MA261 Assignment 1 2106983

1.1 (1) $y' = f(y) = \lambda y$

$y_{n+1} = y_n + \frac{h}{2}(f(y_n) + f(y_n + hf(y_n)))$

$= y_n + \frac{h}{2}(\lambda y_n + \lambda y_n + \lambda h f(y_n))$

$= y_n + \frac{h}{2}(2\lambda y_n + h\lambda^2 y_n)$

$= y_n + \lambda h y_n + \frac{h^2\lambda^2 y_n}{2}$

$y_{n+1} = \left(1 + h\lambda + \frac{h^2\lambda^2}{2}\right) y_n$

Stable if $\left|1 + h\lambda + \frac{h^2\lambda^2}{2}\right| < 1$

$\Leftrightarrow$

$-1 < 1 + h\lambda + \frac{h^2\lambda^2}{2} < 1 \Leftrightarrow$

$-4 < 2h\lambda + h^2\lambda^2 < 0 \Leftrightarrow$

$\frac{-2}{h} < \lambda + \frac{h\lambda^2}{2} < 0 \Leftrightarrow$

$\frac{-2}{h} < \lambda\left(1 + \frac{h\lambda}{2}\right) < 0 \Leftrightarrow$

$\Rightarrow \frac{-2}{h} < \lambda < 0$ so method stable for $\lambda > \frac{-2}{h}$ //

(2) Solve inequality $\left|1 + h\lambda + \frac{h^2\lambda^2}{2}\right| < 1$

$\Leftrightarrow \left|1 + hi\alpha - \frac{h^2\alpha^2}{2}\right| < 1$

$\Leftrightarrow \sqrt{\left(1 - \frac{h^2\alpha^2}{2}\right)^2 + h^2\alpha^2} < 1$

$\Rightarrow \left(1 - \left(\frac{h^2\alpha^2}{2}\right)\right)^2 + h^2\alpha^2 < 1$

$\Leftrightarrow 1 - h^2\alpha^2 + \frac{h^4\alpha^4}{4} + h^2\alpha^2 < 1.$

$\Leftrightarrow \frac{h^4\alpha^4}{4} < 0$

There are no values for which this inequality holds

but if $\alpha = 0$, $\lambda = 0$, $y' = 0$ so $y = C \in \mathbb{R}$ which is bounded

So method is A stable for $\alpha = 0$ only.

1.2

(i) $\underline{Y}(t+h) = \underline{Y}(t) + \frac{h}{2}\left(f(\underline{Y}(t)) + f(\underline{Y}(t) + hf(\underline{Y}(t))\right) + T_n$ by defn of T_n

Taylor expand around $\underline{Y}(t)$

$= \underline{Y}(t) + \frac{h}{2}\left(f(\underline{Y}(t)) + f(\underline{Y}(t)) + f'(\underline{Y}(t))hf(\underline{Y}(t)) + \frac{1}{2}f''(\varepsilon_1)(hf(\underline{Y}(t)))^2\right.$

$+ T_n$ for $\varepsilon_1 \in (\underline{Y}(t), \underline{Y}(t) + hf(\underline{Y}(t)))$

$= \underline{Y}(t) + hf(\underline{Y}(t)) + \frac{h^2}{2}f'(\underline{Y}(t))\underline{Y}'(t) + \frac{h^3}{4}f''(\varepsilon_1)(\underline{Y}'(t))^2 + T_n$

By Taylor expansion

$\underline{Y}(t+h) = \underline{Y}(t) + h\underline{Y}'(t) + \frac{h^2}{2}\underline{Y}''(t) + \frac{h^3}{6}\underline{Y}'''(\varepsilon_2)$ for $\varepsilon_2 \in (t, t+h)$

Comparing equations and given that $\underline{Y}''(t) = f'(\underline{Y}(t))\underline{Y}'(t)$ we have

$$T_n = \frac{h^3}{6}\underline{Y}'''(\varepsilon_2) - \frac{h^3}{4}f''(\varepsilon_1)(\underline{Y}'(t))^2$$

$$= h^3\left(\frac{1}{6}\underline{Y}'''(\varepsilon_2) - \frac{1}{4}f''(\varepsilon_1)(\underline{Y}'(t))^2\right)$$

$$\leq h^3\left(\frac{C_3}{6} - \frac{L}{4}C_1^2\right)$$ as $\underline{Y} \in C^3$ and f is Lipschitz so derivates are bounded.

$\Rightarrow T_n = O(h^3)$

(2) $e_{n+1} = |y_{n+1} - \underline{y}(t_{n+1})|$

$= |y_n + \frac{h}{2}(f(y_n) + f(y_n + hf(y_n))) - (\underline{y}(t_n) + \frac{h}{2}(f(\underline{y}(t_n)) +$
$f(\underline{y}(t_n) + hf(\underline{y}(t_n))) + T_n)|$

$= |y_n - \underline{y}(t_n) + \frac{h}{2}(f(y_n) - f(\underline{y}(t_n))) + \frac{h}{2}(f(y_n + hf(y_n)) - f(\underline{y}(t_n) +$
$hf(\underline{y}(t_n)))) - T_n|$

Δineq.
$\leq |y_n - \underline{y}(t_n)| + \frac{h}{2}|f(y_n) - f(\underline{y}(t_n))| + \frac{h}{2}|f(y_n + hf(y_n)) - f(\underline{y}(t_n) +$
$hf(\underline{y}(t_n)))| + |T_n|$

f lipschitz
$\leq e_n + \frac{h}{2}L_f|y_n - \underline{y}(t_n)| + \frac{h}{2}L_f|y_n - \underline{y}(t_n) + hf(y_n) - hf(\underline{y}(t_n))|$
$+ |T_n|$

Δineq and f lipschitz
$\leq e_n + \frac{h}{2}L_f e_n + \frac{h}{2}L_f(e_n + hL_f e_n) + |T_n|$

$= e_n + \frac{h}{2}L_f e_n + \frac{h}{2}L_f e_n + \frac{h^2 L_f^2}{2} e_n + |T_n|$

$= (1 + hL_f + \frac{h^2}{2}L_f^2)e_n + |T_n|$

So $R(\mu) = 1 + \mu + \frac{\mu^2}{2}$

Using useful version of the Gronwall lemma. $\left(e_n \leq \frac{e^{cn} - 1}{c} O(h^{p})\right.$
$\left.\text{if } e_{n+1} \leq (c+1)e_n + O(h^p)\right)$

$$e_n \leq \frac{e^{n(hL_f + \frac{h^2}{2}L_f^2)} - 1}{hL_f + \frac{h^2}{2}L_f^2} O(h^3)$$

$$= \frac{e^{nhL_f + \frac{nh^2L_f^2}{2}} - 1}{L_f + \frac{hL_f^2}{2}} O(h^2) \leq \frac{e^{L_f T + \frac{L_f^2 T h}{2}} - 1}{L_f + \frac{hL_f^2}{2}} O(h^2)$$

$\Rightarrow e_n = O(h^2)$ ~~$\frac{L_f^2 h}{2}$~~

as $$\frac{e^{L_f T + \frac{L_f^2 T h}{2}} - 1}{L_f + \frac{h L_f^2}{2}} \longrightarrow \frac{e^{L_f T} - 1}{L_f} \text{ as } h \to 0$$

1.3 stable if $|R(\mu)| < 1 \quad \forall \mu \in \mathbb{R}$ however polynomials tend to $\pm\infty$ for $\mu \longrightarrow \pm\infty$ so $\exists \mu \in \mathbb{R}$ st. $|R(\mu)| \geq 1$ and hence the approximation is never unconditionally stable.

Follows as $\mu = \lambda h$ so if $\exists \mu \in \mathbb{R}$ st. it is unstable then $\exists \lambda \in \mathbb{R}$ namely μ/h st it is unstable. //