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# Subjective Utility, Anticipated Utility, and the Allais Paradox

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One of the most notable counterexamples to expected utility theory is the "Allais paradox" (M. Allais, 1953, *Econometrica*, **31**, 503–546). A number of alternative theories have been proposed in an attempt to resolve this paradox, notably including U. S. Karmarkar's subjectively weighted utility (SWU) theory (*Organizational Behavior and Human Performance*, 1978, **21**, 61–72; 1979, **24**, 67–72). It is shown that SWU theory necessarily involves violations of dominance, but that the theory can be modified to avoid these violations. The result is a special case of J. Quiggin's anticipated utility theory (1982, *Journal of Economic Behaviour and Organisation*, **3**, 323–343). 

Since it was first advanced, the expected utility (EU) theory of von Neumann and Morgenstern (1944) has been subject to a wide range of criticism. Perhaps the best-known criticism has been that of Allais (1953). Allais proposed a decision problem, now known as the Allais paradox, for which most people make choices which violate the EU theory and, in particular, the axiom of "irrelevance of independent alternatives."

In two recent articles, Karmarkar (1978, 1979) has put forward a subjectively weighted utility (SWU) theory which, he claims, yields a resolution of the Allais paradox. Karmarkar criticizes previous alternatives to EU theory, such as that of Handa (1977), because they admit the phenomena of subcertainty and supercertainty of weights, whereas the approach used in SWU theory requires coherence of weights.

The purpose of the present paper is to show that, like Handa's theory, the SWU theory must admit violations of dominance unless the weighting system is linear. In the latter case, the SWU theory is identical to EU. It is further shown how SWU theory can be modified to avoid this problem and related difficulties of aggregation. The resulting theory is a special case of the anticipated utility (AU) theory, described in Quiggin (1982). The relationship of this theory to the Allais paradox is also discussed.

The term "anticipated utility" is used to indicate that the process of forming estimates of future utility differs from that of taking a mathematical expectation. The AU theory retains the plausible EU axioms of

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continuity, transitivity, and avoidance of dominance violations, but uses a much weaker form of the controversial "independence of irrelevant alternatives" axiom. Because AU theory retains the requirement for transitivity, it is more consistent with the general economic theory of choice than are alternatives, such as the prospect theory of Kahneman and Tversky (1979), which do not possess this property.

A more radical solution to problems such as the Allais paradox is offered by Machina (1982), who presents a theory in which preferences are expressed over probability distribution functions, rather than as a weighted average of the values of a utility function. While the solution offered here is a special case of Machina's, it has the advantages of operational simplicity and of involving a comparatively minor modification of EU theory.

## NOTATION

The discussion will be confined to lotteries with monetary outcomes $x_1, x_2, \ldots, x_n$ having probabilities $p_1, p_2 \ldots p_n$, respectively. Thus, a lottery

$$L = \{(x_1, x_2, \ldots, x_n); (p_1, p_2, \ldots, p_n)\},$$

where $\Sigma_i\, p_i = 1$, would yield outcome $x_i$ with probability $p_i$. The SWU approach transforms each probability $p_i$ into a weight

$$w(p_i) = p_i^\alpha/[p_i^\alpha + (1 - p_i)^\alpha]; \qquad 0 < \alpha \leq 1.$$

The subjectively weighted utility of a lottery $L$ is given by

$$\mathrm{SWU}(L) = \Sigma_i\, w(p_i)U(x_i)/\Sigma_i\, w(p_i)$$

where $U$ is a von Neumann–Morgenstern utility function.

## VIOLATIONS OF THE DOMINANCE IN THE SWU THEORY

The purpose of this section is to show that SWU theory exhibits violations of dominance in the strong sense that a lottery $L_1$ may be preferred to an alternative $L_2$, even though $L_1$ yields a worse outcome with probability 1.

It may be shown that if $\alpha < 1$, then for any $p_i > 1/2$

$$w(p_i) < 2\, w(p_i/2).$$

An example will illustrate the resulting violations of dominance. Let

$$L = \{(0, x); (1/4, 3/4)\}$$

be a very generous lottery with free tickets and a prize of \$$x$ awarded with probability 3/4. Now suppose that the lottery is operated in two states, and the government of state 1 decides to tax the winners. However, instead of imposing a uniform tax, the government injects a further element of gambling. Fifty percent of winners, drawn at random, are levied a tax of $\delta_1$, while the other fifty percent are levied a tax of $\delta_2$. Meanwhile, the government of State 2 does nothing. Investors thus have a choice of two lotteries:

$$L_1 = \{(0, x - \delta_1, x - \delta_2); (1/4, 3/8, 3/8)\}$$

and

$$L_2 = \{(0, x); (1/4, 3/4)\}.$$

The SWU theory gives

$$\text{SWU } (L_1) = w(3/8)(U(x - \delta_1) + U(x - \delta_2))/(w(1/4) + 2w(3/8)).$$

$$\text{SWU } (L_2) = w(3/4)U(x)/(w(1/4) + w(3/4)).$$

Since $2w(3/8)$ is greater than $w(3/4)$, it may be shown that, provided $\delta_1$ and $\delta_2$ are sufficiently small, $\text{SWU}(L_1)$ will be greater than $\text{SWU}(L_2)$. Thus, investors would prefer the lottery in State 1, despite the tax. We might even imagine that this increase in demand would encourage the government of State 1 to levy a charge $\delta_3$ on lottery tickets. Provided this charge $\delta_3$ is sufficiently small, $L_1$ would still be preferred to $L_2$, even though its outcome is strictly worse with probability 1.

The fundamental problem here is that probability weights have been derived from individual probabilities without reference to the distribution of outcomes. What is required is an approach which would distinguish extreme outcomes from intermediate ones. This is examined in the following section.

## WEIGHTING INTERMEDIATE OUTCOMES

It is apparent that, in order for the problem of "intermediate" outcomes to be meaningful, there must be at least three possible outcomes. It can also be seen that violation of dominance in Karmarkar's theory can only arise if more than two outcomes are involved. With only two outcomes, we always have

$$w(p) + w(1 - p) = 1.$$

For more than two outcomes, the probability weights $w_i$ will not, in general, add to 1. Karmarkar solves this problem by dividing each weight by the sum of the $w_i$. This "averaging" procedure generates the difficulties of aggregation noted in Karmarkar (1979). An alternative approach, which avoids these difficulties, is based on the observation that the ordering of the outcomes imposes limits on the aggregations which are possible.

In order to assist in this, it will be assumed from now on that the prizes $x_i$ are ordered from worst to best, i.e.,

$$x_1 \leqslant x_2 \leqslant x_3 \leqslant x_n.$$

It is apparent that if $x_2$ were "disaggregated," the resulting outcomes, say, $x_{2a}$ and $x_{2b}$, would still lie between $x_1$ and $x_3$. Conversely, while "small" changes in outcome values could lead to "adjacent" outcomes such as $x_2$ and $x_3$ being lumped together, they could not result in $x_2$ and $x_4$ being combined, with $x_3$ unaffected.

These considerations may help to motivate the following alternative extension of Karmarkar's two-outcome weights to the multioutcome case. Define a "probability weighting function," $h$, and an anticipated utility function AU($L$).

$$\mathrm{AU}(L) = \Sigma_i\, h_i\,(\mathbf{p})\; U(x_i)$$

$$h_j(\mathbf{p}) = w(\Sigma_{i \leqslant j} p_i) - w(\Sigma_{i<j} p_i).$$

For the three-outcome case, this formula reduces to

$$h_1(p_1,p_2,p_3) = w(p_1)$$

$$h_2(p_1,p_2,p_3) = w(p_1 + p_2) - w(p_1) = w(p_2 + p_3) - w(p_3)$$

$$h_3(p_1, p_2, p_3) = w(p_3).$$

It is apparent in the three-outcome case that the weight placed on outcomes 1 and 2 together will be the same whether or not they are aggregated. Quiggin (1982) gives a formal proof that, given a formula to determine the weights in the two-outcome case, this "additive" method is the only extension procedure which will avoid the possibility of violations of dominance.

Alternative weighting functions $w$ can be extended in the same way. Quiggin (1982) provides a set of axioms, weaker than those of von Neumann and Morgenstern's EU theory, which imposes the following conditions on $w$:

$$w(0) = 0,\ w(1/2) = 1/2,\ w(1), = 1$$

$$w(p) + w\,(1 - p) = 1,\ 0 < p < 1$$

and requires that the probability weighting function $h$ should be derived from $w$ as shown above.

It will be noted that, whereas previous approaches have used weights depending on individual probabilities, the probability weighting function $h$ depends on the entire probability distribution. That is, $h_i(\mathbf{p})$ is not determined by $p_i$ alone, but by all the values $p_1, p_2, \ldots, p_n$. Thus, an "extreme" event need not have the same weight as an "intermediate" event with the same objective probability.

## THE ALLAIS PARADOX AND THE INDEPENDENCE AXIOM

The Allais paradox consists of two pairs of choices. First, the individual is asked to choose between

$$L_1 = \{(0, \$1m, \$5m); (.01, .89, .1)\}$$

and

$$L_2 = \{(\$1m); (1)\}$$

and then between

$$L_3 = \{(0, \$5m); (.9, .1)\}$$

and

$$L_4 = \{(0, \$1m); (.89, .11)\}.$$

Most individuals prefer $L_2$ to $L_1$ but choose $L_3$ over $L_4$. As Karmarkar (1979) shows, this is inconsistent with any possible utility function satisfying EU theory. Karmarkar attributes this result to the fact that the low probability event is "overweighted." However, it is fairly easy to generate a similar example which contradicts the SWU theory.
Let

$$L_5 = \{(-\$10, 0, \$5m); (.01, .89, .1)\}$$

and

$$L_6 = \{(0, \$1m, \$1.1m); (.89, .1, .01)\}.$$

Most individuals who prefer $L_3$ to $L_4$ would still prefer $L_5$ to $L_6$. However, this preference cannot be reconciled with SWU theory. Following Karmarkar's notation, let $U(\$5m) = 1$, $U(0) = 0$, and $U(\$1m) = \pi$, and write the probability weights on .1, .11 and .01 as $a'$, $b'$ and $\epsilon'$, respectively.

As Karmarkar shows, the choice of $L_2$ over $L_1$ implies that

$$(a' + \epsilon')\, \pi > a'.$$

However, the choice of $L_6$ over $L_5$ means that

$$\begin{aligned} a'/(1 - b' + a' + \epsilon') &> (a' + \epsilon' U(-\$10))/(1 - b' + a' + \epsilon') \\ &> (a'\, \pi + \epsilon'\, U(\$1.1m))/(1 - b' + a' + \epsilon') \\ &> (a' + \epsilon')\, \pi/(1 - b' + a' + \epsilon'). \end{aligned}$$

That is, $a > (a' + \epsilon')\pi$, contradicting the previous bound on $\pi$.

By contrast, the AU theory admits all of the choices described above. Suppose that $\pi = .8 \simeq U(\$1.1m)$, and that

$$w(.01) = .05$$

$$w(.1) = .12$$

$$w(.11) = .13.$$

Thus, low-probability "extreme" events are heavily overweighted, while the utility difference between $\$1m$ and $\$5m$ is much smaller than that between $\$1m$ and 0. Preferences of this general kind are clearly required to justify the choice of $L_1$ over $L_2$. Using the approach described above, we have

$$h(.01, .89, .1) = (.05, .83, .12),$$

and this yields $\mathrm{AU}(L_1) = .78$, whereas $\mathrm{AU}(L_2) = .8$. On the other hand, $L_3$ is chosen over $L_4$, since $\mathrm{AU}(L_3) = .12$, whereas $\mathrm{AU}(L_4) = .13 \times .8 = .104$.

Thus, the AU theory, like the SWU theory, can justify the choices suggested by Allais. The difference between the two becomes apparent when we consider the choice between $L_5$ and $L_6$. We have already evaluated $h(.01, .89, .1)$. Using the same procedure,

$$h(.89, .1, .01) = (.87, .08, .05).$$

This yields $\mathrm{AU}(L_5) \simeq .12$, and $\mathrm{AU}(L_6) \simeq .104$ just as in the choice between $L_3$ and $L_4$. The disaggregation of the $\$1m$ outcome in $L_4$ into two similar outcomes in $L_6$ has no great effect on its evaluation, whereas in SWU theory its weight is greatly increased by this.

In order to observe the real basis of the Allais paradox, it is necessary to recall that Allais produced the paradox in order to criticize the EU axiom of the irrelevance of independent alternatives. Roughly speaking, this axiom states that if outcome $x$ occurs with a probability of $p$ in each of two gambles $L_1$ and $L_2$, then the ranking of $L_1$ and $L_2$ will be unaffected by changes in the value of $x$. The change from $L_1$ to $L_3$ and from $L_2$ to $L_4$ may be fitted into this category, since a .89 chance of $\$1m$ in $L_1$ and $L_2$ is replaced with a .89 chance of zero in $L_3$ and $L_4$. Yet, as Allais notes, the preference ranking is usually reversed by this shift. Thus, the independence axiom is not validated by observed preferences.

The second counterexample shows that the independence axiom also applies in SWU theory, though in a more restricted set of circumstances—when the set of outcome probabilities, regardless of order, is the same in both gambles. For example, if the zero outcomes in $L_5$ and $L_6$ are replaced with $\$1m$, this cannot affect the ranking in SWU theory. However, just as in the Allais paradox, it is likely to yield a reversal of the preference rankings for many individuals.

The independence properties of AU theory are much weaker. They derive from the fact that $h(p_1, p_2, \ldots, p_n)$ is independent of the values $x_1, x_2, \ldots, x_n$ (but not of their ranking). As is shown by Quiggin (1982), this independence property may be derived from the requirement that strict violations of dominance are avoided and that $w(1/2) = 1/2$. Essentially, this independence property applies only when two lotteries have identical outcomes over a given part of the probability distribution, such as the bottom tail. By contrast, the counterexamples above rely on the property that the bottom tail of one distribution is equal to an intermediate part of another.

More general theories, such as that of Machina (1982), dispense with even this weak independence axion. However, whereas such theories may be quite hard to make operational, AU theory, with restrictions on functional form such as those proposed by Karmarkar, can be applied in practice. For example, Quiggin (1981) gives estimates of the degree of overweighting of small probabilities by Australian farmers. Data used for this purpose were not even specially collected; instead, the data used in an EU theory study (Bond & Wonder 1980) were reanalyzed.

## CONCLUDING COMMENTS

The "averaging" method used by Karmarkar to impose coherence on SWU theory leads to difficulties in aggregation and violations of dominance. It also fails to accord with the intuitively plausible suggestion that overweighting of low-probability events should apply only when outcomes are extreme. The AU theory, using an "additive" method of achieving coherence, avoids these difficulties and presents a resolution of the Allais paradox which refutes the EU axiom of the irrelevance of independent alternatives.

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