\varepsilon(t,x)$ by

$$v^\varepsilon(t,x) = \sum_{i=1}^N \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^N \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon\left|\ln \varepsilon\right|. \tag{5.3}$$

Lemma 5.1. *For sufficiently small ε, v^ε is a strict subsolution to*

$$\varepsilon \partial_t v^\varepsilon - \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\left(\varepsilon \mathcal{I}_n[v^\varepsilon] - W'(v^\varepsilon)\right) < -\frac{\sigma}{2}. \tag{5.4}$$

Moreover, for ε sufficiently small, v^ε satisfies

$$-2\tilde{\sigma}\varepsilon\left|\ln\varepsilon\right| \leq v^\varepsilon(t,x) - \sum_{i=1}^N \mathbb{1}_{\{d_i(t,\cdot) \geq \tilde{\sigma}/2\}}(x) \leq -\frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right|. \tag{5.5}$$

Proof. We will break the proof into four main steps. First, we estimate the equation for $v^\varepsilon(t,x)$ for any (t,x). Then, we will show that $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is near a single front $\Gamma_t^{i_0}$ and then when (t,x) is far from all fronts Γ^i_t, $i = 1, \dots, N$. Lastly, we establish (5.5) for all (t,x).

For convenience, we use the following notation throughout the proof.

$$\begin{aligned}
\phi_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
\psi_i &:= \psi_i\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x\right)\\
\tilde{\phi}_i &:= \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) - H\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right)\\
a^i_\varepsilon &:= a_\varepsilon\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}; t, x\right)\\
\bar{a}^i_\varepsilon &:= \bar{a}_\varepsilon(t,x) \quad \text{corresponding to } a^i_\varepsilon\\
b^i_\varepsilon &:= \varepsilon\mathcal{I}_n\left[\phi\left(\frac{d_i(t,\cdot)-\tilde{\sigma}}{\varepsilon}\right)\right](x) - C_n \mathcal{I}_1[\phi]\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right).
\end{aligned} \tag{5.6}$$

We note that it will be important for the reader to remember the dependence of ψ_i and a_ε on the variables t, x and $\xi = d_i(t,x)/\varepsilon$ when taking derivatives in t, x.

<u>**Step 1**</u>. Computation for $v^\varepsilon(t,x)$ in (1.1) for an arbitrary $(t,x) \in [t_0, t_0+h] \times \mathbb{R}^n$.

First, the time derivative of v^ε at (t,x) is given by

$$\varepsilon \partial_t v^\varepsilon(t,x) = \sum_{i=1}^N \dot{\phi}_i\, \partial_t d_i(t,x) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^N \left[(\psi_i)_\xi \partial_t d_i(t,x) + \varepsilon(\psi_i)_t\right].$$

By Lemma 3.8,

$$\varepsilon \partial_t v^\varepsilon = \sum_{i=1}^N \dot{\phi}_i\, \partial_t d_i(t,x) + \mathcal{O}(\varepsilon^{1/2}).$$

Next, we consider the nonlocal term. For each $i = 1, \ldots, N$, we use that ϕ satisfies (1.3) to find

$$\varepsilon \mathcal{I}_n[\phi_i](x) = \varepsilon \mathcal{I}_n[\phi_i](x) - C_n \mathcal{I}_1[\phi] \left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon} \right) + W'(\phi_i).$$

Also, using that ψ satisfies (3.8) and Lemma 3.9, we find that

$$\begin{aligned}
\varepsilon \mathcal{I}_n[\psi_i](x) &= \varepsilon \mathcal{I}_n[\psi_i](x) - C_n \mathcal{I}_1[\psi_i] \left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon} \right) - \mathcal{L}[\psi] \left(\frac{d_i(t,\cdot) - \tilde{\sigma}}{\varepsilon} \right) + W''(\phi_i)\, \psi_i \\
&= \mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon |\ln \varepsilon|} a_\varepsilon^i + \dot{\phi}_i c_0 \left(\bar{a}_\varepsilon^i - \sigma \right) - \tilde{\sigma} \left(W''(\phi_i) - W''(0) \right) + W''(\phi_i)\, \psi_i.
\end{aligned}$$

Therefore, the 1/2-Laplacian of v^ε can be written as

$$\begin{aligned}
\varepsilon \mathcal{I}_n[v^\varepsilon](x) = \sum_{i=1}^N & \left[\varepsilon \mathcal{I}_n[\phi_i](x) - C_n \mathcal{I}_1[\phi] \left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon} \right) + W'(\phi_i) \right] \\
&+ \varepsilon |\ln \varepsilon| \sum_{i=1}^N \left[\mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon |\ln \varepsilon|} a_\varepsilon^i + \dot{\phi}_i c_0 \left(\bar{a}_\varepsilon^i - \sigma \right) \right. \\
&\qquad \left. - \tilde{\sigma} \left(W''(\phi_i) - W''(0) \right) + W''(\phi_i)\, \psi_i \right].
\end{aligned}$$

Recall the definitions of $\tilde{\phi}_i$ and b_ε^i introduced in (5.6). Since W is periodic, we have that $W'(\phi_i) = W'(\tilde{\phi}_i)$ and similarly $W''(\phi_i) = W''(\tilde{\phi}_i)$. Using this, rearranging, and utilizing the notation b_ε^i, we can equivalently write

$$\begin{aligned}
\varepsilon \mathcal{I}_n[v^\varepsilon](x) = & \sum_{i=1}^N \left[(b_\varepsilon^i - a_\varepsilon^i) + W'(\tilde{\phi}_i) \right] \\
& + \varepsilon |\ln \varepsilon| \sum_{i=1}^N \left[\mathcal{O}(\varepsilon^{1/2}) + W''(\tilde{\phi}_i) \psi_i + \dot{\phi}_i c_0 \left(\bar{a}_\varepsilon^i - \sigma \right) - \tilde{\sigma} \left(W''(\tilde{\phi}_i) - W''(0) \right) \right].
\end{aligned}$$

Then, the equation for v^ε at (t,x) can be written as

$$\begin{aligned}
\operatorname{Eqn}(v^\varepsilon) &:= \varepsilon \partial_t v^\varepsilon(t,x) - \frac{1}{\varepsilon |\ln \varepsilon|} \left(\varepsilon \mathcal{I}_n[v^\varepsilon(t,\cdot)](x) - W'(v^\varepsilon(t,x)) \right) \\
&= \mathcal{O}(\varepsilon^{1/2}) + \sum_{i=1}^N \dot{\phi}_i \partial_t d_i(t,x) \\
&\quad - \frac{1}{\varepsilon |\ln \varepsilon|} \left\{ \sum_{i=1}^N \left[(b_\varepsilon^i - a_\varepsilon^i) + W'(\tilde{\phi}_i) \right] \right. \\
&\quad + \varepsilon |\ln \varepsilon| \sum_{i=1}^N \left[\mathcal{O}(\varepsilon^{1/2}) + W''(\tilde{\phi}_i) \psi_i + \dot{\phi}_i c_0 \left(\bar{a}_\varepsilon^i - \sigma \right) - \tilde{\sigma} \left(W''(\tilde{\phi}_i) - W''(0) \right) \right] \\
&\quad \left. - W' \left(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon |\ln \varepsilon| \sum_{i=1}^N \psi_i - \tilde{\sigma} \varepsilon |\ln \varepsilon| \right) \right\}.
\end{aligned}$$

Grouping the error terms, the $\dot{\phi}_i$ terms, and the nonlinear terms together, we have

$$\begin{aligned}\operatorname{Eqn}(v^\varepsilon) = \mathcal{O}(\varepsilon^{1/2}) - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i \left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
+ \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{W'\left(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
- \sum_{i=1}^N \left(W'(\tilde{\phi}_i) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right]\right)\Bigg\}.\end{aligned}$$

Fix an index $i_0 \in \{1,\dots,N\}$. For the remainder of Step 1, we will conveniently isolate every term indexed with i_0 to help with Step 2. First, we do a Taylor expansion for W' around $\tilde{\phi}_{i_0}$ to obtain

$$\begin{aligned}W'\Bigg(\sum_{i=1}^N \tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\Bigg)\\
= W'(\tilde{\phi}_{i_0}) + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
+ \mathcal{O}\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)^2\right).\end{aligned}$$

By Lemma 3.8, we can write

$$\frac{1}{\varepsilon\,|\ln\varepsilon|}\mathcal{O}\left(\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)^2\right) = \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|).$$

Hence, we have that

$$\begin{aligned}\operatorname{Eqn}(v^\varepsilon) = \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right)\\
- \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right]\\
+ \frac{1}{\varepsilon\,|\ln\varepsilon|}\Bigg\{W'(\tilde{\phi}_{i_0}) + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\tilde{\phi}_i + \varepsilon\,|\ln\varepsilon|\sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\right)\\
- \left(W'(\tilde{\phi}_{i_0}) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_{i_0})\psi_{i_0} - \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\right]\right)\\
- \sum_{i\neq i_0}\left(W'(\tilde{\phi}_i) + \varepsilon\,|\ln\varepsilon|\left[W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right]\right)\Bigg\}\end{aligned}$$

where in the last two lines we extracted the i_0 term. Cancelling the $W'(\tilde{\phi}_{i_0})$ and $W''(\tilde{\phi}_{i_0})\psi_{i_0}$ terms then distributing $1/(\varepsilon\,|\ln\varepsilon|)$, we simplify to

$$\begin{aligned}
\operatorname{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0} \mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) \\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right] \\
&\quad + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + \sum_{i\neq i_0}\psi_i - \tilde{\sigma}\right) + \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right) \\
&\quad - \sum_{i\neq i_0}\left[\frac{W'(\tilde{\phi}_i)}{\varepsilon\,|\ln\varepsilon|} + W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

Next, we do a Taylor expansion for W' around 0 and recall that $W'(0) = 0$ to write

$$W'(\tilde{\phi}_i) = W'(0) + W''(0)\tilde{\phi}_i + \mathcal{O}((\tilde{\phi}_i)^2) = W''(0)\tilde{\phi}_i + \mathcal{O}((\tilde{\phi}_i)^2).$$

With this, we now have that

$$\begin{aligned}
\operatorname{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0} \mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) \\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right] \\
&\quad + W''(\tilde{\phi}_{i_0})\left(\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + \sum_{i\neq i_0}\psi_i - \tilde{\sigma}\right) + \tilde{\sigma}\left(W''(\tilde{\phi}_{i_0}) - W''(0)\right) \\
&\quad - \sum_{i\neq i_0}\left[\frac{W''(0)\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} + W''(\tilde{\phi}_i)\psi_i - \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

We rearrange to group the terms with $W''(\tilde{\phi}_{i_0}) - W''(0)$ together

$$\begin{aligned}
\operatorname{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0} \mathcal{O}\left(\frac{(\tilde{\phi}_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) \\
&\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b_\varepsilon^i - a_\varepsilon^i) + \sum_{i=1}^N \dot{\phi}_i\left[\partial_t d_i(t,x) - c_0\left(\bar{a}_\varepsilon^i - \sigma\right)\right] \\
&\quad + \left(W''(\tilde{\phi}_{i_0}) - W''(0)\right)\sum_{i\neq i_0}\frac{\tilde{\phi}_i}{\varepsilon\,|\ln\varepsilon|} - \tilde{\sigma}W''(0) \\
&\quad + \sum_{i\neq i_0}\left[\left(W''(\tilde{\phi}_{i_0}) - W''(\tilde{\phi}_i)\right)\psi_i + \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right)\right].
\end{aligned}$$

Looking at the last line, we Taylor expand W'' around 0 to find, for $i \neq i_0$,

$$\begin{aligned}
(W''(\tilde{\phi}_{i_0}) - W''(\tilde{\phi}_i))\psi_i + \tilde{\sigma}\left(W''(\tilde{\phi}_i) - W''(0)\right) &= \mathcal{O}(\psi_i) - \tilde{\sigma}\left(W'''(0)\tilde{\phi}_i + \mathcal{O}(\tilde{\phi}_i^2)\right) \\
&= \mathcal{O}(\psi_i) + \mathcal{O}(\tilde{\phi}_i)
\end{aligned}$$

and also

$$W''(\tilde\phi_{i_0}) - W''(0) = W'''(0)\tilde\phi_{i_0} + \mathcal{O}((\tilde\phi_{i_0})^2) = \mathcal{O}(\tilde\phi_{i_0}).$$

Therefore, we have

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2}) + \sum_{i\neq i_0}\left[\mathcal{O}\left(\frac{(\tilde\phi_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde\phi_i) + \mathcal{O}(\psi_i)\right]\\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b^i_\varepsilon - a^i_\varepsilon) + \sum_{i=1}^N \dot\phi_i\left[\partial_t d_i(t,x) - c_0\left(\bar a^i_\varepsilon - \sigma\right)\right] + \mathcal{O}(\tilde\phi_{i_0})\sum_{i\neq i_0}\frac{\tilde\phi_i}{\varepsilon\,|\ln\varepsilon|} - \sigma\end{aligned}$$

where we also used that $\sigma = W''(0)\tilde\sigma$. Hence, we conclude that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|) + \mathcal{O}(|\ln\varepsilon|^{-1}) + \mathcal{O}(\varepsilon^{1/2})\\ &\quad + \sum_{i\neq i_0}\left[\mathcal{O}\left(\frac{(\tilde\phi_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde\phi_i) + \mathcal{O}(\dot\phi_i) + \mathcal{O}(\psi_i) + \frac{\mathcal{O}(\tilde\phi_{i_0})\tilde\phi_i}{\varepsilon\,|\ln\varepsilon|}\right]\\ &\quad - \frac{1}{\varepsilon\,|\ln\varepsilon|}\sum_{i=1}^N (b^i_\varepsilon - a^i_\varepsilon) + \sum_{i=1}^N \dot\phi_i\left[\partial_t d_i(t,x) - c_0\left(\bar a^i_\varepsilon - \sigma\right)\right] - \sigma.\end{aligned}\tag{5.7}$$

<u>**Step 2**</u>. $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is near the front $\Gamma^{i_0}_t$.

Assume that $|d_{i_0}(t,x) - \sigma| \le |\ln\varepsilon|^{-1/2}$ for some index $1 \le i_0 \le N$. Then, by (5.1), for ε sufficiently small,

$$|d_i(t,x) - \sigma| \ge |\ln\varepsilon|^{-1/2} \quad \text{for all } i \neq i_0.$$

We begin by estimating the error terms in (5.7) for $i \neq i_0$. First, we use (3.2) to estimate

$$\begin{aligned}|\tilde\phi_i| &\le \left|\phi\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right) - H\left(\frac{d_i(t,x)-\sigma}{\varepsilon}\right) + \frac{\varepsilon}{\alpha(d_i(t,x)-\sigma)}\right| + \left|\frac{\varepsilon}{\alpha(d_i(t,x)-\sigma)}\right|\\ &\le \frac{C\varepsilon^2}{|d_i(t,x)-\sigma|^2} + \frac{\varepsilon}{\alpha\,|d_i(t,x)-\sigma|}\\ &\le C\varepsilon^2\,|\ln\varepsilon| + \frac{\varepsilon\,|\ln\varepsilon|^{1/2}}{\alpha}\\ &= \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}),\end{aligned}$$

from which it follows that

$$\frac{|\tilde\phi_i|}{\varepsilon\,|\ln\varepsilon|} \le \mathcal{O}(|\ln\varepsilon|^{-1/2}) \quad\text{and}\quad \frac{|\tilde\phi_i|^2}{\varepsilon\,|\ln\varepsilon|} \le \mathcal{O}(\varepsilon).\tag{5.8}$$

Next, we use (3.3) to find that

$$|\dot\phi_i| \le \frac{C\varepsilon^2}{|d_i(t,x)-\sigma|^2} \le \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|)\tag{5.9}$$

By Lemma 3.8, $\mathcal{O}(\psi_i) = o(1)$ for $i \neq i_0$. Combining the above estimates in view of (5.7), we have

$$\begin{aligned}&\sum_{i\neq i_0}\left(\mathcal{O}\left(\frac{(\tilde\phi_i)^2}{\varepsilon\,|\ln\varepsilon|}\right) + \mathcal{O}(\tilde\phi_i) + \mathcal{O}(\dot\phi_i) + \mathcal{O}(\psi_i) + \frac{\mathcal{O}(\tilde\phi_{i_0})\tilde\phi_i}{\varepsilon\,|\ln\varepsilon|}\right)\\ &\qquad\le \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\,|\ln\varepsilon|^{1/2}) + \mathcal{O}(\varepsilon^2\,|\ln\varepsilon|) + o(1) + \mathcal{O}(|\ln\varepsilon|^{-1/2})\end{aligned}$$

Next, we check the terms with $\bar{a}^i_\varepsilon$ and a^i_ε. For $i \neq i_0$, we use that d_i is smooth, Lemma 3.3, and (5.9) to obtain

$$\sum_{i\neq i_0} \dot{\phi}_i \left[\partial_t d_i(t,x) - c_0\left(\bar{a}^i_\varepsilon - \sigma\right)\right] = \sum_{i\neq i_0} \mathcal{O}\left(\frac{\dot{\phi}_i}{\varepsilon^{1/2}\left|\ln\varepsilon\right|}\right) = \mathcal{O}(\varepsilon^{3/2}) \leq \mathcal{O}(\varepsilon). \tag{5.10}$$

For $i = i_0$, we use that $\dot{\phi}_{i_0} \geq 0$, (5.2), and Lemma 3.5 to estimate

$$\begin{aligned}\dot{\phi}_{i_0}[\partial_t d_{i_0}(t,x) - c_0(\bar{a}^{i_0}_\varepsilon - \sigma)] &= \dot{\phi}_{i_0}\left([\partial_t d_{i_0}(t,x) - \mu\Delta d_{i_0}(t,x) + c_0\sigma] + [\mu\Delta d_{i_0}(t,x) - c_0\bar{a}^{i_0}_\varepsilon]\right)\\ &\leq \dot{\phi}_{i_0}(0 + o(1)) = o(1).\end{aligned}$$

Lastly, by Lemma 3.6 and Lemma 3.7,

$$\begin{aligned}\frac{1}{\varepsilon\left|\ln\varepsilon\right|}\sum_{i=1}^N \left|b^i_\varepsilon - a^i_\varepsilon\right| &= \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\sum_{i\neq i_0}\left|b^i_\varepsilon - a^i_\varepsilon\right|\\ &\leq \frac{1}{\varepsilon\left|\ln\varepsilon\right|}\sum_{i\neq i_0}\frac{C\varepsilon}{\left|\ln\varepsilon\right|^{-1/2}} = \mathcal{O}(\left|\ln\varepsilon\right|^{-1/2}).\end{aligned} \tag{5.11}$$

Consequently, in (5.7), we have that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) \leq\ & \mathcal{O}(\varepsilon\left|\ln\varepsilon\right|) + \mathcal{O}(\left|\ln\varepsilon\right|^{-1}) + \mathcal{O}(\varepsilon^{1/2})\\ &+ \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\left|\ln\varepsilon\right|^{1/2}) + \mathcal{O}(\varepsilon^2\left|\ln\varepsilon\right|) + o(1) + \mathcal{O}(\left|\ln\varepsilon\right|^{-1/2}) - \sigma.\end{aligned}$$

Taking ε sufficiently small, (5.3) holds.

Step 3. $v^\varepsilon(t,x)$ satisfies (5.4) when (t,x) is away from all fronts Γ^i_t.

Assume that for all $i = 1,\dots,N$,

$$\left|d_i(t,x) - \sigma\right| \geq \left|\ln\varepsilon\right|^{-1/2}.$$

Then, we estimate exactly as in Step 1 but we include $i = i_0$ in (5.10) and do not drop $i = i_0$ in (5.11). Consequently, we have that

$$\begin{aligned}\mathrm{Eqn}(v^\varepsilon) \leq\ & \mathcal{O}(\varepsilon\left|\ln\varepsilon\right|) + \mathcal{O}(\left|\ln\varepsilon\right|^{-1}) + \mathcal{O}(\varepsilon^{1/2})\\ &+ \mathcal{O}(\varepsilon) + \mathcal{O}(\varepsilon\left|\ln\varepsilon\right|^{1/2}) + \mathcal{O}(\varepsilon^2\left|\ln\varepsilon\right|) + o(1) + \mathcal{O}(\left|\ln\varepsilon\right|^{-1/2}) - \sigma.\end{aligned}$$

Taking ε sufficiently small, (5.3) holds.

Step 4. $v^\varepsilon(t,x)$ satisfies (5.5).

It is enough to show v^ε satisfies the following.

(1) In the set $\{d_N(t,x) \geq \tilde{\sigma}/2\}$:

$$N - 2\tilde{\sigma}\varepsilon\left|\ln\varepsilon\right| \leq v^\varepsilon \leq N - \frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right|.$$

(2) In the set $\{d_{i+1}(t,x) < \tilde{\sigma}/2 \leq d_i(t,x)\}$ for $i = 1,\dots,N-1$:

$$i - 2\tilde{\sigma}\varepsilon\left|\ln\varepsilon\right| \leq v^\varepsilon \leq i - \frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right|.$$

(3) In the set $\{d_1(t,x) < \tilde{\sigma}/2\}$:

$$-2\tilde{\sigma}\varepsilon\left|\ln\varepsilon\right| \leq v_\varepsilon \leq -\frac{\tilde{\sigma}}{2}\varepsilon\left|\ln\varepsilon\right|.$$

We begin with (2). For a fixed $1 \le i_0 \le N-1$, let (t,x) be such that $d_{i_0+1}(t,x) < \frac{\tilde\sigma}{2} \le d_{i_0}(t,x)$. Note that

$$-\frac{1}{d_i(t,x)-\tilde\sigma} < \frac{2}{\tilde\sigma} \quad \text{for all } i_0+1 \le i \le N.$$

Then, by (3.2) and Lemma 3.8, for ε small we have

(5.12)

$$\begin{aligned}
v^\varepsilon(t,x) &= \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\psi_i - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&= \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \sum_{i=i_0+1}^{N} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\mathcal{O}(\psi_i) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\le \sum_{i=1}^{i_0} 1 + \sum_{i=i_0+1}^{N} \left(0 - \frac{\varepsilon}{\alpha(d_i(t,x)-\tilde\sigma)} + \frac{C\varepsilon^2}{(d_i(t,x)-\tilde\sigma)^2}\right) + \mathcal{O}(\varepsilon^{1/2}) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\le i_0 + \sum_{i=1_0+1}^{N}\left(\frac{\varepsilon}{\alpha(\tilde\sigma/2)} + \frac{C\varepsilon^2}{(\tilde\sigma/2)^2}\right) + \mathcal{O}(\varepsilon^{1/2}) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\le i_0 - \frac{\tilde\sigma}{2}\varepsilon\left|\ln\varepsilon\right|.
\end{aligned}$$

On the other hand,

$$-\frac{1}{d_i(t,x)-\tilde\sigma} \ge \frac{2}{\tilde\sigma} \ge 0 \quad \text{for all } 1 \le i \le i_0.$$

Hence, for ε small, we similarly estimate from below using that $\phi \ge 0$ to find

(5.13)

$$\begin{aligned}
v^\varepsilon(t,x) &= \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\psi_i - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&= \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \sum_{i=i_0+1}^{N} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\mathcal{O}(\psi_i) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\ge \sum_{i=1}^{i_0} \left(1 - \frac{\varepsilon}{\alpha(d_i(t,x)-\tilde\sigma)} - \frac{C\varepsilon^2}{(d_i(t,x)-\tilde\sigma)^2}\right) + \sum_{i=i_0+1}^{N} 0 + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\mathcal{O}(\psi_i) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\ge i_0 + \sum_{i=1}^{i_0}\left(\frac{\varepsilon}{\alpha(\tilde\sigma/2)} + \frac{C\varepsilon^2}{(\tilde\sigma/2)^2}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\mathcal{O}(\psi_i) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\ge i_0 - 2\tilde\sigma\varepsilon\left|\ln\varepsilon\right|.
\end{aligned}$$

We now look at (1). Let (t,x) be such that $d_N(t,x) \ge \frac{\tilde\sigma}{2}$. Then, using that $\phi \le 1$, Lemma 3.8, and taking ε sufficiently small,

$$\begin{aligned}
v^\varepsilon(t,x) &= \sum_{i=1}^{N} \phi\left(\frac{d_i(t,x)-\tilde\sigma}{\varepsilon}\right) + \varepsilon\left|\ln\varepsilon\right| \sum_{i=1}^{N}\psi_i - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\le \sum_{i=1}^{N} 1 + \mathcal{O}(\varepsilon^{1/2}) - \tilde\sigma\varepsilon\left|\ln\varepsilon\right| \\
&\le N - \frac{\tilde\sigma}{2}\varepsilon\left|\ln\varepsilon\right|.
\end{aligned}$$

On the other hand, since $d_i(t,x) \geq \tilde{\sigma}/2$ for all $1 \leq i \leq N$, we have

$$-\frac{1}{d_i(t,x)-\tilde{\sigma}} \geq \frac{2}{\tilde{\sigma}} \geq 0 \quad \text{for all } 1 \leq i \leq N.$$

Estimating as in (5.13), we find that, for ε small,

$$v^\varepsilon(t,x) \geq N - 2\tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.$$

Finally, we show (3). Let (t,x) be such that $d_1(t,x) < \frac{\tilde{\sigma}}{2}$. Note that

$$-\frac{1}{d_i(t,x)-\tilde{\sigma}} < \frac{2}{\tilde{\sigma}} \quad \text{for all } 1 \leq i \leq N.$$

Estimating as in (5.12) for sufficiently small ε, we get

$$v^\varepsilon(t,x) \leq 0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.$$

On the other hand, using that $\phi \geq 0$, for ε small,

$$\begin{aligned} v^\varepsilon(t,x) &= \sum_{i=1}^N \phi\left(\frac{d_i(t,x)-\tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^N \psi_i - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\ &\geq \sum_{i=1}^N 0 + \mathcal{O}(\varepsilon^{1/2}) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon| \\ &\geq -2\tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,. \end{aligned}$$

This completes the proof. ∎

6. Proof of Theorem 1.1

Proof. We apply an adaptation of the abstract method introduced in [2], see also [1].

Begin by defining the families of open sets $(D^i)_{i=1}^N$ and $(E^i)_{i=1}^N$ by

$$D^i = \operatorname{Int}\left\{(t,x) \in (0,\infty)\times\mathbb{R}^n : \liminf_{\varepsilon\to0}{}_* \frac{u^\varepsilon - i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\} \subset (0,\infty)\times\mathbb{R}^n$$

$$E^i = \operatorname{Int}\left\{(t,x) \in (0,\infty)\times\mathbb{R}^n : \limsup_{\varepsilon\to0}{}_* \frac{u^\varepsilon - (i-1)}{\varepsilon\,|\ln\varepsilon|} \leq 0\right\} \subset (0,\infty)\times\mathbb{R}^n.$$

To define the traces of D^i and E^i, we first define the functions $\underline{\chi}^i, \overline{\chi}^i : (0,\infty)\times\mathbb{R}^n \to [-1,1]$, respectively, by

$$\underline{\chi}^i = \mathbb{1}_{D_i} - \mathbb{1}_{(D_i)^c} \quad \text{and} \quad \overline{\chi}^i = \mathbb{1}_{(E_i)^c} - \mathbb{1}_{E_i}.$$

Since D^i is open, $\underline{\chi}^i$ is lower semicontinuous, and since $(E_i)^c$ is closed, $\overline{\chi}^i$ is upper semicontinuous. To ensure that $\overline{\chi}^i$ and $\underline{\chi}^i$ remain lower and upper semicontinuous, respectively, at $t=0$, we set

$$\underline{\chi}^i(0,x) = \liminf_{t\to0,\ y\to x} \underline{\chi}^i(t,y) \quad \text{and} \quad \overline{\chi}^i(0,x) = \limsup_{t\to0,\ y\to x} \overline{\chi}^i(t,y).$$

Define the traces D_0^i and E_0^i by

$$D_0^i = \{x \in \mathbb{R}^n : \underline{\chi}^i(0,x) = 1\} \quad \text{and} \quad E_0^i = \{x \in \mathbb{R}^n : \overline{\chi}^i(0,x) = -1\}.$$

To apply the abstract method, we need the following propositions. We delay their proofs.

Proposition 6.1 (Initialization)**.** *For each $i = 1, \ldots, N$,*

$$\Omega_0^i \subset D_0^i \quad \textit{and} \quad (\overline{\Omega}_0^i)^c \subset E_0^i.$$

Proposition 6.2 (Propagation)**.** *For each $i = 1, \ldots, N$, the set D^i is a generalized super-flow, and the set $\overline{E^i}$ is a generalized sub-flow.*

For $t > 0$, define the sets D_t^i and E_t^i by

$$D_t^i = D^i \cap (\{t\} \times \mathbb{R}^n) \quad \text{and} \quad E_t^i = E^i \cap (\{t\} \times \mathbb{R}^n).$$

By the abstract method (see [1,2]), it follows from Propositions 6.1 and 6.2 that

$${}^+\Omega_t^i \subset D_t^i \subset {}^+\Omega_t^i \cup \Gamma_t^i \quad \text{and} \quad {}^-\Omega_t^i \subset E_t^i \subset {}^-\Omega_t^i \cup \Gamma_t^i.$$

The conclusion readily follows; we provide the details for completeness.

First, since ${}^+\Omega_t^i \subset D_t^i$, we use the definition of D_t^i to see that

$$\liminf_{\varepsilon\to 0} {}_* u^\varepsilon(t,x) \geq i \quad \text{for } x \in {}^+\Omega_t^i. \tag{6.1}$$

Using that ${}^-\Omega_t^{i+1} \subset E_t^{i+1}$, we similarly get

$$\limsup_{\varepsilon\to 0} {}_* u^\varepsilon(t,x) \leq (i+1) - 1 = i \quad \text{for } x \in {}^-\Omega_t^{i+1}. \tag{6.2}$$

Therefore, for $i = 1, \ldots, N-1$,

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = i \quad \text{in } {}^+\Omega_t^i \cap {}^-\Omega_t^{i+1}.$$

Next, by the comparison principle, $0 \leq u^\varepsilon \leq N$. Consequently,

$$0 \leq \liminf_{\varepsilon\to 0} {}_* u^\varepsilon \quad \text{and} \quad \limsup_{\varepsilon\to 0} {}_* u^\varepsilon \leq N.$$

Hence, together with (6.2) and respectively (6.1) we have

$$\lim_{\varepsilon\to 0} u^\varepsilon(t,x) = 0 \quad \text{in } {}^-\Omega_t^1 \quad \text{and} \quad \lim_{\varepsilon\to 0} u^\varepsilon(t,x) = N \quad \text{in } {}^+\Omega_t^N.$$

□

It remains to prove Propositions 6.1 and 6.2. We begin with the initialization.

6.1. **Proof of Proposition 6.1.**

Proof. We will prove that $\Omega_0^{i_0} \subset D_0^{i_0}$ for all $1 \leq i_0 \leq N$. The proof of $(\overline{\Omega}_0^{i_0})^c \subset E_0^{i_0}$ is similar.

Fix i_0, a point $x_0 \in \Omega_0^{i_0}$, and a small constant $\tilde\sigma > 0$. To prove that $x_0 \in D_0^{i_0}$, it is enough to show that, for all (t,x) in a neighborhood of $(0, x_0)$,

$$\liminf_{\varepsilon\to 0} {}_* \frac{u^\varepsilon(t,x) - i_0}{\varepsilon\,|\ln\varepsilon|} \geq 0.$$

For this, we will use (5.3) to construct a suitable subsolution $v^\varepsilon \leq u^\varepsilon$, depending on $\tilde\sigma$.

We begin by defining smooth functions φ_i for each $i = 1, \ldots, i_0$ that satisfy conditions *(i),(ii),(iii)* in Definition 2.1. Indeed, first let $r_i > 0$ be given by

$$r_i = d_i(x_0) - \frac{\tilde\sigma}{2}, \quad i = 1, \ldots, i_0,$$

where d_i is given in (1.4). Note that $B_{r_i}(x_0) \subset\subset \Omega_0^i$ and

$$r_i - r_{i+1} = d_i(x_0) - d_{i+1}(x_0) \geq d(\Gamma_0^i, \Gamma_0^{i+1}).$$

Define the smooth functions $\varphi_i(x)$, $i = 1, \ldots, i_0$, by

$$\varphi_i(t,x) = (r_i - Ct)_+^2 - |x - x_0|^2$$

for a large constant $C > 0$, to be determined. It is easy to check that the signed distance function $\tilde{d}_i(t,x)$ associated to $\{x : \varphi_i(t,x) > 0\}$ is

$$\tilde{d}_i(t,x) = r_i - Ct - |x - x_0| \tag{6.3}$$

and that

$$\{(t,x) : \varphi_i(t,x) > 0\} = \bigcup_{t\geq 0} \{t\} \times B(x_0, r_i - Ct).$$

Hence, *(i)* in Definition 2.1 is satisfied. Next, we see that

$$\nabla\varphi_i(t,x) = (-2C(r_i - Ct), -2(x - x_0))$$

and, for $t < r_{i_0}/(2C)$, we have

$$\begin{aligned}
\partial_t\varphi_i - \mu \operatorname{tr}\left((I - \widehat{\nabla_x\varphi_i} \otimes \widehat{\nabla_x\varphi_i})D_x^2\varphi_i\right) &= -2C(r_i - Ct) + 2\mu\,(n-1) \\
&\leq -Cr_{i_0} + 2\mu\,(n-1) \\
&\leq -c_0\sigma
\end{aligned}$$

for $C > 0$ sufficiently large. Hence, *(ii),(iii)* in Definition 2.1 are also satisfied.

Let ρ and d_i be such that d_i is a smooth, bounded extension of $\tilde{d}_i$ outside of Q^i_ρ as in Definition 3.4. Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{i_0} \psi_i\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.$$

By Lemma 5.1 (with $N = i_0$), we have that v^ε is a subsolution to (5.4) in $[0, r_{i_0}/(2C)] \times \mathbb{R}^n$.

We claim that $v^\varepsilon \leq u^\varepsilon$ in a neighborhood $\mathcal{N}(0,x_0) \subset \{d_{i_0}(t,x) \geq \tilde{\sigma}/2\}$. Let x be such that $d_{i_0}(0,x) \geq \tilde{\sigma}/2$. Then $d_i(x) \geq \tilde{\sigma}/2$ for all $i = 1, \ldots, i_0$, and we use (3.2) to estimate

$$\begin{aligned}
u^\varepsilon(0,x) &\geq \sum_{i=1}^{i_0} \phi\left(\frac{d_i(x)}{\varepsilon}\right) \\
&\geq \sum_{i=1}^{i_0} \left(1 - \frac{\varepsilon}{\alpha d_i(x)} - \frac{C\varepsilon^2}{(d_i(x))^2}\right) \\
&\geq \sum_{i=1}^{i_0} \left(1 - \frac{2\varepsilon}{\alpha\tilde{\sigma}} - \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right) \\
&\geq i_0 - \left(\frac{2\varepsilon}{\alpha\tilde{\sigma}} + \frac{4C\varepsilon^2}{\tilde{\sigma}^2}\right) N \\
&\geq i_0 - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,,
\end{aligned}$$

for ε sufficiently small. For each $1 \leq i < i_0$, we can similarly show that

$$u^\varepsilon(0,x) \geq i - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \quad \text{in } \left\{x : d_i(0,x) \geq \frac{\tilde{\sigma}}{2}\right\}.$$

On the other hand, when $d_1(0,x) < \frac{\tilde{\sigma}}{2}$, we simply have

$$u^\varepsilon(0,x) = \sum_{i=1}^{N} \phi\left(\frac{d_i(x)}{\varepsilon}\right) \geq 0 > -\frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon|\,.$$

Therefore, by the second inequality in (5.5),

$$u^\varepsilon(0,x) \geq \sum_{i=1}^{i_0} \mathbb{1}_{\{d_i(0,\cdot)\geq\tilde{\sigma}/2\}}(x) - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \geq v^\varepsilon(0,x).$$

By the comparison principle, the claim holds. Consequently, by the first inequality in (5.5), we have

$$\liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \geq \liminf_{\varepsilon\to 0}{}_* \frac{v^\varepsilon(t,x)-i_0}{\varepsilon\,|\ln\varepsilon|} \geq -2\tilde{\sigma} \quad \text{in } \mathcal{N}(0,x_0).$$

Letting $\tilde{\sigma}\to 0$, the result follows. □

6.2. **Proof of Proposition 6.2.**

Proof. Fix $1\leq i_0\leq N$. We will show that D^{i_0} is a generalized super-flow. The proof that $\overline{E^{i_0}}$ is a generalized sub-flow is similar.

Let $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^n$, $h>0$, and $\varphi_{i_0}:[t_0,t_0+h]\times\mathbb{R}^n\to\mathbb{R}$ be a smooth function satisfying *(i)-(iv)* in Definition 2.1. If $i_0\neq 1$, we will construct a smooth test function φ_i for the generalized flows D^i, $1\leq i<i_0$. If $i_0=1$, then we omit this step.

Consider the sequence of smooth functions φ_1^k given by

$$\varphi_1^k(t,x)=\varphi_{i_0}(t,x)+\frac{1}{k}.$$

Since $\nabla\varphi_{i_0}\neq 0$ on $\{\varphi_{i_0}=0\}$ and φ_{i_0} is smooth,

$$\nabla\varphi_1^k=\nabla\varphi_{i_0}\neq 0 \quad \text{in } \left\{-\frac{1}{k}\leq\varphi_{i_0}\leq\frac{1}{k}\right\}$$

for sufficiently large k. Consequently,

$$\partial\{\varphi_1^k\geq 0\}=\{\varphi_1^k=0\}$$

for large enough k. Then, the sequence of sets $(\{\varphi_1^k\geq 0\})_{k\in\mathbb{N}}$ is strictly decreasing and

$$\lim_{k\to\infty}\{\varphi_1^k\geq 0\}=\bigcap_{k\geq 1}\{\varphi_1^k\geq 0\}=\{\varphi_{i_0}\geq 0\}.$$

Recall from *(i)* that $\{\varphi_{i_0}\geq 0\}\subset[t_0,t_0+h]\times B(x_0,r)$. Since $\{\varphi_{i_0}\geq 0\}$ is closed and $B(x_0,r)$ is open, for k sufficiently large, we have

$$\{\varphi_{i_0}\geq 0\}\subset\{\varphi_1^k\geq 0\}\subset[t_0,t_0+h]\times B(x_0,r).$$

Moreover, since the mean curvature equation is geometric and φ_{i_0} satisfies *(iii)*, we have

$$\partial_t\varphi_1^k+F^*(\nabla\varphi_1^k,D^2\varphi_1^k)=\partial_t\varphi_{i_0}+F^*(\nabla\varphi_{i_0},D^2\varphi_{i_0})\leq-\delta$$

in $[t_0,t_0+h]\times\overline{B}(x_0,r)$. Therefore, φ_1^k satisfies *(i),(ii),(iii)* in Definition 2.1 when k is large.

For $i=1,\dots,i_0-1$, we define the smooth functions $\varphi_i:[t_0,t_0+h]\times\mathbb{R}^n\to\mathbb{R}$ by

$$\varphi_i(t,x):=\varphi_{i_0}(t,x)+\frac{1}{k}\left(\frac{i_0-i}{i_0-1}\right),$$

for a sufficiently large k. As a consequence of our previous discussion, each φ_i satisfies *(i),(ii),(iii)* in Definition 2.1. Moreover, (5.1) holds and, since $D^i_{t_0}$ is open,

$$\{x\in\overline{B}(x,r):\varphi_i(t_0,x)\geq 0\}=\left\{x\in\overline{B}(x,r):\varphi_{i_0}(t_0,x)\geq-\frac{1}{k}\left(\frac{i_0-i}{i_0-1}\right)\right\}\subset D^i_{t_0}$$

by making k larger, if necessary. Therefore, each φ_i also satisfies *(iv)* in Definition 2.1.

Let $\tilde{d}_i(t,x)$ be the signed distance function associated to $\{(t,x) : \varphi_i(t,x) \geq 0\}$. Let ρ and d_i be such that d_i is a smooth, bounded extension of $\tilde{d}_i$ outside of Q^i_ρ as in Definition 3.4. Let $v^\varepsilon = v^\varepsilon(t,x)$ be given by

$$v^\varepsilon(t,x) = \sum_{i=1}^{i_0} \phi\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}\right) + \varepsilon\,|\ln\varepsilon| \sum_{i=1}^{i_0} \psi_i\left(\frac{d_i(t,x) - \tilde{\sigma}}{\varepsilon}; t, x\right) - \tilde{\sigma}\varepsilon\,|\ln\varepsilon|\,.$$

By Lemma 5.1, v^ε is a subsolution to (5.4) in $[t_0, t_0+h] \times \mathbb{R}^n$. We next show that $u^\varepsilon \geq v^\varepsilon$.

By the initial condition *(iv)* in Definition 2.1, we have that

$$\{x : d_i(t_0,x) \geq 0\} = \{x : \varphi_i(t_0,x) \geq 0\} \subset D^i_{t_0} = \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0,x) - i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\}.$$

Therefore,

$$\{x : d_i(t_0,x) \geq \tilde{\sigma}/2\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0,x) - i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\},$$

which further gives that

$$u^\varepsilon(t_0,x) \geq i - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \quad \text{in } \{x : d_i(t_0,x) \geq \tilde{\sigma}/2\}.$$

In particular,

$$u^\varepsilon(t_0,x) \geq \sum_{i=1}^{i_0} \mathbb{1}_{\{d_i(t_0,\cdot)\geq\tilde{\sigma}/2\}}(x) - \frac{\tilde{\sigma}}{2}\varepsilon\,|\ln\varepsilon| \geq v^\varepsilon(t_0,x),$$

by the second inequality in (5.5).

By the comparison principle, $u^\varepsilon \geq v^\varepsilon$ on $[t_0, t_0+h] \times \mathbb{R}^n$. By the first inequality in (5.5), we have that

$$\frac{u^\varepsilon(t_0+h,x) - i}{\varepsilon\,|\ln\varepsilon|} \geq \frac{v^\varepsilon(t_0+h,x) - i}{\varepsilon\,|\ln\varepsilon|} \geq -2\tilde{\sigma} \quad \text{in } \{d_i(t_0+h,x) > \tilde{\sigma}/2\}.$$

Taking $\tilde{\sigma} \to 0$, it follows that

$$\{x : \varphi_i(t_0+h,x) \geq 0\} = \{x : d_i(t_0+h,x) \geq 0\} \subset \left\{x : \liminf_{\varepsilon\to 0}{}_* \frac{u^\varepsilon(t_0+h,x) - i}{\varepsilon\,|\ln\varepsilon|} \geq 0\right\},$$

as desired. □

7. Appendix

In this section, we prove the estimates stated in Section 3. We begin with the lemmas pertaining to a_ε and $\bar{a}_\varepsilon$ before addressing the results for the corrector ψ.

7.1. **Proof of Lemmas 3.3 and 3.7.**

First, we prove the general estimate in Lemma 3.3.

Proof of Lemma 3.3. Begin by writing

$$\begin{aligned} a_\varepsilon &= \int_{|z|<1/\varepsilon^{1/2}} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\ &\quad + \int_{|z|>1/\varepsilon^{1/2}} \left(\phi\left(\xi + \frac{d(t,x+\varepsilon z) - d(t,x)}{\varepsilon}\right) - \phi\left(\xi + \nabla d(t,x)\cdot z\right)\right)\frac{dz}{|z|^{n+1}} \\ &=: I + II. \end{aligned}$$

For the long-range interactions,

$$|II| \leq 2\|\phi\|_\infty \int_{|z|>1/\varepsilon^{1/2}} \frac{dz}{|z|^{n+1}} = C\varepsilon^{1/2}.$$

For the short range interactions, we use the mean value theorem and Taylor's theorem to estimate

$$\begin{aligned} |I| &\leq \|\dot\phi\|_\infty \int_{|z|<1/\varepsilon^{1/2}} \frac{|d(t,x+\varepsilon z) - d(t,x) - \nabla d(t,x)\cdot \varepsilon z|}{\varepsilon} \frac{dz}{|z|^{n+1}} \\ &\leq \|\dot\phi\|_\infty \left\|D^2 d\right\|_\infty \int_{|z|<1/\varepsilon^{1/2}} \varepsilon\,|z|^2 \frac{dz}{|z|^{n+1}} \\ &= \frac{C\varepsilon}{\varepsilon^{1/2}} = C\varepsilon^{1/2}. \end{aligned}$$

The conclusion for a_ε follows.

Consequently, using the behavior of ϕ at $\pm\infty$ in (1.3), we estimate

$$\begin{aligned} |\bar a_\varepsilon(t,x)| &\leq \frac{1}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} |a_\varepsilon(\xi;t,x)|\,\dot\phi(\xi)\,d\xi \\ &\leq \frac{C\varepsilon^{1/2}}{\varepsilon\,|\ln\varepsilon|}\int_{\mathbb{R}} \dot\phi(\xi)\,d\xi = \frac{C}{\varepsilon^{1/2}\,|\ln\varepsilon|}. \end{aligned}$$

□

Next, we prove the estimates referenced in the proof of Lemma 3.7.

Lemma 7.1. *Let d be as in Defintion 3.4. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left| a_\varepsilon\left(\frac{d(t,x)}{\varepsilon};t,x\right)\right| \leq \frac{C\varepsilon}{\rho}.$$

Lemma 7.2. *Let d be as in Defintion 3.4. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left|\mathcal{I}_n\left[\phi\left(\frac{d(t,\cdot)}{\varepsilon}\right)\right](x)\right| \leq \frac{C}{\rho}.$$

Lemma 7.3. *Let d be as in Defintion 3.4. If $|d(t,x)| > \rho$, then there is a constant $C = C(n,\phi,d) > 0$ such that*

$$\left|\mathcal{I}_1[\phi]\left(\frac{d(t,x)}{\varepsilon}\right)\right| \leq \frac{C\varepsilon}{\rho}.$$

The proofs are all very similar, but we provide the details for completeness.

Proof of Lemma 7.1. For convenience, we stop the notation in t and write $a_\varepsilon = a_\varepsilon\left(d(t,x)/\varepsilon;t,x\right)$ as

$$\begin{aligned} a_\varepsilon &= \mathrm{P.V.}\int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x) + \nabla d(x)\cdot\varepsilon z}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} \\ &= \mathrm{P.V.}\int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} \\ &\quad - \mathrm{P.V.}\int_{\mathbb{R}^n} \left(\phi\left(\frac{d(x) + \nabla d(x)\cdot\varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}} \\ &:= I + II. \end{aligned}$$

For I, we have

$$
\begin{aligned}
I &:= I_1 + I_2 \\
&= \int_{|z|<\frac{\rho}{2\varepsilon}} \left(\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(x)}{\varepsilon}\right) \nabla d(x)\cdot \varepsilon z \right) \frac{dz}{|z|^{n+1}} \\
&\quad + \int_{|z|>\frac{\rho}{2\varepsilon}} \left(\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) \right) \frac{dz}{|z|^{n+1}}.
\end{aligned}
$$

For the long-range interactions,

$$
|I_2| \leq 2\,\|\phi\|_\infty \int_{|z|>\frac{\rho}{2\varepsilon}} \frac{dz}{|z|^{n+1}} = \frac{C\varepsilon}{\rho}.
$$

For the short-range interactions, fix z such that $|z| < \frac{\rho}{2\varepsilon}$. By Taylor's Theorem,

$$
\phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(x)}{\varepsilon}\right) \nabla d(x)\cdot \varepsilon z = \frac{1}{2} D^2 \left[\phi\left(\frac{d(x+(\cdot))}{\varepsilon}\right)\right]\Big|_{\varepsilon\tau z} (\varepsilon z)\cdot(\varepsilon z),
$$

for some $0 \leq \tau \leq 1$. Notice that

$$
\begin{aligned}
D^2 \left[\phi\left(\frac{d(x+(\cdot))}{\varepsilon}\right)\right]\Big|_{\varepsilon\tau z} &= \ddot{\phi}\left(\frac{d(x+\varepsilon\tau z)}{\varepsilon}\right) \frac{\nabla d(x+\varepsilon\tau z)\otimes \nabla d(x+\varepsilon\tau z)}{\varepsilon^2} \\
&\quad + \dot{\phi}\left(\frac{d(x+\varepsilon\tau z)}{\varepsilon}\right)\frac{D^2 d(x+\varepsilon\tau z)}{\varepsilon}
\end{aligned}
$$

Since

$$
\left|\frac{d(x+\varepsilon\tau z)}{\varepsilon}\right| \geq \frac{|d(x)|}{\varepsilon} - \tau|z| \geq \frac{\rho}{\varepsilon} - \frac{\rho}{2\varepsilon} = \frac{\rho}{\varepsilon},
$$

we can use the asymptotics on $\dot{\phi}$ and $\ddot{\phi}$ in (3.3) to obtain

$$
\left|\dot{\phi}\left(\frac{d(x+\varepsilon\tau z)}{\varepsilon}\right)\right| \leq \frac{C\varepsilon^2}{\rho^2} \quad \text{and} \quad \left|\ddot{\phi}\left(\frac{d(x+\varepsilon\tau z)}{\varepsilon}\right)\right| \leq \frac{C\varepsilon^2}{\rho^2}.
$$

Therefore, since the first and second derivatives of d are bounded,

$$
\left| \phi\left(\frac{d(x+\varepsilon z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(x)}{\varepsilon}\right) \nabla d(x)\cdot \varepsilon z \right| \leq \frac{C\varepsilon^2}{\rho^2}|z|^2.
$$

Hence,

$$
|I_1| \leq \frac{C\varepsilon^2}{\rho^2}\int_{|z|<\frac{\rho}{2\varepsilon}} |z|^2 \frac{dz}{|z|^{n+1}} = \frac{C\varepsilon^2}{\rho^2}\frac{\rho}{\varepsilon} = \frac{C\varepsilon}{\rho}.
$$

Next, notice that $II = 0$ when $\nabla d(t,x) = 0$. Assume then $\delta := 2\,\|\nabla d\|_\infty > 0$ and write

$$
\begin{aligned}
II &:= II_1 + II_2 \\
&= \int_{|z|<\frac{\rho}{\delta\varepsilon}} \left(\phi\left(\frac{d(x)+\nabla d(x)\cdot\varepsilon z}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot{\phi}\left(\frac{d(x)}{\varepsilon}\right) \nabla d(x)\cdot z \right) \frac{dz}{|z|^{n+1}} \\
&\quad + \int_{|z|>\frac{\rho}{\delta\varepsilon}} \left(\phi\left(\frac{d(x)+\nabla d(x)\cdot z}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) \right) \frac{dz}{|z|^{n+1}}
\end{aligned}
$$

For the long-range interactions,

$$
|II_2| \leq 2\,\|\phi\|_\infty \int_{|z|>\frac{\rho}{\delta\varepsilon}} \frac{dz}{|z|^{n+1}} = \frac{C\delta\varepsilon}{\rho}.
$$

For the short-range interactions, fix z such that $|z| < \frac{\rho}{\delta\varepsilon}$. By Taylor's Theorem, we have

$$\begin{aligned}
&\phi\left(\frac{d(x)+\nabla d(x)\cdot\varepsilon z}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\nabla d(x)\cdot z\\
&\quad=\left(\ddot{\phi}\left(\frac{d(x+\nabla d(x)\cdot\varepsilon\tau z)}{\varepsilon}\right)\frac{\nabla d(x)\otimes\nabla d(x)}{\varepsilon}+\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)D^2d(x)\right)\varepsilon z\cdot z
\end{aligned}$$

for some $0\le\tau\le 1$. Since

$$\left|\frac{d(x+\nabla d(x)\cdot\varepsilon\tau z)}{\varepsilon}\right|\ge\frac{|d(x)|}{\varepsilon}-2\,\|\nabla d\|_\infty\,|z|\ge\frac{\rho}{\varepsilon}-\|\nabla d\|_\infty\,\frac{\rho}{\delta\varepsilon}=\frac{\rho}{2\varepsilon},$$

we can use the asymptotics on $\dot{\phi}$ and $\ddot{\phi}$ in (3.3) to obtain

$$\left|\dot{\phi}\left(\frac{d(x+\nabla d(x)\cdot\varepsilon\tau z)}{\varepsilon}\right)\right|\le\frac{C\varepsilon^2}{\rho^2}\quad\text{and}\quad\left|\ddot{\phi}\left(\frac{d(x+\nabla d(x)\cdot\varepsilon\tau z)}{\varepsilon}\right)\right|\le\frac{C\varepsilon^2}{\rho^2}.$$

Therefore, since the first and second derivatives of d are bounded,

$$\left|\phi\left(\frac{d(x)+\nabla d(x)\cdot\varepsilon z}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\nabla d(x)\cdot z\right|\le\frac{C\varepsilon^2}{\rho^2}\,|z|^2.$$

Hence,

$$|II_1|\le\frac{C\varepsilon^2}{\rho^2}\int_{|z|<\frac{\rho}{\delta\varepsilon}}|z|^2\,\frac{dz}{|z|^{n+1}}=\frac{C\varepsilon^2}{\rho^2}\,\frac{\rho}{\delta\varepsilon}=\frac{C\varepsilon}{\delta\rho}.$$

The result follow by combining the estimates for I_1, I_2, II_1, II_2. □

Proof of Lemma 7.2. We drop the notation in t and begin by writing

$$\begin{aligned}
&\mathcal{I}_n\left[\phi\left(\frac{d(\cdot)}{\varepsilon}\right)\right](x)\\
&\quad=\mathrm{P.V.}\int_{\mathbb{R}^n}\left(\phi\left(\frac{d(x+z)}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}\\
&\quad=\int_{|z|<\frac{\rho}{2}}\left(\phi\left(\frac{d(x+z)}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\frac{\nabla d(x)}{\varepsilon}\cdot z\right)\frac{dz}{|z|^{n+1}}\\
&\qquad+\int_{|z|>\frac{\rho}{2}}\left(\phi\left(\frac{d(x+z)}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^{n+1}}\\
&\quad=:I_1+I_2.
\end{aligned}$$

Looking first at the long-range interactions, we have

$$|I_2|\le 2\,\|\phi\|_\infty\int_{|z|>\frac{\rho}{2}}\frac{dz}{|z|^{n+1}}=\frac{C}{\rho}.$$

For the short-range interactions, fix z such that $|z|<\rho/2$. Then, by Taylor's theorem,

$$\begin{aligned}
&\phi\left(\frac{d(x+z)}{\varepsilon}\right)-\phi\left(\frac{d(x)}{\varepsilon}\right)-\dot{\phi}\left(\frac{d(x)}{\varepsilon}\right)\frac{\nabla d(x)}{\varepsilon}\cdot z\\
&=\left(\ddot{\phi}\left(\frac{d(x+\tau z)}{\varepsilon}\right)\frac{\nabla d(x+\tau z)\otimes\nabla d(x+\tau z)}{\varepsilon^2}+\dot{\phi}\left(\frac{d(x+\tau z)}{\varepsilon}\right)\frac{D^2d(x+\tau z)}{\varepsilon}\right)z\cdot z
\end{aligned}$$

for some $0\le\tau\le 1$. Since

$$\left|\frac{d(x+\tau z)}{\varepsilon}\right|\ge\frac{|d(x)|}{\varepsilon}-\frac{|z|}{\varepsilon}>\frac{\rho}{\varepsilon}-\frac{\rho}{2\varepsilon}=\frac{\rho}{2\varepsilon},$$

we can apply (3.3) to estimate

$$\left|\dot\phi\left(\frac{d(x+\tau z)}{\varepsilon}\right)\right| \le \frac{C\varepsilon^2}{\rho^2} \quad \text{and} \quad \left|\ddot\phi\left(\frac{d(x+\tau z)}{\varepsilon}\right)\right| \le \frac{C\varepsilon^2}{\rho^2}.$$

Therefore, using that the first and second derivatives of d are bounded,

$$\left|\phi\left(\frac{d(x+z)}{\varepsilon}\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot\phi\left(\frac{d(x)}{\varepsilon}\right)\frac{\nabla d(x)}{\varepsilon}\cdot z\right| \le \frac{C}{\rho^2}|z|^2.$$

Thus, we have that

$$|I_1| \le \frac{C}{\rho^2}\int_{|z|<\frac{\rho}{2}} |z|^2 \frac{dz}{|z|^{n+1}} = \frac{C}{\rho}.$$

The conclusion follows by combining the estimates for I_1 and I_2. □

Proof of Lemma 7.3. Omitting the notation in t, we write

$$\begin{aligned}
\mathcal{I}_1[\phi]\left(\frac{d(x)}{\varepsilon}\right) &= \text{P. V.}\int_{\mathbb{R}}\left(\phi\left(\frac{d(x)}{\varepsilon}+z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^2}\\
&= \int_{|z|<\frac{\rho}{2\varepsilon}}\left(\phi\left(\frac{d(x)}{\varepsilon}+z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot\phi\left(\frac{d(x)}{\varepsilon}\right)z\right)\frac{dz}{|z|^2}\\
&\quad + \int_{|z|>\frac{\rho}{2\varepsilon}}\left(\phi\left(\frac{d(x)}{\varepsilon}+z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right)\right)\frac{dz}{|z|^2}\\
&=: I_1 + I_2.
\end{aligned}$$

For the long-range interactions,

$$|I_2| \le 2\|\phi\|_\infty \int_{|z|>\frac{\rho}{2\varepsilon}}\frac{dz}{|z|^2} = \frac{C\varepsilon}{\rho}.$$

For the short-range interactions, fix z such that $|z| < \rho/(2\varepsilon)$. Then, by Taylor's theorem,

$$\phi\left(\frac{d(x)}{\varepsilon}+z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot\phi\left(\frac{d(x)}{\varepsilon}\right)z = \ddot\phi\left(\frac{d(x)}{\varepsilon}+\tau z\right)z^2$$

for some $0 \le \tau \le 1$. Since

$$\left|\frac{d(x)}{\varepsilon}+\tau z\right| \ge \frac{|d(x)|}{\varepsilon} - |z| > \frac{\rho}{\varepsilon} - \frac{\rho}{2\varepsilon} = \frac{\rho}{2\varepsilon},$$

we use (3.3) to estimate

$$\left|\ddot\phi\left(\frac{d(x)}{\varepsilon}+\tau z\right)\right| \le \frac{C\varepsilon^2}{\rho^2}.$$

Therefore,

$$\left|\phi\left(\frac{d(x)}{\varepsilon}+z\right) - \phi\left(\frac{d(x)}{\varepsilon}\right) - \dot\phi\left(\frac{d(x)}{\varepsilon}\right)z\right| \le \frac{C\varepsilon^2}{\rho^2}|z|^2,$$

so that

$$|I_1| \le \frac{C\varepsilon^2}{\rho^2}\int_{|z|<\frac{\rho}{2\varepsilon}}|z|^2\frac{dz}{|z|^2} = \frac{C\varepsilon}{\rho}.$$

Combining the estimates for I_1 and I_2, we have the desired result. □

7.2. **Proof of Lemmas 3.8 and 3.9.** This section is in preparation.

References

[1] G. Barles and F. Da Lio, A geometrical approach to front propagation problems in bounded domains with Neumann-type boundary conditions, *Interfaces Free Bound.* **5** (2003), 239–274.
[2] G. Barles and P. E. Souganidis, A new approach to front propagation problems: theory and applications, *Arch. Rational Mech. Anal.* **141** (1998), 237–296.
[3] L. Brown, The self-stress of dislocations and the shape of extended nodes, *Phil. Mag.* **10** (1964), p. 441.
[4] X. Cabré and Y. Sire, Nonlinear equations for fractional Laplacians II: existence, uniqueness, and qualitative properties of solutions, *Trans. Amer. Math. Soc.* **367** (2015), 911–941.
[5] X. Chen, Generation and propagation of interfaces for reaction-diffusion equations *J. Differential Equations* **96** (1992), 116-141.
[6] M. G. Crandall, H. Ishii, and P.-L. Lions, User's guide to viscosity solutions of second order partial differential equations, *Bull. Amer. Math. Soc.* **27** (1992), 1–67.
[7] F. Da Lio, N. Forcadel, and R. Monneau, Convergence of a non-local eikonal equation to anisotropic mean curvature motion. Application to dislocation dynamics, *J. Eur. Math. Soc.* **10** (2008), 1061–1104.
[8] E. Di Nezza, G. Palatucci, and E. Valdinoci, Hitchhiker's guide to the fractional Sobolev spaces, *Bull. Sci. Math.* **136** (2012), 521–573.
[9] S. Dipierro, G. Palatucci and E. Valdinoci, Dislocation dynamics in crystals: a macroscopic theory in a fractional Laplace setting, *Comm. Math. Phys.* **333** (2015), 1061–1105.
[10] S. Dipierro, S. Patrizi, and E. Valdinoci, A fractional glance to the theory of edge dislocations, *Geometric and Functional Inequalities and Recent Topics in Nonlinear PDE's* (2022), pp 27.
[11] L. C. Evans and J. Spruck, Motion of level sets by mean curvature. I. *J. Differential Geom.* **33** (1991), 635–681.
[12] M. González and R. Monneau, Slow motion of particle systems as a limit of a reaction-diffusion equation with half-Laplacian in dimension one, *Discrete Contin. Dyn. Syst.* **32** (2012), 1255–1286.
[13] J. R. Hirth and L. Lothe, Theory of Dislocations, Second Edition. Malabar, Florida: Krieger, 1992.
[14] D. Hull and D. J. Bacon, Introduction to Dislocations, ButterworthHeinemann, Oxford, 2011.
[15] C. Imbert and P. E. Souganidis, Phasefield theory for fractional diffusion-reaction equations and applications, preprint. (2009), pp 41.
[16] L. Modica and S. Mortola, Un esempio di Γ^--convergenza, *Boll. Un. Mat. Ital. B (5)* **14** (1997), 285–299.
[17] R. Monneau and S. Patrizi, Derivation of the Orowan's law from the Peierls–Nabarro model, *Commun. Partial. Differ. Equ.* **37** (2012), 1887–1911.
[18] R. Monneau and S. Patrizi, Homogenization of the Peierls-Nabarro model for dislocation dynamics, *J. Differential Equations* **253** (2012), 2064–2105.
[19] F. R. N. Nabarro, Dislocations in a simple cubic lattice, *Proc. Phys. Soc.* **59** (1947), 256–272.
[20] F. R. N. Nabarro, Fifty-year study of the Peierls–Nabarro stress, *Mater. Sci. Engrg. A* **234–236** (1997), 67–76.
[21] S. Osher and J. A. Sethian, Fronts propagating with curvature dependent speed: algorithms based on Hamilton-Jacobi formulations, *J. Computational Phys.* **79** (1988), 12–49.
[22] G. Palatucci, O. Savin and E. Valdinoci, Local and global minimizers for a variational energy involving a fractional norm, *Ann. Mat. Pura Appl.* **192** (2013), 673–718.
[23] R. Peierls, The size of a dislocation, *Proc. Phys. Soc.* **52** (1940), 34–37.
[24] O. Savin and E. Valdinoci, Γ-convergence for nonlocal phase transitions *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **29** (2012), 479–500.
[25] P. R. Stinga, User's guide to fractional Laplacians and the method of semigroups, In Anatoly Kochubei, Yuri Luchko (Eds.), *Fractional Differential Equations*, 235–266, Berlin, Boston: De Gruyter, 2019.

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