



# Stationary states of opinion diffusion

Project for the exam: AMS (DSE)

Paola Serra and Marzio De Corato

January 8, 2022

# Theoretical Framework

# Statistical Mechanics

*“Ludwig Boltzmann, who spent much of his life studying statistical mechanics, died in 1906, by his own hand. Paul Ehrenfest, carrying on the work, died similarly in 1933. Now it is our turn to study statistical mechanics.” States of Matter (1975), by David L. Goodstein*

# Concepts of statistical mechanics: entropy [4]

**Central problem of thermodynamics:** characterize the actual state of equilibrium among all virtual states

**Entropy postulate:** there exist a function  $S$  of the extensive variables  $(X_0, X_1 \dots X_r)$  called entropy, that assumes the maximum value for a state of equilibrium among all virtual states and that possesses the following properties:

- Extensivity  $S^{(1 \cup 2)} = S^1 + S^2$
- Convexity  $S((1 - \alpha)X^1 + \alpha X^2) \geq (1 - \alpha)S(X^1) + \alpha S(X^2)$
- Monotonicity  $\frac{\partial S}{\partial E}|_{X_1 \dots X_r} = \frac{1}{T} > 0$

The equilibrium state corresponds to the maximum entropy compatible with the constraints

# Concepts of statistical mechanics: entropy [4]

- **Fundamental postulate of statistical mechanics**  $S = k_b \ln |\Gamma|$   
Where  $S$  is the thermodynamic entropy,  $k_b$  is Boltzmann constant and  $|\Gamma|$  the volume in the phase space

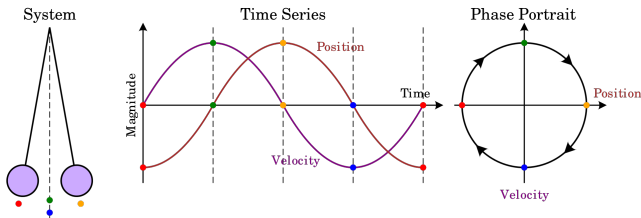


Image taken from [1]

# Concepts of statistical mechanics: ensemble [4]

- **Statistical ensemble:** a large number of virtual copies of a system ; each of them is a possible state of the real system. It is the formalization of a repeated experiment proposed by Gibbs
- **Microcanonical ensemble:**  $p = 1/W$   $W$  is the number of microstates
- **Canonical ensemble:**  $p = \frac{1}{Z} \exp\left(-\frac{E}{kT}\right)$  where  $Z = \sum_i \exp\left(\frac{-E_i}{k_b T}\right)$

$$\langle A(x) \rangle = \frac{1}{Z} \int dx_s A(x_s) \exp \left[ -\frac{H^S(x_s)}{k_b T} \right] \quad (1)$$

$$Z = \int dx_s \exp \left[ -\frac{H^S(x_s)}{k_b T} \right] \quad (2)$$

$$\langle H(x)^2 \rangle - \langle H(x) \rangle^2 = k_b T^2 C \quad (3)$$

# Ising Model

## Ising model [3, 2]

- An array of atoms that can take states  $\pm 1$
- $E(\mathbf{x}, J, H) = - \left[ \frac{1}{2} \sum_{m,n} J_{mn} x_m x_n + \sum_n H x_n \right]$
- $P(x|\beta, J, H) = \frac{1}{Z(\beta, J, H)} \exp[-\beta E(x, J, H)]$
- $\beta = 1/k_b T \quad Z(\beta, J, H) = \sum_x \exp[-\beta E(x, J, H)]$
- $g(m) = \frac{\langle \sigma_i \sigma_{i+m} \rangle - \langle \sigma_i \rangle \langle \sigma_{i+m} \rangle}{1 - \langle \sigma_i \rangle \langle \sigma_{i+m} \rangle} = \langle \sigma_i \sigma_{i+m} \rangle$  if  $H = 0$



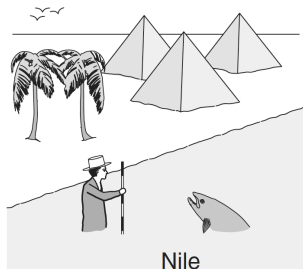
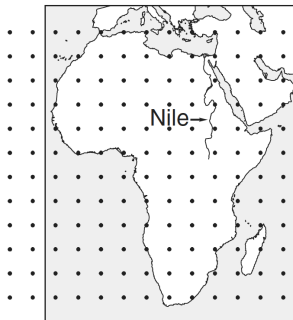
# Numerical simulations

*"Never make a calculation until you know the answer. Make an estimate before every calculation, try a simple physical argument (symmetry! invariance! conservation!) before every derivation, guess the answer to every paradox and puzzle. Courage: No one else needs to know what the guess is. Therefore make it quickly, by instinct. A right guess reinforces this instinct. A wrong guess brings the refreshment of surprise. In either case life as a spacetime expert, however long, is more fun!" John Archibald Wheeler*

# Numerical simulation [4]

- **Simulation** *the evolution equations of a system are defined to such a detail that a computer can make its behaviour explicit, and one looks to see what takes place*
- Numerical experiment
- **Molecular dynamics:** the equation of motion are solved numerically  
PROS: information of both the dynamical and static properties of the system are explored
- **Monte Carlo:** a fictitious evolution process of the system is solved in order to get the equilibrium distribution  
PROS 1) also the systems whose dynamics is not defined can be explored 2) a fictitious dynamics can be considered in order to reach the equilibrium faster
- Molecular dynamics may provide a proof of statistical mechanics, Monte Carlo methods presuppose its validity

# Monte Carlo method [4]



# Monte Carlo method [4]

- We would calculate an integral of type  $\langle A \rangle = \int_0^1 dx A(x) \rho(x)$  where  $\rho$  is the probability distribution.
- Evaluate the integrand in  $N+1$  points uniformly arranged between 0 and 1  $\langle A \rangle \approx \frac{1}{N+1} \sum_{i=0}^N A(x_i) \rho(x_i)$
- A better convergence is reached if the  $x_i$  density is proportional to  $\rho(x)$ . In this case we have  $\langle A \rangle = 1/N \sum_{i=1}^N A(\mathbf{x}_i)$ .

# Monte Carlo method [4]

- The quantity  $\langle A \rangle$  is evaluated over a distribution  $\rho(x)$  where  $x_i$  are random, independent and distributed with a probability density  $\rho(x)$
- The independence conditions can be relaxed with the condition that the correlations between  $x_i$  and  $x_{i+l}$  go to zero fairly rapidly as  $l$  grows
- The true dynamics is replaced with a fictitious stochastic dynamics. The state at  $t + 1$  depends only from the state at  $t$ .  $\rightarrow$  Markov Chain

# Monte Carlo method [4]

- The evolution of probability is described by the **master equation**  
$$\Delta p_a(t) = \sum'_{b \neq a} [W_{ab} p_b(t) - W_{ba} p_a(t)]$$
- stationary  $\rightarrow \sum'_{b \neq a} [W_{ab} p_b(t) - W_{ba} p_a(t)] = 0 \quad \forall a$
- Detailed balance property  $W_{ab} W_{bc} W_{ca} = W_{ac} W_{cb} W_{ba}$
- $W_{ab} p_b(t) - W_{ba} p_a(t) = 0 \quad \forall a, b$

# Monte Carlo method [4]

- We would sample  $p_a^{eq}$
- This can be performed as long as the  $W_{ab}$  is ergodic and the detailed balance property holds
- The transition between any two arbitrary states can take place as long as one waits for a sufficient amount of time

## Application [4]

- $H(\sigma) = - \sum_{\langle ij \rangle} J \sigma_i \sigma_j - \sum_i h \sigma_i$
- $P_\sigma = \frac{e^{-H(\sigma)/k_b T}}{Z} \quad Z = \sum_\sigma e^{-H(\sigma)/k_b T}$
- The observable are calculated as
$$E = \langle H \rangle = \sum_{\sigma} H(\sigma) P_\sigma^B \quad M = \sum_{\sigma} (\sigma_i \sigma) P_\sigma^B$$
- The markov chain states should be the microstates  $\sigma$  and  $P_\sigma$  the stationary distribution
- $W_{\sigma\sigma'} = W_{\sigma'\sigma} \frac{P_\sigma}{P_{\sigma'}} = W_{\sigma'\sigma} \exp - \frac{H(\sigma) - H(\sigma')}{k_b T}$
- The Z term is no more present !!!



## Application [4]

- $W_{\sigma'\sigma} = \begin{cases} \kappa H(\sigma') < H(\sigma) \\ \kappa \exp \{ - [H(\sigma') - H(\sigma)] / k_b T \} \end{cases} \quad \{A\} \neq [A]_T = \frac{1}{T}$
- $A(\sigma) = \langle A \rangle [1 + O(N^{-1/2})] \quad \forall \sigma \in \Gamma$
- $S = 0.5 N k_b \ln 2 \quad \frac{|\Gamma|}{2^N} \approx 2^{-0.5N} \quad N = 100 \quad 10^{-15}$
- $\approx [A]_T = \frac{1}{T} \sum_{t=T_0}^{T\langle A \rangle_0 + T} A_{\sigma(t)}$
- $\langle \Delta A_T^2 \rangle \approx \frac{1}{T} (A_{\sigma(t)} - [A]_T)^2$
- $\sigma_t$  and  $\sigma_{t'}$  are independent if  $|t - t'|$  is larger than a characteristic time  $\tau_0$
- $\langle \Delta A_T^2 \rangle \approx \frac{\tau_0}{T} (A_{\sigma(t)} - [A]_T)^2$

# Goals and methods

# Goals and methods

- Reproduce the main result for a 2D anti ferromagnetic lattice ( $J = -1$ ) with no external magnetic field ( $H = 0$ ) with the montecarlo-metropolis
- Once checked that the script provide the correct results apply it to a lattice ( $J = +1$ ). In this case the spins represent an opinion and the sites people. The goal is to find the stationary states (at  $T = 0$  and  $T \neq 0$ )
- Introduce in the lattice some blocks that never change their status. These islands represent groups that never change mind and only diffuse their ideas. (at  $T = 0$  and  $T \neq 0$ )

# Results

# Simulation features

- 10x10 lattice
- Periodic boundary conditions  $\rightarrow$  the topology of a torus (genus equal to 1)
- 6000 steps

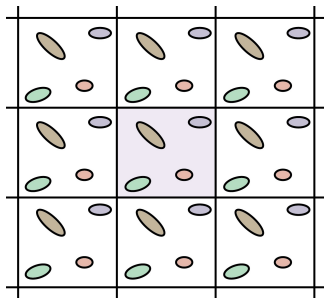


Image taken from [1]

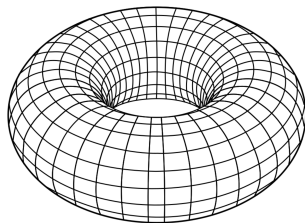
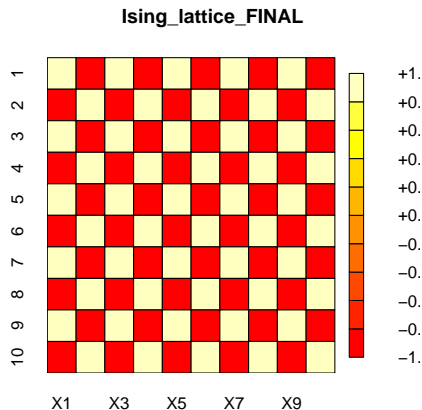
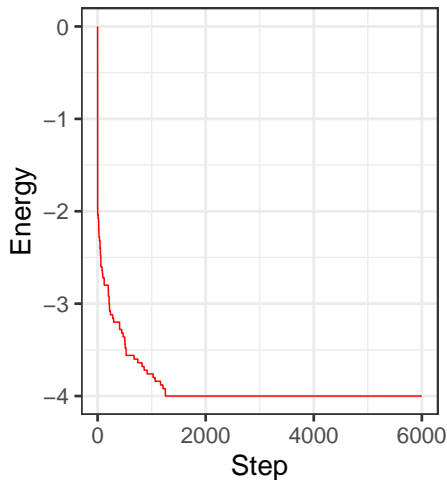
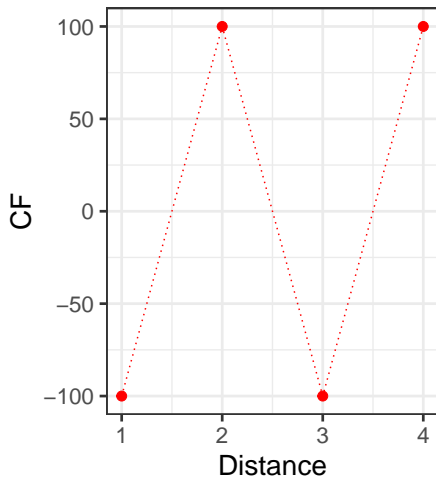
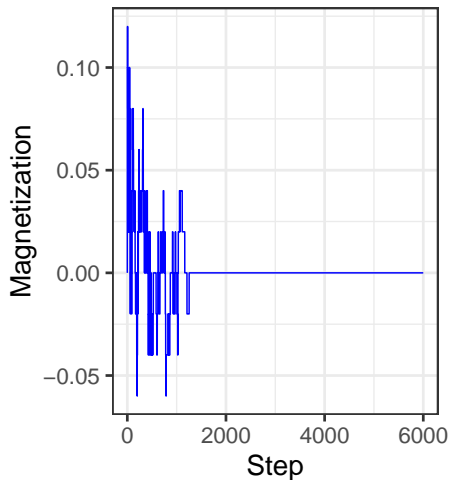


Image taken from [1]

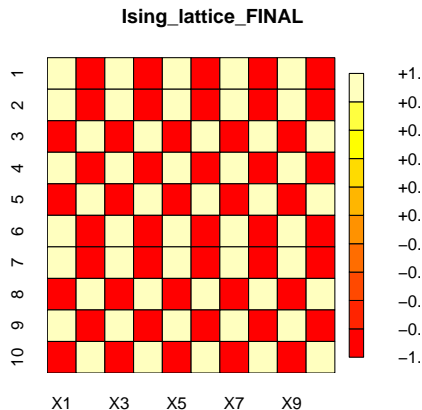
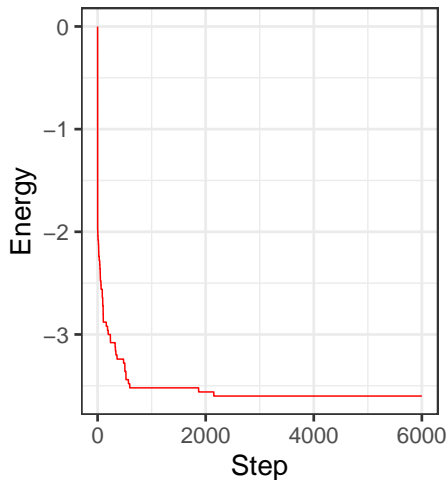
# Antiferromagnetic $J=-1$ , $T=0$



# Antiferromagnetic $J=-1$ , $T=0$

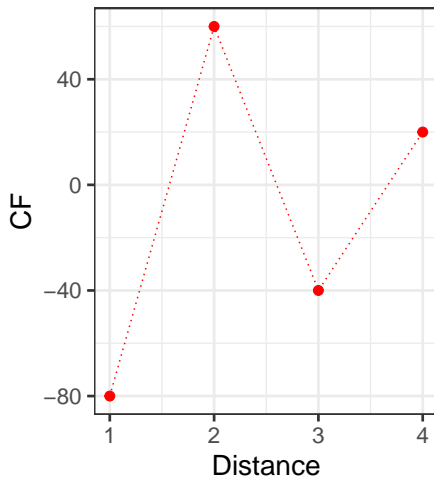
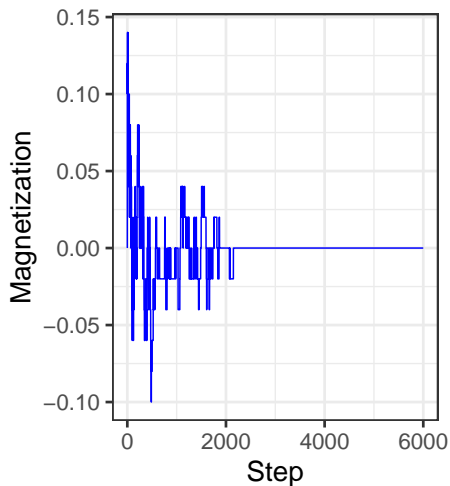


# Antiferromagnetic $J=-1$ , $T=1.5$

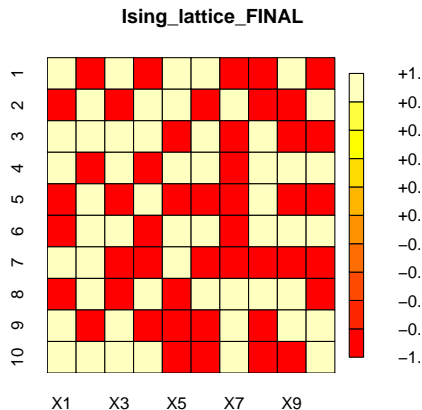
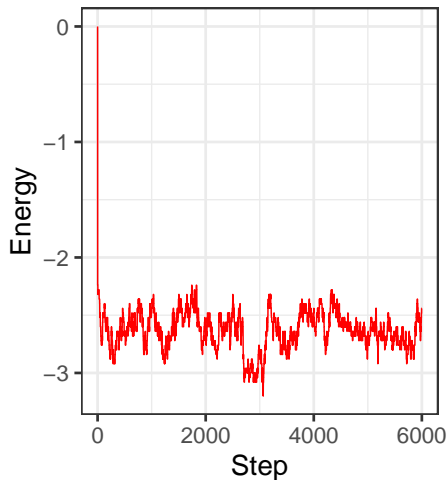




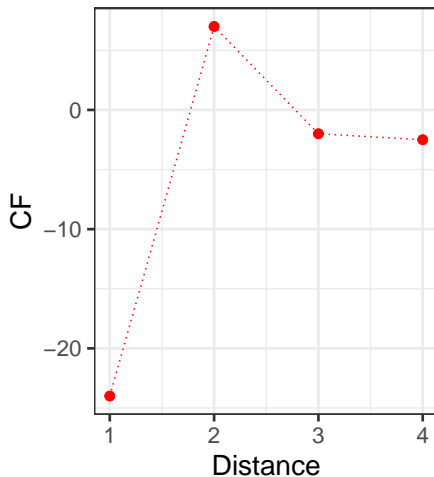
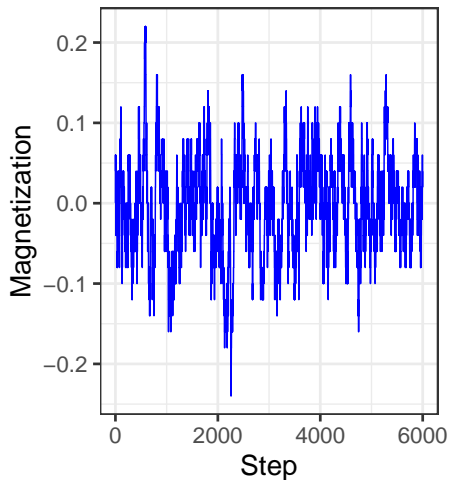
# Antiferromagnetic $J=-1$ , $T=1.5$



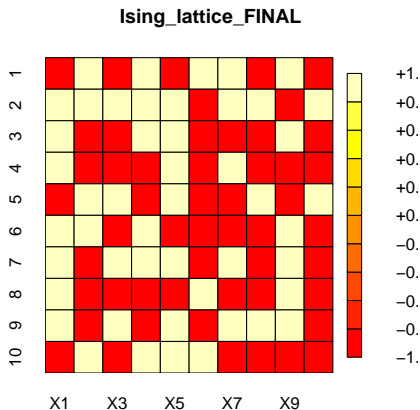
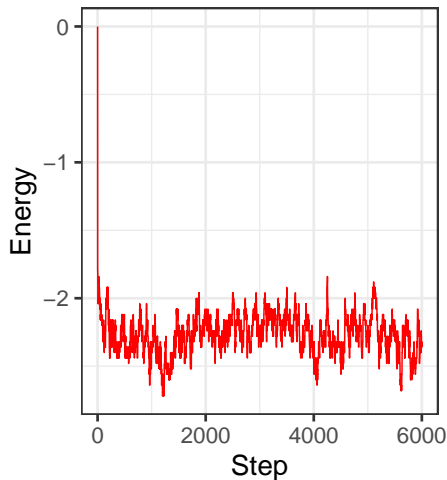
# Antiferromagnetic $J=-1$ , $T=4$



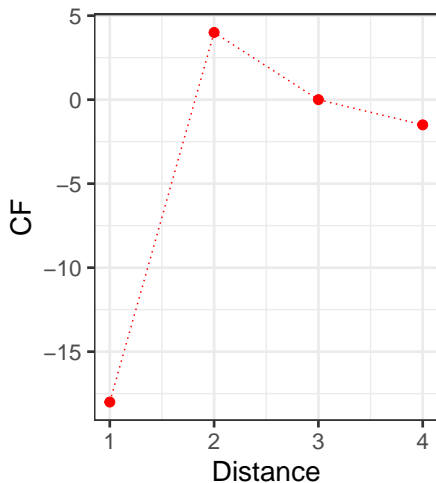
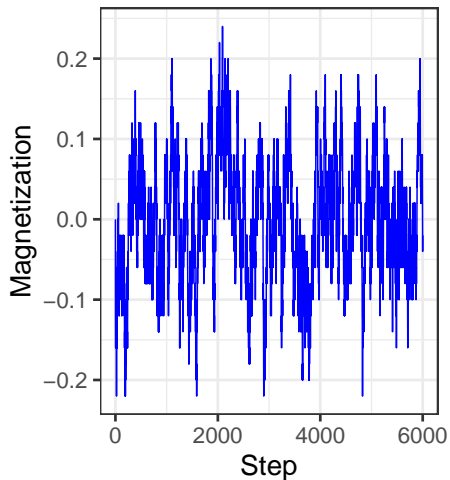
# Antiferromagnetic $J=-1$ , $T=4$



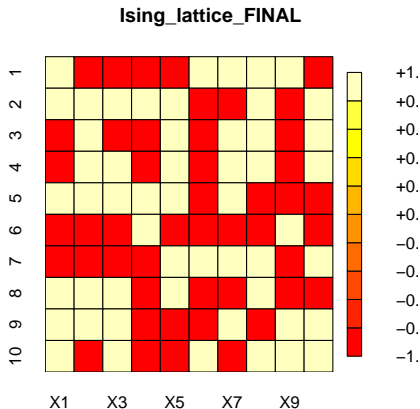
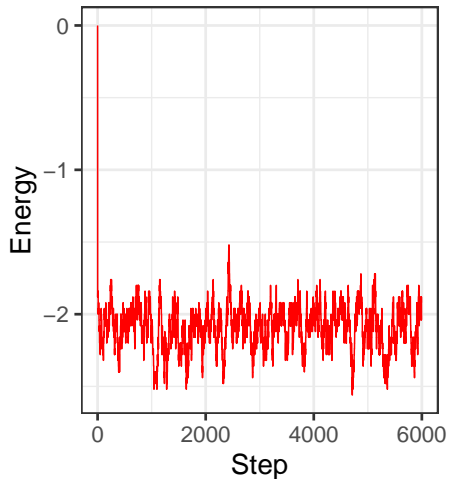
# Antiferromagnetic $J=-1$ , $T=8$



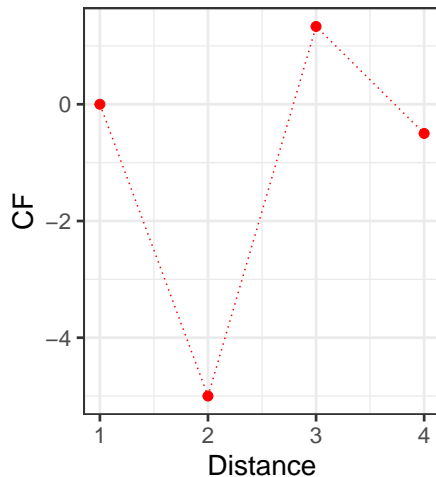
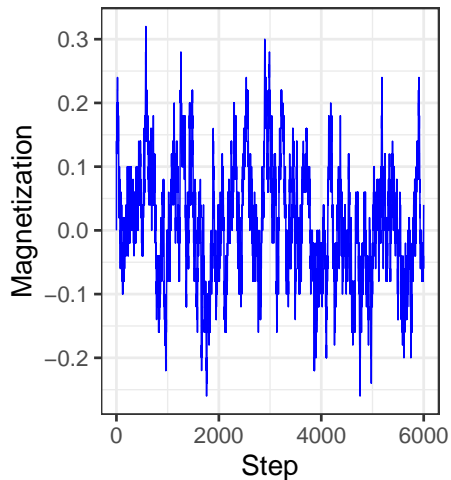
# Antiferromagnetic $J=-1$ , $T=8$



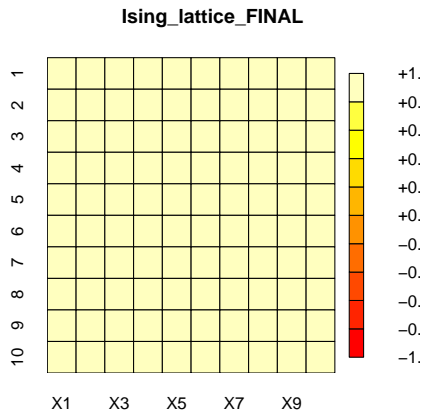
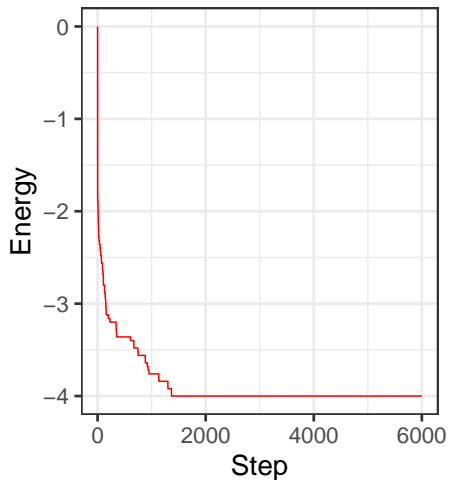
# Ferromagnetic $J=-1$ , $T=30$



# Ferromagnetic $J=-1$ , $T=30$

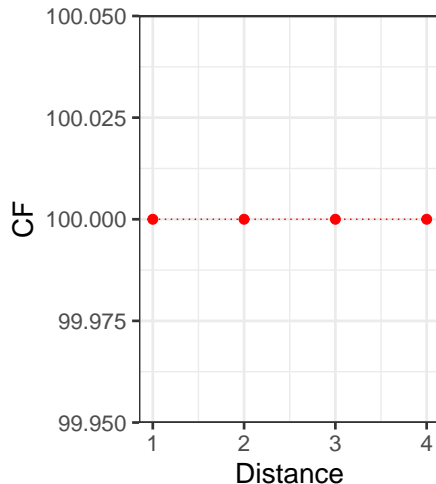
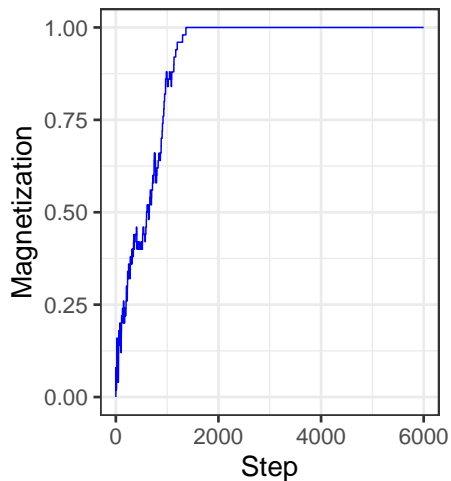


# Ferromagnetic $J=+1$ , $T=0$

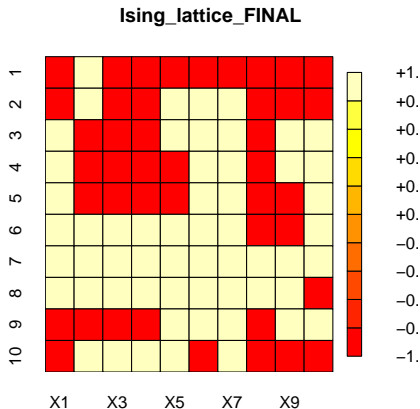
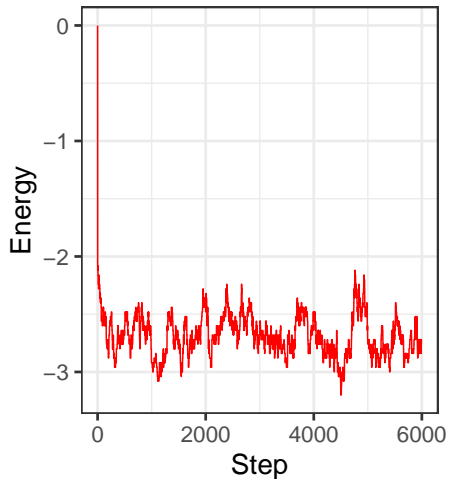




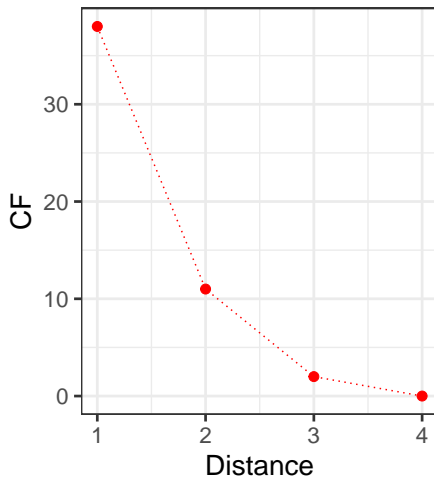
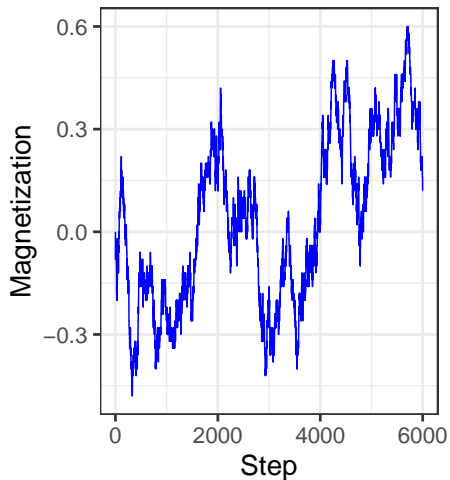
# Ferromagnetic $J=+1$ , $T=0$



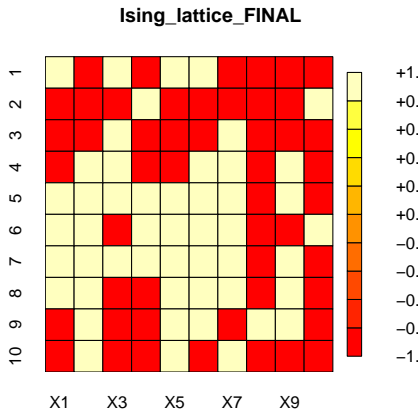
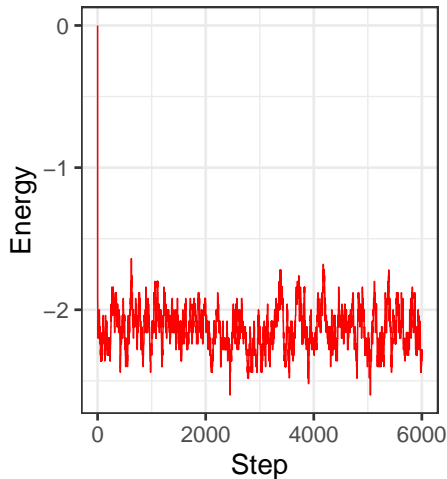
# Ferromagnetic $J=+1$ , $T=8$



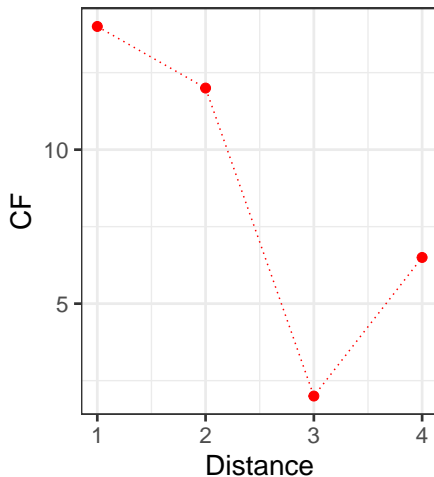
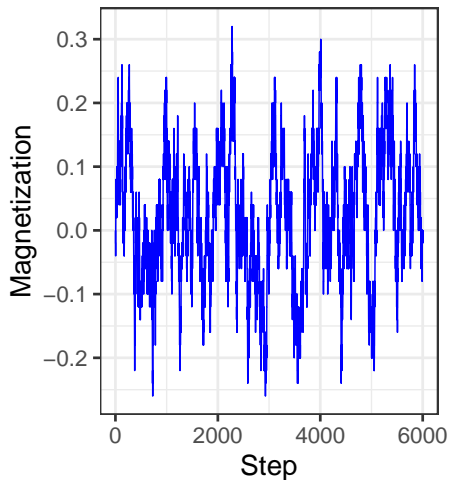
# Ferromagnetic $J=+1$ , $T=8$



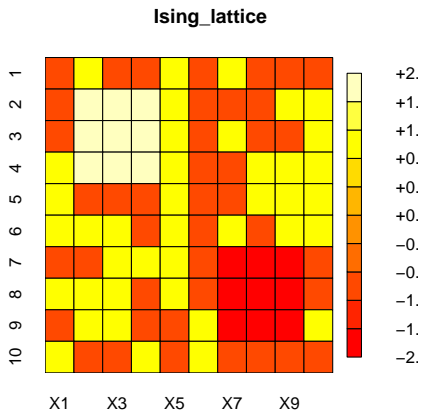
# Ferromagnetic $J=+1$ , $T=16$



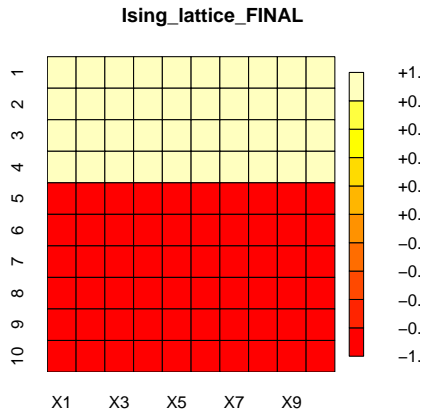
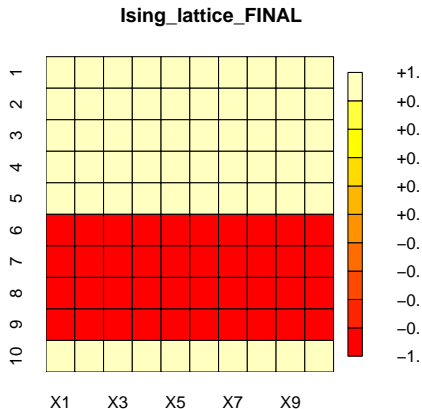
# Ferromagnetic $J=+1$ , $T=16$



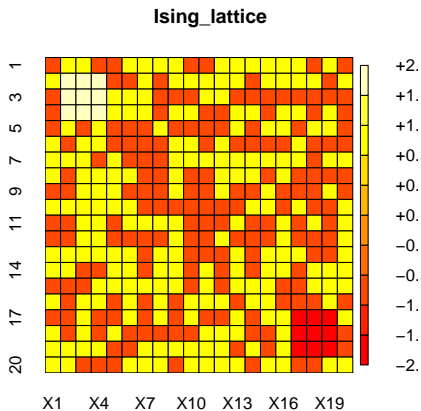
# Ferromagnetic with two centers $T=0$



# Ferromagnetic with two centers $T=0$



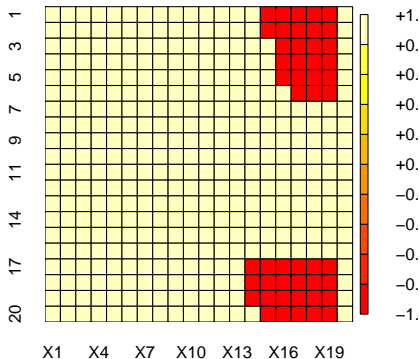
# Ferromagnetic with two centers $T=0$ , $20 \times 20$



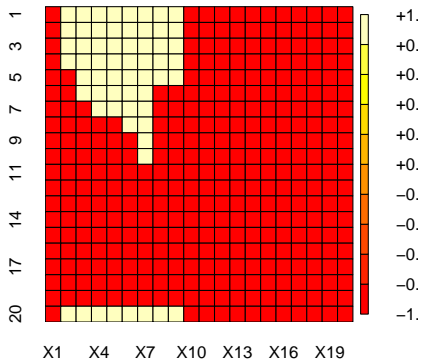


# Ferromagnetic with two centers $T=0$ , $20 \times 20$

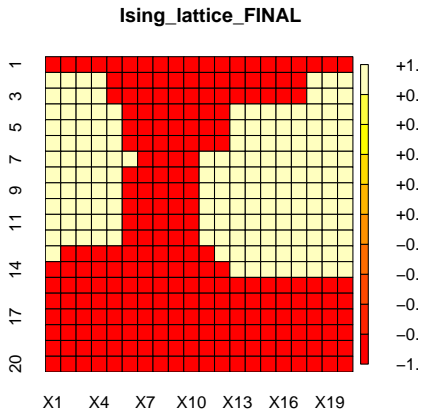
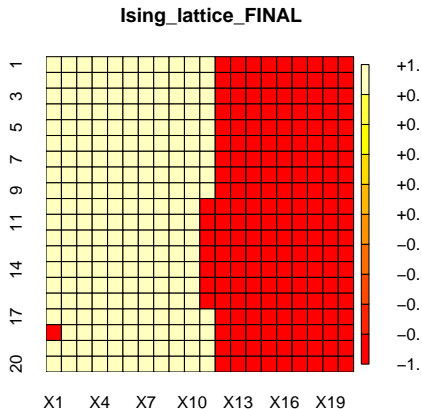
Ising\_lattice\_FINAL



Ising\_lattice\_FINAL

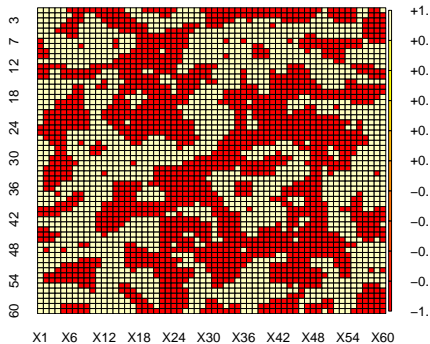


# Ferromagnetic with two centers $T=0.25$ , $20 \times 20$

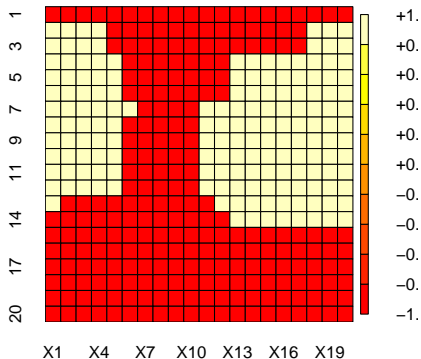


# Ferromagnetic with two centers $T=0$ , $60 \times 60$

Ising\_lattice\_FINAL



Ising\_lattice\_FINAL



# Bibliography I

- [1] <https://commons.wikimedia.org/wiki.>
- [2] Jeffrey C. Grossman and Elif Ertekin, *NSE C242 Phys C203, Spring 2008, U.C. Berkeley*, howpublished = <https://nanohub.org/resources/4579/download/05-01-08-forweb.pdf>,
- [3] David JC MacKay and David JC Mac Kay. *Information theory, inference and learning algorithms*. Cambridge university press, 2003.
- [4] Luca Peliti. *Statistical mechanics in a nutshell*. Princeton University Press, 2011.

# Concepts of statistical mechanics: entropy [4]

**Central problem of thermodynamics:** characterize the actual state of equilibrium among all virtual states

**Entropy postulate:** there exist a function  $S$  of the extensive variables  $(X_0, X_1 \dots X_r)$  called entropy, that assumes the maximum value for a state of equilibrium among all virtual states and that possesses the following properties:

- Extensivity  $S^{(1 \cup 2)} = S^1 + S^2$
- Convexity  $S((1 - \alpha)X^1 + \alpha X^2) \geq (1 - \alpha)S(X^1) + \alpha S(X^2)$
- Monotonicity  $\frac{\partial S}{\partial E}|_{X_1 \dots X_r} = \frac{1}{T} > 0$

The equilibrium state corresponds to the maximum entropy compatible with the constraints

# Concepts of statistical mechanics: entropy [4]

- **Fundamental postulate of statistical mechanics**  $S = k_b \ln |\Gamma|$
- Where  $S$  is the thermodynamic entropy,  $k_b$  is Boltzmann constant and  $|\Gamma|$  the volume in the phase space

$$S(X_0, \dots, X_r) = k_b \ln \int_{\Gamma} dx = k_b \int dx \prod_{i=0}^r [\theta(X_i(x) - (X_i - \Delta X_i))\theta(X_i - X_i(x))] \quad (4)$$

- $(X_0, \dots, X_r)$  are the extensive variables
- The  $\theta$  functions assures that the integrand is not null only in the interval  $X_i - \Delta X_i \leq X_i(x) \leq X_i$

# Concepts of statistical mechanics: micro-canonical ensemble

- Lets focus on a particular observable  $A$  (extensive)

$$S(X; a) = k_b \ln \int_{\Gamma} dx \delta(A(x) - a) \quad (5)$$

$$S(X) = S(X; a^*) \geq S(X; a) \quad (6)$$

$$\begin{aligned} \frac{|\Gamma(a)|}{|\Gamma|} &= \frac{1}{|\Gamma|} \int_{\Gamma} dx \delta(A(x) - a) \\ &= \exp \left\{ \frac{1}{k_b} [S(X; a) - S(X; a^*)] \right\} \\ &\simeq \exp \left\{ \frac{1}{k_b} \left[ \frac{\partial^2 S}{\partial A^2} \Big|_{a^*} (a - a^*)^2 \right] \right\} \end{aligned} \quad (7)$$

$$a^* = \langle A(x) \rangle = \frac{1}{|\Gamma|} \int_{\Gamma} dx A(x) \quad (8)$$

# Concepts of statistical mechanics: canonical ensemble

$$a^* = \frac{1}{|\Gamma|} \int_{\Gamma} dx_S dx_R A(x_S) \quad (9)$$

$$\langle A(x) \rangle = \int dx_S dx_R A(x_S) \delta(H^{(s)}) \quad (10)$$

$$\langle A(x) \rangle = \frac{1}{|\Gamma|} \int dx_S dx_R A(x_S) \delta(H^S(x_S) + H^R(x_R) - H^S(x_S)) \quad (11)$$

$$\langle A(x) \rangle = \frac{1}{|\Gamma|} \int dx_S A(x_S) \times \int dx_R \delta(H^R(x_R) - (E - H^S(x_S))) \quad (12)$$

$$\int dx_R \delta(H^R(x_R) - (E - H^S(x_S))) \simeq \exp \left\{ \frac{1}{k_b} S^R(E - H^S) \right\} \quad (13)$$



# Concepts of statistical mechanics: canonical ensemble

$$\exp \left\{ \frac{1}{k_b} S^R(E - H^S) \right\} \simeq \exp \left[ \frac{1}{k_b} S^R(E) \right] \exp \left[ -\frac{1}{k_b} \frac{\partial S^R}{\partial E} \Big|_E H^S(x_S) \right] \quad (14)$$

$$\langle A(x) \rangle = \frac{1}{Z} \int dx_S A(x_S) \exp \left[ -\frac{H^S(x_S)}{k_b T} \right] \quad (15)$$

$$Z = \int dx_S \exp \left[ -\frac{H^S(x_S)}{k_b T} \right] \quad (16)$$

$$\langle A(x) \rangle = \frac{1}{Z} \int dE \int dx \delta(H(x) - E) A(x) \exp\left(-\frac{E}{k_b T}\right) \quad (17)$$

$$\langle A(x) \rangle = \frac{1}{Z} \int dE' a^*(E') \exp \left[ -\frac{E' - TS(E')}{k_b T} \right] \quad (18)$$

## Concepts of statistical mechanics: canonical ensemble[4]

$$Z \simeq \exp \left[ -\frac{E^* - TS(E^*)}{k_b T} \right] = \exp \left( -\frac{F}{k_b T} \right) \quad (19)$$

$$\frac{\partial \ln Z(\beta)}{\partial \beta} = -\frac{1}{Z} \int dx H(x) \exp \left[ -\frac{H}{k_b T} \right] = -\langle H(x) \rangle = -E \quad (20)$$

$$\frac{\partial^2 \ln Z(\beta)}{\partial \beta^2} = \langle H(x)^2 \rangle - \langle H(x) \rangle^2 \quad (21)$$

$$\langle H(x)^2 \rangle - \langle H(x) \rangle^2 = -\frac{\partial E}{\partial (1/k_b T)} = k_b T^2 \frac{\partial E}{\partial T} = k_b T^2 C \quad (22)$$

In this way a statistical quantity the variance has been connected to a thermodynamic quantity: the temperature.