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Author(s): David Schmeidler

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SUBJECTIVE PROBABILITY AND EXPECTED UTILITY WITHOUT ADDITIVITY

BY DAVID SCHMEIDLER¹

An act maps states of nature to outcomes; deterministic outcomes as well as random outcomes are included. Two acts f and g are comonotonic, by definition, if it never happens that $f(s) > f(t)$ and $g(t) > g(s)$ for some states of nature s and t . An axiom of comonotonic independence is introduced here. It weakens the von Neumann-Morgenstern axiom of independence as follows: If $f > g$ and if f , g , and h are comonotonic, then $\alpha f + (1 - \alpha)h > \alpha g + (1 - \alpha)h$.

If a nondegenerate, continuous, and monotonic (state independent) weak order over acts satisfies comonotonic independence, then it induces a unique non-(necessarily-)additive probability and a von Neumann-Morgenstern utility. Furthermore, one can compute the expected utility of an act with respect to the nonadditive probability, using the Choquet integral.

This extension of the expected utility theory covers situations, as the Ellsberg paradox, which are inconsistent with additive expected utility. The concept of *uncertainty aversion* and interpretation of comonotonic independence in the context of social welfare functions are included.

KEYWORDS: Comonotonic independence, uncertainty aversion, expected utility, subjective probability.

1. INTRODUCTION

BAYESIAN STATISTICAL TECHNIQUES are applicable when the information and uncertainty with respect to the parameters or hypotheses in question can be expressed by a probability distribution. This prior probability is also the focus of most of the criticism against the Bayesian school. My starting point is to join the critics in attacking a certain aspect of the prior probability: The probability attached to an uncertain event does not reflect the heuristic amount of information that led to the assignment of that probability. For example, when the information on the occurrence of two events is symmetric they are assigned equal prior probabilities. If the events are complementary the probabilities will be $1/2$, independently of whether the symmetric information is meager or abundant.

There are two (unwritten?) rules for assigning prior probabilities to events in case of uncertainty. The first says that symmetric information with respect to the occurrence of events results in equal probabilities. The second says that if the space is partitioned into k symmetric (i.e., equiprobable) events, then the probability of each event is $1/k$. I agree with the first rule and object to the second. In the example above, if each of the symmetric and complementary

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uncertain events is assigned the index $3/7$, the number $1/7$, $1/7 = 1 - (3/7 + 3/7)$, would indicate the decision maker's confidence in the probability assessment. Thus, allowing nonadditive (not necessarily additive) probabilities enables transmission or recording of information that additive probabilities cannot represent.

The idea of nonadditive probabilities is not new. Nonadditive (objective) probabilities have been in use in physics for a long time (Feynman (1963)). The nonadditivity describes the deviation of elementary particles from mechanical behavior toward wave-like behavior. Daniel Ellsberg (1961) presented his arguments against necessarily additive (subjective) probabilities with the help of the following "mind experiments": There are two urns each containing one hundred balls. Each ball is either red or black. In urn *I* there are fifty balls of each color and there is no additional information about urn *II*. One ball is chosen at random from each urn. There are four events, denoted *IR*, *IB*, *IIR*, *IIB*, where *IR* denotes the event that the ball chosen from urn *I* is red, etc. On each of the events a bet is offered: \$100 if the event occurs and zero if it does not. According to Ellsberg most decision makers are indifferent between betting on *IR* and betting on *IB* and are similarly indifferent between bets on *IIR* and *IIB*. It may be that the majority are indifferent among all four bets. However, there is a nonnegligible proportion of decision makers who prefer every bet from urn *I* (*IB* or *IR*) to every bet from urn *II* (*IIB* or *IIR*). These decision makers cannot represent their beliefs with respect to the occurrence of uncertain events through an additive probability.

The most compelling justification for representation of beliefs about uncertain events through additive prior probability has been suggested by Savage. Building on previous work by Ramsey, de Finetti, and von Neumann-Morgenstern, Savage suggested axioms for decision theory that lead to the criterion of maximization of expected utility. The expectation operation is carried out with respect to a prior probability derived uniquely from the decision maker's preferences over acts. The axiom violated by the preference of the select minority in the example above is the "sure thing principle", i.e., Savage's P2.

In this paper a simplified version of Savage's model is used. The simplification consists of the introduction of objective or physical probabilities. An act in this model assigns to each state an objective lottery over deterministic outcomes. The uncertainty concerns which state will occur. Such a model containing objective and subjective probabilities has been suggested by Anscombe and Aumann (1963). They speak about roulette lotteries (objective) and horse lotteries (subjective). In the presentation here the version in Fishburn (1970) is used. The von Neumann-Morgenstern, (N-M), utility theorem used here can also be found in Fishburn (1970).

The concept of objective probability is considered here as a physical concept like acceleration, momentum, or temperature; to construct a lottery with given objective probabilities (a roulette lottery) is a technical problem conceptually not different from building a thermometer. When a person has constructed a "perfect" die, he assigns a probability of $1/6$ to each outcome. This probability is

objective in the same sense as the temperature measured by the thermometer. Another person can check and verify the calibration of the thermometer. Similarly, he can verify the perfection of the die by measuring its dimensions, scanning it to verify uniform density, etc.... Rolling the die many times is not necessarily the exclusive test for verification of objective probability.

On the other hand, the subjective or personal probability of an event is interpreted here as the number used in calculating the expectation (integral) of a random variable. This definition includes objective or physical probabilities as a special case where there is no doubt as to which number is to be used. This interpretation does not impose any restriction of additivity on probabilities, as long as it is possible to perform the expectation operation which is the subject of this work.

Subjective probability is derived from a person's preferences over acts. In the Anscombe-Aumann type model usually five assumptions are imposed on preferences to define unique additive subjective probability and von Neumann-Morgenstern utility over outcomes. The first three assumptions are essentially von Neumann-Morgenstern's—weak order, independence, and continuity—and the fourth assumption is equivalent to Savage's P3, i.e., state-independence of preferences. The additional assumption is nondegeneracy; without it uniqueness is not guaranteed.

The example quoted earlier can be embedded in such a model. There are four states: (IB, IIB) , (IB, IIR) , (IR, IIB) , (IR, IIR) . The deterministic outcomes are sums of dollars. For concreteness of the example, assume that there are 101 deterministic outcomes: $\$0, \$1, \$2, \dots, \100 . An act assigns to each state a probability distribution over the outcomes. The bet " $\$100$ if IIB " is an act which assigns the (degenerate objective) lottery of receiving " $\$100$ with probability one" to each state in the event IIB and "zero dollars with probability one" to each state in the event IIR . The bet on IIR is similarly interpreted. Indifference between these two acts (bets), the independence condition, continuity, and weak order imply indifference between either of them and the constant act which assigns to each state the objective lottery of receiving $\$100$ with probability $1/2$ and receiving zero dollars with probability $1/2$. The same considerations imply that the constant act above is indifferent to either of the two acts (bets): " $\$100$ if IB " and " $\$100$ if IR ". Hence the indifference between IB and IR and the indifference between IIB and IIR in Ellsberg's example, together with the von Neumann-Morgenstern conditions, imply indifference between all four bets. The nonnegligible minority of Ellsberg's example does not share this indifference: they are indifferent between the constant act (as above) and each bet from urn I , and prefer the constant act to each bet from urn II .

Our first objective consists of restatement, or more specifically of weakening, of the independence condition such that the new assumption together with the other three assumptions can be consistently imposed on the preference relation over acts. In particular the special preferences of the example become admissible. It is obvious that the example's preferences between bets (acts) do not admit additive subjective probability. Do they define in some consistent way a unique nonaddi-

tive subjective probability, and if so, is there a way to define the expected utility maximization criterion for the nonadditive case?

An affirmative answer to this problem is presented in the third section. Thus the new model rationalizes nonadditive (personal) probabilities and admits the computation of expected utility with respect to these probabilities. It formally extends the additive model and it makes the expected utility criterion applicable to cases where additive expected utility is not applicable.

Before turning to a precise and detailed presentation of the model, another heuristic observation is made. The nomenclature used in economics distinguishes between risk and uncertainty. Decisions in a risk situation are precisely the choices among roulette lotteries. The probabilities are objectively given; they are part of the data. For this case the economic theory went beyond von Neumann-Morgenstern utility and defined concepts of risk aversion, risk premium, and certainty equivalence. Translating these concepts to the case of decisions under uncertainty we can speak about uncertainty aversion, uncertainty premium, and risk equivalence. Returning to the example, suppose that betting \$100 on *IIR* is indifferent to betting \$100 on a risky event with an (objective) probability of $3/7$. Thus, the subjective probability of an event is its risk equivalent ($P(IIR) = 3/7$). In this example the number $1/7$ computed earlier expresses the uncertainty premium in terms of risk. Note that nonadditive probability may not exhibit consistently either uncertainty aversion or uncertainty attraction. This is similar to the case of decisions in risk situations where von Neumann-Morgenstern utility (of money) may be neither concave nor convex.

2. AXIOMS AND BACKGROUND

Let X be a set and Y be the set of distributions over X with finite supports

$$Y = \left\{ y: X \rightarrow [0,1] \mid y(x) \neq 0 \text{ for finitely many } x \text{'s in } X \right. \\ \left. \text{and } \sum_{x \in X} y(x) = 1 \right\}.$$

For notational simplicity we identify X with the subset $\{y \in Y \mid y(x) = 1 \text{ for some } x \text{ in } X\}$ of Y .

Let S be a set and let Σ be an algebra of subsets of S . Both sets, X and S are assumed to be nonempty. Denote by L_0 the set of all Σ -measurable finite valued functions from S to Y and denote by L_c the constant functions in L_0 . Let L be a convex subset of Y^S which includes L_c . Note that Y can be considered a subset of some linear space, and Y^S , in turn, can then be considered as a subspace of the linear space of all functions from S to the first linear space. Whereas it is obvious how to perform convex combinations in Y it should be stressed that convex combinations in Y^S are performed pointwise. I.e., for f and g in Y^S and α in $[0,1]$, $\alpha f + (1 - \alpha)g = h$ where $h(s) = \alpha f(s) + (1 - \alpha)g(s)$ on S .

In the neo-Bayesian nomenclature, elements of X are (deterministic) *outcomes*, elements of Y are random outcomes or (roulette) *lotteries*, and elements of L are

acts (or horse lotteries). Elements of S are *states* (of nature) and elements of Σ are events.

The primitive of a neo-Bayesian decision model is a binary (preference) relation over L to be denoted by \succeq . Next are stated several properties (axioms) of the preference relation, which will be used in the sequel.

(i) **WEAK ORDER:** (a) *For all f and g in L : $f \succeq g$ or $g \succeq f$.* (b) *For all f , g , and h in L : If $f \succeq g$ and $g \succeq h$, then $f \succeq h$.*

The relation \succeq on L induces a relation also denoted by \succeq on Y : $y \succeq z$ iff $y^S \succeq z^S$ where y^S denotes the constant function y on S (i.e., $\{y\}^S$). As usual, \succ and \sim denote the asymmetric and symmetric parts, respectively, of \succeq .

DEFINITION: Two acts f and g in Y^S are said to be *comonotonic* if for no s and t in S , $f(s) \succ f(t)$ and $g(t) \succ g(s)$.

A constant act f , i.e., $f = y^S$ for some y in Y , and any act g are comonotonic. An act f whose statewise lotteries $\{f(s)\}$ are mutually indifferent, i.e., $f(s) \sim y$ for all s in S , and any act g are comonotonic. If X is a set of numbers and preferences respect the usual order on numbers, then any two X -valued functions f and g are comonotonic iff $(f(s) - f(t))(g(s) - g(t)) \geq 0$ for all s and t in S .

Clearly, *IIR* and *IIB* of the Introduction are *not* comonotonic. (Comonotonicity stands for common monotonicity.)

Next our new axiom for neo-Bayesian decision theory is introduced.

(ii) **COMONOTONIC INDEPENDENCE:** *For all pairwise comonotonic acts f , g and h in L and for all α in $]0, 1[$: $f \succ g$ implies $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$. ($]0, 1[$ is the open unit interval.)*

Elaboration of this condition is delayed until after condition (vii).

Comonotonic independence is clearly a less restrictive condition than the independence condition stated below.

(iii) **INDEPENDENCE:** *For all f , g and h in L and for all α in $]0, 1[$: $f \succ g$ implies $\alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h$.*

(iv) **CONTINUITY:** *For all f , g and h in L : If $f \succ g$ and $g \succ h$, then there are α and β in $]0, 1[$ such that $\alpha f + (1 - \alpha)h \succ g$ and $g \succ \beta f + (1 - \beta)h$.*

Next, two versions of state-independence are introduced. The intuitive meaning of each of these conditions is that the preferences over random outcomes do not depend on the state that occurred. The first version is the one to be used here. The second version is stated for comparisons since it is the common one in the literature.

(v) MONOTONICITY: *For all f and g in L : If $f(s) \succeq g(s)$ on S then $f \succeq g$.*

(vi) STRICT MONOTONICITY: *For all f and g in L , y and z in Y and E in Σ : If $f \succ g$, $f(s) = y$ on E and $g(s) = z$ on E , and $f(s) = g(s)$ on E^c , then $y \succ z$.*

OBSERVATION: If $L = L_0$, then (vi) and (i) imply (v).

PROOF: Let f and g be finite step functions such that $f(s) \succeq g(s)$ on s . There is a finite chain $f = h_0, h_1, \dots, h_k = g$ where each pair of consecutive functions h_{i-1}, h_i are constant on the set on which they differ. For this pair (vi) and (i) imply (v). Transitivity (i)(b) of \succeq concludes the proof.

Clearly (i) and (v) imply (vi).

For the sake of completeness we list as axiom:

(vii) NONDEGENERACY: *Not for all f and g in L , $f \succeq g$.*

Out of the seven axioms listed here the completeness of the preferences, (i)(a), seems to me the most restrictive and most imposing assumption of the theory. One can view the weakening of the completeness assumption as a main contribution of all other axioms. Imagine a decision maker who initially has a partial preference relation over acts. After an additional introspection she accepts the validity of several of the axioms. She can then extend her preferences using these axioms. For example, if she ranks $f \succ g$ and $g \succ h$, and if she accepts transitivity, then she concludes that $f \succ h$. From this point of view, the independence axiom, (iii), seems the most powerful axiom for extending partial preferences. Given $f \succ g$ and independence we get for all h in L and α in $]0, 1[$: $f' \equiv \alpha f + (1 - \alpha)h \succ \alpha g + (1 - \alpha)h \equiv g'$. However after additional retrospection this implication may be too powerful to be acceptable. For example, consider the case where outcomes are real numbers and $S = [0, 2\pi]$. Let f and g be two acts defined: $f(s) = \sin(s)$ and $g(s) = \sin(s + \pi/2) = \cos(s)$. The preferences $f \succ g$ may be induced by the rough evaluation that the event $[\pi/3, 4\pi/3]$ is more probable than its complement. Define the act h by $h(s) = \sin(77s)$. In this case the structure of the acts $f' = \frac{1}{2}f + \frac{1}{2}h$ and $g' = \frac{1}{2}g + \frac{1}{2}h$ is far from transparent and the automatic implication of independence, $f' \succ g'$, may seem doubtful to the decision maker. More generally: the ranking $f \succ g$ implies some rough estimation by the decision maker of the probabilities of events (in the algebra) defined by the acts f and g . If mixture with an arbitrary act h is allowed, the resulting acts f' and g' may define a much finer (larger) algebra (especially when the algebra defined by h is qualitatively independent of the algebras of f and g). Careful retrospection and comparison of the acts f' and g' may lead them to the ranking $g' \succ f'$ (as in the case of the Ellsberg paradox) contradictory to the implication of the independence axiom. Qualifying the comparisons and the application of independence to comonotonic acts rules out the possibility of contradiction. If f , g , and h are pairwise comonotonic, then the comparison of f to g is not very

different from the comparison of f' to g' . Hence the decision maker can accept the validity of the implication: $f \succ g \Leftrightarrow f' \succ g'$, without fear of running into a contradiction. Note that accepting the validity of comonotonic independence, (ii), means accepting the validity of the above implication without knowing the specific acts f, g, h, f', g' , but knowing that all five are pairwise comonotonic.

Before presenting the von Neumann-Morgenstern theorem we point out that stating the axioms of (i) weak order, (iii) independence, and (iv) continuity do not require that the preference relation \succeq be defined on a set L containing L_c . Only the convexity of L is required for (ii) and (iii).

VON NEUMANN-MORGENSTERN THEOREM: *Let M be a convex subset of some linear space, with a binary relation \succeq defined on it. A necessary and sufficient condition for the relation \succeq to satisfy (i) weak order, (iii) independence, and (iv) continuity is the existence of an affine real valued function, say w , on M such that for all f and g in M : $f \succeq g$ iff $w(f) \geq w(g)$. (Affinity of w means that $w(\alpha f + (1 - \alpha)g) = \alpha w(f) + (1 - \alpha)w(g)$ for $0 < \alpha < 1$.) Furthermore, an affine real valued function w' on M can replace w in the above statement iff there exist a positive number α and a real number β such that $w'(f) = \alpha w(f) + \beta$ on M .*

As mentioned earlier, for proof of this theorem and the statement and proof of Anscombe-Aumann Theorem below, the reader is referred to Fishburn (1970).

IMPLICATION: Suppose that a binary relation \succeq on some convex subset L of Y^S with $L_c \subset L$ satisfies (i) weak order, (ii) comonotonic independence, and (iv) continuity. Suppose also that there is a convex subset M of L with $L_c \subset M$ such that any two acts in M are comonotonic. Then by the von Neumann-Morgenstern Theorem there is an affine function on M , to be denoted by J , which represents the binary relation \succeq on M . I.e., for all f and g in M : $f \succeq g$ iff $J(f) \geq J(g)$. Clearly, if $M = L_c \equiv \{y^S | y \in Y\}$ any two acts in M are comonotonic. Hence, if a function u is defined on Y by $u(y) = J(y^S)$, then u is affine and represents the induced preferences on Y . The affinity of u implies $u(y) = \sum_{x \in X} y(x)u(x)$.

When subjective probability enters into the calculation of expected utility of an act, an integral with respect to a finitely additive set function has to be defined. Denote by P a finitely additive probability measure on Σ and let a be a real valued Σ -measurable function on S . For the special case where a is a finite step function, a can be uniquely represented by $\sum_{i=1}^k \alpha_i E_i^*$ where $\alpha_1 > \alpha_2 > \dots > \alpha_k$ are the values that a attains and E_i^* is the indicator function on S of $E_i \equiv \{s \in S | a(s) = \alpha_i\}$ for $i = 1, \dots, k$. Then

$$\int_S a dP = \sum_{i=1}^k P(E_i) \alpha_i.$$

The more general case where a is not finitely valued is treated as a special case of nonadditive probability.

ANSCOMBE-AUMANN THEOREM: *Suppose that a preference relation \succeq on $L = L_0$ satisfies (i) weak order, (iii) independence, (iv) continuity, (vi) strict monotonicity, and (vii) nondegeneracy. Then there exist a unique finitely additive probability measure P on Σ and an affine real valued function u on Y such that for all f and g in L_0 :*

$$f \succeq g \quad \text{iff} \quad \int_S u(f(\cdot)) dP \geq \int_S u(g(\cdot)) dP.$$

Furthermore, if there exist P and u as above, then the preference relation they induce on L_0 satisfied conditions (i), (iii), (iv), (vi), and (vii). Finally, the function u is unique up to a positive linear transformation.

There are three apparent differences between the statement of the main result in the next section and the Anscombe-Aumann Theorem above: (i) Instead of strict monotonicity, monotonicity is used. It has been shown in the Observation that it does not make a difference. However, for the forthcoming extension, monotonicity is the natural condition. (ii) Independence is replaced with comonotonic independence. (iii) The finitely additive probability measure P is replaced with a nonadditive probability v .

3. THEOREM

A real valued set function v on Σ is termed *nonadditive probability* if it satisfies the normalization conditions $v(\phi) = 0$ and $v(S) = 1$, and monotonicity, i.e., for all E and G in Σ : $E \subset G$ implies $v(E) \leq v(G)$. We now introduce the definition of $\int_S a dv$ for v nonadditive probability and $a = \sum_{i=1}^k \alpha_i E_i^*$ a finite step function with $\alpha_1 > \alpha_2 > \dots > \alpha_k$ and $(E_i)_{i=1}^k$ a partition of S . Let $\alpha_{k+1} = 0$ and define

$$\int_S a dv = \sum_{i=1}^k (\alpha_i - \alpha_{i+1}) v \left(\bigcup_{j=1}^i E_j \right).$$

For the special case of v additive the definition above coincides with the usual one mentioned in the previous section.

THEOREM: *Suppose that the preference relation \succeq on $L = L_0$ satisfies (i) weak order, (ii) comonotonic independence, (iv) continuity, (v) monotonicity, and (vii) nondegeneracy. Then there exist a unique nonadditive probability v on Σ and an affine real valued function u on Y such that for all f and g in L_0 :*

$$f \succeq g \quad \text{iff} \quad \int_S u(f(\cdot)) dv \geq \int_S u(g(\cdot)) dv.$$

Conversely, if there exist v and u as above, u nonconstant, then the preference

relation they induce on L_0 satisfies (i), (ii), (iv), (v), and (vii). Finally, the function u is unique up to positive linear transformations.

PROOF: From the Implication of the N-M Theorem we get a N-M utility u representing the preference relation \succeq induces on Y . By nondegeneracy there are f^* and f_* in L_0 with $f^* \succ f_*$. Monotonicity, (v), implies existence of a state s in S such that $f^*(s) \equiv y^* \succ f_*(s) \equiv y_*$. Since u is given up to a positive linear transformation, suppose from now on $u(y^*) = 1$ and $u(y_*) = -1$. Denote $K = u(Y)$. Hence K is a convex subset of the real line including the interval $[1, -1]$.

For an arbitrary f in L_0 denote

$$M_f = \{ \alpha f + (1 - \alpha) y^S | y \in Y \text{ and } \alpha \in [0, 1] \}.$$

Thus M_f is the convex hull of the union of f and L_c . It is easy to see that any two acts in M_f are comonotonic. Hence, there is an affine real-valued function on M_f , which represents the preference relation \succeq restricted to M_f . After rescaling, this function, J_f satisfies $J_f(y^{*S}) = 1$ and $J_f(y_*^S) = -1$. Clearly, if $h \in M_f \cap M_g$, then $J_f(h) = J_g(h)$. So, defining $J(f) = J_f(f)$ for f in L_0 , we get a real valued function on L_0 which represents the preferences \succeq on L_0 and satisfies for all y in Y : $J(y^S) = u(y)$. Let $B_0(K)$ denote the Σ -measurable, K -valued finite step function on S . Let $U: L_0 \rightarrow B_0(K)$ be defined by $U(f)(s) = u(f(s))$ for s in S and f in L_0 . The function U is onto, and if $U(f) = U(g)$, then by monotonicity $f \sim g$, which in turn implies $J(f) = J(g)$.

We now define a real valued function I on $B_0(K)$. Given a in $B_0(K)$, let f in L_0 be such that $U(f) = a$. Then define $I(a) = J(f)$. I is well defined since as mentioned earlier J is constant on $U^{-1}(a)$:

$$\begin{array}{ccc} L_0 & \xrightarrow{U} & B_0 \\ & \searrow J \quad \swarrow I & \\ & R & \end{array}$$

We now have a real valued function I on $B_0(K)$ which satisfies the following three conditions: (i) For all α in K : $I(\alpha S^*) = \alpha$. (ii) For all pairwise comonotonic functions a , b , and c in $B_0(K)$ and α in $[0, 1]$: if $I(a) > I(b)$ then $I(\alpha a + (1 - \alpha)c) > I(\alpha b + (1 - \alpha)c)$. (iii) If $a(s) \geq b(s)$ on S for a and b in $B_0(K)$, then $I(a) \geq I(b)$.

To see that (i) is satisfied, let y in Y be such that $u(y) = \alpha$. Then $J(y^S) = \alpha$ and $U(y^S) = \alpha S^*$. Hence $I(\alpha S^*) = \alpha$. Similarly (ii) is satisfied because comonotonicity is preserved by U and J represents \succeq which satisfies comonotonic independence. Finally (iii) holds because U preserves monotonicity.

The Corollary of Section 3 and the Remark following it in Schmeidler (1986) say that if a real valued function I on $B_0(K)$ satisfies conditions (i), (ii), and (iii), then the nonadditive probability v on Σ defined by $v(E) = I(E^*)$ satisfies for all

a and b in $B_0(K)$:

$$(*) \quad I(a) \geq I(b) \quad \text{iff} \quad \int_S a \, dv \geq \int_S b \, dv.$$

Hence, for all f and g in L_0 :

$$f \succeq g \quad \text{iff} \quad \int_S U(f) \, dv \geq \int_S U(g) \, dv,$$

and the proof of the main part of the theorem is completed.

To prove the opposite direction note first that in Schmeidler (1986) it is shown and referenced that if I on $B_0(K)$ is defined by (*), then it satisfies conditions (i), (ii), and (iii). (Only (ii) requires some proof.) Secondly, the assumptions of the opposite direction say that J is defined as a combination of U and I in the diagram. Hence the preference relation on L_0 induced by J satisfies all the required conditions. (U preserves monotonicity and comonotonicity and $\int_S a \, dv$ is a (sup) norm continuous function of a .)

Finally, uniqueness properties of the expected utility representation will be proved. Suppose that there exist an affine real valued function u' on Y and a nonadditive probability v' on Σ such that for all f and g in L_0 :

$$(**) \quad f \succeq g \quad \text{iff} \quad \int_S u'(f(s)) \, dv' \geq \int_S u'(g(s)) \, dv'.$$

Note that monotonicity of v' can be derived instead of assumed. When considering (**) for all f and g in L_c we immediately obtain, from the uniqueness part of the N-M Theorem, that u' is a positive linear transformation of u . On the other hand it is obvious that the inequality in (**) is preserved under positive linear transformations of the utility. Hence, in order to prove that $v' = v$ we may assume without loss of generality that $u' = u$. For an arbitrary E in Σ let f in L_0 be such that $U(f) = E^*$. (For example, $f(s) = y^*$ on E and $f(s) = y^*/2 + y_*/2$ on E^c . Then $\int_S U(f) \, dv = v(E)$ and $\int_S U(f) \, dv' = v'(E)$.) Let y in Y be such that $u(y) = v(E)$. (For example, $y = v(E)y^* + (1 - v(E))(y^*/2 + y_*/2)$.) Then $f \sim y^S$ which in turn implies $u(y) = u'(y) = \int_S u'(y^S) \, dv' = v'(E)$. The last equality is implied by (**). Q.E.D.

In order to extend the Theorem to more general acts, we have to specify precisely the set of acts L on which the extension holds and we have to extend correspondingly the definition of the integral with respect to nonadditive probability. We start with the latter.

Denote by B the set of real valued, bounded Σ -measurable functions on S . Given a in B and a nonadditive probability v on Σ we define

$$\int_S a \, dv = \int_{-\infty}^0 (v(a \geq \alpha) - 1) \, d\alpha + \int_0^{\infty} v(a \geq \alpha) \, d\alpha.$$

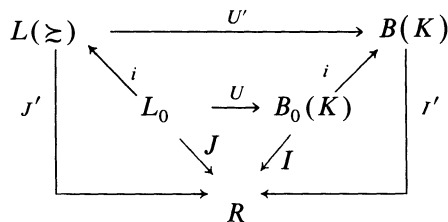
Each of the integrands above is monotonic, bounded and identically zero where $|\alpha| > \lambda$ for some number λ . This definition of integration for nonnegative

functions in B has been suggested by Choquet (1955). A more detailed exposition appears in Schmeidler (1986). It should be mentioned here that this definition coincides, of course, with the one at the beginning of this section when a obtains finitely many values.

For the next definition, existence of weak order \succeq over L_c is presupposed. An act $f: S \rightarrow Y$ is said to be Σ -measurable if for all y in Y the sets $\{s|f(s) > y\}$ and $\{s|f(s) \succeq y\}$ belong to Σ . It is said to be bounded if there are y and z in Y such that $y \succeq f(s) \succeq z$ on S . The set of all Σ -measurable bounded acts in Y^S is denoted by $L(\succeq)$. Clearly, it contains L_0 .

COROLLARY: (a) Suppose that a preference relation \succeq over L_0 satisfies (i) weak order, (ii) comonotonic independence, (iv) continuity, and (v) monotonicity. Then it has a unique extension to all of $L(\succeq)$ which satisfies the same conditions (over $L(\succeq)$). (b) If the extended relation, also to be denoted by \succeq , is nondegenerate, then there exist a unique nonadditive probability v on Σ and an affine real valued function u (unique up to positive linear transformations) such that for all f and g in $L(\succeq)$: $f \succeq g$ iff $\int_S u(f(\cdot)) dv \geq \int_S u(g(\cdot)) dv$.

PROOF: The case of degeneracy is obvious, so assume nondegenerate preferences. Consider the following diagram:



The inner triangle is that of the proof of the Theorem. $B(K)$ is the set of K -valued, Σ -measurable, bounded functions on S , and i denotes identity. U' is the natural extension of U and is also onto. Because $B_0(K)$ is (sup) norm dense in $B(K)$ and I satisfies condition (iii), I' is the unique extension of I that satisfies on $B(K)$ the three conditions that I satisfies on $B_0(K)$.

The functional J' , defined on $L(\succeq)$ by: $J'(f) = I'(U'(f))$, extends J . Hence, the relation \succeq on $L(\succeq)$ defined by $f \succeq g$ iff $J'(f) \geq J'(g)$ extends the relation \succeq on L_0 , and satisfies the desired properties.

By the Corollary of Section 3 in Schmeidler (1986) there exists a nonadditive probability v on Σ such that for all f and g in $L(\succeq)$: $I'(f) \geq I'(g)$ iff $\int_S U'(f) dv \geq \int_S U'(g) dv$.

Hence, the expected utility representation of the preference relation has been shown. To complete the proof of (b), uniqueness of v and uniqueness up to a positive linear transformation of u have to be established. However, it follows from the corresponding part of the Theorem. The uniqueness properties also imply that the extension of \succeq from L_0 to $L(\succeq)$ is unique. Q.E.D.

REMARK 1: Instead of first stating the Theorem for L_0 and then extending it to $L(\succeq)$, one can state directly the extended theorem. More precisely a preference relation on L , $L_0 \subset L \subset Y^S$ is defined such that in addition to the conditions (i), (ii), (iv), and (vii) it satisfies $L = L(\succeq)$. It can then be represented by expected utility with respect to nonadditive probability. However, the first part of the Corollary shows that in this case the preference relation of $L(\succeq)$ is overspecified: The preferences of L_0 dictate those over $L(\succeq)$.

REMARK 2: If Σ does not contain all subsets of S , and $\#X \geq 3$ then $L(\succeq)$ contains finite step functions that do not belong to L_0 . Let y and z in Y be such that $y \sim z$ but $y \neq z$, and let $E \subset S$ but $E \notin \Sigma$. Define $f(s) = y$ on E and $f(s) = z$ on E^C . Clearly $f \notin L_0$. The condition $\#X \geq 3$ is required to guarantee existence of y and z as above.

REMARK 3: It is an elementary exercise to show that under the conditions of the Theorem, v is additive iff \succeq satisfies (iii) independence (instead of or in addition to (ii) comonotonic independence). Also an extension of an independent relation, as in Corollary (a), is independent. Hence our results formally extend the additive theory.

We now introduce formally the concept of uncertainty aversion alluded to in the Introduction. A binary relation \succeq on L is said to reveal *uncertainty aversion* if for any three acts f , g , and h in L and any α in $[0, 1]$: If $f \succeq h$ and $g \succeq h$, then $\alpha f + (1 - \alpha)g \succeq h$. Equivalently we may state: If $f \succeq g$, then $\alpha f + (1 - \alpha)g \succeq g$. For definition of *strict* uncertainty aversion the conclusion should be a strict preference $>$. However, some restrictions then have to be imposed on f and g . One such obvious restriction is that f and g are not comonotonic. We will return to this question in a subsequent remark.

Intuitively, uncertainty aversion means that "smoothing" or averaging *utility* distributions makes the decision maker better off. Another way is to say that substituting objective mixing for subjective mixing makes the decision maker better off. The definition of uncertainty aversion may become more transparent when its full mathematical characterization is presented.

PROPOSITION: Suppose that \succeq on $L = L(\succeq)$ is the extension of \succeq on L_0 according to the Corollary. Let v be the derived nonadditive subjective probability and I (the I' of the Corollary) be the functional on B , $I(a) = \int_S a dv$. Then the following conditions are equivalent: (i) \succeq reveals uncertainty aversion. (ii) For all a and b in B : $I(a + b) \geq I(a) + I(b)$. (iii) For all a and b in B and for all α in $[0, 1]$:

$$I(\alpha a + (1 - \alpha)b) \geq \alpha I(a) + (1 - \alpha)b.$$

(iv) For all a and b in B and for all α in $[0, 1]$:

$$I(\alpha a + (1 - \alpha)b) \geq \min \{ I(a), I(b) \}.$$

(v) For all α in R the sets $\{a \in B | I(a) \geq \alpha\}$ are convex. (vi) There exists an $\bar{\alpha}$ in

R s.t. the set $\{a \in B | I(a) \geq \bar{\alpha}\}$ is convex. (vii) For all a and b in B and for all α in $[0, 1]$: If $I(a) = I(b)$, then $I(\alpha a + (1 - \alpha)b) \geq I(a)$. (viii) For all a and b in B : If $I(a) = I(b)$, then

$$I(a + b) \geq I(a) + I(b).$$

(ix) v is convex. I.e., for all E and F in Σ :

$$v(E) + v(F) \leq v(EF) + v(E + F).$$

(x) For all a in B : $I(a) = \min \{ \int_S a \, dp | p \in \text{core}(v) \}$, where $\text{core}(v) = \{ p: \Sigma \rightarrow R | p \text{ is additive, } p(S) = v(S) \text{ and for all } E \text{ in } \Sigma, p(E) \geq v(E) \}$.

PROOF: For any functional on B : (iii) implies (iv), (iv) implies (vii), (iv) is equivalent to (v), and (v) implies (vi). The positive homogeneity of degree one of I results in: (ii) equivalent to (iii) and (vii) equivalent to (viii). (vi) implies (v) because for all β in R , ($\beta = \alpha - \bar{\alpha}$), $I(a + \beta S^*) = I(a) + \beta$, and because adding βS^* preserves convexity.

(viii) implies (ix). Suppose, without loss of generality, that $v(E) \geq v(F)$. Then there is $\gamma \geq 1$ such that $v(E) = \gamma v(F)$. Since $I(E^*) = v(E) = \gamma v(F) = I(\gamma F^*)$, we have by (viii), $v(E) + \gamma v(F) \leq I(E^* + \gamma F^*)$. But $E^* + \gamma F^* = (EF)^* + (\gamma - 1)F^* + (E + F)^*$, which implies $I(E^* + \gamma F^*) = v(EF) + (\gamma - 1)v(F) + v(E + F)$. Inserting the last equality in the inequality above leads to the inequality in (ix). The equivalence of (ix), (x), and (ii) is stated as Proposition 3 in Schmeidler (1986).

Last but not least, (i) is equivalent to (iv). This becomes obvious after considering the mapping U' from the diagram in the proof of the Corollary. *Q.E.D.*

The basic result of the proposition is the equivalence of (i), (iii), (iv), (ix), and (x). (iv) is quasiconcavity of I and it is the translation of (i) by U' from L to B . (iii) is concavity, which usually is a stronger assumption. Here I is concave iff it is quasiconcave. Concavity captures best the heuristic meaning of uncertainty aversion.

REMARK 4: The Proposition holds if all the inequalities are strict and in (i) it is strict uncertainty aversion. To show it precisely, null or dummy events in Σ have to be defined. An event E in Σ is termed *dummy* if for all F in Σ : $v(F + E) = v(F)$. In (ii)–(vii), in order to state strict inequality one has to assume that a and b' are not comonotonic for any b' which differs from b on a dummy set. To have a strict inequality in (ix) one has to assume that $(E - F)^*$, $(EF)^*$, and $(F - E)^*$ are not dummies. In (x) a geometric condition on the core of v has to be assumed.

REMARK 5: The point of view of this work is that if the information is too vague to be represented by an additive prior, it still may be represented by a

nonadditive prior. Another possibility is to represent vague information by a set of priors. Condition (x) and its equivalence to other conditions of the Proposition point out when the two approaches coincide.

REMARK 6: The concept of uncertainty appeal can be defined by: $f \succeq g$ implies $f \succeq \alpha f + (1 - \alpha)g$. In the Proposition then *all* the inequalities have to be reversed and maxima have to replace minima. Obviously, additive probability or the independence axiom reveal uncertainty neutrality.

4. CONCLUDING REMARKS

4.1. In the introduction a point of view distinguishing between objective and subjective probabilities has been articulated. It is not necessary for the results of this work. What matters is that the lotteries in Y be constructed of additive probabilities. These probabilities can be subjectively arrived upon. This is the point of view of Anscombe and Aumann (1963). They describe their result as a way to assess complicated probabilities, “horse lotteries”, assuming that the probabilities used in the simpler “roulette lotteries” are already known. The Theorem here can also be interpreted in this way, and one can consider the lotteries in Y as derived within the behavioristic framework as follows:

Let Ω be a set (a roulette). An additive probability P on all subsets of Ω is derived via Savage’s Theorem. More specifically, let Z be a set of outcomes with two or more elements. (Suppose that the sets Z and X are disjoint.) Let F denote the set of Savage’s acts, i.e., all functions from Ω to Z . Postulating existence of a preference relation on F satisfying Savage’s axioms leads to an additive probability P on Ω . Next we identify a lottery, say y , in Y with all the acts from Ω to X which induce the probability distribution y . Thus we have a two step model within the framework of a behavioristic (or personal or subjective) theory of probability. Since the motivation of our Theorem is behavioristic (i.e., derivation of utility and probability from preference), the conceptual consistency of the work requires that the probabilities in Y could also be derived from preferences. We will return to the question of conceptual consistency in the next remark.

Instead of the two step model of the previous paragraph one can think of omitting the roulette lotteries from the model. One natural way to do this is to try to extend Savage’s Theorem to nonadditive probability. This has been done by Gilboa (1987). Another approach has been followed by Wakker (1986), wherein he substituted a connected topological space for the linear structure of Y .

4.2. In recent years many articles have been written which challenged the expected utility hypothesis in the von Neumann-Morgenstern model and in the model with state-dependent acts. We restrict our attention to models that (i) introduce functional representation of a preference relation derived from axioms, and (ii) separate “utilities” from “probabilities” (in the representation). Furthermore (iii) we consider functional representations which are sums of products of two numbers; one number has a “probability” interpretation and the other number has a “utility” interpretation. (For recent works disregarding restriction

(iii) the reader may consult Fishburn (1985) and the reference there.) Restriction (iii) is tantamount to the functional representation used in the Theorem (the Choquet integral). An article that preceded the present work in this kind of representation using nonadditive probability is Quiggin (1982). (Thanks for this reference are due to a referee.) His result will be introduced here somewhat indirectly.

4.2.1. Consider a preference relation over acts satisfying the assumptions, and hence the conclusions, of the theorem. Does there exist an *additive* probability P on Σ and a nondecreasing function f from the unit interval onto itself such that $v(E) = f(P(E))$ on Σ ? (Such a function f is referred to as a distortion function.) Conditions leading to a positive answer when the function f is increasing are well known. (They are stated as a step in the proof in Savage (1954); see also Fishburn (1970).) In this case v represents qualitative (or ordinal) probability, and the question we deal with can be restated as follows: Under what conditions does a qualitative probability have an additive representation? The problem is much more difficult when f is just nondecreasing but not necessarily increasing. A solution has been provided by Gilboa (1985).

4.2.2. The set of nonadditive probabilities which can be represented as a composition of a distortion function f and an additive probability P is "small" relative to all nonadditive probabilities. For example, consider the following version of the Ellsberg paradox. There are 90 balls in an urn, 30 black, B , balls and all the other balls are either white, W , or red, R . Bets on the color of a ball drawn at random from the urn are offered. A correct guess is awarded by \$100. There are six bets: " B ", " R ", " W ", " B or W ", " R or W ", and " B or R ". The following preferences constitute an Ellsberg paradox: $B \succ R \sim W$, R or $W \succ B$ or $R \sim B$ or W . It is impossible to define an additive probability on the events B , R , and W such that this probability's (nondecreasing) distortion will be compatible with the above preferences.

4.2.3. In Quiggin's model X is the set of real numbers. An act is a lottery of the form $y = (x_i, p_i)_{i=1}^k$ where $k \geq 1$, $x_1 \geq x_2 \geq \dots \geq x_k$, $p_i \geq 0$ and $\sum p_i = 1$. Quiggin postulates a weak order over all such acts which satisfies several axioms. As a result he gets a unique distortion function f and a monotonic, unique up to a positive linear transformation, utility function u on X such that the mapping $y \rightarrow \sum_{i=1}^k (x_i - x_{i+1}) f(\sum_{j=1}^i p_j)$ represents the preferences. However, $f(1/2) = 1/2$. Quiggin's axioms are not immediate analogues of the assumptions in Section 2. For example he postulates the existence of certainty equivalence for each act, i.e., for every y there is x in X such that $y \sim x$.

Yaari (1987) simplified Quiggin's axioms and got rid of the restriction $f(1/2) = (1/2)$ on the distortion function. However Yaari's main interest was the uncertainty aversion properties of the distortion function f . Hence his simplified axioms result in linear utility over the set of incomes, X . He explored the duality between concavity of the utility functions in the theory of risk aversion and the convexity of the distortion function in the theory of uncertainty aversion. Quiggin extended his results from distributions over the real numbers with finite support to distributions over the real line having density functions. Yaari dealt

with arbitrary distribution functions over the real line. Finally, Segal (1984) and Chew (1984) got the most general representation for Quiggin's model.

I conclude my remark on the works of Quiggin, Yaari, and Segal with a criticism from a normative, behavioristic point of view: It may seem conceptually inconsistent to postulate a decision maker who, while computing anticipated utility, assigns weight $f(p)$ to an event known to him to be of probability p , $p \neq f(p)$. His knowledge of p is derived, within the behavioristic model, from preferences over acts (as in 4.1 above). The use of the terms "anticipation" and "weight", instead of "expectation" and "probability" does not resolve, in my opinion, the inconsistencies. One way out would be to follow paragraph 4.2.1 above and to try to derive simultaneously distorted and additive probabilities of events.

4.3. The first version of this work (Schmeidler (1982)) includes a slightly extended version of the present Theorem. First recall that Savage termed an event E null if for all f and g in L : $f = g$ on E^c implies $f \sim g$. Clearly, if the conditions of the theorem are satisfied then an event is null iff it is dummy. The extended version of the Theorem includes the following addition:

The nonadditive probability ν of the Theorem satisfies the following condition: $\nu(E) = 0$ implies E is dummy, if and only if the preference relation also satisfies: E is not null, $f = g$ on E^c and $f(s) > g(s)$ on E imply $f > g$.

4.4. The expected utility model has in economic theory two other interpretations in addition to decisions under uncertainty. One interpretation is decisions over time: s in S represents time or period. The other interpretation of S is the set of persons or agents in the society, and the model is applied to the analysis of social welfare functions. Our extension of the expected utility model may have the same uses.

Consider the special case where $f(s)$ is s person's income. Two income allocations f and g are comonotonic if the social rank (according to income) of any two persons is not reversed between f and g . Comonotonic f , g , and h induce the same social rank on individuals and then $f > g$ implies $\gamma f + (1 - \gamma)h > \gamma g + (1 - \gamma)h$. This restriction on independence is, of course, consistent with strict uncertainty aversion which can here be interpreted as *inequality* (or *inequity*) *aversion*. In other words we have here an "Expected Utility" representation of a concave Bergson-Samuelson social welfare function.

4.5. One of the puzzling phenomena of decisions under uncertainty is people buying life insurance and gambling at the same time.² This behavior is compatible with the model of this paper. Let $S^0 = S^1 \times S^2 \times S^3$, where s^1 in S^1 describes a possible state of health of the decision maker, s^2 in S^2 describes a possible resolution of the gamble, and s^3 in S^3 describes a possible resolution of all other relevant uncertainties. Let ν^i be a nonadditive probability on S^i , $i = 0, 1, 2, 3$. Suppose that ν^1 is strictly convex (i.e., satisfying strict uncertainty aversion), ν^2 is strictly concave (i.e., $\nu^2(E) + \nu^2(F) > \nu(E \cup F) + \nu(E \cap F)$ if $E \setminus F$ and $F \setminus E$

² It is not puzzling, as a referee pointed out, if one accepts the Friedman-Savage (1948) explanation of this phenomenon.

are nonnull). Furthermore, if $E^0 = E^1 \times E^2 \times E^3$, and $Ei \subset S^i$, then $v^0(E^0) = v^1(E^1)v^2(E^2)v^3(E^3)$. To simplify matters suppose that X is a bounded interval of real numbers (representing an income in dollars), and the utility u is linear on X . Let the preference relation over acts on S^0 be represented by $f \rightarrow \int u(f) dv^0$. In this case buying insurance and gambling (betting) simultaneously is preferred to buying insurance only or gambling only, ceteris parabus. Also either of these last two acts is preferred to "no insurance no gambling."

School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel

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