Econometrica, Vol. 77, No. 5 (September, 2009), 1575–1605

AMBIGUITY AND SECOND-ORDER BELIEF

BY KYOUNGWON SEO[1]

Anscombe and Aumann (1963) wrote a classic characterization of subjective expected utility theory. This paper employs the same domain for preference and a closely related (but weaker) set of axioms to characterize preferences that use second-order beliefs (beliefs over probability measures). Such preferences are of interest because they accommodate Ellsberg-type behavior.

KEYWORDS: Ambiguity, Ellsberg paradox, second-order belief.

1. INTRODUCTION

THE ELLSBERG (1961) PARADOX has raised questions about the subjective expected utility model and has stimulated development of a number of more general theories. In one version of the paradox (Ellsberg (2001, p. 151)), there is an urn known to contain 200 balls of four colors: R_{I}, B_{I}, R_{II}, and B_{II}. R_{I} and R_{II} denote two different shades of red; similarly, B_{I} and B_{II} denote two different shades of blue. The urn is known to contain 50 R_{II} balls and 50 B_{II} balls, but the number of R_{I} (or B_{I}) balls is unknown. One ball is to be drawn from the urn. Consider the following six bets on the color of the ball that is drawn:

	100		50	50
	R_{I}	B_{I}	R_{II}	B_{II}
A	\$100	\$0	\$0	\$0
B	\$0	\$100	\$0	\$0
C	\$0	\$0	\$100	\$0
D	\$0	\$0	\$0	\$100
AB	\$100	\$100	\$0	\$0
CD	\$0	\$0	\$100	\$100

Bet A gives \$100 if the drawn ball is R_{I} and \$0 otherwise. The other bets are interpreted similarly. Many subjects rank $C \sim D \succ A \sim B$ and $AB \sim CD$. Subjective expected utility (SEU) cannot accommodate this behavior.

[1]I am indebted to Larry G. Epstein for his illuminating guidance and invaluable advice. I thank D. Ahn, P. Barelli, P. Ghirardato, F. Gul, I. Kopylov, B. Lipman, M. Marinacci, K. Nehring, B. Polak, and P. Wakker for helpful discussions and suggestions. I also thank audiences at the University of Rochester, Boston University, the University of Chicago, Washington University in St. Louis, Rice University, Yale University, Northwestern University, the Canadian Economic Theory Conference, the RUD'06 Workshop on Risk, Utility and Decision, and the 2006 NBER/NSF/CEME Conference on General Equilibrium. Finally, the detailed comments of the co-editor and the referees were extremely valuable.

 DOI: 10.3982/ECTA6727

One explanation of this behavior is that the agent has in mind a second-order belief or a probability measure on probability measures. The agent forms a belief on the proportion of the R_{I} balls or the type of the urn, and then translates bets into two-stage lotteries. Segal (1987) adopted this approach to accommodate the above ranking. He used anticipated utility theory (see Quiggin (1982), for example). However, he was well aware that "modelling the Ellsberg paradox as a two-stage lottery does not depend on anticipated utility theory," but on nonreduction of two-stage lotteries. Klibanoff, Marinacci, and Mukerji (2005) (henceforth KMM) proposed and axiomatized a utility representation of preference involving the standard expectations of utilities, where nonreduction of a second-order belief is key to accommodating the Ellsberg paradox. Nau (2006) and Ergin and Gul (2009) characterized utility representations that, at least in special cases of their models, can be interpreted similarly to that of KMM.

This paper provides a new axiomatization for a model of preference involving a second-order belief. An important difference from the cited models lies in what is assumed about the domain of preference. Models of preference typically model the ranking not only of bets, but also of all other acts—an *act* over a state space S is a (measurable) function from S into the set of outcomes. In the Ellsberg case, the natural state space is

$$S_E = \{R_{\mathrm{I}}, B_{\mathrm{I}}, R_{\mathrm{II}}, B_{\mathrm{II}}\}$$

and bets are binary acts over S_E. I now use the Ellsberg setting to highlight the noted difference in assumptions about the domain.

KMM assumed two subdomains and two corresponding preferences. One subdomain consists of acts on S_E and a preference is given over this set of acts. For the other subdomain, they introduced another state space $\Delta(S_E)$, the set of all probability measures over S_E. Each probability measure over S_E corresponds to a particular number of R_{I} balls in the Ellsberg urn. KMM called an act over $\Delta(S_E)$ a second-order act. They assumed that the preference over second-order acts is an SEU preference, which leads immediately to second-order belief.[2]

Ergin and Gul permitted issue preference, called source dependence in Nau.[3] They assumed two issues and their state space is a product space. In the Ellsberg context, one issue is which ball is drawn and the other issue is what color is each ball. The second issue determines the type of the urn and hence a probability measure over S_E. Given preference on acts over the product state space they proved a representation involving a first-order belief over the product state space that can be interpreted as a second-order belief over the first issue.

[2]See Section 6 for further discussion of the relation between the KMM representation result and the main theorem in this paper.

[3]Nau allowed state dependence.

Therefore, KMM, Nau, and Ergin and Gul assume state spaces bigger than S_E. They presumed that the analyst can observe more than just the ranking of acts over the color of the drawn ball—the ranking of acts over the "type of the urn" must also be observable. Similar remarks apply to their model in general (not only Ellsbergian) settings.

The importance of the domain assumption can be illustrated in the context of an asset market. Consider a simple model where the asset price may go up (H) or go down (L). In this setting, a bet on H corresponds to buying the asset and a bet on L corresponds to selling the asset—decisions that are observed in many data sets. On the other hand, a second-order act (or a bet on the second issue) is a bet on the true nature of the market—the *probability* that the price goes up. But we do not observe bets on the true probability; that is, the payoffs of real-world securities depend on realizations of prices, and not separately on the mechanism that generates these realizations.

This paper adopts a domain consisting of lotteries over acts defined over a basic state space, which is S_E in the Ellsberg case. Arguably, this domain is closer to the set of choices involved in the Ellsberg paradox than are the domains of KMM, and Ergin and Gul. In addition, the domain in this paper is the same as that in Anscombe and Aumann (1963), one of the classic papers on SEU. Frequently, "the Anscombe–Aumann domain" is taken to be the set of all acts whose prizes are lotteries (see Kreps (1988), for example). Note, however, that in their paper, Anscombe–Aumann used the set of all lotteries over such acts. Thus, the choice objects in this paper and Anscombe–Aumann have three stages.

The model in this paper, referred to as second-order subjective expected utility (SOSEU), has the representation[4]

$$V(P) = \int U(f)\,dP(f) \quad \text{and} \quad U(f) = \int_{\Delta(S)} v\left(\int_S u(f)\,d\mu\right) dm(\mu),$$

where P is a lottery over acts, f is an act, and m is a second-order belief (a probability measure on $\Delta(S)$). Degenerate lotteries can be identified with acts and thus V induces the utility function U over acts. When v is linear, V collapses to Anscombe–Aumann's SEU.

SOSEU has axiomatic foundations different from SEU. In their characterization of SEU, Anscombe and Aumann assumed order, continuity, independence, reversal of order, and dominance. I drop reversal of order and modify dominance to characterize SOSEU.

The functions u and v in SOSEU are cardinally unique. Thus it is possible to discuss connections between the properties of the functions and preference. In particular, v captures ambiguity attitude: if v is concave, then preference exhibits Ellsbergian behavior. As usual, u characterizes attitude toward risk.

[4]Technical details are provided later.

The interpretations of u and v are considered in detail in KMM; Nau, and Ergin and Gul have related results and discussions.

The domain in this paper makes it possible to analyze attitudes toward ambiguity and two-stage lotteries at the same time. Specifically, SOSEU has the property that if the agent reduces two-stage lotteries to one-stage lotteries in the usual way, then he does not exhibit Ellsberg-type behavior. This prediction is confirmed in the experiment by Halevy (2007). He claimed that a descriptive theory of ambiguity aversion "should account, at the same time, for violation of reduction of compound *objective* lotteries."

The paper is organized as follows: Section 2 introduces the setup. In Section 3, Anscombe and Aumann's axioms and theorem are presented. Section 4 motivates dropping their axiom reversal of order, and modifying dominance. This leads to the SOSEU representation theorem. Section 5 examines the connection between nonindifference to ambiguity and violation of reduction of two-stage lotteries. Section 6 discusses related literature. Proofs are contained in the Appendices.

2. THE SETUP

For any topological space X, let $\Delta(X)$ be the set of all Borel probability measures on X, and let $C_b(X)$ be the set of all bounded continuous functionals on X. Endow $\Delta(X)$ with the weak convergence topology, that is, for $\nu_n, \nu \in \Delta(X)$, $\nu_n \to \nu$ if $\int \eta\, d\nu_n \to \int \eta\, d\nu$ for every $\eta \in C_b(X)$. If X is a separable metric space, so is $\Delta(X)$. (See Aliprantis and Border (1999, p. 482); these authors are henceforth denoted AB.) Let $\mathcal{B}_X$ denote the Borel σ-algebra on X and denote by $\delta_x \in \Delta(X)$ a point mass on X, defined by $\delta_x(A) = 0$ if $x \notin A$ and by $\delta_x(A) = 1$ if $x \in A$.

Let $S = \{s_1, s_2, \dots, s_{|S|}\}$ be a finite set of states. Let Z denote a set of outcomes or prizes, where Z is a separable metric space. An act f is a function from S into $\Delta(Z)$. Let $\mathcal{H}$ be the set of all acts endowed with the product topology. Preference $\succeq$ is defined on $\Delta(\mathcal{H})$.

I refer to an element in $\Delta(Z)$ as a one-stage lottery and refer to an element in $\Delta(\Delta(Z))$ as a two-stage lottery (or a compound lottery). A constant act (an act taking the same value for every $s \in S$) is viewed also as a one-stage lottery. Moreover, any act f is identified with δ_f. Then it is immediate that $\Delta(Z) \subset \mathcal{H} \subset \Delta(\mathcal{H})$ and hence $\Delta(\Delta(Z)) \subset \Delta(\mathcal{H})$. Therefore, the preference $\succeq$ induces rankings on $\mathcal{H}$, $\Delta(Z)$, and $\Delta(\Delta(Z))$.

Typical elements in $\Delta(\mathcal{H})$ are denoted by P, Q, and R. I use f, g, and h for elements in $\mathcal{H}$. In addition, $\bar{P}$, $\bar{Q}$, and $\bar{R}$ are typical elements for $\Delta(\Delta(Z))$, and p, q, and r are typical elements for $\Delta(Z)$. Denote by $(x_1, \alpha_1; \dots; x_n, \alpha_n)$ a lottery that gives x_1 with probability α_1 and so on, where $x_1, x_2, \dots, x_n$ can be outcomes, lotteries, or acts.

A typical object P in $\Delta(\mathcal{H})$ is depicted in Figure 1.

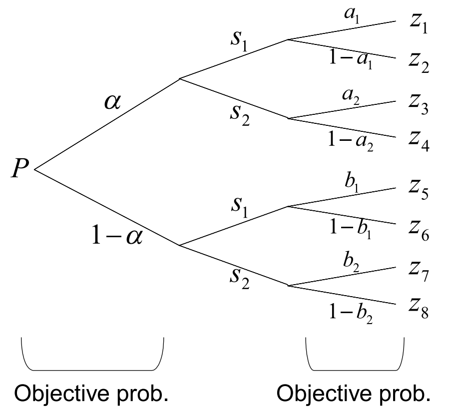

FIGURE 1.—A typical element in $\Delta(\mathcal{H})$. The first and the last nodes are governed by objective probabilities α, a_1, a_2, b_1, and b_2. The second node is selected according to the realized state s_1 or s_2.

3. THE ANSCOMBE–AUMANN MODEL

Preferences having an SEU form on $\Delta(\mathcal{H})$ were characterized by Anscombe and Aumann (1963) (henceforth AA). Using the notations and definitions of this paper, AA's axioms and theorem can be restated.[5]

AXIOM 1—Order: *$\succeq$ is complete and transitive.*

AXIOM 2—Continuity: *$\succeq$ is continuous.*

DEFINITION 1: For $f, g \in \mathcal{H}$ and $\alpha \in [0, 1]$, $\alpha f \oplus (1-\alpha)g \in \mathcal{H}$ is a componentwise mixture, that is, for every $s \in S$ and every $B \in \mathcal{B}_Z$, $(\alpha f \oplus (1-\alpha)g)(s)(B) = \alpha f(s)(B) + (1-\alpha)g(s)(B)$. This operation is referred to as a second-stage mixture.

AXIOM 3—Second-Stage Independence: *For any $\alpha \in (0, 1]$ and one-stage lotteries $p, q, r \in \Delta(Z)$,*

$$\alpha p \oplus (1-\alpha)r \succeq \alpha q \oplus (1-\alpha)r \quad \Longleftrightarrow \quad p \succeq q.$$

Consider two lotteries $\alpha p \oplus (1-\alpha)r$ and $\alpha q \oplus (1-\alpha)r$. Both give the same prize r with probability $(1-\alpha)$. The two lotteries differ only in the α-probability event. So it is intuitive that the agent's ranking between them depends only on the ranking between p and q, regardless of the common prize r.

[5]Actually, they do not state the first four axioms—order, continuity, second-stage independence, and first-stage independence. Instead, they assume expected utility functions on $\Delta(\mathcal{H})$ and $\Delta(Z)$, respectively.

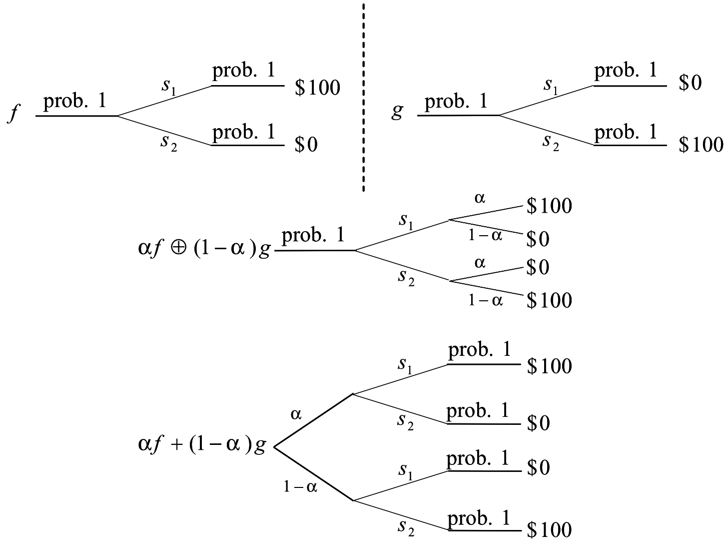

FIGURE 2.—Examples of mixture operations: $f \in \mathcal{H}$ gives \$100 if s_1 is realized and \$0 if s_2 is realized; $g \in \mathcal{H}$ gives \$0 for s_1 and \$100 for s_2. The second-stage mixture $\alpha f \oplus (1-\alpha)g \in \mathcal{H}$ is an act that gives the lottery $(\$100, \alpha; \$0, 1-\alpha)$ for s_1 and the lottery $(\$0, \alpha; \$100, 1-\alpha)$ for s_2. The first-stage mixture $\alpha f + (1-\alpha)g \in \Delta(\mathcal{H})$ is the lottery $(f, \alpha; g, 1-\alpha)$.

DEFINITION 2: For $P, Q \in \Delta(\mathcal{H})$ and $\alpha \in [0,1]$, $\alpha P + (1-\alpha)Q \in \Delta(\mathcal{H})$ is a lottery such that $(\alpha P + (1-\alpha)Q)(B) = \alpha P(B) + (1-\alpha)Q(B)$ for $B \in \mathcal{B}_{\mathcal{H}}$. This operation is called a first-stage mixture. For simplicity, I write $\alpha f + (1-\alpha)g$ instead of $\alpha\delta_f + (1-\alpha)\delta_g$ for any acts f and g.

See Figure 2 for examples illustrating the mixture operations.

AXIOM 4—(First-Stage Independence): *For any* $\alpha \in (0,1]$ *and lotteries* $P, Q, R \in \Delta(\mathcal{H})$,

$$\alpha P + (1-\alpha)R \succeq \alpha Q + (1-\alpha)R \iff P \succeq Q.$$

First-stage independence can be interpreted in a way similar to second-stage independence.

AXIOM 5—Reversal of Order: *For every* $f, g \in \mathcal{H}$ *and* $\alpha \in [0,1]$, $\alpha f \oplus (1-\alpha)g \sim \alpha f + (1-\alpha)g$.

Reversal of order assumes that the agent is not concerned about whether the mixture operation is taken before or after the realization of the state. Later, I will discuss an argument against this axiom.

AXIOM 6—AA Dominance: *Let* $f, g \in \mathcal{H}$ *and* $s \in S$. *If* $f(s') = g(s')$ *for all* $s' \neq s$ *and* $f(s) \succeq g(s)$, *then* $f \succeq g$.

This axiom says that when two acts give the identical prizes except in one state s, the prizes in state s determine the agent's ranking between the two acts.

DEFINITION 3: An SEU representation is a bounded continuous mixture linear function $u : \Delta(Z) \to \mathbb{R}$ and a probability measure $\mu \in \Delta(S)$ such that V^{AA} represents $\succeq$ on $\Delta(\mathcal{H})$, where

$$V^{\mathrm{AA}}(P) = \int_{\mathcal{H}} U^{\mathrm{AA}}(f)\, dP(f) \quad \text{and} \quad U^{\mathrm{AA}}(f) = \int_S u(f)\, d\mu.$$

AA's theorem can be restated.[6]

THEOREM 3.1—AA (1963): *Preference $\succeq$ on $\Delta(\mathcal{H})$ satisfies order*, *continuity*, *second-stage independence*, *first-stage independence*, *reversal of order*, *and AA dominance if and only if it has an SEU representation*.

An SEU representation cannot accommodate Ellsberg-type behavior. Therefore, I proceed to develop a generalization of this model.

4. MAIN REPRESENTATION THEOREM

Here I show that by dropping reversal of order and modifying AA dominance, one obtains a model of preference that can accommodate nonindifference to ambiguity.

Consider the following example that illustrates that reversal of order is problematic given ambiguity. In the Ellsberg example described in the Introduction, let f be the act that gives \$100 if the chosen ball is red (R_{I} or R_{II}) and nothing otherwise; g gives \$100 if the ball drawn is blue (B_{I} or B_{II}) and nothing otherwise. Let p be (\$100, 1/2; \$0, 1/2). As Ellsberg predicted and later experiments confirmed, many people feel indifferent between f and g, but strictly prefer p to f and p to g.

Compare $\frac{1}{2}f + \frac{1}{2}g$ and $\frac{1}{2}f \oplus \frac{1}{2}g$ (see Figure 3). The first-stage mixture $\frac{1}{2}f + \frac{1}{2}g$ gives ambiguous acts f or g. If the agent strictly prefers p to f and p to g, it is reasonable to assume that he strictly prefers p to $\frac{1}{2}f + \frac{1}{2}g$ by the intuition of first-stage independence. On the other hand, the second-stage mixture $\frac{1}{2}f \oplus \frac{1}{2}g$ has no ambiguity and can be identified with p because it yields the lottery p whichever state is realized. Therefore, the agent will strictly prefer $\frac{1}{2}f \oplus \frac{1}{2}g$ to $\frac{1}{2}f + \frac{1}{2}g$. Under reversal of order, $\frac{1}{2}f \oplus \frac{1}{2}g$ and $\frac{1}{2}f + \frac{1}{2}g$ must be indifferent. This illustrates the intuition against adopting reversal of order.[7]

[6]Under reversal of order, one of the two independence axioms is redundant. I leave both of them for comparison with the next section.

[7]The preceding intuition translates to the present setting Gilboa and Schmeidler's (1989) rationale for their axiom "uncertainty aversion," namely, that "hedging" across ambiguous states can increase utility.

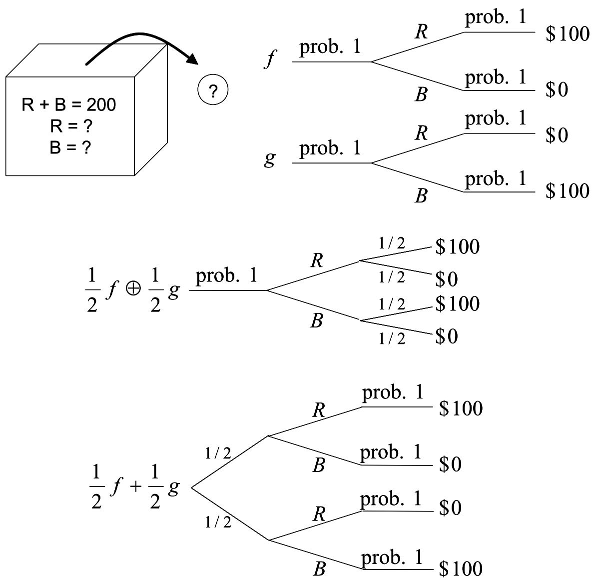

FIGURE 3.—One ball is randomly drawn from the Ellsberg urn which contains 200 balls that are either red or blue. The exact number of red (or blue) balls is unknown. An act f is a bet on red and act g is a bet on blue. The second-stage mixture $\frac{1}{2}f \oplus \frac{1}{2}g$ is unambiguous, but the first-stage mixture is not.

However, one may think in a different way. For any number of blue balls in the urn, the final probability of getting \$100 is 1/2 not only for $\frac{1}{2}f \oplus \frac{1}{2}g$, but also for $\frac{1}{2}f + \frac{1}{2}g$. Hence the agent may be indifferent between $\frac{1}{2}f \oplus \frac{1}{2}g$ and $\frac{1}{2}f + \frac{1}{2}g$, while preferring $\frac{1}{2}f \oplus \frac{1}{2}g$ to f and g. Implicit in this argument is that $\frac{1}{2}f + \frac{1}{2}g$ becomes a two-stage lottery when the number of blue balls is given and that the agent reduces the two-stage lottery to a one-stage lottery.[8]

The preceding argument supporting reversal of order is normatively appealing, but Halevy (2007) reported that most people who reduce compound lotteries are ambiguity neutral (see the next section). Since the argument to maintain reversal of order requires reduction, it may not be acceptable at a descriptive level. In this paper, I drop reversal of order and suggest a descriptive model to explain Ellsberg-type behavior.

Recall that AA dominance deals only with $\mathcal{H}$, not with $\Delta(\mathcal{H})$. Under reversal of order, stating properties on $\mathcal{H}$ is enough to describe properties on $\Delta(\mathcal{H})$. Since I drop reversal of order, AA dominance must be modified.

Each $f \in \mathcal{H}$ and $\mu \in \Delta(S)$ induces a one-stage lottery, namely $\Psi(f, \mu) \equiv \mu(s_1)f(s_1) \oplus \mu(s_2)f(s_2) \oplus \cdots \oplus \mu(s_{|S|})f(s_{|S|}) \in \Delta(Z)$; that is, $\Psi(f, \mu)$ is the one-stage lottery, or constant act, obtained by "reducing" the act f using the prob-

[8] See Ellsberg (2001, p. 230) for a similar argument by Pratt and Raiffa.

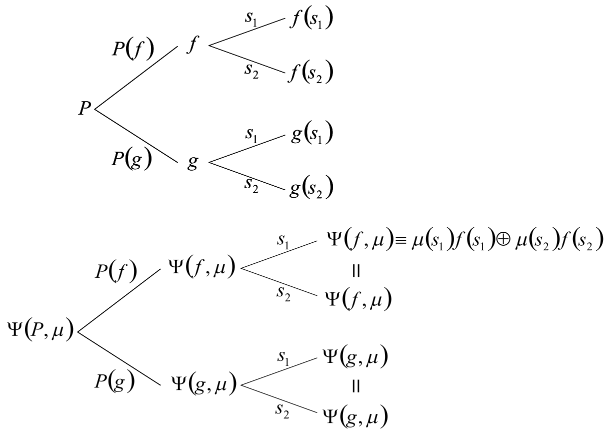

FIGURE 4.—If μ is assumed to be the true probability law, the decision maker translates P to $\Psi(P, \mu)$.

ability law μ. For $P \in \Delta(\mathcal{H})$, define the two-stage lottery $\Psi(P, \mu) \in \Delta(\Delta(Z))$ by $\Psi(P, \mu)(B) = P(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\})$ for each $B \in \mathcal{B}_{\mathcal{H}}$. See Figure 4 for an example of $\Psi(P, \mu)$, and recall that the preference $\succeq$ on $\Delta(\mathcal{H})$ directly induces preferences over two-stage lotteries (including the object $\Psi(P, \mu)$) by its restriction to lotteries over constant acts.

AXIOM 7—Dominance: *For any* $P, Q \in \Delta(\mathcal{H})$, *if* $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ *for all* $\mu \in \Delta(S)$, *then* $P \succeq Q$.

To interpret dominance, consider an agent who is not certain of the true probability law over states, but who believes that there is a *true law*. Now suppose that $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for *every* $\mu \in \Delta(S)$, that is, for *every* probability law, he prefers the two-stage lottery induced by P to the one induced by Q. Then he must prefer P to Q.

There is an implicit assumption behind this interpretation. When the probability law is given, one may interpret the choice object $P \in \Delta(\mathcal{H})$ as a three-stage lottery which is out of the domain. Dominance implicitly assumes that the agent reduces stages two and three of the three-stage lottery: the two-stage lottery $\Psi(P, \mu)$ is formed by reducing the acts in the support of P to constant acts using the probability law μ. See the end of this section for more discussion of dominance and for an example illustrating that dominance excludes some interesting preferences.

It is instructive to compare dominance with AA dominance. Since the latter deals only with acts in $\mathcal{H}$, extend AA dominance to lotteries over acts. Then the extended AA dominance states that $P \succeq Q$ if $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for all Dirac measures $\mu = \delta_s$, $s \in S$. Dominance posits the stronger hypothesis that

$\Psi(P,\mu) \succeq \Psi(Q,\mu)$ for all measures μ in $\Delta(S)$. Thus dominance is weaker than the extended AA dominance.

It is also instructive to observe that the extended AA dominance implies Kreps' reversal-of-order-style axiom (Kreps (1988, p. 107)). It states that all lotteries over Savage acts (prizes of the acts are elements in Z) that, for each state s, map naturally to the same lottery over outcomes, must be indifferent.[9] The Kreps axiom is essential to SEU. As previously mentioned, dominance is a weaker axiom than the extended AA dominance, thereby allowing for more general representation than SEU.

A more complete and formal comparison of dominance and AA dominance is provided in the next lemma.

LEMMA 4.1: (i) *Order*, *continuity*, *reversal of order*, *and AA dominance imply dominance*.

(ii) *Dominance and second-stage independence imply AA dominance*.

The main utility representation is defined as follows.

DEFINITION 4: A second-order subjective expected utility (SOSEU) representation is a probability measure $m \in \Delta(\Delta(S))$, a bounded continuous mixture linear function $u : \Delta(Z) \to \mathbb{R}$, and a bounded continuous and strictly increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ such that V represents $\succeq$ on $\Delta(\mathcal{H})$, where

$$V(P) = \int_{\mathcal{H}} U(f)\, dP(f), \tag{4.1}$$

$$U(f) = \int_{\Delta(S)} v\left(\int_S u(f)\, d\mu\right) dm(\mu).$$

The probability measure m is called a second-order belief.

SOSEU can accommodate nonindifference to ambiguity. When the second-order belief m is nondegenerate and v is nonlinear, the implied behavior cannot be explained by a unique (subjective) probability on S. Instead the agent behaves as though he has multiple priors on S and assigns a probability to each prior on S. SEU is the special case when v is linear.[10]

The new representation theorem follows.

[9]To see that the extended AA dominance implies the Kreps axiom, consider two lotteries P and Q over Savage acts. Assume that, for each s, the two lotteries induce the same lottery over Z. Then $\Psi(P,\delta_s) = \Psi(Q,\delta_s)$ for every $s \in S$ and the extended AA dominance implies that P is indifferent to Q.

[10]The functional form of an SOSEU representation is similar to that of KMM (2005). Many properties of the functional form are investigated in their paper.

THEOREM 4.2: *Preference $\succeq$ on $\Delta(\mathcal{H})$ satisfies order, continuity, second-stage independence, first-stage independence, and dominance if and only if it has an SOSEU representation.*

Appendix A provides a sketch of the proof and also some examples to demonstrate the tightness of the theorem. The complete proof is given in Appendix B.

Lemma 4.1 suggests that reversal of order is the crucial difference between an SEU representation and an SOSEU representation. This is summarized in the next corollary.

COROLLARY 4.3: *Preference $\succeq$ on $\Delta(\mathcal{H})$ has an SEU representation if and only if it has an SOSEU representation and satisfies reversal of order.*

AA assumed reversal of order. Under reversal of order, the agent does not care when the objective uncertainty is resolved and he collapses the two objective uncertainties into one objective uncertainty. Thus Corollary 4.3 says that if the agent collapses the two objective probabilities into one, he also collapses the second-order belief (on $\Delta(S)$) into the belief (on S).

Briefly consider uniqueness of the representation in Theorem 4.2. It is easy to show that u and $v \circ u$ are unique up to a positive affine transformation (see Appendix C). The second-order belief m is unique in some special cases—for example, if $v(z) = \exp(z)$, the representation has a form similar to a moment generating function and m is unique. However, m is not unique in general. For example, suppose that v is linear. Then any second-order belief that has the same first moment will show the same behavior. Similarly, a polynomial v of degree n implies that if two second-order beliefs, m and m', represent the same preference, they have the same moments up to nth order. See Appendix C for a characterization of the uniqueness class of measures for any given u and v.

I conclude this section with an interesting example of preference that violates dominance. Gilboa and Schmeidler (1989) axiomatized the multiple-priors (MP) model on $\mathcal{H}$. Consider two alternative extensions to $\Delta(\mathcal{H})$:

$$V^{\mathrm{MP1}}(P) = \min_{\mu \in C} \int_{\mathcal{H}} \left(\int_S u(f)\, d\mu \right) dP(f),$$

$$V^{\mathrm{MP2}}(P) = \int_{\mathcal{H}} \left(\min_{\mu \in C} \int_S u(f)\, d\mu \right) dP(f)$$

for a closed set $C \subset \Delta(S)$. Both representations induce the same preference on $\mathcal{H}$, but only V^{MP1} satisfies dominance (see examples in Appendix A). The reason can be understood as follows: agents who have either representation might behave as if they were playing a game with a malevolent nature. They suspect that nature will choose a probability law μ that is most unfavorable to them. The difference between V^{MP1} and V^{MP2} lies in the agent's view of the timing of nature's move—it is before the first randomization (corresponding to P) in V^{MP1} and afterward in V^{MP2}. But when evaluating $P \in \Delta(\mathcal{H})$, an agent who satisfies dominance uses two-stage lotteries $\Psi(P, \mu)$ as if μ, though still unknown, has already been chosen by nature. Therefore, V^{MP1} satisfies dominance and V^{MP2} does not.

5. AMBIGUITY AND COMPOUND LOTTERIES

Here I discuss the relations between ambiguity attitude and a two-stage lottery. A two-stage lottery deals only with objective probabilities; ambiguity attitude deals with the situation where objective probabilities are unknown. The two may seem conceptually distinct, but in an SOSEU representation, they are closely related.

An axiom on compound lotteries is introduced.[11]

AXIOM 8—Reduction of Compound Lotteries (ROCL): *For any* $p, q \in \Delta(Z)$ *and* $\alpha \in [0, 1]$, $\alpha p \oplus (1-\alpha)q \sim \alpha p + (1-\alpha)q$.

Since p and q are one-stage lotteries, $\alpha p + (1-\alpha)q$ constitutes a two-stage lottery. Observe that $\alpha p \oplus (1-\alpha)q$ and $\alpha p + (1-\alpha)q$ have the same final outcome distribution. Thus, under ROCL, the agent considers only the final distribution and he does not care about the timing of risk resolution.

An SOSEU representation does not satisfy ROCL unless v is linear. When v is nonlinear,

$$\begin{aligned} V(\alpha p + (1-\alpha)q) &= \alpha v(u(p)) + (1-\alpha)v(u(q)) \\ &\neq v(\alpha u(p) + (1-\alpha)u(q)) \\ &= V(\alpha p \oplus (1-\alpha)q). \end{aligned}$$

Under SOSEU, the utility of any act f is given by

$$U(f) = \int_{\Delta(S)} v\left(\int_S u(f)\, d\mu\right) dm(\mu),$$

which suggests the interpretation that the agent processes an act in a two-stage fashion. This suggests further a connection between the evaluations of acts and two-stage lotteries. In the following, I will show that, given other axioms, ROCL is equivalent to reversal of order and that ROCL implies neutrality to ambiguity.

[11] Segal (1990) had a slightly different form of ROCL, but it is not difficult to see that the two axioms are equivalent.

LEMMA 5.1: *ROCL and reversal of order are equivalent under dominance*.

PROOF: Since $\Delta(Z) \subset \mathcal{H}$, it is straightforward that reversal of order implies ROCL. Conversely, assume ROCL. Then, for any $\mu \in \Delta(S)$,

$$\begin{aligned}\Psi(\alpha f + (1-\alpha)g, \mu) &= \alpha\Psi(f, \mu) + (1-\alpha)\Psi(g, \mu) \\ &\sim \alpha\Psi(f, \mu) \oplus (1-\alpha)\Psi(g, \mu) \\ &= \Psi(\alpha f \oplus (1-\alpha)g, \mu).\end{aligned}$$

Applying dominance leads to $\alpha f + (1-\alpha)g \sim \alpha f \oplus (1-\alpha)g$. *Q.E.D.*

COROLLARY 5.2: *Preference $\succeq$ has an SEU representation if and only if it has an SOSEU representation and satisfies ROCL*.

PROOF: By Lemma 5.1 and Corollary 4.3, this is straightforward. *Q.E.D.*

An SOSEU representation reduces to SEU if and only if ROCL is satisfied. In particular, ROCL implies neutrality to ambiguity. This is consistent with Halevy's (2007) experimental findings.

Halevy designed the following experiment. There are three urns, each containing 10 balls which can be red or black.[12] One ball is to be drawn. Urn 1 contains 5 red balls and 5 black balls. In urn 2, the proportion is unknown. For urn 3, a ticket is drawn from a bag containing 11 tickets with numbers 0 to 10 written on them. The number on the drawn ticket determines the number of red balls in urn 3. Each participant is asked to place a bet on the color of the drawn ball from each urn. Before any ball is drawn, the participant is given the option to sell each bet. The subject is asked the minimal price at which he/she is willing to sell the bet. Let V_i be the reservation price for urn i, $i = 1, 2, 3$.

Ambiguity neutrality implies $V_1 = V_2$ and ROCL implies $V_1 = V_3$. In Halevy's experiment, 18 subjects set $V_1 = V_3$ and 17 out of them set $V_1 = V_2$. Moreover, out of 86 subjects who showed $V_1 \neq V_3$, 80 showed $V_1 \neq V_2$. Halevy concluded that "there is a very tight association between ambiguity neutrality and reduction of compound lotteries" and that "a descriptive theory that accounts for ambiguity aversion should account—at the same time—for violation of reduction of compound *objective* lotteries." The domain in this paper includes both acts and two-stage lotteries, and an SOSEU representation relates ambiguity attitude to ROCL.[13]

[12]In his experiment, there were four urns. The fourth urn is omitted here because it is not relevant to my point.

[13]Klibanoff, Marinacci, and Mukerji (2005) dealt with acts, but not with compound lotteries. Segal's (1990) model has two-stage lotteries, but no acts.

6. RELATED LITERATURE

The idea of second-order probabilities appeared in Savage (1972, p. 58), but he implicitly assumed reduction and argued that second-order probabilities accomplish nothing more than first-order probabilities do. Segal (1987, 1990) allowed nonreduction of two-stage lotteries. In the former paper, he showed that a model of preferences that allow nonreduction of two-stage lotteries can accommodate the Ellsberg paradox. He used anticipated utility theory and thus his model can accommodate also the Allais paradox. Gärdenfors and Sahlin (1982, 1983) also suggested a second-order probability measure. They used it to determine the set of all "satisfactorily reliable" first-order beliefs and applied the maximin criterion for expected utilities (MMEU) rule.

The violation of ROCL and the recursive structure of utility in the present model bring to mind the closely related model of Kreps and Porteus (1978), who provided axiomatic foundations for recursive expected utility with *objective* temporal lotteries. Their model not only has a functional form similar to SOSEU, but also takes a similar approach: Kreps and Porteus assumed independence at each stage and relaxed reduction. However, precise probabilities are not given in most real-world problems. Klibanoff and Ozdenoren (2007) incorporated subjective uncertainty to characterize subjective recursive expected utility, which does not deal with ambiguity. SOSEU is also defined on a domain that involves subjective uncertainty, but features second-order beliefs that can accommodate Ellsbergian behavior.

Violations of ROCL have been documented in some experimental literature. See, for example, Ronen (1971), Snowball and Brown (1979), Schoemaker (1989), Bernasconi and Loomes (1992), Bernasconi (1994), and Busescu and Fischer (2001). However, Cubitt, Starmer, and Sugden (1998) did not find significant violation of ROCL in an experiment that tested several well known accounts of the common ratio effect. Keller (1985) reported that the framing of the problems may affect the degree of the violations, and Güth, van Damme, and Weber (2005) found that the level of econometrics education may also have similar effects.

Some experiments deal with ambiguous urns and two-stage risky urns at the same time. Yates and Zukowski (1976), Chow and Sarin (2002), and Halevy (2007) designed experiments involving one-stage risky urns, ambiguous urns, and two-stage risky urns. Those urns are similar to urns 1, 2, and 3 mentioned in the previous section, respectively. All three papers report that one-stage risky urns are most preferred, ambiguous urns are least preferred, and two-stage risky urns are intermediate to the others. Halevy found a tight association between ambiguity neutrality and ROCL, while this connection is not addressed in the other papers.

Kreps (1988, pp. 105–110) noted that order, continuity, and first-stage independence deliver the representation (4.1) with no further structure on the function U. He also noted that adding a reversal-of-order-style axiom and a monotonicity assumption (p. 109, Axiom 7.16) guarantees AA's subjective expected utility model. Theorem 4.2 (suitably modified) provides a similar result: order, continuity, first-stage independence, and dominance (instead of the reversal-of-order-style and monotonicity axioms) are equivalent to the utility representation (4.1) with $U(f) = \int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\, dm(\mu)$, where $\widehat{U}$ is the restriction of U to the constant acts $\Delta(Z)$.[14]

Finally, consider (further to the discussion in the Introduction) the relation to KMM. They additionally assumed preference $\succeq^2$ over the set of all (second-order) acts on $\Delta(S)$ and imposed three axioms. The natural question is how their axioms are related to those in this paper.

One needs to define $\succeq^2$ to discuss the connection to KMM. For $f \in \mathcal{H}$, let $f^2 : \Delta(S) \to \Delta(Z)$ be the act on $\Delta(S)$ satisfying $f^2(\mu) = \Psi(f, \mu)$ and define[15]

$$f^2 \succeq^2 g^2 \quad \text{if and only if} \quad f \succeq g. \tag{6.1}$$

Then preference $\succeq$ over $\mathcal{H}$ induces $\succeq^2$ over the subset $\{f^2 : f \in \mathcal{H}\}$ of all second-order acts. Similarly, preference $\succeq$ over $\Delta(\mathcal{H})$ induces $\succeq^2$ over $\Delta(\{f^2 : f \in \mathcal{H}\})$, the set of lotteries over the second-order acts f^2, having the form $f^2(\mu) = \Psi(f, \mu)$.

Now their axioms can be considered. First, (6.1) is KMM's consistency axiom. Second, their expected utility on lotteries axiom is equivalent to second-stage independence under order and continuity. Finally, their subjective expected utility on second order acts axiom, is the counterpart of first-stage independence and dominance, again given order and continuity. One can see the last connection by observing that Theorem 4.2 may be viewed as proving that $\succeq^2$ has the representation

$$P^2 \longmapsto \int \left[\int \widehat{U}(f^2(\mu))\, dm(\mu) \right] dP^2(f^2) \quad \text{for} \quad P^2 \in \Delta(\{f^2 : f \in \mathcal{H}\})$$

for some bounded continuous function $\widehat{U}$ on $\Delta(Z)$ and measure $m \in \Delta(\Delta(S))$. Thus the restriction of $\succeq^2$ to $\{f^2 : f \in \mathcal{H}\}$ is a subjective expected utility preference.

APPENDIX A: PROOF SKETCH AND EXAMPLES

This section sketches the sufficiency proof of Theorem 4.2 and provides examples to demonstrate the tightness of the theorem.

[14]See Lemma B.8, where second-stage independence does not play a role in constructing m.

[15]If f and f' induce the same second-order act, the two acts must be the same, because $\Psi(f, \mu) = \Psi(f', \mu)$ for all μ implies $f = f'$. Thus, $\succeq^2$ is well defined.

Proof Sketch

First-stage independence implies that preference can be represented by $V(P) = \int_{\mathcal{H}} U(f)\,dP(f)$. The key part of the proof is to construct the second-order belief $m \in \Delta(\Delta(S))$ satisfying

$$\int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\,dm(\mu) = U(f) \quad \text{for all } f \in \mathcal{H},$$

where $\widehat{U}$ is the restriction of U to the constant acts $\Delta(Z)$. For intuition about existence of such a measure m, consider the discretized version where there are n available acts and k possible priors: $Am = b$, where

$$A = \begin{pmatrix} \widehat{U} \circ \Psi(f_1, \mu_1) & \cdots & \widehat{U} \circ \Psi(f_1, \mu_k) \\ \vdots & \ddots & \vdots \\ \widehat{U} \circ \Psi(f_n, \mu_1) & \cdots & \widehat{U} \circ \Psi(f_n, \mu_k) \end{pmatrix},$$

$$m = \begin{pmatrix} m_1 \\ \vdots \\ m_k \end{pmatrix}, \quad b = \begin{pmatrix} U(f_1) \\ \vdots \\ U(f_n) \end{pmatrix}.$$

By Farkas' lemma, $Am = b$ has a nonnegative solution m if and only if, for all $y \in \mathbb{R}^n$,

$$A^T y \geq 0 \quad \Longrightarrow \quad b^T y \geq 0.$$

By the infinite dimensional version of Farkas' lemma (see Theorem B.1), it suffices to show that, for all signed measures t' on $\mathcal{H}$,

$$\text{(A.1)} \qquad \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\,dt'(f) \geq 0 \quad (\text{for all } \mu \in \Delta(S))$$

$$\text{(A.2)} \qquad \Longrightarrow \quad \int_{\mathcal{H}} U(f)\,dt' \geq 0.$$

To show that, under dominance, (A.1) implies (A.2), first notice that t' can be decomposed into $\alpha P - \beta Q$, where $P, Q \in \Delta(\mathcal{H})$ are in the domain of objects of choice and $\alpha, \beta \geq 0$. Then rearranging (A.1) gives[16]

$$\text{(A.3)} \qquad \alpha \int_{\Delta(Z)} \widehat{U}\,d\Psi(P, \mu) \geq \beta \int_{\Delta(Z)} \widehat{U}\,d\Psi(Q, \mu) \quad \text{for all } \mu \in \Delta(S).$$

[16]Recall that $\Psi(P, \mu)(B) = P(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\})$ for $B \in \mathcal{B}_{\mathcal{H}}$. Thus, by the change of variables theorem,

$$\int_{\Delta(Z)} \widehat{U}(p)\,d\Psi(P, \mu)(p) = \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\,dP(f).$$

The same is true for Q.

Normalize U such that $\int_{\mathcal{H}} U\,d\bar{R} = 0$ for some $\bar{R} \in \Delta(\Delta(Z))$. Consider the case $\alpha > \beta \geq 0$. Other cases can be proved similarly. Recall that $\bar{P} \longmapsto \int_{\Delta(Z)} \widehat{U}\,d\bar{P}$ represents preference on $\Delta(\Delta(Z))$. Then (A.3) implies

$$\begin{aligned}\Psi(P,\mu) &\succeq \frac{\beta}{\alpha}\Psi(Q,\mu) + \left(1-\frac{\beta}{\alpha}\right)\Psi(\bar{R},\mu)\\ &= \Psi\left(\frac{\beta}{\alpha}Q + \left(1-\frac{\beta}{\alpha}\right)\bar{R},\mu\right) \quad \text{for all } \mu \in \Delta(S).\end{aligned}$$

Now apply dominance to get

$$P \succeq \frac{\beta}{\alpha}Q + \left(1-\frac{\beta}{\alpha}\right)\bar{R}.$$

Since $V(P) = \int_{\mathcal{H}} U(f)\,dP(f)$ represents preference, (A.2) follows and thus a second-order belief m exists.

Second-stage independence is used only to derive $\widehat{U} = v \circ u$, where u is a mixture-linear function on $\Delta(Z)$ and v is a strictly increasing function on $u(\Delta(Z))$. Since u is mixture linear, $U(f) = \int_{\Delta(S)} v \circ u \circ \Psi(f,\mu)\,dm(\mu) = \int_{\Delta(S)} v(\int_S u(f)\,d\mu)\,dm(\mu)$ follows.

Examples for the Tightness of Theorem 4.2

Each example satisfies all but one of the axioms characterizing an SOSEU representation.

EXAMPLE 1—All but Second-Stage Independence: Let

$$V(P) = \int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dP(f)$$

for some fixed $\mu \in \Delta(S)$ and a bounded continuous but non-mixture-linear $u : \Delta(Z) \to \mathbb{R}$.

EXAMPLE 2—All but First-Stage Independence: Let

$$V(P) = \min_{\mu \in C}\int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dP(f),$$

where u is bounded, continuous, and mixture linear, and $C \subset \Delta(S)$ is a closed subset. To show dominance, note that

$$\Psi(P,\mu) \succeq \Psi(Q,\mu) \quad (\text{for all } \mu \in \Delta(S))$$

$$
\begin{aligned}
\implies\quad & \int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dP(f) \\
& \qquad \geq \int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dQ(f) \quad (\text{for all } \mu \in \Delta(S)) \\
\implies\quad & \min_{\mu\in C}\int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dP(f) \geq \min_{\mu\in C}\int_{\mathcal{H}}\left(\int_S u(f)\,d\mu\right)dQ(f).
\end{aligned}
$$

EXAMPLE 3—All but Dominance: Modify Example 2 by taking

$$V(P) = \int_{\mathcal{H}}\left(\min_{\mu\in C}\int_S u(f)\,d\mu\right)dP(f).$$

This violates (only) dominance. Let $S = \{1, 2\}$, $P = \frac{1}{2}f + \frac{1}{2}g$, $Q = \delta_h$, $u(f(1)) = 1$, $u(f(2)) = 2$, $u(g(1)) = 1$, $u(g(2)) = 0$, $u(h(1)) = 1$, $u(h(2)) = 1$, and $C = \Delta(\Delta(S))$. Then $V(\Psi(P, \mu)) = 1 = V(\Psi(Q, \mu))$ for all $\mu \in \Delta(S)$, but $V(P) = 1/2 < 1 = V(Q)$.

APPENDIX B: PROOFS

B.1. *Preliminaries*

Notations and definitions follow AB (1999) and Craven and Koliha (1977).

For any real vector space $\mathcal{M}$, let $\mathcal{M}^{\#}$ be the algebraic dual of $\mathcal{M}$, that is, the set of all linear functionals on $\mathcal{M}$. Denote by $\langle m, m^{\#}\rangle$ an evaluation of $m^{\#} \in \mathcal{M}^{\#}$ at the point $m \in \mathcal{M}$. Suppose that $A : \mathcal{M} \to \mathcal{T}$ is a linear map between two vector spaces $\mathcal{M}$ and $\mathcal{T}$. The algebraic adjoint $A^{\#} : \mathcal{T}^{\#} \to \mathcal{M}^{\#}$ of A is the linear map satisfying

$$\langle m, A^{\#}t^{\#}\rangle = \langle Am, t^{\#}\rangle \quad \text{for all } m \in \mathcal{M} \text{ and } t^{\#} \in \mathcal{T}^{\#}.$$

A dual pair is a pair $\langle \mathcal{M}, \mathcal{M}'\rangle$ of two vector spaces together with a function $(m, m') \longmapsto \langle m, m'\rangle$, from $\mathcal{M} \times \mathcal{M}'$ into $\mathbb{R}$, satisfying (i) $m \longmapsto \langle m, m'\rangle$ is linear, (ii) $m' \longmapsto \langle m, m'\rangle$ is linear, (iii) if $\langle m, m'\rangle = 0$ for each $m' \in \mathcal{M}'$, then $m = 0$, and (iv) if $\langle m, m'\rangle = 0$ for each $m \in \mathcal{M}$, then $m' = 0$. I will refer to (iii) and (iv) as separation properties. Given a dual pair $\langle \mathcal{M}, \mathcal{M}'\rangle$, the weak topology on $\mathcal{M}$ is denoted by $\sigma(\mathcal{M}, \mathcal{M}')$. Under $\sigma(\mathcal{M}, \mathcal{M}')$, a sequence $m_n \in \mathcal{M}$ converges to $m \in \mathcal{M}$ if and only if $\langle m_n, m'\rangle \to \langle m, m'\rangle$ for all $m' \in \mathcal{M}'$. It is well known that the topological dual of $(\mathcal{M}, \sigma(\mathcal{M}, \mathcal{M}'))$ may be identified with $\mathcal{M}'$. In other words, for each $\sigma(\mathcal{M}, \mathcal{M}')$-continuous linear functional ϕ on $\mathcal{M}$, there is a unique $m' \in \mathcal{M}'$ such that $\phi(m) = \langle m, m'\rangle$ for all $m \in \mathcal{M}$. The weak topology $\sigma(\mathcal{M}', \mathcal{M})$ is defined symmetrically for $\mathcal{M}'$. From now on, for any dual pair $\langle \mathcal{M}, \mathcal{M}'\rangle$, $\mathcal{M}$ and $\mathcal{M}'$ are topological vector spaces equipped with the weak topologies.

Given dual pairs $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$, the continuity of a linear mapping $A : \mathcal{M} \to \mathcal{T}$ can be checked by using $A^\#$; A is continuous if and only if $A^\#(\mathcal{T}') \subset \mathcal{M}'$. The restriction A' of $A^\#$ to $\mathcal{T}'$ is called the topological adjoint of A with respect to $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$, or simply the adjoint of A.

A nonempty set $K \subset \mathcal{M}$ is called a convex cone if $K + K \subset K$ and $\alpha K \subset K$ for every $\alpha \geq 0$. The polar cone $K' \subset \mathcal{M}'$ of the convex cone $K \subset \mathcal{M}$ is defined as $K' = \{m' : \langle m, m' \rangle \geq 0 \text{ for all } m \in K\}$.

The main tool used in the paper is the following result from Craven and Koliha (1977, Theorem 2).

THEOREM B.1—Generalized Farkas Theorem: *Let $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$ be dual pairs, let K be a convex cone in $\mathcal{M}$, and let $A : \mathcal{M} \to \mathcal{T}$ be a continuous linear map. Let $A(K)$ be closed and $\tau \in \mathcal{T}$. Then the following conditions are equivalent.*[17]

(a) *The equation $Am = \tau$ has a solution $m \in K$.*
(b) $A't' \in K' \Longrightarrow \langle \tau, t' \rangle \geq 0$.

B.2. *Proof of Theorem 4.2*

LEMMA B.2: *The map $(f, \mu) \longmapsto \Psi(f, \mu)$ from $\mathcal{H} \times \Delta(S)$ into $\Delta(Z)$ is continuous.*

PROOF: Suppose that (f_n, μ_n) converges to (f, μ) in the product space $\mathcal{H} \times \Delta(S)$. Note that S is finite. Then, for any $\eta \in C_b(Z)$,

$$\begin{aligned}
&\int_Z \eta \, d\Psi(f_n, \mu_n) \\
&\quad = \int_Z \eta d\big[\mu_n(s_1) f_n(s_1) \oplus \mu_n(s_2) f_n(s_2) \oplus \cdots \oplus \mu_n(s_{|S|}) f_n(s_{|S|})\big] \\
&\quad = \sum_{s \in S} \left(\mu_n(s) \int_Z \eta \, df_n(s) \right) \\
&\quad \to \sum_{s \in S} \left(\mu(s) \int_Z \eta \, df(s) \right) = \int_Z \eta \, d\Psi(f, \mu).
\end{aligned}$$

Q.E.D.

PROOF OF THEOREM 4.2—Necessity: Completeness, transitivity, and continuity are clear.

Second-Stage Independence: For $p \in \Delta(Z)$, $V(p) = v(u(p))$ because p does not depend on the probability measure $m \in \Delta(\Delta(S))$. Since v is strictly increasing, preference on $\Delta(Z)$ is represented by u. Thus second-stage independence is satisfied because u is mixture linear.

[17]It is easy to see $(a) \Rightarrow (b)$. Suppose that $Am = \tau$, $m \in K$, and $A't' \in K'$. Then $\langle \tau, t' \rangle = \langle Am, t' \rangle = \langle m, A't' \rangle \geq 0$, because $A't' \in K'$.

First-Stage Independence: Let $\alpha \in (0,1]$ and $P, R \in \Delta(\mathcal{H})$. Then it is easy to see that

$$V(\alpha P + (1-\alpha)R) = \alpha V(P) + (1-\alpha)V(R).$$

First-stage independence is clear.

Dominance: Let P be any element in $\Delta(\mathcal{H})$. By Lemma B.2 and continuity of $v \circ u$, $v[u(\Psi(f,\mu))]$ is jointly continuous on $\mathcal{H} \times \Delta(S)$ and hence is $P \times m$-measurable. Since $v \circ u$ is bounded, $v[u(\Psi(f,\mu))]$ is $P \times m$-integrable. Then, apply the Fubini theorem (AB (1999, p. 411)) to get

$$\begin{aligned} V(P) &= \int_{\mathcal{H}} \int_{\Delta(S)} v\left(\int_S u(f)\, d\mu\right) dm(\mu)\, dP(f) \\ &= \int_{\mathcal{H}} \int_{\Delta(S)} v\big[u(\Psi(f,\mu))\big]\, dm(\mu)\, dP(f) \\ &= \int_{\Delta(S)} \int_{\mathcal{H}} v\big[u(\Psi(f,\mu))\big]\, dP(f)\, dm(\mu). \end{aligned}$$

Note that by the change of variables theorem (AB (1999, p. 452)),

$$\int_{\mathcal{H}} v\big[u(\Psi(f,\mu))\big]\, dP(f) = \int_{\Delta(Z)} v \circ u(p)\, d\Psi(P,\mu)(p) = V(\Psi(P,\mu)).$$

Thus,

$$V(P) = \int_{\Delta(S)} V(\Psi(P,\mu))\, dm(\mu).$$

Since m is a nonnegative measure, this completes the necessity part of the proof.

PROOF OF THEOREM 4.2—Sufficiency: When $P \sim Q$ for all $P, Q \in \Delta(\mathcal{H})$, the representation is trivial. Thus assume that $\succeq$ satisfies the following statement.

AXIOM 9—Nondegeneracy: *$P \succ Q$ for some $P, Q \in \Delta(\mathcal{H})$.*

Follow Lemmas B.3–B.10 to prove sufficiency.

LEMMA B.3: (i) *Preference $\succeq$ restricted to $\Delta(Z)$ is represented by a bounded continuous mixture linear function $u : \Delta(Z) \to \mathbb{R}$. Moreover, u is unique up to positive affine transformation*. (ii) *Preference $\succeq$ is represented on $\Delta(\mathcal{H})$ by*

$$V(P) = \int_{\mathcal{H}} U(f)\, dP(f)$$

for $P \in \Delta(\mathcal{H})$, where $U : \mathcal{H} \to \mathbb{R}$ is a bounded continuous function and is unique up to positive affine transformation.

PROOF: (i) Since $\mathcal{H}$ is a metric space, the mapping $f \mapsto \delta_f$ from $\mathcal{H}$ into $\Delta(\mathcal{H})$ is an embedding (AB (1999 p. 480)). Moreover, $\mathcal{H}$ is a product space of $\Delta(Z)$'s. Thus, the weak convergence topology on $\Delta(Z)$ coincides with the relative topology on $\Delta(Z)$ induced by $\Delta(\mathcal{H})$. Hence, continuity implies that the restriction of $\succeq$ to $\Delta(Z)$ is continuous under the weak convergence topology on $\Delta(Z)$. Moreover, preference $\succeq$ restricted to $\Delta(Z)$ satisfies order and (second-stage) independence. Therefore, (i) holds (see Grandmont (1972), for example).

(ii) can be proved in a similar way. *Q.E.D.*

Let $\widehat{U}$ be the restriction of U to $\Delta(Z)$.

LEMMA B.4: *There exist $p, q \in \Delta(Z)$ such that $p \succ q$. Consequently, there is a one-stage lottery p such that $\widehat{U}(p) \neq 0$.*

PROOF: Suppose that $p \sim q$ for all p and q in $\Delta(Z)$. This means that $\bar{P} \sim \bar{Q}$ for all $\bar{P}$ and $\bar{Q}$ in $\Delta(\Delta(Z))$, by Lemma B.3(ii). Then, for any $P, Q \in \Delta(\mathcal{H})$ and $\mu \in \Delta(S)$, $\Psi(P, \mu) \sim \Psi(Q, \mu)$ because $\Psi(P, \mu), \Psi(Q, \mu) \in \Delta(\Delta(Z))$. By the dominance axiom, $P \sim Q$ for all $P, Q \in \Delta(\mathcal{H})$, contradicting nondegeneracy. *Q.E.D.*

To apply the generalized Farkas theorem, let

$$\mathcal{M} = ca(\Delta(S)), \quad \mathcal{M}' = C_b(\Delta(S)),$$
$$\mathcal{T} = C_b(\mathcal{H}), \quad \mathcal{T}' = ca(\mathcal{H})$$

where $ca(X)$ denotes the set of all Borel signed measures on X having bounded variation. Both $\langle \mathcal{M}, \mathcal{M}' \rangle$ and $\langle \mathcal{T}, \mathcal{T}' \rangle$ are dual pairs with bilinear operations $\langle m, m' \rangle = \int_{\Delta(S)} m' \, dm$ and $\langle t, t' \rangle = \int_{\mathcal{H}} t \, dt'$ for $m \in \mathcal{M}$, $m' \in \mathcal{M}'$, $t \in \mathcal{T}$, and $t' \in \mathcal{T}'$ (AB (1999, p. 475)). Let

$$K = ca_+(\Delta(S))$$

be the subset of $\mathcal{M}$ consisting of all nonnegative Borel measures on $\Delta(S)$. K is clearly a convex cone. Recall that $\widehat{U}$ is the restriction of U to $\Delta(Z)$ and define a linear mapping A from $\mathcal{M}$ into the set of all functionals on $\mathcal{H}$ by

$$(Am)(f) = \int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu) \, dm(\mu) \quad \text{for} \quad f \in \mathcal{H}. \tag{B.1}$$

The premises of the generalized Farkas theorem will be verified.

LEMMA B.5: *The mapping A is a linear mapping from $\mathcal{M}$ into $\mathcal{T}$.*

PROOF: It suffices to show that $A(\mathcal{M}) \subset \mathcal{T} = C_b(\mathcal{H})$. Let $m \in \mathcal{M}$ and assume that $f_n \to f$ for $f_n, f \in \mathcal{H}$. Note that $\widehat{U}$ is bounded and $\widehat{U} \circ \Psi(f_n, \mu) \to \widehat{U} \circ \Psi(f, \mu)$ by Lemma B.2. By the Lebesgue dominated convergence theorem, $\int_{\Delta(S)} \widehat{U} \circ \Psi(f_n, \mu)\, dm(\mu) \to \int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\, dm(\mu)$. Hence $f \mapsto (Am)(f)$ is continuous. Boundedness of $f \mapsto (Am)(f)$ comes from boundedness of $\widehat{U}$. *Q.E.D.*

LEMMA B.6: *The mapping A is continuous.*

PROOF: It suffices to show that $A^{\#}(\mathcal{T}') \subset \mathcal{M}'$. Let $t' \in \mathcal{T}'$. Then $A^{\#}t'$ lies in $\mathcal{M}^{\#}$, that is, $A^{\#}t'$ is a linear functional on $\mathcal{M}$. Hence,

$$\begin{aligned}\langle m, A^{\#}t' \rangle = \langle Am, t' \rangle &= \int_{\mathcal{H}} (Am)(f)\, dt'(f) \\ &= \int_{\Delta(S)} \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dt'(f)\, dm(\mu) = \langle m, m' \rangle,\end{aligned}$$

where $m' \in \mathcal{M}^{\#}$ is defined by $m'(\mu) = \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dt'(f)$. The order of integration has changed in the third equality by the Fubini theorem. Since $\widehat{U} \circ \Psi$ is bounded continuous, $\widehat{U} \circ \Psi$ is $t' \times m$-integrable and the Fubini theorem can be applied.

Now, it suffices to show that $m' \in \mathcal{M}' = C_b(\Delta(S))$. Since $\widehat{U} \circ \Psi$ is bounded, m' is bounded. To see continuity, let $\mu_n \to \mu$ for $\mu_n, \mu \in \Delta(S)$. Since $\mu \mapsto \widehat{U} \circ \Psi(f, \mu)$ is continuous for each $f \in \mathcal{H}$, it follows that $\widehat{U} \circ \Psi(f, \mu_n) \to \widehat{U} \circ \Psi(f, \mu)$. Observing that $\widehat{U} \circ \Psi$ is bounded,

$$m'(\mu_n) = \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu_n)\, dt'(f) \to \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dt'(f) = m'(\mu)$$

by the Lebesgue dominated convergence theorem. Hence $m' \in \mathcal{M}'$. *Q.E.D.*

LEMMA B.7: *$A(K)$ is closed.*

PROOF: Suppose that $\theta_n = A(\lambda_n m_n) \in A(K)$ converges to $\theta \in \mathcal{T}$, where $\lambda_n \in \mathbb{R}_+$ and $m_n \in \Delta(\Delta(S))$.

Step 1: *m_n has a subsequence $m_{k(n)}$ that converges to some $m \in \Delta(\Delta(S))$.* Since S is finite, $\Delta(S)$ is a compact metric space and so is $\Delta(\Delta(S))$ (AB (1999, p. 482)). Hence, m_n has a converging subsequence.

Step 2: *$\langle Am_{k(n)}, t' \rangle \to \langle Am, t' \rangle$ for any $t' \in \mathcal{T}'$.* By Step 1 and the continuity of A, $Am_{k(n)} \to Am$. Thus this step is proved.

Step 3: $\lambda_{k(n)}$ *converges to some* $\lambda \geq 0$. By Lemma B.4, take $p \in \Delta(Z)$ such that $\widehat{U} \circ \Psi(p, \mu) = \widehat{U}(p) \neq 0$ for any $\mu \in \Delta(S)$. Then

$$(Am)(p) = \int_{\Delta(S)} \widehat{U} \circ \Psi(p, \mu)\, dm(\mu) = \widehat{U}(p) \neq 0,$$

which implies that $Am \neq 0$. Therefore, by the separation property of a dual pair, $\langle Am, \bar{t}' \rangle \neq 0$ for some $\bar{t}' \in \mathcal{T}'$. Note that $\lambda_{k(n)} \langle Am_{k(n)}, \bar{t}' \rangle = \langle \theta_{k(n)}, \bar{t}' \rangle \to \langle \theta, \bar{t}' \rangle$. Then by Step 2, it follows that $\lambda_{k(n)} \to \lambda \equiv \langle \theta, \bar{t}' \rangle / \langle Am, \bar{t}' \rangle$. Since $\lambda_n \geq 0$ for all n, then $\lambda \geq 0$.

Step 4: $\theta \in A(K)$. For all $t' \in \mathcal{T}'$, $\langle \theta_{k(n)}, t' \rangle = \lambda_{k(n)} \langle Am_{k(n)}, t' \rangle \to \lambda \langle Am, t' \rangle = \langle A(\lambda m), t' \rangle$. Moreover, by the hypothesis, $\langle \theta_{k(n)}, t' \rangle \to \langle \theta, t' \rangle$ for all $t' \in \mathcal{T}'$. Note that $\langle \theta_{k(n)}, t' \rangle$ is a sequence in $\mathbb{R}$ and converges to at most one point. Thus, $\langle A(\lambda m), t' \rangle = \langle \theta, t' \rangle$ for all $t' \in \mathcal{T}'$ and $\theta = A(\lambda m) \in A(K)$ by the separation property of a dual pair. *Q.E.D.*

The following lemma uses the generalized Farkas theorem to prove the existence of second-order belief.

LEMMA B.8: *There exists* $m \in \Delta(\Delta(S))$ *such that* $\int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\, dm(\mu) = U(f)$ *for all* $f \in \mathcal{H}$.

PROOF: It is enough to show that $Am = U$ for some $m \in \Delta(\Delta(S))$, where A is defined in (B.1).

First, I will prove that there exists $m \in K = ca_+(\Delta(S))$ that solves $Am = U$. I have already shown in Lemmas B.5–B.7 that the premises of the generalized Farkas theorem are satisfied. Therefore, it suffices to show that if $\langle m, A't' \rangle \geq 0$ for all $m \in K$, then $\langle U, t' \rangle \geq 0$.

Assume that $\langle m, A't' \rangle \geq 0$ for all $m \in K$ and show that

$$\langle U, t' \rangle \geq 0. \tag{B.2}$$

By the hypothesis, $\langle m, A't' \rangle = \langle Am, t' \rangle = \int_{\mathcal{H}} Am\, dt' = \int_{\mathcal{H}} \int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\, dm(\mu)\, dt'(f) \geq 0$ for all $m \in K$. Since $\delta_\mu \in K$ for each $\mu \in \Delta(S)$, it follows that

$$\int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dt'(f) \geq 0 \quad \text{for all } \mu \in \Delta(S). \tag{B.3}$$

Let $t' = \alpha P - \beta Q$ by the Hahn decomposition theorem, where $\alpha, \beta \geq 0$ and $P, Q \in \Delta(\mathcal{H})$. Let $\alpha \geq \beta$. The other case, $\alpha < \beta$, can be proved similarly. If $\alpha = 0$, the statement (B.2) is trivial because $\alpha = \beta = 0$.

Let $\alpha > 0$. Note that (B.3) implies

$$\int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dP(f) \geq \gamma \int_{\mathcal{H}} \widehat{U} \circ \Psi(f, \mu)\, dQ(f) \quad \text{for all } \mu \in \Delta(S), \tag{B.4}$$

where $\gamma = \beta/\alpha$.

Recall from Lemma B.3(ii) that U is unique up to positive affine transformation. Normalize U such that $\int_{\mathcal{H}} U\, d\bar{R} = 0$ for some $\bar{R} \in \Delta(\Delta(Z))$. Since $\widehat{U}$ is the restriction of U, $\int_{\Delta(Z)} \widehat{U}\, d\bar{R} = 0$. Observe that, for all $B \in \mathcal{B}_{\mathcal{H}}$ and $\mu \in \Delta(S)$,

$$\begin{aligned}\Psi(\bar{R}, \mu)(B) &= \bar{R}\big(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\}\big) \\ &= \bar{R}\big(\{p \in \Delta(Z) : p \in B\}\big) \\ &= \bar{R}(B \cap \Delta(Z)) = \bar{R}(B).\end{aligned}$$

The second equality comes from the fact that $\bar{R}$ assigns zero probability outside of $\Delta(Z)$. Thus, $\bar{R} = \Psi(\bar{R}, \mu)$ and $\int_{\Delta(Z)} \widehat{U}\, d\Psi(\bar{R}, \mu) = 0$ for all $\mu \in \Delta(S)$. Then, by the argument in footnote 16, (B.4) implies

$$\int_{\Delta(Z)} \widehat{U}\, d\Psi(P, \mu) \geq \gamma \int_{\Delta(Z)} \widehat{U}\, d\Psi(Q, \mu) + (1-\gamma) \int_{\Delta(Z)} \widehat{U}\, d\Psi(\bar{R}, \mu)$$
$$\text{for all } \mu \in \Delta(S).$$

Hence by Lemma B.3(ii), it follows that

$$\Psi(P, \mu) \succeq \gamma \Psi(Q, \mu) + (1-\gamma)\Psi(\bar{R}, \mu) \quad \text{for all } \mu \in \Delta(S). \tag{B.5}$$

Moreover, for any $B \in \mathcal{B}_{\mathcal{H}}$,

$$\begin{aligned}&[\gamma \Psi(Q, \mu) + (1-\gamma)\Psi(\bar{R}, \mu)](B) \\ &\quad = \gamma \Psi(Q, \mu)(B) + (1-\gamma)\Psi(\bar{R}, \mu)(B) \\ &\quad = \gamma \cdot Q\big(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\}\big) + (1-\gamma) \cdot \bar{R}\big(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\}\big) \\ &\quad = (\gamma Q + (1-\gamma)\bar{R})\big(\{f \in \mathcal{H} : \Psi(f, \mu) \in B\}\big) \\ &\quad = \Psi(\gamma Q + (1-\gamma)\bar{R}, \mu)(B).\end{aligned}$$

Therefore, by (B.5),

$$\Psi(P, \mu) \succeq \Psi(\gamma Q + (1-\gamma)\bar{R}, \mu) \quad \text{for all } \mu \in \Delta(S).$$

By dominance, it follows that

$$P \succeq \gamma Q + (1-\gamma)\bar{R}.$$

Therefore, by Lemma B.3(ii),

$$\int_{\mathcal{H}} U\, dP \geq \int_{\mathcal{H}} U\, d(\gamma Q + (1-\gamma)\bar{R}) = \gamma \int_{\mathcal{H}} U\, dQ. \tag{B.6}$$

Then, by (B.6),

$$\langle U, t' \rangle = \int_{\mathcal{H}} U\, dt' = \int_{\mathcal{H}} U\, d[\alpha(P - \gamma Q)] \geq 0.$$

This completes the proof of (B.2).

Now, apply the generalized Farkas theorem to obtain $m \in K = ca_+(\Delta(S))$ satisfying the equation $Am = U$ or, equivalently,

$$\int_{\Delta(S)} \widehat{U} \circ \Psi(f, \mu)\, dm(\mu) = U(f) \quad \text{for each } f \in \mathcal{H}. \tag{B.7}$$

To prove that m is a probability measure, let $p \in \Delta(Z)$ be such that $\widehat{U}(p) \neq 0$ as in Lemma B.4 and let f be the constant act giving p in every state. Since $U(p) = \widehat{U}(p) \neq 0$, (B.7) becomes

$$\int_{\Delta(S)} dm(\mu) = 1.$$

Q.E.D.

Now, I will show a general property about utility representation.

LEMMA B.9: *Let X be a connected topological space. If two bounded continuous functions $u : X \to \mathbb{R}$ and $w : X \to \mathbb{R}$ represent the same preference on X, then there exists a continuous and strictly increasing function $v : u(X) \to \mathbb{R}$ such that $w = v \circ u$.*

PROOF: Define v on $u(X)$ by

$$v(y) = w(x) \quad \text{if} \quad u(x) = y.$$

Then v is well defined and strictly increasing, and $w = v \circ u$.

To show the continuity of v, note that X is connected and w is continuous. Hence, $v(u(X)) = w(X)$ is connected. Since v is (strictly) increasing, it must be continuous. *Q.E.D.*

LEMMA B.10: *There exists a bounded continuous and strictly increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ such that $\widehat{U} = v \circ u$.*

PROOF: Observe that u and $\widehat{U}$ represent the same preference on $\Delta(Z)$. By Lemma B.9, a continuous and strictly increasing function $v : u(\Delta(Z)) \to \mathbb{R}$ exists such that $\widehat{U} = v \circ u$. Boundedness comes from the fact that $\widehat{U}$ is bounded. *Q.E.D.*

Finally, by Lemma B.3(ii), $V(P) = \int_{\mathcal{H}} U(f)\,dP(f)$ represents $\succeq$ on $\Delta(\mathcal{H})$, and by Lemmas B.3(i), B.8, and B.10, it follows that

$$\begin{aligned} U(f) &= \int_{\Delta(S)} \widehat{U} \circ \Psi(f,\mu)\,dm(\mu) \\ &= \int_{\Delta(S)} v \circ u \circ \Psi(f,\mu)\,dm(\mu) \\ &= \int_{\Delta(S)} v\left(\int_S u(f)\,d\mu\right) dm(\mu). \end{aligned}$$

This completes the sufficiency part of the proof.

B.3. *Proof of Lemma 4.1*

PROOF: (i) Suppose that $\succeq$ satisfies order, continuity, reversal of order, and AA dominance. For any $P \in \Delta(\mathcal{H})$, let $\Pi(P)$ be the act obtained by collapsing all the objective probabilities into $\Delta(Z)$, that is, Π is a function from $\Delta(\mathcal{H})$ into $\mathcal{H}$ such that for every $B \in \mathcal{B}_Z$ and $s \in S$, $\Pi(P)(s)(B) = \int_{\mathcal{H}} f(s)(B)\,dP(f)$. For Π to be well defined, $f(s)(B)$ must be P-integrable as a function of f.

Step 1: *Π is well defined*. Since Z is metrizable, the function $p \mapsto p(B)$ from $\Delta(Z)$ into $\mathbb{R}$ is measurable (AB (1999, p. 485)). Moreover, the function $f \mapsto f(s)$ is measurable. Thus, $f \mapsto f(s)(B)$ is measurable. Since $f(s)(B)$ is bounded, $f \mapsto f(s)(B)$ is P-integrable.

Step 2: *$\int_Z \eta(z)\,d\Pi(P)(s)(z) = \int_{\mathcal{H}} \int_Z \eta(z)\,df(s)(z)\,dP(f)$ for any $s \in S$, $\eta \in C_b(Z)$ and $P \in \Delta(\mathcal{H})$*. When η is a measurable step function (i.e., $\eta(Z)$ is a finite set), this is clear. For any $\eta \in C_b(Z)$, take a sequence η_n of step functions such that $\eta_n(z)$ converges to $\eta(z)$ for each $z \in Z$. Then, by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \int_Z \eta(z)\,d\Pi(P)(s)(z) &= \lim \int_Z \eta_n(z)\,d\Pi(P)(s)(z) \\ &= \lim \int_{\mathcal{H}} \int_Z \eta_n(z)\,df(s)(z)\,dP(f) \\ &= \int_{\mathcal{H}} \int_Z \eta(z)\,df(s)(z)\,dP(f). \end{aligned}$$

Step 3: *Π is continuous*. Fix $s \in S$. Suppose that $P_n \to P$. Note that $f \mapsto \int_Z \eta(z)\,df(s)(z)$ is continuous. Then, by Step 2,

$$\int_Z \eta(z)\,d\Pi(P_n)(s)(z) = \int_{\mathcal{H}} \left(\int_Z \eta(z)\,df(s)(z)\right) dP_n(f)$$

$$\to \int_{\mathcal{H}} \left(\int_Z \eta(z)\, df(s)(z) \right) dP(f)$$
$$= \int_Z \eta(z)\, d\Pi(P)(s)(z).$$

Thus, $P \mapsto \Pi(P)(s)$ is continuous for every $s \in S$. Therefore, Π is continuous.

Step 4: $\Pi(P) \sim P$ *for any* $P \in \Delta(\mathcal{H})$. Reversal of order implies that $\Pi(P) \sim P$ when P has a finite support. Since $\mathcal{H}$ is metrizable, the set of all probability measures on $\mathcal{H}$ with finite support is dense in $\Delta(\mathcal{H})$ (AB (1999, p. 481)). For any $P \in \Delta(\mathcal{H})$, take $P_n \in \Delta(\mathcal{H})$ with finite support such that $P_n \to P$. Then $\Pi(P_n) \sim P_n$ for all n. By continuity and Step 3,

$$\Pi(P) = \lim \Pi(P_n) \sim \lim P_n = P.$$

Step 5: $\Pi(\Psi(P, \mu)) = \Psi(\Pi(P), \mu)$ *for any* $P \in \Delta(H)$ *and* $\mu \in \Delta(S)$. For any $B \in \mathcal{B}_Z$,

$$\begin{aligned}
&\Pi(\Psi(P, \mu))(s)(B) \\
&\quad = \int_{\mathcal{H}} f(s)(B)\, d\Psi(P, \mu)(f) \\
&\quad = \int_{\Delta(Z)} p(B)\, d\Psi(P, \mu)(p) = \int_{\mathcal{H}} \Psi(f, \mu)(B)\, dP(f) \\
&\quad = \int_{\mathcal{H}} \big[\mu(s_1) f(s_1) \oplus \cdots \oplus \mu(s_{|S|}) f(s_{|S|})\big](B)\, dP(f) \\
&\quad = \int_{\mathcal{H}} \mu(s_1)[f(s_1)(B)] + \cdots + \mu(s_{|S|})\big[f(s_{|S|})(B)\big]\, dP(f) \\
&\quad = \mu(s_1) \int_{\mathcal{H}} f(s_1)(B)\, dP(f) + \cdots + \mu(s_{|S|}) \int_{\mathcal{H}} f(s_{|S|})(B)\, dP(f) \\
&\quad = \mu(s_1)[\Pi(P)(s_1)(B)] + \cdots + \mu(s_{|S|})\big[\Pi(P)(s_{|S|})(B)\big] \\
&\quad = \big[\mu(s_1) \cdot \Pi(P)(s_1) \oplus \cdots \oplus \mu(s_{|S|}) \cdot \Pi(P)(s_{|S|})\big](B) \\
&\quad = \Psi(\Pi(P), \mu)(B).
\end{aligned}$$

The third equality is obtained by the change of variables theorem.

Step 6: $\succeq$ *satisfies dominance*. Suppose that $\Psi(P, \mu) \succeq \Psi(Q, \mu)$ for all $\mu \in \Delta(S)$. By Steps 4 and 5,

$$\Psi(P, \mu) \sim \Pi(\Psi(P, \mu)) = \Psi(\Pi(P), \mu).$$

Therefore, $\Psi(\Pi(P), \mu) \succeq \Psi(\Pi(Q), \mu)$ for all $\mu \in \Delta(S)$. Since $\Psi(\Pi(P), \delta_s) = \Pi(P)(s)$, it follows that $\Pi(P)(s) \succeq \Pi(Q)(s)$ for all $s \in S$. For $k = 0, 1, \ldots, |S|$, define $h_k \in \mathcal{H}$ by

$$h_k(s) = \begin{cases} \Pi(P)(s), & \text{if } s > k, \\ \Pi(Q)(s), & \text{if } s \leq k. \end{cases}$$

Then, by AA dominance and Step 4,

$$P \sim \Pi(P) = h_0 \succeq h_1 \succeq \cdots \succeq h_{|S|} = \Pi(Q) \sim Q,$$

which completes the proof of (i).

(ii) Let $f, g \in \mathcal{H}$ and suppose that $f(s) = g(s)$ for all $s \neq s'$ and $f(s') \succeq g(s')$ for some $s' \in S$. By second-stage independence, $\Psi(f, \mu) \succeq \Psi(g, \mu)$ for any $\mu \in \Delta(S)$. Dominance implies $f \succeq g$. *Q.E.D.*

APPENDIX C: UNIQUENESS OF THE SOSEU REPRESENTATION

The following lemma provides some uniqueness properties.

LEMMA C.1: *Suppose that $P \succ Q$ for some $P, Q \in \Delta(\mathcal{H})$ and let the two triples (u, v, m) and (u', v', m') represent $\succeq$ on $\Delta(\mathcal{H})$. Then the following statements hold*:

(i) *u and u' are the same up to positive affine transformation, and so are $v \circ u$ and $v' \circ u'$.*

(ii) *$\int_{\Delta(S)} \varphi \, dm = \int_{\Delta(S)} \varphi \, dm'$ for all $\varphi \in D$, where*

$$D = \left\{ \varphi \in C(\Delta(S)) : \exists \lambda \in ca(T) \text{ such that } \varphi(\mu) = \int_T v(\mu \cdot t) \, d\lambda(t) \text{ for all } \mu \right\}$$

and

$$T = \left[u(\Delta(Z))\right]^{|S|} \subset \mathbb{R}^{|S|}.$$

PROOF: (i) Note that u and u' represent the same preference on $\Delta(Z)$, and so do $\bar{P} \mapsto \int_{\Delta(Z)} v \circ u \, d\bar{P}$ and $\bar{P} \mapsto \int_{\Delta(Z)} v' \circ u' \, d\bar{P}$ on $\Delta(\Delta(Z))$.

(ii) Note that $u' = au + b$ for some $a > 0$ and $b \in \mathbb{R}$, and $v' \circ u' = cv \circ u + d$ for some $c > 0$ and $d \in \mathbb{R}$. Thus, $v'(ax + b) = cv(x) + d$ for any $x \in u(\Delta(Z))$. Then

$$\begin{aligned} \int_{\Delta(S)} v'\left(\int_S u'(f) \, d\mu\right) dm'(\mu) &= \int_{\Delta(S)} v'\left(a \int_S u(f) \, d\mu + b\right) dm'(\mu) \\ &= c \int_{\Delta(S)} v\left(\int_S u(f) \, d\mu\right) dm'(\mu) + d. \end{aligned}$$

Since $\int_{\Delta(S)} v'(\int_S u'(f)\,d\mu)\,dm'(\mu)$ and $\int_{\Delta(S)} v(\int_S u(f)\,d\mu)\,dm(\mu)$ represent the same preference, then

$$\int_{\Delta(S)} v\left(\int_S u(f)\,d\mu\right) dm(\mu) = \int_{\Delta(S)} v\left(\int_S u(f)\,d\mu\right) dm'(\mu) \quad \text{for all} f \in \mathcal{H}.$$

Since S is finite, it follows that

$$\int_{\Delta(S)} v(\mu \cdot t)\,dm(\mu) = \int_{\Delta(S)} v(\mu \cdot t)\,dm'(\mu) \quad \text{for all } t \in u(\Delta(Z))^{|S|}.$$

Integrating both sides gives

$$\int_{u(\Delta(Z))^{|S|}} \int_{\Delta(S)} v(\mu \cdot t)\,dm(\mu)\,d\lambda(t) = \int_{u(\Delta(Z))^{|S|}} \int_{\Delta(S)} v(\mu \cdot t)\,dm'(\mu)\,d\lambda(t)$$

for any $\lambda \in ca(u(\Delta(Z))^{|S|})$. Observe that $(\mu, t) \mapsto v(\mu \cdot t)$ is jointly continuous and bounded, and hence $m \times \lambda$-integrable. By the Fubini theorem,

$$\int_{\Delta(S)} \int_T v(\mu \cdot t)\,d\lambda(t)\,dm(\mu) = \int_{\Delta(S)} \int_T v(\mu \cdot t)\,d\lambda(t)\,dm'(\mu) \quad \text{for all } \lambda \in ca(T)$$

for $T = [u(\Delta(Z))]^{|S|}$, which completes the proof. *Q.E.D.*

By Lemma C.1, characterizing D is crucial in determining the class of m that represents the same preference.

REFERENCES

ALIPRANTIS, C. D., AND K. C. BORDER (1999): *Infinite Dimensional Analysis*. Berlin: Springer. [1578,1592,1594-1596,1600,1601]

ANSCOMBE, F. J., AND R. J. AUMANN (1963): "A Definition of Subjective Probability," *Annals of Mathematical Statistics*, 34, 199–205. [1577,1579,1581]

BERNASCONI, M. (1994): "Nonlinear Preferences and Two-Stage Lotteries: Theories and Evidence," *The Economics Journal*, 104, 54–70. [1588]

BERNASCONI, M., AND G. LOOMES (1992): "Failures of the Reduction Principle in an Ellsberg-Type Problem," *Theory and Decision*, 32, 77–100. [1588]

BUSESCU, D., AND I. FISCHER (2001): "The Same but Different: An Empirical Investigation of the Reducibility Principle," *Journal of Behavioral Decision Making*, 14, 187–206. [1588]

CHOW, C. C., AND R. SARIN (2002): "Known, Unknown and Unknowable Uncertainties," *Theory and Decision*, 52, 127–138. [1588]

CRAVEN, B. D., AND J. J. KOLIHA (1977): "Generalizations of Farkas' Theorem," *SIAM Journal on Mathematical Analysis*, 8, 983–997. [1592,1593]

CUBITT, R., C. STARMER, AND R. SUGDEN (1998): "Dynamic Choice and the Common Ratio Effect: An Experimental Investigation," *Economic Journal*, 108, 1362–1380. [1588]

ELLSBERG, D. (1961): "Risk, Ambiguity and the Savage Axioms," *Quarterly Journal of Economics*, 75, 643–669. [1575]

——— (2001): *Risk, Ambiguity and Decision*. New York: Gerland Publishing. [1575,1582]

ERGIN, H., AND F. GUL (2009): "A Subjective Theory of Compound Lotteries," *Journal of Economic Theory*, 144, 899–929. [1576]

GÄRDENFORS, P., AND N. E. SAHLIN (1982): "Unreliable Probabilities, Risk Taking, and Decision Making," *Synthese*, 53, 361–386. [1588]

——— (1983): "Decision Making With Unreliable Probabilities," *British Journal of Mathematical and Statistical Psychology*, 36, 240–251. [1588]

GILBOA, I., AND D. SCHMEIDLER (1989): "Maxmin Expected Utility With Non-Unique Prior," *Journal of Mathematical Economics*, 18, 141–153. [1581,1585]

GRANDMONT, J. (1972): "Continuity Properties of a von Neumann–Morgenstern Utility," *Journal of Economic Theory*, 4, 45–57. [1595]

GÜTH, W., E. VAN DAMME, AND M. WEBER (2005): "Risk Aversion on Probabilities: Experimental Evidence of Deciding Between Lotteries," *Homo Oeconomicus*, 22, 191–209. [1588]

HALEVY, Y. (2007): "Ellsberg Revisited: An Experimental Study," *Econometrica*, 75, 503–536. [1578,1582,1587,1588]

KELLER, L. R. (1985): "Testing of the 'Reduction of Compound Alternatives' Principle," *Omega*, 13, 349–358. [1588]

KLIBANOFF, P., AND E. OZDENOREN (2007): "Subjective Recursive Expected Utility," *Economic Theory*, 30, 49–87. [1588]

KLIBANOFF, P., M. MARINACCI, AND S. MUKERJI (2005): "A Smooth Model of Decision Making Under Ambiguity," *Econometrica*, 73, 1849–1892. [1576,1584,1587]

KREPS, D. M. (1988): *Notes on the Theory of Choice*. Boulder, CO: Westview Press. [1577,1584, 1588]

KREPS, D. M., AND E. L. PORTEUS (1978): "Temporal Resolution of Uncertainty and Dynamic Choice Theory," *Econometrica*, 46, 185–200. [1588]

NAU, R. F. (2006): "Uncertainty Aversion With Second-Order Utilities and Probabilities," *Management Science*, 52, 136–145. [1576]

QUIGGIN, J. (1982): "A Theory of Anticipated Utility," *Journal of Economic Behavior and Organization*, 3, 323–343. [1576]

RONEN, J. (1971): "Some Effects of Sequential Aggregation in Accounting on Decision-Making," *Journal of Accounting Research*, 9, 307–332. [1588]

SAVAGE, L. J. (1972): *The Foundations of Statistics*. New York: Dover (original ed. Wiley, 1954). [1588]

SCHOEMAKER, P. (1989): "Preferences for Information on Probabilities versus Prizes—the Role of Risk-Taking Attitudes," *Journal of Risk and Uncertainty*, 2, 27–60. [1588]

SEGAL, U. (1987): "The Ellsberg Paradox and Risk Aversion: An Anticipated Utility Approach," *International Economic Review*, 28, 175–202. [1576,1588]

——— (1990): "Two-Stage Lotteries Without the Reduction Axiom," *Econometrica*, 58, 349–377. [1586-1588]

SNOWBALL, D., AND C. BROWN (1979): "Decision Making Involving Sequential Events: Some Effects of Disaggregated Data and Dispositions Toward Risk," *Decision Sciences*, 10, 527–546. [1588]

YATES, F. J., AND L. G. ZUKOWSKI (1976): "Characterization of Ambiguity in Decision Making," *Behavioral Science*, 21, 19–25. [1588]

Dept. of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University, 2001 Sheridan Road, Evanston, IL 60208, U.S.A.; k-seo@kellogg.northwestern.edu.

Manuscript received September, 2006; final revision received May, 2009.