CSE 103: Probability and statistics **Fall 2010**

Topic 2 — Multiple events, conditioning, and independence, II

2.1 Conditional probability, continued

2.1.1 The Monty Hall problem

This probability puzzle is weakly related to a game on an old TV show called *Let's Make a Deal*, and has been renamed after the host of that show. The host brings the game player to a room with three closed doors. One of the doors leads to a treasure chest while the other two doors each lead to a goat. The player picks a door (at random, presumably), hoping for the best. Now something interesting happens. Instead of opening that door, the host opens one of the other two doors to reveal a goat. The player is then allowed to either stick to his original guess, or to switch to the other unopened door. Which should he do?

In surveys, the majority of people feel intuitively that it doesn't make a difference. They reason that there are two unopened doors, and the treasure could be behind either of them, so each has a 50-50 chance of being the lucky door. But this is incorrect. The player should switch to the other door: by doing so, he will double his chances of getting the treasure! A conditional probability calculation shows why:

$$\begin{aligned}\Pr(\text{treasure in other door}) &= \Pr(\text{treasure in other door}|\text{initial guess correct})\Pr(\text{initial guess correct}) + \\ &\quad \Pr(\text{treasure in other door}|\text{initial guess wrong})\Pr(\text{initial guess wrong}) \\ &= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{3} \;=\; \frac{2}{3}.\end{aligned}$$

2.1.2 Another summation rule

There is also a summation rule for conditional probabilities. Suppose $A_1, \ldots, A_k$ are disjoint events whose union is Ω; and let $E, F \subset \Omega$ be any two events. Then

$$\Pr(E|F) \;=\; \sum_{i=1}^{k} \Pr(E \cap A_i|F) \;=\; \sum_{i=1}^{k} \Pr(E|A_i, F)\Pr(A_i|F).$$

(It's just like the regular summation rule, but operating within F rather than Ω.)

2.1.3 Sex bias in graduate admissions

In 1973, the dean of graduate admissions at Berkeley noticed something alarming: a male applicant had a higher chance of being admitted than a female applicant! Specifically,

$$\Pr(\text{admitted}|\text{male}) = 0.44, \quad \Pr(\text{admitted}|\text{female}) = 0.35.$$

This is a significant difference given that the sample space (applicants to graduate programs at Berkeley) was very large, of size 12,763. The dean wondered, which departments were responsible for this bias?

To determine this, data was collected for each of the departments on campus. And it was found that no department (with a few insignificant exceptions) had any bias against females! Specifically, for each department d,

$$\Pr(\text{admitted}|\text{male, department} = d) \leq \Pr(\text{admitted}|\text{female, department} = d).$$

How could this be?

Let's go back to the summation formulas for conditional probabilities.

$$\begin{aligned}
\Pr(\text{admit}|\text{male}) &= \sum_d \Pr(\text{admit}|\text{male, dept} = d)\Pr(\text{dept} = d|\text{male}) \\
\Pr(\text{admit}|\text{female}) &= \sum_d \Pr(\text{admit}|\text{female, dept} = d)\Pr(\text{dept} = d|\text{female})
\end{aligned}$$

The answer lies in the second term. Some departments (engineering) are easier to get into than others (humanities). Males tended to apply to the easier departments, while females on average applied to the harder departments. So while no individual department had bias, on the whole it looked like women had a lower probability of acceptance.

2.2 Bayes' rule

The following example is taken from *Probabilistic Reasoning in Intelligent Systems* by Judea Pearl:

> You wake up in the middle of the night to the shrill sound of your burglar alarm. What is the chance that a burglary attempt has taken place?

The relevant facts are:

- There is a 95% chance that an attempted burglary attempt will trigger the alarm. That is,
$$\Pr(\text{alarm}|\text{burglary}) = 0.95.$$
- There is a 1% chance of a false alarm.
$$\Pr(\text{alarm}|\text{no burglary}) = 0.01.$$
- Based on local crime statistics, there is a one-in-10,000 chance that a house will be burglarized on a given night.
$$\Pr(\text{burglary}) = 10^{-4}.$$

We are interested in the chance of a burglary given that the alarm has sounded. We can use the conditional probability formula for this:

$$\Pr(\text{burglary}|\text{alarm}) = \frac{\Pr(\text{burglary, alarm})}{\Pr(\text{alarm})} = \frac{\Pr(\text{alarm}|\text{burglary})\Pr(\text{burglary})}{\Pr(\text{alarm})}.$$

The one term we don't immediately know is Pr(alarm). By the summation rule,

$$\Pr(\text{alarm}) = \Pr(\text{alarm}|\text{burglary})\Pr(\text{burglary}) + \Pr(\text{alarm}|\text{no burglary})\Pr(\text{no burglary}).$$

Putting it all together,

$$\Pr(\text{burglary}|\text{alarm}) = \frac{0.95 \times 10^{-4}}{0.95 \times 10^{-4} + 0.01 \times (1 - 10^{-4})} = 0.00941,$$

about 0.94%. Thus our belief in a burglary has risen approximately a hundredfold from its default value of 10^{-4}, on account of the alarm.

It is frequently the case, as in this example, that we wish to update the chances of an event H based on new evidence E. In other words, we wish to know $\Pr(H|E)$. The derivation above implicitly uses the following formula, called **Bayes' rule**:

$$\Pr(H|E) = \frac{\Pr(E|H)\Pr(H)}{\Pr(E)}.$$

As another example, let's look at the Three Prisoner's Paradox, which is actually just a reformulation of the Monty Hall problem. The story goes that there are three prisoners A, B, C in a jail, and one of them is to be declared guilty and executed the following morning. As the night progresses, prisoner A is racked with worry, and calls the prison guard over. He wants to know whether he is the unlucky one. The guard replies, "I am not allowed to tell you whether you will be declared guilty. But I can say that prisoner B will be declared innocent." Prisoner A thinks about this for a little while and then starts worrying even more. Before he asked the question, it seemed like his chances of dying were one-in-three. But after his innocuous question, the chance seems to have risen to one-in-two!

Actually, Prisoner A's chances are still one in three. The two events of interest are

$$\begin{aligned} G_A &= A \text{ will be declared guilty} \\ I_B &= \text{the guard, when prompted, will declare } B \text{ to be innocent.} \end{aligned}$$

Using the summation rule,

$$\Pr(I_B) = \Pr(I_B|G_A)\Pr(G_A) + \Pr(I_B|G_A^c)\Pr(G_A^c) = \frac{1}{2} \cdot \frac{1}{3} + \frac{1}{2} \cdot \frac{2}{3} = \frac{1}{2}.$$

Therefore, by Bayes' rule,

$$\Pr(G_A|I_B) = \frac{\Pr(I_B|G_A)\Pr(G_A)}{\Pr(I_B)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{2}} = \frac{1}{3}.$$

2.3 Independence

Two events are called *independent* if the outcome of one (that is, whether or not it occurs) does not affect the probability that the other will occur. For instance, suppose you flip two fair coins. The outcome of either coin does not influence the other; therefore the two outcomes are independent.

Formally, we say events A and B (defined on some sample space Ω) are independent if

$$\Pr(A \cap B) = \Pr(A)\Pr(B).$$

Can you show that this definition implies the following?

- $\Pr(A|B) = \Pr(A)$.
- $\Pr(B|A) = \Pr(B)$.

- $\Pr(A|B^c) = \Pr(A)$.

In the following examples, are events A and B independent or not?

1. You have two children. $A = \{\text{first child is a boy}\}$, $B = \{\text{second child is a girl}\}$.
2. You throw two dice. $A = \{\text{first is a } 6\}$, $B = \{\text{sum} > 10\}$.
3. You get dealt two cards at random from a deck of 52. $A = \{\text{first is a heart}\}$, $B = \{\text{second is a club}\}$.
4. Same sample space. $A = \{\text{first is a heart}\}$, $B = \{\text{second is a } 10\}$.
5. The three scenarios depicted in these Venn diagrams.