MA261 Assignment 3

1.1

(1)

$$1 - \frac{B}{K} \Rightarrow [K] = [B] = \text{fish}$$

$$\alpha^2 + B^2 \Rightarrow [\alpha]^2 = [B]^2 \Rightarrow [\alpha] = \text{fish}$$

$$rB\left(1 - \frac{B}{K}\right) - \beta \frac{B^2}{\alpha^2 + B^2} \Rightarrow [rB] = [\beta]$$

$$B'(t) = RHS \Rightarrow \frac{[B]}{[t]} = [rB] = [\beta]$$

so $[\beta] = \frac{[B]}{[t]} = \text{fish week}^{-1}$, $[r] = [t]^{-1} = \text{week}^{-1}$

(2) $\tau = \frac{t}{T}$, $x(\tau) = \frac{B(\tau T)}{F} = \frac{B(t)}{F}$

$$B'(t) = \frac{d}{dt} F x\left(\frac{t}{T}\right) = \frac{F}{T} x'(\tau)$$

So $B'(t) = rB\left(1 - \frac{B}{K}\right) - \beta \frac{B^2}{\alpha^2 + B^2}$

$$\Rightarrow \frac{F}{T} x'(\tau) = Frx\left(1 - \frac{F}{K}x\right) - \beta \frac{F^2 x^2}{\alpha^2 + F^2 x^2}$$

$$\Rightarrow x' = Trx\left(1 - \frac{F}{K}x\right) - T\beta \frac{F x^2}{\alpha^2 + F^2 x^2}$$

$$\Rightarrow x' = Trx\left(1 - \frac{F}{K}x\right) - \frac{T\beta}{F} \frac{x^2}{\frac{\alpha^2}{F^2} + x^2} = x(1-x) - \Pi_1 \frac{x^2}{\Pi_2 + x^2}$$

$Tr = 1 \Rightarrow T = \frac{1}{r}$, $\frac{F}{K} = 1 \Rightarrow F = K$

$\Pi_1 = \frac{T\beta}{F} = \frac{\beta}{rK}$, $\Pi_2 = \frac{\alpha^2}{F^2} = \frac{\alpha^2}{K^2}$

$$[\Pi_1] = \frac{[\beta]}{[r][K]} = \frac{\text{fish week}^{-1}}{\text{week}^{-1}\,\text{fish}} = 1$$

$$[\Pi_2] = \frac{[\alpha]^2}{[K]^2} = \frac{\text{fish}^2}{\text{fish}^2} = 1$$

$$(3) \quad x' = T r x\left(1 - \frac{F}{K}x\right) - \frac{T\beta}{F}\frac{x^2}{\frac{\alpha^2}{F^2}+x^2} = \Pi_3 x(1 - \Pi_4 x) - \frac{x^2}{1+x^2}$$

$$\frac{\alpha^2}{F^2} = 1 \Rightarrow F = \alpha$$

$$\frac{T\beta}{F} = 1 \Rightarrow T = \frac{F}{\beta} = \frac{\alpha}{\beta}$$

$$\Pi_3 = Tr = \frac{\alpha r}{\beta}, \quad \Pi_4 = \frac{F}{K} = \frac{\alpha}{K}$$

$$[\Pi_3] = \frac{[\alpha][r]}{[\beta]} = \frac{\text{fish week}^{-1}}{\text{fish week}^{-1}} = 1, \quad [\Pi_4] = \frac{[\alpha]}{[K]} = \frac{\text{fish}}{\text{fish}} = 1$$

1.2

(1)

$$\underline{Y}(t_{n+1}) = \exp(D_n(t_{n+1}-t_n))\underline{Y}(t_n) + \int_{t_n}^{t_{n+1}} \exp(D_n(t_{n+1}-\tau))\, g_n(\underline{Y}(\tau))\, d\tau$$

$\Leftrightarrow$

$$y_{n+1} = \exp(D_n h) y_n + \int_{t_n}^{t_{n+1}} \exp(D_n(t_{n+1}-\tau))\, \cancel{g_n(y_n)}\, g_n(y_n)\, d\tau$$

$\Leftrightarrow$

$$y_{n+1} = \exp(D_n h) y_n + \left(\int_{t_n}^{t_{n+1}} \exp(D_n(t_{n+1}-\tau))\, d\tau\right) g_n(y_n)$$

$\Leftrightarrow$

$$= \exp(D_n h) y_n + \Big[-D_n^{-1}\left(\exp(D_n(t_{n+1}-\tau))\right)\Big]_{t_n}^{t_{n+1}} g_n(y_n)$$

$$= \exp(D_n h) y_n + \left(-D_n^{-1} + D_n^{-1}\exp(D_n h)\right) g_n(y_n)$$

$$= \exp(D_n h) y_n + \left(-D_n^{-1} + D_n^{-1}\exp(D_n h)\right)\left(f(y_n) - D_n y_n\right)$$

$$= \exp(D_n h) y_n - D_n^{-1} f(y_n) + D_n^{-1}\exp(D_n h) f(y_n) + y_n - \exp(D_n h) y_n$$

$$= y_n + \underset{\exp(D_n h) D_n^{-1}}{\cancel{D_n^{-1}\exp(D_n h)}} f(y_n) - D_n^{-1} f(y_n)$$

$$= y_n + \left(\exp(D_n h) - I_m\right) D_n^{-1} f(y_n)$$

$$= y_n + \left(\exp(h d_y f(y_n)) - I_m\right)\left(d_y f(y_n)\right)^{-1} f(y_n) \quad //$$

(2) $f(y) = Ay \qquad D_n := d_y f(\cancel{y_n}\ \underline{Y}(t_n)) = A$

$$y_{n+1} = y_n + \left(\exp(hA) - I_m\right) A^{-1} A y_n$$

$$= y_n + \left(\exp(hA) - I_m\right) y_n = \left(I_m + \exp(hA) - I_m\right) y_n$$

$$= \exp(hA) y_n$$

~~Next~~ $y' = Ay$ so $\underline{Y}(t) = \exp(At)$

$$e_n := |y_n - \underline{Y}(t_n)| = |\exp(hA) y_{n-1} - \exp(At_n)| \qquad t_n - t_{n-1} = h$$

$$= |\exp(hA) y_{n-1} - \exp(A(h + t_{n-1}))|$$

$$= |\exp(hA) y_{n-1} - \exp(\overset{hA}{\cancel{AA}}) \exp(At_{n-1})|$$

$$= \exp(hA) |y_{n-1} - \underline{Y}(t_{n-1})|$$

$$= \exp(hA) e_{n-1}$$

Iterating we see

$$e_n = (\exp(hA))^n e_0 = (\exp(hA))^n |y(0) - \underline{Y}(0)| = 0$$

So as e_n equals 0 $\quad E(h) := \max\limits_{0 \le k \le N} e_k = 0$

So method is exact.

Q3) 1.3

(1) $H(y) = c \cdot y \qquad \nabla H(y) = c$

$$\nabla H(y) \cdot f(y) = 0 \quad \forall y \in \mathbb{R}^m$$
$$\Leftrightarrow \quad c \cdot f(y) = 0$$

$$H(y_{n+1}) = c \cdot y_{n+1} = c \cdot \left(y_n + h \sum_{i=1}^{s} \gamma_i k_i\right)$$
$$= c \cdot y_n + h \sum_{i=1}^{s} \gamma_i \, c \cdot k_i$$
$$= H(y_n) + h \sum_{i=1}^{s} \gamma_i \underbrace{c \cdot f(\tilde{y}_i)}_{=0} \quad \text{for some } \tilde{y}_i \in \mathbb{R}^m \text{ for each } i.$$
$$= H(y_n) \quad \text{as } c \cdot f(y) = 0$$

Iterating we see that $H(y_{n+1}) = H(0)$ so $H(y_n) = H(0)$ //

(2) $H(y) = y \cdot Cy \qquad \nabla H(y) = (C + C^T) y$

$$\nabla H(y) \cdot f(y) = 0$$
$$\Leftrightarrow \quad (C + C^T) y \cdot f(y) = 0$$
$$\Leftrightarrow \quad y \cdot (C + C^T) f(y) = 0$$

$$H(y_{n+1}) = y_{n+1} \cdot C y_{n+1} = \left(y_n + h \sum_{i=1}^{s} \gamma_i k_i\right) \cdot C \left(y_n + h \sum_{i=1}^{s} \gamma_i k_i\right)$$
$$= y_n \cdot C y_n + y_n \cdot (C + C^T) h \sum_{i=1}^{s} \gamma_i k_i + h \sum_{i=1}^{s} \gamma_i k_i \cdot C h \sum_{i=1}^{s} \gamma_i k_i$$
$$= H(y_n) + h \sum_{i=1}^{s} \gamma_i \, y_n \cdot (C + C^T) k_i + h^2 \sum_i \sum_j \gamma_i \gamma_j \, k_j \cdot C k_i$$
$$= H(y_n) + h \sum_{i=1}^{s} \gamma_i \left(Y_{n,i} - h \sum_{j=1}^{s} B_{i,j} k_j\right) \cdot (C + C^T) k_i + h^2 \sum_i \sum_j \gamma_i \gamma_j \, k_j \cdot C k_i$$
$$= H(y_n) - h^2 \sum_{i} \sum_{j} \gamma_i B_{i,j} k_j \cdot (C + C^T) k_i + h^2 \sum_i \sum_j \gamma_i \gamma_j \, k_j \cdot C k_i$$
$$= H(y_n) + h^2 \left(\sum_i \sum_j \gamma_i \gamma_j \, k_j \cdot C k_i - \gamma_i B_{i,j} k_j \cdot (C + C^T) k_i \right)$$

$= H(y_n) + h^2\left(\sum_{i}\sum_{j} \gamma_i \gamma_j k_j \cdot Ck_i - \gamma_i B_{i,j} k_j \cdot Ck_i - \gamma_i B_{i,j} k_j \cdot C^T k_i\right)$

$= H(y_n) + h^2\left(\sum_{i}\sum_{j} (\gamma_i \gamma_j - \gamma_i B_{i,j}) k_j \cdot Ck_i - \gamma_i B_{i,j} k_j \cdot C^T k_i\right)$

$= H(y_n) + h^2\left(\sum_{i}\sum_{j} \gamma_j B_{j,i} k_j \cdot Ck_i - \gamma_i B_{i,j} k_j \cdot C^T k_i\right)$

As $k_j \cdot C^T k_i = k_j^T C^T k_i$
$= k_i^T C k_j$
$= k_i \cdot Ck_j$ //

$= H(y_n) + h^2 \sum_{i}\sum_{j} \gamma_j B_{j,i} k_j \cdot C k_i - \gamma_i B_{i,j} k_i \cdot Ck_j$

$= H(y_n) + h^2 \sum_{i}\sum_{j} \gamma_j B_{j,i} k_j \cdot Ck_i - h^2 \sum_{i}\sum_{j} \gamma_i B_{i,j} k_i \cdot Ck_j$

$= H(y_n) + h \sum_{j=1}^{s} \gamma_j (Y_{n,j} - y_n) \cdot Ck_i - h \sum_{i=1}^{s} \gamma_i k_i \cdot C(Y_{n,i} - y_n)$, $Y_{n,i} - y_n = h\sum_{j=1}^{s} B_{i,j} k_j$

$j = i$ in first sum

$= H(y_n) + h \sum_{i=1}^{s} \gamma_i (Y_{n,i} - y_n) \cdot Ck_i - \gamma_i k_i \cdot C(Y_{n,i} - y_n)$

$= H(y_n)$

So iterating $H(y_{n+1}) = H(0)$ so $H(y_n) = H(0)$ //

$\gamma_i B_{i,j} + \gamma_j B_{j,i} = \gamma_i \gamma_j$

(3)

$j \backslash i$	1	2
1	$\frac{d}{2} + \frac{d}{2} = \frac{1}{4}$ *	$\frac{1}{2}(1-2d) + 0\frac{1}{2} = \frac{1}{4}$
2	$\frac{1}{2}0 + \frac{1}{2}(1-2d) = \frac{1}{4}$	$\frac{d}{2} + \frac{d}{2} = \frac{1}{4}$

* $d = \frac{1}{4}$, $\frac{1}{2}(1-2d) = \frac{1}{4} \Rightarrow 1 - 2d = \frac{1}{2} \Rightarrow d = \frac{1}{4}$

So $d = \frac{1}{4}$ is the only value such that the method has quadratic invariance.