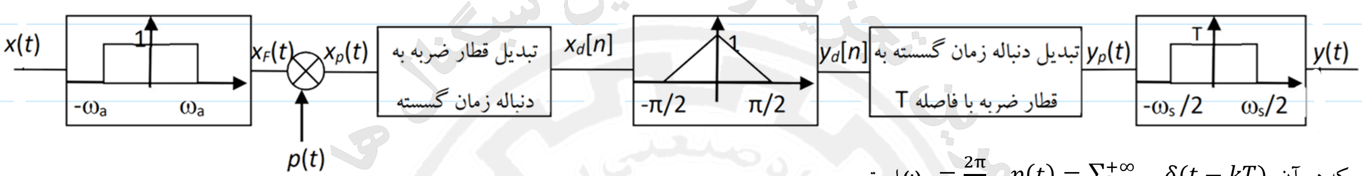
حل تمرین سری ششم-تجزیه و تحلیل سیگنال ها و سیستم ها-دانشگاه صنعتی اصفهان

۱- سیستم زیر طراحی یک پردازشگر دیجیتال برای سیگنال‌های پیوسته را نشان می‌دهد:

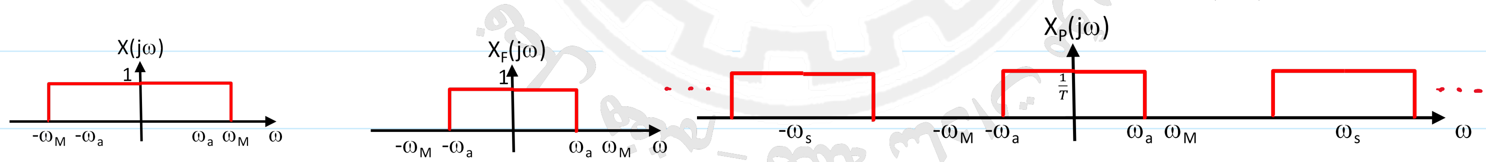



که در آن $p(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT)$ و $\omega_s = \frac{2\pi}{T}$ است.

الف) حداکثر مقدار فرکانس قطع $\omega_a$ برحسب T چقدر باید باشد تا اختلاط فرکانسی (aliasing) در طیف $x_p(t)$ رخ ندهد؟

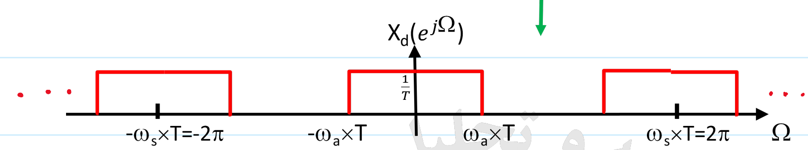
$$\omega_s > 2\omega_a \Rightarrow \frac{2\pi}{T} > 2\omega_a \Rightarrow \omega_a < \frac{\pi}{T} \qquad (\omega_a)_{max} = \frac{\pi}{T}$$

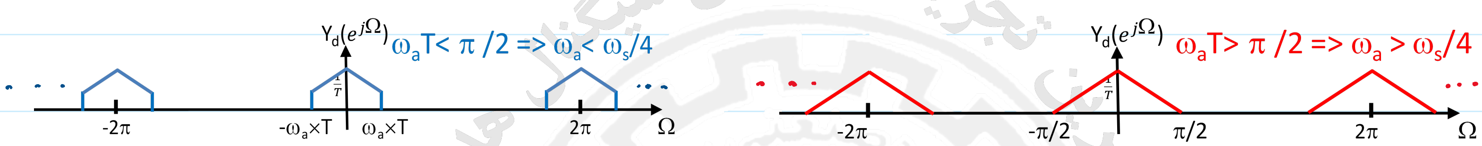
ب) اگر طیف $x(t)$ به صورت $\Pi\left(\frac{\omega}{2\omega_M}\right)$ و $\omega_M > \omega_a$ باشد، با فرض عدم تداخل، طیف سیگنال‌های $x(t)$، $x_F(t)$، $x_p(t)$، $x_d[n]$، $y_d[n]$، $y_p(t)$ و $y(t)$ را رسم کنید.



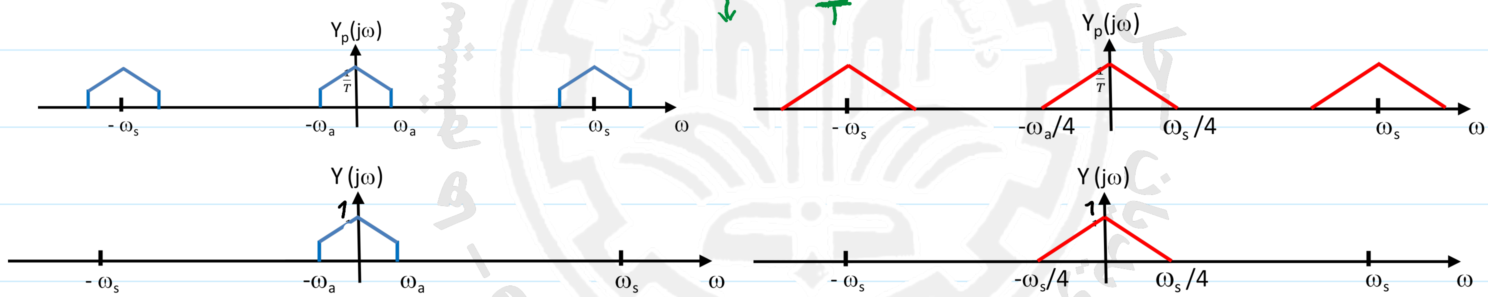

$\Omega = \omega \times T$

$$X_p(j\omega) = \frac{1}{2\pi} X_F(j\omega) * 2\pi \times \frac{1}{T} \sum \delta(\omega - k\omega_s)$$
$$= \frac{1}{T} \sum_{k=-\infty}^{+\infty} X_F(j(\omega - k\omega_s))$$



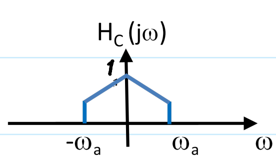




$\omega = \frac{\Omega}{T}$

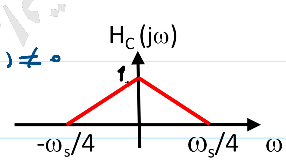



ج) پاسخ فرکانسی سیستم پردازشگر زمان پیوسته معادل سیستم فوق را در دو حالت $\omega_a > \frac{\omega_s}{4}$ و $\omega_a < \frac{\omega_s}{4}$ رسم کنید.

$|\omega| < \omega_M$



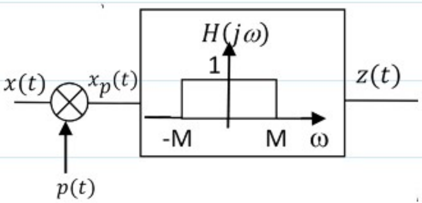

$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \quad X(j\omega) \neq 0$



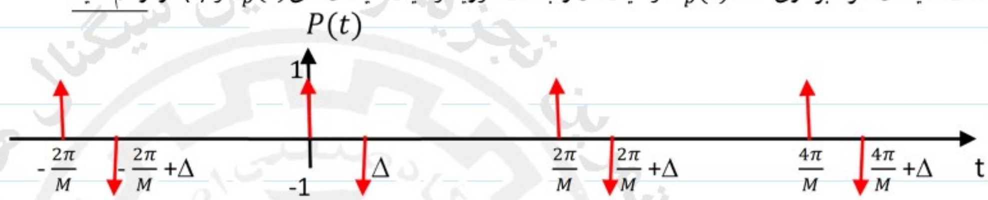

$|\omega| < \omega_M$

۲- سیگنال باند محدود x(t) با طیف $X(j) = \Lambda(\frac{\omega}{2M})$، با سیگنال p(t) به صورت زیر نمونه برداری می‌شود.

($\Lambda\left(\frac{\omega}{2M}\right)$ یک سیگنال مثلثی با پایه 2M و ارتفاع واحد است. $\Delta = \pi/(2M)$)




الف) سیگنال نمونه برداری شده $x_p(t)$ و طیف آن را بدست اورید و طیف سیگنال های $x_p(t)$ و z(t) را رسم کنید.




$$x_p(t) = x(t)\,p(t) \Rightarrow X_p(j\omega) = X(j\omega) * P(j\omega) \times \frac{1}{2\pi}$$

$$p(t) = \sum_{k=-\infty}^{+\infty} \delta\left(t - k\frac{2\pi}{M}\right) - \sum_{k=-\infty}^{+\infty} \delta\left(t - \Delta - k\frac{2\pi}{M}\right)$$

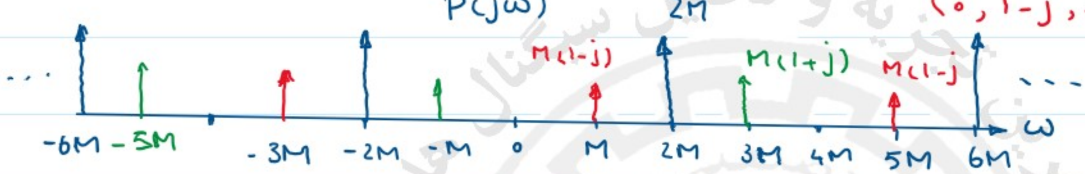
$$P(j\omega) = 2\pi \times \frac{1}{2\pi/M} \times \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) - e^{-j\omega\Delta} \times 2\pi \times \frac{1}{2\pi/M} \times \sum_{k=-\infty}^{+\infty} \delta(\omega - k\omega_s) \qquad \omega_s = \frac{2\pi}{T} = M$$

$$= M\left(1 - e^{-j\omega\Delta}\right) \sum_{k=-\infty}^{+\infty} \delta(\omega - k\cancel{\omega_s}^{M}) = M \sum_{k=-\infty}^{+\infty} \left(1 - e^{-j\omega\Delta}\right) \delta(\omega - kM)$$

$$= M \sum_{k=-\infty}^{+\infty} \left(1 - e^{-jkM\Delta}\right) \delta(\omega - kM) \qquad \Delta \rightarrow \frac{\pi}{2M}$$

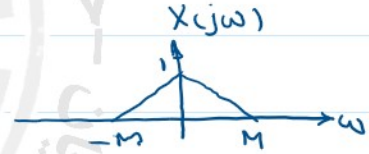
$$= M \sum_{k=-\infty}^{+\infty} \underbrace{\left(1 - e^{-jk\frac{\pi}{2}}\right)} \delta(\omega - kM)$$

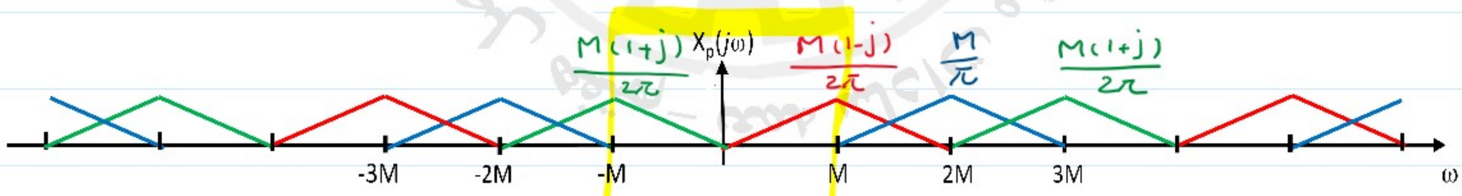
$(0, 1-j, +2, 1+j)$

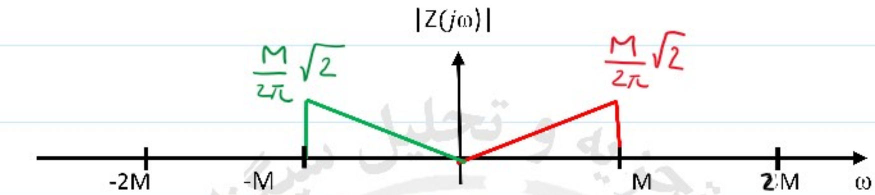



$$X_p(j\omega) = \frac{M}{2\pi} X(j\omega) * \sum_{k=-\infty}^{+\infty} \left(1 - e^{-jk\pi/2}\right) \delta(\omega - kM)$$

$$= \frac{M}{2\pi} \sum_{k=-\infty}^{+\infty} \left(1 - e^{-jk\pi/2}\right) X(j(\omega - kM))$$

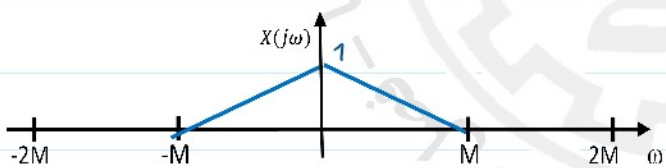


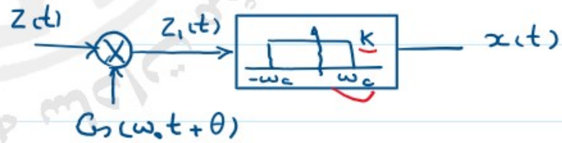





ب) سیستمی طراحی کنید که سیگنال x(t) را از روی سیگنال z(t) بازسازی کند.



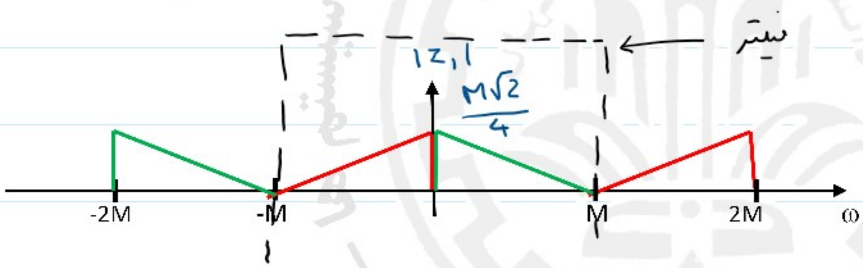




$$z_1(t) = z(t)\cos(\omega_0 t + \theta) \longrightarrow \frac{1}{2}\left[e^{j\theta}e^{j\omega_0 t} + e^{-j\theta}e^{-j\omega_0 t}\right]$$

$$Z_1(j\omega) = \frac{1}{2\pi}Z(j\omega) * \pi\left[e^{j\theta}\delta(\omega-\omega_0) + e^{-j\theta}\delta(\omega+\omega_0)\right]$$

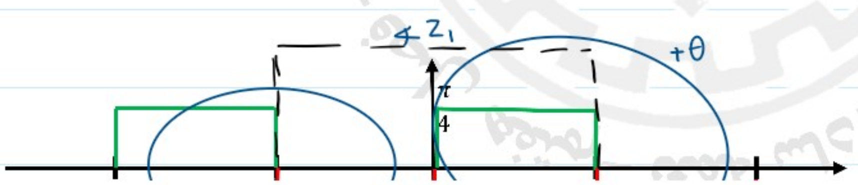
$$= \frac{1}{2}\left[e^{j\theta}Z(j(\omega-\omega_0)) + e^{-j\theta}Z(j(\omega+\omega_0))\right] \quad \Rightarrow \omega_0 = M$$

$$= \frac{1}{2}\left[e^{j\theta}Z(j(\omega-M)) + e^{-j\theta}Z(j(\omega+M))\right]$$

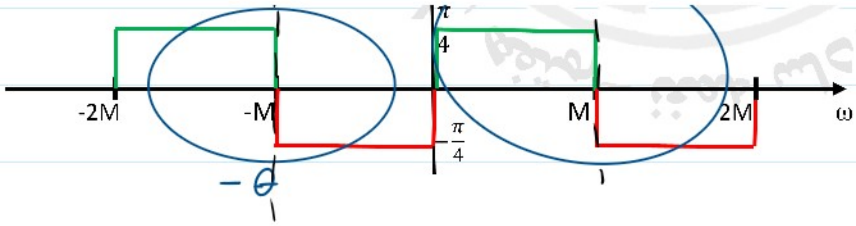



$\omega_c = M$

$\frac{M\sqrt{2}}{4} \times K = 1 \Rightarrow K = \frac{4}{M\sqrt{2}} = \frac{2\sqrt{2}}{M}$



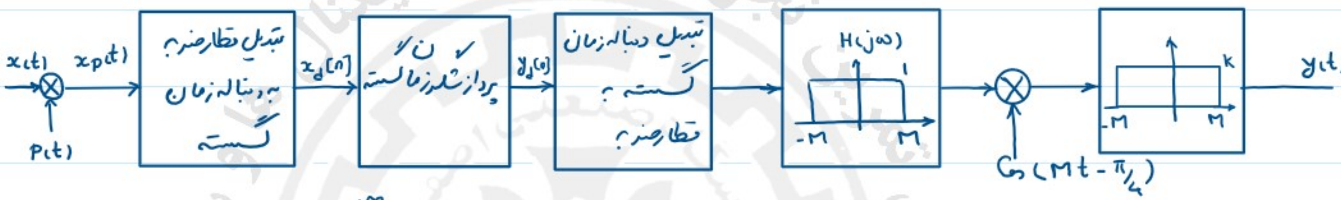

$\frac{\pi}{4} + \theta = 0 \Rightarrow \theta = -\frac{\pi}{4}$




$\frac{\pi}{4}+\theta=0 \quad \Rightarrow \quad \theta=-\frac{\pi}{4}$

$-\frac{\pi}{4}-\theta=0$

ج) یک پردازشگر زمان گسسته طراحی کنید که معادل فیلتر پایین گذر با فرکانس قطع M/2 برای سیگنال زمان پیوسته ورودی باشد.




$$x_p(t)=x(t)\,p(t)=\sum_{k=-\infty}^{+\infty} x(t)\left[\delta\left(t-k\frac{2\pi}{M}\right)-\delta\left(t-\Delta-k\frac{2\pi}{M}\right)\right]$$

$$=\sum_{k=-\infty}^{+\infty} x\left(k\frac{2\pi}{M}\right)\delta\left(t-k\frac{2\pi}{M}\right)-x\left(\Delta+k\frac{2\pi}{M}\right)\delta\left(t-\Delta-k\frac{2\pi}{M}\right) \quad ; \quad \Delta=\frac{\pi}{2M}$$

$$X_p(j\omega)=\sum_{k=-\infty}^{+\infty} \underline{x\left(k\frac{2\pi}{M}\right)}\, e^{-jk\frac{2\pi}{M}\omega}\;\underline{-\,x\left(\Delta+k\frac{2\pi}{M}\right)}\, e^{-j\left(\Delta+k\frac{2\pi}{M}\right)\omega} \qquad (1)$$

$$x_d[n]=\begin{cases} x\left(\frac{n}{2}\,\frac{2\pi}{M}\right) & n \text{ زوج} \\ -x\left(\frac{n-1}{2}\,\frac{2\pi}{M}+\Delta\right) & n \text{ فرد} \end{cases}$$

$$X_d(e^{j\Omega})=\sum_{n=-\infty}^{+\infty} x_d[n]\,e^{-j\Omega n}=\sum_{n\text{ زوج}}\cdots+\sum_{n\text{ فرد}}$$

$n=2m$ ، $n=2m+1$

$$=\sum_{m=-\infty}^{+\infty} \underline{x\left(m\frac{2\pi}{M}\right)}\, e^{-j\Omega 2m}+\sum_{m=-\infty}^{+\infty} \underline{-x\left(m\frac{2\pi}{M}+\Delta\right)}\, e^{-j\Omega(2m+1)} \qquad (2)$$

$$\Omega=\frac{\pi}{M}\,\omega$$

$$\left(\Delta+k\frac{2\pi}{M}\right)\omega=(2m+1)\Omega$$

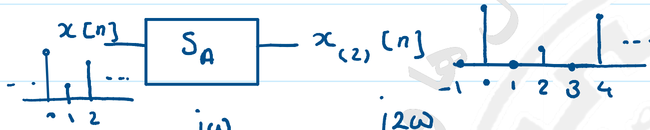
$$\left(\Delta\frac{M}{\pi}+2k\right)\omega\times\frac{\pi}{M}=(2m+1)\Omega$$

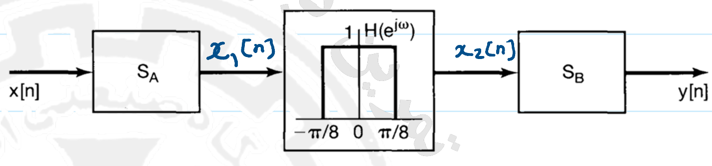
تنها در حالتی تساوی را داریم که $\begin{cases}\Delta\frac{M}{\pi}=1\\ \Omega=\omega\times\frac{\pi}{M}\end{cases}$ $\Leftarrow$ $\Delta=\frac{\pi}{M}$ ولی در این مسئله $\Delta=\frac{\pi}{2M}$

Two discrete-time systems SI and S2 are proposed for implementing an ideal lowpass filter with cutoff frequency π/4. System S1 is depicted in Figure(a). System S2 is depicted in Figure (b). In these figures, SA corresponds to a zero-insertion system that inserts one zero after every input sample, while SB corresponds to a decimation system that extracts every second sample of its input.

(a) Does the proposed system S1 correspond to the desired ideal lowpass filter?

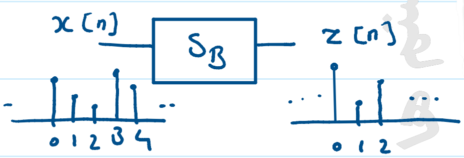
(b) Does the proposed system S2 correspond to the desired ideal lowpass filter?

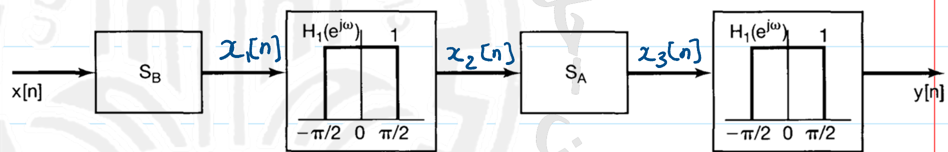





(a)

$$X_{(2)}(e^{j\omega}) = X(e^{j2\omega})$$






(b)

$$x[n]\times\sum_{k=-\infty}^{+\infty}\delta[n-ka] = z_{(a-1)}[n] \xrightarrow{\mathcal{F}} \frac{1}{2\pi}X(e^{j\omega}) \circledast 2\pi\times\frac{1}{a}\sum_{k=-\infty}^{+\infty}\delta\left(\omega-k\frac{2\pi}{a}\right) = Z(e^{ja\omega})$$

$$\Rightarrow \frac{1}{a}X(e^{j\omega}) * \sum_{k=0}^{a-1}\delta\left(\omega-k\frac{2\pi}{a}\right) = Z(e^{ja\omega})$$

$$\Rightarrow \frac{1}{a}\sum_{k=0}^{a-1}X\left(e^{j(\omega-k\frac{2\pi}{a})}\right) = Z(e^{j\overset{\omega}{a\omega}}) \Rightarrow Z(e^{j\omega}) = \frac{1}{a}\sum_{k=0}^{a-1}X\left(e^{j\frac{\omega-2k\pi}{a}}\right)$$

$$Z(e^{j\omega}) = \frac{1}{2}\left[X(e^{j\frac{\omega}{2}}) + X\left(e^{j(\frac{\omega}{2}-\pi)}\right)\right]$$ $a=2$ در این مسئله

تحلیل سیستم (a)

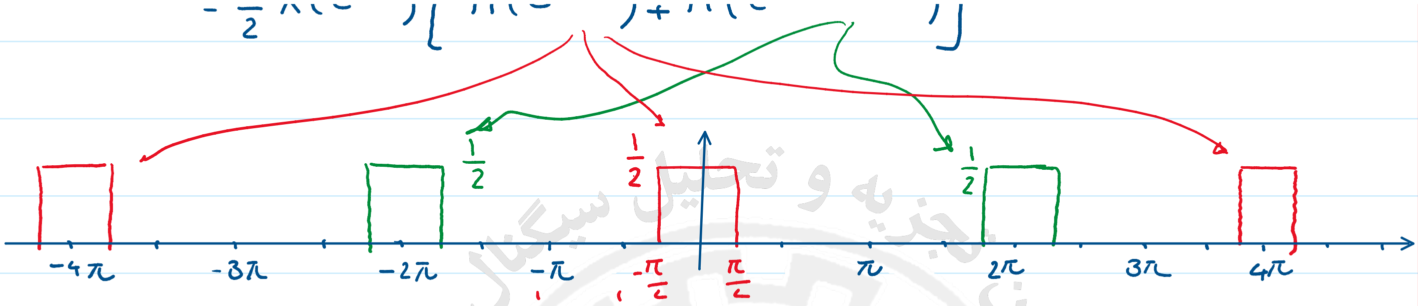
$$x_1[n] = x_{(2)}[n] \Rightarrow X_1(e^{j\omega}) = X(e^{j2\omega})$$

$$X_2(e^{j\omega}) = X_1(e^{j\omega})H(e^{j\omega}) = X(e^{j2\omega})H(e^{j\omega})$$

$$y(e^{j\omega}) = \frac{1}{2}\left[X_2(e^{j\frac{\omega}{2}}) + X_2\left(e^{j(\frac{\omega}{2}-\pi)}\right)\right] = \frac{1}{2}\left[X(e^{j\omega})H(e^{j\frac{\omega}{2}}) + X(e^{j(\omega-2\pi)})H(e^{j(\frac{\omega}{2}}\right.$$

$$= \frac{1}{2}X(e^{j\omega})\left[H(e^{j\frac{\omega}{2}}) + H\left(e^{j(\frac{\omega}{2}-\pi)}\right)\right]$$

$$\frac{1}{2} X(e\ ) [ H(e\ ) + H(e\ ) ]$$



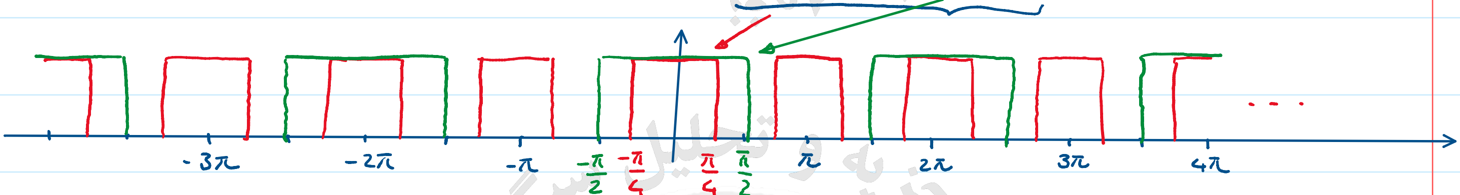

پاسخ نهایی دارای دوره تناوب $2\pi$ است $\Rightarrow$ یک فیلتر پایین‌گذر با فرکانس قطع $\frac{\pi}{4}$ است.

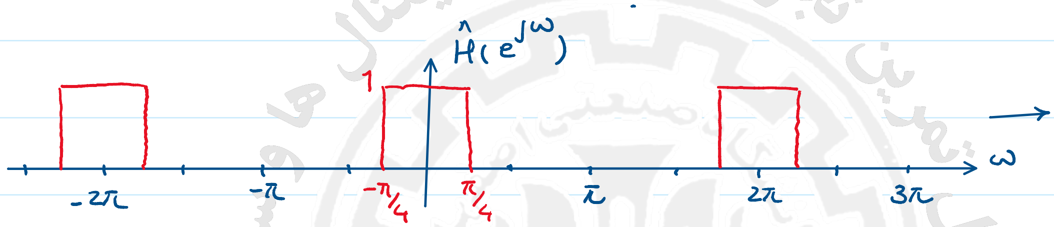
(b) تحلیل سیستم

$$X_1(e^{j\omega}) = \frac{1}{2}\left[ X(e^{j\frac{\omega}{2}}) + X(e^{j(\frac{\omega}{2}-\pi)}) \right]$$

$$X_2(e^{j\omega}) = X_1(e^{j\omega}) H_1(e^{j\omega}) \quad ; \quad X_3(e^{j\omega}) = X_2(e^{j2\omega}) = X_1(e^{j2\omega}) H_1(e^{j2\omega})$$

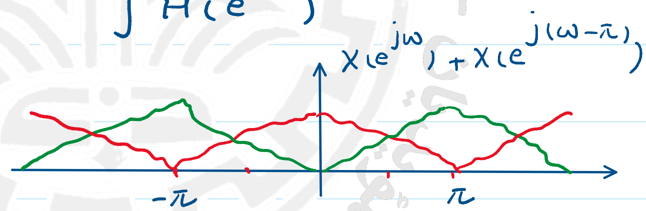
$$y(e^{j\omega}) = X_3(e^{j\omega}) H_1(e^{j\omega}) = X_1(e^{j2\omega}) \underbrace{H_1(e^{j2\omega}) H_1(e^{j\omega})}$$




$\rightarrow$ LPF

$$y(e^{j\omega}) = \frac{1}{2}\left[ X(e^{j\omega}) + X(e^{j(\omega-\pi)}) \right] \hat{H}(e^{j\omega})$$

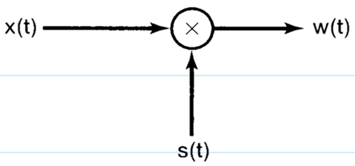




سیستم (b) عملیات فیلترینگ با فرکانس قطع $\frac{\pi}{4}$ را روی ورودی انجام نمی‌دهد.

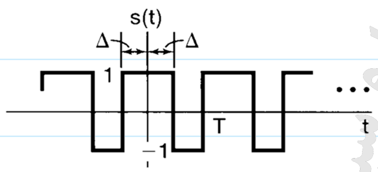
The input signal is band limited with $|X(jw)| = 0$ for $|w| >= W_M$.

(a) For $\Delta = T/3$, determine, in terms of $W_M$, the maximum value of T for which there is no aliasing among the replicas of X(jw) in W(jw ).

(b) For $\Delta = T/4$, determine, in terms of $w_M$, the maximum value of T for which there is no aliasing among the replicas of X(jw) in W(jw ).




$$W(t) = x(t) \cdot s(t) \Rightarrow W(j\omega) = \frac{1}{2\pi} X(j\omega) * S(j\omega)$$




$$\hat{S}(t) = S(t) + 1$$

متناوب

$$a_k = \frac{1}{T}\int_{\langle T\rangle} \hat{S}(t)\, e^{-jk\omega_s t}\, dt = \frac{1}{T}\int_{-\Delta}^{\Delta} 2 \times e^{-jk\omega_s t}\, dt$$

$$= \frac{2}{T} \times \frac{-1}{jk\omega_s t}\, e^{-jk\omega_s t}\Big|_{-\Delta}^{\Delta} = \frac{2}{k\pi}\sin(k\omega_s\Delta) = \frac{4\Delta}{T}\,\mathrm{sinc}\left(\frac{2k\Delta}{T}\right) \qquad k \neq 0$$

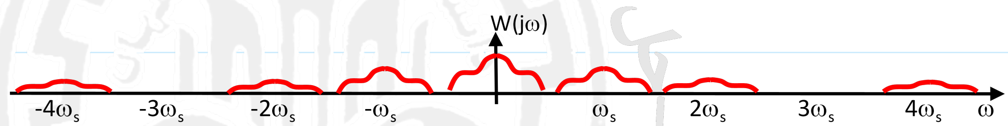
برای سیگنال اصلی $a_0 = \dfrac{2\Delta - (T - 2\Delta)}{T} = \dfrac{4\Delta}{T} - 1$

$$S(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k\, \delta(\omega - k\omega_s) = \sum_{k=-\infty}^{+\infty} \frac{8\pi\Delta}{T}\,\mathrm{sinc}\left(\frac{2k\Delta}{T}\right)\delta(\omega - k\omega_s) - 2\pi\,\delta(\omega)$$

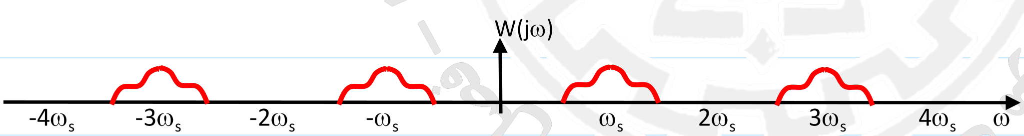
$$W(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{4\Delta}{T}\,\mathrm{sinc}\left(\frac{2k\Delta}{T}\right) X(j(\omega - k\omega_s)) - X(j\omega)$$

a) $\Delta = T/3 \Rightarrow W(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{4}{3}\,\mathrm{sinc}\left(\frac{2k}{3}\right) X(j(\omega - k\omega_s)) - X(j\omega)$ $\quad k = 0, \pm 1, \pm 2, \pm 4$

$\omega_s > 2\omega_M \Rightarrow T < \dfrac{\pi}{\omega_M}$ $\Leftarrow$




b) $\Delta = T/4 \Rightarrow W(j\omega) = \sum_{k=-\infty}^{+\infty} \mathrm{sinc}(k/2)\, X(j(\omega - k\omega_s)) - X(j\omega)$ $\quad k = \pm 1, \pm 3, \pm 5, \ldots$




$\Rightarrow \omega_s > \omega_M \Rightarrow T < \dfrac{2\pi}{\omega_M}$