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Rank- and Sign-Dependent Linear Utility Models for Binary Gambles*

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Binary SEU, prospect theory, and rank-dependent utility are generalized to event weights that depend on the event and on whether or not its consequence is preferred to the status quo and to the alternative consequence. The special cases are characterized, and the representation is axiomatized qualitatively. Assuming utility is additive over joint receipt of gambles, several topics are investigated: differences among buying and selling prices and choice indifference; differences between judged and choice indifferences; a reason people may buy both lotteries and insurance; and the form of utility for money, namely, power functions on either side of the status quo. *Journal of Economic Literature* Classification Numbers: 026, 213.

1. INTRODUCTION

It is widely conceded that among money outcomes, the value of there being no change from the status quo is unlike any other value. Whether an outcome is perceived to be a gain or loss matters, and it is not mere convention what is considered to be neither a gain nor a loss, which outcome I denote by e. At the same time, most formally developed theories of utility result in an interval scale of utility in which it is purely conventional which element is assigned the value of 0 utility. Among the recent theories of this character are the various rank dependent ones [6, 12, 14, 16, 18–22, 26]. Notable exceptions are the early discussion by Edwards [3], prospect theory [8], and the skew-symmetric, additive theories [4, 5, 9, 10].

Here I wish to explore in greater detail than exists in the literature the possibilities of weighted-average theories for binary gambles, not unlike classical expected utility theory except that the outcome e plays a highly significant role.

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Luce and Narens [16] showed that the most general interval-scale version of utility over binary gambles is only slightly more general than SEU. Let a and b denote outcomes—e.g., sums of money or other gambles—and E a chance or uncertain event. The object $a \bigcirc_E b$ is interpreted to be the gamble in which a is the outcome if E occurs and b otherwise. Then, they showed the representation to be of the rank-dependent form[1]

$$U(a \bigcirc_E b) = \begin{cases} S_>(E)\, U(a) + [1 - S_>(E)]\, U(b), & \text{if } \ a \gtrsim b \\ S_<(E)\, U(a) + [1 - S_<(E)]\, U(b), & \text{if } \ a \lesssim b, \end{cases} \tag{1}$$

where U is defined over all pure consequences and all gambles, and the weighting functions S_i, $i = >, <$, are from events into $[0, 1]$. The model is *rank dependent* in the sense that the weighting assigned to an event depends upon the preference order assigned to a and b. I provided a generalization of the binary rank-dependent model to gambles with finitely many outcomes [12].

Observe that the representation is unchanged under affine transformations of the form $U \to rU + s$, where $r > 0$. Thus, in this model 0 does not have a different role from any other value.

Sign-Dependent Utility

The Representation

Looking at the form of the rank-dependent model, Eq. (1), there is a fairly natural sign-dependent (but not rank-dependent) model in which the weights depend on the sign of the outcome relative to no change in the status quo, e, as well as the event. It is

$$U(a \bigcirc_E b) = \begin{cases} U(a)\, S^+(E) + U(b)[1 - S^+(E)], & a \gtrsim e,\ b \gtrsim e \\ U(a)\, S^+(E) + U(b)[1 - S^-(E)], & a \gtrsim e,\ b \lesssim e \\ U(a)\, S^-(E) + U(b)[1 - S^+(E)], & a \lesssim e,\ b \gtrsim e \\ U(a)\, S^-(E) + U(b)[1 - S^-(E)], & a \lesssim e,\ b \lesssim e. \end{cases} \tag{2}$$

This representation, which I shall refer to as *sign-dependent utility*, obviously reduces to the standard SEU one provided $S^+ = S^- =$ a probability measure. Indeed, if the question of whether the weights are additive

[1] For rank-dependent utility, we earlier used S^+ for $S_>$ and S^- for $S_<$, but in the present context it seems better to reserve the $+$ and $-$ notation for the relation of outcomes to 0 rather than for their relation to each other.

over disjoint events is ignored, it looks just like SEU in the positive domain and, separately, in the negative domain, with different weights attached to the event in the two domains. However, in the mixed positive–negative case the two weights are used and so definitely do not add to 1.

Like the binary rank-dependent model, two functions over events are involved rather than the single one of SEU. The question is whether this is sufficient to accommodate the empirical anomalies that show SEU to be descriptively inadequate. That is taken up below.

An interesting alternative to Eq. (2) is proposed in [7] for the case of known probabilities. In that theory, the role of an overall status quo is replaced by a local aspiration level, namely, the certainty equivalent (CE) to the gamble. Consequences better than the CE are weighted differently than those that are inferior to the CE. An axiomatization of the representation is provided in which application of the well-known independence assumption is limited to consequence changes that do not result in a changed relation to the CE.

Additivity of Utility over the Joint Receipt of Consequences

As will become apparent, I am able to develop the theory involving sign dependence only by taking seriously the direct addition (in contrast to the probability weighting for gambles) of utilities. Therefore, it is necessary to postulate a qualitative operation $\oplus$ over the underlying space of alternatives, with $a \oplus b$ interpreted to mean the joint receipt of a and b, whether they are gambles or pure consequences. Thus, I am assuming that U is additive over $\oplus$, i.e., $U(a \oplus b) = U(a) + U(b)$. This may strike many readers as a surprising assumption, one they may feel has widely been repudiated. For example, if the domain is money and if one assumes, for sums of money x and y, that $x \oplus y = x + y$, then it follows immediately that U must be directly proportional to money. Beyond doubt, the whole thrust of modern utility theory has been that non-linear utility is needed to account for much of the observed behavior, and so either $x \oplus y \neq x + y$ or the assumption of additivity is a step backward. I can only ask the reader to bear with me until we see what can be accounted for by accepting additivity and investigating what comes of it.

An Empirical Prediction of Additive, Sign-Dependent Weighted Utility

Consider the pair of independent gambles $a \bigcirc_E b$ and $b \bigcirc_E a$, where $a > e > b$. In ordinary SEU theory, their joint receipt would be treated as their convolution: $a \oplus a$ if E occurs when the first gamble is run and $\neg E$ occurs when the second is run, abbreviated $(E, \neg E)$; $a \oplus b = b \oplus a$ if (E, E) or $(\neg E, \neg E)$; and $b \oplus b$ if $(\neg E, E)$. And then the SEU would be calculated for this gamble with three consequences. By contrast, if people simply add the utilities of the separate gambles, and the representation of Eq. (2) applies, then the utility of the pair is

$$\begin{aligned} U(a \bigcirc_E b) + U(b \bigcirc_E a) &= U(a)\, S^+(E) + U(b)[1 - S^-(E)] \\ &\quad + U(b)\, S^-(E) + U(a)[1 - S^+(E)] \\ &= U(a) + U(b), \end{aligned}$$

which says that the total utility of the two gambles is unaffected by the choice of the event E. This prediction, which of course is also valid for additive SEU, strikes me as very counterintuitive. Consider the pair of money gambles (\$1000, p, $-$\$200) and ($-$\$200, p, \$1000). When $p = 0$ or 1, the outcome from the pair is exactly \$800. When $p = \frac{1}{2}$, then the possible outcomes are \$2000 with probability $\frac{1}{4}$, \$800 with probability $\frac{1}{2}$, and $-$\$400 with probability $\frac{1}{4}$. Anyone who is risk averse is likely to prefer the sure outcome to the risky one. So the theory seems inadequate. What is not clear is whether the problem lies in the additivity or in the sign-dependent model, which includes SEU as a special case. Jointly, they appear to be wrong descriptively as well as normatively.

By contrast, additivity together with the rank-dependent utility model, Eq. (1), does not entail this prediction;

$$\begin{aligned} U(a \bigcirc_E b) + U(b \bigcirc_E a) &= U(a)\, S_>(E) + U(b)[1 - S_>(E)] \\ &\quad + U(b)\, S_{<-}(E) + U(a)[1 - S_<(E)] \\ &= U(a) + U(b) + [U(a) - U(b)][S_>(E) - S_<(E)] \end{aligned}$$

clearly depends on E. Assuming that $S_>$ and $S_<$ agree at the sure and null events, then the intermediate gamble is preferred to the sure-thing ones if and only if $S_>(E) > S_<(E)$. Put another way, the risk-averse decision maker weights the event associated with the poorer outcome more heavily than when it entails the better outcome.

Thus, despite the evidence that some form of sign-dependent utility is needed, it appears that the simplest model, at least when coupled with additivity of utility over joint receipt of consequences, leads to an implausible prediction that is avoided by the simplest rank-dependent model. This suggests that we should try to combine features of both representations.

Rank- and Sign-Dependent Weighted Utility Representation

The principle for combining the two ideas of sign- and rank-dependence within the general context of weighted average utility model is not difficult.

The weight assigned to an event is affected by the relation of its associated outcome both to e and to the other possible outcome. Thus, there will be four weighting functions into $[0, 1]$ of the general form S_i^j, where $i = >, <$ and $j = +, -$. It is easy to write down the resulting representation:

$$U(a \bigcirc_E b) = \begin{cases} U(a)\, S_>^+(E) + U(b)[1 - S_<^+(E)], & \text{if } a \gtrsim b \gtrsim e \\ U(a)\, S_<^+(E) + U(b)[1 - S_>^+(E)], & \text{if } b \gtrsim a \gtrsim e \\ U(a)\, S_>^+(E) + U(b)[1 - S_>^-(E)], & \text{if } a \gtrsim e \gtrsim b \\ U(a)\, S_<^-(E) + U(b)[1 - S_<^+(E)], & \text{if } b \gtrsim e \gtrsim a \\ U(a)\, S_>^-(E) + U(b)[1 - S_>^-(E)], & \text{if } e \gtrsim a \gtrsim b \\ U(a)\, S_<^-(E) + U(b)[1 - S_<^-(E)], & \text{if } e \gtrsim b \gtrsim a. \end{cases} \tag{3}$$

This binary rank- and sign-dependent (RSD) weighted utility representation combines in the simplest way I know of the advantages of both the sign-dependent and rank-dependent representations, but it does so at considerable expense. There are four weighting functions, as against two in either of the component models. In the next section, we explore four ways in which the number of functions can be reduced.

Relation of RSD Utility to Other Theories

There are several different ways in which the four weighting functions of Eq. (3) can be collapsed into fewer. Three of these arise by considering the 2×2 table of rank versus sign. Either the rows can be equated, or the columns, or the diagonals. Each corresponds to an important special case. A fourth reduction occurs if we impose an accounting equivalence involving event complements.

First, consider equating the signs:

$$S_i^+ = S_i^-, \qquad i = >, <. \tag{4}$$

It is easy to verify that in the presence of Eq. (3), Eq. (4) is equivalent to the rank-dependent representation, Eq. (1).

It will be recalled [12] that this rank-dependent model entails the accounting equivalence

$$(a \bigcirc_E b) \bigcirc_F b \sim (a \bigcirc_F b) \bigcirc_E b. \tag{5}$$

Some recent, unpublished, empirical work of A. Brothers suggests that a sizeable fraction of subjects violate Eq. (5), which is one of the reasons generalizations such as Eq. (3) have been considered. Indeed, one can show that, on the assumption of Eq. (3), Eq. (5) is equivalent to Eq. (1). For example, consider a, b, E, and F such that

$$a > a \bigcirc_E b > e > a \bigcirc_F b > b.$$

Then, applying Eq. (3) to Eq. (5) and collecting terms, it follows readily that

$$S^{+}_{>}(Z) = S^{-}_{>}(Z), \qquad \text{for} \quad Z = E, F.$$

The other case is similar.

Thus, the full model of Eq. (3) does not imply Eq. (5). It is of interest that the predicted failure of Eq. (5) is selective; it only occurs when a and b are of opposite signs, but when they are of the same sign the equivalence of Eq. (5) still obtains.

Second, consider equating the order:

$$S^{j}_{>} = S^{j}_{<}, \qquad j = +, -. \tag{6}$$

In the presence of Eq. (3), it is easily verified that Eq. (6) is equivalent to the sign-dependent model of Eq. (2). Of course, as we saw earlier, this representation coupled with additivity is certainly suspect.

Third, consider equating the diagonals:

$$S^{+}_{>} = S^{-}_{<} \qquad \text{and} \qquad S^{+}_{<} = S^{-}_{>}. \tag{7}$$

This is a slight generalization of Kahneman and Tversky's [8] prospect theory for binary gambles. To see this, recall that they dealt with risky alternatives of the form $(x, p; y, q)$, where $p + q \leqslant 1$, which is interpreted as meaning that x is the consequence with probability p, y with probability q, and 0 with probability $1 - p - q$. Using V for subjective value, essentially our U, they postulated two equations:

(KT 1) $V(x, p; y, q) = \pi(p)\, V(x) + \pi(q)\, V(y)$,
if either $p + q < 1$ or $x \geqslant 0 \geqslant y$ or $x \leqslant 0 \leqslant y$

(KT 2) $V(x, p; y, q) = \pi(p)\, V(x) + [1 - \pi(p)]\, V(y)$,
if either $x > y > 0$ or $x < y < 0$.

They do not explicitly say what happens for $y > x > 0$ and $y < x < 0$, but apparently they are assuming that $(x, p; y, q) \sim (y, q; x, p)$, in which case (KT 2) yields

(KT 2′) $V(x, p; y, q) = [1 - \pi(q)]\, V(x) + \pi(q)\, V(y)$,
if either $y > x > 0$ or $y < x < 0$.

Omitting the case $p + q < 1$, and letting $p = \Pr(E)$ and $q = \Pr(\neg E) = 1 - p$, it is easy to verify that Eq. (3) reduces to the KT equations iff

$$\pi(p) = S^{+}_{>}(E) = S^{-}_{<}(E) \qquad \text{and} \qquad 1 - \pi'(q) = S^{+}_{<}(E) = S^{-}_{>}(E). \tag{8}$$

This identification is not exactly their model since $\pi' \neq \pi$. That is why I said it was a generalization of prospect theory. Below, we arrive at $\pi' = \pi$.

The fourth reduction arises by considering the transparent accounting equivalence:

$$a \bigcirc_E b \sim b \bigcirc_{\neg E} a. \tag{9}$$

It is easy to verify, for Eq. (4), that Eq. (9) is equivalent to

$$S^j_>(E) + S^j_<(\neg E) = 1, \qquad j = +, -. \tag{10}$$

Again, this reduces the number of weighting functions to two.

An additional reduction is sometimes possible by combining Eq. (10) with each of the three earlier conditions, Eqs. (4), (6), and (7). When added to the condition for rank dependence, Eq. (4), Eq. (10) results in the simplest weighted utility model. When added to the sign-dependent model, Eq. (6), it leads to additivity over complementary events, i.e., $S^j_i(\neg E) = 1 - S^j_i(E)$. And when added to Eq. (7) it reduces Eq. (8) to prospect theory since

$$\pi'(q) = 1 - S^+_<(E) = S^+_>(\neg E) = \pi(q),$$

so reducing it to the single function π, as in prospect theory.

A natural additional simplification of Eq. (10) is to suppose that the effects of rank and of sign are additively[2] independent:

$$S^j_i = S_i + S^j. \tag{11}$$

From Eqs. (10) and (11), it follows readily that

$$S_>(E) - S_<(E) = S_>(\neg E) - S_<(\neg E),$$
$$S^+(E) + S^-(E) = S^-(\neg E) + S^+(\neg E).$$

Clearly, the roles of S_i and S^j are some what asymmetric, suggesting that we might haive the two equations

$$S_>(E) + S_<(\neg E) = 1 \qquad \text{and} \qquad S^j(E) - S^j(\neg E) = 0,$$

but I know of no condition that forces such a normalization.

[2] Although a multiplicative form seems a priori possible, it does preclude the weights being additive over disjoint events. Although there is no compelling reason to postulate additivity, surely it should not be totally precluded.

Implications of Additive RSD Weighted Utility

Asymmetry of Reflected Gambles Need Not Imply Asymmetry of the Utility Function

In contrast to prospect theory [8], RSD utility does not necessarily require a fundamental difference in the shape of the utility function for positive and negative outcomes in order to account for observed asymmetries. For suppose, as in Theorem 4 below which axiomatizes Eq. (3), that utility is additive over joint receipt of consequences. If, in addition, we suppose for sums of money x and y that $x \oplus y = x + y$, then for some constant K, $U(x) = Kx$ and so $U(-x) = -U(x)$. Nevertheless, a positive/negative asymmetry is quite possible in responses to gambles because different weights apply depending upon the sign of the outcome. Suppose that $x > y > e$; then by Eq. (3) and $U(-x) = -U(x)$,

$$U(a \bigcirc_E b) + U(-a \bigcirc_E -b) = [U(a) - U(b)][S^+_>(E) - S^-_<(E)].$$

This quantity is zero, and so $-U(x \bigcirc_E y) = U[-(x \bigcirc_E y)] = U(-x \bigcirc_E -y)$, if and only if the basic property of prospect theory, Eq. (7), obtains.

Thus, prospect theory is inconsistent with additive utility, the assumption that $x \oplus y = x + y$, and the existence of an asymmetry between a gamble and its reflection in which the signs of the consequences reversed. But if prospect theory is not assumed, then gambling decisions may exhibit such an asymmetry without having to impose an asymmetry on the utility function itself because the weights applied to events with positive and negative outcomes do not satisfy Eq. (7). In prospect theory, it was necessary for U to carry the burden of asymmetry because the weights were chosen so as to force a certain symmetry between positive and negative outcome gambles. By abandoning this, and at the expense of doubling the number of weights, RSD utility can admit symmetry of the utility function which is forced if the utility of independent gambles is additive.

Buying Price, Selling Price, and Choice Indifference

What is a gamble worth to a person? That question does not seem to have a simple answer, as has long been recognized. We may define at least three different concepts of worth: the choice indifference, which is the sum of money that is indifferent to the gamble when choices are made; the buying price, which is the most one will pay in order to gain the gamble; and the selling price, which is the least one will accept to part with the gamble. Within the context of the present theory, these need not be the same, and it is interesting to see how they relate.

Let $a > e > -b$ and consider the gamble $a \bigcirc_E -b$. Denote by CI, BP, and SP the choice indifference, the buying price, and the selling price, respectively. Clearly, CI is defined by

$$\mathrm{CI} \sim a \bigcirc_E -b, \tag{12}$$

and so by the RSD representation, Eq. (3),

$$U(\mathrm{CI}) = U(a)\, S^+_>(E) + U(-b)[1 - S^-_>(E)].$$

The definitions of BP and SP are more subtle. As earlier, let $\oplus$ denote the joint receipt of two consequences, and define $\ominus$ by $a \ominus b \sim c$ iff $a \sim b \oplus c$. If U is additive over $\oplus$, then it is subtractive over $\ominus$. Now, if one buys the gamble for the amount c, then one is saying that having the gamble $(a \ominus c) \bigcirc_E (-b \ominus c)$ is at least as good as no change in the status quo e, and since the BP is the largest such c, by monotonicity we have the definition

$$(a \ominus \mathrm{BP}) \bigcirc_E (-b \ominus \mathrm{BP}) \sim e. \tag{13}$$

By monotonicity, $a \gtrsim \mathrm{BP} \gtrsim -b$, and so $a \ominus \mathrm{BP} \gtrsim e \gtrsim -b \ominus \mathrm{BP}$. Therefore, by Eq. (3) and rearrangement of terms,

$$U(\mathrm{BP}) = U(\mathrm{CI})/[1 - S^-_>(E) + S^+_>(E)].$$

We see immediately that

$$\mathrm{BP} \gtrsim \mathrm{CI} \Leftrightarrow U(\mathrm{BP}) \geqslant U(\mathrm{CI}) \Leftrightarrow S^-_>(E) \geqslant S^+_>(E).$$

Clearly, the BP and CI agree in a sign-independent situation, but need not in a sign-dependent one.

If one possesses the gamble and opts to sell it for the amount c, then one is saying, in effect, that $(c \ominus a) \bigcirc_E (c \ominus -b) \gtrsim e$. Note, that, by definition, $c \sim (c \ominus -b) \oplus -b$. This with monotonicity, associativity, and commutativity of $\oplus$ implies

$$c \oplus b \sim (c \ominus -b) \oplus -b \oplus b \sim (c \ominus -b) \oplus e \sim c \ominus -b.$$

Since the selling price is the smallest c for which one will sell, monotonicity implies that SP is defined by

$$(\mathrm{SP} \ominus a) \bigcirc_E (\mathrm{SP} \oplus b) \sim e. \tag{14}$$

Again, by monotonicity $a \gtrsim \mathrm{SP} \gtrsim -b$, and so $\mathrm{SP} \oplus b \gtrsim e \gtrsim \mathrm{SP} \ominus a$. Thus, by Eq. (3) and some rearrangement of terms,

$$U(\mathrm{SP}) = \frac{U(a)\, S^-_<(E) + U(-b)[1 - S^+_<(E)]}{1 - S^+_<(E) + S^-_<(E)}.$$

It is routine to show that

$$\mathrm{SP} \gtrsim \mathrm{BP} \Leftrightarrow [1 - S_>^-(E)]/S_>^+(E) \geqslant [1 - S_<^+(E)]/S_<^-(E).$$

Thus, if prospect theory, Eq. (7), obtains, $\mathrm{SP} \sim \mathrm{BP}$, which of course is well known to be contrary to fact. Clearly a sufficient condition for $\mathrm{SP} > \mathrm{BP}$, which is the observed behavior, is that

$$S_>^-(E) < S_<^+(E) \qquad \text{and} \qquad S_>^+(E) < S_<^-(E).$$

This holds if the person always assigns more weight to an event when it leads to the poorer outcome than when it leads to the better one.

The relation between SP and CI does not lead to anything very simple. However, it is possible to show that incorporating the similar inequalities

$$S_>^+(E) < S_<^+(E) \qquad \text{and} \qquad S_>^+(E) < S_<^-(E)$$

implies $\mathrm{SP} > \mathrm{CI}$. Again, this is a matter of assigning more weight when the event is associated with the poor outcome.

A Possible Reason Why Judged and Choice Indifferences Disagree

It has been usual to assume that the stated monetary equivalence to a gamble is a persons' estimate of his/her choice indifference value for that gamble. For example, only by making this assumption can one interpret the so-called preference reversal experiments as falsifying the transitivity of preference. There now is evidence that this assumption probably is wrong [2, 24, 25]. When asked to establish a monetary equivalence to a gamble, people do not always provide a value that is selected 50% of the time when a choice is offered between the alleged indifference value and the gamble [2]. When the gamble is structured to provide a large outcome with a low probability, the judged value grossly overestimates the choice value. So we must assume that judged indifference points are determined in some way other than as a choice indifference.

This can be accounted for by supposing that judgments of indifference lead the subject to imagine either ownership of the gamble for which s/he establishes a selling price or purchase of the gamble for which s/he sets a buying price. In either event, the judged value will not agree with the choice indifference if the full additive, sign- and rank-dependent model is correct. Apparently, these values are overestimates of CI.

In order to account for the fact that the discrepancy is observed only for the low probability gambles, we must assume that the weights disagree substantially for improbable events, but not for those in the intermediate range of probabilities.

Purchases of Both Lotteries and Insurance

Within the context of classical SEU, it has proved awkward to account for the fact that some people buth buy insurance and lotteries. For the sake of discussion, suppose that the lottery is $a \bigcirc_E e$, where $a > e$, and the risk that one wishes to insure against is $-a \bigcirc_E e$. According to the above analysis, one is willing to buy the lottery for the amount c if and only if

$$(a \ominus c) \bigcirc_E -c \gtrsim e. \tag{15}$$

Although one commonly speaks of "buying" insurance, in truth one is actually selling to an insurance company a negative lottery that one possesses, such as the possible loss of a house due to fire. So one is willing to pay the amount c, i.e., sell the lottery for $-c$, if and only if

$$(-c \ominus -a) \bigcirc_E -c \gtrsim e. \tag{16}$$

As noted earlier, $-c \ominus -a = a \oplus -c$. Using this and the associativity and monotonicity of $\oplus$,

$$a \sim (a \oplus -c) \oplus c \sim (-c \ominus -a) \oplus c,$$

whence by definition $-c \ominus -a \sim a \ominus c$. Therefore, Eq. (16) is identical to Eq. (15): The decision to purchase lottery $a \bigcirc_E e$ for the amount c rather than maintain the status quo is identical to the decision to sell the risky alternative $-a \bigcirc_E e$ for the amount $-c$ rather than do nothing. Thus, it is hardly surprising that people do both.

An Empirical Prediction of Additive RSD Utility

Using the RSD representation of Eq. (3), it is trivial to derive[3] the following: $(\forall a > e,\ b > e)$ and $(\forall a < e,\ b < e)$

$$(a \oplus b) \bigcirc_E e \sim (a \bigcirc_E e) \oplus (b \bigcirc_E e).$$

This is strong property that, if wrong, should be easy to reject. Here is the sort of prediction that it makes. Let E denote the occurrence of a six on the throw of a die. Consider the two alternatives:

(1) Two independent dice, one white and one red, are thrown, and the decision maker receives \$100 if the white six comes up and \$20 if the red six comes up; in all other cases, no money is exchanged.

[3] This prediction also follows from the general model, discussed in the next major section, in which no change from the status quo e is a generalized zero and the translations of $\langle X, \gtrsim, \bigcirc_A \rangle$ are also ones of $\langle X, \gtrsim, \oplus \rangle$. For, if this is true, then for some translation τ,

$$(a \oplus b) \bigcirc_A \sim \tau(a \oplus b) \sim \tau(a) \oplus \tau(b) \sim (a \bigcirc_A e) \oplus (b \bigcirc_A e).$$

(2) One die is thrown and both \$100 and \$20 are received for the six and nothing otherwise.

The assertion is that these two alternatives are indifferent. Note that in testing this proposed property it is important in case (2) to treat the two sums as distinct, separate consequences that both happen to depend upon the same event. Put another way, we do not wish to assume that for money, $x \oplus y$ is necessarily the same as $x + y$.

Estimating the Weighting Functions Using Money Gambles

If the domain of pure consequences is money and if the joint receipt of x and y is equivalent to $x + y$, then U is simply proportional to money.[4] Thus, if we let z denote a sum of money such that $z \sim x \bigcirc_E y$, then from Eq. (3) and the proportionality of U with money, the weights are given by the formula:

$$S_i^j(E) = (z - y)/(x - y),$$

where $i = >$ if $x < y$ and $<$ if $x < y$, and $j = +$ if $x > 0$, $y > 0$ and $-$ if $x < 0$, $y < 0$. These ratios should, of course, be independent of the values selected for x and y provided that the relevant inequalities are maintained. Further, once the weights are estimated, the mixed positive and negative cases provide a way to test the theory.

Should the theory be confirmed, the procedure just outlined offers an obvious way for decision analysts to estimate the weighting functions in real situations. However, analysts should be very cautions, as was discussed earlier, about treating judged indifference points as estimates of choice ones. Exactly how best to by-pass this difficulty is not clear to me, because choice indifference points are tedious and tricky to estimate. The ideal would be to develop instructions that induce people actually to provide good estimates of their choice indifference points, but experience suggests that this is difficult to do.

AXIOMATIZATIONS

The question of axiomatizing the SD and RSD utility representations can be approached in two distinctly different ways. The first is to make use of known results about concatenation structures with singular points. The second is to give a direct axiomatization of the additive RSD representation. I do not have, at present, a way to axiomatize RSD without invoking additivity.

[4] As was noted earlier, this follows from the additivity of U over $\oplus$ together with the monetary assumption that $x \oplus y = x + y$.

Concatenation Structures With Singular Points

The strategy to be followed is much like the one that led to the rank-dependent model proposed in [16], but it is based instead on recent results in [13] about generalized concatenation structures with singular points. I remind the reader of that strategy.

A structure $\mathscr{A} = \langle A, \succsim, \bigcirc \rangle$ is called a (*binary*) *concatenation structure*[5] provided $\succsim$ is a simple order and $\bigcirc$ is an operation that is monotonic relative to $\succsim$. We considered the general class of concatenation structures that exhibit two properties, homogeneity and finite uniqueness, both stated in terms of the automorphisms (symmetries) of the structure, not in terms of the primitives directly. *Homogeneity* obtains if for each pair of points there exists an automorphism that takes the one point into the other. *Finite uniqueness* obtains if there exists an integer N such that whenever an automorphism has N distinct fixed points, then that automorphism is necessarily the identity. Sometimes, when useful, we explictly speak of *N-point uniqueness.* There are three inherently different general types of homogeneous, finitely unique concatenation structures, and the possible numerical ones on the real numbers have been fully characterized [16]. Within the utility context, where the operation is interpreted as a chance mixture of outcomes and there are distinct operations associated with different events, but a common utility function obtains, additional results lead us to expect that the only cases that can arise are the interval scale ones—the so-called cases of cardinal utility—and these were shown to be equivalent to the binary RD representation.

In [13] I explored the general theory of concatenation structures in which some points are *singular* in the sense that they remain fixed under all automorphisms of the structure. I continued to assume structures that are highly regular everywhere except at the singular points. In particular, between two adjacent singularities homogeneity is assumed to hold. This is referred to as *homogeneity between adjacent singular points.* More homogeneity than this is not possible because if an automorphism α were to map a into b where $a < e < b$ for e singular, then we would have the contradiction $b = \alpha(a) < \alpha(e) = e < b$. The concept of *finite uniqueness* is unchanged.

Under the conditions of homogeneity between adjacent singular points and finite uniqueness, at most three singular points can exist: a maximum, a minimum, and a unique interior singularity. Moreover, the interior one, if one exists, exhibits "zero-like" properties. In particular, for an interior singularity e, there exist functions θ and η such that on either side of e they

[5] Sometimes one assumes that $\succsim$ is a weak order, but by working with equivalent classes one can replace it by a simple order. More importantly, in some definitions $\bigcirc$ is taken to be a partial operation, in which case some form of local definability is introduced.

agree with translations and $a \bigcirc e = \theta(a)$ and $e \bigcirc a = \eta(a)$, where by a *translation* one means an automorphism with no fixed points other than the singular ones. When e satisfies the above property, it is said to be a *generalized zero.* An extremum, if any, may exhibit either an "infinity-like" property or be an infinity on one side and a generalized zero on the other side.

The canonical, real representation of such structures that have only an interior sigularity was also determined. The most general real representation $\mathscr{R} = \langle \mathrm{Re}, \geqslant, \otimes \rangle$ can be chosen so that the translations are all multiplication by positive constants, and there exist positive constants C_1, C_2, L, and R, and functions f_i: Re $\rightarrow$ (onto) Re, $i = 1, 2$, such that:

(i) f_i is strictly increasing.

(ii) f_i/ι, where ι is the identity, is strictly decreasing.

(iii) for all $x, y \in \mathrm{Re}$,

$$x \otimes y = \begin{cases} yf_1(x/y), & y > 0 \\ C_1 x, & y = 0,\ x \geqslant 0, \\ C_2 x, & y = 0,\ x < 0, \\ yf_2(x/y), & y < 0. \end{cases}$$

(iv) $f_1(u)/u > C_1$ for $u > 0$
$< C_2$ for $u < 0$.

(v) $f_2(u)/u > C_2$ for $u > 0$
$< C_1$ for $u < 0$.

(vi) $f_i(u)/u$ approaches $-\infty$ as u approaches 0 from below and $+\infty$ as u approaches 0 from above.

(vii) $f_1(-L) = f_2(-1/R) = 0$.

(viii) Every automorphism of $\mathscr{R}$ is a translation.

(ix) If the structure is idempotent, i.e., $(\forall x)\ x \otimes x = x$, then $f_i(1) = 1$.

Thus, we know the general form of concatenation structures with an interior singular point and no extreme points, and by part (viii) we see that we are dealing basically with a ratio scale case—affine transformations simply are not possible when there is a singular point. One consequence of this obvious fact is that the weighted average form is not forced as it is in the case of interval scales. Thus, a far richer set of possibilities can arise. For the purposes of this paper I wish to single out the simplest cases, the weighted average ones.

Linearity Condition

Assuming the canonical real representation described above of a concatenation structure with only an interior singularity, consider the following *linearity condition*: ($\forall x, y, z$ such that either $x>0$ and $y>0$ or $x<0$ and $y<0$),

$$(x+y) \otimes z + 0 \otimes z = x \otimes z + y \otimes z. \tag{17}$$

Within a gambling context, this asserts that a pair of independent gambles based on an event, say E, in which the possible pattern of outcomes are $x+y$, $x+y+z$, z, or $2z$ depending on whether (E, E), $(E, \neg E)$, $(\neg E, E)$, or $(\neg E, \neg E)$, respectively, occurs is seen as equivalent to $x+y$, $x+z$, $y+z$, $2z$ under the same conditions. This amounts to saying that the decision maker does not really distinguish between $(\neg E, E)$ and $(E, \neg E)$, so they are equally likely and the outcome associated with one E and one $\neg E$ is $\frac{1}{2}[(x+y+z)+z] = \frac{1}{2}(x+z) + \frac{1}{2}(y+z)$.

THEOREM 1. *Suppose $\mathscr{R} = \langle \mathrm{Re}, \geqslant, \otimes \rangle$ is a canonical, unbounded, idempotent, concatenation structure for which* 0 *is an interior singular point, and that $\mathscr{R}$ is homogeneous and finitely unique. Then, the linearity condition, Eq.* (17), *holds iff there are constants S^+ and S^- in* [0, 1] *such that for* ($\forall x, y \in \mathrm{Re}$)

$$x \otimes y = \begin{cases} S^+ x + (1 - S^+) y, & x \geqslant 0,\ y \geqslant 0 \\ S^+ x + (1 - S^-) y, & x \geqslant 0,\ y \leqslant 0 \\ S^- x + (1 - S^+) y, & x \leqslant 0,\ y \geqslant 0 \\ S^- x + (1 - S^-) y, & x \leqslant 0,\ y \leqslant 0. \end{cases} \tag{18}$$

Proof. For $z>0$, substitute the canonical form into the linearity condition and let $u = x/z$ and $v = y/z$, where $uv > 0$:

$$f_1(u+v) + f_1(0) = f_1(u) + f_1(v).$$

Consider $u>0$, $v>0$; it is well known that the only monotonic increasing solutions are of the form $f_1(u) = f_1(0) + S^+ u$ for some $S^+ > 0$. By idempotence, $1 = f_1(1) = f_1(0) + S^+$, whence the first line of the representation. For $u<0$, $v<0$, for some $S^- > 0$, by a similar argument $f_1(u) = 1 - S^+ + S^- u$.

The same line of argument for $z<0$ yields the form of f_2 with constants T^+ and T^-. However, since the f functions must be continuous at zero, it is easy to see that they reduce to S^+ and S^-, respectively.

Conversely, the representation, Eq. (18), trivially implies the linearity condition, Eq. (17). ∎

The RSD representation can be derived in a similar fashion by adding somewhat different constraints to the general canonical representation of a concatenation structure with a generalized zero. The following versions of linearity do the job:

$$(\forall x, y \text{ with } xy>0) \qquad x\otimes y=\begin{cases}(x-y)\otimes 0+y, & \text{if } x>y\\ 0\otimes(y-x)+x, & \text{if } x<y\end{cases} \tag{19}$$

$$(\forall x, y \text{ with } x>0>y \text{ or } x<0<y) \qquad x\otimes y=x\otimes 0+0\otimes y. \tag{20}$$

The first of these equations says that for consequences on the same side of zero, one can simply subtract away from both alternatives the amount with the smaller absolute magnitude, thereby reducing the problem to a gamble having only one non-zero consequence. The second says that a gamble involving both positive and negative consequences is thought of as the sum of two gambles, with each consequence pitted against the null one.

THEOREM 2. *Suppose* $\mathscr{R}=\langle \mathrm{Re}, \geqslant, \otimes\rangle$ *is a canonical, unbounded, idempotent, concatenation structure for which* 0 *is an interior singular point and that is homogeneous and finitely unique. Then, the linear conditions of Eqs.* (19) *and* (20) *hold iff the canonical form is given by*

$$x\otimes y=\begin{cases}S^+_>x+(1-S^+_>)y, & \text{if } x\geqslant y\geqslant 0\\ S^+_<x+(1-S^+_<)y, & \text{if } y\geqslant x\geqslant 0\\ S^+_>x+(1-S^-_>)y, & \text{if } x\geqslant 0\geqslant x\\ S^-_<x+(1-S^+_<)y, & \text{if } y\geqslant 0\geqslant x\\ S^-_>x+(1-S^-_>)y, & \text{if } 0\geqslant x\geqslant y\\ S^-_<x+(1-S^-_<)y, & \text{if } 0\geqslant y\geqslant x.\end{cases} \tag{21}$$

Proof. Using property (iii) of the canonical representation for $x\otimes y$ and imposing Eqs. (19) and (20) as appropriate

$$\begin{aligned}
&(x-y)\otimes 0+y=C_1(x-y)+y, && \text{if } x\geqslant y\geqslant 0\\
&0\otimes(y-x)+x=(y-x)f_1(0)+x, && \text{if } y>x\geqslant 0\\
&(x\otimes 0)+(0\otimes y)=\begin{cases}C_1x+yf_2(0),\\ C_2x+yf_1(0),\end{cases} && \begin{aligned}&\text{if } x\geqslant 0\geqslant y\\ &\text{if } y\geqslant 0\geqslant x\end{aligned}\\
&(x-y)\otimes 0+y=C_2(x-y)+y, && \text{if } 0\geqslant x\geqslant y\\
&0\otimes(y-x)+x=(y-x)f_2(0)+x, && \text{if } 0\geqslant y\geqslant x.
\end{aligned}$$

Setting $S^+_>=C_1$, $S^+_<=1-f_1(0)$, $S^-_>=C_2$, $S^-_<=1-f_2(0)$, and introducing utilities and events, we see that Eq. (3) follows.

It is easy to verify that Eq. (21) implies Eqs. (19) and (20). ∎

Utility as a Function of Money

The next question to be raised is the degree to which having a utility function that is additive over an operation $\oplus$ of joint receipt constrains both the form of U as a function of money and of the form of the operation $\otimes$. As has been remarked several times, it is trivial to show that U is proportional to money if one assumes that for pure sums of money x and y, $x \oplus y = x + y$. If that is not assumed, then additional possibilities exist, which need to be worked out. The major conclusion is that U must be a two-sided power function of money and that the mixing operation must be of the weighted linear form of Eq. (3).

THEOREM 3. *Suppose* $\langle \mathrm{Re}^j, \succsim, \oplus, \otimes \rangle$, $j = +, -$, *is such that*

(i) $\langle \mathrm{Re}^j, \succsim, \oplus \rangle$ *is a unit structure, i.e., multiplications by positive constants are translations of the structure, that is, isomorphic to* $\langle \mathrm{Re}^j, \geqslant, + \rangle$;

(ii) $\langle \mathrm{Re}^j, \succsim, \otimes \rangle$ *is a unit structure*;

(iii) (Distribution) $(\forall x, y, z \in \mathrm{Re}^j \cup \{0\})$

$$(x \otimes y) \oplus z \sim (x \oplus z) \otimes (y \oplus z). \tag{22}$$

Then the isomorphism is of the following form: There exist positive constants $k(j)$ *and* $\beta(j)$, $j = +, -$, *such that for real* x, y

$$U(x) = \begin{cases} k(+)\, x^{\beta(+)}, & \text{if } x > e, \\ -k(-)(-x)^{\beta(-)}, & \text{if } x < e. \end{cases} \tag{23}$$

The operations are of the form: There exist constants S_i^j, $i = >, <$, *and* $j = +, -$, *such that* $(\forall x, y \in \mathrm{Re}^j)$,

$$x \oplus y = (x^{\beta(j)} + y^{\beta(j)})^{1/\beta(j)} \tag{24}$$

$$x \otimes y = \begin{cases} [x^{\beta(j)} S_>^j + y^{\beta(j)}(1 - S_>^j)]^{1/\beta(j)}, & \text{if } x \succsim y \\ [x^{\beta(j)} S_<^j + y^{\beta(j)}(1 - S_<^j]^{1/\beta(j)}, & \text{if } x \prec y. \end{cases} \tag{25}$$

Proof. We work out the proof in detail for the case $j = +$; the argument is similar for $j = -$. Since $\oplus$ is isomorphic to the additive positive reals and is a unit structure, it is known [17, Theorem 2.7] that the isomorphism must be a power function, and the form for $\oplus$, Eq. (24) follows immediately. To get the form for $\otimes$ one substitutes Eq. (24) into the distribution equation, Eq. (22). Writing the operation $\otimes$ as a function G of two variables yields the functional equation

$$G(x, y)^{\beta(+)} + z^{\beta(+)} = G[(x^{\beta(+)} + z^{\beta(+)})^{1/\beta(+)}, (y^{\beta(+)} + z^{\beta(+)})^{1/\beta(+)}]^{\beta(+)}.$$

Observe that since $\otimes$ is a unit structure, for positive u,

$$G(xu, yu) = G(x, y)u,$$

Setting $X = (x/z)^{\beta(+)}$, $Y = (y/z)^{\beta(+)}$, $F(x, y) = G[x^{1/\beta(+)}), \; y^{1/\beta(+)}]^{\beta(+)}$, these two equations reduce to

$$F(XU, YU) = F(X, Y)U, \tag{26}$$

$$F(X, Y) + 1 = F(X+1, Y+1). \tag{27}$$

By induction on Eq. (27), for all positive integers

$$F(X, Y) + M = F(X+M, Y+M),$$

whence for all positive integers M and N

$$F(NX, NY) + M = F(NX + M, NY + M).$$

Using Eq. (26), this is true if and only if

$$F(X, Y) + M/N = F(X + M/N, Y + M/N).$$

By Assumption (i), F is from $\mathrm{Re}^+ \times \mathrm{Re}^+$ onto Re^+ and it is strictly increasing in each variable, so it is continuous, whence for all $Z \in \mathrm{Re}^+$,

$$F(X, Y) + Z = F(X+Z, Y+Z). \tag{28}$$

Using continuity, extend Eqs. (26) and (28) to include 0 in the domain. Following the method on p. 235 of [1] observe that by Eq. (28)

$$F(X, Y) = \begin{cases} F(X - Y + Y, 0 + Y) = F(X - Y, 0) + Y, & \text{if } X \geqslant Y \\ F(0 + X, Y - X + X) = F(0, Y - X) + X, & \text{if } X < Y. \end{cases}$$

Applying Eq. (26),

$$F(X, Y) = \begin{cases} F(1, 0)(X - Y) + Y, & \text{if } X \geqslant Y \\ F(0, 1)(Y - X) + X, & \text{if } X < Y. \end{cases}$$

Setting $S_>^+ = F(1, 0)$ and $S_<^+ = 1 - F(0, 1)$ and substituting back through G yields the form asserted. ∎

A Direct Axiomatization of Additive RSD Utility

The only way I know to axiomatize the RSD utility representation of Eq. (3) in a direct fashion is to involve, in addition to the operations $\bigcirc_E$ for each event E, an operation $\oplus$ of jointly receiving two gambles and to postulate that utility is additive over that operation. It would be desirable to find an axiomatization that only involves the mixtures of gambles.

THEOREM 4. *Let* $\langle X, \mathscr{E}, \succsim, \oplus, \bigcirc_E, e\rangle_{E\in\mathscr{E}}$ *be a structure in which X is a set;* $\mathscr{E}$ *is a collection of subsets of that excludes the universal set and is closed under complementation;* $\succsim$ *is a binary relation on X;* $\oplus$ *and, for each* $E\in\mathscr{E}$, $\bigcirc_E$ *are binary operations on X; and* $e\in X$. *Let* X^i, $i=+,-$, *denote, respectively, the positive and the negative elements of the structure. Suppose the following axioms are met:*

(i) $\mathscr{X}=\langle X, \succsim, \oplus, e\rangle$ *is a Dedekind complete, ordered group with identity e that is order dense.*

(ii) $(\forall E\in\mathscr{E})$ $\bigcirc_E$ *is monotonic relative to* $\succsim$.

(iii) (Distribution) $(\forall E\in\mathscr{E})$ *and either* $(\forall a, b, c\in X^+\cup\{e\})$ *or* $(\forall a, b, c\in X^-\cup\{e\})$, *then*

$$(a \bigcirc_E b)\oplus c \sim (a\oplus c) \bigcirc_E (b\oplus c) \tag{29a}$$

$$(a\oplus b) \bigcirc_E e \sim (a \bigcirc_E e)\oplus (b \bigcirc_E e) \tag{29b}$$

$$e \bigcirc_E (a\oplus b) \sim (e \bigcirc_E a)\oplus (e \bigcirc_E b) \tag{29c}$$

(iv) $(\forall E\in\mathscr{E})$ $(\forall a, b\in X)$ *if either* $a\succsim e\succsim b$ *or* $b\succsim e\succsim a$, *then*

$$a \bigcirc_E b \sim (a \bigcirc_E e)\oplus (e \bigcirc_E b). \tag{30}$$

(v) $(\forall E\in\mathscr{E})$ $(\forall a\in X)$ *there exists some* $b\in X$ *such that* $b \bigcirc_E e\sim a$ *and some* $c\in X$ *such that* $e \bigcirc_E c\sim a$.

(vi) *The group of automorphisms of* $\mathscr{X}^i_E=\langle X^i, \succsim, \bigcirc_E\rangle$, $i=+,-$, *is finitely unique.*

Then, there is an order preserving representation U that is additive over $\oplus$ *and for which there exist functions* S^j_i *in* $[0, 1]$ *such that Eq.* (3) *holds.*

LEMMA 1. *There exists an isomorphism U between* $\mathscr{X}$ *and* $\langle \mathrm{Re}, \geqslant, +, 0\rangle$.

Proof. First, we establish that $\mathscr{X}$ is positive, i.e., if $a, b\in X^+$, then $a\oplus b\geqslant \max(a, b)$. By monotonicity, $a\oplus b> a\oplus e=a$ and $a\oplus b> e\oplus b=b$. Further, it is left solvable since if $a>b$, then using associativity, $a=(a\oplus -b)\oplus b$, where $-b$ is the inverse of b under $\oplus$. Thus, by [16, Theorem 2.1], $\mathscr{X}$ is an Archimedean ordered group. By Hölder's theorem, there is an isomorphic representation U into the additive reals. By order density, Dedekind completeness, and positiveness, it is onto the reals. ∎

LEMMA 2. $(\forall E\in\mathscr{E})$ *e is a generalized zero of* $\mathscr{X}_E=\langle X, \succsim, \bigcirc_E\rangle$ *in the* *sense that there are functions σ and η, dependent on E, from X onto X such that* $(\forall a \in X)$

$$a \bigcirc_E e \sim \sigma(a) \qquad \text{and} \qquad e \bigcirc_E a \sim \eta(a),$$

and their restrictions to $X^i \cup \{e\}$, $i = +, -$, *agree with translations of* $\mathscr{X}$.

Proof. By assumption (ii), $a \bigcirc_E e \in X^i \cup \{e\}$ iff $a \in X^i \cup \{e\}$. Thus, for some function σ from $X^i \cup \{e\}$ into $X^i \cup \{e\}$, $a \bigcirc_E e \sim \sigma(a)$. By assumption (ii), σ is order preserving. By assumptions (ii) and (v) σ is onto. Moreover, it agrees with an automorphism, and so a translation, of $\mathscr{X}^i$ since, using Eq. (29b),

$$\sigma(a \oplus b) \sim (a \oplus b) \bigcirc_E e \sim (a \bigcirc_E e) \oplus (b \bigcirc_E e) \sim \sigma(a) \oplus \sigma(b).$$

The proof for $e \bigcirc_E a$ is similar. ∎

Transform $\oplus$, restricted to X^i, $i = +, -$, into a conjoint structure in the usual way:

$$(a, b) \succsim' (c, d) \qquad \text{iff} \quad a \oplus b \succsim c \oplus d.$$

LEMMA 3. $\bigcirc_E$ *restricted to* X^i *distributes in* $\langle X^i \times X^i, \succsim' \rangle$.

Proof. It is easy to verify that this conjoint structure is Archimedean, satisfies the Thomsen condition, and is solvable in both coordinates because of assumption (i). To show distribution, suppose $(a, f) \sim' (c, g)$ and $(b, f) \sim' (d, g)$; then, by definition, $a \oplus f \sim c \oplus g$ and $b \oplus f \sim d \oplus g$. By assumption (ii),

$$(a \oplus f) \bigcirc_E (b \oplus f) \sim (c \oplus g) \bigcirc_E (d \oplus g).$$

Applying Eq. (29a) to both sides and translating back into the conjoint notation, we have $(a \bigcirc_E b, f) \sim' (c \bigcirc_E d, g)$. ∎

LEMMA 4. $(\forall E \in \mathscr{E})$ $\bigcirc_E$ *is idempotent; i.e.,* $(\forall a \in X) a \bigcirc_E a \sim a$.

Proof. Since e is an identity of $\oplus$, Eq. (29c) yields

$$e \bigcirc_E e \sim e \bigcirc_E (e \oplus e) \sim (e \bigcirc_E e) \oplus (e \bigcirc_E e),$$

which by Lemma 1 is possible only if $e \bigcirc_E e \sim e$. Using this, the fact e is an identity of $\oplus$, and Eq. (29a),

$$a \sim e \oplus a \sim (e \bigcirc_E e) \oplus a \sim (e \oplus a) \bigcirc_E (e \oplus a) \sim a \bigcirc_E a. \quad ∎$$

LEMMA 5. *The translations of* $\mathscr{X}_E^i$ *are closed under function composition and so form a group.*

Proof. Suppose the translations of $\mathscr{X}_E^i$ are not closed under function composition, i.e., for some translations σ and τ and for some $a \in X^i$, $\sigma\tau(a) \sim a$. By assumption (v), let b be such that $b \bigcirc_E e \sim a \sim \sigma\tau(a)$. Thus, $\tau^{-1}\sigma^{-1}(b) \bigcirc_E e \sim a \sim b \bigcirc_E e$. By assumption (ii), $\tau^{-1}\sigma^{-1}(b) \sim b$ and so b is a fixed point of $\sigma\tau$. By Lemma 4, $b \bigcirc_E e \sim a \sim a \bigcirc_E a$, whence by assumption (ii) either $b > a > e$ or $b < a < e$. By induction, $\sigma\tau$ has countably many fixed points, which is impossible by assumption (vi). ∎

LEMMA 6. *The translations of $\mathscr{X}_E$ are translations of $\mathscr{X}$.*

Proof. Consider first just the positive part. By Lemmas 3 and 5, [11, Theorem 5.1] establishes that $\mathscr{X}_E^+$ is a homogeneous unit structure and its translations form an Archimedean ordered group, and so its translations commute. By [11, Theorem 5.2] these translations are automorphisms of the conjoint structure and so of $\mathscr{X}^+$. The argument is similar over the negative elements.

So we turn to the case where $a > e$ and $b > e$. Let α be any translation of $\mathscr{X}_E$ and let σ and η be the functions defined in Lemma 2:

$$\begin{aligned}
\alpha(a \oplus b) &\sim \alpha[\sigma\sigma^{-1}(a) \oplus \eta\eta^{-1}(b)] && \text{(definition of inverse)}\\
&\sim \alpha[(\sigma^{-1}(a) \bigcirc_E e) \oplus (e \bigcirc_E \eta^{-1}(b))] && \text{(Lemma 2)}\\
&\sim \alpha[\sigma^{-1}(a) \bigcirc_E \eta^{-1}(b)] && \text{(Eq. (30))}\\
&\sim \alpha\sigma^{-1}(a) \bigcirc_E \alpha\eta^{-1}(b) && (\alpha \text{ is a translation of } \mathscr{X}_E)\\
&\sim \sigma^{-1}\alpha(a) \bigcirc_E \eta^{-1}\alpha(b) && (\alpha \text{ is a translation of } \mathscr{X}_E \text{ and so of } \mathscr{X}^i, \text{ whence } \alpha \text{ commutes with } \sigma^{-1} \text{ and } \eta^{-1})\\
&\sim (\sigma^{-1}\alpha(a) \bigcirc_E e) \oplus (e \bigcirc_E \eta^{-1}\alpha(b)) && \text{(Eq. (30))}\\
&\sim \alpha(a) \oplus \alpha(b) && \text{(Lemma 2).}
\end{aligned}$$ ∎

Proof of Theorem. Suppose $E \in \mathscr{E}$ and $a \in X^+$. By Lemmas 1, 2, and 6, there exists a constant, denote it $S_>^+(E)$, such that

$$U(a \bigcirc_E e) = U[\sigma(a)] = S_>^+(E)\, U(a).$$

Now, consider any $a, b \in X^+$ with $a > b$. Since U is onto the reals, there exists some c with $U(c) = U(a) - U(b)$. So $c > e$ and $a = b \oplus c$. Using (ii) and Eq. (29a)

$$a \bigcirc_E b \sim (c \oplus b) \bigcirc_E (e \oplus b) \sim (c \bigcirc_E e) \oplus b.$$

Thus,

$$\frac{U(a \bigcirc_E b) - U(b)}{U(a) - U(b)} = \frac{U[(c \bigcirc_E e) \oplus b] - U(b)}{U(c)} = \frac{U(c \bigcirc_E e)}{U(c)} = S_{>}^{+}(E).$$

By assumption (ii), $a \gtrsim a \bigcirc_E b \gtrsim b$, and so the ratio lies in $[0, 1]$.

Similar definitions and proofs can be given for the three other cases, $(<, +)$, $(>, -)$, and $(<, -)$. This completes the representation in the positive and negative domains. The mixed ones are treated as follows. By Eq. (30), if $a \gtrsim e \gtrsim b$, then

$$\begin{aligned} U(a \bigcirc_E b) &= U(a \bigcirc_E e) + U(e \bigcirc_E b) \\ &= U(a)\, S_{>}^{+}(E) + U(e)[1 - S_{>}^{+}(E)] \\ &\quad + U(e)\, S_{>}^{-}(E) + U(b)[1 - S_{>}^{-}(E)] \\ &= U(a)\, S_{>}^{+}(E) + U(b)[1 - S_{>}^{-}(E)]. \end{aligned}$$

The case with $b \gtrsim e \gtrsim a$ is similar. ▮

It is worth observing that the two crucial assumptions, Eqs. (29) and (30), are both implied by the representation of Eq. (3) provided that additivity holds over $\oplus$. Moreover, they are the qualitative analogues of the conditions Eqs. (19) and (20) imposed earlier on the canonical representation of a concatenation structure with a generalized zero. And Eq. (21) imposed in Theorem 3 is the specialization of Eq. (29a) to money gambles. It is clear that these are, in principle, testable assumptions. As we have done in the proof, the main role of these assumptions is to permit the analysis of gambles into ones of the form either $a \bigcirc_E e$ or $e \bigcirc_E a$, which in turn are just translations of a (Lemma 2). For gambles that involve only positive consequences, one factors out the minimum of a and b using Eq. (29a). For those that involve only negative ones, one factors out the maximum. And for those that involve both positive and negative outcomes, a and $-b$, Eq. (30) says that the decision maker thinks of this as the simple sum of two gambles: one involving a and e and the other involving e and $-b$. These simplifications, which do not seem implausible although they certainly are not rational, are the primary basis of the representation of Eq. (3).

In the empirical literature, the right side of Eq. (30) is called a "duplex gamble," and Slovic [23, p. 223] remarks "It can be argued that this type of gamble is as faithful an abstraction of real life decision situations as its more commonly studied counterpart in which [the probability of a loss is 1 minus the probability of a win]." He does not comment on the degree to which he believes people treat them as indifferent.

It is also worth observing that assumption (vi) of Theorem 3 is neither really testable nor in a very desirable form. One would like an axiomatization in which additional qualitative axioms permitted it to be proved as a lemma.

COROLLARY 1. *Under the assumptions of Theorem* 4, *suppose further that*

(i) $\mathrm{Re}^j \subseteq X^j$, $j = +, -$;

(ii) $\langle \mathrm{Re}^j, \geqslant, \oplus \rangle$ *is a unit structure;*

(iii) $\langle \mathrm{Re}^j, \geqslant, \bigcirc_E \rangle$ *is a unit structure.*

Then U *over* Re^j *is of the form given in Eq.* (23), *and* $U(e) = 0$.

Proof. Immediate from Theorems 3 and 4. ∎

The following corollary makes clear which qualitative conditions correspond to the three special subcases discussed earlier.

COROLLARY 2. *Under the conditions of Theorem* 4,

(i) *the sign-dependent representation,* $S^j_> = S^j_<$, $j = +, -$, *obtains iff* $(\forall a \in X)$

$$(a \bigcirc_E e) \oplus (e \bigcirc_E a) \sim a; \tag{31}$$

(ii) *the rank-dependent representation,* $S^+_i = S^-_i$, $i = >, <$, *obtains iff* $(\forall a \in X)$

$$(a \bigcirc_E e) \sim a \oplus (e \bigcirc_E -a) \qquad \text{and} \qquad (e \bigcirc_E a) \sim a \oplus (-a \bigcirc_E e), \tag{32}$$

where $a \oplus -a \sim e$;

(iii) *The prospect theory representation,* $S^+_> = S^-_<$ *and* $S^-_> = S^+_<$, *obtains iff* $(\forall a \in X)$

$$(a \bigcirc_E e) \oplus (-a \bigcirc_E e) \sim e \sim (e \bigcirc_E a) \oplus (e \bigcirc_E -a). \tag{33}$$

Proof. (i) Setting $S^+_> = S^+_<$, we see that for $a > e$

$$U(a \bigcirc_E e) = U(a)\, S^+_> = U(a)\, S^+_< = U(a) - U(e \bigcirc_E a),$$

from which the conclusion, Eq. (31), follows immediately. The argument for $a < e$ is similar and leads to the same conclusion.

(ii) Setting $S^+_> = S^-_>$ and using the fact $U(-a) = -U(a)$, we see that for $a > e$,

$$U(a \bigcirc_E e) = U(a)\, S^+_> = U(a)\, S^-_> = U(a) + U(e \bigcirc_E -a),$$

whence $(a \bigcirc_E e) \sim a \oplus (e \bigcirc_E -a)$. Beginning with $e \bigcirc_E a$ and $S^+_< = S^-_<$ yields the other equivalence. The assumption $a < e$ yields the same conclusions.

(iii) Similar to (i) and (ii). ▌

Observe that Eq. (31) is a special case of the property that we earlier noted makes suspect the pure sign-dependent theory coupled with additivity. Similarly, Eq. (33) of prospect theory together with additive utility seems doubtful empirically. It says that the status quo e is indifferent to a gamble in which the consequences are e iff either (E, E) or $(\neg E, \neg E)$ occur, a iff $(\neg E, E)$, and $-a$ iff $(E, \neg E)$. This does not seem likely for either risk-averse or risk-seeking decision makers. In contrast, the property characterizing the purely rank-dependent theory does not seem implausible: it says that the gamble (a if E and e otherwise) is indifferent to ($a \oplus e = a$ if E and $a \oplus -a = e$ otherwise).

CONCLUSIONS

The paper explored a weighted linear utility representation for binary gambles in which the weights depend on both the rank order and the sign of the consequences, Eq. (3). Three specializations of this additive RSD utility model reduced the number of weighting functions from four to two. The pure sign-dependent model and prospect theory were both shown to lead to highly unlikely predictions. In practice, prospect theory avoids the problem by not accepting the additivity of utility over independent gambles. The pure rank-dependent model does not imply the same unacceptable prediction; however, it suffers from its implication of the accounting equivalence (5), $(a \bigcirc_E b) \bigcirc_F b \sim (a \bigcirc_F b) \bigcirc_E b$, which empirically seems not to be widely true.

Assuming an operation $\oplus$ of joint receipt of gambles and utility U additive over $\oplus$, relations among buying price, selling price, and choice indifference were explored. A caution was sounded over the fact that choice and judged monetary indifferences do not always agree, and an as yet untested theory for that difference was offered in terms of buying and selling prices. A simple explanation, also in terms of buying and selling prices, was provided for the fact that people may buy both lotteries and insurance. Assuming further that for money $x \oplus y = x + y$, which forces utility to be proportional to money, a simple method was provided for estimating the four weighting functions.

An axiomatization of additive, rank- and sign-dependent utility (Theorem 4) was based on the family of mixing operations over events and an operation of jointly receiving two independent gambles. Because of the asymmetric way in which people respond to gambles having both positive and negative outcomes, it is possible to admit additivity of utility only by either rejecting the hypothesis that the joint receipt of two sums of money is the same as receiving their sum or having the full complexity of Eq. (3). The key axioms, Eqs. (29) and (30), both involve methods for transforming a gamble with two non-null consequences into the consideration of gambles having just one non-null consequence. Furthermore, if money gambles are included, it follows that the utility of money is a two-sided power function (Corollary 1 of Theorem 4).

It should be remarked that binary theories that are weaker than SEU do not automatically deal with more complex gambles. A more general RD theory was suggested in [12], but no general RSD theory has yet been published, although P.C. Fishburn and I [15] have a paper in press on such a theory.

Note added in proof. Subsequent to this paper having been accepted, several people have reported in Volts axiomatizations of the RSD representation that do not involve the additional operation of joint receipt. Those people are: Chew So-Hong, A. Tversky and D. Kahneman, and P. P. Wakker.

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