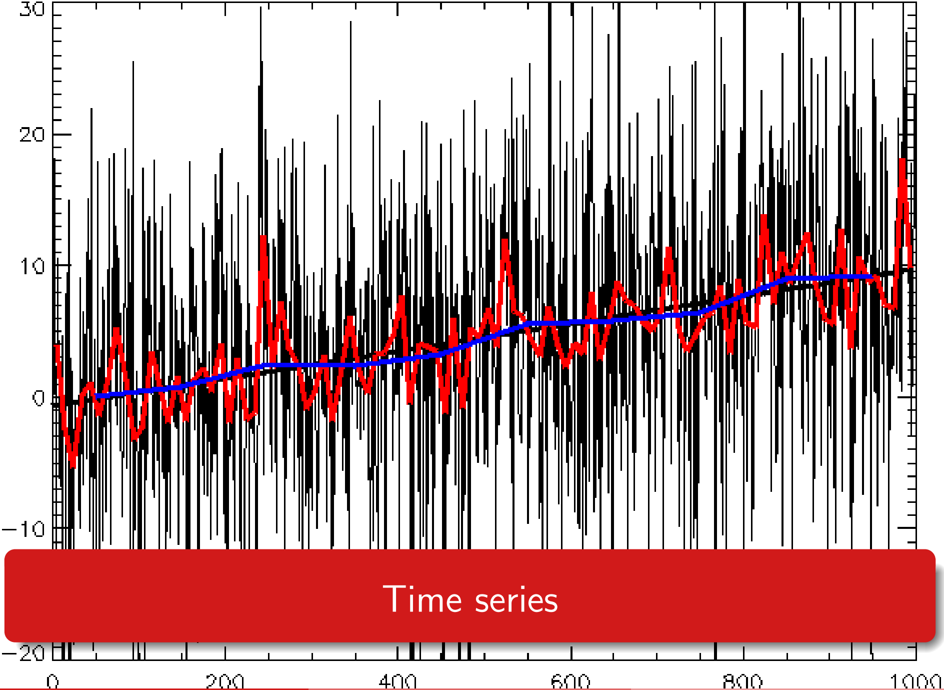


Time series

Preliminary definitions

Time series definition [4]

Informal definition

INTUITIVE DEFINITION: A time-series is a set of observation x_t each one being recorded at a specific time t.
FORMAL DEFINITION: A time series model for the observed data x_t is a specification of the joint distribution (or possible only the means covariance) of a sequence of random variable X_t of which x_t is postulated to be a realization

A binary process

Consider the sequence of iid random variables, with $P[X_t = 1] = p$ and $P[X_t = -1] = 1 - p$

Random walk

The random walk is obtained by cumulatively summing iid random variables. Thus a random walk with zero mean is obtained by defining $S_0 = 0$ and $S_t = X_1 + X_2 + ... + X_t$ for $t = 1, 2, ..$ where X_t a iid noise.

Stationarity, autocovariance and autocorrelation[4]

Mean Function

Let X_t be a time series with $E(x_t^2) < \infty$ The mean function of X_t is $\mu_X(t) = E(X_t)$. The covariance function of X_t is $\gamma_X(r,s) = Cov(X_r, X_S) = E[(X_r - \mu_X(r))(X_s - \mu_X(s))] \quad \forall r, s$

Weakly stationary TS

X_t is weakly stationary if i) $\mu_X(t)$ is independent from time t and ii) $\gamma_X(t+h)$ is indipendent of t $\forall h$

Autocovariance function

At lag h the auto-covariance function is defined as $\gamma_X(h) = Cov(X_{t+h}, X_t)$

Autocorrelation function

At lag h the autocorrelation function is defined as

$$\rho(h)_X = \frac{\gamma_X(h)}{\gamma_X(0)} = Cor(X_{t+h}, X_t)$$

Stationarity, autocovariance and autocorrelation [4]

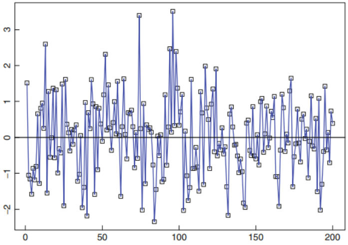


Figure 1-12
200 simulated values of IID N(0,1) noise

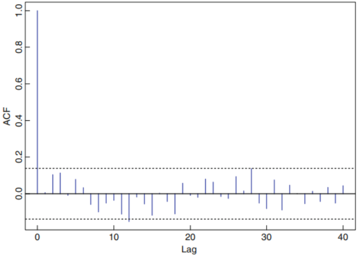
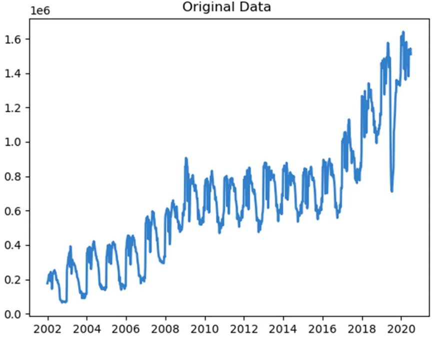
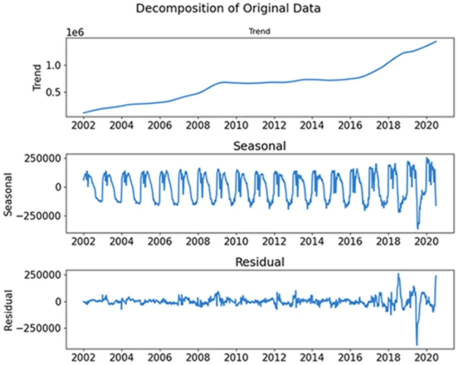


Figure 1-13
The sample autocorrelation function for the data of Figure 1-12 showing the bounds $\pm 1.96/\sqrt{n}$

Stationarity, autocovariance and autocorrelation [6]



(a)



(b)

Definition [4]

Linear process

Time series X_t is a linear process if it has the representation

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j}$$

for all t, where $Z_t \approx \mathrm{WN}(0, \sigma^2)$ and ψ_j is a sequence of costants with $\sum_{j=-\infty}^{\infty} \psi_j < \infty$. In terms of backward shift operator B $BX_t = X_{t-1}$ we have $X_t = \phi(B)Z_t$ Therefore the previous definition can be rewritten as $X_t = \phi(B)Z_t$ in which $\phi(B)$ can be thought as a linear filter that when applied to the white noise input series Z_t produces the output X_t

Definition [4]

autoregressive–moving-average models (ARMA)

The time series X_t is an ARMA(1,1) process if its stationary and satisfiy (for every t)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

where $Z_t \approx \mathrm{WN}(0, \phi^2)$ and $\phi + \theta \neq 0$ or in terms of filters ϕ and θ

$$\phi(B) X_t = \theta(B) Z_t$$

Causality [4]

Remarks

- A stationary solution of the ARMA(1,1) equation exists if and only if $\phi \neq \pm 1$
- If $|\phi| < 1$, then the unique stationary solution is given by $X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$. In this case we say that X_t is causal or a causal function of Z_t or a causal function of Z_t since X_t can be expressed in terms of the current and past values Z_s, $s \leq t$
- If $|\phi| > 1$, then the unique stationary solution is given by $X_t = -\theta\phi^{-1} Z_t - (\phi + \theta) \sum_{j=1}^{\infty} \phi^{-j-1} Z_{t-j}$. The solution is non-causal, since X_t is then a functional of Z_s $s \geq t$

Definition [4]

ARMA (p,q)

X_t is an ARMA(p,q) process if X_t is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - ... - \phi_1 X_{t-q} - = Z_t + \theta Z_t + \theta_1 Z_{t-1} + ... + \theta_q Z_{t-q}$$

where $Z_t \approx WN(0, \sigma^2)$ and the polynomials $(1 - \phi_1 z - ... - \phi_p z^p)$ and $(1 + \phi_1 z + ... + \phi_p z^p)$ have no common factor.

Definition [4]

Causality

An ARMA(p,q) process X_t is causal, or a causal function of Z_t if there exist constants ϕ_j such that $\sum_{j=0}^{\infty} |\phi_j| < \infty$ and

$$X_t = \sum_{j=0}^{\infty} \sum \phi_j Z_{t-j} \quad \forall \quad t$$

this is equivalent to the condition

$$\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p \neq \quad |z| < 1$$

Spectral Analysis

Spectral Analysis [4]

Spectral density

Given a zero mean stationary time series X_t with autocovariance function $\gamma()$ satfying $\sum_{h=-\infty} |\gamma(h)| < \infty$, the spectral density of X_t is the function defined by

$$f(\lambda) = \frac{1}{2\pi} \sum e^{-ih\lambda} \gamma(h) \quad -\infty < \lambda < \infty$$

with the condition that

$$f(\lambda) \geq 0 \quad \forall \lambda \in (-\pi, \pi]$$

$$\gamma(h) = \int_{-\pi}^{\pi} e^{ih\lambda} f(\lambda) d\lambda = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$$

Spectral Analysis [3]

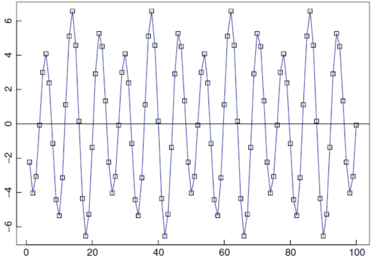


Figure 4-1
A sample path of size 100 from the time series in Example 4.1.2

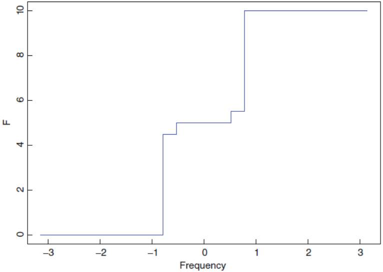


Figure 4-2
The spectral distribution function $F(\lambda)$, $-\pi \leq \lambda \leq \pi$, of the time series in Example 4.1.2

Time-Invariant Linear Filters [4]

Linear process

The process Y_t is the output of a linear filter $C = \{c_{t,k}, t, k = 0 \pm 1, ...\}$ applied to an input process X_t if $Y_t = \sum_{k=-\infty}^{\infty} c_{t,k} X_k \quad t = 0, \pm 1, ...$

Time invariant

The filter is said to be time-invariant if the weights $c_{t,t-k}$ are independent of t e.g. $c_{t,t-k} = \psi_k$

$$Y_t = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

$$Y_{t-s} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-s-k}$$

The TLF ϕ is to be causal if $\phi_j = 0$ for $j < 0$

Time-Invariant Linear Filters [4]

Transfer function

Let X_t be a stationary time series with mean zero and spectral density $f_x(\lambda)$. Suppose that $\Phi = \{\phi_j, j = 0, \pm 1, ...\}$ is an absolutely summable TLF. Then the time series

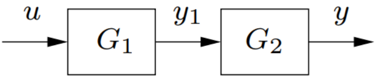
$$Y_t = \sum_{j=-\infty}^{\infty} \phi_j X_{t-j}$$

is stationary with mean zero and spectral density

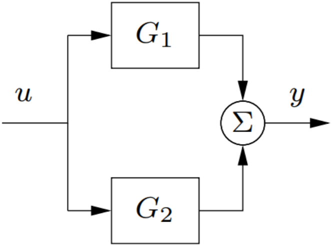
$$f_Y(\lambda) = |\Psi(e^{-i\lambda})|^2 f_X(\lambda) = \Psi(e^{-i\lambda})\Psi(e^{i\lambda}) f_X(\lambda)$$

where $\Psi(e^{-i\lambda}) = \sum_{j=-\infty}^{\infty} \phi_j e^{-ij\lambda}$ where $\Psi(e^{-i})$ is called the transfer function of the filter, and the squared modulus $|\Psi(e^{-i})|$ is referred to as the power transfer function of the filter.

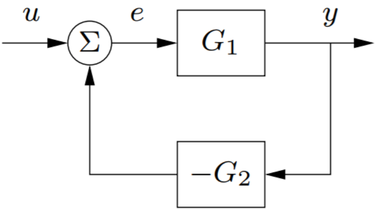
Transfer Functions [3]



$$G_{yu}(s) = G_2(s)G_1(s)$$



$$G_{yu}(s) = G_1(s) + G_2(s)$$



$$G_{yu}(s) = \frac{G_1(s)}{1 + G_1(s)G_2(s)}$$

Time-Invariant Linear Filters [4]

Simple moving average

$$Y_t = \frac{1}{2q+1} \sum_{|j|<q} X_{t-j}$$

where $\psi = (2q+1)^{-1}, j = -q, ..., q$ and ψ_j

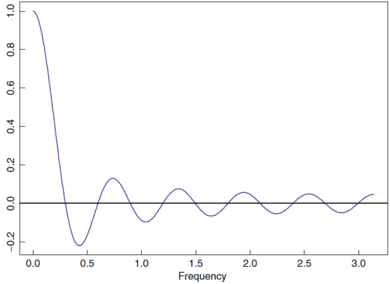


Figure 4-12
The transfer function $D_{10}(\lambda)$ for the simple moving-average filter

Gibbs phenomenon

The poor approximation in the neighbourhood of cut-off frequency (ω_c)

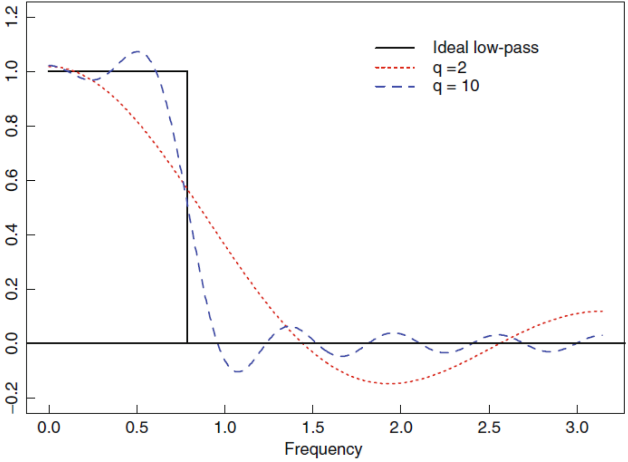


Figure 4-13
The transfer function for the ideal low-pass filter and truncated Fourier approximations $\Psi^{(q)}$ for $q = 2, 10$

Estimation

Esimation [4]

Gaussian Likehood

$$L(\hat{\boldsymbol{\phi}}, \hat{\theta}, \boldsymbol{\sigma}^2) = \frac{1}{\sqrt{(2\pi\sigma)^n r_0 \ldots r_{n-1}}} \exp(-\frac{1}{2\sigma^2} \sum \frac{(X_j - \hat{X}_j)^2}{r_{j-1}})$$

Maximum Likehood Estimators

Differentiating $\ln L(\phi, \theta\, \sigma^2)$ partially with respect to σ^2 and noting that $\hat{X}_j$ and r_j are independent of σ

$$\hat{\sigma}^2 = n^{-1} S(\hat{\boldsymbol{\phi}}, \hat{\theta})$$

$$S(\hat{\boldsymbol{\phi}}, \hat{\theta}) = \sum_{j=1}^{n} (X_j - \hat{X}_j)^2 / r_{j-1}$$

where $\hat{\boldsymbol{\phi}}$,$\hat{\theta}$ are the values of $\boldsymbol{\phi}$,θ that minimize:

$$\mathcal{L}(\phi, \theta) = \ln(n^{-1} S(\phi, \theta)) + n^{-1} \sum_{j=1}^{n} \ln r_{j-1}$$

Estimation [4]

Akaike information criterion

$$AICC = -2 \ln L(\hat{\phi}, \hat{\theta}, S((\phi_p, \theta_q)/n) + 2(p+q+1)n/(n-p-q-2)$$

Residual

$$\hat{W}_t = \left(X_t - \hat{X}_t \phi, \theta\right) / \left(r_{t-1}(\hat{\phi}, \hat{\theta})\right)^{1/2} \quad t = 1, ..., n$$

This should have properties similar to the white noise sequence, thus one can define the rescaled residuals

$$\hat{R}_t = \hat{W}_t / \hat{\sigma}$$

$$\hat{\sigma} = \sqrt{\left(\sum_{t=1}^{n} \hat{W}_t^2\right) / n}$$

Residuals [4]

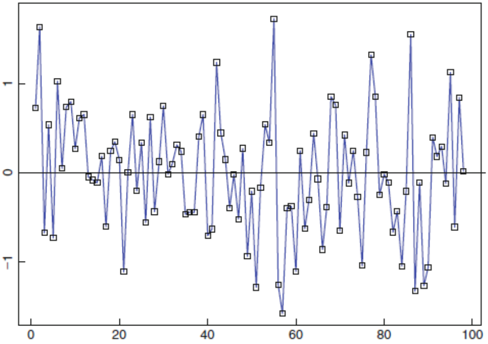
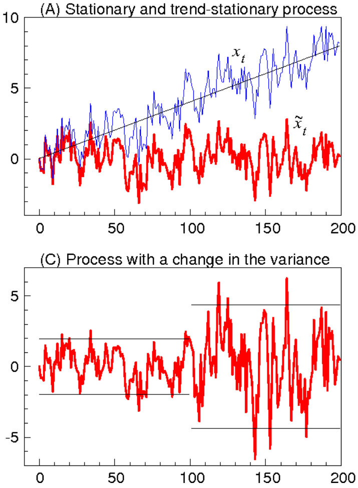


Figure 5-5
The rescaled residuals after fitting the ARMA(1,1) model of Example 5.2.5 to the lake data

Nonstationary time series

Examples [4, 1]



(A) Stationary and trend-stationary process
10
5
0
x_t
$\tilde{x}_t$
0 50 100 150 200
(B) Process with a level shift
5
0
0 50 100 150 200
(C) Process with a change in the variance
5
0
-5
0 50 100 150 200
(D) Unit root process
10
5
0
0 50 100 150 200

Autoregressive integrated moving average [4]

ARIMA definitio

If d is a non-negative integer, then X_t is an ARIMA(p,d,q) process if

$$Y_t = (1-B)^d X_t$$

is causal ARMA(p,q) process $BX_t = X_{t-1}$. Note that $\nabla X_t = X_t - X_{t-1}$

Multivariate time series

Multivariate time series [4]

Definition

$\{\mathbf{X}_t\}$ is an ARMA (p,q) process if $\{\mathbf{X}_t\}$ is stationary and if for every t,

$$\boldsymbol{X}_t - \phi_1 \boldsymbol{X}_{t-1} - ... - \phi_p X_{t-p} = Z_t + \Theta_1 Z_{t-1} + ... + \Theta_q Z_{t-q}$$

where $Z_t \approx$ WN(0,Σ)

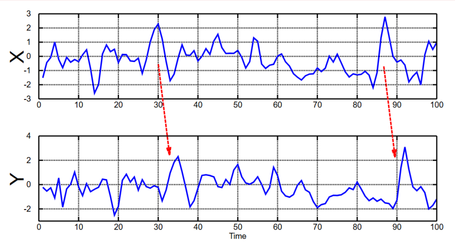
Vector Autoregression [5]

VAR(1)

$$\begin{pmatrix} y_{1,t} \\ y_{2,t} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \begin{pmatrix} a_{1,2} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \begin{pmatrix} y_{1,t-1} \\ y_{2,t-1} \end{pmatrix} + \begin{pmatrix} e_{1,t} \\ e_{2,t} \end{pmatrix}$$

Granger causality test (hamilton2020time)

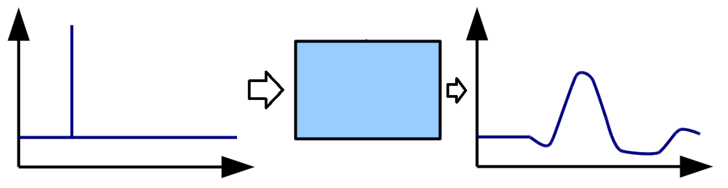
y_2 cause (in Granger sense) y_1 if the coefficent $a_{1,2}$ is signficantly not equal to 0



Vector Autoregression [5]

Impulse response function

y_2 cause (in Granger sense) y_1 if the coefficent $a_{1,2}$ is signficantly not equal to 0

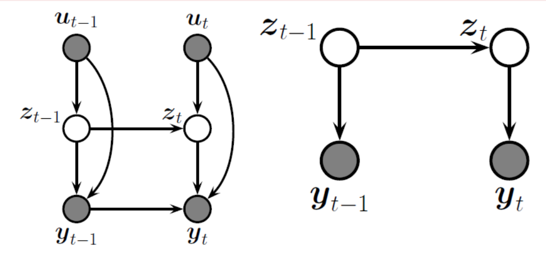


State-space models

Space State Models [7]

Definition

A state-space model (SSM) is a partially observed Markov model in which the hidden state z_t evolves over time according to a Markov process and each hidden state generates some observations y_t at each step. The main goal is to infer the hidden states given the observations (and also to predict the future observations)



Space State Models [7]

Non linear dynamical system

A state-space model (SSM) can be represented as a stochastic discrete time nonlinear dynamical system of the form

$$
\begin{aligned}
z &= f(\mathbf{z}_{t-1}, \mathbf{u}_t, \mathbf{q}_t) \\
y_t &= h(\mathbf{z}_t, \mathbf{u}_t, \mathbf{y}_{1:t-1}, \mathbf{r}_t)
\end{aligned}
$$

where $z_t \in \mathbb{R}^N$ are the hidden states, $\mathbf{u}_t \in \mathbb{R}^N$ are the optional observed inputs, $y\mathbf{y}_t \in \mathbb{R}^N$ are observed output and $\mathbf{f}$ is the transition function, $\mathbf{q}_t$ is the process noise, $\mathbf{h}$ is the observation function and $\mathbf{r}_t$ is the observation noise. The transition model and the observational model are

$$
\begin{aligned}
p(\mathbf{z}_t|\mathbf{z}_{t-1}, \mathbf{u}_t) &= p(\mathbf{z}_t|\mathbf{f}(\mathbf{z}_{t-1}, \mathbf{u}_t)) \\
p(\mathbf{y}_t|\mathbf{z}_t, \mathbf{u}_t, \mathbf{y}_{1:t-1}) &= p(\mathbf{y}_t|\mathbf{h}(\mathbf{z}_t, \mathbf{u}_t, \mathbf{y}_{1:t-1}))
\end{aligned}
$$

Space State Models [7]

Non linear dynamical system

Hidden Markov Model (HMM) $\rightarrow$ a SSM in which the hidden states are discrete, thus $z_t \in 1, ..., K$

Time series segmentation

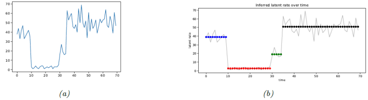


Figure 29.6: (a) A sample time series dataset of counts. (b) A segmentation of this data using a 4 state HMM. Generated by poisson_hmm_changepoint.ipynb.

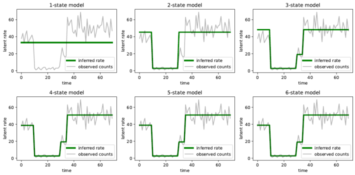


Figure 29.7: Segmentation of the time series using HMMs with 1–6 states. Generated by poisson_hmm_changepoint.ipynb.

Structural time series models [7]

Structural time series (STS)

Defined in terms of linear-Gaussian SSMs. Differently from ARMA method, they have much more flexibility: one can create non-linear, non-Gaussian and even hierarchical extension. Represent the observed scalar time series as a sum of C individual components

$$f(t) = f_1(t) + f(t) + ... + f_C(t) + \epsilon_t$$

Each single component (latent process) $f_c(t)$ is modeled by a linear Gaussian state-space model which is also called dynamic linear model. Since these are linear, one can combine in to a single state-space model.

$$\begin{aligned} p(z_t|z_{t-1}, \theta) =& \mathcal{N}(z_t|Fz_{t-1}, Q) \\ p(y_t|z_t, \theta) =& \mathcal{N}(y_t|Hz_t + \beta^T u_t, \sigma_y^2) \end{aligned}$$

where $\boldsymbol{F}$ and $\boldsymbol{Q}$ are block structure matrices, with one block per component. The vector $\boldsymbol{H}$ then adds up all the relevant piecies from each component to generate the overall mean.

Structural time series models [7]

Local level model

The observation $y_t \in \mathbb{R}^N$ are generated by a Gaussian with (latent) mean μ_t , which evolves over over time according to a random walk

$$y_t = \mu_t + \epsilon_{y,t} \quad \epsilon_{y,t} \approx \mathcal{N}(0, \sigma_y^2)$$
$$\mu_t = \mu_{t-1} + \epsilon_{t-1\mu,t} \quad {}_{\mu,t} \approx \mathcal{N}(0, \sigma_\mu^2)$$

One can also assume that $\mu_1 \in \mathcal{N}(0, \sigma_\mu^2)$. Thus the latent mean at any future step has distribution $\mu_t \in \mathcal{N}(0, t\sigma_\mu^2$ so the variance grows with time. Once can also use an autoregressive model in which $\mu_t = \rho\mu_{t-1} + \epsilon_{\mu,t}$ where $|\rho < 1|$. In this case we have $\mu_\infty \approx \mathcal{N}(0, \frac{\sigma_\mu^2}{1-\rho^2})$: thus the uncertainty grows to finite asymptote instead of undoubtedly

Structural time series models [7]

Space representation of ARMA models

In this case we have

$$Z = \begin{pmatrix} Y_{t-p+1} \\ Y_{t-p+2} \\ ... \\ Y_t \end{pmatrix} \tag{1}$$

And the observation equation is

$$Y_t = [0, 0, 0, ..., 1]X_t \quad t = \pm 0, \pm 1, ... \tag{2}$$

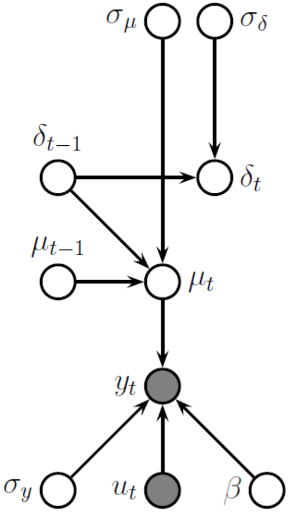
And the state equation is

$$X_{t+1} = \begin{pmatrix} 0 & 1 & 0 & ... & 0 \\ 0 & 0 & 1 & ... & 0 \\ ... & & & & \\ 0 & 0 & 0 & ... & 1 \\ \phi_p & \phi_{p-1} & \phi_{p-2} & ... & \phi_1 \end{pmatrix} \boldsymbol{X}_t + \begin{pmatrix} 0 \\ 0 \\ ... \\ 0 \\ 1 \end{pmatrix} \mathrm{WN}(0, \sigma^2) \quad t = 0, \pm 1, ... \tag{3}$$

Bayesian structural time series [7]

Traking and state estimation

A BSTS model with local linear trend and linear regression on inputs. The observed output is y_t. The latent state vector is defined by $z_t = (\mu_t, \delta_t)$. The (static) parameters are $\theta = (\sigma_y, \sigma_\mu, \sigma_\delta, \beta)$. The covariates are u_t



The Kalman Recursions

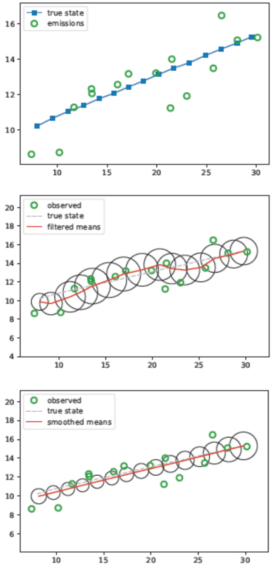
Intuitive idea [7]

Traking and state estimation

We want to track an object moving in 2D. The hidden state x_t encodes the location (x_{t1}, x_{t2}) and the velocity $\dot{x}_{t1}, \dot{x}_{t2}$ of the moving object. The observation $\mathbf{y}_t$ is a noisy version of the location. Suppose that that the sampling period is Δ, thus

$$\boldsymbol{x}_t = \begin{pmatrix} 1 & 0 & \Delta & 0 \\ 0 & 1 & 0 & \Delta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \boldsymbol{x}_{t-1} + \mathbf{WN} \quad (4)$$

$$\boldsymbol{y}_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \boldsymbol{x}_{t-1} + \mathbf{WN} \quad (5)$$

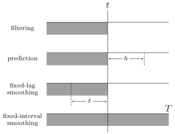


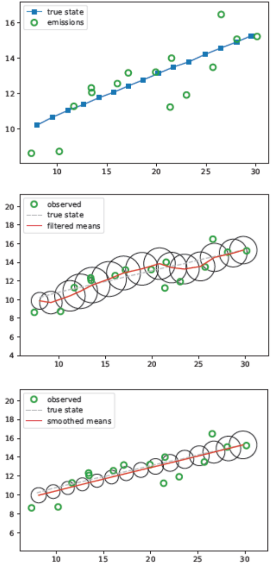
Intuitive idea [7]

Traking and state estimation

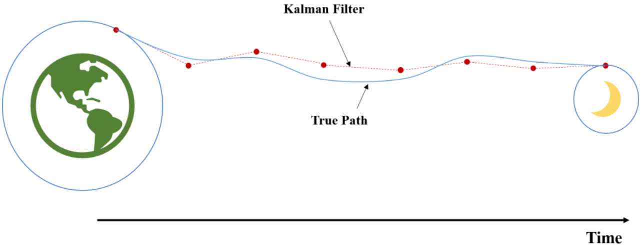
Possible tasks

- Prediction
- Filtering: estimate the unknown location (and velocity) of the object given the noisy obstervation (on-line)
- Smoothing: inference of the locations, given all the observations (off-line)





Intuitive idea [2]



The Kalman Recursions [4]

General idea

Finding the best (in the sense of minimum square error) linear estimates of the state vector X_t in terms of observations Y_1, Y_2 and a random vector Y_0 that is orthogonal to V_t and W_t for all $t \geq 1$ for these three cases:

- $Y_0, ..., Y_{t-1}$ defines the prediction problem
- $Y_0, ..., Y_t$ defines the filtering problem
- $Y_0, ..., Y_n$ $(n > t)$ defines the smoothing problem

Best linear predictor

For the random vector $X = (X_1, ..., X_v)'$

$$P(X) = (P_t(X_1), ..., P_t(X_v))'$$

where $P_t(X) = P(X_i | \boldsymbol{Y}_0, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_t)$ is the best linear predictor of X_i in terms of all components of $\boldsymbol{Y}_0, \boldsymbol{Y}_1, ..., \boldsymbol{Y}_t$

The Kalman Recursions [4]

Kalman prediction

For the state-space model

$$Y_t = G_t X_t + W_t$$
$$X_{t+1} = F_t X_t + V_t$$

where W and V are two uncorrelated noises. The one-step predictors $\hat{X}_t = P_{t-1}(X_t)$ and their error covariance matrices $\Omega_t = E\left[(X_t - \hat{X}_t)(X_t - \hat{X}_t)'\right]$ are uniquely determined by the initial conditions

$$\hat{\boldsymbol{X}}_t = P(X_t|Y_0) \quad \Omega_1 = E\left[\left(\boldsymbol{X}_1 - \hat{\boldsymbol{X}}_1\right)\left(\boldsymbol{X}_1 - \hat{\boldsymbol{X}}_1\right)'\right]$$

and the recursion, for $t = 1, ...,$

$$\hat{\boldsymbol{X}}_{t+1} = F_t \hat{\boldsymbol{X}}_t + \Theta_t \Delta_t^{-1}\left(\boldsymbol{Y}_t - G_t \hat{\boldsymbol{X}}_t\right)$$
$$\Omega_{t+1} = F_t \Omega F_t' + Q_t - \Theta_t \Delta_t^{-1} \Theta_t'$$

where $\Delta_t = G_t \Omega_t G_t' + R_t \quad \Theta_t = F_t \Omega_t G_t'$

The Kalman Recursions [4]

Kalman Filtering

The filtered estimates $X_{t|t} = P_t(X_t)$ and their error covariance matrices $\Omega_{t|t} = E\left[\left(X_t - X_{t|t}\right)\left(X_t - X_{t|t}\right)'\right]$ are determinated by the relations

$$P_t\mathbf{X}_t = P_{t-1}\mathbf{X}_t + \Omega_t G_t' \Delta_t^{-1}\left(\mathbf{Y}_t - G_t\hat{\mathbf{X}}_t\right)$$

and

$$\Omega_{t|t} = \Omega_t - \Omega_t G_t' \Delta_t^{-1} G_t \Omega_t'$$

The Kalman Recursions [4]

The Kalman Fixed-Point Smoothing

The smoothed estimates $X_{t|n} = P_n X_t$ and the error covariance matrices $\Omega_{t|n} = E\left[(X_t - X_{t|n})(X_t - X_{t|n})'\right]$ are determinated for fixed t by the following recursions, which can be solved successively for $n = t, t+1, ..$

$$P_n \mathbf{X}_t = P_{n-1}\mathbf{X}_t + \Omega_t G_t' \Delta_t^{-1}\left(\mathbf{Y}_n - G_n \hat{\mathbf{X}}_n\right)$$

and

$$\Omega_{t|n+1} = \Omega_{t,n}\left[F_n - \Theta_n \Delta_n^{-1} G_n\right]$$

$$\Omega_{t|n} = \Omega_{t|n-1} - \Omega_{t,n} G_n' \Delta_n^{-1} G_n \Omega_{t,n}'$$

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