1) A die is thrown twice, let X_1 and X_2 denote the outcomes, and define random variable X to be the minimum of X_1 and X_2. Determine the distribution of X.

$$X \begin{cases} X_1 \\ X_2 \end{cases}$$

X is min of X_1 & X_2

X_1	1	2	3	4	5	6
X_2	1	1	1	1	1	1

X_1	1	2	3	4	5	6
X_2	4	4	4	4	4	4

X_1	1	2	3	4	5	6
X_2	2	2	2	2	2	2

X_1	1	2	3	4	5	6
X_2	5	5	5	5	5	5

X_1	1	2	3	4	5	6
X_2	3	3	3	3	3	3

X_1	1	2	3	4	5	6
X_2	6	6	6	6	6	6

$P(X=1) = 11/36$

$P(X=2) = 9/36$

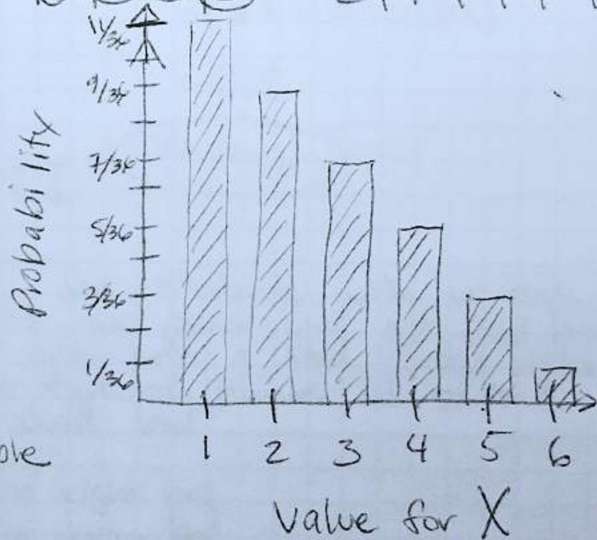
$P(X=3) = 7/36$

$P(X=4) = 5/36$

$P(X=5) = 3/36$

$P(X=6) = 1/36$

Discrete Random Variable



2) A fair die is rolled repeatedly until a six is seen. What is the expected number of rolls?

With every roll there's a 1/6 probability that the roll will be 6.

$Pr(X=6) = 1/6$

$$X = \begin{cases} 1, & \text{roll is } 6 \\ 0, & \text{roll not } 6 \end{cases}$$

total # rolls is $X = X_1 + X_2 + \dots X_K$

first roll: $E(X_1) = 1/6(1) + 5/6(0)$
$= 1/6$

$\mathbb{E}(X) = \frac{1}{p} = \frac{1}{1/6} = 6$

$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \dots \mathbb{E}(X_K)$

since each is 1/6, it will take 6 rolls (expected) untill a six is seen

3) On any given day, the probability it will be sunny is 0.8.
the probability you have a nice dinner 0.25
the probability that you go to bed early 0.5

Assume these events are independent. What is the expected number of days before all three of them happen together?

$Pr(\text{sunny}) = 0.8$
$Pr(\text{nice dinner}) = 0.25$
$Pr(\text{bed early}) = 0.5$

$Pr(X=x, Y=y, Z=z) = Pr(X=x)Pr(Y=y)Pr(Z=z)$

$$Pr(X=\text{sunny}, Y=\text{nice dinner}, Z=\text{bed early}) = Pr(\text{sunny}) \cdot Pr(\text{nice din}) \cdot Pr(\text{bed early})$$
$$= (0.8)(0.25)(0.5)$$
$$= 0.1$$

$$E(X,Y,Z) = \frac{1}{p} = \frac{1}{0.1} = 10 \text{ days}$$

6) There is a dormitory for students with n beds for n students. One night the power goes out, and because its dark, each student gets into a bed chosen uniformly at random. What is the expected number of students who end up in their own bed?

$$X = \begin{cases} 1, & \text{students in right bed} \\ 0, & \text{students in wrong bed} \end{cases}$$

If there is 1 student and 1 bed $(n=1)$, the student would have a bed. $E(x) = 1$

If there are 2 students and 2 beds $(n=2)$, the first student chooses a bed. That student is either in the right bed or wrong bed. The next student's ~~choice~~ placement is directly related (the same) as the first student - they are either in the right bed if the first student is in the right bed, or the wrong bed if the first student in the wrong bed.

$$X \begin{cases} 1, & \text{right bed} \\ 0, & \text{wrong bed} \end{cases} \quad Pr(X=1) = 0.5 \quad Pr(X=0) = 0.5$$

$E(x) = 1 \cdot (0.5) + 1 \cdot (0.5)$

$E(x) = 1$

6) cont. For any given n students and n beds,

$$X = \begin{cases} 1 \\ 2 \\ 3 \\ \cdots \\ n \end{cases}$$

$$X = X_1 + X_2 + X_3 + \cdots X_n$$

$$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \ldots \mathbb{E}(X_n) = np$$

In this case, probability, p, is $\frac{1}{n}$

$$\mathbb{E}(X) = np$$

$$\mathbb{E}(x) = n\left(\tfrac{1}{n}\right)$$

$$\mathbb{E}(x) = 1$$

7) Dependent or Independent

a) Randomly permute $(1, 2, \ldots n)$. X is the number in the first position, Y is the number in the second position.

Check $\Pr(X=x, Y=y) = \Pr(X=x)\Pr(Y=y)$

Dependent

If x is the number in the first position, and is in the set $(1, 2, \ldots n)$, the number in the second position is not x and the probability that it is not x ~~increases~~ increases.

b) Randomly pick a card out of 52. $X = 1$ if the card is 9 and $\emptyset$ otherwise. Y is 1 if the card is a heart, $\emptyset$ if otherwise

$$X \begin{cases} 1, \text{ card is } 9 \\ \emptyset, \text{ card not } 9 \end{cases} \qquad Y \begin{cases} 1, \text{ card is } \heartsuit \\ 0, \text{ } \emptyset \text{ otherwise} \end{cases}$$

$$\Pr(X=1, Y=1) = \Pr(X=1)\Pr(Y=1)$$

$$1/52 = (1/4)(1/4)$$

$$\frac{1}{52} \neq \frac{1}{16} \quad \text{dependent}$$

#8 A die has 6 sides with different probabilities:

$Pr(1) = Pr(2) = Pr(3) = Pr(4) = 1/8$

$Pr(5) = Pr(6) = 1/4$

a) Z is outcome after 1 roll. Find $\mathbb{E}(Z)$ & $var(z)$

z	1	2	3	4	5	6
Pr(Z=z)	1/8	1/8	1/8	1/8	1/4	1/4

$\mathbb{E}(Z) = 1 \cdot 1/8 + 2 \cdot 1/8 + 3 \cdot 1/8 + 4 \cdot 1/8 + 5 \cdot 1/4 + 6 \cdot 1/4$

$\mathbb{E}(Z) = 1/8 + 2/8 + 3/8 + 4/8 + 10/8 + 12/8 = 32/8$

$\mathbb{E}(Z) = 4$

Z	1	2	3	4	5	6
Pr(z)	1/8	1/8	1/8	1/8	2/8	2/8
z^2	1	4	9	16	25	36
$(z-\mu)^2$	9	4	1	0	1	4

$\mu = \mathbb{E}(z)$

$var(z) = \mathbb{E}(z-\mu)^2$

$= \frac{1}{8} \cdot 9 + \frac{1}{8} \cdot 4 + \frac{1}{8} \cdot 1 + \frac{1}{8} \cdot 0 + \frac{2}{8} \cdot 1 + \frac{2}{8} \cdot 4$

$var(z) = 3$

b) Roll die 10 times, X is sum of rolls. $\mathbb{E}(X)$ & $var(X)$?

$\mathbb{E}(X) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \mathbb{E}(X_3) + \dots \mathbb{E}(X_{10})$

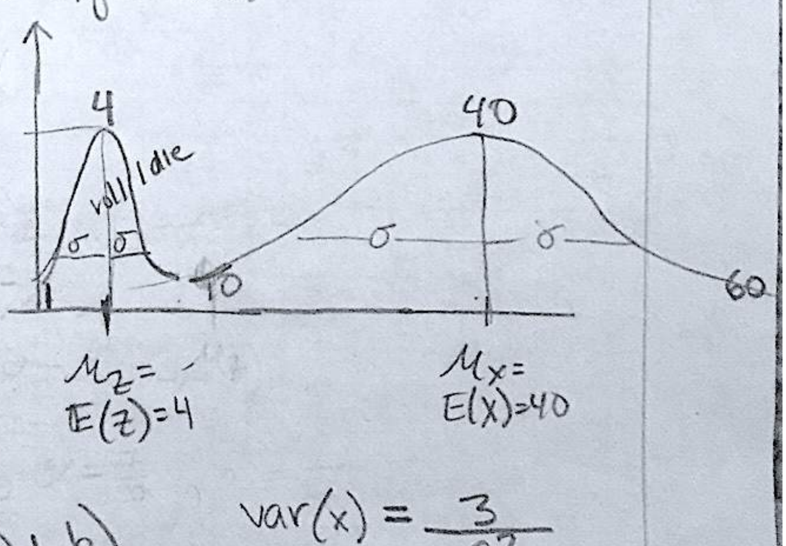
linearity of expectation

$\mathbb{E}(X) = 10 \cdot \mathbb{E}(Z)$

$\mathbb{E}(X) = 40$

$\mu_X = 10 \mu_Z$

$\mathbb{E}(X) = 10 \mathbb{E}(Z)$



$var(aX+b) = a^2 var(X)$

$var(X) = \frac{var(aX+b)}{a^2}$

if $Z = aX+b$, $X = \frac{Z-b}{a}$

$var(X) = \frac{var\left(a \cdot \left(\frac{Z-b}{a}\right) + b\right)}{a^2}$

$var(X) = \frac{var(Z)}{a^2}$

$var(X) = \frac{3}{a^2}$

$a^2 = 1/10$

$var(X) = 30$

#8 c) You roll the dice n times and take an average of all the rolls: call this A. What is $\mathbb{E}(A)$? What is $var(A)$

$$\mathbb{E}(X) = \sum_{i=1}^{n} \mathbb{E}(X_i)$$

$$\mathbb{E}(X) = n \cdot \mathbb{E}(Z)$$

$$\mathbb{E}(x) = 4n$$

$$\mathbb{E}(A) = \frac{\mathbb{E}(X)}{n} = \frac{4n}{n}$$

$$\mathbb{E}(A) = 4$$

$$var(x) = \frac{var(Z)}{a^2}$$

$$a^2 = \frac{1}{n}$$

$$var(x) = \frac{3}{n}$$

12) Suppose a fair coin is tossed repeatedly until the same outcome occurs twice in a row (2 heads in a row or two tails in a row). What is the expected number of tosses?

Fair coin means:

$$Pr(\text{heads}) = 0.50$$
$$Pr(\text{tails}) = 0.50$$

Let X_1 be the first toss $X_1 \begin{cases} 1, & \text{either heads or tails} \\ 0, & \text{not possible} \end{cases}$

$\mathbb{E}(X_1) = 1$ which means that whatever the toss showed is expected

Let X_2 be the second toss $X_2 \begin{cases} 1, & \text{same as first toss} \\ 0, & \text{not the same as first toss} \end{cases}$

$$Pr(X_2 = 1) = 1/2$$

$$\mathbb{E}(X_2) = 2$$

$$\mathbb{E}(X) = E(X_1) + E(X_2)$$

$$\mathbb{E}(X) = 1 + 2 = 3$$

Generalizing, any toss X_i and subsequent toss X_{i+1} will be $\mathbb{E}(X_i) = 1$

$$\mathbb{E}(X_{i+1}) = 2$$