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Ground State Degeneracy from Geometry in 3D

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Abstract: we need an abstract...

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Contents

1 Introduction

- Topological Order
- Fractons
- extensive degeneracy

The formalism developed in [1] allows for the obtention of a large class of models where there are degrees of freedom at vertices in addition to the usual gauge fields at links. From the persepctive of higher gauge theory, such models are understood as lattice realizations of $0, 1$-gauge theories, see [2, 3]. Since degrees of freedom are associated to both 0-cells (vertices) and 1-cells (links). In [4, 5] a similar formalism is constructed for (3+1) topological phases using a 2-group as the gauge group. This is, a $1, 2$-gauge theory for topological phases of matter was constructed. In this case, degrees of freedom are associated to the 1-cells (links) and 2-cells (plaquettes). In [6, 7] the same models were constructed indepently. Taking these ideas into consideration, in [2] we constructed a class of Hamiltonian models for topological phases of matter in arbitrary dimensions that come from higher gauge theories. In this setup the notion of gauge group is promoted to a more general mathematical structure, namely a *chain complex of Abelian groups*. Furthermore, in [2] we also show that the ground state degeneracy is a topological invariant of an abstract co-chain complex. In this sense, the ground state of such models is completely characterized and understood from the 0-th cohomology group associated to such co-chain complex. Moreover, a theorem due to Brown [8] relates this rather abstract cohomology group with the usual cohomology groups of chain complexes, allowing for a natural geometrical interpretation of the ground state

subspace. This is, the Hamiltonian models we obtain from Abelian higher gauge theories have topologically ordered ground states since the GSD is a shown to be a topological invariant. Furthermore, this is confirmed by calculating the entanglement entropy of these models, see [3].

Recently, we have found that by considering some of these fields as having only *classical configurations*, the otherwise purely topological models now become sensitive to the geometry of the manifold upon which they are defined. This is showcased, as in the purely topological case, in the ground states of the Hamiltonian models, as we will show in the next section for a particular example. This fact has turned our attention to these new class of models that, we believe, are related to some other new findings in the topological phases of matter community. In particular, there is a strong interest in a new class of quantum states called *fracton* phases [9–19]. Some characteristic of fracton phases include elementary excitations with restricted mobility and in some cases a ground state degeneracy that explicitly depends on the geometry of the manifold [20]. The latter has caught our attention as the higher gauge theories constructed in [2] can be modified to obtain a source of models with a geometrical dependence of the ground state degeneracy. The systematic construction of such models consists on restricting some l-constituents of the n-gauge configurations to be classical; This leads to equivalence classes of Hamiltonian models, some of which, have a geometrical dependence of the ground state degeneracy.

2 Geometric Quantum Models

In this section we introduce a pair of quantum models whose ground state subspace is related to geometric features of the underlying lattice, as we will precisely show. We begin by introducing a very simple 2-dimensional model which will be used as a warm up for the 3-dimensional version that comes next.

2.1 2D Geometric Quantum Model

Consider then a 2D lattice that can be described by a simplicial complex $K = \bigcup_{n=0}^{2} K_n$, where K_0 is the set of vertices, K_1 of links and K_2 the set of plaquettes. At each vertex $v \in K_0$ we place a local Hilbert space $\mathcal{H}_v$ with basis $\{|x\rangle\,,\, x \in G_0\}$, where G_0 is a finite discrete group. The global Hilbert space is then taken to be the tensor product of all local spaces, namely, $\mathcal{H} : \bigotimes_v \mathcal{H}_v$. Up to this point the model resembles a pure 0-gauge theory (2D Ising model), in the sense of [1–3]. However, in addition to the degrees of freedom at vertices we will label the links of the lattice with elements of a second group G_1. It is important to highlight the fact that we are not placing G_1 degrees of freedom at links, instead we consider these labels as a *coloring* of the lattice. We do this by means of a set of maps $\xi : C_1 \to G_1$, where C_1 is the group of 1-chains, these maps are completely determined bu their action on the generators $l \in K_1$. Moreover, we can think of them as being the *classical configurations* of a 1-gauge field, see [2, 3]. The interaction between the *coloring* of the lattice and the dynamical G_0 degrees of freedom at vertices is determined by a group homomorphism, $\partial : G_1 \to G_0$. Again, the only task the G_1 labels are playing is to serve as parameters for the Hamiltonian operator we are about to define. For each map

$\xi : C_1 \to G_1$ there is a Hamiltonian operator given by:

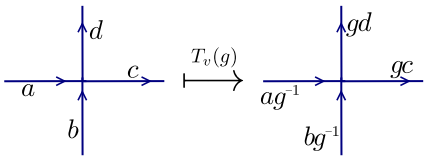
$$H(\xi) = -\sum_l B_l(\xi), \tag{2.1}$$

where $l \in K_1$ are the links of the lattice and the $B_l(\xi)$ operator measures the *0-holonomy* along a link given a coloring $\xi : C_1 \to G_1$, as follows:

$$B_l(\xi)\,|\rangle = \delta(xy^{-1}, \partial(g))\,|\rangle\,, \tag{2.2}$$

where $g = \xi(l) \in G_1$ is determined by the actual coloring of the lattice. The above shows that we are actually dealing with a family of Hamiltonians, parametrized by the ξ maps. Nevertheless, we will see that these Hamiltonians are naturally classified into unitary equivalence classes. To that intent let us give a pair of definitions.

Definition 2.1 (Elementary color transformation)**.** Given an arbitrary vertex $v \in K_0$ and an arbitrary group element $g \in G_1$, we define the *elementary color transformation* $T_v(g) : \xi \to \xi'$; where ξ and ξ' are two *colorings*, as follows:



Where $a, b, c, d \in G_1$ are the colorings of links around the vertex $v \in K_0$; determined by $\xi : C_1 \to G_1$. In other words, all colors of links around the vertex v get transformed by the parameter $g \in G_1$ to get the new coloring $\xi' : C_1 \to G_1$.

2.2 3D Geometric Quantum Model

Roughly speaking, the model consists in a 1-gauge theory of the type considered in [2, 3] whose Hilbert space gets now decorated by classical 2-gauge configurations, as we will see. So, we consider a 1-gauge theory defined over an arbitrary 3-dimensional lattice that can come from a simplicial complex $K = \bigcup_{n=0}^{3} K_n$, for instance. We stick to three dimensions but, in fact, the model could easily be defined on any (finite) d-dimensional lattice. The symmetry of the models is encoded in a pair of Abelian groups G_1 and G_2. More precisely, we will associate G_1-valued degrees of freedom to the links $l \in K_1$ of the lattice or, the local Hilbert spaces $\mathcal{H}_l$ are spanned by basis elements $\{|g\rangle\,,\, g \in G_1\}$. The global Hilbert space is taken to be the tensor product of these local spaces, i.e., $\mathcal{H} := \bigotimes_{l \in K_1} \mathcal{H}_l$. Additionally, we will associate G_2 labels to the faces of the lattice that will serve as a *coloring* for the Hamiltonian operator, to this intent we consider maps $\xi : C_2 \to G_2$ that determine a classical configuration (coloring) of G_2-values at plaquettes $p \in K_2$. Moreover, we consider the group homomorphism, $\partial : G_2 \to G_1$, that will serve as coupling between the classical fields at plaquettes and the dynamical fields at links. The G_2-labels are not dynamical degrees of freedom and their only task is to serve as a parameter for the Hamiltonian operator. Actually, we have one Hamiltonian for each such *coloring* and given by:

$$H(\xi) = -\sum_{p \in K_2} B_p(\xi) - \sum_{v \in K_0} A_v, \tag{2.3}$$

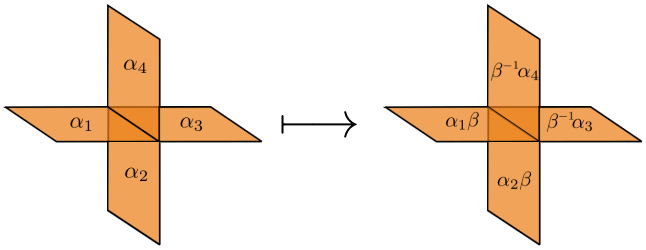
where the A_v is the usual gauge transformation of a 1-gauge theory, eq. (**??**).On the other hand, the holonomy measurement operator gets modified because of the classical configurations living at plaquettes of the complex. The operator will be, instead, similar to its 2-gauge counterpart, of eq. (2.4), i.e.,

$$B_p(\xi) \quad \left| \begin{array}{ccc} & c & \\ d & \alpha & b \\ & a & \end{array} \right\rangle = \delta(a+b-c-d, \partial(\alpha)) \left| \begin{array}{ccc} & c & \\ d & \alpha & b \\ & a & \end{array} \right\rangle, \tag{2.4}$$

notice that in this case the 2-gauge parameter, $\alpha \in G_2$ is now fixed by the classical configuration $\xi(p) = \alpha$, and it is not dynamical as in the full 2-gauge theory (cf. §**??**).

The model can be thought of as being a 2-gauge theory whose plaquette degrees of freedom are now fixed. The fixing of the 2-gauge fields is carried out via the ξ maps. Thus, we are actually dealing with a family of Hamiltonians parametrized by the ξ maps. The role that was being played by the 2-gauge transformation, eq. (**??**), is now split into two transformations that divide the family Hamiltonians into equivalence classes.

Definition 2.2. Given a 1-cell (link) $l \in K_1$ and an element $\beta \in G_2$, we define the *color transformation* $T_l(\beta) : \xi \to \xi'$, where ξ and ξ' are two 2-gauge classical configurations that differ by $T_l(\beta)$ as follows:



This is, the colorings of the faces around the link $l \in K_1$ are all transformed by the parameter β according to the orientations.

It is important to emphasize that the above transformation is NOT a 2-gauge transformation, as it is actually mapping between different Hamiltonians. So, in general, for two different colorings ξ and ξ' their corresponding Hamiltonians are different, i.e., $H(\xi) \neq H(\xi')$. However, if the two colorings, ξ and ξ', differ from one another by a color transformation $T_l(\beta)$ (or a sequence of them), it can be shown that the two corresponding Hamiltonians are unitarily equivalent. To achieve this goal, we define the following unitary transformation.

Definition 2.3. Let $l \in K_1$ be an arbitrary 1-cell, a parameter $\beta \in G_2$ and a basis element $|g\rangle \in \mathcal{H}_l$, $g \in G_1$; The *unitary* transformation $U_l(\beta) : \mathcal{H}_\xi \to \mathcal{H}_{\xi'}$ is given by:

$$U_l(\beta)\,|g\rangle = |\partial(\beta)g\rangle\,,$$

where $\partial : G_2 \to G_1$.

It is worthwhile pointing out that, in the above definition, both $\mathcal{H}_\xi$ and $\mathcal{H}_{\xi'}$ are actually the same Hilbert space, we just write the subscript to highlight the fact that the transformation corresponds to a mutation of colorings from ξ to ξ'.

Proposition 2.4. *Let ξ and ξ' are two colorings, connected by $T_l(\beta)$ for a given $l \in K_1$. Then, the two Hamiltonians $H(\xi)$ and $H(\xi')$ are unitarily equivalent, this is,*

$$U_l^{-1}(\beta)H(\xi')U_l(\beta) = H(\xi).$$

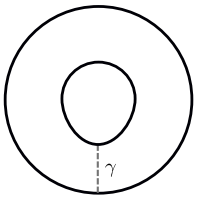
Proof. To show that the above proposition is true, we only need to prove the following:

$$U_l^{-1}(\beta)B_p(\xi')U_l(\beta) = B_p(\xi),$$

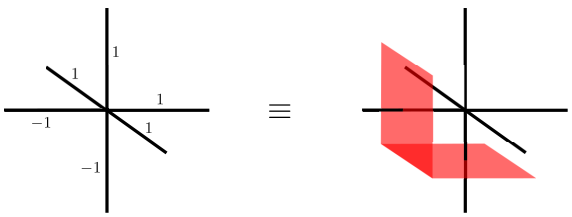
for all plaquettes $p \in K_2$ adjacent to the link l, or more precisely, that have the link l as a boundary. Which follows straightforwardly from applying the definition 2.3 and the definition of the plaquette operator of eq. (2.4). □

From this unitary equivalence we are able to identify the different equivalence classes of models. We now turn into an specific example in order to see the geometrical properties of the models.

Example 2.5. We define the model over a lattice that can come from a simplicial complex of a 3 dimensional manifold $\mathcal{M}$, homeomorphic to $S_2 \times I$, such that it has a shortest (dual) path γ between the two boundaries:

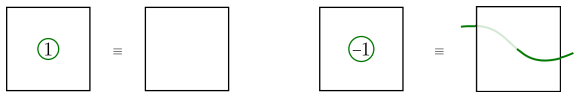


The requirement of having this shortest path is arbitrary and we make this assumption in order to precisely exhibit the main feature of the model. Furthermore, we consider the groups as being $G_1 = G_2 = \mathbb{Z}_2 = \{1, -1\}$ together with the group homomorphism $\partial : \mathbb{Z}_2 \xrightarrow{id} \mathbb{Z}_2$. So we're dealing with a $\mathbb{Z}_2$ 1-gauge theory with classical fields at faces taking values in $\mathbb{Z}_2$, as well. Consequently, the Hilbert space of the model is $\mathcal{H} := \bigotimes_l \mathbb{C}[\mathbb{Z}_2]_l$; An arbitrary basis state $|\psi\rangle \in \mathcal{H}$ consists of spins configurations that will be represented by red (dual) surfaces or the lack of them. We choose this representation as it is best suited for our purpose of exhibiting the ground state in terms of surfaces and strings. For instance, the following configuration:



where the $|-1\rangle$ configurations are represented by dual red surfaces. On the same spirit, we will also use a graphical notation for the coloring of the Hamiltonian represented in the lattice as green (dual) strings, this is, strings going through faces of the lattice; We

would like to emphasize that in this case, we are thinking of the maps $\xi : C_2 \to G_2$ as decorating the lattice with (dual) strings; These strings are NOT representations of states as the local Hilbert spaces live on the links of the lattice only, for instance, the two possible color configurations on a plaquette are shown in Fig.1.



(a) (b)

Figure 1. The two possible colorings at a single plaquette are shown.

The vertex operator, A_v, that implements local gauge transformations is composed of two parts:

$$A_v = \frac{1}{2}(A_v^1 + A_v^{-1}),$$

where $A_v^1 = \mathbb{1}$ and the action of A_v^{-1} is to flip the configurations from $|1\rangle$ to $|-1\rangle$ and vice versa, around the vertex in question. Concerning our representation of surfaces, the action of local gauge transformations is to either create red spheres or to modify surfaces by isotopy.

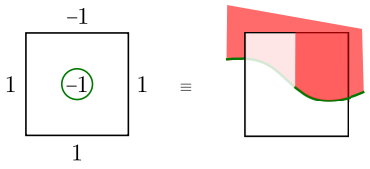
The plaquette operator, $B_p(\xi)$, projects into the "flat"-holonomy subspace of $\mathcal{H}$. Notice that, since they depend on the coloring ξ, the configurations that are considered "flat" depend now on the coloring of the lattice. When the coloring is trivial, this is, $\xi(p) = 1,\ \forall p \in K_2$, the plaquette operator favors configurations with and even number (or zero) of surfaces around a plaquette. A consequence of this is that, overall, the "flat" sector corresponds to configurations of closed surfaces in the bulk that can open in the each of the boundaries of $\mathcal{M}$. We will refer to this case ($\xi(p) = 1,\ \forall p \in K_2$) as *trivial* as it corresponds to defining the Toric Code [22] on $\mathcal{M}$. There are, however, colorings that are different from the trivial one, where some or many plaquettes might have a -1 coloring attached to it. In the case a particular plaquette $p \in K_2$ has such a coloring, the corresponding plaquette operator will project states different from the trivial case. For instance, the following state:

$$B_p \left| \begin{array}{ccc} & -1 & \\ 1 & -1 & 1 \\ & 1 & \end{array} \right\rangle = \delta(1(-1)11,-1) \left| \begin{array}{ccc} & -1 & \\ 1 & -1 & 1 \\ & 1 & \end{array} \right\rangle,$$

$$= \left| \begin{array}{ccc} & -1 & \\ 1 & -1 & 1 \\ & 1 & \end{array} \right\rangle,$$

the state has a single link configuration with $|-1\rangle$ yet the plaquette operator is giving a eigenvalue of 1. This particular configuration would not be considered as "flat" in the trivial theory. In our graphical notation, this state would look like:

The above shows that open surfaces in the bulk are allowed, as long as they end on a green (dual) string, this modifies the dynamics of the model with respect to the trivial case and in particular, the nature of the ground states. To understand this let us look at the



different unitary classes of Hamiltonians there are. Hamiltonian of eq.(2.3) depends on the way the lattice is decorated, given by the maps ξ. In principle, different colorings of the lattice give rise to different models; This is true, up to equivalence classes. The reason is the existence of both the color transformation of Def. **??** and the unitary transformation of Def.2.3, to see this let us study the action of both transformations on the colorings and the states. By looking at Def.**??** we note that the color transformations are labeled by $G_2 = \mathbb{Z}_2$; Therefore, for a given link $l \in K_1$ there are two elementary color transformations, namely, $T_l(1) = \mathrm{id}$ and $T_l(-1)$ that flips the configurations at plaquettes around l, as shown in Def. **??**. Thus, the only non trivial action of the color transformation is given by $T_l(-1)$ and can thought of as either creating a closed (dual) string around the link l or extending an existing string by local isotopy, as shown in Fig.(2.2). With this information we are able to explore the equivalence classes of Hamiltonians. The color transformation can only add closed dual strings or extend existing ones by isotopy; Also, considering that closed dual strings on the bulk can be opened at the two boundaries of $\mathcal{M}$ it is nor difficult to realize that there are two unitary equivalence classes of Hamiltonians.

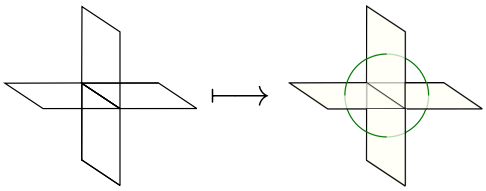


Figure 2. Color transformation around a link.

Class 1.- This class corresponds to the trivial coloring model, i.e., all colorings that are unitarily equivalent to the model with trivial coloring. The equivalence class corresponds to having an even number of dual strings connecting the two boundaries, as it is shown in Fig. 3. This class of models correspond to defining the Toric Code on the manifold $\mathcal{M}$. We wont dwell onto the details of this class of models since they are all known as they correspond to the usual topological models. A detailed description will be found on [**?**].

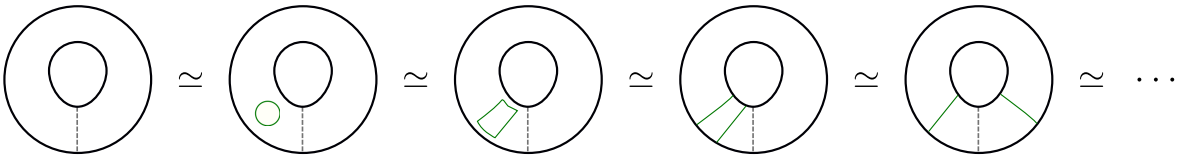


Figure 3. Some equivalent colorings of class 1 are shown.

Class 2.- This class consists on all lattice colorings that have an odd number of strings connecting the two boundaries of $\mathcal{M}$, as shown in Fig.4. This is the class of models that

we are interested in, consequently we will give a preliminary analysis of the ground state subspace of such models. First of all, we know is sufficient to take a representative of the class and we will work with the coloring with one single dual string connecting the two surfaces.

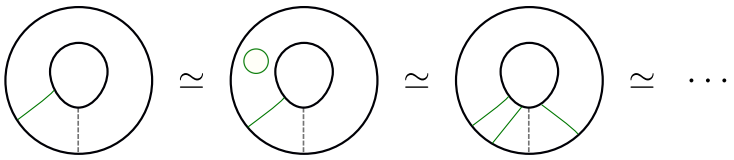


Figure 4. Some equivalent colorings of class 2 are shown.

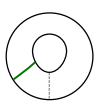
3 Geometric Ground State Degeneracy

3.1 2D Model

3.2 3D Model

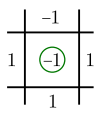
To study the ground state of the model we first notice that all plaquettes being crossed by the string will project to the holonomy value of -1. Therefore, if we take the coloring of the representative in Fig.4 and consider the seed state with $|1\rangle$ at every link, the following state:

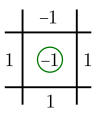
$$|\psi\rangle = \prod_v A_v \left| \quad \right\rangle ,$$



has an energy proportional to the length[1], l_p, of the string. This can be seen from the eigenvalues of the plaquette operators affected by the coloring:

$$B_p \left| \quad \right\rangle = \delta(1111,-1) \left| \quad \right\rangle = 0 ,$$



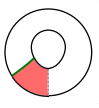


therefore evaluating the Hamiltonian operator on this state we obtain:

$$H\,|\psi\rangle = -(N_v - (N_p - l_p))\,|\psi\rangle\,, \tag{3.1}$$

where $N_v = |K_0|$ and $N_p = |K_2|$ are the number of vertices and plaquettes, respectively. hence, it is clear that the energy of such a state depends linearly on the length of the string. Naturally, arises the question of whether it exists a state with less energy than $|\psi\rangle$. We recall that the minimum energy in the topological case (class 1) is $E_0 = -N_v - N_p$, thus we can also ask if whether this is the energy minimum for the Hamiltonians of class 2. The answer to both questions is obtained by looking at the details of $\mathcal{M}$, recall that the lattice we over which the model is defined has a shortest dual path γ that joins the two surfaces. Thus, if we consider the following state,

$$|\psi_0\rangle = \prod_v A_v \left| \quad \right\rangle ,$$



[1]By length we mean the number of plaquettes being crossed by the string.

that consists on a red surface extending from the coloring string to the shortest path. Evaluating the Hamiltonian operator over this state we obtain,

$$H\left|\psi_0\right\rangle = -(N_v - (N_p - \gamma_p))\left|\psi_0\right\rangle, \tag{3.2}$$

now, we know that $\gamma_p < l_p$ for any dual string length l_p. Therefore, $|\psi_0\rangle$ is the state with least energy in $\mathcal{H}$. We want to stress that no matter what representative of the class of Hamiltonians we choose, the ground state will still perform the same task, this is, to find the shortest path between the two surfaces. Notice that, to localize this path it is enough to act with plaquette operators and look for the ones that get excited. As one can see in eqns. (3.1) and (3.2), the energy of these states is bigger than the ground state energy of the Hamiltonians in class 1, this is due to the existence of a single dual string on the lattice. To understand this let us look at the concrete action of the coloring ξ on the dynamics of the models; The dual string modifies the plaquette operators on its path making them project to the -1 sector, this can be thought of as condensing flux excitations at these plaquettes. However, flux excitations in the (trivial) theory always come in pairs, concretely pairs of dual strings (up to gauge transformations). Then by condensing one of the strings at the modified plaquettes there is one single dual string of excitations left "free". The ground state of the models consist on placing this "free" dual string at the shortest (dual) path from one inner to the outer surfaces, thus minimizing the energy of the state.

We see that even on the simplest example we get rich new behavior both for the ground states and the excited states. The geometric property of the ground state makes us place these models into what is known as fracton phases [9, 19]. Moreover, the procedure of making classical some degrees of freedom of a higher gauge theory we have a plethora of models with similar phenomena. The construction can be carried out for arbitrary dimensions and gauge fields.

4 Conclusions

A Some title

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Acknowledgments

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