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Carnap and de Finetti on Bets and the Probability of Singular Events: The Dutch Book Argument Reconsidered*

by KLAUS HEILIG

INTRODUCTION

Recent contributions by Baillie and Ellis[1] will have made readers of this *Journal* familiar with the so-called 'Dutch Book Argument'. This provides the rationality constraints for a widely accepted procedure of numerically measuring subjective degrees of belief in the occurrence of singular events by means of personal betting quotients in such a way that they can be taken as a concrete application of the abstract calculus of probability. The basic term here—probability—is defined by several axioms which state, roughly speaking, that the probabilities of mutually exclusive events are real numbers in the range from 0 to 1, both bounds included, which sum up to 1. A betting quotient q is the ratio of the sum hazarded (here called the 'stake') by the individual in question on the occurrence of the event E to the sum S (here called the 'total stake' or the 'gross gain') which will be won if E occurs. The more the individual believes in the occurrence of E, the more he will be prepared to risk for a particular sum S. Hence the betting quotient increases with degree of belief. Since the minimal degree of belief will result in risking nothing, and since the maximal degree will lead to a sum hazarded which does not exceed the expected gross gain, betting quotients satisfy by definition the first axiom of the calculus of probability. And it has been argued that the Dutch Book

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[1] Baillie [1973] and Ellis [1973].

argument shows that the second axiom, called the 'sum condition' by Vickers,[1] is also satisfied: the betting quotients must satisfy the sum condition, for if not a very plausible minimal premise of economic rationality will be violated. This premise is encapsulated in the motto: do not throw money out of the window. Betting quotients and the Dutch Book argument are connected with bookmaking as, for example, it is pursued professionally in Great Britain in connection with horse racing. In this case the motto for our individual would be: do not bet in such a manner that it is already clear before the race that a loss will result. The term 'Dutch Book' was adopted from bookmakers' parlance by Lehmann in 1955 to describe such an unfavourable situation.[2] The Dutch Book argument states conversely, then, that a Dutch Book against the individual is not possible if and only if the betting quotients fulfil the sum condition. Since it would be unreasonable to allow such a Dutch Book against oneself, rationality requires that at the very least one should only declare such betting quotients which fulfil the sum condition. Subjective degrees of belief measured by individual betting quotients could, therefore, indeed be understood as probabilities in the sense of the probability calculus.

My thesis is not that this argument is invalid or false in a formal or logical sense, but that because of its very restrictive *conditions of validity* it is neither of empirical significance nor should it be normative. In other words, neither *can* all conditions be empirically fulfilled simultaneously, nor would it be reasonable to demand that they *should* be fulfilled by rational individuals. Hence it is completely illusory to attempt to prove, with the help of betting quotients, that subjective degrees of belief in the occurrence of singular events fulfil the axioms of the probability calculus.

This thesis follows from a comparison of the main points of the essay in which the Dutch Book argument was presented in full detail for the first time, de Finetti's [1931], with the practice of British bookmakers. Never having visited a race course nor a bookmaker, my knowledge of the latter is restricted to several published sources, whose arguments, however, appear to be correct. Ramsey had already briefly hinted at the Dutch Book argument in his critique of the Keynesian 'degree of partial belief' in 1926.[3] Nevertheless, these isolated comments, which were posthu-

[1] Vickers [1976], specifically chapter 4: 'Coherence and the sum condition', pp. 78–95.

[2] Lehmann [1955], p. 251. Unfortunately, I have not succeeded in discovering the origin of that tradition of British (or American?) bookmakers (mentioned by Lehmann) of designating such a foolish business practice as 'Dutch'.

[3] See Ramsey [1931]. He writes: 'Having degrees of belief obeying the laws of probability implies a further measure of consistency, namely, such a consistency between the odds acceptable on different propositions as shall prevent a book being made against you' (p. 81), 'by a cunning bettor', which would mean that you 'would then stand to lose in any event' (p. 80).

mously published in 1931, cannot be considered as the point of departure of the Dutch Book argument. Rather it was first developed at length by de Finetti in his [1931]. This article will thus be the starting point of my essay: in it, all of the conditions of the validity of the Dutch Book argument are presented (if not discussed) in detail, and in a far clearer fashion than in any of his many later publications, certainly clearer than in his frequently cited article of 1937.[1] The same holds true for the presentations of de Finetti's theory by other authors with which I am familiar. To be sure, de Finetti did not speak of Dutch Books, but rather of 'coherent' betting quotients. Nevertheless, betting quotients are coherent when they do not allow for a Dutch Book, and they do not allow for one when they fulfil the sum condition. So, 'Coherence-theorem' and 'Dutch Book argument' are two different names for the same thing.

Lehmann's article and two others by Kemeny and Shimony[2] all reply to Rudolf Carnap's attempt in his [1950] to introduce the concept of inductive or logical probability$_1$ (a quantitative concept completely separated from that of statistical probability$_2$) as a degree of confirmation.[3] This was done with the help of *fair* betting quotients. These fair betting quotients did not originally have anything to do with de Finetti's coherent betting quotients. But Lehmann, Kemeny and Shimony attempted to provide Carnap's original idea with a different foundation by applying axioms for degrees of confirmation based on the 1937 version of de Finetti's coherence-theorem. Carnap[4] himself accepted this suggestion and began (before 1961) to give up his original attempt in favour of de Finetti's theorem. Stegmüller, who has argued strongly for Carnap's ideas, introduced a distinction between Carnap I and Carnap II as a result of other evident modifications in his positions.[5] I shall adopt this convention. Although Carnap II no longer supports the arguments of Carnap I which were connected with fair betting quotients, he still speaks of them. Furthermore Carnap II continues to uphold several of the consequences of the original exposition, consequences which have affected the reception of de Finetti's theorem by other authors. For the sake of clarity, therefore, it would be helpful to first investigate briefly fair betting quotients and inductive probabilities in Carnap I, and to offer some reasons for the transition to coherent betting quotients (1). The latter, together with de Finetti's subjective probabilities, will then be considered in detail (2). A comparison with the economic behaviour of bookmakers will then be

[1] de Finetti [1937].
[2] Kemeny [1955] and Shimony [1955].
[3] Carnap [1950], Preface, p. viii and *passim*.
[4] See Carnap and Jeffrey [1971]. Introduction, p. 2.
[5] Stegmüller [1973], where the new theory of Carnap II is presented and discussed in detail.

undertaken in 3. The reasons for the failure of the Dutch Book argument will then become evident: rational betting quotients are not probabilities at all. This is pointed out in 4. In 5 I show that even the sharpening of the concept of coherence to strict coherence by Shimony and Carnap II does not affect this conclusion.

1 FAIR BETTING QUOTIENTS AND INDUCTIVE PROBABILITY IN CARNAP I

In his *Logical Foundations of Probability* Carnap attempts to distinguish sharply between probability$_2$, a relative frequency in the statistical sense, and the logical or inductive probability$_1$ which, as a degree of confirmation, indicates how strongly an hypothesis is confirmed by the experimental data. This degree of confirmation is measured by the betting quotient which indicates those conditions under which an individual who has the experimental data at his disposal is willing to bet on the correctness of the hypothesis. These betting quotients should be *fair*, i.e., they should not favour either partner. Whether the individual considers a quotient to be fair or not can be ascertained by whether or not he would be willing to exchange roles. Technically, this would mean that he would be equally willing to bet *on* the hypothesis with the betting quotient q, as he would be to bet *against* the hypothesis and consequently *on* the anti-hypothesis, with the betting quotient $(1-q)$.[1]

One can, in general, easily ascertain which betting partner has bet on an hypothesis or an event and which against it by noting which of the two achieves a (positive) net gain when the hypothesis proves to be correct or when the event occurs. In general, the following pattern is yielded:

Table I

	Net result of the bet on E	Net result of the bet against E
E occurs	$S-qS=(1-q)S$	$-S+qS=-(1-q)S$
E does not occur	$-qS$	$+qS$

Given positive values of S, which we can presuppose here with Carnap I, it can be established that that person who achieves a positive net gain when the event occurs bets *on* the event and that person who achieves a positive net gain when it fails to occur bets *against* the event. These characteristics could be called the *direction* of the bet.

[1] See Carnap [1950], p. 164.

Thus, objective criteria not subjective notions are decisive in determining the fairness of betting quotients. However, in his attempt to explicate these objective criteria Carnap falls back into the conceptual world of statistical probability$_2$. Although he assumes that betting on *singular* hypotheses is 'the most important kind of betting situation',[1] he nevertheless determines the fair betting quotient for the hypothesis that an object b exhibits the characteristic M as the relative frequency of objects possessing the characteristic in question within a reference class of otherwise similar objects. The degree of confirmation of the supposedly singular hypothesis is then interpreted as the degree of confirmation of an hypothesis which is *repeatable* under the same conditions. And this latter is itself explained by the relative frequency of objects possessing the characteristic in question. Instead of the elliptical formulation: 'The probability$_1$ that the object b possesses the characteristic M amounts to $\frac{1}{5}$', one could say equivalently, but more clearly: 'The probability$_2$ that objects of the sort b exhibit the characteristic M amounts to $\frac{1}{5}$.' Hence, every probability$_1$-proposition could be replaced by a probability$_2$-proposition.

It seems clear, then, that the concept of fairness in Carnap I is identical with the concept of a fair lottery, where the weighted arithmetical average of the prizes, normally known as 'mathematical expectation', is equal to the fair price of a ticket. If the relative frequency within the reference class of objects with the characteristic M amounts to $\frac{1}{5}$, so that the fair betting quotient also equals $\frac{1}{5}$, then in a situation of n bets with this betting quotient where n times 1 pound is staked, in exactly $\frac{1}{5}$ of all bets 5 pounds will be won, and in $\frac{4}{5}$ of all bets 1 pound will be lost. The sum of the stakes equals the sum of the winnings. The betting is fair in the well-known sense that with the conclusion of all bets none of the participants gains or loses, so that each could quite happily take the role of the other. Fairness in a game of chance, and hence also with Carnap's fair betting quotients, is a requirement which is only defined for probability$_2$ and only for 'very many' or all bets. For there is no rational argument which could explain why anyone, in using a betting quotient which is fair in the long run, should be prepared to reverse his bet, to change its direction, in the case of a single bet. Plausible counter-examples are provided by extreme betting quotients. In a situation where one could bet 1 pound only once on an event whose probability$_2$ amounted to 1/1000, one could, with a fair betting quotient of 1/1000, attain a net win of 999 pounds if the event actually occurred and otherwise would lose 1 pound. Probably no one in this situation would be willing to exchange roles with his opponent who can lose 999 pounds but only win 1 pound.

[1] *Op. cit.*, p. 167.

To show as clearly as possible the difference between this and de Finetti's coherent betting quotients, it should be mentioned that, whereas according to Carnap I one can speak of the fairness of a single betting quotient, for de Finetti only all betting quotients together can be coherent or not coherent. Furthermore, the problem of the sum condition does not exist for Carnap I, since probability$_1$ can always be explained by probability$_2$, that is by a concept which no one would deny fulfils the axioms of the calculus of probability. This reversion had already been clearly established by von Mises in 1951:

> I do not see why one cannot admit to begin with that any numerical statements about a probability$_1$, about plausibility, degree of confirmation, etc., are actually statements about relative frequencies.[1]

Carnap's bets are not bets in the proper sense, *i.e.*, contracts which are *not* governed by the laws of chance,[2] but rather they are just as much games of chance as are lotteries.

2 COHERENT BETTING QUOTIENTS AND SUBJECTIVE PROBABILITY IN DE FINETTI

Whereas in his [1950], and even in the second edition of 1962, probability was primarily intended to be applied to inductive logic, Carnap's interests subsequently shifted in the direction of rational decision theory.[3] This was developed in economics for the purpose of rationalizing non-repeatable decisions and therefore its central maxim of the maximization of expected utility essentially deals with the application of a subjective concept of the probability of singular events. Consequently Carnap no longer pursues his previous argument involving fair betting quotients, presumably because he recognised the obvious weaknesses of his original theory. Rather he adopts de Finetti's theory of subjective probability together with his coherence-theorem.

In de Finetti, there is no implicit connection with probability$_2$. Rather subjective probabilities relate to single events such as horse races which are not repeatable under the same external conditions. His attempt to

[1] von Mises [1957], p. 97. Within the framework of recent attempts to explicate probability$_2$ by means of propensities to produce relative frequencies rather than by relative frequencies themselves, Hacking ([1973], p. 485) comes to the same conclusion: 'I shall maintain that all degrees of confirmation, P_1, depend on statements of propensity using P_2.' Stegmüller [1973] levels a supposedly 'lethal' objection against the work of von Mises, which I attempted to neutralise in my [1975] article by reviving the frequency theory using an antidote. This article also contains an initial argument against de Finetti, which however, as I have since discovered, is not correct.

[2] Sidney [1976], p. 3.

[3] This occurred at the latest in 1965 with the essay 'Inductive Logic and Inductive Intuition'; see Carnap [1968], p. 260.

measure a subjective degree of belief (*grado di fiducia*) is, in terms of his original claim, an empirical, descriptive or operational one. Subjective probabilities must also, in principle, be measurable by means of betting quotients which arise empirically.[1] But the results of empirical research have apparently not agreed with the theory very well, for, in a new footnote to the English translation, de Finetti shifts to the normative level (which is the only level upon which Carnap II argues) when he writes that (his) 'probability theory is not an attempt to describe actual behaviour; its subject is coherent behaviour, and the fact that people are only more or less coherent is inessential.'[2]

The three validity conditions of de Finetti's coherence-theorem can be most simply understood by way of extracts from the original. I have indicated which condition is being described by inserting the relevant numbers in parentheses following the pertinent passages.

> In order to measure the degree of belief that a given subject O is supposed to have in the occurrence of an event E, we must presume . . . that he could be obliged to hold a bank of bets (1) for or against (3) a certain number of events which would include event E among them. . . . The conditions of the bet are determined in the following manner: It is the task of subject O, who holds the bank, to determine the price p of a voucher or bond(1) which entitles one to receive one Lira when a particular event E occurs. This means that he is obliged to buy or sell as many vouchers at the given price(3) as the public wishes(2). If a bettor appears at O's bank and wishes to bet on the event E, he buys a voucher at the price of pS which gives him the right to demand a particular sum S if he wins the bet, i.e., if E occurs. (Conversely, if he wishes to bet against the event he can commit himself, in exchange for the payment of an amount pS from his opponent, to pay the sum S if he loses the bet, i.e., if E occurs. This case is transformed into the previous one if we consider negative values of S there (3).)

and:

> The evaluation of the probability of the event E with p signifies that the one who holds the bet-bank(1) declares himself prepared to accept every bet with any bettor who can fix the gross gain S as he wishes(2), whether positively or negatively(3).[3]

We can, accordingly, tentatively formulate the three conditions for determining whether an individual's evaluation of probabilities is coherent or non-coherent as follows. The individual must state that he is prepared:

(Condition 1) to strike bets on all events at the betting quotients which he himself fixed;

(Condition 2) to allow the opponent to fix the amounts of the gross gains and thereby the stakes; and

[1] See de Finetti [1937], p. 150.
[2] *Ibid.*, p. 111. See also Carnap [1971*a*], p. 13.
[3] de Finetti [1931], p. 304 and p. 308.

(Condition 3) to allow the opponent to determine the sign of the gross gains according to his wishes.

Disregarding condition 3 for the moment, it is apparent that the first two conditions of the coherence-theorem do not describe the behaviour of a normal race goer, but rather that of a professional bookmaker. The coherence-theorem, then, can only be valid for bookmakers who, as is well-known, declare themselves willing to accept any amount as stakes, under conditions which they themselves have fixed. Thus when Hacking, in his account of the matter, writes, 'we dispense with the irrelevant custom of leaving our money in the hands of a bookmaker or arbiter until the issue is settled', since this custom 'is due to human dishonesty and has nothing essential to do with betting',[1] he misunderstands the role which bookmakers play in connection with the Dutch Book argument. It may be sensible in private betting to engage a third person as referee who arbitrates in cases of doubt or conflict and pays out the total stake to the winner. In professional betting, however, the bookmakers are *not* arbiters of the bets of others but are betting partners themselves, inasmuch as the result of the race and, accordingly, of the bet is established by others, namely, the organisers of the race (the Jockey Club).

Since the coherence-theorem can only be valid for bookmakers, it seems reasonable to inquire into the principles according to which they conduct their business, in particular, whether or not they fix their betting quotients so that they fulfil the sum condition.

3 REAL BETTING QUOTIENTS AND THE ECONOMICS OF BOOKMAKING

First of all, it is important to make clear the difference between totalisator-betting and betting at fixed odds. In totalisator-betting the win dividends are established after the race by dividing the total stakes among those bettors who bet on the winning horse. The bettors bet among and against each other, which is why this sort of betting is called *pari mutuel* in France, where it appears to have been developed.[2] Hence, the organiser of the betting takes no risks since he has no liabilities. On the contrary, if the rules of the game allow him to keep a certain percentage of the sums paid in he does a safe business. He can, without hesitation, accept any amount as a stake on any horse.

With betting at fixed odds on the other hand the bookmaker commits himself before the race to paying his opponents a sum if a particular

[1] Hacking [1968], p. 284 and [1967], p. 316.
[2] See p. 1103 of vol. 8, of the 15th edition of the *Encyclopaedia Britannica* (article on "Wagering", pp. 1103–5).

horse wins, a sum which is calculated by means of the betting quotient from the opponents individual stake. This form of betting, unusual in Germany, is the dominant form in Great Britain. It is evident that as opposed to mechanical totalisator-betting, betting at fixed odds demands the skilful consideration of certain economic factors if the bookmaker is to make a living.

British bookmakers do not usually give the conditions under which they wish to make bets in the form of betting quotients, but rather in the form of 'odds *against* a particular horse winning a race',[1] which are also known as 'prices'.[2] These indicate the relation between the bookmaker's stake and the opponent's stake, amounting to the total stake S_i, and hence, can be written and dealt with as fractions. Odds of '4 to 1' signify, then, that for every money unit of the opponent's stake which is bet *on* the victory of a particular horse, the bookmaker adds another four money units and pays out a total of five (the take-out) if the particular horse chosen wins the race. A betting quotient q is therefore related to the corresponding price p as follows:

$$q = \frac{1}{p+1}, \quad \text{and conversely} \quad p = \frac{1}{q} - 1$$

The bookmaker always bets *against* a horse, while the opponent always bets on it. The bookmaker says 'this horse will not win'; the opponent, called the backer or the punter in standard racing terminology, says 'it will win'. Bookmakers offer prices for all horses and accordingly offer bets of any magnitude wished (below particular limits which are determined by their financial reserves). They thereby fulfil conditions 1 and 2 of the coherence-theorem; not, however, condition 3, since they only accept positive S_i.

What do they do, however, in order to secure themselves against a Dutch Book? For bookmakers must above all avoid a Dutch Book against themselves, a situation which they call 'over-broke', which is 'worse than just being broke'.[3] This danger is always present if the sum of the betting quotients is smaller than 1. An opponent can then make a system of bets by means of a skilful combination of stakes on all horses which guarantees him a gain. He can most simply calculate how much he must hazard for each bet if he undertakes to achieve a gross gain S of the same magnitude in every single bet. This amount is determined, when only whole

[1] Chenery [1963], p. 25.

[2] Sidney [1976], p. 4. It is from this book, particularly chapter 6, pp. 85–106, that I have learned most about bookmaking. For the backer's point of view, Mitchell [1972], chapters 1 and 2, pp. 1–43, is a valuable source.

[3] Sidney [1976], p. 96.

money units are used, as a (lowest) common multiple either of the sums of the numerators and denominators of the odds (prices) or of the reciprocals of the betting quotients. If, for example, the prices (odds) are 4/1, 3/1 and 1/1, and therefore the corresponding betting quotients 1/5, 1/4 and 1/2, then 20 is the lowest common multiple. Consequently the opponent must stake 4 on the first horse, 5 on the second, and 10 on the third, a total of 19, in order to win a gross gain of 20, independently of the outcome of the race, and, therefore, to be certain of a total gain of 1.

In order to prevent such a Dutch Book (the over-broke-stake) the bookmaker will raise the sum of the betting quotients to 1 or to more than 1. That a Dutch Book against the bookmaker is impossible in either of these cases can be proved as follows.

Presuppositions

(1) The opponent must bet on all events. Otherwise a Dutch Book is impossible already, independently of the sum of the betting quotients, since the opponent must suffer a loss if an event occurs on which he had not bet. Hence, all stakes q_iS_i must be > 0, and consequently, since the betting quotients q_i cannot by definition be < 0, all q_i and all S_i must be > 0.

(2) The sum of the stakes $\Sigma q_iS_i = C$.

(3) If the horse i wins the total gain $G_i = S_i - C$.

Proposition

If $\Sigma q_i \geqslant 1$, not all G_i can be > 0.

Proof

Assume, on the contrary, that all $G_i > 0$, then it follows that all $q_iG_i > 0$ and that $\Sigma q_iG_i > 0$. Using (3), one gets $\Sigma q_i(S_i - C) > 0$ or $\Sigma q_iS_i - \Sigma q_iC > 0$. By (2), $C - \Sigma q_iC > 0$ and hence $\Sigma q_i < 1$. This means, however, that all G_i can be > 0, only if $\Sigma q_i < 1$. Conversely not all G_i can be > 0 if $\Sigma q_i \geqslant 1$. Q.E.D.

This theorem, which underlies the usual practice of bookmakers, guarantees that they adhere to the premise of rationality mentioned in the *Introduction* above. Because this states only negatively what should be avoided and not what should be done positively, I shall term it *a negative* rationality premise for the economic behaviour of bookmakers.[1] Furthermore, because further premises will emerge in the course of the exposition, I shall term the above the *first negative rationality premise.*

[1] The expression 'rationality premise' comes from Baillie [1973], p. 393.

If actual bookmakers can avoid a Dutch Book by means of betting quotients which fulfil the sum condition as well of those which add up to more than one, the question arises as to which of these alternatives they will choose. Certainly every bookmaker, as a businessman, acts with the aim of making a gain through his activity. This *positive rationality premise* is the most general economic principle for any professional business activity and hence in particular for bookmaking. Consequently, all bookmakers together will strive to pay out less in winnings than they had collected from all of the opponents. As with the totalisator, they therefore retain a certain percentage of the stakes. They do not do this, however, by simply subtracting their own profit from the sum of the stakes and paying out the rest; they have, after all, made bets at fixed odds. Rather, they determine the odds from the beginning so that if all bets are made sufficiently often this will be the result. The group of all bookmakers will, however, only achieve this goal if every individual strives for it. Consequently, every bookmaker fixes his betting quotients in such a fashion that they add up to more than one, usually to 118 per cent.[1] The excess percentage represents the bookmaker's margin of gain. For a single race, this is only a potential gain, because he cannot be certain of having set the right relation on all horses. But over the years, a margin of this order is realised; 10 to 20 per cent 'is probably a good indicator both of bookmakers' gross margins and what backers collectively can expect to lose in the long term'.[2] If a bookmaker is lucky he can already be certain of a gain before the race, just as the opponent can if, conversely, the sum of betting quotients is less than 1. Whereas, to be sure, the clever opponent cannot be prevented from enjoying the benefits of the positive effect of a Dutch Book when the sum of the betting quotients is favourable for him, the bookmaker really must have luck since he has no direct influence on how much is staked on what. Of course, by changing the betting quotients, he can try to make his customers strike just those bets which he still needs in order to 'bet over'. This agreeable state of affairs which is the ideal position for any bookmaker,[3] was known in the days of the pioneers of British race course betting as 'making a book' or 'round betting'.[4] In this way we have also uncovered the proper meaning of the term 'bookmaking'.

If the bookmaker, however, were to fix his betting quotients in accordance with the sum condition, he would just be a pure 'stakeholder' in Hacking's sense. Like the totalisator, he would only collect the bets of the backers and mediate them against one another, without having a fixed

[1] According to Sidney [1976], p. 96 and to information I received by telephone from a specialist in a large firm of British bookmakers.

[2] Mitchell [1972], p. 13.

[3] *Ibid.*, p. 7.

[4] Sidney [1976], p. 96.

percentage or even retaining anything in the long run. Therefore, although he would fulfil the *first negative rationality premise* he would *not* fulfil the *positive* one. Certainly under these conditions no-one would make bookmaking his profession, since it would be impossible to earn anything.

In the practical business of the bookmaker, which takes place at a race course with little more than fifteen minutes between two races,[1] the initial betting quotients are only specified bids for betting with which to open the market. In this market, in which many backers and several bookmakers participate, the bookmaker constantly changes his prices, for two reasons.

In the first place, the various bookmakers offer bets on the same horse at initially varying prices. So long as a bookmaker is interested in concluding bets on a particular horse, he must adjust his price to that of the other bookmakers, since the bettors will give preference to longer odds and consequently they will strike their bets with the bookmaker who offers them the best terms.

In addition to these *external* reasons for price changes, there are also some important *internal* reasons. If, after the betting market opens, the bookmaker assumes high liabilities by accepting several bets on the horse X with the longest odds so that the amount to be payed out is very high in relation to the stakes paid in by the backers, he must then attempt to lessen his risk. He does this by diverting as much money as possible onto other horses with up to this point shorter odds. The liability arising from the bets on horse X (in other words, the amount which the bookmaker would have to pay from the cash which he brought to the races to cover the difference between the sum taken in as stakes and that which would have to be paid out at that stage if X won) can only be diminished if more money is played on horses other than X. So, to control his liability, the bookmaker will reduce the price for X from, *e.g.*, 10/1 to 5/1 and lengthen that for the other horses. The new betting quotients must, of course, also add up to at least one in order to exclude a Dutch Book.

Having sketched the basics of bookmaking, I now want to examine more closely the opponent, whom Ramsey called the 'cunning bettor', a figure essential for de Finetti's argument.[2] The 'cunning bettor' does not, at first glance, conform to the picture which one has of a normal bettor who bets on the victory of *one* horse, but not on the victory of each horse simultaneously. Our opponent has probably no chance to make a Dutch Book against *one* bookmaker at *one* point in time, since no bookmaker would be stupid enough to offer betting quotients with a sum of

[1] Mitchell [1972], p. 34.

[2] In the meantime, de Finetti has suggested a second procedure supposedly equivalent to the first. Thus the consideration of the older and generally accepted argument does not lose its significance; *cf.* de Finetti [1974], pp. 83–9, and Gillies [1972], pp. 140–1.

less than one. This would simply be throwing money out of the window. However, when one considers the changes in prices given by the various bookmakers, it becomes apparent that the system bettor can succeed in betting over in two ways.

If the prices are different from bookmaker to bookmaker at one point in time, he might be able to bet on, say, three horses with one bookmaker and on the rest with the other bookmakers in such a way that the betting quotients of *his* bets add up to less than one. In this case, he would have made a Dutch Book in his favour, if not against one, then against several bookmakers. Mitchell[1] gives an example of such a *static* Dutch Book, made against several bookmakers at one point in time (though the example is atypical for horse race betting since it deals with a match, a race of only two competitors). For the 1971 Boat Race, two bookmakers offered bets according to the following conditions:

Table II

Bookmaker	Cambridge	Oxford	Sum
A	$p = 2/5,\ q = 5/7 = 0.71$	$p = 2/1,\ q = 1/3 = 0{\cdot}33$	1·04
B	$p = 1/6,\ q = 6/7 = 0.86$	$p = 5/1,\ q = 1/6 = 0.17$	1·03

Although each bookmaker could avoid a Dutch Book against himself, the skilful opponent could, for example, stake 30 on Cambridge with *A*, and 7 on Oxford with *B*. He would then win something in either case since his betting quotients (0.71 and 0.17) add up to less than one. His stakes then amount to 37, whereas the gross gain in either case would amount to 42.

But an opponent can also complete a Dutch Book, a *dynamic* one, against *one* bookmaker if he initially bets on several horses and then bets on the remainder afterwards, if the bookmaker is forced by internal reasons to change his prices. Such a case has been indicated by Sidney[2]: a financially strong opponent bets a great deal at the beginning on the horse with the longest odds, that is with the smallest betting quotients. This potential liability forces the bookmaker to shorten these odds and correspondingly lengthen those on the other horses. Since longer odds mean smaller betting quotients, the opponent can again attain a Dutch Book by now betting on the remaining horses if the sum of the betting quotients of *his* bets is less than one. It is precisely because the bookmaker wants to avoid

[1] Mitchell [1972], p. 37–8. [2] Sidney [1976], p. 97.

a static Dutch Book that he must allow a dynamic one, and there is no protection against such a risk.

Thus, the opponent who simultaneously bets on all horses, at first glance a strange figure, turns out to be the most dangerous backer, not only for the single bookmaker, but for all bookmakers together since he tries systematically to take advantage of the inconsistencies that inevitably occur in the hectic routine of the bookmaking business.

4 RATIONAL BETTING QUOTIENTS ARE NOT PROBABILITIES AT ALL

From the first two conditions of validity of the coherence-theorem it followed that it can only be meaningful for professional bookmakers. However, since the betting quotients of actual bookmakers total more than one, the successful practice of the British bookmakers would stand in opposition to the Dutch Book argument, if it were not for the third condition underlying it. The opponent cannot only freely determine the amount, but also the *sign* of the gross gain, and thereby of the stakes. This means, however, that he can choose the *direction* of the betting, which becomes clear by considering the following. In Table I on p. 328, the first column with positive values of S is the result column of the bettor who bets *on* an event. By changing the positive values of S to negative ones, the first column is transformed into the second, which represents the results of the bet *against* the event. Thus, $(1-q)S$ becomes $(1-q)(-S)$ and hence $-(1-q)S$, and $-qS$ becomes $(-q)$ $(-S)$ and thus $+qS$. (This is in effect how it is expressed by de Finetti: see the last sentence of the first quotation on page 331 above.)

It is now easy to see which choice an opponent would make in order to attain a Dutch Book to his advantage; of course, he would always choose stakes with the same signs for all bets, since otherwise at least one G_i would have a different sign from the remainder and this would immediately exclude a Dutch Book:

If the individual fixes the betting quotients in such a manner that their sum is *smaller* than one, the opponent would choose *positive* gross gains S_i, and thereby positive stakes q_iS_i. The individual then assumes the role of a bookmaker, betting *against* the events and foolishly giving his opponent the opportunity to make a Dutch Book.

If, however, the individual fixes the betting quotients in such a manner that their sums are *greater* than one, then the opponent would choose *negative* gross gains S_i, and thereby negative stakes q_iS_i. The individual then bets *on* the events as does an actual opponent, but can be forced by his opponent to place bets on all the events in such a way that the opponent assures himself of a gross gain.

The following example should demonstrate the case concerning negative stakes. Given betting quotients of 1/2, 1/3 and 1/4, the sum of which is

$$\frac{6+4+3}{12}=\frac{13}{12},$$

then the betting opponent would choose stakes of -6, -4, and -3, totalling -13, in order to 'receive' a 'win' of -12 in any case. He is then assured of a gross gain of $G_{1/2/3}=-12-(-13)=+1$.

In practice this means as de Finetti made clear in the first passage quoted above (p. 331) that the opponent forces the individual to place bets of 6, 4, and 3 *on* the various horses with him. He himself, however, must pay out only 12. From this we must draw the same conclusion that de Finetti does: if an individual fulfils *all three* conditions, he excludes the possibility of a Dutch Book against himself if and only if the betting quotients fulfil the sum condition.

Hence the coherence-theorem and its explicit conditions can now be summarized as follows.

If an individual is willing to:

(Condition 1) make bets on all events at betting quotients that he himself has fixed;

(Condition 2) have the amounts of the gross gains and the stakes

(Condition 3) as well as their signs, and thereby the direction of the betting, arbitrarily fixed by the opponent,

then a Dutch Book against the individual is excluded only if the sum of the betting quotients is exactly one,

because otherwise the opponent can make a Dutch Book against the individual. If the sum of the betting quotients is *less* than one, he can do this through *positive* stakes; if it is *more* than one, through *negative* stakes.

Since the individual is obliged by these three conditions to accept every bet, Shimony's and Baillie's objections which suggest that bets could be rejected do not apply. Thus Shimony has written:

In spite of the elegance of de Finetti's theorem, however, the argument in toto has a loophole: a person may be unwilling to have a Dutch book made against him and yet willing to forego the automatic protection against this contingency which laying bets in conformity with the laws of probability provides; in other words, he may be willing to pay the price of vigilance against certain wily proposals of betting combinations in order to preserve his freedom to arrange a betting strategy with other purposes in mind.

And, Baillie similarly writes:

We would have to suppose that our partner would attempt to take advantage of us by setting stakes for which he was bound to win, and further, that we would be silly enough to accept.[1]

My own criticism of the coherence-theorem is now easily formulated. If an individual fulfils the sum condition by satisfying the conditions of the theorem, then he certainly avoids a Dutch Book and hence fulfils the first negative rationality premise. However, he also violates the positive one, since in the long run he can attain no wins and he becomes merely a 'stakeholder', receiving nothing in return for his efforts. It is therefore most unreasonable to expect all conditions to be fulfilled. It follows that no rational person would fulfil all of them and that it would also be unreasonable to demand normatively their fulfilment from a rational person. Under the conditions of de Finetti's theorem, rational betting quotients are not possible in the sense of the negative as well as of the more comprehensive positive rationality premise. Conversely, betting quotients which do fulfil both premises do not fulfil the sum condition. Therefore, although the Dutch Book argument is logically valid *under* its own conditions, *because* these conditions are unrealistic it has no descriptive or prescriptive significance. Hence it is illusion to want to measure subjective probabilities of singular events by means of betting quotients so that they fulfil the axioms of the calculus of probability.

5 STRICTLY COHERENT BETTING QUOTIENTS IN CARNAP II

This criticism of de Finetti applies equally to Carnap's position in his last published paper, for here, stimulated by the articles of Kemeny, Lehmann and Shimony, Carnap adopts de Finetti's theorem (though with certain technical modifications). Carnap discusses[2] how, by choosing negative gross gains S_i instead of positive ones, a bet on an event or an hypothesis can be transformed into a bet against. However, he is even more removed from the practice of actual bookmaking than is de Finetti. He no longer requires that the opponent first bets and then receives gross gains later (or, in the case of negative stakes, that the individual pays in, later receiving the gross gains from the opponent), so that one would first have to calculate the net gains. Instead, the net gains change hands only *after* the betting has been decided. The individual now should be willing to change betting direction, in accordance with condition 3 of the coherence-theorem, if he considers one (!) betting quotient to be fair. Thus, the *consequences* of the fairness of betting quotients are the same in

[1] Shimony [1967], p. 331, and Baillie [1973], p. 395.
[2] See Carnap [1971*b*] pp. 106–7.

Carnap II as they are in Carnap I. But since he does not, and probably cannot indicate the *requirements* of fairness any longer, the mention of fair betting quotients is just a vague reminder of his earlier exposition, particularly because the willingness of the individual to exchange roles is no longer important. While de Finetti, in accordance with condition 3, allows the alert opponent to choose positive or negative stakes to his advantage, depending upon whether the sum was less than or greater than one, Carnap II eliminates the opponent completely as an active being. He does this by introducing the trick of making the signs of the gross gains dependent automatically upon the sum of the betting quotients. For this purpose he introduces the auxiliary quantity $d = \Sigma q_i - 1$ and, to prove that coherent betting quotients fulfil his basic axiom A 3 (corresponding to the sum condition) continues: 'Then either $d > 0$ or $d < 0$; in the latter case $S < 0$.'[1] The second part of the sentence can be understood only if one has learned from a diagram[2] that the gross gain is defined as $S = 1/d$, so that it is negative if $\Sigma q_i < 1$, and positive if $\Sigma q_i < 1$. This would seem to contradict de Finetti's maxims. But Carnap speaks about stakes and gains of the *individual*, and if these are seen from the perspective of de Finetti's opponent, then it is clear that the signs are inverted. Hence, Carnap's results are identical here with those of de Finetti.

In conclusion I shall discuss a *second negative rationality premise* which up until now has not been considered. In his review of de Finetti in 1955 Shimony suggests a sharpening of the concept of coherence. Since that time it has been dealt with under various names, collected together by Carnap II under the terms 'coherence I' and 'coherence II'.[3] De Finetti's coherent betting quotients prohibit a situation in which all bets together could lead to a certain loss by the individual. This would happen in actual bookmaking, when the sum of the stakes is smaller than the smallest gross gain and, hence, all $G_i > 0$. Following Ellis,[4] I shall designate such a Dutch Book, the only kind that we have considered until now, as a Dutch Book$_1$ parallel to the distinction between coherence I and coherence II. If this situation is avoided, however, it does not necessarily follow that the individual can win in at least one case. Rather, even when not all G_i can be > 0 because of $\Sigma q_i \geqslant 1$, it is still possible to imagine another case whereby the individual 'can at best lose nothing, and in at least one possible eventuality he will suffer a positive loss'. Shimony who wrote this in his [1955], seems to have been the one to discover this possibility.[5] This case, which is not quite so unfavourable for the bookmaker, but does not allow a positive gain in the long run, occurs if the

[1] *Ibid.*, p. 110.
[2] *Ibid.*, p. 112.
[3] *Ibid.*, p. 114.
[4] Ellis [1973], p. 131.
[5] See *op. cit.*, p. 9.

sum of the stakes is exactly the same as the smallest gross gain. Thus (at least) one G_i is equal to zero, while the others are positive. But of course such a Dutch Book$_2$ should also be avoided. And if betting quotients fulfil this condition, that is, of excluding not only Dutch Books$_1$, but Dutch Books$_2$ as well, they are, as Carnap II calls them, *strictly coherent*. Kemeny, alluding to Shimony's idea, spoke in 1954 of 'fair' and 'strictly fair' (probably having the fair betting quotients of Carnap I in mind) and found these terms 'more suggestive'.[1] Carnap II on the contrary finds them 'misleading'.[2]

Since for Dutch Books$_2$ there is less possibility for varying the gross gains than with Dutch Books$_1$, one would expect that coherence II imposes more conditions on the betting quotients than coherence I. But even with Shimony and Kemeny, if I am correct, strictly coherent betting quotients fulfil exactly the same standard axioms of the probability calculus as quotients which are simply coherent. To be sure, Carnap II introduces an additional axiom A 5 of *regularity* which is supposed to be a necessary prerequisite of coherence II: 'If (the credence function) C is not regular, it is not strictly coherent,' and 'Regularity . . . is equivalent to strict coherence.'[3] But the regularity axiom in effect only requires that each individual betting quotient be greater than zero, and this was already (as indicated above on page 334) necessary for the proof of the simple coherence-theorem, since the opponent must bet on all events. If not, Dutch books are impossible regardless of the sum of the betting quotients because the opponent would suffer a loss if an event occurs on which he had *not* placed a bet. Therefore, coherent as well as strictly coherent betting quotients must all be greater than zero. However, this in a certain sense contradicts the axioms of probability, in particular, Carnap's Axiom of the lower bound (which is zero), following his A 1.[4]

The prescription that Dutch Books$_2$, as well as Dutch Books$_1$ should rationally be avoided, has led to the plausible idea of trying to specify the requirements of betting quotients. This idea, however, has not been realised, and is probably not realisable since it is not possible to distinguish strictly coherent betting quotients from simple coherent ones—whether in Carnap's case, or with actual bookmakers. This is a point that will become clear from the following.

Let us return to the proof of coherence I in bookmaking. The proposition states that if $\Sigma q_i \geqslant 1$ then not all G_i can be > 0. The proof shows that all G_i and therefore $\Sigma q_i G_i$ can be > 0 only if $\Sigma q_i < 1$. The corresponding proposition for coherence II says that if $\Sigma q_i \geqslant 1$ not all G_i can be $\geqslant 0$

[1] Kemeny [1963], p. 719.
[2] Carnap [1971*b*], p. 115.
[3] *Ibid.*, p. 110 and p. 114.
[4] *Ibid.*, p. 38.

and at least one $G_i > 0$ *and* at least one $G_i = 0$. To prove this one again assumes the contrary—namely, that all G_i are $\geqslant 0$, and at least one $G_i > 0$ and at least one $G_i = 0$, from which it follows that $\Sigma q_i G_i$ must be > 0; the remainder of the proof is then identical with the earlier one.

The precondition for a Dutch Book$_1$ as well as for a Dutch Book$_2$ is, therefore, that $\Sigma q_i < 1$. Conversely, the precondition for coherence I and coherence II is that $\Sigma q_i \geqslant 1$. If the latter is not the case, then the opponent can not only make a Dutch Book$_2$, but also a Dutch Book$_1$ which, clearly, he would prefer. This explains why the risk of a Dutch Book$_2$ plays absolutely no part in the considerations of a real bookmaker. According to de Finetti's conditions only $\Sigma q_i = 1$ remains for coherence II from which, to repeat, it does not follow even from strictly coherent betting quotients that the positive rationality premise is fulfilled, or that it even could be fulfilled. Since the observance of all three conditions would be economically quite senseless, I consider it, incidentally, doubtful that Keynes (who in his 'classical' *Treatise on Probability* insisted that not all probability statements can assume a quantitative form and whose book was, nevertheless, a source of inspiration for Carnap) 'might not be unsympathetic' to the betting quotient theory, as Braithwaite assumes in his editorial foreword to the new edition of 1973.[1] But since we cannot question Keynes, both views necessarily remain idle speculation.

CONCLUSION

The coherence-theorem has fundamental significance for the subjective theory of probability which is, in turn, important for inductive logic as developed by Carnap II and for the theory of rational decision under conditions of risk or uncertainty. If my thesis that the Dutch Book argument fails is accepted, then this could constitute a strong *technical* objection to these three theories. However, one would have to be very cautious as regards the impact of this thesis, since many people support these theories for *philosophical* reasons, about which, certainly as regards inductive logic, I have nothing new or significant to say. As far as decision theory is concerned, the reader is referred to several arguments which I developed in my dissertation.[2] Thus, I will conclude with a few remarks concerning subjective probability.

The appeal of the theory of subjective probability derives largely from its basis in the concept of probability found in everyday life, a concept which includes the probability of singular events. To be sure, in everyday life it is primarily a comparative concept to be made precise by means of coherent betting quotients. Such naive ordinary language approaches always begin

[1] Braithwaite [1973], p. xxii. [2] Heilig [1977].

anew, that is, without regard to the *history* of the subject under consideration. Hence, we find that de Finetti makes very few remarks concerning precursors, while Carnap's pertinent comments have less to do with the study of the sources than of Keynes' *Treatise*. The main historical claim made by Carnap is that the explicitly frequentist approach (developed after 1840 by Cournot, Ellis and Fries, and later by Venn, as a reaction to the subjectivism of Laplace) is an 'illegitimate usurpation',[1] since the founding fathers must have meant something like probability$_1$.

If one traces the origins of what today is called the 'probability calculus',[2] one finds the following. On the one hand, even well before 1654–7 (which is usually taken as the starting point with the works of Pascal, Fermat and Huygens) 'probabilitas' was a well-known concept that served, for example, in ranking the credibility of historical events or theological writings. And on the other hand neither Pascal nor Huygens applied either this, or an objective, or *any* kind of concept of probability, not even a purely formal one in the sense of the mathematical equipossibility definition. As Ivo Schneider has shown,[3] it was first Leibnitz in 1678 (in his 'De incerti aestimatione'), and then James Bernoulli in 1685 (in his still unpublished 'Meditationes et Annotationes') who tried to make the newly developed calculations of expectations in games of chance fruitful for the old concept of probability. Therefore, if one wishes to be pedantic, Hacking's thesis that probability emerged around 1660[4] is not correct. On the contrary, the calculus of probability emerged without any concept of probability. Since games of chance clearly lead to a frequentist concept of probability, one might say that the 'usurpation' was in the opposite direction. But, considering the very loose terminology of the seventeenth century, it is doubtful whether this assertion makes any sense.

Bernoulli later explicitly formulated the mathematical concept of probability in his [1713], giving it a subjective interpretation because of the deterministic metaphysics, as it were, of his day. He defined probability as the measure of our knowledge or ignorance of future events, which, due to divine providence and predestination, are just as certain as those of the past or of the present, for 'if that which is in the future is not certain to occur, then it is not comprehensible why the highest Creator should be accorded the glory of omniscience and omnipotence'.[5] At the same time, he developed the programme of our 'modern' decision theory based on his subjective concept; for his 'Art of Conjecturing' was intended

[1] Carnap [1950], p. xviii of the Preface to the second edition, 1962.
[2] Important new contributions are by Byrne [1968], Schneider [1972] and Hacking [1975].
[3] Schneider [1972], pp. 34 ff.
[4] Hacking [1975], pp. 9, 11, 18 and *passim*.
[5] Bernoulli [1713], p. 211.

as an aid in making better judgments and decisions, particularly in economic questions.[1]

Bernoulli did not complete his *opus*, but his ideas continue to have an effect today. It is probably time that a history was written of the attempts, beginning with Bernoulli's, to combine the calculus of probability with the concept of subjective probability. Since it will presumably be a history of many failures, it might then be possible using both critical *and historical* arguments, to convert some subjectivists.

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[1] *Ibid.*, p. 212.

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