assuming no flow perpendicular to the top and bottom of the aquifer, we obtain

$$\frac{\partial}{\partial t}\int_0^H \phi s_\alpha \,\mathrm{d}z + \nabla_\parallel \cdot \int_0^H \mathbf{u}_\alpha^H \,\mathrm{d}z = \int_0^H q_\alpha \,\mathrm{d}z, \tag{2}$$

where $\mathbf{u}_\alpha^H = (u_\alpha^x, u_\alpha^y)$ and $\nabla_\parallel = (\partial/\partial x, \partial/\partial y)$ are two-dimensional vectors in the aquifer plane. The second term on the left hand side includes the vertical integral of the horizontal velocity of the fluid. Applying the generalized Darcy's law we have that $\mathbf{u}_\alpha^H = -k\lambda_\alpha\left(\nabla_\parallel p_\alpha - \rho_\alpha \mathbf{g}^H\right)$, so that,

$$\int_0^H \mathbf{u}_\alpha^H \,\mathrm{d}z = -\int_0^H k\lambda_\alpha(\nabla_\parallel p_\alpha - \rho_\alpha \mathbf{g}^H)\,\mathrm{d}z. \tag{3}$$

Here k is the permeability of the medium, λ_α and ρ_α are the mobility and density of phase α, respectively; and $\mathbf{g}^H$ is the projection of gravity onto the aquifer plane. To evaluate (3), we assume that [3]: i) the velocity component perpendicular to the aquifer plane is very small, and ii) the fluid density in each phase is constant. Hence the fluids are in hydrostatic equilibrium in the vertical direction. Then, pressure in each fluid phase can be written in terms of the fluid pressure at the top of the aquifer and the elevation of the of the top of the aquifer (z_T), i.e. we take the caprock surface as a datum level to measure fluid pressures. Then, the pressure gradient in the aquifer plane can be evaluated as,

$$\nabla_\parallel p_\alpha = \nabla_\parallel P_\alpha - g_z \rho_\alpha \nabla_\parallel z_T. \tag{4}$$

Next, we define the set of vertically integrated variables and parameters listed in Table 2. Substituting (4)

Table 1: Vertically-averaged variables and parameters.

Parameter	**Expression**	**Parameter**	**Expression**
Gravity	$\mathbf{G} = g_z \nabla_\parallel z_T + \mathbf{g}^H$	Velocities	$\mathbf{U}_\alpha = \frac{1}{H}\int_0^H \mathbf{u}_\alpha^H \,\mathrm{d}z$
Porosity	$\Phi = \frac{1}{H}\int_0^H \phi \,\mathrm{d}z$	Saturations	$S_\alpha = \frac{1}{\Phi H}\int_0^H \phi s_\alpha \,\mathrm{d}z$
Permeability	$K = \frac{1}{H}\int_0^H k \,\mathrm{d}z$	Pressures	$P_\alpha = p_\alpha(z_T)$
Mobilities	$\Lambda_\alpha = \frac{1}{KH}\int_0^H k\lambda_\alpha \,\mathrm{d}z$	Sources/Sinks	$Q_\alpha = \frac{1}{H}\int_0^H q_\alpha \,\mathrm{d}z$

into (3) and the vertically integrated parameters into (2), we obtain a mass conservation equation for the vertically integrated fluid saturations S_α. Table 2 shows a comparison between the original 3D equations and their vertically integrated equivalents.

Table 2: Equations that define the full 3D and 2D vertical equilibrium (VE) models.

3D	**2D**
$\frac{\partial(\phi s_\alpha)}{\partial t} + \nabla \cdot \mathbf{u}_\alpha = q_\alpha$	$\Phi \frac{\partial S_\alpha}{\partial t} + \nabla_\parallel \cdot \mathbf{U}_\alpha = Q_\alpha$
$\mathbf{u}_\alpha = -k\lambda_\alpha(\nabla p_\alpha - \rho_\alpha \mathbf{g})$	$\mathbf{U}_\alpha = -K\Lambda_\alpha\left(\nabla_\parallel P_\alpha - \rho_\alpha \mathbf{G}\right)$
$s_w + s_n = 1$	$S_w + S_n = 1$
$\lambda_\alpha = \lambda_\alpha(s_w)$	$\Lambda_\alpha = \Lambda_\alpha(S_w)$
$p_c = p_n - p_w = p_c(s_w)$	$P_c = P_n - P_w = P_c(S_w)$

The last step in the derivation of the vertically integrated model is to evaluate the vertically integrated mobilities (Λ_α) and capillary pressure (P_c) as function of the vertically integrated saturations (S_α). Assuming hydrostatic pressure distribution, so that $p_n(z) = P_n - \rho_n g_z(z_T - z)$ and $p_w(z) = P_w - \rho_w g_z(z_T - z)$, we have that by definition capillary pressure as function of elevation can be computed as [3],

$$p_c(z) = p_n(z) - p_w(z) = P_n - P_w - \Delta\rho g_z(z_T - z) \tag{5}$$

where the capillary pressure at the top of the aquifer is a function of the wetting saturation at z_T, $P_c = P_n - P_w = p_c(s_w(z_T))$. Then, given the wetting saturation at the top of the aquifer, $s_w^T = s_w(z_T)$, we can get a reconstruction of the fine scale saturation as function of z evaluation the inverse function of $p_c(z)$, to obtain,

$$\hat{s}_\alpha(z) = p_c^{-1}(p_c(z; s_w^T)) \tag{6}$$

Notice that $\hat{s}_w(z)$ is not the true fine scale saturation but the one by assuming hydrostatic fluid pressure distribution in the vertical direction. Now, the vertically integrated constitutive relations can be directly computed by evaluating, $S_\alpha = S_\alpha(\hat{s}_\alpha(s))$, $\Lambda_\alpha = \frac{1}{KH}\int_0^H k\lambda_\alpha(\hat{s}_w)\,\mathrm{d}z$ and $P_c = p_c(s_w^T)$.