CSE 103: Probability and statistics **Fall 2010**

Topic 3 — Random variables, expectation, and variance, I

3.1 Random variables

A *random variable* (r.v.) is defined on a probability space $(\Omega, \Pr)$ and is a mapping from Ω to $\mathbb{R}$.

The value of the random variable is fully determined by the outcome $\omega \in \Omega$. Thus the underlying probability space (probabilities $\Pr(\omega)$) induces a probability distribution over the random variable. Let's look at some examples.

Suppose you roll a fair die. The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$, all outcomes being equally likely. On this space we can then define a random variable

$$X = \begin{cases} 1 & \text{if die is} \geq 3 \\ 0 & \text{otherwise} \end{cases}$$

In other words, the outcomes $\omega = 1, 2$ map to $X = 0$, while the outcomes $\omega = 3, 4, 5, 6$ map to $X = 1$. The r.v. X takes on values $\{0, 1\}$, with probabilities $\Pr(X = 0) = 1/3$ and $\Pr(X = 1) = 2/3$.

Or say you roll this same die n times, so that the sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}^n$. Examples of random variables on this larger space are

$$\begin{aligned} X &= \text{the number of 6's rolled,} \\ Y &= \text{the number of 1's seen before the first 6.} \end{aligned}$$

The sample point $\omega = (1, 1, 1, 1, \ldots, 1, 6)$, for instance, would map to $X = 1, Y = n - 1$. The variable X takes values in $\{0, 1, 2, \ldots, n\}$, with

$$\Pr(X = k) = \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k}$$

(do you see why?).

As a third example, suppose you throw a dart at a dartboard of radius 1, and that it lands at a random location on the board. Define random variable X to be the distance of the dart from the center of the board. Now X takes values in $[0, 1]$, and for any x in this range, $\Pr(X \leq x) = x^2$.

Henceforth, we'll follow the convention of using capital letters for r.v.'s.

3.2 The mean, or expected value

For a random variable X that takes on a finite set of possible values, the *mean*, or *expected value*, is

$$\mathbb{E}(X) = \sum_x x \Pr(X = x)$$

(where the summation is over all the possible values x that X can have). This is a direct generalization of the notion of *average* (which is typically defined in situations where the outcomes are equally likely). If X

is a continuous random variable, then this summation needs to be replaced by an equivalent integral; but we'll get to that later in the course.

Here are some examples.

1. *Coin with bias (heads probability) p.*

   Define X to be 1 if the outcome is heads, or 0 if it is tails. Then

   $$\mathbb{E}(X) \;=\; 0 \cdot \Pr(X=0) + 1 \cdot \Pr(X=1) \;=\; 0 \cdot (1-p) + 1 \cdot p \;=\; p.$$

   Another random variable on this space is X^2, which also takes on values in $\{0,1\}$. Notice that $X^2 = X$, and in fact $X^k = X$ for all $k = 1,2,3,\ldots$! Thus, $\mathbb{E}(X^2) = p$ as well. This simple case shows that in general, $\mathbb{E}(X^2) \neq \mathbb{E}(X)^2$.

2. *Fair die.*

   Define X to be the outcome of the roll, so $X \in \{1,2,3,4,5,6\}$. Then

   $$\mathbb{E}(X) \;=\; 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} \;=\; 3.5.$$

3. *Two dice.*

   Let X be their sum, so that $X \in \{2,3,4,\ldots,12\}$. We can calculate the probabilities of each possible value of X and tabulate them as follows:

x	2	3	4	5	6	7	8	9	10	11	12
$\Pr(X=x)$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

   This gives $\mathbb{E}(X) = 7$.

4. *Roll n die; how many sixes appear?*

   Let X be the number of 6's. We've already analyzed the distribution of X, so

   $$E(X) \;=\; \sum_{k=0}^{n} k \Pr(X=k) \;=\; \sum_{k=0}^{n} k \binom{n}{k} \left(\frac{1}{6}\right)^k \left(\frac{5}{6}\right)^{n-k} \;=\; \frac{n}{6}.$$

   The last step is somewhat mysterious; just take our word for it, and we'll get back to it later!

5. *Toss a fair coin forever; how many tosses to the first heads?*

   Let $X \in \{1,2,\ldots\}$ be the number of tosses until you first see heads. Then

   $$\Pr(X=k) \;=\; \Pr((T,T,T,\ldots,T,H)) \;=\; \frac{1}{2^k}.$$

   It follows that

   $$\mathbb{E}(X) \;=\; \sum_{k=1}^{\infty} \frac{k}{2^k} \;=\; 2.$$

   We saw in class how to do this summation. The technique was based on the formula for the sum of a geometric series: if $|r| < 1$, then

   $$a + ar + ar^2 + \cdots \;=\; \frac{a}{1-r}.$$

6. *Toss a coin with bias p forever; how many tosses to the first heads?*

   Once again, $X \in \{1, 2, \ldots\}$, but this time the distribution is different:

$$\Pr(X = k) \;=\; \Pr((T, T, T, \ldots, T, H)) \;=\; (1-p)^{k-1}p.$$

   Using the same technique as before, we get $\mathbb{E}(X) = 1/p$.

   There's another way to derive this expectation. We always need at least one coin toss. If we're lucky (with probability p), we're done; otherwise (with probability $1-p$), we start again from scratch. Therefore $\mathbb{E}(X) = 1 + (1-p)\mathbb{E}(X)$, so that $\mathbb{E}(X) = 1/p$.

7. *Pascal's wager: does God exist?*

   Here was Pascal's take on the issue of God's existence: if you believe there is some chance $p > 0$ (no matter how small) that God exists, then you should behave as if God exists.

   Why? Well, let the random variable X denote your amount of suffering.

   Suppose you behave as if God exists (that is, you are good). This behavior incurs a significant but finite amount of suffering (you are not able to do some of the things you would like to). Say $X = 10$.

   On the other hand, suppose you behave as if God doesn't exist – that is, you do all the things you want to do. If God really doesn't exist, you're fine, and your suffering is $X = 0$. But if God exists, then you go straight to hell and your suffering is $X = \infty$. Thus your *expected* suffering if you behave badly is $\mathbb{E}(X) = 0 \cdot (1-p) + \infty \cdot p = \infty$.

   So: to minimize your expected suffering, behave as if God exists!