Solution to the 3n+m optimisation problem

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Problem

Let:

- $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ be vectors with known values $\in \mathbb{N}_+$.
- $\mathbf{p} = (p_x, p_y)$ be a point with known values $\in \mathbb{N}_+$;
- $n, m \in \mathbb{N}_+$.

The system of equations is as follows:

$$na_x + mb_x = p_x$$

$$na_y + mb_y = p_y$$

We are tasked with finding the lowest value of $3n + m \in \mathbb{N}_+$.

Solution

Let:

$$A = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}$$

Rewrite the system as a matrix equation:

$$A \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

Case 1: If $\det(A) \neq 0$

Because $\det(A) \neq 0$, we can rewrite to move $\begin{pmatrix} n \\ m \end{pmatrix}$ to the LHS:

$$\begin{pmatrix} n \\ m \end{pmatrix} = A^{-1} \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

Then:

$$n = \frac{b_y p_x - a_y p_y}{\det(A)}$$
$$m = \frac{-b_x p_x + a_x p_y}{\det(A)}$$

These values are guaranteed to give the lowest possible value of $3n+m$, as they are the only pair that satisfies the system. If $n, m \notin \mathbb{N}_+$, there is no solution because these do not satisfy the initial requirement, and we cannot pick another pair.

Case 2: If $\det(A) = 0$

2.1 No pairs n, m satisfy the system

For this to happen, $\mathbf{p}$ must not satisfy $\text{rank}([A|\mathbf{p}]) = \text{rank}(A)$.

2.2 Infinitely many pairs n, m satisfy the system

Assuming that $\mathbf{p}$ satisfies $\text{rank}([A|\mathbf{p}]) = \text{rank}(A)$, we first check whether there exists at least one pair (n, m) where both n and m are in $\mathbb{N}_+$. This can be done by checking the particular solution $\mathbf{v_p} = \begin{pmatrix} n_p \\ m_p \end{pmatrix}$.

- If both n_p and m_p are positive integers, we proceed to find the general solution.
- If either n_p or m_p is not a positive integer, the system has no solution $n, m \in \mathbb{N}_+$.

If a valid pair (n_p, m_p) is found, the general solution is given by:

$$\mathbf{v} = \mathbf{v_p} + t\mathbf{v_n}$$

where $\mathbf{v_p}$ is a particular solution, $\mathbf{v_n}$ is a vector in the null space of A, and $t \in \mathbb{Z}$. The objective function to minimize is:

$$f(t) = 3n_p + m_p + t(3n_n + m_n)$$

where (n_p, m_p) is a particular solution and (n_n, m_n) is a basis vector for the null space. To minimize $f(t)$, perform the following steps:

1. Compute $3n_n + m_n$. If it equals 0, then $f(t)$ is constant, and all solutions have the same value. In this case, yield all pairs n, m.
2. If $3n_n + m_n \neq 0$, find the integer t that minimizes $f(t)$. This depends on the sign of $3n_n + m_n$:
   - If $3n_n + m_n > 0$, decrease t until the result no longer satisfies the integer constraint (i.e., $n_p + tn_n > 0$ and $m_p + tm_n > 0$). Yield that pair.
   - If $3n_n + m_n < 0$, increase t similarly and yield the resulting pair.