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SUBJECTIVE PROBABILITIES ON SUBJECTIVELY UNAMBIGUOUS EVENTS

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This paper suggests a behavioral definition of (subjective) ambiguity in an abstract setting where objects of choice are Savage-style acts. Then axioms are described that deliver probabilistic sophistication of preference on the set of unambiguous acts. In particular, both the domain and the values of the decision-maker's probability measure are derived from preference. It is argued that the noted result also provides a decision-theoretic foundation for the Knightian distinction between risk and ambiguity.

KEYWORDS: Ambiguity, Knightian uncertainty, subjective probability, probabilistic sophistication.

1. INTRODUCTION

1.1. *Objectives*

AT LEAST SINCE FRANK KNIGHT, economists have often referred to a distinction between risk and ambiguity.[2] Roughly, risk refers to situations where the likelihoods of relevant events may be represented by a probability measure, while ambiguity refers to situations where there is insufficient information available for the decision-maker to assign probabilities to events. Ellsberg demonstrated that such a distinction is empirically meaningful. In particular, his Paradox showed that behavior reflecting aversion to ambiguity is inconsistent with Savage's subjective expected utility theory. In response, some generalizations of the Savage model have been developed (most notably, by Schmeidler (1989) and Gilboa and Schmeidler (1989)) in order to resolve the Ellsberg Paradox and, more generally, to accommodate nonindifference to ambiguity.

All of this has proceeded without a formal definition of ambiguity for a general abstract setting of the sort considered by Savage. The main objective of this paper is to provide such a definition.

In order to motivate the paper further and to cast some light on the desiderata for a definition of ambiguity, we describe two classes of applications that are developed below.

[1] This paper was written in part while Epstein was affiliated with U. Toronto and visiting HKUST and while Zhang was affiliated with U. Western Ontario. We are indebted to the Canadian SSHRC, to the NSF (Grant SES-9972442), and to Carleton University for financial support, and to Michelle Cohen, Werner Ploberger, David Schmeidler, Peter Wakker, and especially Mark Machina, Massimo Marinacci, and Uzi Segal for valuable discussions. Two (very patient) referees provided extensive and thoughtful comments that led to major improvements. We are grateful to them and to an editor. A version of this paper was first circulated in 1997.

[2] Knight (1921) and many other authors use the term uncertainty rather than ambiguity. We follow Ellsberg (1961) and use uncertainty as a comprehensive term. Thus in our terminology, every event/act/prospect is uncertain and each is either risky or ambiguous (but not both).

1.2. *A Fully Subjective Theory of Probability*

Savage's (1954) expected utility theory is typically referred to as providing a *subjective theory of probability*. That is because the probability measure underlies choice behavior. More precisely, it is derived from axioms on the preference ordering of uncertain prospects (acts defined on a state space S) and serves as a component in the representation of that preference. We begin by noting two critiques of the Savage model as a subjective theory of probability. Each claims that the Savage model delivers 'too much' to be completely satisfactory.

The first sense in which Savage delivers too much is that his axioms deliver not only the fact that preference is based on probabilities, but also the expected utility functional form. Because the use of probabilities seems more basic than any particular functional form, this aspect of the Savage theory is unattractive as a theory of probability. This critique is due to Machina and Schmeidler (1992) where it has been addressed by these authors through axioms that deliver *probabilistically sophisticated* preferences. Roughly speaking, probabilistic sophistication entails a two-stage procedure for evaluating any act. First, the decision-maker uses a probability measure on the state space in order to translate the act into an induced distribution over outcomes (a lottery); and second, she uses a (not necessarily expected) utility function defined on lotteries to evaluate the induced lottery and produce a utility level for the act. Thus preference is based on probabilities, but in a way that does not impose superfluous functional form restrictions.[3]

The second critique of the Savage theory that applies also to the Machina and Schmeidler extension is more pertinent to this paper. Both theories deliver too much in that they derive probabilities for *all* measurable events, that is, for all events in some prespecified σ-algebra Σ (that could be the power set). Consequently, there is an important sense in which their theories fail to be subjective. They are subjective in the sense that the probability measure on Σ that is delivered is derived from the decision-maker's preference ordering over Σ-measurable acts. However, the domain Σ of the measure is *exogenous* to the model rather than being derived from preference.

Exogeneity of the σ-algebra is not a limitation of a 'subjective theory' if it is believed that decision-makers assign probabilities to all events that are relevant to the context being modeled. In that case, the modeling context may dictate the appropriate specification for Σ, independently of preference. But choice behavior such as that exhibited in the Ellsberg Paradox and related evidence have demonstrated that many decision-makers do not assign probabilities to all events. In situations where some events are 'ambiguous,' decision-makers may not assign probabilities to those events, though the likelihoods of 'unambiguous' events are represented in the standard probabilistic way. For example, in the case of the Ellsberg urn with balls of 3 possible colors, R, B, and G, where the only objective information is that $R + B + G = 90$ and $R = 30$, events in the

[3] That is not quite true as explained in Section 4.1.

class

$$(1.1) \qquad \mathscr{A} = \{\varnothing, \{R\}, \{B, G\}, \{R, B, G\}\}$$

are intuitively unambiguous. Most decision-makers would presumably assign them the obvious probabilities in deciding on how to rank bets based on the color of a ball to be drawn at random. However, the use of probabilities for other events is inconsistent with the common 'ambiguity averse' preference ranking of such bets, namely a preference to bet on R (drawing a red ball) rather than on B and also a preference for betting on $\{B, G\}$ rather than on $\{R, G\}$.

On the other hand, aversion to ambiguity is not universal. Some decision-makers are indifferent to ambiguity and behave in the nonparadoxical and fully probabilistic fashion. The lesson we take from this is that decision-makers may differ not only in the probabilities assigned to given events (an aspect not well illustrated by this example), but also in the identity of the events to which they assign probabilities. Thus to be *fully subjective* a theory of probability should derive *both* the domain *and* the values of the probability measure from preference.

The starting point for such a theory is the definition of 'unambiguous'. Using our definition, we identify the class $\mathscr{A}$ of (*subjectively*) *unambiguous* events. Then we show (Theorem 5.2) that the decision-maker is probabilistically sophisticated on the domain of unambiguous ($\mathscr{A}$-measurable) acts, given suitable axioms on preference. This representation result constitutes a contribution towards a fully subjective theory of probability, because both the domain $\mathscr{A}$ of the decision-maker's probability measure and the values assigned by the measure to events in $\mathscr{A}$ are derived from preference.

1.3. *The Knightian Distinction*

The Knightian distinction provides another perspective on the value of Theorem 5.2. Knight used risk to refer to situations where (possibly subjective) probabilities apply and ambiguity to refer to all other situations. Our definition of ambiguity also leads to the dichotomous characterization of all 'situations', events or acts, as either unambiguous or ambiguous. Then Theorem 5.2 delivers probabilities on the class of unambiguous events. In this way, it provides precise expression to and a *choice-theoretic foundation for the Knightian distinction between risk and ambiguity*. The resulting sharp distinction between (probabilistic) risk and ambiguity also permits a unified treatment of attitudes towards risk and attitudes towards ambiguity, as described in Section 6.[4]

[4] Schmeidler (1989) proposes a definition of ambiguity aversion for the case where preference is defined over two-stage Anscombe-Aumann (1963) acts rather than merely over Savage-style acts as here. See Epstein (1999) for a discussion of the limitations of Schmeidler's definition and of applications that have been made of it.

1.4. *Desiderata*

The preceding applications suggest some (overlapping) desiderata that guide our choice of a definition of ambiguity. We seek a definition of ambiguity that, in addition to capturing intuition, is:

D1. *Behavioral or expressed in terms of preference*: Following the choice-theoretic tradition of Savage, we insist that ambiguity be defined in terms of preference. This ensures verifiability, at least in principle given suitable data on behavior, and hence potential empirical relevance. The choice of a definition thus amounts to answering the question "what behavior would indicate that the decision-maker perceives a specific event or act as ambiguous?"

D2. *Model-free*: Ambiguity and attitudes towards ambiguity seem more basic than the use of any particular functional form for utility. Thus to be satisfactory, a definition should not be tied to particular models such as those in Schmeidler (1989) or Gilboa and Schmeidler (1989).

D3. *Explicit and constructive*: Given an event, it should be possible to check whether or not it is ambiguous. This would aid verifiability. As well, an explicit definition would ensure that the definition produces a *unique* class $\mathscr{A}$ of unambiguous events.[5]

D4. *Consistent with probabilistic sophistication on unambiguous acts*: We have mentioned two reasons for seeking such a representation (as delivered by our Theorem 5.2). First, it delivers a fully subjective theory of probability. Second, it delivers the Knightian distinction between (probabilistic) risk and ambiguity, which in turn permits modeling, within a unified framework, both attitudes towards risk and attitudes towards ambiguity.

We provide a definition that performs fairly well in terms of these criteria. However, two significant limitations should be acknowledged at the outset. First, as explained in Section 4.1, our definition makes sense only for preferences satisfying Savage's axiom P3. Moreover, probabilistic sophistication on unambiguous acts is delivered only after assuming some additional axioms. In our defense, assuming these axioms falls far short of assuming a parametric model of preference such as Choquet expected utility or multiple-priors.

A second limitation that warrants emphasis concerns D1 and the subjective nature of our theory of (un)ambiguity and probability. We define ambiguity in terms of the preference ranking of acts over an exogenously given state space (and outcome set). Thus the state space and its associated universal σ-algebra Σ of events are presumed 'objective.' The way in which our theory is more subjective than other theories is that we endogenize the subclass of unambiguous events in Σ. The importance of the assumed objectivity of S is that state spaces and Savage-style acts are constructs used to model the choice of (physical) actions. Thus the empirically relevant question is "which *actions* are

[5] See Section 8.2 for more on "uniqueness."

unambiguous?" We can answer such a question only *given* a translation of actions into acts over some state space.[6]

1.5. *Outline*

The paper proceeds as follows. We conclude the introduction by describing a by-product of the search for a fully subjective theory of probability that also casts light on technical aspects of the paper. Next we define the key notion of a λ-system and introduce some notation. Section 3 introduces our definition of ambiguity and Section 4 describes axioms that restrict the class of preferences that we admit. These axioms deliver our main result, Theorem 5.2, in Section 5. The Theorem is applied in Section 6 to define attitudes towards risk and ambiguity and it is illustrated in Section 7 in the context of the Choquet expected utility model. Our definition of ambiguity is contrasted with alternatives in Section 8 and Section 9 provides further perspective on the definition by examining it in the context of Ellsberg-style situations. Proofs are relegated to Appendices.

1.6. *A By-Product*

Typically, probability theory posits that any probability measure is defined on an algebra or σ-algebra, constructs that seem natural from a mathematical point of view. In a fully subjective theory, the domain $\mathscr{A}$ of the subjective probability measure, including its mathematical properties, are derived. This permits the appropriateness of the standard assumptions to be evaluated from a decision-theoretic point of view. This argument is due to Zhang (1997), whose major finding in this regard we proceed to outline.

The major point is that $\mathscr{A}$ is typically not a σ-algebra or even an algebra. Moreover, at an intuitive level, while the class of unambiguous events is naturally taken to be closed with respect to complements and disjoint unions, it may not be closed with respect to intersections. This point may be illustrated by borrowing Zhang's example of an Ellsberg-type urn with 4 possible colors—R, B, G, and W. Suppose that the only objective information is that the total number of balls is 100 and that $R+B=G+B=50$. Then it is intuitive that the class of unambiguous events is

$$\mathscr{A}=\{S,\varnothing,\{B,R\},\{B,G\},\{G,W\},\{R,W\}\}. \tag{1.2}$$

Observe that $\mathscr{A}$ fails to be an algebra, because while $\{B,R\}$ and $\{B,G\}$ are unambiguous, their intersection $\{B\}$ is not. As pointed out by Zhang, the appropriate mathematical structure for $\mathscr{A}$ is a λ-system (defined below), also sometimes called a *Dynkin system* in the measure theory literature.

[6] It is an open question whether two representations of the same choice environment, using different state spaces and preferences satisfying our axioms, can lead to different conclusions about ambiguity.

For this paper, the fact that we cannot take $\mathscr{A}$ to be an algebra complicates the derivation of a probability measure on $\mathscr{A}$ and, in particular, prevents us from simply invoking existing results from Savage (1954), Fishburn (1970), and Machina and Schmeidler (1992). The arguments in these studies exploit the fact that the relevant class of events is closed with respect to intersections. We rely instead on a recent representation result in Zhang (1999) for qualitative probabilities on λ-systems.

2. PRELIMINARIES

Let (S, Σ) be a measurable space where S is the set of states and Σ is a σ-algebra. All events in this paper are assumed to lie in Σ; we repeat this explicitly below only on occasion.

Say that a nonempty class of subsets $\mathscr{A} \subset \Sigma$ of S is a λ-*system* if:

λ.1. $S \in \mathscr{A}$;
λ.2. $A \in \mathscr{A} \Rightarrow A^c \in \mathscr{A}$; and
λ.3. $A_n \in \mathscr{A}$, $n = 1, 2, \ldots$ and $A_i \cap A_j = \varnothing$, $\forall i \neq j \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

This definition and terminology appear in Billingsley (1986). A λ-system $\mathscr{A}$ is closed with respect to complements and countable disjoint unions. The intuition for these properties is clear if we think of $\mathscr{A}$ as a class of events to which the decision-maker attaches probabilities. If she can assign a probability to event A, then the complementary probability is naturally assigned to A^c. Similarly, if she can assign probabilities to each of the disjoint events A and B, then the sum of these probabilities is naturally assigned to $A \cup B$. On the other hand, there is no such intuition supporting closure with respect to intersections, or equivalently, with respect to arbitrary unions. Lack of closure with respect to intersections differentiates λ-systems from algebras or σ-algebras. As illustrated by the above example of an Ellsberg-type urn with 4 colors, λ-systems are more appropriate for modeling families of 'unambiguous' events.

We have the following equivalent definition:[7]

LEMMA 2.1: *A nonempty class of subsets $\mathscr{A} \subset \Sigma$ of S is a λ-system if and only if*:

λ.1′. $\varnothing$, $S \in \mathscr{A}$;
λ.2′. A, $B \in \mathscr{A}$ *and* $A \subseteq B \Rightarrow B \setminus A \in \mathscr{A}$; *and*
λ.3′. $A_n \in \mathscr{A}$ *and* $A_n \subseteq A_{n+1}$, $n = 1, 2, \ldots, \Rightarrow \bigcup_{n=1}^{\infty} A_n \in \mathscr{A}$.

Although $\mathscr{A}$ is not an algebra, a probability measure can still be defined on $\mathscr{A}$. Say that $p : \mathscr{A} \mapsto [0,1]$ is a (finitely additive) *probability measure* on $\mathscr{A}$ if:

P.1. $p(\varnothing) = 0$, $p(S) = 1$; and
P.2. $p(A \cup B) = p(A) + p(B)$, $\forall A, B \in \mathscr{A}$, $A \cap B = \varnothing$.

[7] See Billingsley (1986, p. 43). Collections of sets satisfying the following conditions are frequently called alternatively Dynkin systems or d-systems (see, for example, Williams (1991, p. 193)).

Countable additivity of p is defined in the usual way and will be stated explicitly where needed. Given a probability measure p on $\mathscr{A}$, call p *convex-ranged* if for all $A \in \mathscr{A}$ and $0 < r < 1$, there exists $B \subset A$, $B \in \mathscr{A}$, such that $p(B) = rp(A)$.[8]

As in Savage, we assume a set of outcomes $\mathscr{X}$. Prospects are modeled via (simple) acts, Σ-measurable maps from S to $\mathscr{X}$ having finite range. The set of acts is $\mathscr{F} = \{\ldots, f, f', g, h, \ldots\}$. Given a λ-system $\mathscr{A}$, define $\mathscr{F}^{ua}$ by

$$\mathscr{F}^{ua} = \{f \in \mathscr{F} : f \text{ is } \mathscr{A}\text{-measurable}\}, \tag{2.1}$$

where f is $\mathscr{A}$-measurable if $\{s \in S : f(s) \in X\} \in \mathscr{A}$ for any subset X of $\mathscr{X}$. Thinking of $\mathscr{A}$ as the set of unambiguous events, $\mathscr{F}^{ua}$ is naturally termed the set of *unambiguous acts*.

3. UNAMBIGUOUS EVENTS

3.1. *Ellsberg-Based Intuition*

As suggested in the context of the first desideratum, a definition of ambiguity must answer the question "which behavior would reveal that the decision-maker views a given event as 'ambiguous'?' The Ellsberg urn with three colors illustrates our approach. The typical choices described earlier take the form

$$\begin{aligned} &\begin{pmatrix} 100 & \text{if } s = R \\ 0 & \text{if } s = B \\ 0 & \text{if } s = G \end{pmatrix} \succ \begin{pmatrix} 0 & \text{if } s = R \\ 100 & \text{if } s = B \\ 0 & \text{if } s = G \end{pmatrix} \qquad \text{and} \\ &\begin{pmatrix} 100 & \text{if } s = R \\ 0 & \text{if } s = B \\ 100 & \text{if } s = G \end{pmatrix} \prec \begin{pmatrix} 0 & \text{if } s = R \\ 100 & \text{if } s = B \\ 100 & \text{if } s = G \end{pmatrix}. \end{aligned} \tag{3.1}$$

Thus the preference to bet on red rather than blue is reversed by the change in the outcome associated with a green ball. The intuition for the reversal is the complementarity between G and B—there is imprecision regarding the likelihood of B whereas $\{B, G\}$ has precise probability $2/3$. As a result, the change to 100 if G (drawing a green ball) provides an entirely different perspective on the relative valuation of 100 if R as opposed to 100 if B. In alternative notation that renders the complementarity more transparent,

$$R \succ_{\ell} B \qquad \text{but} \qquad R \cup G \prec_{\ell} B \cup G,$$

where $\succcurlyeq_{\ell}$ is interpreted as 'would rather bet on'.

We take such nonseparability as indicative of ambiguity. More precisely, we view such a reversal in rankings as the behavioral manifestation of the intuitively

[8] When $\mathscr{A}$ is a σ-algebra and p is countably additive, this is equivalent to nonatomicity (Rao and Rao (1983, pp. 142–143)).

ambiguous nature of the event G and we use it as the basis for our definition of ambiguity in a general setting.

In fact, the above intuition is more general than may be apparent from (3.1). To see this, expand the urn to contain 150 balls in total. There are still 30 red and 60 that are either blue or green but in addition, there are 60 balls that are either white or yellow. The decision-maker may be given some (possibly perfect) information about the relative numbers of white and yellow balls. Then the following rankings are intuitive for reasons similar to those underlying (3.1):

$$\begin{pmatrix} 100 & \text{if } s=R \\ 0 & \text{if } s=B \\ h(s) & \text{if } s\in\{W,Y\} \\ 0 & \text{if } s=G \end{pmatrix} \succ \begin{pmatrix} 0 & \text{if } s=R \\ 100 & \text{if } s=B \\ h(s) & \text{if } s\in\{W,Y\} \\ 0 & \text{if } s=G \end{pmatrix} \quad \text{and}$$

$$\text{(3.2)}$$

$$\begin{pmatrix} 100 & \text{if } s=R \\ 0 & \text{if } s=B \\ h(s) & \text{if } s\in\{W,Y\} \\ 100 & \text{if } s=G \end{pmatrix} \prec \begin{pmatrix} 0 & \text{if } s=R \\ 100 & \text{if } s=B \\ h(s) & \text{if } s\in\{W,Y\} \\ 100 & \text{if } s=G \end{pmatrix},$$

where the prizes $h(W)$ and $h(Y)$ for drawing white and yellow balls are arbitrary (but common to all four facts). Such a reversal of rankings, which once again admits interpretation as a form of nonseparability, is a behavioral manifestation of the intuitive ambiguity of G in the context of the present modified Ellsbergian setting with enlarged state space $\{R, B, G, W, Y\}$.

One final point of clarification concerns the fact that the story we have just told suggests that the ambiguity of G leads to preference reversals as above for *all* subacts h. This is the case only because of our (implicit) assumption that the added and unspecified information about W and Y does not also provide further information about B and G. In such a case, complementarities between the two pairs of colors may be such that the preference reversal would not occur for all acts h, but it would still occur for some h.[9] This 'explains' completely the definition to follow.

3.2. *Definition*

The primitives (S, Σ) and $\mathscr{X}$ are defined as above. The decision-maker has a preference order $\succcurlyeq$ on the set of acts $\mathscr{F}$. Unambiguous events are now defined from the perspective of $\succcurlyeq$.

[9] If the added information is that $W=B$, then the reversal of rankings is intuitive as above if $h(\cdot)=0$. However, it is plausible that for $h(W)=100$ and $h(Y)=0$, the first ranking in (3.2) is satisfied and this ranking is not affected by the change in outcome from 0 to 100 on G.

DEFINITION: An event T is *unambiguous* if: (a) for all disjoint subevents A, B of T^c, acts h, and outcomes $x^*, x, z, z' \in \mathscr{X}$,

$$
\begin{aligned}
&\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \succcurlyeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \\
&\Rightarrow \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z' & \text{if } s \in T \end{pmatrix} \succcurlyeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z' & \text{if } s \in T \end{pmatrix};
\end{aligned} \tag{3.3}
$$

and (b) the condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, T is *ambiguous*.

The set of unambiguous events is denoted $\mathscr{A}$. It is nonempty because $\varnothing$ and S are unambiguous. Observe that the defining invariance condition is required to be satisfied even if A or B is empty. The requirement that both T and T^c satisfy the indicated invariance builds into the definition the intuitive feature that an event is unambiguous if and only if its complement is unambiguous.

Turn to further interpretation, assuming that $x^* \succ x$.[10] The first two acts being compared yield identical outcomes if the true state lies in $(A \cup B)^c$. Thus the comparison is between 'bets conditional on $(A \cup B)$' with stakes x^* and x and the outcomes shown for $(A \cup B)^c$. The indicated ranking reveals that the decision-maker views A as conditionally more likely than B. Suppose now that the outcome on event T is changed from z to z'. If T is 'unambiguous', then this conditional likelihood ranking should not be affected because 'unambiguous' means or at least entails such separability or invariance.

A moment's reflection on this intuition may help to alleviate concerns regarding features of the definition that may seem arbitrary. After all, why is it the case that acts are restricted to be constant within each of the events A, B, and T, even though outcomes are allowed to vary within $T^c \setminus (A \cup B)$? In fact, these restrictions are vital for the preceding intuition. First, the interpretation in terms of the invariance of the relative conditional likelihoods of A and B is justified only because of the constancy of outcomes on each of A and B. More general comparisons of the form

$$
\begin{pmatrix} f(s) & \text{if } s \in A \\ g(s) & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z & \text{if } s \in T \end{pmatrix} \text{ vs } \begin{pmatrix} g(s) & \text{if } s \in A \\ f(s) & \text{if } s \in B \\ h(s) & \text{if } s \in T^c \setminus (A \cup B) \\ z & \text{if } s \in T \end{pmatrix}
$$

[10] Any $x \in \mathscr{X}$ denotes both the outcome and the constant act producing the outcome x in every state. Thus "$x^* \succ x$" has the obvious meaning.

reflect not only assessments of conditional likelihoods but also other aspects of preference such as attitudes toward uncertainty (risk plus ambiguity).[11] Thus invariance of rankings of this sort has nothing apparent to do with the ambiguity of T.

Similarly, constancy of acts on T is vital for the above intuition. It is T in its entirety that is unambiguous and this does not imply anything about subsets. This leads naturally to the restriction to acts that are constant on T.

On the other hand, there is no good reason to restrict outcomes within $T^c \setminus (A \cup B)$. The previous subsection illustrated the intuitive plausibility of requiring invariance even when outcomes vary within $T^c \setminus (A \cup B)$. More to the point, we are not aware of any intuition for a definition with h restricted to be constant that does not also suggest that the indicated invariance should hold for all h.

A similar point is relevant to another question that may have occurred to some readers. Though in the motivating Ellsberg-style examples, the outcomes z and z' equal either x^* or x, the intuition just described applies equally to general z and z'.[12] Moreover, this generality is important below in that (our proof of) Theorem 5.2 exploits it.

Conclude with two brief observations regarding the intuitive performance of the definition. First, all events are unambiguous if $\succcurlyeq$ is probabilistically sophisticated, *a fortiori* if $\succcurlyeq$ is a subjective expected utility order.[13] Second, in the motivating example of an Ellsberg urn with 3 colors, the events $\{G\}$ and $\{B\}$ are subjectively ambiguous for any preference order that predicts the typical choices (3.1). (The designation of $\{R\}$ and other events depends on more information about preference than is contained in (3.1).) See Section 9 for other Ellsberg-type settings.

4. AXIOMS

Though the definition seems to capture some aspects of ambiguity, our proposal is that it be adopted only subject to some arguably weak axioms for preference that are specified in this section. These axioms serve both to strengthen intuitive appeal (this is the primary role of the first axiom, Savage's P3) and to satisfy desideratum D4 by delivering probabilistic sophistication on the domain of unambiguous acts (this is the role of the remaining axioms).

4.1. *Savage's P3, Separability and Ambiguity*

With regard to intuitive appeal, a natural concern regarding our definition is that the nonseparability reflected by a violation of the invariance in (3.3) may

[11] For elaboration, see the parallel discussion in Machina and Schmeidler (1992, Section 4.2).

[12] More elaborate examples are readily constructed to illustrate this.

[13] That $\mathscr{A} = \Sigma$ given probabilistic sophistication, follows immediately from Machina and Schmeidler's central axiom P4*.

fail for reasons that have nothing to do with ambiguity. To see this, consider Savage's axiom P3.[14]

AXIOM 1 (Savage's P3): *For all non-null events A in Σ and for all x^*, x and f, $x^* \succcurlyeq x$ if and only if $(x^* \text{ if } A;\ f(\cdot) \text{ if } A^c) \succcurlyeq (x \text{ if } A;\ f(\cdot) \text{ if } A^c)$.*

The axiom imposes a form of monotonicity and also state independence (the preference for x^* over x is independent of the common event where they are realized). The necessity of this axiom in our approach implies, in particular, that we have nothing to say about the meaning of ambiguity when preferences are state dependent.

One connection between P3 and our definition of ambiguity is apparent on examination of the intuition provided following the definition. That intuition, regarding the invariance of relative conditional likelihoods, implicitly assumes that if $x^* \succcurlyeq x$, then x^* would also be weakly preferred conditionally in the sense that, for any act f,

$$(x^* \text{ if } A;\ f(\cdot) \text{ if } A^c) \succcurlyeq (x \text{ if } A;\ f(\cdot) \text{ if } A^c),$$

which is one direction in P3.

Another connection stems from the fact that P3 implies: For all non-null A and for all x^*, x, f, and g,

$$(4.1) \qquad (x^* \text{ if } A;\ f(\cdot) \text{ if } A^c) \succcurlyeq (x \text{ if } A;\ f(\cdot) \text{ if } A^c) \Leftrightarrow (x^* \text{ if } A;\ g(\cdot) \text{ if } A^c) \succcurlyeq (x \text{ if } A;\ g(\cdot) \text{ if } A^c).$$

In other words, rankings are invariant to changes in common outcomes for the comparisons shown, which admits interpretation as a form of separability across outcomes. Therefore, if in (3.3) the switch from outcome z to z' on T causes a reversal of rankings, 'then' this reversal is due to complementarities between events (as in the intuition suggested for our definition) rather than between outcomes (the noted switch on T causing a change in the relative evaluation of x^* versus x).

For the above reasons, our definition is appropriate only for preference orders satisfying P3, which we assume in the sequel. We have evidently *not* succeeded in capturing ambiguity for *all* preference orders and desideratum D2 is not fulfilled. However, P3 is a common and arguably mild assumption and is much weaker than imposing a parametric model of preference. In addition, it is useful for perspective to note that there is a parallel limitation associated with the notion of probabilistic sophistication. As defined by Machina and Schmeidler (and as adopted below), probabilistic sophistication captures not only the reliance of preference on probabilities but also monotonicity as embodied in P3, which axiom is clearly not germane to the use of probabilities.[15]

[14] The event A is null (with respect to $\succcurlyeq$) if $f' \sim f$ whenever $f'(\cdot) = f(\cdot)$ on $S \setminus A$.

[15] The generalization by Grant (1995) also assumes some monotonicity, though a weaker form.

The role of P3 is partly interpretational, but it also plays a role in the formal analysis (Theorem 5.2). We turn next to other axioms that play a formal role.

4.2. *Further Axioms*

The further preference axioms specified here will deliver not only the λ-system properties for $\mathscr{A}$ and a probability measure on these unambiguous events, but also the probabilistic sophistication of preference restricted to unambiguous acts. Reasons for seeking such a representation result were offered in the introduction.

The set $\mathscr{F}^{ua}$ of unambiguous acts is defined by (2.1). Also useful, for any given $A \in \mathscr{A}$, is the set of acts

$$\mathscr{F}_A^{ua} = \{f \in \mathscr{F} : f^{-1}(X) \cap A \in \mathscr{A} \text{ for all } X \subset \mathscr{X}\}.$$

Some of the axioms for $\succcurlyeq$ are slight variations of Savage's axioms, with names adapted from Machina and Schmeidler. Though the axioms are expressed in terms of $\mathscr{A}$, they constitute assumptions about $\succcurlyeq$ because $\mathscr{A}$ is derived from $\succcurlyeq$. A final remark is that the remaining axioms relate primarily to $\succcurlyeq$ restricted to $\mathscr{F}^{ua}$.

AXIOM 2 (Nondegeneracy): *There exist outcomes x^* and x such that $x^* \succ x$.*

AXIOM 3 (Weak Comparative Probability): *For all events A, $B \in \mathscr{A}$ and outcomes $x^* \succ x$ and $y^* \succ y$*

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succcurlyeq \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix}$$
$$\Leftrightarrow \begin{pmatrix} y^* & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} \succcurlyeq \begin{pmatrix} y^* & \text{if } s \in B \\ y & \text{if } s \in B^c \end{pmatrix}.$$

This is Savage's axiom P4 restricted to unambiguous events. It requires that for unambiguous events A and B, the preference to bet on A rather than on B is independent of the stakes. The axiom delivers the complete and transitive likelihood relation $\succcurlyeq_\ell$ on $\mathscr{A}$, where $A \succcurlyeq_\ell B$ if $\exists x^* \succ x$ such that

$$\begin{pmatrix} x^* & \text{if } x \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succcurlyeq \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix}.$$

The next axiom imposes suitable richness of the set of unambiguous events. It is clear from Savage's analysis that some richness is required to derive a probability measure on $\mathscr{A}$. Savage's axiom P6 (suitably translated) is not adequate here because $\mathscr{A}$ is not a σ-algebra. However, the spirit of Savage's P6 is retained in the next axiom.

AXIOM 4 (Small Unambiguous Event Continuity): *Let* $f,\ g \in \mathscr{F}^{ua}$, $f \succ g$, *with* $f = (x_1, A_1; x_2, A_2; \ldots; x_n, A_n)$, $g = (y_1, B_1; y_2, B_2; \ldots; y_m B_m)$, *where each* A_i *and* B_i *lies in* $\mathscr{A}$. *Then for any* x *in* $\mathscr{X}$, *there exist two partitions* $\{C_i\}_{i=1}^N$ *and* $\{D_j\}_{j=1}^M$ *of* S *in* $\mathscr{A}$ *that refine* $\{A_i\}_{i=1}^n$ *and* $\{B_j\}_{j=1}^m$ *respectively, and satisfy*:

$$(4.2)\qquad f \succ \begin{pmatrix} x & \text{if } s \in D_k \\ g(s) & \text{if } s \in D_k^c \end{pmatrix}, \quad \text{for all } k \in \{1, \ldots, N\};$$

and

$$(4.3)\qquad \begin{pmatrix} x & \text{if } s \in C_j \\ f(s) & \text{if } s \in C_j^c \end{pmatrix} \succ g, \quad \text{for all } j \in \{1, \ldots, M\}.$$

Very roughly, the axiom requires that unambiguous events can be decomposed into suitably 'small' unambiguous events. When $\mathscr{A}$ is closed with respect to intersections, as in the Savage or Machina-Schmeidler models where it is taken to be the power set, then the axiom is implied by Savage's P6, given Axioms 1–3.[16]

The preceding axioms are largely familiar, at least when imposed on all of $\mathscr{F}$, rather than just on $\mathscr{F}^{ua}$ as here. The remaining two axioms are 'new' and are needed to accommodate the fact that $\mathscr{A}$ may not be a σ-algebra.

Say that a sequence $\{f_n\}_{n=1}^\infty$ in $\mathscr{F}^{ua}$ *converges in preference* to $f_\infty \in \mathscr{F}^{ua}$ if: for any two acts f_*, f^* in $\mathscr{F}^{ua}$ satisfying $f_* \prec f_\infty \prec f^*$, there exists an integer N such that

$$f_* \prec f_n \prec f^*, \quad \text{whenever} \quad n \geq N.$$

AXIOM 5 (Monotone Continuity): *For any* $A \in \mathscr{A}$, *outcomes* $x^* \succ x$, *act* $h \in \mathscr{F}_{A^c}^{ua}$, *and decreasing sequence* $\{A_n\}_{n=1}^\infty$ *in* $\mathscr{A}$ *with* $A_1 \subseteq A$, *define*

$$f_n = \begin{pmatrix} x^* & \text{if } s \in A_n \\ x & \text{if } s \in A \setminus A_n \\ h(s) & \text{if } s \in A^c \end{pmatrix} \qquad \text{and}$$

$$f_\infty = \begin{pmatrix} x^* & \text{if } s \in \bigcap_{n=1}^\infty A_n \\ x & \text{if } s \in A \setminus (\bigcap_{n=1}^\infty A_n) \\ h(s) & \text{if } s \in A^c \end{pmatrix}.$$

If $f_n \in \mathscr{F}^{ua}$ *for all* $n = 1, 2, \ldots,$ *then* $\{f_n\}_{n=1}^\infty$ *converges in preference to* f_∞ *and* $f_\infty \in \mathscr{F}^{ua}$.

[16] Savage's P6 applied to $\mathscr{F}^{ua}$ would require that, given x, if $f = (x_i, A_i)_{i=1}^n \succ g = (y_i, B_i)_{i=1}^n$, where every A_i and B_i lies in $\mathscr{A}$, then there exists a partition $\{G_i\}_{i=1}^N$ of S in $\mathscr{A}$ such that $f \succ (x, G_k; g, G_k^c)$ for all k. Given such a partition, and given that $\mathscr{A}$ is closed with respect to intersections, then the collection of events $D_{ij} = G_i \cap B_j$ satisfies the requirements in Axiom 4.

The name Monotone Continuity describes one aspect of the axiom, that requiring the indicated convergence in preference.[17] The second component of the axiom is the requirement that the limit f_∞ lie in $\mathscr{F}^{ua}$ whenever each f_n is unambiguous. This will serve in particular to ensure that $\mathscr{A}$ satisfies the 'countable' closure condition $\lambda.3$ or $\lambda.3'$ required by the definition of a λ-system.

It might be felt that given the correct definition of 'unambiguous', the derivation of a probability measure on $\mathscr{A}$ should be possible with little more than some richness requirements. The axioms stated thus far can arguably be interpreted as constituting such minimal requirements. However, they do not suffice and we need one final axiom. This may reflect the fact that only some aspects of 'unambiguous' are captured in our definition. In any event, the final axiom is intuitive and arguably weak. Its statement requires some preliminaries.

A finite partition with component events from $\mathscr{A}$ is denoted $\{A_i\}$. Henceforth all partitions have unambiguous components, even where not stated explicitly. Given such a partition, use the obvious abbreviated notation (x_i, A_i). For any permutation σ of $\{1,\dots,n\}$, $(x_{\sigma(i)}, A_i)$ denotes the act obtained by permuting outcomes between the events. Say that the finite partition $\{A_i\}$ is a *uniform partition* if $A_i \sim_\ell A_j$ for all i and j and call $\{A_i\}$ *strongly uniform* if in addition it satisfies: for all outcomes $\{x_i\}$ and for all permutations σ,

$$(x_{\sigma(i)}, A_i) \sim (x_i, A_i). \tag{4.4}$$

In particular, if $\{A_i\}_{i=1}^n$ is a strongly uniform partition, then for all index sets I and J, subsets of $\{1,2,\dots,n\}$,

$$\bigcup_{i\in I} A_i \sim_\ell \bigcup_{i\in J} A_i \quad \text{if} \quad |I| = |J|.$$

AXIOM 6 (Strong-Partition Neutrality): *For any two strongly uniform partitions $\{A_i\}_1^n$ and $\{B_i\}_1^n$, if $A_i \sim_\ell B_i$ for all i, then for all $\{x_i\}$,*

$$\begin{pmatrix} x_1 & \text{if } s\in A_1 \\ x_2 & \text{if } s\in A_2 \\ \cdots & \cdots \\ x_n & \text{if } s\in A_n \end{pmatrix} \sim \begin{pmatrix} x_1 & \text{if } s\in B_1 \\ x_2 & \text{if } s\in B_2 \\ \cdots & \cdots \\ x_n & \text{if } s\in B_n \end{pmatrix}. \tag{4.5}$$

The hypothesis that the A_i's and B_i's satisfy (4.4) expresses another sense in which these events are unambiguous. This makes the conclusion (4.5) natural and much weaker than if the indifference in (4.5) were required for all uniform partitions. The latter axiom would go a long way towards explicitly imposing probabilistic sophistication, an unattractive feature in the present exercise

[17] A related axiom with the same name is used by Arrow (1970) to deliver the countable additivity of the subjective probability measure. Here as well, countable additivity will follow from Monotone Continuity, but as an unintentional by-product.

where the intention is that probabilistic sophistication on unambiguous acts should result primarily from the definition of 'unambiguous.' Axiom 6 is less vulnerable to such a criticism.

To support the claim that Strong-Partition Neutrality is a relatively weak axiom, observe that it is satisfied by *all* Choquet-expected-utility functions as defined in Section 7, proving that it falls far short of imposing probabilistic sophistication.[18]

We turn next to the implications of these axioms.

5. PROBABILISTIC SOPHISTICATION ON UNAMBIGUOUS ACTS

Define "probabilistic sophistication on unambiguous acts $\mathscr{F}^{ua}$" by extending the definition of Machina and Schmeidler. Their definition is obtained if one replaces $\mathscr{F}^{ua}$ and $\mathscr{A}$ by $\mathscr{F}$ and Σ respectively in the formulation to follow. It is occasionally distinguished here terminologically by referring to "global probabilistic sophistication."

Some preliminary notions are required. Denote by $D(\mathscr{X})$ the set of probability distributions on $\mathscr{X}$ having finite support. A probability distribution $P = (x_1, p_1; \ldots; x_m, p_m)$ is said to *first-order stochastically dominate* $Q = (y_1, q_1; \ldots; y_n, q_n)$ with respect to the order $\succcurlyeq$ over the outcome set $\mathscr{X}$ if

$$\sum_{\{i : x_i \preccurlyeq x\}} p_i \leq \sum_{\{j : y_j \preccurlyeq x\}} q_j \quad \text{for all } x \in \mathscr{X}.$$

Use the term *strict dominance* if the above holds with strict inequality for some $x \in \mathscr{X}$.

Given a real-valued function W defined on a mixture subspace dom(W) of $D(\mathscr{X})$, say that W is *mixture continuous* if for any distributions P, Q, and R in dom(W), the sets

$$\{\lambda \in [0,1] : W(\lambda P + (1-\lambda)Q) \geq W(R)\} \qquad \text{and}$$
$$\{\lambda \in [0,1] : W(\lambda P + (1-\lambda)Q) \leq W(R)\}$$

are closed. Say that W is *monotonic* (with respect to stochastic dominance) if $W(P)(>) \geq W(Q)$ whenever P (strictly) stochastically dominates Q, for any P and Q in dom(W).

Given a probability measure p on $\mathscr{A}$, denote by $P_{f,p} \in D(\mathscr{X})$ the distribution over outcomes induced by the act f. Define

$$D_p^{ua}(\mathscr{X}) = \{P_{f,p} : f \in \mathscr{F}^{ua}\}.$$

When p is convex-ranged, then $D_p^{ua}(\mathscr{X})$ is a mixture space.

We can finally state the desired definition. Say that $\succcurlyeq$ is *probabilistically sophisticated on* $\mathscr{F}^{ua}$ if there exists a convex-ranged probability measure p on $\mathscr{A}$

[18] If $\{A_i\}$ is a strongly uniform partition, then the capacity v must be additive on the algebra generated by the partition. Thus the indifference (4.5) is implied.

and a real-valued, mixture continuous and monotonic function W on $D_p^{ua}(\mathscr{X})$ such that $\succcurlyeq$ has utility function U of the form

$$U(f) = W(P_{f,p}). \tag{5.1}$$

Roughly speaking, the probability measure p is used to translate acts in $\mathscr{F}^{ua}$ into (purely risky) lotteries and these are evaluated by means of the risk preference functional W. No stand is taken on the functional form of W, apart from monotonicity and mixture continuity, thus capturing primarily the decision-maker's reliance on probabilities for the evaluation of unambiguous acts. Subjective expected utility is merely one example, albeit an important one, in which W is an expected utility function on lotteries $D_p^{ua}(\mathscr{X})$ and thus U has the familiar form

$$U(f) = \int_S u(f)\, dp. \tag{5.2}$$

We remind the reader that because $\mathscr{A}$ and $\mathscr{F}^{ua}$ are derived from the given primitive preference relation $\succcurlyeq$ on $\mathscr{F}$, probabilistic sophistication so-defined is a property of $\succcurlyeq$ exclusively and does not rely on an exogenous specification of 'unambiguous acts.'

Probabilistic sophistication with measure p implies that likelihood (or the ranking of unambiguous bets) is represented by p; that is,

$$A \succcurlyeq_\ell B \Leftrightarrow pA \geq pB, \quad \text{for all } A, B \in \mathscr{A}.$$

But (5.1) is much stronger, requiring that the ranking of all (not necessarily binary) unambiguous acts be based on p.

Turn to the implications of our axioms. A preliminary result (proven in Appendix A) is that they imply that $\mathscr{A}$ is a λ-system.

LEMMA 5.1: *Under Axioms* 2, 4, *and* 5, $\mathscr{A}$ *is a* λ*-system*.

The following is our main result:

THEOREM 5.2: *Let* $\succcurlyeq$ *be a preference order on* $\mathscr{F}$ *satisfying* $P3$ *and* $\mathscr{A}$ *the corresponding set of unambiguous events. Then the following two statements are equivalent*:

(a) $\succcurlyeq$ *satisfies Axioms* 2–6.

(b) $\mathscr{A}$ *is a* λ*-system and there exists a* (*unique*) *convex-ranged and countably additive probability measure* p *on* $\mathscr{A}$ *such that* $\succcurlyeq$ *is probabilistically sophisticated on* $\mathscr{F}^{ua}$ *with underlying measure* p.

The bulk of the proof is found in Appendices B and C. The arguments used by Savage, Fishburn, and Machina-Schmeidler must be modified because only in the present setting is the relevant class of events $\mathscr{A}$ not closed with respect to intersections. A key step is to show (Appendix B) that our axioms for preference deliver the conditions for the implied likelihood relation that are used by Zhang (1999) in order to obtain a representing probability measure. The proof of probabilistic sophistication is completed in Appendix C.

The first step in discussion of the theorem is a comparison with the main result in Machina and Schmeidler (1992). If their axioms are imposed, then all events are subjectively unambiguous.[19] Therefore, one can view their result as a special case where $\mathscr{A} = \Sigma$ and global probabilistic sophistication prevails.[20] Here, in contrast, $\mathscr{A}$ is allowed to be smaller than Σ, in a way that depends on preference $\succcurlyeq$. This relaxation renders Theorem 5.2 a contribution towards a fully subjective theory of probability as discussed in the introduction. Also claimed there (and summarized in desideratum D4) is that our representation result provides foundations for the Knightian distinction between *probabilistic* risk, represented by prospects in $\mathscr{F}^{ua}$, and ambiguity, represented by all other acts. In Section 6, we exploit this aspect of the theorem to describe a unified treatment of attitudes towards each of risk and ambiguity.

A noteworthy feature of our theorem is that it is silent on the nature of preference on the domain of ambiguous acts. While at first glance this may seem like a weakness, we feel to the contrary that it is a strength.[21] As argued in the introduction, a theory of probability should not deliver structure, such as the expected utility functional form restriction, that is not germane to the use of probabilities. In this sense, restrictions on the ranking of ambiguous acts constitute excess baggage and it is a virtue of our representation theorem to have avoided them (apart from P3).

It is decidedly *not* the case that the noted silence renders the theorem irrelevant to the analysis of choice when there are ambiguous acts. The contrary is true. As just mentioned and as described further in Section 6, the theorem provides the basis for a unified and coherent treatment of risk aversion and ambiguity aversion. It is precisely *because* our axioms do not restrict the structure of preference over ambiguous acts that the scope of the theorem is not unduly limited. The theorem and the use we make of it apply to the Choquet and multiple-priors models and to any other model of choice among ambiguous acts, subject only to the satisfaction of our axioms regarding (for the most part) the ranking of unambiguous acts.

Finally, consider another possible route to deriving from preference both domains and probability measures on those domains.[22] Simply identify domains $\mathscr{A}^*$ where the Machina-Schmeidler axioms (suitably modified to allow for the fact that $\mathscr{A}^*$ may be only a λ-system and not an algebra) are satisfied and apply their result to derive corresponding probabilities. Evidently, such a process delivers probabilistic sophistication on an endogenous subdomain $\mathscr{F}^*$ of acts.

[19] See the end of Section 3.2.

[20] A qualification is that our theorem delivers a countably additive probability measure, while theirs deals with the more general class of finitely additive subjective priors.

[21] We are indebted to Michelle Cohen and Mark Machina for helping to clarify our thinking on this point.

[22] This approach is discussed further in Section 8.

One might view acts in $\mathscr{F}^*$ as risky and use them to support the Knightian distinction. A problem is that in general there will be several such domains $\mathscr{A}^*$ and $\mathscr{F}^*$ without any guidance for selecting the right one as the foundation for *the* Knightian distinction. Our approach makes such a selection and one that has the merit of being based on an intuitive and behavioral characterization of ambiguity.

Another point of comparison is that we are able to deliver a probability measure on $\mathscr{A}$ with relatively weak axioms. In particular, we do *not assume* the Sure-Thing-Principle on $\mathscr{F}^{ua}$ nor that the counterpart axiom used by Machina and Schmeidler is satisfied on $\mathscr{F}^{ua}$. We are able to avoid assuming these because, unlike these earlier authors, we are interested in deriving probabilities only on unambiguous events *and because 'unambiguous' has been defined in an appropriate way*.

6. ATTITUDES TOWARD RISK AND AMBIGUITY

Once one accepts the distinction between risk and ambiguity as meaningful, it follows that attitudes toward risk and toward ambiguity constitute conceptually distinct aspects of preference. Therefore, one would like a modeling approach that affords a clear distinction between them. The key to the latter is the sharp distinction between risk and ambiguity that is provided by Theorem 5.2. In this section, we show how the theorem leads to natural definitions for attitudes towards risk and ambiguity.[23]

Consider first risk attitudes.[24] The prior question is what constitutes risk? From Theorem 5.2, unambiguous acts are probabilistic prospects and it would seem natural to define risk attitudes in terms of the ranking of these acts. Moreover, given that $\succcurlyeq$ is probabilistically sophisticated on $\mathscr{F}^{ua}$, with unique probability measure p on $\mathscr{A}$, the familiar definition can be adapted. Therefore, say that $\succcurlyeq$ is *risk averse* if

$$\text{(6.1)} \qquad \int_S h(s)\, dp \succcurlyeq h, \quad \text{for all } h \text{ in } \mathscr{F}^{ua},$$

where the vector-valued integrals are interpreted as constant acts. Risk loving is defined by reversing the direction of preference; risk neutrality is the conjunction of risk aversion and risk loving.

Implicit in this definition of risk aversion are two normalizations: (i) the identification of riskless prospects with degenerate lotteries or constant acts, and (ii) the identification of risk-neutrality with expected value preferences. A

[23] The approach is adapted from Epstein (1999), where it is discussed at greater length. The difference here is that the reference class of unambiguous events is derived from preference, while it is specified exogenously by the modeler in the cited paper. The latter study discusses also the comparative notions "more risk averse than" and "more ambiguity averse than." Given the length of this paper, we do not pursue these here.

[24] For what follows assume that $\mathscr{X}$ is a convex subset of $\mathscr{R}^n$.

definition of ambiguity aversion (or affinity) can be formulated in a similar fashion once one adopts a parallel pair of normalizations that specify (i) unambiguous prospects and (ii) ambiguity neutral preferences. Our analysis suggests a natural choice for each—acts in $\mathscr{F}^{ua}$ constitute the class of unambiguous prospects and (globally) probabilistically sophisticated preferences define the benchmark of ambiguity neutrality.

This leads to the following definition: Say that the preference order $\succcurlyeq$ on $\mathscr{F}$ is *ambiguity averse* if there exists another order $\succcurlyeq^{ps}$ on $\mathscr{F}$, that is probabilistically sophisticated there, such that

$$(6.2) \qquad h \succcurlyeq^{ps} (\succ^{ps}) f \Rightarrow h \succcurlyeq (\succ) f,$$

for all h in $\mathscr{F}^{ua}$ and f in $\mathscr{F}$. Here $\mathscr{F}^{ua}$ denotes the set of subjectively (for $\succcurlyeq$) unambiguous acts as defined in this paper. The interpretation begins with the view that any probabilistically sophisticated order $\succcurlyeq^{ps}$ is indifferent to ambiguity. Accordingly, if $\succcurlyeq^{ps}$ prefers h to f, then so should the ambiguity averse $\succcurlyeq$, because h is unambiguous for $\succcurlyeq$ while $\succcurlyeq$ discounts f further due to its being ambiguous.

Ambiguity loving is defined by reversing the directions of preference in (6.2); ambiguity neutrality is the conjunction of ambiguity aversion and ambiguity loving.

Because of the subjective nature of ambiguity, one might expect a confounding between the absence of ambiguity on the one hand, and the presence of ambiguity combined with indifference to it on the other. The source of such confounding in our approach is in the identification of probabilistically sophisticated preference as ambiguity neutral even though probabilistically sophisticated decision-makers do not perceive any ambiguity. There is a similar confounding of the absence of risk with indifference to it. If $\mathscr{A} = \{\varnothing, S\}$, then only constant acts are unambiguous and perceived as risky. Yet such a decision-maker is deemed to be risk neutral.[25]

However, a more important separation *is* afforded by our model, namely that between attitudes towards risk and attitudes towards ambiguity. Informally, such a separation is possible because risk attitudes concern the nature of preference *within* $\mathscr{F}^{ua}$, while attitudes toward ambiguity concern primarily the way in which acts in $\mathscr{F}^{ua}$ are related to acts *outside* $\mathscr{F}^{ua}$ (see (6.2)). The separation that is achieved is the following: For each of risk and ambiguity, the following three (exhaustive but overlapping) attitudes are possible—aversion, affinity and 'none of the above' (that is, neither aversion nor affinity holds globally). All nine logical possibilities are admitted by our definitions; for example, it is possible to model a decision-maker who loves risk but dislikes ambiguity, and so on. This is amply demonstrated by the Choquet-expected-utility model to which we now turn (see Corollary 7.4).

[25] Strictly speaking, this extreme is ruled out by our axioms.

7. CHOQUET EXPECTED UTILITY

Here we focus on preference that conforms to Schmeidler's (1989) Choquet expected utility (CEU) model. This is one of the major alternatives to subjective expected utility theory in the context of 'ambiguity.' Therefore, it is appropriate to illustrate our definitions and to demonstrate their tractability in the context of this specific model. We have much less to say for the multiple-priors model and therefore we confine ourselves to CEU.

For CEU, preference is represented by U^{ceu}, where[26]

$$U^{ceu}(f) = \int_S u(f)\, dv. \tag{7.1}$$

Here $v : \Sigma \to [0,1]$ is a capacity and $u : \mathscr{X} \to \mathscr{R}^1$, with $u(\mathscr{X})$ having nonempty interior. Preference so defined satisfies P3 if and only if v satisfies: For all disjoint events E and B,

$$v(E \cup B) = v(B) \Leftrightarrow v(E) = 0. \tag{7.2}$$

Define core(v), the core of v, as in cooperative game theory, that is, as the set of all finitely additive probability measures p on S satisfying

$$p(A) \geq v(A), \quad \text{for all } A \subset S.$$

Say that v is *exact* if

$$v(A) = \min_{m \in \text{core}(v)} m(A), \quad \text{for all } A \subset S. \tag{7.3}$$

A stronger property that is widely assumed is convexity, where v is *convex* if

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B),$$

for all events A and B.[27]

A special case of CEU preferences that is particularly relevant here has

$$v = \phi(p), \tag{7.4}$$

for some probability measure p and some strictly increasing $\phi : [0,1] \to [0,1]$. The resulting preference order is probabilistically sophisticated and thus all events in Σ are unambiguous in this case, that is, $\mathscr{A} = \Sigma$. (Probabilistic sophistication follows by verifying that the risk preference functional W required by the definition in Section 5 can be taken to be

$$W(P) = \int_{\mathscr{X}} ud(\phi(F_P)),$$

[26] v is a capacity if it maps Σ into $[0,1]$, $v(E') \geq v(E)$ whenever $E' \supset E$, $v(\varnothing) = 0$, and $v(S) = 1$. The indicated integral is a Choquet integral and equals $\sum_{i=1}^n u_i[v(\bigcup_{j=i}^n E_j) - v(\bigcup_{j=i+1}^n E_j)]$ if $E_i = \{s : u(f(s)) = u_i\}$ and $u_1 < \cdots < u_n$.

[27] See Schmeidler (1972) for the relation between convexity and exactness. Note that the CEU function (7.1) is a member of the multiple-priors class if and only if v is convex.

for any lottery P, where F_P is the cumulative distribution function corresponding to the probability measure P on $\mathscr{X}$. Such functionals W correspond to the rank-dependent-expected-utility model that has been studied in the theory of preference over lotteries.)[28]

For general capacities, the characterization of $\mathscr{A}$ is of interest. On purely formal (or mechanical) grounds, thinking of the capacity as analogous to a probability measure, albeit nonadditive, each of the following classes of events seems plausible as a conjecture for how to characterize unambiguous events:

$$\mathscr{A}_0 = \{T \subset S : v(T + A) = vT + vA \text{ for all } A \subset T^c\},$$
$$\mathscr{A}_1 = \{T \subset S : vT + vT^c = 1\}, \qquad \text{and}$$
$$\mathscr{A}_2 = \bigcap_{m \in \text{core}(v)} \{A \subset S : mA = vA\}.$$

The first two classes may seem natural because they capture forms of additivity of the capacity. The class $\mathscr{A}_2$ consists of those events on which all measures in the core of v agree, which also seems to reflect a lack of ambiguity.

In general, $\mathscr{A}_0 \subset \mathscr{A}_1$. When v is exact, these three sets coincide (see Ghirardato and Marinacci (1999)).

LEMMA 7.1: *If v is exact, then $\mathscr{A}_0 = \mathscr{A}_1 = \mathscr{A}_2$.*

However, none of the above classes coincides with the class $\mathscr{A}$ of subjectively unambiguous events. This is illustrated starkly by taking v to be a distortion of a probability measure as in (7.4), with the distortion ϕ being strictly convex. Then[29]

$$\mathscr{A} = \Sigma \qquad \text{and} \qquad \mathscr{A}_0 = \mathscr{A}_1 = \{\varnothing, S\}. \tag{7.5}$$

The next lemma provides a complete characterization of $\mathscr{A}$ (see Appendix D for a proof).[30]

LEMMA 7.2: *T is unambiguous if and only if: for all events A and B contained in T^c and for all $C \subset T^c \setminus (A \cup B)$,*

$$v(A) \geq v(B) \Leftrightarrow v(A \cup T) \geq v(B \cup T); \qquad \textit{and if} \tag{7.6}$$
$$(v(A) - v(B))(v(A \cup C) - v(B \cup C)) < 0, \qquad \textit{then} \tag{7.7}$$
$$v(A) - v(B) = v(A \cup T) - v(B \cup T) \qquad \textit{and} \tag{7.8}$$
$$v(A \cup C) - v(B \cup C) = v(A \cup C \cup T) - v(B \cup C \cup T); \tag{7.9}$$

and the above conditions are satisfied also by T^c.

[28] See Chew, Karni, and Safra (1987).

[29] Under these assumptions, v is a convex capacity, hence exact.

[30] The characterizing conditions on the capacity may appear ugly; they are more complex than those used to define $\mathscr{A}_0$ and $\mathscr{A}_1$, for example. However, it is clearly the appeal of the underlying behavioral definition of ambiguity that is the critical esthetic factor.

If the 'reversal' in (7.7) never occurs, then v is (almost) a qualitative probability within T^c.[31] In that case, the ordinal condition (7.6) alone corresponds to 'T unambiguous.' However, when v fails to be a qualitative probability within T^c, then 'T unambiguous' requires that the cardinal conditions in (7.8) and (7.9) obtain.

Observe that the above conditions do not involve u, which therefore has nothing to do with ambiguity.

Next we relate Theorem 5.2 to the CEU model. Say that a capacity v is *convex-ranged on* $\mathscr{A}$ if for every A in $\mathscr{A}$,

$$[0, vA] = \{vB : B \in \mathscr{A}, B \subset A\}.$$

Say that v is *continuous* if for all events in S,

$$vA_n \searrow v(\cap A_n) \quad \text{if} \quad A_n \searrow \qquad \text{and}$$
$$vA_n \nearrow v(\cup A_n) \quad \text{if} \quad A_n \nearrow .$$

The following is largely a corollary of Theorem 5.2:

COROLLARY 7.3: *Let* $\succcurlyeq$ *be a CEU preference order with capacity* v *satisfying* (7.2) *and subjectively unambiguous events* $\mathscr{A}$.

(a) *Suppose that* v *is continuous and convex-ranged on* $\mathscr{A}$. *Then* $\succcurlyeq$ *satisfies the axioms in Theorem* 5.2. *Thus* $\succcurlyeq$ *is probabilistically sophisticated on* $\mathscr{F}^{ua}$ *relative to a convex-ranged and countably additive probability measure* p *on* $\mathscr{A}$. *Moreover*, $v = \phi(p)$ *on* $\mathscr{A}$ *for some strictly increasing and onto map* $\phi : [0,1] \to [0,1]$.

(b) *Suppose in addition that* $v(\mathscr{A}_0) = [0,1]$, *and that* $\mathscr{A}_0$ *is closed with respect to complements*. *Then* $v = p$ *on* $\mathscr{A}$ *and* $\succcurlyeq$ *restricted to* $\mathscr{F}^{ua}$ *has an expected utility representation.*

(c) *Suppose that* v *is continuous and convex-ranged on* $\mathscr{A}$, *that* v *is exact*, *and that*

$$(7.10) \qquad v(A) + v(A^c) = 1 \quad \textit{and} \quad 0 < v(A) < 1 \qquad \textit{for some } A \in \Sigma.$$

Then $v = p$ *on* $\mathscr{A}$ *and*

$$(7.11) \qquad \mathscr{A} = \mathscr{A}_0 = \mathscr{A}_1 = \mathscr{A}_2 .$$

Part (a) shows that the scope of Theorem 5.2 extends well beyond globally probabilistically sophisticated preferences. Part (b) gives conditions under which the CEU preference order is expected utility (and not merely probabilistically sophisticated) on the domain of unambiguous acts. Thus under the assumptions made here ambiguity is the only reason for deviating from expected utility. However, only under the stronger conditions of part (c) can we conclude that $\mathscr{A}$ coincides with all classes of events discussed earlier. Differences such as those

[31] The key defining property of a qualitative probability on T^c is ordinal additivity: For all events E, F, and G, subsets of T^c such that G is disjoint from E and F, $vE \geq vF$ if and only if $v(E \cup G) \geq v(F \cup G)$.

described in (7.5) are eliminated, in part, through the assumption that $v(\mathscr{A}_1) \cap (0,1)$ is nonempty.[32] The equalities in (7.11) are valuable also because they provide explicit descriptions of $\mathscr{A}$ under the stated assumptions.

Finally, consider risk and ambiguity attitudes for the CEU model. We make use of the *conjugate* capacity v^*, defined by

$$v^*(E) = 1 - v(E^c), \quad \text{for all } E \text{ in } \Sigma.$$

COROLLARY 7.4: *Let $\succcurlyeq$ be a CEU order with utility index u and capacity v satisfying the conditions in part (a) of the previous corollary. Let p and ϕ be as provided there. Then:*

(a) *$\succcurlyeq$ is ambiguity averse (loving) if and only if there exists a probability measure m on Σ satisfying*

$$(7.12) \qquad m(\cdot) \geq (\leq) \phi^{-1}(v(\cdot)) \quad \text{on } \Sigma \qquad \text{and} \qquad m(\cdot) = p(\cdot) \quad \text{on } \mathscr{A};$$

$\succcurlyeq$ is risk averse (loving) if and only if u is concave and $\phi(t) \leq t$ on $[0,1]$ *(u is convex and $\phi(t) \geq t$ on* $[0,1]$).

(b) *A sufficient condition for ambiguity aversion (loving) is that, in addition to the maintained assumptions, v (its conjugate v^*) be exact and satisfy* (7.10). *In that case, $\succcurlyeq$ is risk averse (loving) if and only if u is concave (u is convex).*

The corollary permits a comparison with Schmeidler's (1989) definition of ambiguity and risk aversion.[33] In his approach, the capacity alone models the attitude towards ambiguity and u alone models risk attitudes. For example, ambiguity aversion (as defined by Schmeidler) is equivalent to convexity of the capacity and risk aversion is equivalent to concavity of u. In contrast, in our approach, part (a) of the corollary shows that the capacity, via ϕ, plays a role in determining risk attitudes. Only in special cases, such as in part (b), are the separate roles claimed by Schmeidler for u and v justified. Moreover, while convexity of v (which implies exactness) is sufficient for ambiguity aversion given the additional assumptions in (b), it is not necessary under those conditions.

One reason for this difference in the two definitions is that Schmeidler's is (formulated and) motivated within the Anscombe-Aumann framework of two-stage, horse-race/roulette-wheel acts, while intuition and interpretations that are appealing in this setting are not easily transferred to the Savage domain. In this connection, Kreps (1988, p. 101) states that adoption of the Anscombe-Aumann domain is a problematic practice in descriptive applications *where only choices between Savage acts are observable*. Epstein (1999) elaborates on this point in the specific context of the meaning of ambiguity aversion. He shows, for

[32] We owe this strong form of the result in (c) to Massimo Marinacci. Earlier versions of the paper assumed $v(\mathscr{A}_1) = [0,1]$ rather than merely (7.10).

[33] Schmeidler does not define ambiguity; thus a comparison with our definition of ambiguous event or act is not pertinent.

example, that convexity of the capacity is neither necessary nor sufficient for choice between Savage acts that intuitively speaking reflects ambiguity aversion. In contrast, our approach is focussed on behavior in the Savage domain and, we would argue, captures better intuition related to that domain.

8. ALTERNATIVE DEFINITIONS

Some alternative approaches to defining ambiguity are examined here in light of the desiderata set out earlier and the limitations of our approach acknowledged in the introduction (Section 1.4). In particular, connections to the literature on ambiguity are described.

8.1. *Linearly Unambiguous*

We motivated our definition in part by the suggestion that a necessary condition for an event to be unambiguous is that it be 'separable' from events in its complement. The following alternative definition embodies a stronger form of separability and therefore warrants some attention.[34]

DEFINITION 8.1: An event T is *linearly unambiguous* if: (a) for all acts f' and f and all outcomes z and z',

$$(8.1)\qquad \begin{pmatrix} f'(s) & \text{if } s \in T^c \\ z & \text{if } s \in T \end{pmatrix} \succcurlyeq \begin{pmatrix} f(s) & \text{if } s \in T^c \\ z & \text{if } s \in T \end{pmatrix}$$
$$\Rightarrow \begin{pmatrix} f'(s) & \text{if } s \in T^c \\ z' & \text{if } s \in T \end{pmatrix} \succcurlyeq \begin{pmatrix} f(s) & \text{if } s \in T^c \\ z' & \text{if } s \in T \end{pmatrix};$$

and (b) the condition obtained if T is everywhere replaced by T^c in (a) is also satisfied. Otherwise, say that T is *linearly ambiguous*.

It is apparent that if T is linearly unambiguous, then it is also unambiguous. The former is a stronger property because the indicated invariance is required for all acts f' and f and not just for the subclass of 'conditional binary acts' as in (3.3). The economic significance of this difference was touched upon in the discussion immediately following our definition and is similar to the discussion in Section 4.2 of Machina and Schmeidler (1992). In any event, it is apparent that linear ambiguity embodies a stronger form of separability than does ambiguity. Why not use it as the key notion?

One answer is that (8.1) is too demanding to correspond to the intuitive notion of ambiguity. The invariance required by (8.1) may be violated because the decision-maker views outcomes in different states as complementary or substitutable for reasons that have nothing to do with ambiguity. For example,

[34] The definition recalls the statement of the Sure-Thing-Principle but differs in that the ranking invariance in (8.1) is required only for subacts z and z' that are constant on T.

she may be probabilistically sophisticated, thus assigning probabilities to all events and translating any act into the induced lottery over outcomes, but then she might evaluate the lottery by a risk-preference utility functional that is not linear in probabilities (violating the Independence Axiom). Decision-makers who behave as in the Allais Paradox are of this sort. Many events would be linearly ambiguous for such decision-makers, though it seems intuitively that ambiguity has nothing to do with their preferences. In contrast, for such (probabilistically sophisticated) decision-makers all events are unambiguous (see the end of Section 3.2). Roughly speaking, our formal definition of ambiguity relates to behavior in the Ellsberg Paradox but not the Allais Paradox, while linear ambiguity confounds the two.[35] A similar point is made below regarding the definition of ambiguity proposed by Ghirardato and Marinacci (1999).

That is not to dispute the potential usefulness of the notion of linear ambiguity. Zhang (1997), who originated the notion, shows that it can help to provide an expected utility theory that is 'fully subjective' in the sense of the introduction. The linearity of the expected utility function explains the choice of terminology.

8.2. '*Equally Ambiguous*'

Another alternative is motivated by the intuition that, if A and B are unambiguous, then it should be the case that

$$(8.2) \qquad A \succcurlyeq_{\ell} B \Leftrightarrow A \cup C \succcurlyeq_{\ell} B \cup C,$$

for all C disjoint from both A and B. It is easy to see that this equivalence is true given our definition when C is also subjectively unambiguous, but it is not true more generally. Therefore, one might employ (8.2) as the basis for a new definition. For example, say that 'A and B are unambiguous' if (8.2) is satisfied for all suitably disjoint C. Evidently, this defines a *binary* relation on events that might be better described in terms such as 'A and B are equally ambiguous.' Such a relation may prove interesting, but it evidently does not deliver a notion of (absolute rather than relative) ambiguity; in particular, it does not deliver a unique class of 'unambiguous' events.[36]

There is another useful perspective on the above approach. A slight (and natural) variation on the above is to say that 'A and B are equally ambiguous' if

[35] Readers who attach little importance to the Allais Paradox may feel that such a view would justify using (8.1) in place of (3.3). We feel that it argues for imposing a form of the Sure-Thing Principle on the subdomain of unambiguous acts, rather than for changing the meaning of unambiguous.

[36] The binary relation will typically not be an equivalence relation; for example, both $\varnothing$ and S are equally ambiguous as any other event. We emphasize that we are not advocating this approach. However, something along the lines of (8.2) may occur to some readers and thus we wish to clarify its relation to our definition.

for all x^*, x, f, and g,

$$\begin{aligned}&\left(x^* \text{ if } A;\ x \text{ if } B;\ f(\cdot) \text{ if } (A \cup B)^c\right)\\ &\quad \succcurlyeq \left(x \text{ if } A;\ x^* \text{ if } B;\ f(\cdot) \text{ if } (A \cup B)^c\right)\\ &\quad \Leftrightarrow \left(x^* \text{ if } A;\ x \text{ if } B;\ g(\cdot) \text{ if } (A \cup B)^c\right)\\ &\quad \succcurlyeq \left(x \text{ if } A;\ x^* \text{ if } B;\ g(\cdot) \text{ if } (A \cup B)^c\right).\end{aligned}$$

(Here we are restricting attention to A and B that are mutually disjoint. Apart from this restriction, one obtains (8.2) as the special case $x^* \succ x$, $f(\cdot) = x$, $g(\cdot) = x^*$ on C and $g(\cdot) = x$ on $(A \cup B \cup C)^c$, where C is disjoint from $A \cup B$.) In words, this equivalence expresses the intuition that ambiguity has to do with "the effect on own likelihood of a change in the outcomes on other events." Given that "own likelihood" refers to the *ordinal* likelihood relation underlying preference, and we see no alternative possible interpretation, then it is evident why this intuition relates at best to a notion of relative ambiguity.

Another approach that leads to multiple collections of 'unambiguous' events is the following.[37] Though probabilistic sophistication may not prevail globally, because of 'ambiguity,' one might identify subdomains of acts where it does prevail and view each such subdomain as consisting of unambiguous acts. This is essentially what is done by Sarin and Wakker (1992) and Jaffray and Wakker (1994), with the exception that 'expected utility' is used as the reference model rather than 'probabilistic sophistication.' Again, this approach at best relates to 'equally ambiguous.'[38]

Would it make sense to view all acts in the union of the above subdomains as unambiguous? We think not. Probabilistic sophistication on a subdomain of acts $\mathscr{F}_i$ reflects exclusively the nature of the ranking of acts *within* $\mathscr{F}_i$ and similarly for any other subdomain $\mathscr{F}_j$. On the other hand, the designation of $\mathscr{F}_i \cup \mathscr{F}_j$ as a set of unambiguous acts must surely be based on some aspect of the ranking of acts in $\mathscr{F}_i$ relative to those in $\mathscr{F}_j$ (not to mention also some acts that are not in $\mathscr{F}_j$). However, no such implications follow from probabilistic sophistication within each of $\mathscr{F}_i$ and $\mathscr{F}_j$; for example, probabilistic sophistication on the union is not implied. We elaborate and illustrate this discussion in Section 9.2, where it is also shown that defining unambiguous acts by a union as above leads to a counterintuitive designation.

One could alternatively employ the intersection of such subdomains, or a suitable variant of the intersection, as a way to deliver a unique set of 'unambiguous acts.' It would occasionally be 'far too small,' as in the example in Section 9.2. More importantly, the behavioral meaning of such a set seems totally unclear. In particular, such a definition, that is expressed in terms of functional form representation, does not answer the question noted in desidera-

[37] See also the discussion following Theorem 5.2.

[38] We emphasize that the two cited papers are not concerned explicitly with defining ambiguity and thus our comments should not be interpreted as criticisms of these papers.

tum D1, namely "what behavior would indicate that the decision-maker perceives a specific event or act as ambiguous?" Providing an answer to this question seems to us to be the point. (This criticism applies equally to the union-based definition.)

8.3. *Ghirardato and Marinacci*

Ghirardato and Marinacci (1999) define ambiguity in terms of preference.[39] They provide a detailed comparison with our approach, but some comparison here seems appropriate.[40]

We begin with a definition of ambiguity and use it to define attitudes towards ambiguity. Ghirardato and Marinacci adopt the reverse order and define ambiguity aversion (and loving) first. One consequence of this reversal in order is that their definition of subjective ambiguity applies only to preference orders that are either ambiguity averse or ambiguity loving, thus excluding preferences where neither attitude prevails globally. At a conceptual level, we see no justification for such a limitation.

An indication of the difference between the two definitions is that in the Ghirardato-Marinacci approach, the absence of subjective ambiguity ($\mathscr{A} = \Sigma$) is equivalent to preference conforming to subjective expected utility theory, while for us it is equivalent to the probabilistic sophistication of preference.[41] We try to identify the differences in approach that lead to this difference in designations.

The essential problem in both papers is the distinction between risk and ambiguity. Roughly speaking, we make the distinction through our behavioral definition of ambiguous events and acts; all others are risky. Ghirardato and Marinacci make the distinction by identifying the class of purely risky prospects or comparisons through their use of 'cardinal symmetry.' The essence of their approach is that the *only* rankings that clearly involve *only* risk are those of the form

$$(x \text{ if } A;\, x' \text{ if } A^c) \qquad \text{vs} \qquad (y \text{ if } A;\, y' \text{ if } A^c),$$

where $x \succ x'$ and $y \succ y'$, that is, comparisons involving two bets on the same event A but having different stakes. Aspects of preference that are not tied down by such comparisons are interpreted as involving a concern with ambiguity. As a consequence, they impute much more importance to ambiguity as opposed to risk than do we.

To illustrate, consider a probabilistically sophisticated preference order that lies in both the Choquet-expected-utility class and in the multiple-priors class;

[39] Ambiguity is taken as a primitive in Fishburn (1991) and Epstein (1999). Our focus is on models where preference is the only primitive, corresponding to desideratum D1.

[40] We focus exclusively on the definition of ambiguity. There are other major differences; for example, Ghirardato and Marinacci do not provide a counterpart of our Theorem 5.2.

[41] We pointed out at the end of Section 3.2 that (global) probabilistic sophistication implies $\mathscr{A} = \Sigma$. The converse is implied by Theorem 5.2.

namely, take the special Choquet utilities defined by (7.4) where ϕ is convex. According to their definition, *all* events that are non-null and have non-null complements are necessarily ambiguous if preference is not an expected utility order. To see the significance of this designation, consider the choice situation corresponding to the Allais Paradox. This situation is typically described in terms of choice between lotteries. However, Savage (p. 103) translates it into an act framework where objects of choice are acts over a state space. (Roughly, the decision-maker is told there are 100 tickets in an urn, with numbers 1 through 100; and each act promises a specified prize depending on the number of the randomly drawn ticket. For example, one act pays \$0 if the number is 1, \$25 if the number lies between 2 and 11, and \$5 otherwise.) If preference is probabilistically sophisticated as above and 'nonlinear' in a way that produces the paradoxical Allais-type choices, then the Ghirardato-Marinacci definition would designate some events (subsets of ticket numbers) as ambiguous. Essentially, *they identify* (*nonindifference to*) *ambiguity as the explanation for Allais-type choices*.

Of course, it is possible that in the choice situation, the decision-maker does not accept the information provided about the content of the urn and that ambiguity is a factor in her making the 'paradoxical' choices. It is very much an *assumption* on our part, that is implicit in our definition, that ambiguity does not play a role in similar situations. Put another way, an implicit assumption on our part is that a probabilistically sophisticated agent who translates acts into induced lotteries over outcomes, evaluates those lotteries in precisely the same way as she would evaluate those lotteries were they presented to her as objective lotteries and hence as risky prospects. Though undoubtedly an assumption, a similar identification is common practice in the standard models whereby once a subjective expected utility decision-maker has her prior, we typically view her as a vNM decision-maker having the same vNM index. In any event, we are led to interpret an evaluation of induced lotteries that is 'nonlinear' in the sense of deviating from vNM theory as reflective exclusively of preferences over risky prospects. From our perspective, the Ghirardato-Marinacci approach confounds the Allais and Ellsberg Paradoxes, much as occurs with the notion of linear ambiguity, and attributes 'too much' importance to ambiguity as opposed to risk.[42]

9. ELLSBERG URNS

Further perspective on our approach to ambiguity is provided by examining choice situations involving Ellsberg urns.

[42] See Lemma D.1 in the Appendix for a further illustration, within the concrete setting of CEU, of the difference between 'ambiguity' according to the two definitions.

9.1. *Two Colors*

A single urn containing balls that are either red or blue is informative regarding our definition. The decision-maker is told only the total number of balls in the urn and is to rank bets on the color of a randomly drawn ball. The natural state space consists of the two points R and B. If each is non-null, then both singleton sets are necessarily unambiguous according to the formal definition.[43] As for intuition, it is not clear. On the one hand, one might feel that when no information is provided about the color composition of the urn, then many decision-makers would perceive each singleton event as ambiguous. On the other hand, it is not clear what behavior would reflect such ambiguity; and in the choice-theoretic tradition, it is only such behavior that renders meaningful the designation of an event as ambiguous. Because we have adopted the view that the behavioral manifestation of ambiguity is 'nonseparability', it is not surprising that our definition leads to each singleton event being designated as unambiguous. Just as in consumer demand theory, if there are only two goods (and if preference is suitably monotone), then each good is weakly separable, so too here there is insufficient scope for the separability required by our definition to have any bite.

9.2. *Two Urns*

This section describes an example based on Ellsberg's two-urn experiment. The example adds further support for our definition by illustrating its tractability (the condition defining 'subjectively unambiguous' may be checked in concrete settings) and by providing an intuitive designation of ambiguous events.

However, its main purpose is the following: Theorem 5.2 shows that (given our axioms) preference $\succcurlyeq$ is probabilistically sophisticated on the set $\mathscr{F}^{ua}$ of unambiguous acts. One might wonder about the converse; that is, if $\succcurlyeq$ is probabilistically sophisticated on $\mathscr{F}^*$, the set of all acts measurable with respect to some λ-system $\mathscr{A}^*$, then are events in $\mathscr{A}^*$ unambiguous? The example shows that the answer is no; in general, there may be several λ-systems $\mathscr{A}^*$ as above with the class $\mathscr{A}$ of subjectively unambiguous events being only one of them.[44]

[43] This presumes Savage's Axiom P3.

[44] However, probabilistic sophistication on $\mathscr{F}^*$ implies that all events in $\mathscr{A}^*$ *are* unambiguous in the following restricted sense: The event T in $\mathscr{A}^*$ is $\mathscr{A}^*$-unambiguous if it satisfies our definition when all acts involved in the rankings (3.3) are restricted to be $\mathscr{A}^*$-measurable. Moreover, $\mathscr{A}^*$-unambiguity of all events in $\mathscr{A}^*$ is also sufficient, given suitable reformulations of our axioms, for probabilistic sophistication on $\mathscr{F}^*$. This parallels the result that under suitable conditions, all events being unambiguous characterizes global probabilistic sophistication (see Section 8.3). Such a characterization is a slight restatement of the Machina-Schmeidler characterization of global probabilistic sophistication. The local version outlined above has the added unattractive feature of being self-referential (probabilistic sophistication on $\mathscr{F}^*$ is characterized by means of properties relative to $\mathscr{A}^*$). In contrast, in our Theorem, $\mathscr{A}$ and probabilistic sophistication on $\mathscr{F}^{ua}$ are delivered in an explicit and constructive fashion.

Thus, speaking roughly, unambiguity is strictly stronger than (local) probabilistic sophistication. A related conceptual point that is illustrated by the example is that whether or not an event (or act) is deemed to be ambiguous depends on what is assumed observable by the analyst.

Consider the state space $S = S_1 \times S_2$, where $S_1 = S_2 = \Omega$ and where the finite set Ω represents the possible states in each urn S_i, $i = 1, 2$. Let p be a probability measure on Ω with full support. The decision-maker is told that p describes the distribution of states within the first urn S_1, but she is told less about the second urn S_2. For concreteness, take outcomes $\mathscr{X} \subset \mathscr{R}^1$. Let $\mathscr{F}_i$ be the subset of $\mathscr{F}$ (the set of acts over S), consisting of acts that are measurable with respect to Σ_i, the ith co-ordinate σ-algebra.[45]

In order to address the issues mentioned above, for example, in order to identify unambiguous events or subdomains where probabilistic sophistication prevails, we need to specify a preference order $\succcurlyeq$ on $\mathscr{F}$. In Appendix E, we specify $\succcurlyeq$ having the following features (if $|\Omega| \geq 4$):[46]

1. $\succcurlyeq$ satisfies P3; and the preference to bet on $A_1 \times A_2$ over $B_1 \times B_2$ is independent of the stakes involved (Savage's P4), which implies a complete and transitive likelihood relation $\succcurlyeq_\ell$ on all such rectangles.

2. There is a strict preference for betting on any event $E \subset \Omega$ (satisfying $0 < p(E) < 1$) when it is 'drawn' from S_1 rather than from S_2; that is, $E \times S_2 \succ_\ell S_1 \times E$.

3. The two urns are viewed as independent in the sense that (for all A_i, A'_i, and B_i)

$$A_1 \times A_2 \succcurlyeq_\ell B_1 \times A_2 \Leftrightarrow A_1 \times A'_2 \succcurlyeq_\ell B_1 \times A'_2,$$

$$A_1 \times A_2 \succcurlyeq_\ell A_1 \times B_2 \Leftrightarrow A'_1 \times A_2 \succcurlyeq_\ell A'_1 \times B_2.$$

4. $\succcurlyeq$ is probabilistically sophisticated on $\mathscr{F}_1$ and every set in Σ_1 is *unambiguous*.

5. $\succcurlyeq$ is probabilistically sophisticated on $\mathscr{F}_2$ and every set $S_1 \times B_2$ in Σ_2 is *ambiguous* if $0 < p(B_2) < 1$.

The first three properties show that $\succcurlyeq$ is intuitive, which justifies interest in the lesson to be drawn from the last two properties. The fourth illustrates the intuitive performance of our definition of ambiguity. The final one shows that though preference is probabilistically sophisticated on $\mathscr{F}_2$, every 'nontrivial' act in $\mathscr{F}_2$ (every 'nontrivial' event in Σ_2) is ambiguous. There is clear intuition for the fact that probabilistic sophistication within $\mathscr{F}_2$ does not imply unambiguity. Roughly, the noted structure reflects exclusively the decision-maker's view of events *within* Σ_2. On the other hand, whether or not events in Σ_2 are

[45] $\Sigma_1 = \{B_1 \times S_2 : B_1 \subset S_1\}$ and similarly for Σ_2.

[46] If Ω consists of two elements, then all events are unambiguous for the reasons given in the previous subsection. If Ω consists of three elements, the properties below can be delivered under slightly strengthened assumptions; for example, if the three elements have equal probability under p.

subjectively ambiguous depends also on how they are viewed relative to events *outside* Σ_2 (such as the comparison between drawing the event E from S_2 as opposed to drawing it from S_1).

Suppose, however, that the analyst can observe only the ranking of acts in $\mathscr{F}_2$, which can be identified with acts over the state space S_2. Because preference is probabilistically sophisticated on $\mathscr{F}_2$, our definition applied to the restricted preference and state space would imply that all events in S_2 (equivalently, in Σ_2) are *un*ambiguous. This is perfectly natural in a behavioral approach. Whether or not an event is revealed to the analyst to be ambiguous depends on what is assumed observable by the analyst. In the present case, the subjective ambiguity of $S_1 \times B_2$ in Σ_2 is revealed only through the ranking of some acts not in $\mathscr{F}_2$.

Finally, the example casts some light on alternative definitions of ambiguity. Recall the discussion in Section 8.2 of the use of subdomains where probabilistic sophistication prevails (here $\mathscr{F}_1$ and $\mathscr{F}_2$) as the basis for a definition of unambiguity. The union $\mathscr{F}_1 \cup \mathscr{F}_2$ consists of all acts that are based entirely on one urn, whether the first or the second. There is no intuition for viewing all such acts as unambiguous. The intersection $\mathscr{F}_1 \cap \mathscr{F}_2$ consists of only constant acts. Thus using the intersection to define ambiguity would result in subsets of the first urn, where the probability law is told to the decision-maker, being designated ambiguous. As for the Ghirardato-Marinacci definition, it agrees with ours in this example in designating subsets of the second urn as ambiguous.

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A. APPENDIX: λ-SYSTEM

PROOF OF LEMMA 5.1: The only nontrivial step is to show that if T_1 and T_2 are disjoint unambiguous events, then $T_1 \cup T_2$ is unambiguous. Suppose that for some disjoint subsets A and B of $(T_1 \cup T_2)^c$, act h and outcomes $x^*, x, z, z' \in \mathscr{X}$, that

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \setminus (A \cup B) \\ z & \text{if } s \in T_1 \cup T_2 \end{pmatrix} \succcurlyeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \setminus (A \cup B) \\ z & \text{if } s \in T_1 \cup T_2 \end{pmatrix}.$$

Apply (3.3) in turn for T_2 and then T_1 to deduce

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \setminus (A \cup B) \\ z' & \text{if } s \in T_2 \cup T_1 \end{pmatrix} \succcurlyeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ h & \text{if } s \in (T_1 \cup T_2)^c \setminus (A \cup B) \\ z' & \text{if } s \in T_2 \cup T_1 \end{pmatrix}.$$

Therefore, $T_2 \cup T_1$ satisfies the appropriate form of (3.3).

Next prove that (3.3) is satisfied also by $(T_1 \cup T_2)^c$. By Small Unambiguous Event Continuity (Axiom 4) applied to the unambiguous events T_1 and T_2, there exists a partition $\{A_i\}_{i=1}^n$ of S in $\mathscr{A}$ such that $(T_1 \cup T_2)^c$ equals the finite disjoint union

$$(T_1 \cup T_2)^c = \bigcup_{A_i \subseteq (T_1 \cup T_2)^c} A_i.$$

Thus the first part of this proof establishes (3.3) for $(T_1 \cup T_2)^c$.

To complete the proof, it suffices to show that for any $\{A_n\}_{n=1}^\infty$, a decreasing sequence in $\mathscr{A}$, we have $\bigcap_{i=1}^\infty A \in \mathscr{A}$: By Nondegeneracy, there exist two outcomes $x^* \succ x$. Then

$$f_n = \begin{pmatrix} x^* & \text{if } s \in A_n \\ x & \text{if } s \in A_n^c \end{pmatrix} \in \mathscr{F}^{ua}, \quad \text{for all } n = 1, 2, \dots .$$

By Monotone Continuity (Axiom 5),

$$f_\infty = \begin{pmatrix} x^* & \text{if } s \in \bigcap_{n=1}^\infty A_n \\ x & \text{if } s \in \left(\bigcap_{n=1}^\infty A_n\right)^c \end{pmatrix} \in \mathscr{F}^{ua}.$$

Consequently, $\bigcap_{n=1}^\infty A_n \in \mathscr{A}$. *Q.E.D.*

B. APPENDIX: EXISTENCE OF PROBABILITY

The first step in proving Theorem 5.2 is to prove the existence of a probability measure representing $\succcurlyeq_\ell$. This appendix states a theorem (proven by Zhang (1999)) that delivers such a probability measure given suitable properties for $\succcurlyeq_\ell$. The theorem extends Theorem 14.2 of Fishburn (1970) to the present case of a λ-system of events. Next it is shown that these properties are implied by the axioms adopted for $\succcurlyeq$, as specified in Theorem 5.2.

For the following theorem, $\mathscr{A}$ denotes *any* λ-system and $\succcurlyeq_\ell$ is *any* binary relation on $\mathscr{A}$; that is, they are not necessarily derived from $\succcurlyeq$, though the subsequent application is to that case. Define $\mathscr{N}(\varnothing) = \{A \in \mathscr{A} : A \sim_\ell \varnothing\}$.

THEOREM B.1: *There is a unique finitely additive, convex-ranged probability measure p on $\mathscr{A}$ such that*

$$A \succcurlyeq_\ell B \Leftrightarrow p(A) \geq p(B), \qquad \forall A, B \in \mathscr{A}$$

if (and only if) $\succcurlyeq_\ell$ satisfies the following:

F1 : $\varnothing \preccurlyeq_\ell A$, *for any* $A \in \mathscr{A}$.
F2 : $\varnothing \prec_\ell S$.
F3 : $\succcurlyeq_\ell$ *is a weak order*.
F4 : *If* $A, B, C \in \mathscr{A}$ *and* $A \cap C = B \cap C = \varnothing$, *then* $A \prec_\ell B \Leftrightarrow A \cup C \prec_\ell B \cup C$.
F4′ : *For any two uniform partitions* $\{A_i\}_{i=1}^n$ *and* $\{B_i\}_{i=1}^n$ *of* S *in* $\mathscr{A}$, $\bigcup_{i \in I} A_i \sim_\ell \bigcup_{i \in J} B_i$, *if* $|I| = |J|$.

F5 : (i) *If* $A \in \mathscr{A} \setminus \mathcal{N}(\varnothing)$, *then there is a finite partition* $\{A_1, A_2, \ldots, A_n\}$ *of* S *in* $\mathscr{A}$ *such that* (1) $A_i \subset A$ *or* $A_i \subset A^c$, $i = 1, 2, \ldots, n$; (2) $A_i \prec_\ell A$, $i = 1, 2, \ldots, n$.

(ii) *If* A, B, $C \in \mathscr{A} \setminus \mathcal{N}(\varnothing)$ *and* $A \cap C = \varnothing$, $A \prec_\ell B$, *then there is a finite partition* $\{C_1, C_2, \ldots, C_m\}$ *of* C *in* $\mathscr{A}$ *such that* $A \cup C_i \prec_\ell B$, $i = 1, 2, \ldots, m$.

F6 : If $\{A_n\}$ *is a decreasing sequence in* $\mathscr{A}$ *and if* $A_* \prec_\ell \cap_0^\infty A_n \prec_\ell A^*$ *for some* A_* *and* A^* *in* $\mathscr{A}$, *then there exists* N *such that* $A_* \prec_\ell A_n \prec_\ell A^*$ *for all* $n \geq N$.

Axioms F1, F2, F3, and F4 are similar to those in Fishburn's Theorem 14.2, while F5 strengthens the corresponding axiom there. The additional axioms F4′ and F6 are adopted here to compensate for the fact that $\mathscr{A}$ is not a σ-algebra.

For the remainder of the Appendix, $\mathscr{A}$, $\succcurlyeq_\ell$ and $\succcurlyeq$ are as specified in Theorem 5.2 and the axioms stated in (a) are assumed. By Lemma 5.1, $\mathscr{A}$ is a λ-system. The objective now is to prove that conditions F1–F6 are implied by the axioms given for $\succcurlyeq$. Proofs that are elementary are not provided.

LEMMA B.2: *Let* $\{A_i\}$ *be a uniform partition of* S *in* $\mathscr{A}$. *Then for all outcomes* $\{x_i\}$ *and for all permutations* σ,

$$(B.1) \qquad (x_{\sigma(i)}, A_i)_i \sim (x_i, A_i)_i.$$

In other words, every uniform partition is strongly uniform.

PROOF: Without loss of generality, assume $x_1 \succ x_2$ and that

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ x_3 & \text{if } s \in A_3 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} \succ \begin{pmatrix} x_2 & \text{if } s \in A_1 \\ x_1 & \text{if } s \in A_2 \\ x_3 & \text{if } s \in A_3 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix}.$$

Since $\{A_i\}_{i=3}^n$ are unambiguous, the appropriate form of (3.3) implies

$$\begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_1^c \end{pmatrix} \succ \begin{pmatrix} x_2 & \text{if } s \in A_2^c \\ x_1 & \text{if } s \in A_2 \end{pmatrix};$$

that is, $A_1 \succ_\ell A_2$, a contradiction. Similarly for the other cases. *Q.E.D.*

By showing that the Axioms 2–6 for $\succcurlyeq$ imply properties F1–F6 for $\succcurlyeq_\ell$, we prove the following theorem.

THEOREM B.3: *Let* $\succcurlyeq$ *be a preference order on* $\mathscr{F}$ *and denote by* $\mathscr{A}$ *the set of all unambiguous events. If* $\succcurlyeq$ *satisfies Axioms* 2–6, *then there exists a unique convex-ranged and countably additive probability measure on* $\mathscr{A}$ *such that*

$$A \succcurlyeq_\ell B \Leftrightarrow p(A) \geq p(B), \quad \textit{for all } A, B \in \mathscr{A}.$$

PROOF: Fix outcomes $x^* \succ x$. Properties F1–F3 for $\succcurlyeq_\ell$ are immediate.

F4: (Note the role played here by the specific definition of $\mathscr{A}$; not any λ-system would do.) If $A \prec_\ell B$, then

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \prec \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix}, \quad \text{or}$$

$$\begin{pmatrix} x^* & \text{if } s \in A \setminus B \\ x & \text{if } s \in B \setminus A \\ h & \text{if } s \in (A \cap B) \cup (C^c \setminus (A \cup B)) \\ x & \text{if } s \in C \end{pmatrix} \prec \begin{pmatrix} x & \text{if } s \in A \setminus B \\ x^* & \text{if } s \in B \setminus A \\ h & \text{if } s \in (A \cap B) \cup (C^c \setminus (A \cup B)) \\ x & \text{if } s \in C \end{pmatrix},$$

where $h = (x^* \text{ if } A \cap B;\ x \text{ if } C^c \setminus (A \cup B))$. Since C is unambiguous,

$$\begin{pmatrix} x^* & \text{if } s \in A \setminus B \\ x & \text{if } s \in B \setminus A \\ h & \text{if } s \in (A \cap B) \cup (C^c \setminus (A \cup B)) \\ x^* & \text{if } s \in C \end{pmatrix} \prec \begin{pmatrix} x^* & \text{if } s \in B \setminus A \\ x & \text{if } s \in A \setminus B \\ h & \text{if } s \in (A \cap B) \cup (C^c \setminus (A \cup B)) \\ x^* & \text{if } s \in C \end{pmatrix},$$

or $A \cup C \prec_\ell B \cup C$. Reverse the argument to prove the reverse implication.

F4′ follows from Lemma B.2 and Axiom 6.

F5 (i): Since $A \succ_\ell \varnothing$,

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succ x = \begin{pmatrix} x & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix}.$$

By Small Unambiguous Event Continuity (Axiom 4), there is a partition $\{A_i\}_{i=1}^n$ of S in $\mathscr{A}$, refining $\{A, A^c\}$ and such that

$$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in A_i \\ x & \text{if } s \in A_i^c \end{pmatrix}, \qquad i = 1, 2, \ldots, n.$$

That is, for each i, $A_i \subset A$ or $A_i \subset A^c$ and in addition, $A_i \prec_\ell A$.

F5 (ii): Let A, B, and C be in $\mathscr{A} \setminus \mathscr{N}(\varnothing)$, $A \cap C = \varnothing$, and $A \prec_\ell B$. Then

$$f = \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in A^c \end{pmatrix} = \begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in C \\ x & \text{if } s \in A^c \setminus C \end{pmatrix} = g.$$

By Small Unambiguous Event Continuity, there exists a partition $\{C_1, \ldots, C_n\}$ of S in $\mathscr{A}$, refining $\{A, C, A^c \setminus C\}$ and such that

$$f = \begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in C_i \\ g & \text{if } s \in C_i^c \end{pmatrix}, \qquad i = 1, 2, \ldots, n.$$

If $C_i \subset C$, then

$$\begin{pmatrix} x^* & \text{if } s \in B \\ x & \text{if } s \in B^c \end{pmatrix} \succ \begin{pmatrix} x^* & \text{if } s \in C_i \\ g & \text{if } s \in C_i^c \end{pmatrix}$$

$$= \begin{pmatrix} x^* & \text{if } s \in C_i \\ x^* & \text{if } s \in A \\ x & \text{if } s \in (A \cup C_i)^c \end{pmatrix} = \begin{pmatrix} x^* & \text{if } s \in A \cup C_i \\ x & \text{if } s \in (A \cup C_i)^c \end{pmatrix},$$

implying that $A \cup C_i \prec_\ell B$.

F6: Implied by Monotone Continuity. *Q.E.D.*

C. APPENDIX: PROOF OF MAIN RESULT

Necessity of the Axioms in Theorem 5.2: The necessity of Nondegeneracy and Weak Comparative Probability is routine. Denote by $\succcurlyeq_D$ the order on $D_p^{ua}(\mathscr{X})$ represented by W.

Small Unambiguous Event Continuity: Let $f \succ g$ and x be as in the statement of the axiom. Denote by $P = (x_1, p_1; \ldots; x_n, p_n)$ and Q the probability distributions over outcomes induced by f and g respectively, and let $\underline{x}$ be a least preferred outcome in $\{x\} \cup \{x_1, x_2, \ldots, x_n\}$. Since $P \succ_D Q \succcurlyeq_D \delta_{\underline{x}}$, mixture continuity and monotonicity with respect to stochastic dominance ensure there exists some sufficiently large integer N such that $W((1-(1/N))P + (1/N)\delta_{\underline{x}}) > W(Q)$. Because p is convex-ranged, we can partition each set A_i into N equally probable events $\{A_{ij}\}_{j=1}^N$ in $\mathscr{A}$. Let $C_k = \bigcup_{i=1}^N A_{ik}$ for $k = 1, 2, \ldots, N$. Then $\{C_k\}_{k=1}^N$ is a partition of S in $\mathscr{A}$, $p(C_k) = 1/N$ and $p(A_i \setminus C_k) = (1 - 1/N)p_i$ for each i and k. Consequently, $[\underline{x}$ if $s \in C_k$; f if $s \notin C_k]$ induces the probability distribution $(1/N)\delta_{\underline{x}} + (1 - 1/N)P$ which is strictly preferred to Q. Combined with monotonicity with respect to first-order stochastic dominance, this yields $[x$ if $s \in C_k$; f if $s \notin C_k] \succcurlyeq [\underline{x}$ if $s \in C_k$; f if $s \notin C_k] \succ g$. Similarly for the other part of the axiom.

Monotone Continuity: Given a decreasing sequence $\{A_n\}_{n=1}^\infty$ in $\mathscr{A}$, $p(A_n) \searrow p(\bigcap_1^\infty A_i)$ by the countable additivity of p. The required convergence in preference is implied by mixture continuity of W. The limit f_∞ lies in $\mathscr{F}^{ua}$ because we are given that $\mathscr{A}$ is a λ-system.

Strong-Partition Neutrality: Immediate from (5.1).

Sufficiency of the Axioms in Theorem 5.2: Let p be the measure in Theorem B.3.

LEMMA C.1: *For unambiguous events A and B*:

(a) *A is null iff $A \sim_\ell \varnothing$.*
(b) *If $A \sim_\ell B \sim_\ell \varnothing$ and $A \cap B = \varnothing$, then $A \cup B \sim_\ell \varnothing$.*

PROOF: (a) Fix $x \succ y$. Let A be null. Then

$$\begin{pmatrix} x & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} \sim \begin{pmatrix} y & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} = y = \begin{pmatrix} x & \text{if } s \in \varnothing \\ y & \text{if } s \in S \end{pmatrix},$$

implying that $A \sim_\ell \varnothing$. If A is not null, then by P3,

$$\begin{pmatrix} x & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} \succ \begin{pmatrix} y & \text{if } s \in A \\ y & \text{if } s \in A^c \end{pmatrix} = \begin{pmatrix} x & \text{if } s \in \varnothing \\ y & \text{if } s \in S \end{pmatrix},$$

implying that $A \succ_\ell \varnothing$.

(b) Let $x \succ y$ and $A \cup B \succ_\ell \varnothing$, that is,

$$\begin{pmatrix} x & A \\ x & B \\ y & (A \cup B)^c \end{pmatrix} \succ y.$$

By (a), A and B are null and

$$y = \begin{pmatrix} y & A \\ y & B \\ y & (A \cup B)^c \end{pmatrix} \sim \begin{pmatrix} x & A \\ y & B \\ y & (A \cup B)^c \end{pmatrix} \sim \begin{pmatrix} x & A \\ x & B \\ y & (A \cup B)^c \end{pmatrix} \succ y.$$

This is a contradiction. *Q.E.D.*

For each $f \in \mathscr{F}^{ua}$, define

$$P_f = (x_1, p(f^{-1}(x_1)); \ldots; x_n, p(f^{-1}(x_n))).$$

Because p is fixed, it may be suppressed in the notation. Accordingly, write

$$P_f \in D^{ua}(\mathscr{X}) = \{P_f : f \in \mathscr{F}^{ua}\}.$$

Define the binary relation $\succcurlyeq_D$ on $D^{ua}(\mathscr{X})$ by

$$P \succcurlyeq_D Q \qquad \text{if} \quad \exists\, f \succcurlyeq g, \quad P = P_f, \quad \text{and} \quad Q = P_g.$$

LEMMA C.2: *If $P_f = P_g$, then $f \sim g$. Thus $\succcurlyeq_D$ is complete and transitive.*

PROOF: We must prove that for any two partitions $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ of S in $\mathscr{A}$, if $A_i \sim_\ell B_i$, $i = 1, 2, \ldots, n$, then for all outcomes $\{x_i\}_{i=1}^n$,

$$\text{(C.1)} \qquad \begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} \sim \begin{pmatrix} x_1 & \text{if } s \in B_1 \\ x_2 & \text{if } s \in B_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in B_n \end{pmatrix}.$$

Case 1: $\{A_i\}_{i=1}^n$ and $\{B_i\}_{i=1}^n$ are uniform partitions of S in $\mathscr{A}$. The desired conclusion follows from Lemma B.2 and Axiom 6.

Case 2: All probabilities $\{p(A_i)\}_{i=1}^n$ and $\{p(B_i)\}_{i=1}^n$ are rational. Because p is convex-ranged, there exist $\{E_j\}_{j=1}^m$ and $\{C_j\}_{j=1}^m$, two uniform partitions of S in $\mathscr{A}$, such that

$$A_i = \bigcup_{E_j \subseteq A_i} E_j, \qquad i = 1, 2, \ldots, n \qquad \text{and}$$

$$B_i = \bigcup_{C_j \subseteq B_i} C_j, \qquad i = 1, 2, \ldots, n.$$

Now Case 1 may be applied.

Case 3: This is the general case where some of the probabilities $p(A_i)$ or $p(B_i)$ may be irrational. Suppose contrary to (C.1) that

$$\text{(C.2)} \qquad f = \begin{pmatrix} x_1 & \text{if } s \in A_1 \\ x_2 & \text{if } s \in A_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1 \\ x_2 & \text{if } s \in B_2 \\ \cdots & \cdots \\ x_n & \text{if } s \in B_n \end{pmatrix} = g.$$

Without loss of generality, assume that $x_n \succ \cdots \succ x_2 \succ x_1$ and $p(A_1) = p(B_1)$ is irrational.

By the convex range of p over $\mathscr{A}$, there are rational numbers $r_m \nearrow_m$ and two increasing sequences $\{A_1^m\}_{m=1}^\infty$ and $\{B_1^m\}_{m=1}^\infty$ in $\mathscr{A}$ with $A_1^m \subset A_1$ and $B_1^m \subset B_1$, $m = 1, 2, \ldots,$ such that $p(A_1^m) = p(B_1^m) = r_m \nearrow p(A_1) = p(B_1)$ as $m \to \infty$. Accordingly,

$$p(A_1 \setminus A_1^m) = p(B_1 \setminus B_1^m) = p(A_1) - p(A_1^m) \searrow 0 \qquad \text{as} \qquad m \to \infty.$$

Thus, both $\{A_1 \setminus A_1^m\}_{m=1}^\infty$ and $\{B_1 \setminus B_1^m\}_{m=1}^\infty$ are decreasing sequences in $\mathscr{A}$ and

$$\text{(C.3)} \qquad \bigcap_{m=1}^\infty (A_1 \setminus A_1^m) \sim_\ell \bigcap_{m=1}^\infty (B_1 \setminus B_1^m) \sim_\ell \varnothing.$$

Define

$$g_m = \begin{pmatrix} x_1 & \text{if } s \in B_1^m \\ x_2 & \text{if } s \in B_1 \setminus B_1^m \\ g & \text{if } s \in B_1^c \end{pmatrix}, \qquad g_\infty = \begin{pmatrix} x_1 & \text{if } s \in \bigcap\limits_{m=1}^\infty B_1^m \\ x_2 & \text{if } s \in B_1 \setminus (\bigcap_{m=1}^\infty B_1^m) \\ g & \text{if } s \in B_1^c \end{pmatrix}.$$

By Lemma C.1 and (C.3),

$$g_\infty \sim g \prec f.$$

By Monotone Continuity, g_m converges to g_∞ in preference as $m \to \infty$. Conclude that there exists an integer N_1 such that

$$g_m \prec f \qquad \text{whenever} \qquad m \geq N_1.$$

In particular,

$$g_{N_1} = \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_1 \setminus B_1^{N_1} \\ g & \text{if } s \in B_1^c \end{pmatrix} \prec f.$$

By P3,

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_1 \setminus A_1^{N_1} \\ f & \text{if } s \in A_1^c \end{pmatrix} \succcurlyeq \begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_1 & \text{if } s \in A_1 \setminus A_1^{N_1} \\ f & \text{if } s \in A_1^c \end{pmatrix} = f \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_1 \setminus B_1^{N_1} \\ g & \text{if } s \in B_1^c \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_2 \cup (A_1 \setminus A_1^{N_1}) \\ f & \text{if } s \in (A_1 \cup A_2)^c \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_2 \cup (B_1 \setminus B_1^{N_1}) \\ g & \text{if } s \in (B_1 \cup B_2)^c \end{pmatrix}.$$

Note further that $A_2 \cup (A_1 \setminus A_1^{N_1}) \sim_\ell B_2 \cup (B_1 \setminus B_1^{N_1})$ since $p(A_2 \cup (A_1 \setminus A_1^{N_1})) = p(B_2 \cup (B_1 \setminus B_1^{N_1}))$. Thus a proof by induction establishes that

$$\begin{pmatrix} x_1 & \text{if } s \in A_1^{N_1} \\ x_2 & \text{if } s \in A_2^{N_2} \\ x_3 & \text{if } s \in A_3^{N_3} \\ \cdots & \cdots \\ x_n & \text{if } s \in A_n^{N_n} \end{pmatrix} \succ \begin{pmatrix} x_1 & \text{if } s \in B_1^{N_1} \\ x_2 & \text{if } s \in B_2^{N_2} \\ x_3 & \text{if } s \in B_3^{N_3} \\ \cdots & \cdots \\ x_n & \text{if } s \in B_n^{N_n} \end{pmatrix},$$

where $A_i^{N_i} \sim_\ell B_i^{N_i}$, $i = 1, 2, \ldots, n$, and every $p(A_i^{N_i}) = p(B_i^{N_i})$ is rational, contradicting Case 2. *Q.E.D.*

The rest of the proof is similar to Steps 2–6 in the proof of Machina and Schmeidler's Theorem 2. For example, in the proof of mixture continuity of $\succcurlyeq_D$ on $D^{ua}(\mathscr{X})$ (Step 3), Small Unambiguous Event Continuity may be used in place of Savage's P6 in order to overcome the lack of a σ-algebra structure for $\mathscr{A}$.

D. APPENDIX: CHOQUET EXPECTED UTILITY

Use "+" to denote disjoint union.

PROOF OF LEMMA 7.2: Sufficiency may be proven by a tedious but routine verification. We prove necessity. It is convenient to express the conditions in the following equivalent form: For all pairwise disjoint events A, B, C, and D, each disjoint from T,

(D.1) $\quad v(A \cup D) \geq v(B \cup D) \Leftrightarrow v(A \cup D \cup T) \geq v(B \cup D \cup T)$; and if

(D.2) $\quad (v(A \cup D) - v(B \cup D))(v(A \cup D \cup C) - v(B \cup D \cup C)) < 0$, then

(D.3) $\quad v(A \cup D) - v(B \cup D) = v(A \cup D \cup T) - v(B \cup D \cup T)$ and

(D.4) $\quad v(A \cup D \cup C) - v(B \cup D \cup C) = v(A \cup D \cup C \cup T) - v(B \cup D \cup C \cup T)$;

and the above conditions are satisfied also by T^c.

Let T be unambiguous. If $x^* \succ x$, then $v(A+D) \geq v(B+D)$ if

$$\begin{pmatrix} x^* & A+D \\ x & S\setminus(A+D) \end{pmatrix} \succcurlyeq \begin{pmatrix} x^* & B+D \\ x & S\setminus(B+D) \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x^* & A+D \\ x & B \\ x & T^c\setminus(A+B+D) \\ x^* & T \end{pmatrix} \succcurlyeq \begin{pmatrix} x^* & B+D \\ x & A \\ x & T^c\setminus(A+B+D) \\ x^* & T \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} x^* & A+D+T \\ x & S\setminus(A+D+T) \end{pmatrix} \succcurlyeq \begin{pmatrix} x^* & B+D+T \\ x & S\setminus(B+D+T) \end{pmatrix}$$

$$\Leftrightarrow \nu(A+D+T) \geq v(B+D+T).$$

Suppose next that (D.2) is satisfied. In fact, suppose that

(D.5) $$v(A+D) < v(B+D) \quad \text{and} \quad v(A+D+C) > v(B+D+C).$$

(The other case is similar.)

The event T is unambiguous only if

(D.6) $$\begin{pmatrix} x^* & \text{if } s \in A \\ x & \text{if } s \in B \\ y & \text{if } s \in C \\ h(s) & \text{if } s \in T^c\setminus(A+B+C) \\ z & \text{if } s \in T \end{pmatrix} \succcurlyeq \begin{pmatrix} x & \text{if } s \in A \\ x^* & \text{if } s \in B \\ y & \text{if } s \in C \\ h(s) & \text{if } s \in T^c\setminus(A+B+C) \\ z & \text{if } s \in T \end{pmatrix}$$

$\Leftrightarrow$ a similar ranking obtains for the acts where z' replaces z. Suppose that h equals $\bar{y}$ on D and $\underline{y}$ on $T^c\setminus(A+B+C+D)$ and that

(D.7) $$\bar{y} \succ x^* \succ y \succ x \succ \underline{y}, \quad z = y \quad \text{and} \quad z' = x^*.$$

Then by the definition of Choquet integration, the above equivalence becomes

$$\begin{aligned} &u(x^*)v(A+D) + u(y)[v(C+T+A+D) - v(A+D)] \\ &\quad + u(x)[v(B+C+T+A+D) - v(C+T+A+D)] \\ &\quad \geq u(x^*)v(B+D) + u(y)[v(C+T+B+D) - v(B+D)] \\ &\qquad + u(x)[v(B+C+T+A+D) - v(C+T+B+D)] \end{aligned}$$

if and only if

$$\begin{aligned} &u(x^*)v(T+A+D) + u(y)[v(C+T+A+D) - v(T+A+D)] \\ &\quad + u(x)[v(B+C+T+A+D) - v(C+T+A+D)] \\ &\quad \geq u(x^*)v(T+B+D) + u(y)[v(C+T+B+D) - v(T+B+D)] \\ &\qquad + u(x)[v(B+C+T+A+D) - v(C+T+B+D)]. \end{aligned}$$

Equivalently,

$$\begin{aligned} &[u(x^*) - u(y)](v(A+D) - v(B+D)) \\ &\quad + [u(y) - u(x)](v(C+T+A+D) - v(C+T+B+D)) \geq 0 \end{aligned}$$

iff

$$\begin{aligned} &[u(x^*) - u(y)](v(T+A+D) - v(T+B+D)) \\ &\quad + [u(y) - u(x)](v(C+T+A+D) - v(C+T+B+D)) \geq 0, \end{aligned}$$

where this equivalence obtains for all outcomes.

By (D.5) and appropriate forms of (D.1), which have already been proven, conclude that $(v(A+D)-v(B+D))$ and $(v(T+A+D)-v(T+B+D))$ are both negative, while $(v(C+A+D)-v(C+B+D))$ and $(v(C+T+A+D)-v(C+T+B+D))$ are both positive. Because the range of u has nonempty interior, we can vary the above utility values sufficiently to conclude from the preceding equivalence that

$$v(T+A+D)-v(T+B+D)=v(A+D)-v(B+D). \tag{D.8}$$

Next apply a similar argument for the case

$$\bar{y} \succ x^* \succ y \succ x \succ \underline{y}, \qquad z=x \qquad \text{and} \qquad z'=x^*,$$

in place of (D.7). One obtains the equivalence

$$[u(x^*)-u(y)](v(A+D)-v(B+D)) \\ +[u(y)-u(x)](v(C+A+D)-v(C+B+D)) \geq 0$$

iff

$$[u(x^*)-u(y)](v(T+A+D)-v(T+B+D)) \\ +[u(y)-u(x)](v(C+T+A+D)-v(C+T+B+D)) \geq 0,$$

where this equivalence obtains for all outcomes. Apply (D.8) and conclude that

$$v(C+A+D)-v(C+B+D)=v(C+T+A+D)-v(C+T+B+D). \qquad Q.E.D.$$

The following lemma refers to alternative definitions of ambiguity described in Section 8. Its proof is immediate given Lemma 7.1 and results from Zhang (1997) and Ghirardato and Marinacci (1999).

LEMMA D.1: (a) *If $\mathscr{A}_0$ is closed with respect to complements, then $\mathscr{A}_0$ coincides with the class of linearly unambiguous events and $\mathscr{A}_0 \subset \mathscr{A}$.*

(b) *If v is exact, then $\mathscr{A}_1$ coincides with the class of events unambiguous in the sense of either Ghirardato-Marinacci or linear ambiguity and $\mathscr{A}_1 = \mathscr{A}_0 \subset \mathscr{A}$.*

PROOF OF COROLLARY 7.3: (a) The assumptions on v imply that $\succcurlyeq$ satisfies the axioms in Theorem 5.2. (Continuity implies Monotone Continuity for $\succcurlyeq$ and convex-ranged implies Small Unambiguous Event Continuity.) Therefore, there exists a convex-ranged and countably additive p representing the likelihood relation on $\mathscr{A}$ that is implicit in $\succcurlyeq$. Conclude that p must be ordinally equivalent to v on $\mathscr{A}$.

(b) From (a) and Lemma D.1, $v=\phi(p)$ on $\mathscr{A} \supset \mathscr{A}_0$, where p is convex-ranged on $\mathscr{A}$. Therefore, $v(\mathscr{A}_0)=\phi(p(\mathscr{A}_0))=[0,1]$. Because ϕ is (strictly) increasing and onto, conclude that $p(\mathscr{A}_0)=[0,1]$. Now it is straightforward to prove that ϕ is the identity function. (For any two $x_1, x_2 \in [0,1]$ with $x_1+x_2 \leq 1$, there exist $A_1 \in \mathscr{A}_0$ and $A_2 \in \mathscr{A}$ such that $p(A_1)=x_1$, $p(A_2)=x_2$ and $A_1 \cap A_2 = \varnothing$. From the definition of $\mathscr{A}_0$, $v(A_1 \cup A_2)=v(A_1)+v(A_2)$, or

$$\begin{aligned} v(A_1 \cup A_2) = \phi(p(A_1 \cup A_2)) &= \phi(p(A_1)+p(A_2)) \\ &= \phi(x_1+x_2) = v(A_1)+v(A_2) \\ &= \phi(p(A_1)) + \phi(p(A_2)) \\ &= \phi(x_1) + \phi(x_2). \end{aligned}$$

Since ϕ is continuous, ϕ is linear on $[0,1]$.)

(c) Let p be the measure provided by (a) and fix any measure q in core(v). Continuity of v implies that q is countably additive (Schmeidler (1972)). By Lemma D.1, the event A satisfying (7.10) lies in $\mathscr{A}$. We show that it satisfies also: For any $B \in \mathscr{A}$,

$$p(B)=p(A) \Rightarrow q(B)=q(A). \tag{D.9}$$

Then a recent result by Marinacci (1998) allows us to conclude that $p = q$. Because this is true for any q in the core, conclude further that $\text{core}(v) = \{p\}$ and hence, because v is exact, that

$$v = p \qquad \text{on} \qquad \mathscr{A}.$$

From above, $\mathscr{A} \subset \mathscr{A}_2$, the class of events where all measures in the core agree. In addition, $\mathscr{A}_1 \subset \mathscr{A}$ by Lemma D.1 and $\mathscr{A}_2 = \mathscr{A}_1 = \mathscr{A}_0$ by Lemma 7.1.

Thus it suffices to prove (D.9). Let $p(B) = p(A)$. Then

$$\text{(D.10)} \qquad v(B) = \phi(p(B)) = \phi(p(A)) = v(A).$$

Similarly, $v(B^c) = v(A^c)$. From the hypothesis $v(A) + v(A^c) = 1$, deduce that

$$v(B) + v(B^c) = 1.$$

Because $q(\cdot) \geq v(\cdot)$, deduce further that $q(B) = v(B)$. Similarly, $q(A) = v(A)$. Finally, $q(B) = q(A)$ from (D.10). *Q.E.D.*

PROOF OF COROLLARY 7.4: (a) The proof regarding ambiguity aversion is similar to the proof of Epstein (1999, Lemma 3.4). For sufficiency of (7.12), take the probabilistically sophisticated order $\succcurlyeq^{ps}$ required by (6.2) to be the counterpart of (7.4), that is, the CEU function with utility index u and capacity $\phi(m)$. For risk attitudes, note that on $\mathscr{F}^{ua}$, $\succcurlyeq$ is represented by the utility function $\int u(h)\, d\phi(p)$, which is the rank-dependent functional form familiar from the literature on 'non-expected utility' theories of preference over lotteries. Therefore, the asserted characterization of attitudes towards risk follows from Chew, Karni and Safra (1987), for example.

(b) Prove ambiguity aversion; the other case is similar. Because $\mathscr{A} = \mathscr{A}_2$, p can be extended to a measure m on Σ such that $m \in \text{core}(v)$. For the probabilistically sophisticated order $\succcurlyeq^{ps}$ required by (6.2), take the expected utility order with probability measure m and vNM index u. Then it suffices to show that

$$\int u(f)\, dm \geq \int u(f)\, dv$$

for all acts f in $\mathscr{F}$. This is true because

$$\int u(f)\, dv - \int u(f)\, dm = \sum_{i=1}^{n-1} [u(x_i) - u(x_{i+1})] \left(m\left(\bigcup_{j=1}^{i} E_j \right) - v\left(\bigcup_{j=1}^{i} E_j \right) \right) \geq 0,$$

where $f = x_i$ on E_i, $i = 1, \ldots, n$, and $u(x_1) > \cdots > u(x_n)$ and where the nonnegativity is due to the fact that m lies in the core of v. *Q.E.D.*

E. APPENDIX: A PARTICULAR PREFERENCE ORDER

We define a preference order $\succcurlyeq$ satisfying the properties described in Section 9.2. As a first step in defining utility over $\mathscr{F}$, let $U_2 : \mathscr{F}_2 \to \mathscr{R}^1$ be defined by

$$\text{(E.1)} \qquad U_2(f) = \int_{S_2} u(f)\, d\phi(p), \qquad f \in \mathscr{F}_2,$$

where u is a continuous and strictly increasing vNM index, where $\phi : [0,1] \to [0,1]$ is a strictly increasing and onto map and integration is in the sense of Choquet (Section 7). Assume further that $\phi(t) < t$ for all $t \in (0,1)$.

To define U on $\mathscr{F}$, observe that given any act f over S and $s_1 \in S_1$, the restriction $f(s_1, \cdot)$ can be viewed as an act over S_2, giving meaning to $U_2(f(s_1, \cdot))$. Thus U can be defined as follows: For each $f \in \mathscr{F}$,

$$\text{(E.2)} \qquad U(f) = \int_{S_1} U_2(f(s_1, \cdot))\, dp(s_1) = \int_{S_1} \int_{S_2} u(f)\, d\phi(p(s_2))\, dp(s_1).$$

Let $\succcurlyeq$ be the preference order represented by U. The asserted properties can be verified.

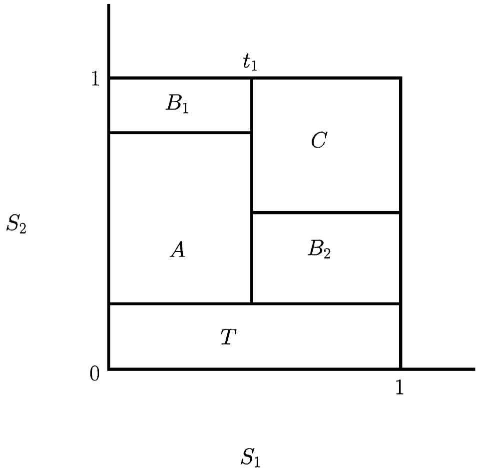

FIGURE 1.—Two-Urn Example

Of particular interest is the assertion regarding the ambiguity of events in Σ_2. Its proof is straightforward, though tedious (it is available upon request from the authors). To show that a given T is ambiguous, one must provide disjoint events A and B, each disjoint from T, an act h and outcomes x^*, x, z, and z' such that the invariance in (3.3) is violated when z is replaced by z'. Figure 1 illustrates schematically the kinds of events that work for the T shown there.[47] To see intuitively why the noted invariance is violated, take the act h to be constant and equal to y on $T^c \setminus (A \cup B)$ (this suffices for the proof), suppose that B_1 is empty and let $x^* \succ y \succ x$. If $z = x^*$, then the best outcome x^* is attained on an event containing $[0, t_1] \times S_2$, and the latter has objective probability $p([0, t_1])$. This precision may lead to the preference for the conditional bet on A rather than on B (equal here to B_2) and hence to the first ranking shown in (3.3). However, if z is replaced by $z' = y$, the above perspective is changed and a reversal in ranking may occur.

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[47] In the figure, each urn is taken to be the unit interval and $B = B_1 \cup B_2$ and $C = T^c \setminus (A \cup B)$.

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