MA261 Assignment 4

1.1

i. $k=2$ so 2 step method

method converges $\iff$ consistent and zero-stable

zero-stable if first characteristic polynomial satisfies root condition which is if $\rho(z)=0$, either $|z|<1$ or $|z|=1$ and $\rho'(z)\neq 0$

$\rho(z) = z^2+az+b$ for $s=0,1,2$

$$\sigma(z) = \begin{cases} c & s=0 \\ cz & s=1 \\ cz^2 & s=2 \end{cases}$$

consistent if $\rho(1)=0$ and $\sigma(1)=\rho'(1)$

$\rho(1)=0 \iff a+b+1=0$ so if $a+b=-1$

$\sigma(1) = c$ for all s

$\rho'(1) = 2+a$ so $\sigma(1)=\rho'(1) \iff c=2+a$

so consistent if $a+b=-1$, $c-a=2$ for all s.

zero-stable: $\rho(z) = z^2+az+b = (z-1)(z-b)$ assuming consistency

zero-stable if $|b|<1$ or $|b|=1$, $\rho'(b)\neq 0$, $\rho'(z) = 2z+a$

Case 1: $b=-1, a=0$ $\rho'(b)=-2\neq 0$ so valid

Case 2: $b=1, a=-2$ $\rho'(b)=0$ so not valid.

so zero-stable if $-1\leq b<1$

as ~~the~~ s has no effect on first characteristic polynomial which determines root condition these conditions are the same for all s again

so conditions for convergence are

$a+b=-1$, $c-a=2$, and $-1 \le b < 1$ for all s//.

ii. conditions in (i) ensure convergence and hence consistency of order 1.

To ensure consistency of order 2 we need $2+\frac{\alpha_1}{2}-2\beta_2-\beta_1=0$

$s=0$: $2+\frac{a}{2}-0-0=0 \Rightarrow a=-4$ so $b=3$ so no longer zero-stable so no method exists for $s=0$.//

$s=1$: $2+\frac{a}{2}-0-c=0 \Rightarrow a=2(c-2)=2a \Rightarrow a=0, b=-1, c=2$

so $(a,b,c)=(0,-1,2)$ gives a method that converges and has consistency order of at least 2. for $s=1$//

$s=2$: $2+\frac{a}{2}-2c-0=0 \Rightarrow a=4c-4$
$\iff c-2=4c-4$
$\iff 2=3c \Rightarrow c=\frac{2}{3}, a=-\frac{4}{3}, b=\frac{1}{3}$/

so $(a,b,c)=\left(-\frac{4}{3},\frac{1}{3},\frac{2}{3}\right)$ gives a method that converges and has consistency order of at least 2. for $s=2$//

1.2

i. $x''(t) + x(t) + \varepsilon(x(t)^2 - 1)x'(t) = 0 \quad x(0) = x_0,\ x'(0) = 0$

$x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2),\ x(0) = x_0 \Rightarrow x_0(0) = x_0,\ x_1(0) = 0$

$x'(t) = x_0'(t) + \varepsilon x_1'(t) + O(\varepsilon^2),\ x'(0) = 0 \Rightarrow x_0'(0) = 0,\ x_1'(0) = 0$

$x''(t) = x_0''(t) + \varepsilon x_1''(t) + O(\varepsilon^2)$

$x'' + x + \varepsilon(x^2 - 1)x' = 0$

$\Leftrightarrow$

$x_0'' + \varepsilon x_1'' + x_0 + \varepsilon x_1 + \varepsilon((x_0 + \varepsilon x_1)^2 - 1)(x_0' + \varepsilon x_1') + O(\varepsilon^2) = 0$

$\Leftrightarrow$

$x_0'' + \varepsilon x_1'' + x_0 + \varepsilon x_1 + \varepsilon(x_0^2 + 2\cdot\varepsilon x_0 x_1 - 1)(x_0' + \varepsilon x_1') + O(\varepsilon^2) = 0$

$\Leftrightarrow$

$x_0'' + \varepsilon x_1'' + x_0 + \varepsilon x_1 + \varepsilon x_0^2 x_0' - \varepsilon x_0' + O(\varepsilon^2) = 0$

so

$x_0'' + x_0 = 0, \quad x_0(0) = x_0,\ x_0'(0) = 0$ and

$x_1'' + x_1 + x_0^2 x_0' - x_0' = 0 \quad x_1(0) = 0,\ x_1'(0) = 0$ //

ii. $x_0(t) = A\cos(t) + B\sin(t)$

$x_0(0) = x_0 \Rightarrow A = x_0,\ x_0'(0) = 0 \Rightarrow B = 0$ so

$x_0(t) = x_0 \cos t$ //, $x_0'(t) = -x_0 \sin t$

$x_1'' + x_1 + x_0^2 x_0' - x_0' = 0 \Leftrightarrow x_1'' + x_1 = x_0' - x_0^2 x_0'$

$\Leftrightarrow$

$x_1'' + x_1 = x_0^3 \cos^2 t \sin t - x_0 \sin t$

$$x_1'' + x_1 = x_0^3 \cos^2 t \sin t - x_0 \sin t$$

$$= \frac{x_0^3}{2}\left(\cos 2t + 1\right)\sin t - x_0 \sin t$$

$$= \frac{x_0^3 \cos 2t \sin t}{2} + \frac{x_0^3}{2}\sin t - x_0 \sin t$$

$$= \frac{x_0^3}{2}\left(\frac{\sin 3t - \sin t}{2}\right) + \frac{x_0^3}{2}\sin t - x_0 \sin t$$

$$= \frac{x_0^3}{4}\sin 3t + \frac{x_0^3}{4}\sin t - x_0 \sin t$$

$$= \frac{x_0^3}{4}\sin 3t + \left(\frac{x_0^3}{4} - x_0\right)\sin t \qquad \text{if } \frac{x_0^3}{4} - x_0 = 0$$

(condition on x_0)

$$= \frac{x_0^3}{4}\sin 3t$$

$$x_1 = A\cos t + B\sin t + C\cos 3t + D\sin 3t$$

$$x_1' = -A\sin t + B\cos t - 3C\sin 3t + 3D\cos 3t$$

$$x_1'' = -A\cos t - B\sin t - 9C\,\overset{\cos}{\cancel{\sin}}\,3t - 9D\sin 3t$$

$$x_1'' + x_1 = \frac{x_0^3}{4}\sin 3t \Rightarrow -8C = 0,\ -8D = \frac{x_0^3}{4}$$

$$\Rightarrow C = 0,\ D = -\frac{x_0^3}{32}$$

so $x_1 = A\cos t + B\sin t - \frac{x_0^3}{32}\sin 3t$, $x_1' = -A\sin t + B\cos t - \frac{3x_0^3}{32}\cos(3t)$

$x_1(0) = 0 \Rightarrow A = 0$, $x_1'(0) = 0 \Rightarrow B - \frac{3x_0^3}{32} = 0 \Rightarrow B = \frac{3x_0^3}{32}$

so $x_1(t) = \frac{3x_0^3}{32}\sin t - \frac{x_0^3}{32}\sin 3t$

and

$x_0(t) = x_0 \cos t$ so $x(t) = x_0\cos t + \varepsilon\left(\frac{3x_0^3}{32}\sin t - \frac{x_0^3}{32}\sin 3t\right) + O(\varepsilon)$

function ↑ number ↑

MAM 1.3

i. $y(\tau) = x\left(\frac{\tau}{\omega}\right) = x(t) = y(\omega t) \qquad \tau = \omega t$

$x'(t) = \frac{d}{dt} x(t) = \frac{d}{dt} y(\omega t) = \omega y'(\tau)$

$x''(t) = \omega^2 y''(\tau)$

so

$x''(t) + x(t) + \varepsilon(x(t)^2 - 1)x'(t) = 0$

$\Leftrightarrow$

$\omega^2 y''(\tau) + y(\tau) + \varepsilon(y(\tau)^2 - 1)\omega y'(\tau) = 0$

$\Leftrightarrow$

$\omega^2 y'' + y + \varepsilon\omega(y^2 - 1)y' = 0 \qquad y(0) = x(0) = x_0$

$\omega y'(0) = x'(0) = 0 \Rightarrow y'(0) = 0$

ii. $\omega = 1 + \varepsilon\omega_1 + O(\varepsilon^2), \quad y = y_0 + \varepsilon y_1 + O(\varepsilon^2), \quad y(0) = x_0 \Rightarrow y_0(0) = x_0, \; y_1(0) = 0$

$y' = y_0' + \varepsilon y_1' + O(\varepsilon^2) \qquad y'(0) = 0 \Rightarrow y_0'(0) = 0$

$y'' = y_0'' + \varepsilon y_1'' + O(\varepsilon^2) \qquad y_1'(0) = 0$

$\omega^2 y'' + y + \varepsilon\omega(y^2 - 1)y' = 0$

$\Leftrightarrow$

$(1 + \varepsilon\omega_1)^2(y_0'' + \varepsilon y_1'') + y_0 + \varepsilon y_1 + \varepsilon(1 + \varepsilon\omega_1)((y_0 + \varepsilon y_1)^2 - 1)(y_0' + \varepsilon y_1') + O(\varepsilon^2) = 0$

$\Leftrightarrow$

$(1 + 2\varepsilon\omega_1)(y_0'' + \varepsilon y_1'') + y_0 + \varepsilon y_1 + \varepsilon(y_0^2 + 2\varepsilon y_0 y_1 - 1)(y_0' + \varepsilon y_1') + O(\varepsilon^2) = 0$

$\Leftrightarrow$ $y_0'' + 2\varepsilon\omega_1 y_0'' + \varepsilon y_1'' + y_0 + \varepsilon y_1 + \varepsilon y_0^2 y_0' - \varepsilon y_0' + O(\varepsilon^2) = 0$

so

$y_0'' + y_0 = 0, \quad y_0(0) = x_0, \; y_0'(0) = 0$ and

$2\omega_1 y_0'' + y_1'' + y_1 + y_0^2 y_0' - y_0' = 0 \qquad y_1(0) = 0, \; y_1'(0) = 0.$

iii. so $y_0(\tau) = x_0\cos\tau$ // as same equation in 1.2(ii)

$y_0'(\tau) = -x_0\sin(\tau)$, $y_0''(\tau) = -x_0\cos(\tau)$

$$y_1'' + y_1 = y_0' - y_0^2 y_0' - 2\omega_1 y_0''$$

$$\Rightarrow$$

$$y_1'' + y_1 = 2\omega_1 x_0\cos(\tau) + x_0^3\cos^2\tau\sin\tau - x_0\sin\tau$$

$$\Rightarrow \quad y_1'' + y_1 = 2\omega_1 x_0\cos(\tau) + \frac{x_0^3}{4}\sin 3\tau + \left(\frac{x_0^3}{4} - x_0\right)\sin\tau \qquad \text{again from 1.2(ii)}$$

to find periodic solution need to remove terms with $\cos\tau$ or $\sin\tau$

Case 1: $x_0 = 0$, then $y_1'' + y_1 = 0$ and $y_1(0) = 0$, $y_1'(0) = 0$ implies that ~~that~~ $y_1 \equiv 0$, and $y_0 \equiv 0$ so $y = O(\varepsilon^2)$, can have any choice of ω_1.

Case 2: $x_0 \neq 0$, then $\omega_1 = 0$ and

$$y_1'' + y_1 = \frac{x_0^3}{4}\sin 3\tau$$ so we obtain same solution from 1.2(ii)

$$y_1 = \frac{3x_0^3}{32}\sin\tau - \frac{x_0^3}{32}\sin 3\tau \text{ //}$$

$$y = x_0\cos\tau + \varepsilon\left(\frac{3x_0^3}{32}\sin\tau - \frac{x_0^3}{32}\sin 3\tau\right) + O(\varepsilon^2)$$

this shows that to have a non-trivial ~~solution~~ periodic expansion of higher order we need to ~~have~~ use expansions of higher order for both ω_1 and y. ~~as seen~~
as seen in the remark.