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# A simplified axiomatic approach to ambiguity aversion

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**Abstract** This paper takes the Anscombe–Aumann framework with horse and roulette lotteries, and applies the Savage axioms to the horse lotteries and the von Neumann–Morgenstern axioms to the roulette lotteries. The resulting representation of preferences yields a subjective probability measure over states and two utility functions, one governing risk attitudes and one governing ambiguity attitudes. The model is able to accommodate the Ellsberg paradox and preferences for reductions in ambiguity.

**Keywords** Ambiguity · Savage axioms · Anscombe–Aumann framework · Independence axiom · Ellsberg paradox

**JEL Classification** D81

One of the famous problems that highlights the difference between risk and ambiguity (or uncertainty) is the two-color Ellsberg problem (see Ellsberg 1961). A decision maker is faced with two urns. The first urn contains 50 red and 50 yellow balls, and the second contains 100 balls but in an unknown mixture of red and yellow. The decision maker will be paid \$10 if she can draw a ball of a specified color and must choose from which urn to draw. The first urn generates a known payoff distribution, so it is risky, but the second urn generates an unknown payoff distribution, so it is ambiguous. Subjects systematically avoid the ambiguous urn in favor of the risky urn,

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regardless of which color generates the payoff. This creates a need for a model of choice behavior which can both accommodate the distinction between risk and ambiguity and separate attitudes toward ambiguity from attitudes toward risk.

The literature contains many models that address this need. Importantly, though, the path taken by this literature, including the literature that preceded Ellsberg's work, includes several forks where different researchers chose divergent modeling approaches. This paper adds to the literature by taking a particularly simple approach to creating a model of ambiguity attitudes through reconnecting some of these divergent paths.

The first important divergence that informs this paper detached subjective expected utility from objective expected utility. In expected utility (EU) the decision maker chooses among lotteries, or objective probability distributions, and von Neumann and Morgenstern (1944) generated the first classic axiomatization of these preferences. The axiomatization identifies a set of properties such that the individual's preferences satisfy those properties if and only if they can be represented by an expected utility function form

$$U(p) = \int_X u(x)dp(x), \tag{1}$$

where $x$ is a payoff in the set $X$, $p$ is the probability distribution which is the object of choice with $p(x)$ the probability placed by $p$ on payoff $x$, and $u(x)$ is the von Neumann–Morgenstern (vNM) utility function that captures risk attitudes. The key contribution here is that the utility function $u$ embodies everything about how an individual feels about risk.

In subjective expected utility (SEU), on the other hand, objective probabilities do not necessarily exist, and the individual forms her own assessment of those probabilities and then chooses among the resulting lotteries. To construct a model around this idea, one must provide an object for the decision maker to form beliefs over, and this object is a state of nature. In subjective expected utility models the decision maker chooses among acts, which are mappings that assign payoffs to states of nature, and the subjective expected utility representation is

$$V(f) = \int_S u(f(s))d\mu(s), \tag{2}$$

where $s$ is a state in the set $S$, $f$ is an act which is the object of choice with $f(s)$ the payoff assigned to state $s$ by act $f$, $u(\cdot)$ is the vNM utility function defined over payoffs, and $\mu(s)$ is the subjective probability assigned to the state $s$. Unlike EU where the only way two individuals can differ is through the shape of the utility function $u$, two SEU maximizers can differ in two ways: the shape of the utility function $u$, and the probabilities $\mu$ they assign to states.

The second important divergence relevant to this paper arises from how researchers have gone about axiomatizing subjective expected utility. The primary challenge lies in deriving subjective probabilities from preferences, and one must use a choice space that is sufficiently rich that it can identify both the utility function and the subjective probability measure. The two seminal approaches can be found in Anscombe and Aumann (1963) and Savage (1954). Savage achieves sufficient richness through an infinite state space where objects of choice are acts, as defined above, and his axioms hold if and only if preferences have the representation in Eq. 2. In comparison, Anscombe and Aumann reduce the state space but expand the choice set by breaking acts into two components. A roulette lottery is an objective probability distribution, while a horse lottery is a mapping from states into roulette lotteries, and individuals choose among horse lotteries. Savage's acts, then, are horse lotteries that map states into degenerate roulette lotteries.[1] Letting $h$ denote a horse lottery and $h_s$ the ensuing roulette lottery in state $s$, Anscombe and Aumann axiomatize a preference representation of the form[2]

$$V(h) = \sum\nolimits_S \left( \int_X u(x) dh_s(x) \right) d\mu(s). \tag{3}$$

Both models treat the Ellsberg urn problem the same way. There are two states of the world, one in which a red ball is drawn and one in which a yellow is drawn. The decision maker is paid \$10 for drawing a red ball. In the unambiguous urn the probability is objective, so $\mu$ must assign probability $\frac{1}{2}$ to a payoff of \$10 (a red ball is drawn) and probability $\frac{1}{2}$ to a payoff of \$0 (a yellow ball is drawn). In the ambiguous urn the subjective probability could be anything, but if $\mu$ assigns probability $q$ to a payoff of \$10 it must assign $1 - q$ to a payoff of \$0.[3] If $q < \frac{1}{2}$ the decision maker prefers the unambiguous urn for betting on a red ball, but both models then predict that the same decision maker prefers the ambiguous urn for betting on a yellow ball, contrary to how experimental subjects behave.

Because subjective expected utility is inconsistent with the Ellsberg paradox evidence of ambiguity aversion, the models must be generalized, and the obvious candidate for generalization is the subjective probability measure $\mu$. Once again the path has split, this time according to how the probability measure is generalized. One approach, pioneered by Gilboa (1987), Schmeidler (1989), and Sarin and Wakker (1992), makes the probability measure nonadditive. The intuition is straightforward. Let $E_1$ and $E_2$ be two disjoint events, such as drawing a red ball from the urn and drawing a yellow ball from the urn. The measure $\mu$ is superadditive if $\mu(E_1) + \mu(E_2) \leq \mu(E_1 \cup E_2)$, or, put differently, the measure is superadditive if it assigns less "probability" to the two events separately than it does to the two events together. In the context of the ambiguous Ellsberg urn, superadditivity makes it possible for a decision maker to assign a "probability" less than a half to

[1] A degenerate lottery places probability one on a single outcome.

[2] The sum in the Anscombe–Aumann representation reflects the fact that their state space is finite, and replaces the integral in Eq. 2.

[3] Note that, given the state, the resulting lottery is degenerate, placing all the probability on the appropriate payoff, so the Savage and Anscombe–Aumann representations, Eqs. 2 and 3, generate the same values.

drawing a red ball *and* a "probability" less than a half to drawing a yellow ball, even though the probability of drawing a ball of one color or the other is one. Because true probabilities must be additive and sum to one, these nonadditive probabilities are typically called capacities, and the decision model is known as Choquet expected utility.

A second approach, developed by Gilboa and Schmeidler (1989), assumes that in the presence of ambiguity the decision maker is unable to form a single probability measure over states of nature, so she forms multiple probability measures and then evaluates acts according to the worst probability measure for that act. This approach is known as the multiple priors approach, and the decision model is known as maxmin expected utility (MEU).[4] Schmeidler (1989) shows that Choquet expected utility is a special case of maxmin expected utility. To see how it handles ambiguity aversion, suppose that the decision maker confronted with the ambiguous Ellsberg urn believes that the urn could have either 40 red balls, 50 red balls, or 60 red balls. If the bet pays \$10 for drawing a red ball, the worst prior is the one that makes the probability of drawing a red ball 0.4, in which case the unambiguous urn dominates. For the bet paying \$10 for a yellow ball, the worst prior is the one that makes the probability of drawing a red ball 0.6, and again the unambiguous urn dominates.

The third approach is the one taken in this paper. I generalize subjective expected utility by assuming Savage's infinite state space, Anscombe and Aumann's formulation of horse and roulette lotteries, and applying Savage's axioms to preferences over horse lotteries instead of acts. In addition, I apply the familiar von Neumann–Morgenstern axioms when preferences are restricted to roulette lotteries. This very simple approach yields a preference representation of the form[5]

$$W(h) = \int_S w\left(\int_X u(x)h_s(x)\right) d\mu(s), \tag{4}$$

where $h$ is a horse lottery, the state of nature $s \in S$ determines the objective probability distribution (roulette lottery) $h_s$ over payoffs $x \in X$, $\mu$ is the subjective probability distribution over states of nature, $u$ is a vNM utility function governing attitudes toward risk, and $w$ is another utility function, this time governing attitudes toward ambiguity. Preferences with the representation in Eq. 4 are called *second-order expected utility* (SOEU) preferences, and the purpose of this paper is to provide an axiomatic foundation for such preferences.

A comparison of Eqs. 4 and 3 highlights the differences between SOEU and SEU preferences. In both expressions the term in large parentheses is the

[4] The model has recently been generalized by Maccheroni et al. (2006), and extended by Gilboa et al. (2010).

[5] This functional specification is also proposed by Kreps and Porteus (1978) for the analysis of dynamic choices under risk. The specification here was first proposed in Neilson (1993).

expected utility of an unambiguous roulette lottery, specifically the roulette lottery determined from the horse lottery $h$ and the state $s$. In the SOEU model the decision maker transforms this expected utility by the *second-order utility function* $w$, and Anscombe–Aumann subjective expected utility is the special case that arises when $w(z) = z$.[6] The paper shows how a concave second-order utility function can accommodate choice patterns in the Ellsberg paradox, and the second-order utility function captures ambiguity attitudes the same way that the von Neumann–Morgenstern utility function captures risk attitudes, with concavity corresponding to aversion.

The key to the construction here is that states correspond to roulette lotteries, as in the Anscombe–Aumann approach, and this paper's contribution to the literature is the simplicity of the axiomatic framework.[7] In particular, no new axioms are proposed, only a different application of the old axioms. The expected utility axioms are applied to preferences over roulette lotteries, and then the Savage axioms are applied to preferences over horse lotteries, thereby combining elements from objective and subjective expected utility, and elements from the Savage and Anscombe–Aumann approaches.

Other researchers have provided axiomatizations of second-order expected utility, but all using different approaches.[8] Klibanoff et al. (2005) axiomatize a more general version of SOEU in which the first integral is taken over the set of probability distributions over states rather than over states themselves, allowing the decision maker to have multiple priors as in the maxmin expected utility model. Their functional form coincides with the one here in the case that all of the probability is placed on a single prior. Nau (2006) and Ergin and Gul (2009) axiomatize more general versions of SOEU using a richer, higher-dimensional state space. Chew and Sagi (2008) show how SOEU can arise when the Savage axioms apply within, but not necessarily across, appropriate subsets of the event space. Grant et al. (2009) present two axiomatizations of SOEU that rely on ambiguity aversion as assumptions, similar to Werner's (2005) axiomatization of (objective) expected utility that relies on risk aversion as an assumption. Their paper can therefore be thought of as an explicit axiomatization of ambiguity averse SOEU. Finally, Strzalecki (2009) shows that the intersection of SOEU preferences with MEU preferences is precisely the set of SEU preferences.

[6] Anscombe and Aumann's original approach, however, only allows for a finite state space. Consequently, the axioms in Section 1, combined with an ambiguity-neutrality axiom, would comprise an infinite state space version of the Anscombe–Aumann model.

[7] This simplicity has a cost, though, as the approach implicitly identifies as ambiguous anything other than a constant horse lottery, or put differently, anything that generates a subjective belief. While this is consistent with Strzalecki (2009), it runs contrary to preference-based attempts to define "ambiguous," such as in Epstein and Zhang (2001), Ghirardato et al. (2004), and Ahn (2008).

[8] Hazen (1987) constructs and axiomatizes a model that is similar in spirit to SEOU, but differs from it in the way that some nonexpected utility models differ from expected utility. Hazen and Lee (1991) then show how Hazen's model accommodates evidence such as the Ellsberg paradox.

Section 1 sets up the Anscombe–Aumann framework, identifies the axioms, and provides the representation theorem. Section 2 shows that the second-order expected utility representation in expression (4) can easily accommodate ambiguity averse behavior when the function $w$ is concave. Section 3 offers a brief conclusion.

## 1 The representation theorem

The model adopts the roulette and horse lottery framework of Anscombe and Aumann (1963). The bounded interval $X$ is the payoff space, and $\Delta(X)$ is the set of all Borel-measurable probability distributions over $X$. Members of $\Delta(X)$ are also called *roulette lotteries*. Let $S$ be the set of states of the world, with generic element $s$. Define $\Sigma$ to be the set of all subsets of $S$, with generic element $E$, which is interpreted as an event. Savage (1954) defines an *act* as a mapping from $S$ to $X$, while Anscombe and Aumann define a *horse lottery* as a mapping from $S$ to $\Delta(X)$, that is, a mapping assigning a roulette lottery to every state. The resolution of an act is an outcome in the payoff space, while the resolution of a horse lottery is a roulette lottery, which is a probability distribution over payoffs. Let $\mathcal{H}$ denote the set of all horse lotteries.

The key to this paper is applying the Savage axioms to horse lotteries instead of acts. To do so, assume that the individual has a preference ordering $\succsim$ defined over $\mathcal{H}$. In what follows, $f, f', h, h' \in \mathcal{H}$ are horse lotteries, $\pi, \pi', \rho, \rho' \in \Delta(X)$ are roulette lotteries, and $E$, $E'$, $E_i \in \Sigma$ are events. Abusing notation when the context is clear, the roulette lottery $\pi$ is also a degenerate horse lottery assigning the same probability distribution to every state in $S$, that is, $h_s = \pi$ for all $s \in S$. These degenerate horse lotteries are called *constant lotteries*. The set $E^c$ is the complement of $E$ in $S$, that is, $S \setminus E$. A set $E$ is *null* if $h \sim f$ whenever $h_s = f_s$ for all $s \in E^c$, and where $\sim$ is the indifference relation. It is said that $h = f$ *on* $E$ if $h_s = f_s$ for all $s \in E$. It is said that $h \succsim f$ *given* $E$ if and only if $h' \succsim f'$ whenever $h_s = h'_s$ for $s \in E$, $f_s = f'_s$ for $s \in E$, and $h'_s = f'_s$ for all $s \in E^c$.

We use the following axioms over $\succsim$. Axioms A1–A7 are the Savage axioms, and axiom A8 and A9 apply Grandmont's (1972) versions of the von Neumann and Morgenstern continuity and independence axioms to constant horse lotteries, which are simply probability distributions.[9]

**A1** (Ordering) $\succsim$ is complete and transitive.

**A2** (Sure-thing principle) If $f = f'$ and $h = h'$ on $E$, and $f = h$ and $f' = h'$ on $E^c$, then $f \succsim h$ if and only if $f' \succsim h'$.

[9] Grandmont's Archimedean axiom A8 could be replaced by the more standard continuity axiom if attention were restricted to horse lotteries that generate simple roulette lotteries in each state, that is, probability distributions with finite support. Doing so would require altering expression (4) to replace the inner integral with a sum.

**A3** (Eventwise monotonicity) If $E$ is not null and if $f = \pi$ and $h = \rho$ on $E$, then $f \succsim h$ given $E$ if and only if $\pi \succsim \rho$.

**A4** (Weak comparative probability) Suppose that $\pi \succsim \rho$, $f = \pi$ on $E$, $f = \rho$ on $E^c$, $h = \pi$ on $E'$, and $h = \rho$ on $E'^c$, and suppose that $\pi' \succsim \rho'$, $f' = \pi'$ on $E$, $f' = \rho'$ on $E^c$, $h' = \pi'$ on $E'$, and $h' = \rho'$ on $E'^c$. Then $f \succsim h$ if and only if $f' \succsim h'$.

**A5** (Nondegeneracy) $\pi \succ \rho$ for some $\pi, \rho \in \Delta(X)$.

**A6** (Small event continuity) If $f \succ h$, for every $\pi \in \Delta(X)$ there is a finite partition of $S$ such that for every $E_i$ in the partition, if $f' = \pi$ on $E_i$ and $f' = f$ on $E_i^c$ then $f' \succ h$, and if $h' = \pi$ on $E_i$ and $h' = h$ on $E_i^c$ then $f \succ h'$.

**A7** (Uniform monotonicity) For all $E \in \Sigma$ and for all $\pi \in h(E)$, if $f \succsim \pi$ given $E$, then $f \succsim h$ given $E$. If $\pi \succsim f$ given $E$, then $h \succsim f$ given $E$.

**A8** (Continuity over risk) For every $\pi_0 \in \Delta(X)$ the sets $\{\pi \in \Delta(X) | \pi \succsim \pi_0\}$ and $\{\pi \in \Delta(X) | \pi_0 \succsim \pi\}$ are closed in $\Delta(X)$.

**A9** (Independence over risk) $\pi \succsim \pi'$ if and only if $a\pi + (1-a)\rho \succsim a\pi' + (1-a)\rho$ for all $\rho \in \Delta(X)$ and all scalars $a \in (0, 1)$.

Axioms A1–A7 are the standard Savage axioms modified so that they govern preferences over horse lotteries instead of preferences over acts. The main difference between these axioms and Savage's, then, is that here probability distributions in $\Delta(X)$ replace outcomes in $X$.

It is worth investigating whether the sure-thing principle (A2) makes the independence axiom (A9) redundant. The independence axiom implies that, for any roulette lotteries $\pi, \rho, \sigma, \tau \in \Delta(X)$ and $a \in [0, 1]$, if $a\pi + (1-a)\sigma \succsim a\rho + (1-a)\sigma$ then $a\pi + (1-a)\tau \succsim a\rho + (1-a)\tau$. These are probability mixtures, and therefore $a\pi + (1-a)\sigma$ and the other three mixtures are all elements of $\Delta(X)$. One can obtain almost the same construction using a sure-thing principle framework, where $E, E^c \in \Sigma$ are events. Construct horse lotteries $f, f', h, h' \in \mathcal{H}$ according to the following table:

| | $E$ | $E^c$ |
|---|---|---|
| $f$ | $\pi$ | $\sigma$ |
| $f'$ | $\pi$ | $\tau$ |
| $h$ | $\rho$ | $\sigma$ |
| $h'$ | $\rho$ | $\tau$ |

The sure-thing principle states that $f \succsim h$ if and only if $f' \succsim h'$, that is, preferences only depend on states in which the two horse lotteries being considered have different outcomes. If the individual assigns subjective probability $a$ to event $E$ , the horse lottery $f$ would generate a subjective probability distribution identical to the probability mixture $a\pi + (1-a)\sigma$. However, the probability in the horse lottery $f$ is subjective, and so the mixture entailed in $f$ is not a true probability mixture, and therefore not in $\Delta(X)$, so $f$ and $a\pi + (1-a)\sigma$ are really two different objects. Furthermore, there is no way to use a horse lottery to build a probability mixture from its component parts, and therefore the horse lottery framework of axiom A2 cannot duplicate the roulette lottery framework of axiom A9. Both are needed.

**Theorem 1** *Preferences satisfy* A1–A9 *if and only if there exists a unique, convex-ranged probability measure* $\mu : \Sigma \to [0, 1]$*, a bounded function* $u : X \to \mathbb{R}$*, and a monotone increasing, bounded function* $w : u(X) \to \mathbb{R}$ *such that for all* $f, h \in \mathcal{H}$*,* $h \succsim f$ *if and only if*

$$\int_S w\left(\int_X u(x)d(h_s(x))\right)d\mu(s) \geq \int_S w\left(\int_X u(x)d(f_s(x))\right)d\mu(s).$$

*Moreover, the function* $u$ *is unique up to increasing affine transformations, and for a given specification of* $u$ *the function* $w$ *is unique up to increasing affine transformations over the domain* $u(X)$.

*Proof* Proof of the "if" direction is standard. For the "only if" direction, by the Savage axioms A1–A7, there exists a unique, convex-ranged[10] probability measure $\mu : \Sigma \to [0, 1]$, and a bounded function $v : \Delta(X) \to \mathbb{R}$ such that the preference ordering $\succsim$ is represented by the functional

$$W(h) = \int_S v\left(h_s\right) d\mu(s). \tag{5}$$

Furthermore, $\mu$ is unique and $v$ is unique up to increasing affine transformations over its relevant domain. By axioms A8 and A9, $\succsim$ restricted to constant horse lotteries can be represented by

$$V(\pi) = \int_X u(x)d\pi(x), \tag{6}$$

where $u$ is bounded and unique up to increasing affine transformations. Axiom A3 implies that $V(\pi)$ and $v(\pi)$ must represent the same preferences over roulette lotteries, so $V$ is bounded and there exists a monotone function $w : \mathbb{R} \to \mathbb{R}$ such that

$$v(\pi) = w(V(\pi)). \tag{7}$$

Because $V$ is bounded so is $w$. Since $v$ is unique up to increasing affine transformations over the relevant domain, so is $w$ for a given specification of $u$, and the relevant domain is $u(X)$. Substituting Eq. 6 into Eq. 7 and then Eq. 7 into Eq. 5 yields

$$W(h) = \int_S w\left(\int_X u(x)dh_s(x))\right)d\mu(s). \tag{8}$$

□

[10]The probability measure $\mu$ is convex-ranged if for any event $E$, for every $\alpha \in [0, \mu(E)]$ there exists an event $E' \subseteq E$ for which $\mu(E') = \alpha\mu(E)$.

It is worth pointing out why the axioms do not get us all the way to subjective expected utility (and ambiguity neutrality). The key is that axioms A1–A7 only link the event space $\Sigma$ to the roulette lottery space $\Delta(X)$, and not all the way to the payoff space $X$. The independence axiom A9 places structure on the link between the roulette lottery space and the payoff space, but not enough to provide that missing link. Consequently, risk attitudes and ambiguity attitudes remain separated.

## 2 Ambiguity aversion

The second-order expected utility specification in Eq. 4 easily accommodates ambiguity attitudes as revealed in patterns such as the Ellsberg paradox. First consider the two-color paradox described in the introduction, where an individual is paid \$10 for drawing a red ball from one of two urns. Urn 1 is unambiguous and has 50 red balls and 50 yellow ones. Urn 2 has an unknown mixture of 100 red and yellow balls. To model this we begin with two state-of-nature variables, $s_1$ corresponding to the number of red balls in the first urn and $s_2$ corresponding to the number of red balls in the second urn. To fit the axiomatic framework assume that these are continuous variables taking values between 0 and 100. In general for state $s$ a horse lottery $h$ would generate the roulette lottery that pays \$10 with probability $s/100$. Since there are only two payoffs we can normalize the von Neumann–Morgenstern utility function $u$ so that $u(10) = 1$ and $u(0) = 0$, and the second-order expected utility of this horse lottery would be

$$W(h) = \int_0^{100} w\left(\frac{s}{100}\right) d\mu(s). \tag{9}$$

With two different state variables there must be two different subjective probability distributions.[11] The distribution $\mu_1(s_1)$ captures the individual's beliefs over the number of red balls in urn 1, and $\mu_2(s_2)$ captures her beliefs over the number of red balls in urn 2. Because urn 1 is unambiguous, $\mu_1$ places probability one on $s_1 = 50$. Let $h_i$ denote the horse lottery corresponding to a bet on urn $i$. Using Eq. 9 one can compute

$$W(h_1) = \int_0^{100} w\left(\frac{s_1}{100}\right) d\mu_1(s_1) = w\left(\frac{50}{100}\right)$$

and

$$W(h_2) = \int_0^{100} w\left(\frac{s_2}{100}\right) d\mu_2(s_2).$$

[11] Alternatively, one could specify a single, joint distrubution over the two state variables. Since they would be statistically independent, though, they can be treated separately.

In the Ellsberg example individuals tend to choose $h_1$ over $h_2$, and Jensen's inequality implies that $W(h_1) \geq W(h_2)$ if the second-order utility function $w$ is concave and the mean of $\mu_2$ is 50. So, for example, the Ellsberg pattern is consistent with subjects having uniform subjective probability distributions over [0, 100] for the second urn.

Concave $w$ can also explain the three-color Ellsberg paradox. In this paradox a single urn contains 90 balls, 30 of which are red and the remaining 60 an unknown mixture of yellow and black. When individuals are given a chance to win \$10 if they draw a red ball or \$10 if they draw a yellow ball, they tend to bet on a red ball. When individuals are given a chance to win \$10 if they draw either a red or black ball, or \$10 if they draw either a yellow or black ball, they tend to bet on the yellow and black combination. In a subjective expected utility context the first choice suggests that fewer than 30 of the balls are yellow, and the second choice suggests that fewer than 30 of the balls are black; hence the paradox.

Because there is only a single urn we can let $s$ define the state according to the number of yellow balls in that urn, in which case $60 - s$ is the number of black balls. Normalize $u$ in the same way as above. The constant horse lottery $h_R$ is a bet on drawing a red ball, the horse lottery $h_Y$ is a bet on drawing a yellow ball, the horse lottery $h_{RB}$ is a bet on drawing a red or black ball, and the constant horse lottery $h_{YB}$ is a bet on drawing a yellow or black ball. The Ellsberg preferences have $h_R \succ h_Y$ but $h_{YB} \succ h_{RB}$. One can compute

$$W(h_R) = \int_0^{60} w\left(\frac{30}{90}\right) d\mu(s) = w\left(\frac{30}{90}\right)$$

and

$$W(h_Y) = \int_0^{60} w\left(\frac{s}{90}\right) d\mu(s).$$

By Jensen's inequality the individual prefers the constant horse lottery $h_R$ if the second-order utility function $w$ is concave and the mean of the subjective probability distribution is 30. Similarly,

$$W(h_{YB}) = \int_0^{60} w\left(\frac{60}{90}\right) d\mu(s) = w\left(\frac{60}{90}\right)$$

and

$$W(h_{RB}) = \int_0^{60} w\left(\frac{90 - s}{90}\right) d\mu(s).$$

By Jensen's inequality she prefers the constant horse lottery $h_{YB}$ to the nonconstant one $h_{RB}$ if $w$ is concave and the mean of $\mu$ is 30. The same conditions predict exactly the three-color Ellsberg pattern.

Finally, consider an Ellsberg-like situation in which there are three urns, all of which contain a mixture of 100 red and yellow balls. Urn 1 is unambiguous, containing 50 red and 50 yellow balls. Urn 2 contains an unknown mixture of 100 balls, but with at least 30 red and at least 30 yellow balls. Urn 3 is completely ambiguous, containing a completely unknown mixture of 100 red and yellow balls. An individual can win \$10 for drawing a red ball from one of the urns. Intuition from the two-color Ellsberg paradox suggests that the individual would prefer urn 1 to urn 2 to urn 3. Because there are three urns there are three different state variables, and let $s_i$ denote the state variable corresponding to the number of balls in urn $i$, let $\mu_i$ denote the subjective probability distribution for state variable $s_i$, and let $h_i$ denote the horse lottery corresponding to a bet on urn $i$.

$$W(h_1) = \int_0^{100} w\left(\frac{s_1}{100}\right) d\mu_1(s_1) = w\left(\frac{50}{100}\right),$$

$$W(h_2) = \int_{30}^{70} w\left(\frac{s_2}{100}\right) d\mu_2(s_2),$$

$$W(h_3) = \int_0^{100} w\left(\frac{s_3}{100}\right) d\mu_3(s_3).$$

If the second-order utility function $w$ is concave and if $\mu_3$ is a mean-preserving spread of $\mu_2$, which in turn has the same mean as $\mu_1$, the intuitive pattern holds.

## 3 Conclusion

This paper applies old axioms from Savage (1954) and the von Neumann and Morgenstern axioms of Grandmont (1972) to an old choice framework developed by Anscombe and Aumann (1963) in which states of the world correspond to objective risks. The axioms lead to second-order expected utility preferences which consist of a subjective probability measure over states of the world, a utility function governing risk attitudes, and another utility function governing ambiguity attitudes. Concavity of the first utility function implies risk aversion, and concavity of the second is consistent with ambiguity aversion in the Ellsberg paradoxes.

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