حل تکلیف شماره ۳ درس تجزیه و تحلیل سیگنالها و سیستم ها
نیمسال اول ۱۴۰۰-۱۳۹۹

۱) الف -

$$x(t) = \underbrace{3}_{\text{با هر دوره تناوب}} + \underbrace{5\cos t}_{T_1 = 2\pi} + \underbrace{6\sin(2t+\pi/4)}_{T_2=\pi}$$

$$\Downarrow$$

$$T = 2\pi \longrightarrow \omega_0 = \frac{2\pi}{T} = 1$$

$$x(t) = 3 + \frac{5}{2}e^{jt} + \frac{5}{2}e^{-jt} + 6\left[\frac{1}{2j}e^{2jt+\frac{\pi}{4}j} - \frac{1}{2j}e^{-2jt-\frac{\pi}{4}j}\right]$$

$$\overset{\omega_0=1}{=} 3 + \frac{5}{2}e^{j\omega_0 t} + \frac{5}{2}e^{-j\omega_0 t} - \underbrace{3je^{j\pi/4}}\,\underbrace{e^{2j\omega_0 t}} + 3je^{-j\pi/4}e^{-2j\omega_0 t}$$

$a_0 = 3$, $a_1 = 5/2$, $a_{-1} = 5/2$, $a_2 = -3je^{j\pi/4} = -3e^{j3\pi/4}$

$a_{-2} = 3e^{j3\pi/4}$

$a_k = 0 \qquad k \neq 0, \pm1, \pm2$

ب -

$$x(t) = 3 + \underbrace{\cos t}_{T_1 = 2\pi} + \underbrace{5\sin 4\pi t}_{T_2 = 1/2}$$

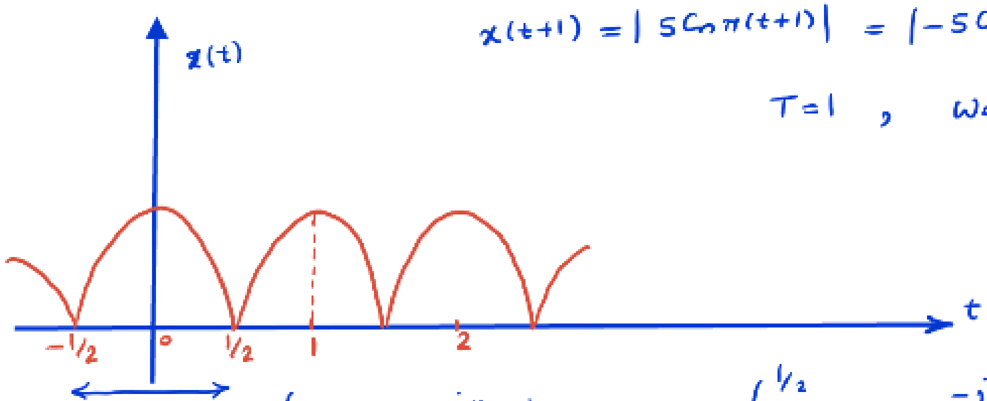
$x(t)$ متناوب نیست.

ج -

$$x(t) = |5\cos \pi t|$$

$$x(t+1) = |5\cos \pi(t+1)| = |-5\cos \pi t| = x(t)$$

$T = 1$, $\omega_0 = 2\pi$



$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt = \frac{1}{1}\int_{-1/2}^{1/2} 5\cos \pi t\, e^{-jk2\pi t}dt$$

$$= \int_{-1/2}^{1/2}\left[\frac{5}{2}e^{j\pi t} + \frac{5}{2}e^{-j\pi t}\right]e^{-jk2\pi t}dt$$

$$a_k = \frac{10(-1)^k}{\pi(1-4k^2)}$$

هم فرکانس های مضارب ω_0 را دارد.

۲)

تابع ویژه همه سیستم های LTI $\leftarrow$ $e^{st} \longrightarrow \boxed{LTI} \longrightarrow \underbrace{H(s)}_{\text{مقدار ویژه}}e^{st}$

$$h(t) = \delta(t-T)$$ الف -

$$\xrightarrow{x(t)} \boxed{h(t)=\delta(t-T)} \longrightarrow y(t) = \alpha \cdot x(t)$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t-kT) \Rightarrow y(t) = x(t) * h(t) = \sum_{k} \delta(t-kT) * \delta(t-T)$$

$$= \sum_{k=-\infty}^{+\infty} \left[\delta(t-kT) * \delta(t-T)\right] = \sum_{k=-\infty}^{+\infty} \delta(t-T-kT) = \sum_{k=-\infty}^{+\infty} \delta(t-(k+1)T)$$

$$\underset{k+1=m}{=} \sum_{m=-\infty}^{+\infty} \delta(t-mT) = x(t) \longrightarrow$$ فقط ضریب تابع ویژه و با مقدار ویژه 1 است.

اما با مقدار ویژه $1/2$:

$$x(t) \triangleq \sum_{k=-\infty}^{+\infty} 2^{k} \delta(t-kT) \Rightarrow y(t) = \sum_{k} \left[2^{k} \delta(t-kT) * \delta(t-T)\right]$$

$$y(t) = \sum_{k} 2^{k} \delta(t-T-kT) = \sum_{k=-\infty}^{+\infty} 2^{k} \delta(t-(k+1)T) \underset{m=k+1}{=} \sum_{m=-\infty}^{+\infty} 2^{m-1} \delta(t-mT)$$

$$y(t) = 2^{-1} \cdot \underbrace{\sum_{m=-\infty}^{+\infty} 2^{m} \delta(t-mT)}_{x(t)}$$

ب - $h(t)$ حقیقی و زوج

$$e^{j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(j\omega)e^{j\omega t}$$

$$H(j\omega) = H(s)\Big|_{s=j\omega} = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt$$ $e^{j\omega t}$ حالت خاصی از e^{st} است.

$$e^{-j\omega t} \longrightarrow \boxed{h(t)} \longrightarrow H(-j\omega)e^{-j\omega t}$$

اما چون $h(t)$ زوج است $H(j\omega)$ هم زوج است پس $H(-j\omega) = H(j\omega)$

$$\underbrace{\tfrac{1}{2}e^{j\omega t} + \tfrac{1}{2}e^{-j\omega t}}_{=\cos\omega t} \longrightarrow \boxed{h(t)} \longrightarrow \tfrac{1}{2}H(j\omega)e^{j\omega t} + \tfrac{1}{2}\underbrace{H(-j\omega)}_{H(j\omega)}e^{-j\omega t}$$

$$= H(j\omega)\cos\omega t \longrightarrow$$ $\cos\omega t$ تابع ویژه است

۳)

$$x(t) \xleftrightarrow{T} a_k \qquad \omega_0 = 2\pi/T$$

$$x(t) \xleftrightarrow{T'=2T} b_k \qquad \omega_0' = \frac{2\pi}{T'} = \pi/T = \omega_0/2$$

$$b_k = \frac{1}{2T}\int_0^{2T} x(t)e^{-jk\omega_0' t}dt = \frac{1}{2T}\int_0^{T} x(t)e^{-jk(\frac{\omega_0}{2})t}dt + \frac{1}{2T}\int_T^{2T} x(t)e^{-jk(\frac{\omega_0}{2})t}dt$$

$$t-T \triangleq \lambda$$

$$b_k = \frac{1}{2T}\int_0^{T} x(t)e^{-jk(\frac{\omega_0}{2})t}dt + \frac{1}{2T}\int_0^{T} x(T+\lambda)e^{-jk\frac{\omega_0}{2}(T+\lambda)}d\lambda$$

$$b_k = \frac{1}{2T}\int_0 \quad \cdots \quad \frac{1}{2T}\int_0 \underbrace{\quad\quad}_{x(\lambda)}$$

$$= \quad // \quad + \frac{1}{2T}\int_0^T x(\lambda) e^{-jk(\frac{\omega_0}{2})\lambda} \underbrace{e^{-jk\frac{\omega_0}{2}T}}_{e^{-jk\pi}} d\lambda$$

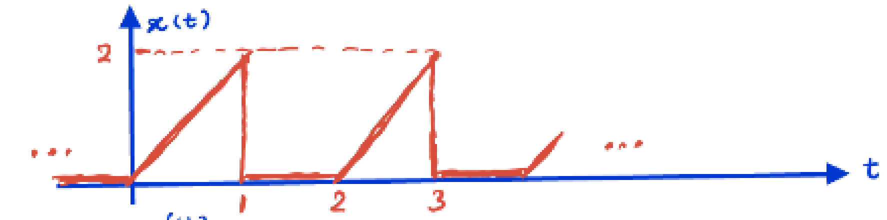
$$= \frac{1}{2T}\int_0^T x(t) e^{-jk(\frac{\omega_0}{2})t}\left[1+(-1)^k\right]dt$$

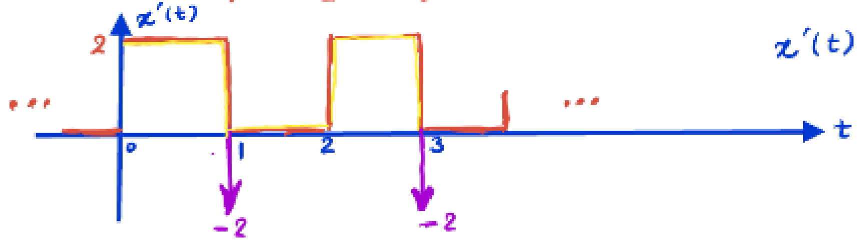
$$= \begin{cases} 0 & k \text{ فرد} \\ \frac{1}{T}\int_0^T x(t) e^{-j\frac{k}{2}\omega_0 t}\,dt & k \text{ زوج} \end{cases} = \begin{cases} 0 & k \text{ فرد} \\ a_{k/2} & k \text{ زوج} \end{cases}$$

$$\boxed{b_k = \begin{cases} 0 & k \text{ فرد} \\ a_{k/2} & k \text{ زوج} \end{cases} \quad \text{یا} \quad a_k = b_{2k}}$$

۴) الف -

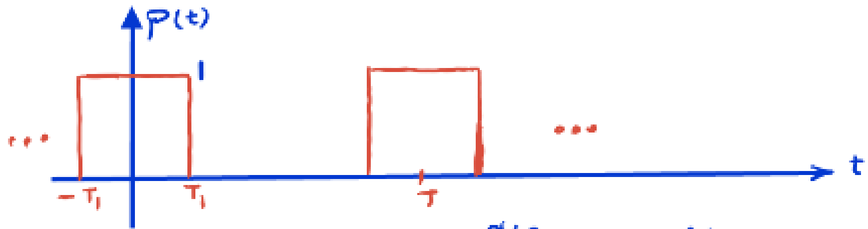
$$x(t) = \begin{cases} 2t & 0 \le t < 1 \\ 0 & -1 \le t < 0 \end{cases} \qquad T=2$$





$x'(t) = x_1(t) + x_2(t)$

مربعی   قطار ضربه



$$P(t) \xleftrightarrow{FS} \frac{2\sin k\omega_0 T_1}{k\omega_0 T} = \frac{\sin k\omega_0 T_1}{k\pi}$$

$$\text{DC مقدار} = 2T_1/T$$

$T_1 = 1/2$

$T = 2$

$$x_1(t) = 2P(t - 1/2) \xleftrightarrow{FS} b_k = 2e^{-jk\omega_0 t_0} \times \frac{\sin k\omega_0 T_1}{k\pi}$$

$t_0 = 1/2$, $T_1 = 1/2$, $\omega_0 = \pi$

$$b_k = 2e^{-jk\pi/2}\frac{\sin k\pi/2}{k\pi}$$

$t_0 = 1$

$$\cdots \sum^{+\infty} \delta(t - mT - 1) \xleftrightarrow{FS} -2 \times \frac{1}{T} \times e^{-jk\omega_0 t_0}$$

$$x_2(t) = -2\sum_{m=-\infty} \delta(t-m\ldots) \longleftrightarrow \quad C_k = -e^{-jk\pi}$$

$T=2$

$$\sum_m \delta(t-mT) \longleftrightarrow \frac{1}{T}$$

$$x'(t) \longleftrightarrow a'_k = b_k + C_k = 2e^{-jk\pi/2}\,\frac{\sin k\pi/2}{k\pi} - e^{-jk\pi}$$

$$= \begin{cases} -1 & \text{زوج } K \\ \frac{2(-j)^k}{k\pi}\sin\frac{k\pi}{2}+1 & \text{فرد } K \end{cases}$$

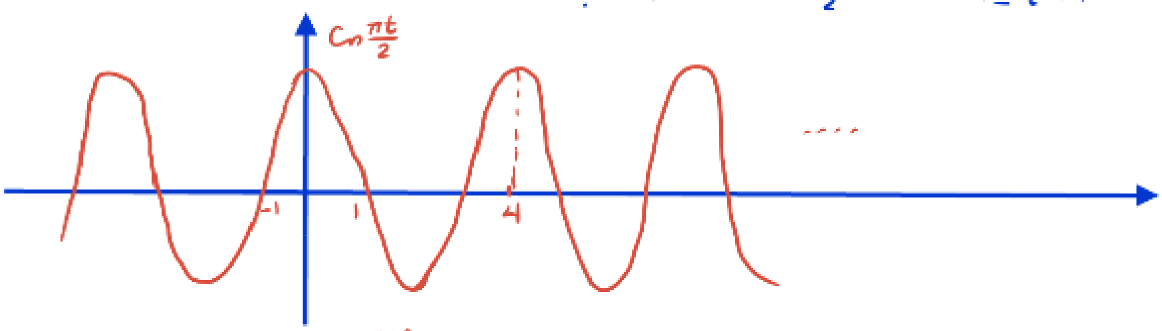
$$\sin\frac{k\pi}{2} \rightarrow \begin{cases} k=1 \rightarrow 1 \\ k=3 \rightarrow -1 \\ \vdots \end{cases} \qquad (-j)^k \sin\frac{k\pi}{2} \rightarrow \begin{cases} k=1 \rightarrow -j \\ k=3 \rightarrow -j \\ \vdots \end{cases}$$

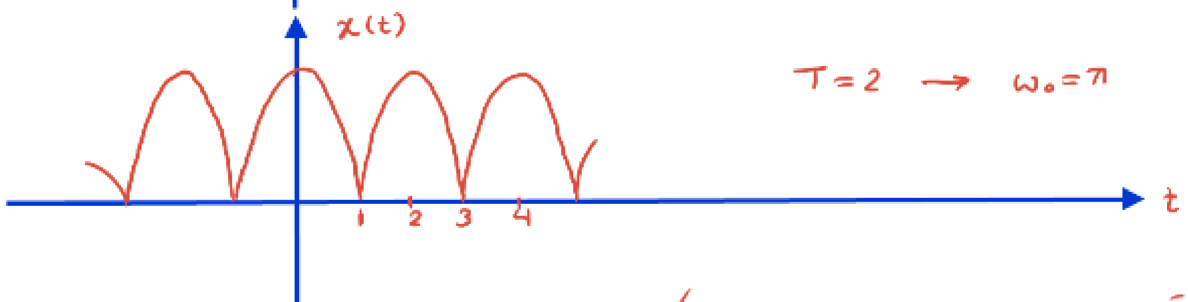
$$a'_k = \begin{cases} -1 & \text{زوج } K \\ \frac{-2j}{k\pi}+1 & \text{فرد } K \end{cases}$$

$$a'_k = a_k(jk\omega_0) \longrightarrow \underbrace{a_k}_{k\neq 0} = \frac{a'_k}{jk\omega_0} = \frac{a'_k}{jk\pi}$$

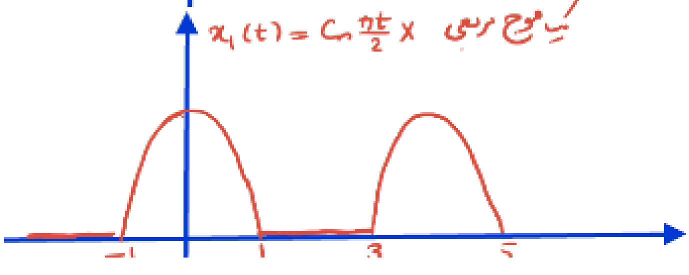
$$\boxed{a_k = \begin{cases} -\frac{1}{jk\pi} & \text{زوج } K \quad k\neq 0 \\ \frac{1}{jk\pi} - \frac{2}{k^2\pi^2} & \text{فرد } K \end{cases} \qquad a_0 = \frac{1}{2}\int_0^1 2t\,dt = 1/2}$$

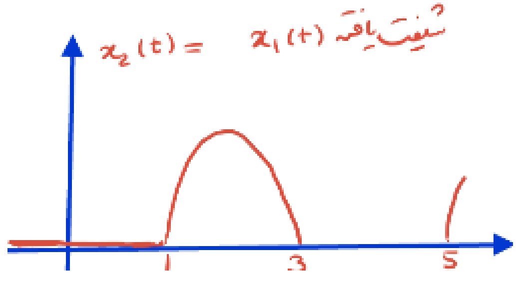
ب) $x(t) = \cos\frac{\pi t}{2} \quad -1 \le t < 1$ با $T=2$

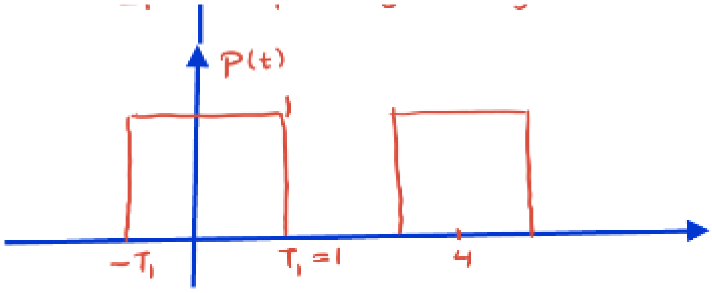




$T=2 \rightarrow \omega_0=\pi$







$$p(t) \longleftrightarrow C_k = \frac{\sin k\pi/2}{k\pi} \qquad \omega_0' = \frac{2\pi}{T} = \pi/2$$
$$T_1 = 1$$

$$x_1(t) = \cos\frac{\pi t}{2} \cdot p(t) = \frac{1}{2} e^{j\frac{\pi}{2}t} p(t) + \frac{1}{2} e^{-j\pi/2 t} p(t)$$

$$e^{jm\omega_0' t} p(t) \longleftrightarrow C_{k-m}$$

$$x_1(t) \xleftrightarrow{FS} \frac{1}{2} C_{k-1} + \frac{1}{2} C_{k+1} \triangleq d_k$$

$$d_k = \frac{1}{2}\left[\frac{\sin (k-1)\pi/2}{(k-1)\pi} + \frac{\sin (k+1)\pi/2}{(k+1)\pi}\right]$$

$$x_2(t) = x_1(t-2) \longleftrightarrow e^{-jk\omega_0'(2)} d_k = (-1)^k d_k$$

$$x(t) \xleftrightarrow{\text{بادوره تناوب } 4} b_k = d_k + (-1)^k d_k = \begin{cases} 0 & \text{فرد } k \\ 2d_k & \text{زوج } k \end{cases}$$

با استفاده از مسئله (۳):

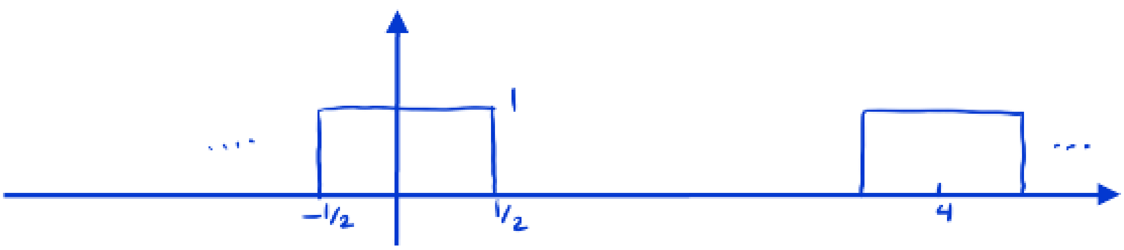
$$x(t) \xleftrightarrow{\text{بادوره تناوب } 2} a_k = b_{2k} = 2d_{2k}$$

$$a_0 = 2/\pi$$

۵) الف - $T=4$, $a_k = \begin{cases} \dfrac{j^k \sin k\pi/4}{k\pi} & k \neq 0 \\ 0 & k=0 \end{cases}$

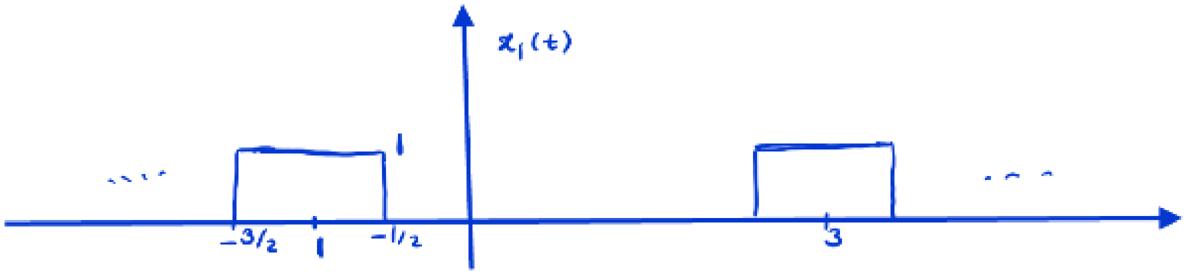
مربعی پریودیک با $\omega_0 T_1 = \pi/4$ $p(t) \longleftrightarrow b_k = \dfrac{\sin k\pi/4}{k\pi}$

$$T=4 \longrightarrow \omega_0 = \pi/2 \longrightarrow T_1 = 1/2$$
$$\omega_0 T_1 = \pi/4$$



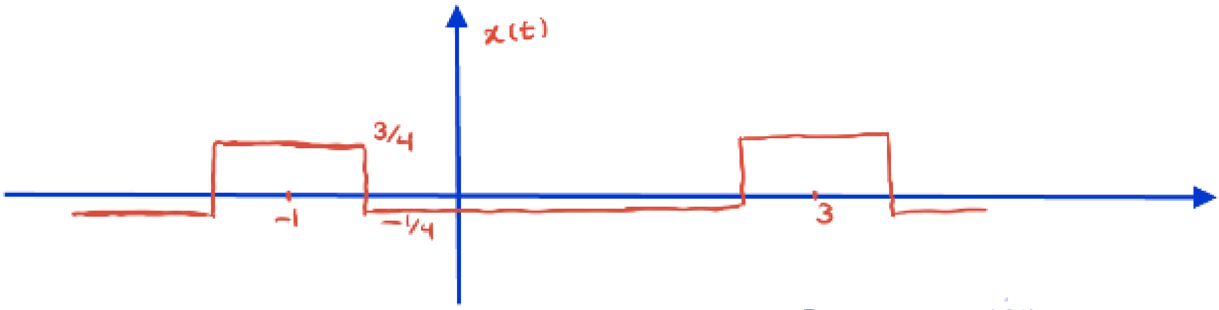
$j^k \quad e^{jk\pi/2} \quad e^{jk\pi/2} b_k \longrightarrow$ شیفت در حوزه زمان

$$x_1(t-t_0) \xleftrightarrow{\mathcal{F}S} e^{-jk\omega_0 t_0}\, b_k \qquad -\omega_0 t_0 = \pi/2 \rightarrow t_0 = -1$$



$$x(t) = x_1(t) - (\text{مقدار dc سیگنال } x_1) \qquad \text{مقدار dc سیگنال } x_1 = \frac{1}{T}\int_0^T x_1(t)\,dt = 1/4$$

$$x(t) = x_1(t) - 1/4$$



ب)

$$T=4 \qquad a_k = \begin{cases} 2 & k \text{ فرد} \\ 1 & k \text{ زوج} \end{cases}$$

حقیقی و زوج $x(t)$ ← حقیقی و زوج a_k

$$a_k = \frac{3}{2} - \frac{1}{2}\underbrace{e^{-jk\pi}}_{(-1)^k} \qquad \omega_0 = \frac{2\pi}{4} = \pi/2$$

$$6 \times \sum_{m=-\infty}^{+\infty} \delta(t-4m) \longleftrightarrow 6 \times \frac{1}{4} = 3/2$$

$\pi/2 \qquad t_0 = 2$

$$-2 \times \sum \delta(t-4m-2) \longleftrightarrow -2 \times e^{-jk\omega_0 t_0} \times \frac{1}{4} = -\frac{1}{2} e^{-jk\pi}$$

$$x(t) = 6\sum_m \delta(t-4m) - 2\sum_m \delta(t-4m-2)$$