JOURNAL OF ECONOMIC THEORY 5, 348–376 (1972)

On Models of Expectations that Arise From Maximizing Behavior of Economic Agents over Time

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Received January 19, 1972

1. INTRODUCTION

In this paper we try to build a microeconomics of expectations. The first task that needs to be done, it seems to us, is to construct an appropriate *equilibrium* concept. We present our concept in one model. In this model, the actors of the model forecast values for a particular variable —price of output, say—, they make plans according to these forecasts, and market conditions at the time that the plans are carried out determine actual prices. The forecast is said to be an *equilibrium forecast* if actual prices turn out to equal expected prices. Demonstration that such an equilibrium exists involves much the same problems as proving that prices exist that equate supply and demand in the Arrow–Debreu–McKenzie world. One way of viewing our equilibrium concept is that it is a logical extension to expectations of the concept of equilibrium price. In fact, this is the approach of Radner [17] in a model built on the classical foundations of Arrow–Debreu–McKenzie. Our approach differs from Radner's [17] in that we try to build a theory that focuses on disequilibrium adjustment of capital stock. By working in what one may call the tradition of the neoclassical theory of investment, we hope to help build a microeconomics that will be useful for building macro-models.

Our model itself may be of some independent interest. It is a continuum of the producers model stimulated by the basic approach of Aumann [1]. We rewrite the standard textbook version of the theory of entry and exit in the context of this model and show that it is useful for modeling the concept of full equilibrium. We introduce entry cost and study the full equilibrium concept that emerges. Expectation in the context of this model takes the form of people forecasting the size of the industry at time t, and people entering at a rate proportional to present value of anticipated profit thereby generating the actual size at time t on the way to full equilibrium where profits are normal. If the actual size at each time t is equal to the expected size then the forecast is said to be equilibrium. For special cases the forecast is shown to be "stable" in the context of a mechanism that is supposed to represent adjustment of a disequilibrium forecast to the equilibrium forecast in the same manner as the usual price tâtonnement. But the microeconomics of this mechanism are no more solid than the microeconomics of the tâtonnement. However, the concept of equilibrium forecast and the "equilibrium" disequilibrium path of industry size associated with it are safe from the charges of ad hockery.

2. A Model without An Expectations Formation Mechanism

Let us consider the simplest model first. We have a continuum of producers, $[0, x]$. Producer $t \in [0, x]$ has a supply function $s_t(p)$. Aggregate supply is $\int_0^x s_t(p)\, dt \equiv S(p)$. Aggregate demand is $D(p)$. The profit function of producer t is given by $\pi_t(p) = ps_t(p) - c_t(s_t(p))$ where c_t is the cost function of producer t.

We take x as a measure of the size of the industry, i.e., the ordinary linear measure of the set of producers is taken as the "size" of the industry.

Let us assume that all firms, presently existing and potentially existing, are the same, i.e., $\pi_t = \pi$, $s_t = s$, $c_t = c$. We take concrete demand and supply functions: $D(p) = Ap^{-\alpha}$, $s(p) = Bp^{\beta}$ where A, B are shift parmaeters and α, β are elasticities of demand and short-run supply for the one good. Under pure competition, the short-run supply function is gotten by equating marginal cost to price. Hence a variable cost function that generates $s(p) = Bp^{\beta}$ is $VC(q) = \beta/(1 + \beta)\, B^{-1/\beta} q^{(\beta+1)/\beta}$ where q is quantity. Total cost is given by $C(q) = VC(q) + F$ where F is fixed cost. Let us assume that each firm, current or potential, knows $D(p)$, $s(p)$ and that for each size x we are in short-run equilibrium, i.e., $Ap^{-\alpha} = xBp^{\beta}$. Thus to each size x there is $p(x)$ given by

$$p(x) = (A^{-1}B)^{-1/(\alpha+\beta)} x^{-1/(\alpha+\beta)}. \tag{1}$$

Notice that (1) does not generate any results that contradict common sense. For as size x increases, the short-run equilibrium price level falls. Furthermore, a decrease in variable cost (increase in B) or a decrease in demand (a decrease in A) leads to a fall in price.

Now we turn to formulating our first model of the dynamics of industrial expansion or contraction. The profit function of producer t is

$$\pi(p) = (1/(\beta + 1))\, Bp^{\beta+1} - F, \tag{2}$$

which is gotten by $\pi(p) = ps(p) - c(s(p))$ and a little algebra. Substitute (1) into (2) to get

$$\pi[p(x)] = \frac{1}{\beta + 1} A^{(1+\beta)/(\alpha+\beta)} B^{(\alpha-1)/(\alpha+\beta)} x^{-(1+\beta)/(\alpha+\beta)} - F. \tag{3}$$

Aggregate profits for an industry of size x at short-run equilibrium is given by $\int_0^x \pi[p(x)]\, ds = x\pi[p(x)]$. Now two obvious formulations of an adjustment mechanism come up.[1] One is $dx/dt = \pi(p(x))$; the other is $dx/dt = x\pi(p(x))$. It seems to us that new entrants are more likely to come in lured by "profits per capita" $x\pi[p(x)]/x = \pi[p(x)]$, rather than aggregate profits $x\pi[p(x)]$. At any rate, $dx/dt = x\pi[p(x)]$ is stable, as can easily be shown.[2] We take $dx/dt = \pi[p(x)]$ for starters. Let us graph profits for the case $\alpha < \infty$.

Notice that if $\alpha < \infty$ we have a unique point $\bar{x}$ such that $\pi = 0$. Whereas if $\alpha = \infty, \pi + F = (1/(\beta + 1))B$ so that if $(1/(\beta + 1))B > F$, the industry is infinitely large at equilibrium, if $(1/(\beta + 1))B = F$, the size is indeterminate, and if $(1/(\beta + 1))B < F$, the equilibrium size is zero. Since $\alpha = \infty$ is really an unrealistic case (no demand is *perfectly* elastic) we concentrate on the case $\alpha < \infty$. Solving explicitly for full industry equilibrium, we obtain

$$\bar{x} = AB^{(\alpha-1)/(1+\beta)}[(\beta + 1)F]^{-(\alpha+\beta)/(1+\beta)}. \tag{4}$$

It is trivial to see that the process $dx/dt = \pi[p(x)]$ is globally stable. Comparative statics may be read off the equilibrium Eq. (4). The most interesting result is that inelastic demand $\alpha < 1$ causes the effect of a decrease in variable cost (i.e., an increase in B) to work against the size of the industry. We explain this in more detail below.

Imagine that the industry is at full equilibrium and B increases to B_1, e.g., this may be due to an improvement in technology; see Fig. 1. An increase to B_1 means that the first effect is to shift the aggregate supply curve $\bar{x}Bp^{\beta}$ to the right. This sets into motion forces leading to a decrease in price received for each firm (we realize we are glossing over the dis-

[1] These mechanisms are completely ad hoc. What is desperately needed is a theory out of which some mechanism would arise naturally. We ignore that for this piece.

[2] This is a good point to discuss the usefulness of the continuum approach some more. We could replace $dx/dt = \pi[p(x)]$ by $x_{t+1} - x_t = \pi[p(x_t)]$ or some such. The discrete time formulation appears a bit more tolerable, i.e., x_t equals the number of firms instead of the linear measure of an interval of firms. But notice the formal resemblance of the two models. It seems to us that here the question boils down to period versus continuous time analysis. We take continuous time for convenience. Furthermore, it seems more reasonable that firms flow in continuously rather than come in by one-period lumps.

equilibrium dynamics of price adjustment but we are concentrating on entry and exit dynamics now). But at the same time, the profit function $\pi(p) = (1/(\beta + 1)) Bp^{\beta+1} - F$ shifts upward as B shifts up to B_1. Now it is clear by Fig. 1 that in the inelastic case, consumers not being all that responsive to a fall in price, competition among the firms drives price down to a large degree. Whereas in the elastic case (the dotted demand curve in Fig. 1) we see that the opposite is the case, i.e., competition drives price down but not all that much. Thus in the inelastic case the upward shift in B does not raise the profit opportunities enough to compensate for the large fall in price. Now let us increase the degree of realism. First, as was suggested to me by my colleague, Edward Zabel, a new entrant must bear an extra cost of formation of his business, e.g., he must learn to operate it first.

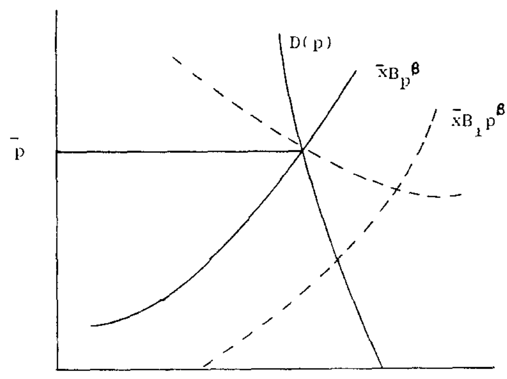

FIGURE 1

Hence there is an entry cost for new operators, and we shall introduce such a cost into the model.

3. A Model with Entry Cost and Expectations Formation

Also an expectation-formation mechanism must be entered into the model. Quite clearly, a potential entrant can count on others entering when the situation looks attractive to him. He must guess how many and how fast. Obviously, if he expects entrants to come in very rapidly he would not expect to cover the entry cost.

Let us put down a model and discuss the modeling in detail.

Let $x^a(t)$ be the actual size of the industry at time t, "a" stands for "actual." Let $x_t^e(s)$ be the size expected at time s, "e" stands for[3] "expected." Then

$$dx^a/dt = \max \left\{ \int_t^\infty e^{-\delta(s-t)} \pi[p(x_t^e(s))]\, ds - F_1, 0 \right\}, \tag{5}$$

when

$$\int_t^\infty e^{-\delta(s-t)} \pi[p(x_t^e(s))]\, ds \geqslant 0,$$

and

$$dx^a/dt = \min \left\{ \int_t^\infty e^{\delta(s-t)} \pi[p(x_t^e(s))]\, ds, 0 \right\}$$

otherwise.

The meaning of (5) is the following. At time t, expectations $x_t^e(s)$, $s \geqslant t$ are held by potential entrants and actual operators. Operators recognize that at time s if industry size is $x_t^e(s)$, $p(x_t^e(s))$ will be determined by the short-run equilibrium condition $D(p) = x_t^e(s)\, s(p)$, i.e., $p(x_t^e(s)) = (A^{-1}Bx_t^e(s))^{-1(\alpha+\beta)}$ for our special demand and supply curves. Profits tomorrow are not worth as much as profits today, so we discount the future from t on. If the anticipated discounted stream of profits $\int_t^\infty e^{-\delta(s-t)} \pi[p(x_t^e(s))]\, ds$ is greater than the cost of forming a new firm F_1, the new entrants pay the fee F_1 and enter in numbers proportional to the net anticipated stream of discounted profits. If the future for the present operators looks bleak, i.e., $\int_t^\infty e^{-\delta(s-t)} \pi[p(x_t^e(s))]\, ds < 0$, then exit takes place proportional to the size of anticipated future losses.

Look at the function

$$h(x) = \int_t^\infty e^{-\delta(s-t)} \pi[p(x)]\, ds = \pi[p(x)] \frac{1}{\delta}. \tag{6}$$

For $\alpha < \infty$ there is only one $\bar{x}_1$, $\bar{x}_2$ such that $\pi[p(\bar{x})](1/\delta) = F_1, 0$ resp., i.e.,

$$\begin{aligned} \bar{x}_1 &= [(\beta + 1)(F + \delta F_1)]^{-(\alpha+\beta)/(1+\beta)} AB^{(\alpha-1)/(1+\beta)}, \\ \bar{x}_2 &= [(\beta + 1)\, F]^{-(\alpha+\beta)/(1+\beta)} AB^{(\alpha+1)/(1+\beta)}. \end{aligned} \tag{7}$$

Let us assume that if industry size $x^a(t)$ is less than $\bar{x}$, then anticipations develop according to the rule $x_t^e(s) = \bar{x}_1 + f_t(s)(x^a(t) - \bar{x}_1)$ and if $x^a(t) > \bar{x}_2$, $x_t^e(s) = \bar{x}_2 + f_t(s)(x^a(t) - \bar{x}_2)$.

[3] Let us emphasize here that "expected" has nothing to do with uncertainty. Our research strategy is to do the simplest case first. It is hoped that techniques of analysis that we develop in studying the certain case will help us in studying the more complicated and much more important uncertainty case. For us the word "expected" will mean that subjective expectations are held with certainty.

The approach we shall take for this section is to prove *qualitative results independent of* $f_t \in F$. Here $F \equiv \{f_t \mid \text{for all } t \geqslant 0\ f_t \text{ is continuous, decreasing, } f_t(t) = 1,\ f_t(\infty) = 0\}$. Thus if compatible expectations exist, our qualitative results will not be disturbed by a bad guess for f_t. From this point on, f_t is fixed once and for all. Set

$$g_{f_t}(x, \bar{x}_i) = \int_t^\infty e^{-\delta(s-t)}\pi[p(\bar{x}_i + f_t(s)(x - \bar{x}_i))]\, ds, \quad i = 1, 2.$$

Take $x_0 < \bar{x}_1$. Since $0 < f_t \leqslant 1$, we have

$$x_0{}^e(s) = \bar{x}_1 + f_0(s)(x_0 - \bar{x}_1) < \bar{x}_1\ ,$$

so that $g_{f_t}(x_0{}^e(s), \bar{x}_1) > F_1$ for all $s \geqslant 0$. Thus

$$dx^a/dt = g_{f_t}(x_t{}^e(s), \bar{x}_1) - F_1 > 0$$

when $t = 0$. Now for arbitrary $t > 0$, $x_t{}^e(s) = \bar{x}_1 + f_t(s)(x^a(t) - \bar{x}_1)$, i.e., expectations are revised by the new information $x^a(t)$. Hence, so long as $g_{f_t}(x^e{}_t(s), \bar{x}_1) \geqslant F_1$ by (5)

$$dx^a/dt = g_{f_t}(x_t{}^e(s), \bar{x}_1) - F_1\,. \tag{8}$$

It is straightforward to show that the differential equation (8) generates for $x_0 < \bar{x}_1$ a trajectory $x^a(t)$ such that $x^a(0) = x_0$, $x^a(\infty) = \bar{x}_1$ and $g_{f_t}(x_t{}^e(s), \bar{x}_1) \geqslant F_1$ for all $s \in [0, \infty)$. By similar argument $x_0 > \bar{x}_2$ implies $dx^a/dt = g_{f_t}(x_t{}^e(s), \bar{x}_2)$ generates a trajectory such that $x^a(0) = x_0$, $x^a(\infty) = \bar{x}_2$ and $g_{f_t}(x_t{}^e(s), \bar{x}_2) \leqslant 0$ for all s where now, $x_t{}^e(s) = \bar{x}_2 + f_t(s)(x^a(t) - \bar{x}_2)$. Hence we get the situation depicted in Fig. 2 for each t.

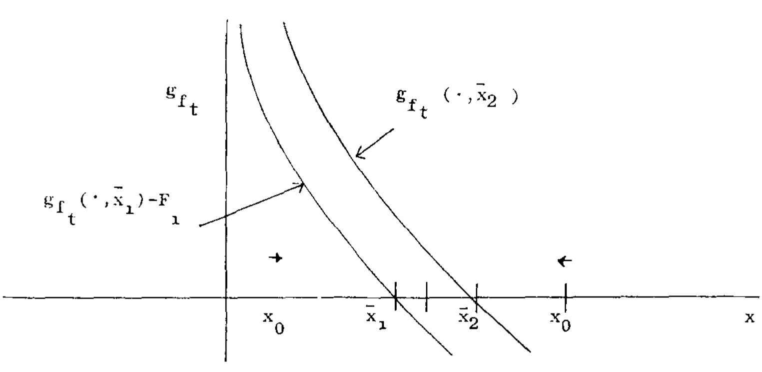

FIGURE 2

Notice that the curve $g_f(\cdot, \bar{x}_2)$ goes below the x axis indicating negative profits for large x. One may disagree with this formation because, presumably, if a firm is making negative profits, it could shut down immediately. But, we want to keep exit dynamics symmetric with entry dynamics, i.e., if firms are losing money, they will exit at a rate proportional to the present value of the anticipated loss. Adjustment cost is implicitly recognized in that they do not exit instantly but rather with a staggered response. For this reason we draw the profit curves with negative profits as a possibility. This does not seem to be too great a sin, for response to a bleak outlook should be staggered over time rather than be instantaneous.

Notice that if $x_0 < \bar{x}_1$, the system (5) generates $x^a(t)$ tending to $\bar{x}_1$. If $x_0 > \bar{x}_2$, $x^a(t)$ tends to $\bar{x}_2$. For $x_0 \in [\bar{x}_1, \bar{x}_2]$, the system remains at rest. New entrants do not come in because the cost of formation is greater than the amount of expected profit. Old operators remain in business because profits are being earned. Notice that an upper limit on profits is given by expectations f_t and costs of formation F_1.

Let us examine closely the functions

$$g_{f_t}(x, \bar{x}_i) = \int_t^\infty e^{-\delta(s-t)}\pi[p(\bar{x}_i + f_t(s)(x - \bar{x}_i))]\, ds, \quad i = 1, 2. \tag{9}$$

It is clear that the functions (9) do not depend on s because s is "integrated out" or because the horizon is open-ended.

Make the substitution $y = s - t$, $dy = ds$. Then

$$\int_{s=t}^{s=\infty} e^{-\delta(s-t)}\pi[p(\bar{x}_i + f_t(s)(x - \bar{x}_i))]\, ds$$

becomes

$$\int_{y=0}^{y=\infty} e^{-\delta y}\pi[p(\bar{x}_i + f_t(y + t)(x - \bar{x}_i))]\, dy.$$

Inserting the short-run price equilibrium $p(x) = (A^{-1}B)^{-1/(\alpha+\beta)}x^{-1/(\alpha+\beta)}$ into $g_{f_t}(x, \bar{x}_i)$, we get

$$g_{f_t}(x, \bar{x}_i) = \int_0^\infty e^{-\delta y}\pi\{(A^{-1}B)^{-1/(\alpha+\beta)}\, [\bar{x}_i + f_t(y + t)(x - \bar{x}_i)]^{-1/(\alpha+\beta)})\}\, dy. \tag{10}$$

Since $\pi(p) = 1/(\beta + 1)\, Bp^{\beta+1} - F$, after some algebra, we get

$$\begin{aligned} g_{f_t}(x, \bar{x}_i) = \int_0^\infty e^{-\delta y} \Big\{ & \frac{1}{\beta + 1} A^{(1+\beta)/(\alpha+\beta)} B^{(\alpha-1)/(\alpha+\beta)} \\ & \times [\bar{x}_i + f_t(y + t)(x - \bar{x}_i)]^{-(1+\beta)/(\alpha+\beta)} - F \Big\}\, dy. \end{aligned} \tag{11}$$

Now recall that the endpoints $\bar{x}_1$, $\bar{x}_2$ of the equilibrium $(\bar{x}_1, \bar{x}_2)$ are defined by $g_0(\bar{x}_1, \bar{x}_1) = F_1$, $g_0(\bar{x}_2, \bar{x}_2) = 0$, where 0 is the zero function, i.e., $f_t \equiv 0$ for all $t \geqslant 0$ the formula (11). More specifically,

$$\int_0^\infty e^{-\delta y} \left\{ \frac{1}{\beta + 1} A^{(1+\beta)/(\alpha+\beta)} B^{(\alpha-1)/(\alpha+\beta)} [\bar{x}_1]^{-(1+\beta)/(\alpha+\beta)} - F \right\} dy = F_1, \quad (12)$$

$$\int_0^\infty e^{-\delta y} \left\{ \frac{1}{\beta + 1} A^{(1+\beta)/(\alpha+\beta)} B^{(\alpha-1)/(\alpha+\beta)} [\bar{x}_2]^{-(1+\beta)/(\alpha+\beta)} - F \right\} dy = 0. \quad (13)$$

Solving (12), (13) explicitly, we get

$$\bar{x}_1 = [(\beta + 1)(F + \delta F_1)]^{-(\alpha+\beta)/(1+\beta)} A B^{(\alpha-1)/(1+\beta)}, \quad (14)$$

$$\bar{x}_2 = [(\beta + 1)F]^{-(\alpha+\beta)/(1+\beta)} A B^{(\alpha-1)/(1+\beta)}. \quad (15)$$

Note that $\bar{x}_2$ is the same as $\bar{x}$ given by Eq. (3), i.e., $\pi[p(\bar{x})] = 0$, the zero-profit industry size. Equation (14) for $\bar{x}_1$ is the same as (15) with F replaced by $F + \delta F_1$. Notice that the discount δ affects $\bar{x}_1$ only. The comparative statics may be read off (14), (15).

Another thing to notice is the dependence of g_{f_t} on f_t. If people expect new entrants to come in faster, i.e., $f_{1,t} \leqslant f_t$ for all t, then g_{f_t} changes in a definite way.

If $x < \bar{x}_i$ then for all $t \geqslant 0$ $f_{1,t}(y + t) < f_t(y + t)$ implies

$$f_{1,t}(y + t)(x - \bar{x}_i) > f_t(y + t)(x - \bar{x}_i).$$

Thus $\pi[p(\bar{x}_i + f_{1,t}(y + t)(x - \bar{x}_i))] < \pi[p(\bar{x}_i + f_t(y + t)(x - \bar{x}_i))]$.

We see that a change in expectations toward the belief that new entrants will come in faster leads the industry to move to $\bar{x}_1$ starting from $x_0 < \bar{x}_1$ more slowly. The same holds for $x_0 > \bar{x}_2$. Notice that only the dynamics is affected by changes in expectations f_t—not the comparative statics.

Let us study the comparative statics closely. Suppose the industry is in equilibrium, the size is $\bar{x} \in [\bar{x}_1, \bar{x}_2]$ and an increase occurs in entry cost F_1, say, due to licensing or the requirement of more human capital for effective entrepreneurship. There is no immediate effect on price because $D(p) = \bar{x}s(p)$ determines price in the short run. But the curve $g_{f_t}(x, \bar{x}_1) - F_1$ depicted in Fig. 3 shifts downward, due to the downward shift in $\bar{x}_1$. An increase in F_1 will cause the industry to remain at rest. If the industry is in $[\bar{x}_1, \bar{x}_2]$ the insider's profits are not effected, but outsiders find it less worthwhile to enter if F_1 increases.

The most interesting comparative statics result is a change in B. Suppose variable cost decreases, i.e., B increases to B_1. Now the one-period profit function $\pi(p)$ increases but competition among firms causes short-run price to fall. It will fall much less if consumers are very responsive to price changes, i.e., demand is quite elastic. Hence positive anticipated profit should appear to potential entrants if the shift is large enough in B. We would expect a price-responsive consuming public to support a larger equilibrium industry size than a public with inelastic demand. In this case the whole equilibrium set $[\bar{x}_1, \bar{x}_2]$ should shift to the right. It does, as is seen by Eq. (14), (15). Now if the industry size happened to be at $\bar{x}_2$ when the increase in B occurred, then there may be no change in equilibrium size at all. Profits would increase for those operators already in business but present value of anticipated profits may not become large enough to cover entry cost. In the inelastic case the equilibrium set shifts to the left and profits fall. Again, forces leading to exit may not be set in motion. If the industry happened to be sitting at $\bar{x}_2$, i.e., producing at minimum average cost and earning zero profit, then exit would take place. If the industry were at $\bar{x}_1$, i.e., producing at the minimum point of the average cost curve adjusted for discount and entry cost, then exit might not take place unless a large enough fall in variable cost took place.

More comparative statics may be read off (14), (15) and the short-run effects may be studied by examining the short-run price given as a function of industry size. But, we shall move on, hoping that we have demonstrated the usefulness of the model along that dimension. Now let us turn to the most difficult part of this paper—expectations. Furthermore, we think that the treatment of expectations is new whereas the rest of the paper is, in the main, old results stated in a sharper form.

4. The Beginnings of a Theory of Expectations Formation

Professor Marc Nerlove, in the Second Henry Schultz Lecture, 1970, has focused attention on the paucity of theoretical justification for the lag structures and expectations formation mechanism commonly used in the mechanisms literature. Even though our theory of expectations formation is developed in a very primitive model, it is based, at least partially, on maximizing behavior over time. While there is still ad hockery present (we have specified the rate of entry by a differential equation, but no theoretical justification is given for it) we feel that the forces of ad hockery are pushed back somewhat.

Presumably any specification of rate of entry $dx/dt = F(\Pi_t)$ where Π_t is the present value of anticipated profit at time t may arise naturally out of a model of adjustment provided that the adjustment cost function as a function over time is the "right" one. It seems that there is some hope of specifications $dx/dt = F(\Pi_t)$ finding[4] theoretical justification. Parenthetically, we remark that the words "theoretical justification" refer to arising out of a sensible model of economic units maximizing over time. At any rate, the upshot of all this is that we have tied the formation of expectations to an information storing process based on the rate of entry. We go on to detailed development. We shall first examine the linear case $\pi[p(x)] = mx + b$, $m < 0$, $b > 0$. It turns out that the equilibrium $f_t(s)$ for this case is independent of t. We would like to acknowledge the help of K. J. Arrow at this point. Our original formulation for the general case was the same as that for the linear case, i.e., we assumed that $f_t(s) = f(s - t)$. He pointed out that, for general profit functions, $\bar{f} = E(\bar{f})$ may not imply that $\bar{x}_t^e(s) = \bar{x}^a(s)$ for all $s, t, s \geqslant t$. He suggested the more general formulation

$$\dot{x} = \int_t^\infty e^{-\delta(s-t)}\pi[p(\bar{x} + f_t(s)(x - \bar{x})], \quad x(0) = x_0 .$$

It turns out that this generates equilibrium forecasts with $\bar{x}_t^e(s) = \bar{x}^a(s)$, for all $s, t, s \geqslant t$. For the linear case it turns out that $f_t(s) = f(s - t)$ generates equilibrium forecasts with the desired property. We are entitled to put the word "the" in front of equilibrium because we have a general uniqueness theorem further on in the paper. Thus we shall only consider f_t of the form $f_t = f$ for the linear case.

Now suppose that profits $\pi(p(x)) = mx + b$, i.e., profits are a linear function of the size x of the industry. Assume that extra fixed cost of entry F_1 is 0. Thus the equation of adjustment given $x_0 < \bar{x}$, $m\bar{x} + b = 0$ is, for any expectations function f_t in F, and specializing to the special case $f_t(s) = f(s - t)$, $f(0) = 1$, $f(\infty) = 0$, we have

$$\begin{aligned}\frac{dx}{dt} &= \int_0^\infty e^{-\delta y}\pi(p[\bar{x} + f(y)(x - \bar{x})])\, dy, \quad x(0) = x_0, \qquad (16)\\ &= \int_0^\infty e^{-\delta y}[m\bar{x} + b + mf(y)(x - \bar{x})]\, dy\\ &= m(x - \bar{x}) \int_0^\infty e^{-\delta y} f(y)\, dy.\end{aligned}$$

Notice that $f_t(y + t) = f(y)$.

[4] What about the Gordon–Hines argument that $dx/dt = F(\Pi_t)$ will be "discovered" by rational actors and, hence, be destroyed? Recall that Π_t denotes present value of *anticipated* profits so that as the system $dx/dt = F(\Pi_t)$ denerates a path $x^a(t)$, the path or "information" $x^a(t)$ is fed back into the expectations formation mechanism thereby effecting anticipations for the future. Thus anticipated profit changes accordingly. The only way rational actoıs can discover $dx/dt = F(\Pi_t)$ is by uncovering the expectations formation mechanism itself.

Equation (16) may be rewritten as

$$\frac{d[\log(x - \bar{x})]}{dt} = m \int_0^\infty e^{-\delta y} f(y)\, dy. \tag{17}$$

Hence

$$x - \bar{x} = ce^{mt} \int_0^\infty e^{-\delta y} f(y)\, dy.$$

Since $c = x_0 - \bar{x}$, we have

$$g(t) \equiv [x(t) - \bar{x}]/(x_0 - \bar{x}) = e^{mt} \int_0^\infty e^{-\delta y} f(y)\, dy.$$

Thus, in the case where $\pi(p(x))$ is linear, the operator $E(f)$ is given by

$$E(f) = e^{mt} \int_0^\infty e^{-\delta y} f(y)\, dy = e^{mtL(f)}, \quad \text{where} \quad L(f) = \int_0^\infty e^{-\delta y} f(y)\, dy. \tag{18}$$

An *equilibrium forecast* $\bar{f}$ is a fixed point of the mapping (18).

One may take the view that an equilibrium forecast $\bar{f}$ is much more important from an economic point of view than the mechanism $E(f_n) = f_{n+1}$, $E(f_0) = f_1$ for the following reasons. First, rational actors will try to anticipate market conditions in the future. If they are good at guessing the future, their forecasts will be fulfilled, i.e., movement from x_0 to $\bar{x}$ will be given by an $\bar{f}$ such that $E\bar{f} = \bar{f}$ if the actors are perfect forecasters. Hence if $\bar{f}$ were unique one could use it as an approximation to the path of adjustment from x_0 to $\bar{x}$. The accuracy of this approximation would be an increasing function of the prediction ability of the actors involved. At any rate, just as it is necessary to establish the existence of a price that equates supply and demand, so also is it necessary to establish existence of a forecast that equates anticipations with actions conditioned on those anticipations.

Secondly, a mechanism such as $E(f_n) = f_{n+1}$ is ad hoc. But the fixed point $\bar{f}$ itself should have a solid microeconomic foundation. As we pointed out, mechanisms $\dot{x}(t) = F(\Pi_t)$, where Π_t is anticipated profit, should arise out of adjustment cost models.

If the reader is willing to go along with us on the microeconomic foundations of $\bar{f}$, then we can get solid microeconomic foundations for an "equilibrium" path of adjustment by the industry from x_0 to $\bar{x}$, i.e., $\bar{x}(t) = \bar{x} + \bar{f}(t)(x_0 - \bar{x})$.

Let $E^n(f)$ denote the n-th iterate of E applied to f, e.g., $E^2(f) = E(E(f))$. One candidate for a learning process is $f_n = E^n f_0$, $n = 1,2,\ldots$, i.e., people form expectations represented by f_0. After some time passes, they realize they are in error. For what actually turns up as they act according to f_0 is $f_1 = Ef_0$. When will they revise? And how will they form expectations after experience causes them to discard f_0 as a poor predictor (they will not discard f_0 if $f_1 \equiv Ef_0 = f_0$). Let us assume that enough time elapses so that they can make a fairly close approximation to the rest of f_1 (remember that at time t_0 they know f_1 on $[0, t_0]$ only). Then a logical candidate to replace f_0 is $f_1 = Ef_0$. The function f_1 has been generated by their experience over time. Now if people use f_1 as a predictor from the time $t_0 > 0$ that f_0 is discarded (in a more complete model, t_0 would be determined by the cost of forecast revision and error), we have

$$dx/dt = \int_0^\infty e^{-\delta y}\pi(\bar{x} + f_1(y)(x - \bar{x}))\, dy, \qquad x(0) = x(t_0). \tag{19}$$

Equation (19) describes the dynamics as of time t_0. Since steady progress is being made toward achieving the equilibrium size $\bar{x}$, errors in forecasts would become negligible once the industry is within epsilon of the long-run equilibrium $\bar{x}$. It is more interesting to model the case where the equilibrium is changing through time. Notice that the operator $E(f) = e^{mt} \int_0^\infty e^{-\delta y} f(y)\, dy$ depends only on m, not on b and $\bar{x}$. Thus the constant term b in the profit function $\pi(x) = mx + b$ has no effect on E. Let us imagine that b shifts in time periods 1,2,3,... and these time periods are long enough relative to the adjustment time represented by $dx/dt = m(x - \bar{x})\, L(f)$, so that E is a good approximation to the process of adjusting expectations. Now each $b(i)$ determines a long-run equilibrium $\bar{x}(i)$. At time 0 people anticipate according to f_0 thereby generating $f_1 \equiv [x - \bar{x}(0)]/[x_0 - \bar{x}(0)]$. At time 1, the new target is $\bar{x}(1)$ and the new initial point is $\bar{x}(0)$. (Let us agree to use $\bar{x}(0)$ as an approximation for the position that the economy finds itself when $\bar{x}(0)$ shifts to $\bar{x}(1)$.) People forecast approach to $\bar{x}(1)$ using f_1 thereby generating

$$f_2 \equiv [x - \bar{x}(1)]/[\bar{x}(0) - \bar{x}(1)]$$

through the differential equation $df_2/dt = m(x - \bar{x}(1))\, L(f_1)$ whose solution is $f_2(t) = e^{mL(f_1)t}$. And so it goes. Notice that we get $f_n = E(f_{n-1})$, as in the simple case. Thus expectations stabilize even though the industry is perpetually chasing a moving target. This argument breaks down, however, if m changes also. Notice that (19) generates $f_2 = E(f_1)$ in the same manner as above. Recall that $f_2 \equiv [x(t) - \bar{x}]/[x(t_0) - \bar{x}]$. Given this discussion, let us admit as a candidate for representation of expectation formation the process

$$f_n = Ef_{n-1}, \qquad f_1 = Ef_0. \tag{20}$$

Now (20) does not pass the tests laid out by Muth [4] and Gordon and Hines [6]. Muth requires that people economize on information, i.e., expectation-formation mechanisms that do not allow people to learn or have people making consistent errors, do not pass the test. Examples are Cournot expectations and adaptive expectations. In Cournot expectations, each economic unit expects the actions of its fellows to be the same as they were last period. Even after being proven wrong, the Cournot unit stubbornly predicts the behavior of its rivals using the same formula.

Let us examine adaptive expectations. Our usage of the term "adaptive" expectations may appear somewhat unorthodox. It is usually used to denote the formula $x_{t+1}^e = x_t^e + \beta(x_t^a - x_t^e)$, $0 < \beta$ or some variant thereof. However, the above formula is usually iterated to produce a moving average:

$$x_{t+1}^e = \beta \sum_{j=0}^{m} (1-\beta)^j x_{t-j}^a + (1-\beta)^{m+1} x_{t-m}^e .$$

The last term is dropped and we have a form

$$x_{t+1}^e = F(x_t^a, x_{t-1}^a, \ldots, x_{t-m}^a).$$

The Gordon–Hines argument says that rational units clinging to F period by period would eventually be just as foolish as their more naive Cournot counterparts. But, yet, researchers have had a fair amount of success using F. See Bailey [13] for a report on empirical successes and a theoretical development. The general formula is $x_t^e = F(x_{t-1}^a, x_{t-2}^a, \ldots, x_{t-k}^a, S_1, \ldots, S_r)$ where r, k are independent of t and $S_1, \ldots, S_n$ are statistics made up from $x_1^a, \ldots, x_t^a$, i.e., mean, variance, etc. Now presumably economic units would predict using F, find that they were consistently wrong, and would discard F. But in adaptive expectations theory, they doggedly stick to F just as the more primitive Cournot unit did.

Now if F fits the data well, then one may justify using it anyway—even though according to Gordon–Hines it has no theoretical justification. However, if one reasons one step further the situation becomes murky. For example, consider Cournot units again. Each unit realizes that it is not learning. But again each tries to outthink the other and it may be rational for them to keep expecting that the others will behave as they did in the last period. In more detail, consider the hog cycle. At first blush one would argue that a stable hog cycle could not exist—because people could predict the ups and downs and behave in such a manner to destroy the regularity. But on the other hand, consider a farmer's problem. He may feel that the dip is coming again next year. Experience has told him so. But if he stays in the hog business and the others exit, then he will make a profit. So he forms a subjective distribution over the possibilities and possibly stays. Although it seems very unlikely that a stable hog cycle could persist, it is not so clear as one would think at first blush that rational behavior over time *automatically* destroys a stable cycle.

Nevertheless, let us accept the Gordon–Hines argument that rational behavior over time will destroy any prediction mechanism based on a time-independent function of past information such as (20). Now after a long period of time the mechanism $f_n = E(f_{n-1})$, $f_1 = Ef_0$ would be eventually discovered and rational actors would behave in such a way so as to invalidate it. However, it seems to us that there is a relation between the level of complexity of a mechanism and the time that actors take to discover it. Another way of looking at (20) is that it is some sort of analog to the usual price tâtonnement

$$dp/dt = f(E(p)), \tag{21}$$

where E is excess demand and p is price. In other words, (20) is to the equilibrium $\bar{f}$ of self-fulfilling expectations as (21) is to equilibrium price $\bar{p}$.

Now there's little hope at all for (20) to describe a tendency to equilibrium if f_n doesn't converge to some $\bar{f}$ independent of f_0. If $f_n \to \bar{f}$, $n \to \infty$, then $E(\bar{f}) = \bar{f}$. Hence $\bar{f}$ is a self-fulfilling expectations function and arbitrary expectations f_0 eventually become self-fulfilling through the process of learning and taking into account past information, i.e., $f_n \to \bar{f}$ where $f_n = E(f_{n-1}), f_1 = E(f_0)$.

We wish to reemphasize here that the process (20) is weakly justified from a microeconomic point of view. However, as a last resort, we defend studying it for the potential usefulness of the techniques thereby generated. The process (20) could be looked upon as a method to calculate the equilibrium $\bar{f}$ for the linear case. Such a process may work for general $\pi(p(x))$ but we have not proved this.

We state and prove our theorems below:

Consider the operator $E(f) = e^{mt} \int_0^\infty e^{-\delta y} f(y)\, dy$. Then,

THEOREM 1. *E has only one fixed point $\bar{f}$ and $\bar{f}$ is of the form $e^{\bar{K}t}$ where $\bar{K}$ is the negative fixed point of the map $H(K) \equiv m/(\delta - K)$. Furthermore, E sends every $f \in F$ into a function g of the form e^{Kt} for some constant $K < 0$. Finally, for all $f_0 \in F$, the sequence $f_n = E(f_{n-1})$, $f_1 = E(f_0)$ converges to $\bar{f}$ as $n \to \infty$.*

Proof. We prove the theorem in a sequence of lemmas.

LEMMA 1. *For all $f \in F$, $E(f)$ is of the form e^{Kt}.*

Proof. Let $f \in F$. Then

$$E(f) = e^{mt} \int_0^\infty e^{\delta y} f(y)\, dy.$$

Put

$$K \equiv m \int_0^\infty e^{-\delta y} f(y)\, dy.$$

Then $E(f) = e^{Kt}$. Q.E.D.

LEMMA 2. *The map $H(K) \equiv m/(\delta - K)$ has only one negative fixed point $\bar{K}$ and for all $K_0 \leqslant 0$, the sequence $K_n = H(K_{n-1})$, $K_1 = H(K_0)$ converges to $\bar{K}$ as $n \to \infty$.*

Proof. A fixed point of $H(K)$ must satisfy $m/(\delta - K) = K$, i.e., $K^2 - \delta K + m = 0$. This quadratic equation has roots

$$\bar{K} = \tfrac{1}{2}[\delta \pm (\delta^2 - 4m)^{1/2}]$$

Since $m < 0$, the roots are real. Only one of these roots is positive. The negative one is $\bar{K} = \frac{1}{2}[\delta - (\delta^2 - 4m)^{1/2}]$. To see that $K_n \to \bar{K}$, $n \to \infty$ where $K_n = H(K_{n-1})$, $K_1 = H(K_0)$, consider the map $G \equiv H(H(K)) = m/[\delta - m/(\delta - K)] = m(\delta - K)/[\delta^2 - K\delta - m]$. Notice that G is increasing on $(-\infty, 0]$ because H is decreasing there. Also notice that $G(0) = \delta m/(\delta^2 - m) < 0$. Fixed points of G are given by

$$m(\delta - K)/[\delta^2 - K\delta - m] = K, \text{ i.e., } \quad m\delta - mK = K\delta^2 - K^2\delta - mK, \text{ i.e.,}$$

$K^2 - K\delta + m = 0$. Thus the fixed points of G are the same as H. Hence there is only one fixed point for G on $(-\infty, 0]$. Thus G must cut the 45° line in a stable manner on $(-\infty, 0]$ because G increases, $G(0) < 0$, and G has only one fixed point on $(-\infty, 0]$. Thus the process $K_{2n} = G(K_{2n-2})$, $K_2 = G(K_0)$ converges to $\bar{K}$, $n \to \infty$. Also $K_{2n+1} = G(K_{2n-1})$, $K_3 = G(K_1)$ converges to $\bar{K}$, $n \to \infty$. Thus $K_n \to \bar{K}$, $n \to \infty$. Q.E.D.

Proof of Theorem. To see that $\bar{f}$ is unique, note first that $E(\bar{f}) = \bar{f}$ implies that $\bar{f} = e^{Kt}$ for some $K \leqslant 0$. Then note that

$$E\bar{f} = e^{mt} \int_0^\infty e^{-\delta y} e^{Ky}\, dy = e^{m/(\delta - K)t}.$$

We have $m/(\delta - K) = K$ and thus $K = \bar{K}$.

The process $f_n = E(f_{n-1})$, $f_1 = E(f_0)$ converges to $\bar{f}$ because $f_1 = e^{K_0 t}$ where $K_0 = m \int_0^\infty e^{-\delta y} f_0(y)\, dy$, $f_2 = E(f_1) = e^{H(K_0)t} = e^{K_1 t}$, $f_n = e^{H(K_{n-2})t} = e^{K_{n-1}t}$. Thus $K_n = H(K_{n-1})$ describes the process $f_n = E(f_{n-1})$, $f_1 = E(f_0)$. Hence $f_n \to \bar{f}$ by Lemma 2. Q.E.D.

Notice that $\bar{f}$ generates a self-fulfilling set of forecasts, i.e., $\bar{x}_t^e(s) \equiv \bar{x} + \bar{f}(s-t)(x_0 - \bar{x}) = \bar{x}^a(s)$ for all s, t. Here $\bar{x}^a(t)$ is the path generated by actors forecasting with $f_t(s) = \bar{f}(s-t)$ for all s, t. The reason that $\bar{f}$ is independent of t can be seen by noticing that

$$\bar{x}(s) = x_t^e(s) = \bar{x} + \bar{f}(s-t)(\bar{x}(t) - \bar{x}) = \bar{x} + \bar{f}(s - t\bar{f}(t)(x_0 - \bar{x})$$

implies that $\bar{f}(s) \equiv (\bar{x}(s) - \bar{x})/(x_0 - \bar{x}) = \bar{f}(s-t)\bar{f}(t)$. But the latter functional equation is solved by the exponential $e^{\bar{K}t}$. Furthermore, the exponential functions are the only continuous ones that solve it. Since linear $\pi(p(x))$ generates exponential equilibrium forecasts, the equilibrium forecast will be independent of time.

We have worked out the case $\pi(p(x)) = mx + b$ thoroughly in the above. Existence of $\bar{f}_t$ is mathematically technical and is established in the appendix for more general profit functions. It would be nice to have convergence results for a larger class of aggregate profit functions. For example, the profit function that arises in Sections 2 and 3 is strictly convex. Unfortunately, this seems more difficult. Now let us turn to the question of uniqueness of $\bar{f}_t$.

The theorem may be extended to a larger class of expectations functions than F. Let $L(f)$ denote $\int_0^\infty e^{-\delta y} f(y)\, dy$. If δ were allowed to take all values in the complex plane than $L(f)$ is just the classical Laplace transform of f (Spiegel [12]). Let $\bar{F} = \{f \mid L(f) < \infty\}$. Let $f_0 \in \bar{F}$ and let $K_0 = L(f_0)$. Then $E(f_0)[t] = e^{K_0 t}$. Let $\bar{K}$ denote the positive fixed point of $H(K)$.

THEOREM 2. *If $K_0 \neq \bar{K}, \delta$ or if there does not exist $n \geqslant 1$ such that $H^n(K_0) = \delta$ then $H_n(K_0) \to \bar{K}$, $n \to \infty$.*

Proof. Look at $G(K) \equiv H^2(K)$. $G(\delta) = 0$, G is stable on $(-\infty, \bar{K})$, and G is unstable on $(\bar{K}, \infty)$. For $K_0 < \delta$, the proof follows the same argument as in Theorem 1. For $K_0 > \delta$ we show that there is n such that $H^n(K_0) < \delta$.

Case a. $\delta < K_0 < \bar{K}$. Since G is stable on $(-\infty, \bar{K})$ there is n such that $G^n(K_0) < \delta$.

Case b. $K_0 > \bar{K}$. When K_0 is larger than the vertical asymptote, $\hat{K}$, of G then $G(K_0) < 0$ and $G^n(K_0) \to \bar{K}$. If $K_0 < \hat{K}$, $G^n(K_0)$ increases in n, then goes negative for some n_0. The reader may construct his own proof upon graphing G.

Notice that in some sense "almost all" initial expectations are stabilized. Only in "measure zero" cases can expectations run away and explode. More precisely, if one introduced a measure μ on $\bar{F}$ (say a probability measure representing a priori beliefs about the correct initial guess (f_0) such that the probability of any particular f is zero (a usual condition in continuum probability models) then $P\{E^n(f_0) \to \bar{f},\ n \to \infty\} = 1$, i.e., the probability that an initial guess f_0 does not become stabilized through time is 0. Furthermore, this conclusion is independent of the probability measure μ on $\bar{F}$ so long as $\mu(\{f\}) = 0$, for all f in $\bar{F}$.

The reader may think that $P\{E^n(f_0) \to \bar{f},\ n \to \infty\} = 1$ independent of μ is an odd definition of stability. However, we feel that a definition that ignores "flukes" is more useful for real world modelling.

To investigate[5] uniqueness of the solution to the functional differential equation

$$\dot{x}(t) = \int_t^\infty e^{-\delta(s-t)} R(x(s))\, ds,\ x(0) = x, \tag{22}$$

where $R(x) \equiv \pi(p(x))$, we need to prove

THEOREM 2. *If $x(t)$ solves* (22) *then*

$$\dot{x}(t+h) = e^{\delta h}\dot{x}(t) - e^{\delta h} \int_t^{t+h} e^{-\delta(s-t)} R(x(s))\, ds \tag{23}$$

for all $t \geqslant 0$, $h \geqslant 0$.

Proof. If $x(t)$ solves (22), then

$$\begin{aligned}
\dot{x}(t+h) &= \int_{t+h}^\infty e^{-\delta(s-t-h)} R(x(s))\, ds \\
&= e^{\delta h} \int_t^\infty e^{-\delta(s-t)} R(x(s))\, ds - e^{\delta h} \int_t^{t+h} e^{-\delta(s-t)} R(x(s))\, ds \\
&= e^{\delta h}\dot{x}(t) - e^{\delta h} \int_t^{t+h} e^{-\delta(s-t)} R(x(s))\, ds \qquad \text{Q.E.D.}
\end{aligned}$$

THEOREM 3. *If $R(x)$ is strictly decreasing, there is only one function $x(t)$ that solves* (22).

Proof. The strategy will be to use (23) to show that if $x(t)$, $y(t)$ solves (22) then $x(t) \leqslant y(t)$ for all t or $x(t) \leqslant y(t)$ for all t. Thus it will turn out that all solutions of (22) are "comparable" in the "component wise"

[5] What we are doing here is giving another method for looking at uniqueness and existence of solutions to second-order differential equations. Equation (22) is second order. This can be seen by differentiating it one more time with respect to t. We present our methods here because they have some hope of being generalized in different directions than the usual methods used to study existence and uniqueness for differential equations.

ordering. Now by taking integrals up to t of both sides of (22) the resulting right hand side defines a mapping φ from functions to functions that is nonincreasing in the component wise ordering. But solutions of (22) are fixed points of φ. Thus suppose x, y solve (22) and, say, $x(t) \leqslant y(t)$ for all t. But this gives us $\varphi(x)(s) \geqslant \varphi(y)(s)$ for all s since φ is nonincreasing. From this we get $x = \varphi(x) \geqslant y = \varphi(y)$ so $x(t) = y(t)$ for all t. The only part of the proof that is not completely straightforward is showing that x, y solve (22) implies that $x(t) \leqslant y(t)$ for all t or $x(t) \geqslant y(t)$ for all t. Let us get on with that. First we show that if $x(t)$ rises above $y(t)$ for some t_0, then x stays above y for all $t \geqslant t_0$, i.e., assume there is $t_0 > 0$, $h > 0$ such that $x(t_0) = y(t_0)$, $x(t_0 + s) > y(t_0 + s)$, $0 \leqslant s \leqslant h$, and $\dot{x}(t_0) \geqslant \dot{y}(t_0)$. We shall show that $\bar{h} \equiv \max\{h \mid x(t_0 + s) > y(t_0 + s)$ for $s \in (0, \bar{h})\} = \infty$. Suppose $\bar{h} < \infty$. Then

$$\dot{x}(t_0 + \bar{h}) = e^{\delta\bar{h}}\dot{x}(t_0) - e^{\delta\bar{h}} \int_{t_0}^{t_0+\bar{h}} e^{-\delta(s-t_0)} R(x(s))\, ds$$

$$> e^{\delta\bar{h}}\dot{y}(t_0) - e^{\delta\bar{h}} \int_{t_0}^{t_0+\bar{h}} e^{-\delta(s-t_0)} R(y(s))\, ds = \dot{y}(t_0 + \bar{h}),$$

since $\dot{x}(t_0) \geqslant \dot{y}(t_0)$, $x(t) > y(t)$, t in $(t_0, t_0 + \bar{h})$. Now $\dot{x}(t)$, $\dot{y}(t)$ are continuous in t. Therefore, there is $\epsilon > 0$ such that $\dot{x}(t_0 + \bar{h} + \epsilon) > \dot{y}(t_0 + \bar{h} + \epsilon)$. But this contradicts $x(t) > y(t)$, t in $(t_0, t_0 + \bar{h})$, $x(t_0 + \bar{h}) = y(t_0 + \bar{h})$; so $\bar{h}$ must be infinite. It will help the reader if he draws a picture. This shows that if $x(t)$ rises above $y(t)$ for some t_0, then x stays above y for all $t \geqslant t_0$. Reversing the roles of x and y in the proof gives us $x(t) \leqslant y(t)$ for all t or $x(t) \geqslant (y(t)$ for all t as the only possibilities. The rest of the proof is easy. Because, as we pointed out before, fixed points of a nonincreasing operator are unique. Q.E.D.

Notice that uniqueness only depends on the "separability" structure of the fundamental Eq. (23) and the monotonicity of the operator defined by the associated integral equation of (22). We present one generalization here.

THEOREM 4. *Let $G_t(z)$ be a strictly increasing differentiable function in z, t such that G_t^{-1} defined for all reals. Then*

$$\dot{x}(t) = G_t\left[\int_t^{\infty} e^{-\delta(s-t)} R(x(s))\, ds\right], \quad x(0) = x \tag{24}$$

has a unique solution if it has any at all.

Proof. The fundamental equality becomes

$$\dot{x}(t+h) = G_{t+h}\left[\int_{t+h}^{\infty} e^{-\delta(s-t-h)}R(x(s))\,ds\right], \quad x(0) = x_0$$

$$= G_{t+h}\left[e^{\delta h}G_t^{-1}(\dot{x}(t)) - e^{\delta h}\int_t^{t+h} R(x(s))\,ds\right]$$

The rest of the proof is much the same as before. Q.E.D.

Theorem 4 shows us that, to a large extent, the property of uniqueness of self-fulfilling expectations is independent of the specification of rate of entry as a function of anticipated present value of profits provided that the rate of entry increases as anticipated present value of profits increases and the rate is 0 if anticipated present value is zero. Theorem 4 is important because it shows that uniqueness is extremely robust. It may be true that analogous results hold for δ, R depending on t, but we have not tried to prove this. But it should be relatively easy.

It is quite easy to obtain some properties of the unique equilibrium. If the profit function shifts or the time rate of discount changes. Proofs of this type of proposition are variations on the uniqueness proof. An example is

THEOREM 5. *Let $S(x) > R(x)$ for all $x > 0$. Suppose that both S, R are strictly decreasing. Let y, x be the equilibrium corresponding to S, R, respectively. Then $y(t) \geqslant x(t)$ for all $t > 0$, and $y(t) > x(t)$ for some t.*

Proof. If not, then there are two cases. Case B: $y(t) \leqslant x(t)$ for all t, and Case A: There is t_0, h such that $x(t_0) = y(t_0)$, $\dot{y}(t_0) \leqslant \dot{x}(t_0)$ and $y(t) \leqslant x(t)$ on $(t_0, t_0 + s)$, $s \leqslant h$. Case A can be ruled out by a similar argument as the existence proof. For

$$\dot{y}(t_0+h) = e^{\delta h}\dot{y}(t_0) - e^{\delta h}\int_{t_0}^{t_0+h} e^{-\delta(s-t_0)}S(y(s))\,ds$$

$$< e^{\delta h}\dot{x}(t_0) - e^{\delta h}\int_{t_0}^{t_0+h} e^{-\delta(s-t_0)}S(x(s))\,ds$$

$$< e^{\delta h}\dot{x}(t_0) - e^{\delta h}\int_{t_0}^{t_0+h} e^{-\delta(s-t_0)}R(x(s))\,ds = \dot{x}(t_0+h).$$

Hence the existence type of argument shows that $y(t) < x(t)$, $t > t_0$. But this is a case B situation, where $t_0 = 0$, $x(0) = x(t_0)$.

Suppose that $y(t) \leqslant x(t)$ for all t. Let ψ, φ be the operators associated with S, R, respectively. Then $y = \psi(y)$, $x = \varphi(x)$. But $\psi(z)(t) > \varphi(z)(t)$

for all z, t. So $y \leqslant x$ implies that $y = \psi(y) \geqslant \psi(x) > \varphi(x) = x$ a contradiction. So $y(t) > x(t)$ for $t > 0$. Q.E.D.

We may read off a number of interesting results from this theorem and the specific profit function that we calculated at the beginning of this paper. This type of proposition is a variation on the uniqueness proof. An example is

THEOREM 5. *Let $S(x) > R(x)$ for all $x > 0$. Suppose that both S, R are strictly decreasing. Let y, x be the equilibrium corresponding to S, R, respectively. Then $y(t) \geqslant x(t)$ for all $t > 0$, and $y(t) > x(t)$ for some t.*

Proof. If not then there are two cases. Case B: $y(t) \leqslant x(t)$ for all t, and Case A: There is t_0, h such that $x(t_0) = y(t_0), \dot{y}(t_0) \leqslant \dot{x}(t_0)$ and $y(t) \leqslant x(t)$ on $(t_0, t_0 + s)$, $s \leqslant h$. Case A can be ruled out by a similar argument as the existence proof. For

$$\begin{aligned}
\dot{y}(t_0 + h) &= e^{\delta h}\dot{y}(t_0) - e^{\delta h} \int_{t_0}^{t_0+h} e^{-\delta(s-t_0)} S(y(s))\, ds \\
&< e^{\delta h}\dot{x}(t_0) - e^{\delta h} \int_{t_0}^{t_0+h} e^{-\delta(s-t_0)} S(x(s))\, ds \\
&< e^{\delta h}\dot{x}(t_0) - e^{\delta h} \int_{t_0}^{t_0+h} e^{-\delta(s-t_0)} R(x(s))\, ds = \dot{x}(t_0 + h).
\end{aligned}$$

Hence the existence type of argument shows that $y(t) < x(t)$, $t > t_0$. But this is a case B situation, where $t_0 = 0$, $x(0) = x(t_0)$.

Suppose that $y(t) \leqslant x(t)$ for all t. Let ψ, φ be the operators associated with S, R, respectively. Then $y = \psi(y)$, $x = \varphi(x)$. But $\psi(z)(t) > \varphi(z)(t)$ for all z, t. So $y \leqslant x$ implies that $y = \psi(y) \geqslant \psi(x) > \varphi(x) = x$ a contradiction. So $y(t) > x(t)$ for $t > 0$. Q.E.D.

We may read off a number of interesting results from this theorem and the specific profit function that we calculated at the beginning of this paper. For example, look at Eq. (3). Any upward shift in $\pi(p(x)) = R(x)$ implies that the equilibrium path $x(t)$ will shift upward also. Furthermore, a similar technique as that used in Theorem 5 gives $\delta < \delta_1$ implies that $x_\delta(t) > x_{\delta_1}(t)$ for all $t > 0$ where x_δ, x_{δ_1} are the equilibria corresponding to δ, δ_1, respectively. In other words, if the future is discounted more heavily, then fewer firms will be present at each point in time in equilibrium. This is reasonable because rate of entry is based on present value of anticipated profit—a function of δ that decreases in δ.

This is a good place to explain why we are interested in the special case of point expectations. In studying this case we confronted the problem of designing an appropriate equilibrium concept. Once the equilibrium concept was laid down, the discussion moved naturally to stability. We think that the discussion we have given, the theorems we have proved, the insights we have acquired, and the techniques we have generated, will be helpful in studying the case of uncertainty.

5. Summary

In this paper we used the continuum of the producers model to study entry and exit of competitive industry. We discussed the effects of changes in expectations, the discount factor, entry cost, and variable cost. Our equilibrium concept turned out to be an interval of sizes whose width increased as entry cost and discount rate increased. By using concrete demand and supply functions throughout we were able to focus on the role that elasticity of demand plays in the effect of a decrease in variable cost. We found that decrease in variable cost works against equilibrium size if demand is inelastic. This occurs because competition between sellers runs down price more in the short run than decrease in variable cost raises the profit function.

We introduced expectations into the model. We showed that there is only one set of consistent expectations. This existence and uniqueness result holds for a broad class of profit functions and rate of entry specifications. The technique developed to analyze this model should be useful for many problems involving "perfect foresight" or self-fulfilling expectations. We have shown that it is possible to work out how the perfect foresight equilibrium path changes as parameters of the problem change.

We feel that this exercise represents a contribution to theory. First, the unique perfect foresight path converged to a steady state. This result shows that there are implicit restrictions when the assumption of perfect foresight is made. For if underlying technology and demand are time stationary, then the perfect foresight path converges to a fixed target. We conjecture that this will turn out to be true for more general models where demand and technology are time stationary.

One may ask whether the concept of equilibrium is meaningful from an economic point of view, i.e., are there any "natural" forces that push the economic system toward such an equilibrium path? We think that there are such forces although they may be extremely weak in some instances. For example, in our model it is hard to believe that there are strong forces pushing x_0^{500} to x_{500}, i.e., one can conceive of situations where the forecast of entrants at time 500 based at period 0 turns out to be incorrect but no losses are sustained. On the other hand, it never hurts to make a correct forecast. Hence, in some broad sense one would expect fluctuations around the perfect foresight path.

A Bayesian process is formulated for Muth's rational expectations (which are related to perfect foresight) by Cyert and DeGroot [22]. Presumably a similar analysis could be carried out for perfect foresight.

Finally, we think that our work represents an attempt to put perfect foresight equilibrium on the same logical basis as general equilibrium. Furthermore, for our simple model, at least, we are able to do much more than simply establish existence.

APPENDIX: Existence of an Equilibrium Forecast for General Profit Functions

THEOREM. *Let $\pi(p(x)) \equiv R(x)$, R be nonincreasing in x and let there be just one $\bar{x}$ such that $R(\bar{x}) = 0$. Also let R be continuous in x and satisfy a Lipchitz condition, i.e., there is $k > 0$ such that for all*

$$x_1, x_2, \ |R(x_1) - R(x_2)| \leqslant k \,|x_1 - x_2|.$$

Then there is an equilibrium, i.e., there is $\bar{x}_t^e(s)$, $t \geqslant 0$, $s \geqslant t$ such that $\bar{x}_t^e(s) = \bar{x}^a(s)$ for all $s \geqslant t$, $t \geqslant 0$, where

$$\dot{\bar{x}}^a(t) = \int_t^\infty e^{-\delta(s-t)} R(\bar{x}_t^e(s))\, ds, \quad t \geqslant 0. \tag{A1}$$

Proof. Write $f_t(s) \equiv f(s, t)$. Our strategy is to take $f \equiv f(s, t)$ in F, solve

$$\dot{y}(t) = \int_t^\infty e^{-\delta(s-t)} R(\bar{x} + f(s, t)(y(t) - \bar{x}))\, ds, \quad y(0) = x_0, \tag{A2}$$

put $g(s, t) \equiv (y(s) - \bar{x})/(y(t) - \bar{x})$, show that $g \in F$, get a map $E\colon F \to F$ in this way, show that E has a fixed point $\bar{f}$, and show that $\bar{x}_t^e(s) \equiv \bar{x} + \bar{f}(s, t)(\bar{y}(t) - \bar{x}) = \bar{y}(s)$, where $\bar{y}(t)$ solves (A2) with F replaced by $\bar{f}$ on the right hand side. Let us continue.

Let $F \equiv \{f(s, t) \mid f(t, t) = 0, f(s, t)$ is nonincreasing in s, f is continuous in both variables, and $0 \leqslant f \leqslant 1\}$. Now it is straightforward, using the standard uniqueness theory for differential equations, e.g., see Coddington and Levinson [18, p. 8] to show that Lipchitz implies that the solution y to (A2) is unique for each f in F. Thus the map E is well defined.

It is also straightforward to show that g is in F. This follows because for each $y < \bar{x}$, $0 \leqslant f(s, t) \leqslant 1$ implies that the right hand side of (A2) is bounded by $\int_t^\infty e^{-\delta(s-t)}R(y)\,ds = R(y)/\delta$ since R is nonincreasing. Since $R > 0$ on $(x_0, \bar{x})$ it is easy to see that the solution $y(t) < \bar{x}$ for $t \geqslant 0$. Thus, $g(s, t) = (y(s) - \bar{x})/(y(t) - \bar{x})$ is in F. Notice that $y(s) \to \bar{x}$ as $s \to \infty$. This is because $y(s)$ is bounded above by the solution of

$$\dot{z}(t) = \int_t^\infty e^{-\delta(s-t)} R(z(t))\,ds = R(z(t))/\delta, \tag{A3}$$

since $R(\bar{x} + f(s, t)(y(t) - \bar{x})) \leqslant R(y(t))$ for $y(t) \leqslant \bar{x}$.

Now the reader has probably guessed that we are going to use a fixed-point argument to produce an $\bar{f}$ in F such that $E(\bar{f}) = \bar{f}$. We will use the following:

SCHAUDER TYCHONOFF THEOREM (Dunford and Schwartz [15, p. 45-6]). *Let F be a compact, convex subset of a locally convex linear topological space X and let E be a continuous mapping from F to F. Then there is an $\bar{f}$ such that $E(\bar{f}) = \bar{f}$.*

Usage of the above theorem is trickier than usage of the Brouwer theorem because the properties of topologies on F that make F compact are not as "nice" as their counterparts in n-dimensional space. For example, $F = \{f \mid f(s, t)$ is continuous in s, t, nonincreasing in s, $f(t, t) = 1\}$ is not compact in the topology of pointwise convergence ($f_n \to f$, $n \to \infty$ iff $f_n(s, t) \to f(s, t)$, $n \to \infty$ for all $s, t \in [0, \infty)$). This is so because continuity is not preserved under pointwise convergence. F is not compact under the topology of uniform convergence because it is possible to find sequences $\{f_n\}_{n=1}^\infty$ that have no uniformly convergent subsequence.

We get around these problems by extending F to $\bar{F} = \{f \mid f(t, t) = 1$, $f \geqslant 0$ is nonincreasing in s, and f is *measurable*$\}$. Measurability of f means that f is continuous over all of the plane except for "small" sets. A good discussion of Lebesque measurability is given in Loéve [19]. Now measurability is a property that is preserved under pointwise convergence whereas continuity is not. Thus $\bar{F}$ is closed under pointwise convergence. Hence it can easily be shown that $\bar{F}$ is compact under pointwise convergence by an application of the Tychonoff theorem on product topologies [15, p. 32].

We show that E can be extended to $\bar{E}$: $\bar{F} \to \bar{F}$, i.e., $\bar{E} = E$ on F—we demonstrate that $\bar{E}$ is continuous, we produce a fixed point $\bar{f}$ in $\bar{F}$, and we show that, indeed, $\bar{f}$ lies in F.

Now that these preliminaries are out of the way let us go on with the proof.

LEMMA 1. *Let $f(s, t)$ be measurable on the plane, nonincreasing in s, $0 \leqslant f \leqslant 1$, and $f(t, t) = 1$ for all $t \geqslant 0$. Then*

$$h(y, t) \equiv \int_t^\infty e^{-\delta(s-t)} R(\bar{x} + f(s, t)(y - \bar{x}))\, ds \tag{A4}$$

is uniformly Lipchitz in $y \geqslant 0$, i.e., there is k independent of t such that $| h(y_1, t) - h(y_2, t)| \leqslant h \mid y_1 - y_2 \mid$ for all nonnegative y_1, y_2, t.

Proof. The Lebesque integral is well defined because R bounded in s, t. Now

$$\begin{aligned} &| h(y_1, t) - h(y_2, t)| \\ &\quad \leqslant \int_t^\infty e^{-\delta(s-t)} \{| R(\bar{x} + f(s, t)(y_1 - \bar{x})) - R(\bar{x} + f(s, t)(y_2 - \bar{x}))|\}\, ds \\ &\quad \leqslant k/\delta \sup_{s,t} | f(s, t)| \mid y_1 - y_2 \mid \leqslant k/\delta \mid y_1 - y_2 \mid \end{aligned}$$

because $0 \leqslant f \leqslant 1$ and R is Lipchitz. Here "sup" means supremum.

From Lemma 1 we are assured by the standard uniqueness theory for differential equations that a unique solution to (A2) exists for each f in $\bar{F}$. Notice that E sends the large set $\bar{F}$ to the much smaller set of f that possess first derivatives in s, t. ◆

LEMMA 2. *$\bar{E}: \bar{F} \to \bar{F}$ is continuous in the topology of pointwise convergence.*

Proof. This is another example of where the notion of measurability and Lebesque integral is more useful than continuity and the Riemann integral, viz., the number of times when one can legally pass the limit under the integral is much greater in the Lebesque case.

Let f_n be in $\bar{F}$, and let $f_n \to f_0$ pointwise. We show that $\bar{E}(f_n) \to \bar{E}(f)$. First f is measurable because it is a pointwise limit of measurable functions. It is trivial to see that f is nonincreasing in s and $f(t, t) = 1$. Thus f is in $\bar{F}$. Hence $\bar{E}(f)$ is well defined.

Integrate both sides of (A2) up to t to obtain

$$y(t) = x_0 + \int_0^t \int_r^\infty e^{-\delta(s-r)} R(\bar{x} + f(s, r)(y(r) - \bar{x}))\, ds\, dr. \tag{A5}$$

The function y solves (A2) iff it solves (A5). Now it is a routine application of Lebesque's dominated convergence theorem (Loéve [19, p. 125]) to show that the right hand side of (A5) is continuous, i.e., let $\varphi(y)$ denote the right hand side of (A5). Then φ sends functions $y(t)$ to functions $z(t)$. Continuity means that $z_n(t) \to z(t)$, $n \to \infty$ for all $t \geqslant 0$ implies $\varphi(z_n)(s) \to \varphi(z)(s)$, $n \to \infty$ for all s.

Now let $f_n \to f$ pointwise; let y_n, y be the corresponding solutions of (A5). We show that every cluster point, $\bar{y}$, of the sequence $\{y_n\}$ is equal to y. To do this let $y_{n_j} \to \bar{y}$, $j \to \infty$ pointwise. Since $f_{n_j} \to f$ another application of the dominated convergence theorem gives us

$$\int_0^t \int_r^\infty e^{-\delta(s-t)} R(\bar{x} + f_{n_j}(s, r)(y_{n_j}(r) - \bar{x}))\, ds\, dr \to \int_0^t \int_r^\infty e^{-\delta(s-t)} \times R(\bar{x} + f(s, r)(\bar{y}(r) - \bar{x}))\, ds\, dr, j \to \infty.$$

Thus $\bar{y}$ solves (A5). Hence the uniqueness theory of differential equations gives us $\bar{y} = y$ because y solves (A5) also. Thus all cluster points are the same. Therefore, $y_n \to y$, $n \to \infty$. Now $\bar{E}f \equiv g$ where $g(s, t) = (y(s) - \bar{x})/(y(t) - \bar{x})$ where y solves (A2). Since

$$(y_n(s) - \bar{x})/(y_n(t) - \bar{x}) \to (y(s) - \bar{x})/(y(t) - \bar{x}), \qquad n \to \infty.$$

We have $\bar{E}(f_n) \to \bar{E}f$, $n \to \infty$.

LEMMA 3. *E has a fixed point in F.*

Proof. By Lemma 2 $\bar{E}: \bar{F} \to \bar{F}$ is continuous. Thus there is $\bar{f}$ in $\bar{F}$ such that

$$\bar{y}(t) = x_0 + \int_0^t \int_r^\infty e^{-\delta(s-r)} R(\bar{x} + \bar{f}(s, r)(\bar{y}(r) - \bar{x})\, ds\, dr. \tag{A6}$$

The right hand side of (A6) is continuous in t. Thus the left hand side must be also. Therefore, $\bar{f}(s, t) = (\bar{y}(s) - \bar{x})/(\bar{y}(t) - \bar{x})$ is continuous in s, t. Thus $\bar{f}$ is in F. Q.E.D.

The three Lemmas give us the proof of the theorem. The proof was rather roundabout. After all, all that was asked for was a solution x to the differential equation

$$\dot{x}(t) = \int_t^\infty e^{-\delta(s-t)} R(x(s))\, ds, \quad x(0) = x_0. \tag{A7}$$

But our method gives another way of looking at existence and uniqueness for second-order differential equations like (A7). Equation (A7) is second order because it can be differentiated again to get $\ddot{x} = \delta\dot{x} - R(x)$, $x(0) = x_0$. A phase diagram can be drawn to study uniqueness and existence in the usual way. However, our methods quickly generate a uniqueness proof for more general cases, viz., Theorem 3. We feel that it is worthwhile to be cognizant of more than one way of looking at uniqueness and existence theory for second order differential equations thrown up by these types of problems. We conjecture that our technique will be useful for many economic problems of this type.

Even if it should turn out that there is an easier way to establish existence for the general case, our method suggests the following economic interpretation. At time 0 economic agents expect entrants to come in according to the scheme $x_0^e(s) = \bar{x} + f(s, 0)(x_0 - \bar{x})$. At time t they revise the rule to $x_t^e(s) = \bar{x} + f(s, t)(x(t) - \bar{x})$.

The schemes or "rules of thumb" $f(s, t)$ are equilibrium if $x_t^e(s) = x^a(s)$ for all t, s. We do not want to make too much of this interpretation but it did seem worth mentioning.

We would like to make one more point. That is, the existence proof may be generalized to more general profit functions and entry mechanisms. Let us just sketch how this might go. First, all that is needed to get uniqueness of the solution to the differential Eq. (A2) is some sort of Lipchitz condition. Then one wants to be able to bound the right hand side of (A2) by a function $B(y)$ that decreases and is zero at $\bar{x}$.

Roughly speaking, this will suffice to keep $[x^a(s) - \bar{x}]/[x^a(t) - \bar{x}]$ in F. The rest of it just amounts to checking when one can pass the limit under the integral sign and such. We leave it to the reader to find a minimal set of assumptions on entry mechanisms and profit functions such that existence goes through.

Since it will probably be useful to have as many techniques available as possible to deal with these long-run perfect foresight paths, let us present a sketch of a uniqueness proof that relies on a phase diagram analysis. Start with the equation

$$\dot{x} = \int_t^\infty e^{-\delta(s-t)} R(x(s))\, ds = e^{\delta t} \int_t^\infty e^{-\delta s} R(x(s))\, ds.$$

Differentiate it with respect to t to get

$$\ddot{x} = \delta \dot{x} - R(x). \tag{A8}$$

Equation (A8) may be rewritten

$$\dot{x}_2 = \delta x_2 - R(x_1), \qquad \dot{x}_1 = x_2\,, \qquad x_1(0) = x_0\,, \tag{A9}$$

where $x_1 \equiv x$.

A phase diagram may be drawn for Eq. (A9). It turns out to be a saddle-point configuration. The unique perfect foresight path starting from x_0 is gotten by picking $x_2(0)$ so that the solution of (A9) converges to the steady state $\bar{x}$, where $\bar{x}$ is defined by $R(\bar{x}) = 0$. This is analogous to the standard phase diagrammatic analysis of optimal growth; see Burmeister and Dobell [25, Chap. 11, p. 395]. The phase diagram analysis is more intuitive than our other methods but it is more difficult to generalize to the nonautonomous of time case.

Recently Hale [24] has analyzed functional differential equations of the form

$$\dot{x} = \int_{-\infty}^{t} a(t-u)\, g(x(u))\, du, \tag{A10}$$

where a is an exponential like function on the line and g is a vector-valued function from n-dimensional space to itself. He uses Lyaponov function techniques to study stability. His analysis may be of some help in an n good generalization of our analysis.

One more method should be mentioned. It is easy to construct a problem

$$\min \int_0^\infty e^{-\delta t} F(x, \dot{x})\, dt \tag{A11}$$

such that the Euler equation

$$d/dt \left(e^{-\delta t} \frac{\partial}{\partial \dot{x}} F(x, \dot{x}) \right) = \frac{\partial}{\partial x} (e^{-\delta t} F(x, \dot{x})) \tag{A12}$$

turns out to be (A8). To see this calculate out (A12), divide by $e^{-\delta t}$ to get

$$-\partial F/\partial x - \delta\, \partial F/\partial \dot{x} + (\partial^2 F/\partial x\, \partial \dot{x})\dot{x} + (\partial^2 F/\partial \dot{x}^2)\ddot{x} = 0. \tag{A13}$$

Solve for $F(x, \dot{x})$ such that $\partial^2 F/\partial \dot{x}^2 = 1$,

$$\partial^2 F/\partial x\, \partial \dot{x} = -\delta \qquad \text{and} \qquad (-\partial F/\partial x) - \delta(\partial F/\partial \dot{x}) = R(x)$$

so that (A8) is reproduced by (A13). Doing this we get

$$F = \dot{x}^2/2 - \delta x \dot{x} + \frac{\delta^2 x^2}{2} - \int_0^x R(y)\, dy.$$

This is convex in $\dot{x}$, x and information can be gotten on (A8) through the fact that it solves a convex minimization problem. Bellman's principle of optimality may be used to gather qualitative information about the optimal solution as in Lucas and Prescott [20].

We have sketched a number of techniques for dealing with long-run perfect foresight paths in this paper. Some of them will be more useful for some purposes than others. We hope that our sketch of all of the techniques that we have been able to find in the literature and invent for ourselves will be useful for workers in this area.

ACKNOWLEDGMENTS

I would like to thank K. Arrow, R. Cyert, M. DeGroot, D. Dasgupta, J. Friedman, S. Chakravarty, R. Ruffin, S. Rosen, and E. Zabel for valuable conversation. My ideas on expectations formation were stimulated in large part by talks with Dasgupta and Friedman. This work also profited a great deal from the help of Professor McKenzie and his seminar group. It was presented at the M.S.S.B. summer workshop on Uncertainty and Sequence Economies at Berkeley in the summer of 1971. Professor Arrow found a conceptual error in this work and suggested a way around it — which we used. I had access to unpublished work by Lester Telser, which was helpful. I would like to thank NSF for Grant GS 3185 which supported this work. R. Lucas and S. Fischer gave me valuable aid for which I thank them. I thank the referee for remarks that led to a substantial revision of the paper. Needless to say, I am responsible for errors that remain.

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