Solution to the $3n + m$ Optimisation Problem

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Problem

Let:

- $\mathbf{a} = (a_x, a_y)$ and $\mathbf{b} = (b_x, b_y)$ be vectors with known values $\in \mathbb{N}_+$.
- $\mathbf{P} = (P_x, P_y)$ be a point with known values $\in \mathbb{N}_+$;
- $n, m \in \mathbb{N}_+$.

The system of equations is as follows:

$$\begin{cases} a_x n + b_x m = P_x \\ a_y n + b_y m = P_y \end{cases}$$

We are tasked with finding the smallest value of $3n + m$ such that $n, m \in \mathbb{N}_+$.

Solution

Let:

$$A = \begin{pmatrix} a_x & a_y \\ b_x & b_y \end{pmatrix}$$

Rewrite the system as a matrix equation:

$$A \begin{pmatrix} n \\ m \end{pmatrix} = \begin{pmatrix} P_x \\ P_y \end{pmatrix}$$

Case 1: If $\det(A) \neq 0$

Because $\det(A) \neq 0$, we can apply Cramér's rule:

$$n = \frac{\det \begin{pmatrix} P_x & a_y \\ P_y & b_y \end{pmatrix}}{\det(A)} = \frac{b_y P_x - a_y P_y}{\det(A)}$$

$$m = \frac{\det \begin{pmatrix} a_x & P_x \\ b_x & P_y \end{pmatrix}}{\det(A)} = \frac{a_x P_y - b_x P_x}{\det(A)}$$

Therefore:

$$3n + m = \frac{P_x(3b_y - b_x) + P_y(-3a_y + a_x)}{\det(A)}$$

However, if $n, m \notin \mathbb{N}_+$, there is no solution to the problem because these values do not satisfy the initial requirement, and no other pair can be selected.

Case 2: If $\det(A) = 0$

We can rewrite m in terms of n (using the first equation, for example):

$$m = \frac{P_x - n a_x}{b_x}$$

Now, substitute into $3n + m$:

$$3n + m = 3n + \frac{P_x - n a_x}{b_x} = \left(\frac{3b_x - a_x}{b_x} \right) n + \frac{P_x}{b_x}$$

The result is a linear function of the form $f(n) = an + b$, where $a, b > 0$.
We now want to minimize the expression:

$$3n + m = \left(\frac{3b_x - a_x}{b_x} \right) n + \frac{P_x}{b_x}$$

subject to the constraint that m must be a positive integer. We already derived that:

$$m = \frac{P_x - n a_x}{b_x}$$

For m to be an integer, the numerator $P_x - n a_x$ must be divisible by b_x. This gives the following condition:

$$P_x - n a_x \equiv 0 \pmod{b_x}$$

or equivalently:

$$n a_x \equiv P_x \pmod{b_x}$$

So, we need to find the smallest $n \in \mathbb{N}_+$ that satisfies the linear congruence:

$$n a_x \equiv P_x \pmod{b_x}$$

Thus, the value we are looking for is $f(n)$ of the previously mentioned n.