A Theory of Decision-Making under Risk as a Tradeoff between Expected Utility, Expected Utility Deviation and Expected Utility Skewness

Pavlo Blavatskyy*

Abstract: This paper presents a new decision theory for modelling choice under risk. The new theory is a two-parameter generalization of expected utility theory. The proposed theory assumes that a decision maker: 1) behaves as if maximizing expected utility; but 2) may experience disappointment (elation) when the utility of a lottery's outcome falls short of (exceeds) the expected utility of the lottery; and 3) may have a preference for gambling (attraction/aversion to positively/negatively skewed lotteries). The proposed theory can rationalize the fourfold pattern of risk attitudes; the common ratio effect and the reverse thereof (in certain types of decision problems); the Allais paradox in classical common consequence problems and the reverse Allais paradox—in common consequence problems with an even split of a probability mass; violations of the betweenness axiom; switching behavior in the Samuelson's example; violations of ordinal, upper and lower cumulative independence (which falsify rank-dependent utility and cumulative prospect theory); and preference reversals between valuations and choice. Behavioral characterization (axiomatization) of the theory is provided. In application to insurance, the theory can rationalize full insurance with an actuarially unfair premium and aversion to probabilistic insurance. In application to optimal portfolio investment, the theory can rationalize the equity premium puzzle.

JEL classification codes: D01; D03; D81

Keywords: Decision Theory; Risk; Expected Utility Theory; Allais paradox; Disappointment; Equity Premium Puzzle

---

* Prof. Pavlo Blavatskyy, School of Management and Governance, Murdoch University, 90 South Street, Murdoch, WA 6150, AUSTRALIA, Ph: (08)9360 2838, Fax: (08)9360 6966, Email: pavlo.blavatskyy@gmail.com
I am grateful to David Butler for helpful comments.

A Theory of Decision-Making under Risk as a Tradeoff between Expected Utility, Expected Utility Deviation and Expected Utility Skewness

Few socio-economic factors can be characterized with certainty and risk is inherently present in many economic transactions. Therefore, the need for a theory of decision-making under risk was already recognized in the early chapters of the history of economic thought. The first criterion proposed for evaluating risky gambles over money was to maximize their expected value (Huygens, 1657). In response to the St. Petersburg paradox, Bernoulli (1738/1954) suggested replacing expected value with expected utility, which became known as expected utility theory. The theory gained momentum in microeconomics after von Neumann and Morgenstern (1947) provided its preference foundation and extension to arbitrary (not necessarily monetary) outcomes.

In response to the Allais (1953) paradox, expected utility theory was further generalized to numerous non-expected utility theories (*cf*. Starmer, 2000). One group of such theories is a family of "disappointment" models developed by Bell (1985), Loomes and Sugden (1986), Gul (1991) and Jia et al. (2001). This family is also related to the financial approach to decision-making based on a tradeoff between the expected return on an asset and the dispersion of the asset's returns (*cf*. Markowitz, 1952; Blavatskyy, 2010). "Disappointment" models can rationalize some behavioral patterns that falsify expected utility theory. Yet, other non-expected utility theories—most notably, a family of rank-dependent models developed by Quiggin (1981), Yaari (1987), Tversky and Kahneman (1992)—apparently can accommodate an even larger set of behavioral regularities falsifying expected utility.

This paper presents a second-generation non-expected (or non-non-expected) utility theory. Our theory generalizes the first-generation "disappointment" models to allow for the possibility that a decision maker may experience not only disappointment/elation but also—misfortune (bitter disappointment) and euphoria (extreme elation). From another perspective, the proposed theory extends the financial approach to decision making as it is based on the tradeoff not only between expected utility and utility dispersion but also—skewness. A decision maker who is prone to gambling may be attracted to positively skewed distributions and averse to negatively skewed distributions.

The paper is organized as follows. The proposed theory of decision making under risk is presented in section 1. Section 2 illustrates how this theory can rationalize behavioral patterns in several well-known examples (some of which falsify other prominent non-expected utility theories such as rank-dependent models). A preference foundation, *i.e*. a characterization of the subjective parameters of the theory in terms of the observed behavior, is given in section 3. In section 4 the proposed theory is applied to the demand for insurance. Section 5 presents another application of the theory to the problem of optimal portfolio investment. Related literature, in particular on the existing models of disappointment aversion, is reviewed in section 6. Limitations and possible extensions of the theory are discussed in section 7. Section 8 concludes.

1. Theory

There is a nonempty abstract topological set *X*. The elements of set *X* are called *outcomes* (consequences). For example, an outcome $x \in X$ can be a monetary amount, a consumption bundle, a financial portfolio, a health state, a marital status, *etc*. Choice alternatives are *lotteries*—probability distributions over set *X*. A (simple) lottery is denoted by $L: X \to [0,1]$, $\sum_{x \in X} L(x) = 1$.

A decision maker has a real-valued utility function $u: X \to \mathbb{R}$ that is unique up to a positive affine transformation (*i.e.*, without the loss of generality, utility function *u*(.) can be normalized for two arbitrary outcomes). *Expected utility* of lottery *L* is defined by equation (1). Expected utility of a degenerate lottery that yields one outcome $x \in X$ for sure (*i.e.*, with probability one) is simply *u*(*x*).

$$EU(L) = \sum_{x \in X} L(x)\, u(x) \tag{1}$$

Expected utility *EU*(*L*) can be interpreted as a reasonable *ex ante* utility level that a decision maker may expect from the *ex post* outcome of lottery *L*. Some decision makers may experience disappointment when the utility of lottery *L*'s *ex post* outcome falls short of its *ex ante* expectation *EU*(*L*). Let X_L^- denote the set of all such disappointing outcomes, *i.e.* $X_L^- \equiv \{x \in X | u(x) \leq EU(L)\}$. On the other hand, some decision makers may experience elation when the utility of lottery *L*'s outcome exceeds its *ex ante* expectation *EU*(*L*). Let X_L^+ denote the set of all such elating outcomes, *i.e.* $X_L^+ \equiv \{x \in X | u(x) > EU(L)\}$. The extent of possible disappointment is measured by *expected utility deviation* below the expected utility of a lottery. It is defined by equation (2). The extent of possible elation is measured by expected utility deviation above the expected utility of a lottery and it is also given by (2). Expected utility deviation of a degenerate lottery is zero. Expected utility deviation of a non-degenerate lottery is strictly positive unless all possible outcomes of this lottery yield the same utility (*i.e.*, this lottery is *de facto* degenerate on the utility scale).

$$\begin{aligned} EUD(L) &= \sum_{x \in X} L(x) \max\{EU(L) - u(x), 0\} = \frac{1}{2} \sum_{x \in X} L(x)\, |u(x) - EU(L)| = \\ &= \sum_{x \in X_L^+} \sum_{y \in X_L^-} L(x) L(y) [u(x) - u(y)] \end{aligned} \tag{2}$$

The second part of equation (2) shows that expected utility deviation is nothing but one half of the mean absolute deviation of utility of lottery's outcomes from the expected utility of the lottery. The mean absolute deviation is a well-known statistical measure of dispersion that is sometimes used in finance to capture financial risk (*cf*. Blavatskyy, 2010). The third part of equation (2) shows that expected utility deviation is similar to Gini (1912) mean difference statistic (on the utility scale). For binary lotteries (when sets X_L^+ and X_L^- are both singleton), expected utility deviation is nothing but one half of Gini (1912) mean difference statistic (which is also known as the second L-moment, or L-scale, *cf*. Hosking, 1990). For lotteries with more than two outcomes, however, expected utility deviation is less than one half of Gini (1912) mean difference statistic. The latter aggregates utility differences over all possible pairs of outcomes, whereas the former aggregates utility differences only over pairs consisting of one disappointing and one elating outcome. Thus, one half of Gini (1912) mean difference statistic is generally greater than expected utility deviation.

Expected utility deviation *EUD*(*L*) can be interpreted as a reasonable *ex ante* bound on how much a decision maker may expect the utility of lottery *L*'s *ex post* outcome to deviate from its *ex ante* expectation *EU*(*L*). In other words, it may be reasonable for a decision maker to form an *ex ante* expectation that the utility of *L*'s *ex post* outcome falls within *EU*(*L*)±*EUD*(*L*). From this point of view, a decision maker may experience euphoria when the *ex post L*'s outcome yields utility higher than *EU*(*L*)+*EUD*(*L*). The extent of such euphoria can be measured by expected utility deviation above *EU*(*L*)+*EUD*(*L*). On the other hand, a decision maker may experience misfortune (bitter disappointment) when the *ex post* lottery *L*'s outcome yields utility lower than *EU*(*L*)-*EUD*(*L*). The extent of such misfortune or bad luck can be measured by expected utility deviation below *EU*(*L*)-*EUD*(*L*).

For lotteries yielding outcomes that are symmetrically distributed on the utility scale these two measures of euphoria and misfortune are exactly equal. For lotteries yielding outcomes that are positively (negatively) skewed on the utility scale, the measure of euphoria is greater (smaller) than the measure of misfortune. Thus, the difference between the measures of euphoria and misfortune can be interpreted as *expected utility skewness*. It is defined in equation (3) below. Alternatively, we can think of measure (3) as utility of gambling since it measures "excessive" happiness that a decision maker can experience from betting on the long shots. Expected utility skewness of a degenerate lottery that yields one outcome for sure is zero.

$$(3)\quad EUS(L) = \sum_{x \in X} L(x) \left[\max\{u(x) - EU(L) - EUD(L), 0\} - \max\{EU(L) - EUD(L) - u(x), 0\} \right]$$

In expected utility theory, a decision maker behaves as if he maximizes only expected utility (1) and does not care about expected utility deviation/skewness (von Neumann and Morgenstern, 1947). In the mean-variance approach to optimal portfolio investment, pioneered by Markowitz (1952), an investor behaves as if trading off the expected return on an asset vs. the dispersion of asset's returns. In the financial literature, dispersion is usually measured through variance or standard error, which leads to normatively unappealing violations of the first-order stochastic dominance, *cf*. Borch (1969). These violations are avoidable if dispersion is measured through the mean absolute (semi-)deviation, *cf*. Blavatskyy (2010). Generalizing such a model to allow for a non-linear utility function, which is rarely done in finance, we obtain a decision theory in which a decision maker cares not only about expected utility (1) but also about expected utility deviation (2). In this paper we go one step further.

We assume that a decision maker may care not only about expected utility (1) and expected utility deviation (2) but also about expected utility skewness (3). The simplest model of multiattribute choice is a model that aggregates various attributes into one (weighted) index. In a decision theory proposed in this paper, preferences over lotteries are represented by utility function (4) that is a weighted sum of expected utility (1), expected utility deviation (2) and expected utility skewness (3):

$$U(L) = EU(L) - \rho \cdot EUD(L) + \tau \cdot EUS(L) \tag{4}$$

where $\rho \in [-1,1]$ and $\tau \in [-1,8/9]$ are two subjective parameters. Expected utility theory is a special case of model (4) when $\rho = \tau = 0$. Parameter ρ captures a decision maker's attitude to expected utility deviation. Parameter τ captures a decision maker's attitude to expected utility skewness. We expect a typical decision maker to be averse to expected utility deviation (the sentiment of disappointment to outweigh the sentiment of elation) but attracted to expected utility skewness. Thus, parameter ρ enters into (4) with a negative sign, whereas parameter τ -- with a positive sign. *Ceteris paribus*, a decision maker with a positive (negative) parameter ρ is averse (attracted) to lotteries yielding outcomes that are widely dispersed on the utility scale. *Ceteris paribus*, a decision maker with a positive (negative) parameter τ is attracted (averse) to lotteries yielding outcomes that are positively skewed on the utility scale and averse (attracted) to negatively skewed lotteries.

A special case of interest in economics is a lottery over monetary outcomes, *i.e.* when $X \subseteq \mathbb{R}$. In this case it is conventional to describe a lottery by its *cumulative distribution function* $F: \mathbb{R} \to [0,1]$. For any $x \in X$, $F(x)$ denotes the probability that the *ex post* outcome of a lottery is less than or equal to x. Expected utility of a lottery is then given by

$$EU(L) = \int_X u(x) dF(x) \tag{5}$$

expected utility deviation is given by

$$EU(L) = \int_X \max\{EU(L) - u(x), 0\}\, dF(x) \tag{6}$$

and expected utility skewness is given by

$$EU(L) = \int_X \left[\max\{u(x) - EU(L) - EUD(L), 0\} - \max\{EU(L) - EUD(L) - u(x), 0\}\right] dF(x) \tag{7}$$

A lottery L with a cumulative distribution function $F(.)$ first-order stochastically dominates lottery L' with a cumulative distribution function $G(.)$ if $G(x) \geq F(x)$ for all $x \in X$. A decision maker does not violate first-order stochastic dominance if $U(L) \geq U(L')$ for all such pairs of lotteries L and L'. This imposes a restriction on subjective parameters ρ and τ of utility function (4). Specifically, for $\tau \in [-1,0]$ parameter ρ must not exceed $1+\tau$ in the absolute value, *i.e.* $|\rho| \leq 1 + \tau$; for $\tau \in [0,⅓]$ parameter ρ must not exceed one in the absolute value, *i.e.* $|\rho| \leq 1$; and, finally, for $\tau \in [⅓,8/9]$ inequality $(\rho \pm \tau)^2 \leq 8\tau(1-\tau)$ must hold. Thus, as long as subjective parameters ρ and τ are not too large, preferences represented by utility function (4) do not violate the first-order stochastic dominance.

2. Examples

2.1 The fourfold pattern of risk attitudes

One of the first descriptive challenges to expected utility theory was an observation that people may reveal opposing risk preferences (*e.g*. Friedman and Savage, 1948). For instance, many individuals insure their house or a car while simultaneously purchasing lotto tickets. Tversky and Kahneman (1992, p. 306) provided another example, known as the fourfold pattern of risk attitudes: individuals reveal risk averse preferences when dealing with probable gains or improbable losses and risk seeking/loving preferences—when dealing with improbable gains or probable losses.

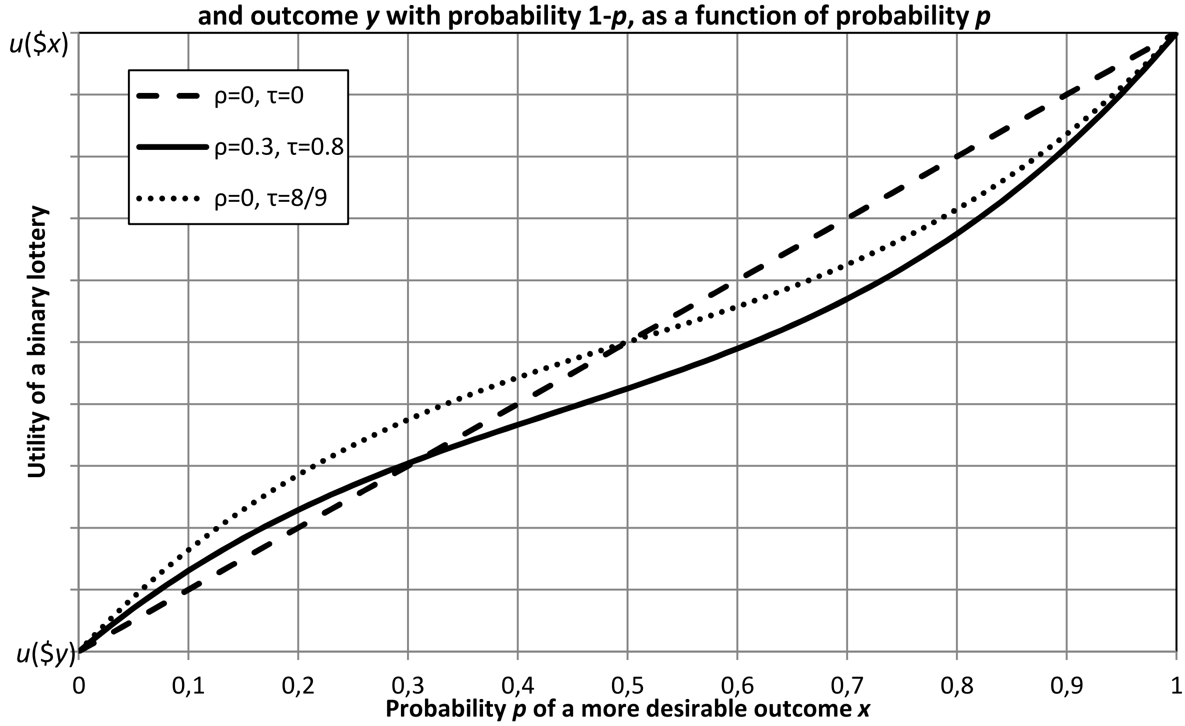
Consider a binary lottery that yields \$50 with a probability $p \in [0,1]$ and \$0 with a probability 1-p. When p=0.1, experimental subjects stated a median certainty equivalent of \$9 for this lottery, which is greater than the lottery's expected value (\$5), revealing risk seeking preferences. On the other hand, when p=0.9, subjects stated a median certainty equivalent of \$37 for this lottery, which is less than its expected value (\$45), revealing risk aversion (Tversky and Kahneman, 1992, Table 3, p. 307).

For a binary lottery that yields a more desirable outcome $x \in X$ with a probability $p \in [0,1]$ and a less desirable outcome $y \in X$ with probability 1-p, utility function (4) can be simplified into (8).

$$U(L) = u(y) + p(1 - (1 - p)[\rho + \tau(2p - 1)])[u(x) - u(y)] \quad (8)$$

When ρ=τ=0, utility (8) is a linear function of probability p, depicted as a dashed 45° line on figure 1. In this case, a decision maker can only reveal either risk-averse preferences (when utility function $u(.)$ is concave) or risk-seeking preferences (when utility function $u(.)$ is convex). When ρ=0 and τ>0, utility (8) is inverse S-shaped (concave in the neighborhood of p=0 and convex in the neighborhood

Figure 1 Utility (8) of a binary lottery that yields outcome *x* with probability *p* and outcome *y* with probability 1-*p*, as a function of probability *p*



of p=1) crossing the 45° line at p=0.5. For example, utility (8) with ρ=0 and τ=8/9 is plotted as a dotted line on figure 1. When ρ>0 and τ>0, utility (8) is inverse S-shaped crossing the 45° line at a point below p=0.5. For example, utility (8) with ρ=0.3 and τ=0.8 is plotted as a solid line on figure 1. When ρ<0 and τ>0, utility (8) is inverse S-shaped crossing the 45° line at a point above p=0.5.

A decision maker with ρ>0 and τ>0 can reveal risk seeking preferences over binary lotteries with a low probability p and risk aversion—over lotteries with a moderate or high p. For illustration, consider the case when utility function u(.) is linear, ρ=0.3 and τ=0.8. Figure 1 shows that such an individual reveals risk seeking preferences when p<0.31 and risk averse preferences—when p>0.32.

Tversky and Kahneman (1992, Table 3, p. 307) also found that experimental subjects stated a median certainty equivalent of -\$8 for a binary lottery that yields -\$50 with probability 0.1 and \$0 with probability 0.9, which is lower than the lottery's expected value (-\$5), revealing risk averse preferences. This finding is consistent with utility representation (8). Note that, in this case, the probability of a more desirable outcome (which is now \$0) is p=0.9 and utility function (8) with τ>0 generates risk aversion for p=0.9. Finally, experimental subjects stated a median certainty equivalent of -\$39 for a binary lottery that yields -\$50 with probability 0.9 and \$0 with probability 0.1, which is higher than the lottery's expected value (-\$45), revealing risk seeking preferences. This is consistent with utility function (8) since the probability of a more desirable outcome (\$0) is p=0.1 in this case.

2.2 The common ratio effect and the reverse common ratio effect

The common ratio effect is another early example of descriptive limitations of expected utility theory. Kahneman and Tversky (1979, p. 266) observed that the majority of experimental subjects choose \$3000 for sure over an 80% chance of \$4000 (with a 20% chance of \$0). The majority of the same subjects, however, also choose a 20% chance of \$4000 (with an 80% chance of \$0) over a 25% chance of \$3000 (with a 75% chance of \$0). Note that all lotteries involved in this example yield not more than two outcomes. In such a case, utility function (4) simplifies into a simpler formula (8). If preferences over binary lotteries are represented by utility function (8) then a decision maker exhibits a systematic common ratio effect in the above example whenever condition (9) holds.

$$25 - 5\rho - 3\tau < \frac{125}{4}\frac{u(\$3000) - u(\$0)}{u(\$4000) - u(\$0)} < 8\frac{25 - 20\rho + 12\tau}{8 - 6\rho + 3\tau} \tag{9}$$

When ρ=τ=0, the left-hand-side and the right-hand-side of inequality (9) are both equal to 25. In such a case, a decision maker (who acts as an expected utility maximizer) cannot exhibit any systematic common ratio effect. Yet, if either ρ or τ is not zero, the common ratio effect may appear. The black area on figure 2 shows the range of parameter values (ρ, τ) such that the left-hand-side of inequality (9) is less than the right-hand-side of inequality (9), *i.e*. when a decision maker may exhibit a systematic common ratio effect in the Kahneman and Tversky (1979) example. Note that a decision maker can never reveal a common ratio effect when both parameters ρ and τ are negative.

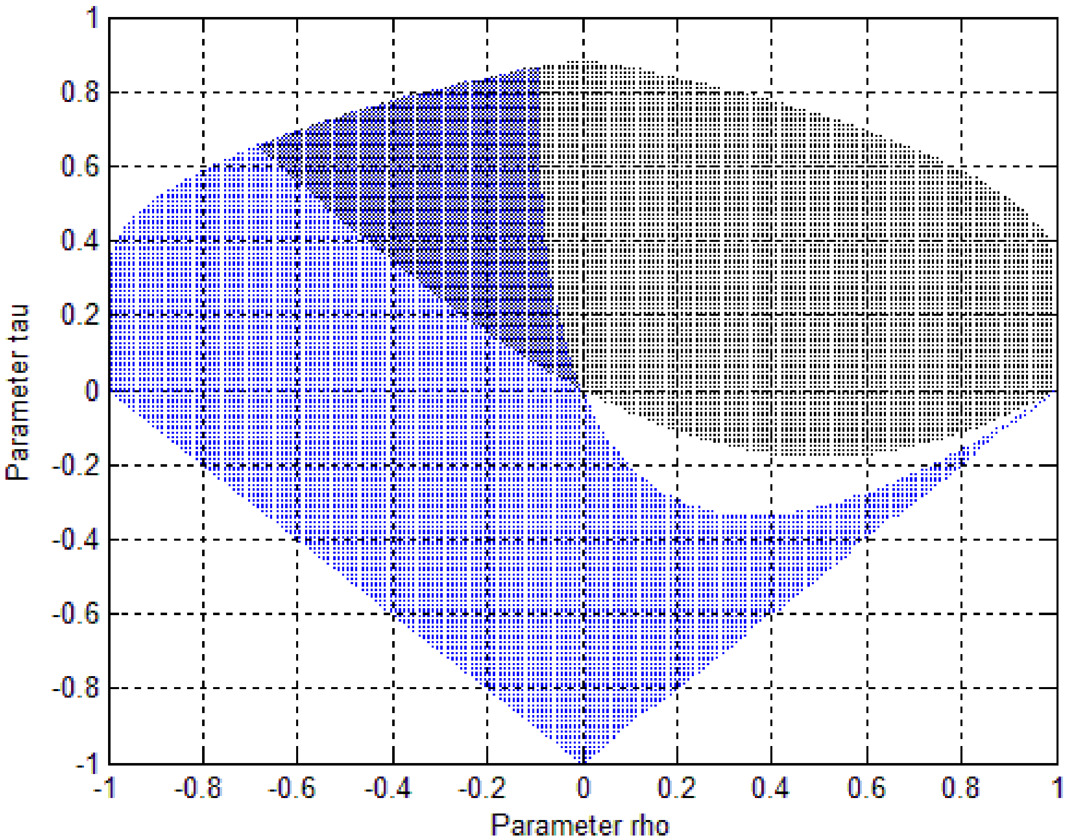


Figure 2 Parameter values (ρ, τ) consistent with the common ratio effect in Kahneman and Tversky (1979) example (black area) and the reverse common ratio effect in Blavatskyy (2010a) example (blue area)

Blavatskyy (2010a, experiment 1, pairs 4-6) recently discovered that the classical common ratio effect may be reversed when a risky lottery in the first binary choice problem yields a more desirable outcome with a probability less than one half. Consider the following example from Blavatskyy (2010a, p. 222, pair #6 in Table 1). First, a decision maker chooses between \$10 for sure and a ¼ chance of \$100 (with a ¾ chance of \$0). Second, a decision maker chooses between a ⅓ chance of \$10 (with a ⅔ chance of \$0) and a 1/12 chance of \$100 (with an 11/12 chance of \$0). Blavatskyy (2010a, table 2, p. 225) found that many subjects choose a risky lottery in the first problem but switch to a safer lottery (*i.e.*, to a ⅓ chance of \$10) in the second problem. Only few subjects choose \$10 for sure in the first problem but switch to a riskier lottery in the second problem. As a result, a systematic reverse common ratio effect is observed. If preferences over binary lotteries are represented by utility function (8) then such an effect appears whenever condition (10) holds.

$$\frac{72-66\rho+55\tau}{9-6\rho+2\tau} < 32\frac{u(\$10)-u(\$0)}{u(\$100)-u(\$0)} < 8-6\rho+3\tau \qquad (10)$$

When ρ=τ=0, the left-hand-side and the right-hand-side of inequality (10) are both equal to 8. Thus, a decision maker who maximizes expected utility (ρ=τ=0) cannot exhibit any systematic reverse common ratio effect. Yet, if either ρ or τ is not zero, the reverse common ratio effect may appear. The blue area on figure 2 shows the range of parameter values (ρ, τ) such that the left-hand-side of inequality (10) is less than the right-hand-side of inequality (10), *i.e*. when a decision maker may exhibit a systematic reverse common ratio effect in the Blavatskyy (2010a) example.

Figure 2 shows that black and blue areas intersect when τ is positive and ρ is slightly negative. Thus, utility function (8) can simultaneously generate the common ratio effect in Kahneman and Tversky (1979) example and the reverse common ratio effect—in Blavatskyy (2010a) example. In contrast, Quiggin (1981) rank-dependent utility or Tversky and Kahneman (1992) cumulative prospect theory with a popular weighting function $w(p) = p^{\gamma}/(p^{\gamma} + (1-p)^{\gamma})^{1/\gamma}$ cannot rationalize both of these effects at the same time (*cf*. Blavatskyy, 2010a, p. 228). Model (8), however, cannot rationalize another type of the reverse common ratio effect, also documented in Blavatskyy (2010a),—when a sure option is well below the expected value of a risky lottery in the first binary choice problem.

2.3 The Allais paradox (the common consequence effect) and the reverse Allais paradox

Allais (1953, p.527) constructed an example challenging the descriptive validity of expected utility theory. This example became known as the Allais paradox or, more generally, the common consequence effect (*cf*., Starmer, 2000, p. 336). The paradox is often illustrated with very large hypothetical outcomes (*e.g*., Conlisk, 1989). Yet, it has been also replicated with small real payoffs. For instance, Starmer and Sugden (1991) found that many experimental subjects exhibit the following fanning-out choice pattern: 1) subjects choose £7 for sure over a lottery yielding a 0.2 chance of £10, a 0.75 chance of £7 and a 0.05 chance of £0; 2) subjects also choose a 0.2 chance of £10 (with a 0.8 chance of £0) over a 0.25 chance of £7 (with a 0.75 chance of £0). If preferences are represented by utility function (4), a necessary condition for such a fanning-out choice pattern is inequality (11).

$$(11) \qquad \frac{10\rho - 21\tau}{25 - 20\rho + 12\tau} < \frac{32(20\rho + 3\tau)}{400 - 20\rho + 297\tau - 7.5(20\rho + 3\tau)[1 + \operatorname{sign}(\rho)]}$$

If ρ=τ=0, the left-hand-side and the right-hand-side of inequality (11) are both equal to zero. In this case, a decision maker behaves as an expected utility maximizer and cannot reveal a fanning-out choice pattern in the Allais example. Yet, if either ρ or τ is not zero, the common consequence effect may appear. The black area on figure 3 shows the range of parameter values (ρ, τ) such that the left-hand-side of inequality (11) is less than the right-hand-side of inequality (11), *i.e*. when a decision maker may reveal a fanning-out choice pattern in the Starmer and Sugden (1991) example. Note that a decision maker can never reveal a fanning-out choice pattern when both ρ and τ are negative.

In all common consequence problems a relatively riskier choice alternative is obtained from a relatively safer one by shifting some probability mass (0.25 in the above example) from the middle outcome to the extreme outcomes. In the classical design of the Allais paradox, this probability mass is divided in uneven proportions between the two extreme outcomes. In the above example from Starmer and Sugden (1991), a probability mass of 0.2 is allocated to the highest outcome £10 and only a probability mass of 0.05 is allocated to the lowest outcome £0 (thus, the proportion is 4:1).

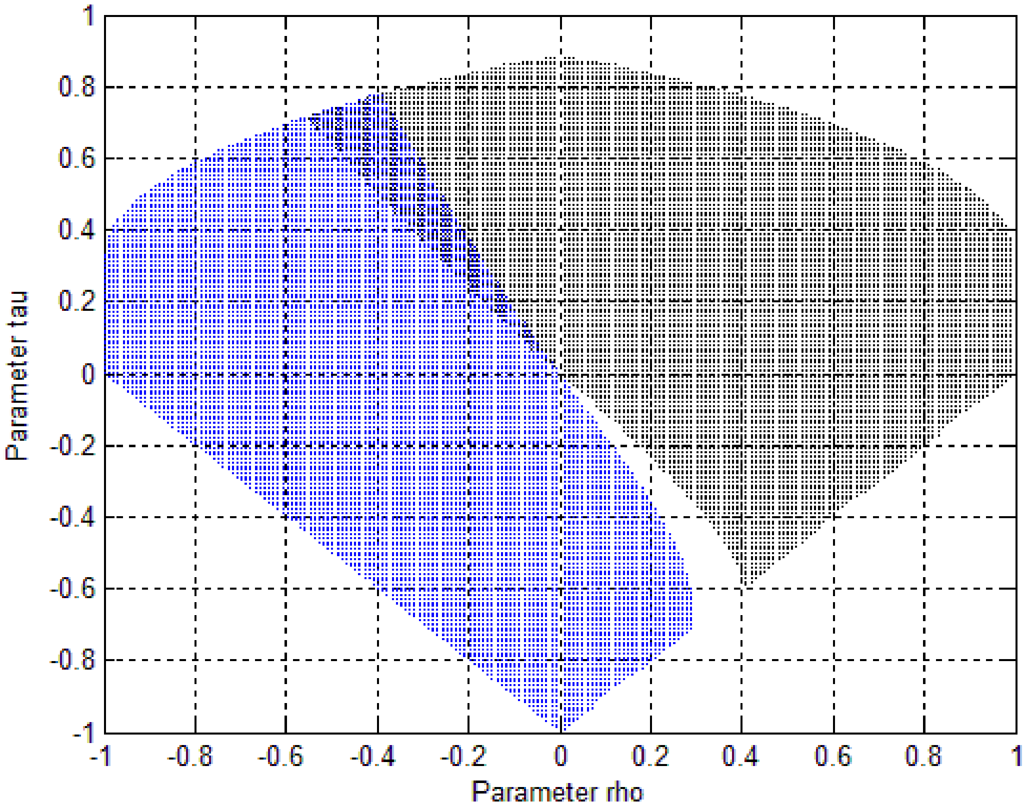


Figure 3 Parameter values (ρ, τ) consistent with the common consequence effect in Starmer and Sugden (1991) example (black area) and the reverse CC effect in Starmer (1992) example (blue area)

Common consequence problems can be also designed with an even split of a probability mass. Several studies investigated such experimental design and documented the reverse Allais paradox (*e.g.*, Starmer, 1992; Loomes and Sugden, 1998; Humphrey and Verschoor, 2004; Blavatskyy, 2013). For instance, Starmer (1992) found that many experimental subjects exhibit the following fanning-in choice pattern: 1) subjects choose a lottery yielding a 0.1 chance of £7, a 0.8 chance of £3 and a 0.1 chance of £0 over £3 for sure; 2) subjects also choose a 0.2 chance of £3 (with a 0.8 chance of £0) over a 0.1 chance of £7 (with a 0.9 chance of £0). If preferences are represented by utility function (4), a necessary condition for such a fanning-in choice pattern is inequality (12).

$$\frac{5\rho}{10-\rho+8\tau-4\rho[1+\mathrm{sign}(\rho)]}<\frac{5\rho-12\tau}{50-45\rho+36\tau} \tag{12}$$

The blue area on figure 3 shows the range of parameter values (ρ, τ) satisfying inequality (12). Note that a decision maker can never reveal a fanning-in choice pattern when both ρ and τ are positive (or equal to zero). Black and blue areas on figure 3 intersect when τ is positive and ρ is negative. Thus, utility function (4) can simultaneously generate the Allais paradox in the classical common consequence problems (as in Starmer and Sugden, 1991) and the reverse Allais paradox—in common consequence problems with an even split of a probability mass (such as in Starmer, 1992).

The intersection area on figure 3 is smaller than the intersection area on figure 2. Thus, according to model (4), the reversal of the Allais paradox can be a relatively rare behavioral phenomenon compared to the reversal of the common ratio effect. Nonetheless, some people can exhibit both reversals. For example, a decision maker with ρ=-0.4 and τ=0.6 can reveal the common ratio effect in Kahneman and Tversky (1979) example, the reverse common ratio effect—in Blavatskyy (2010a) example, the classical Allais paradox—in Starmer and Sugden (1991) example and the reverse Allais paradox—in Starmer (1992) example.

2.4 Violations of the betweenness axiom

The betweenness axiom is a weaker form of the independence axiom of expected utility theory. The betweenness axiom holds if any compound lottery (*i.e.*, a probability mixture) of two simple lotteries is always located between those simple lotteries in terms of desirability (*e.g.*, Dekel, 1986). Several studies found that the majority[1] of subjects violate the betweenness axiom (*e.g.*, Prelec, 1990; Bernasconi, 1994; Camerer and Ho, 1994).

Consider an example from Bernasconi (1994, p. 60, pair 1). Let *S* denote a degenerate lottery that yields £10 for sure. Let *R* denote a risky lottery that offers a 0.8 chance of £16 and a 0.2 chance of £0. Finally let $S\lambda R$ denote a compound lottery that yields *S* with probability $\lambda \in (0,1)$ and *R*—with probability 1-λ. Bernasconi (1994, p. 60) found that the majority of subjects reveal quasi-concave preferences when λ=0.95: subjects choose *S*0.95*R* over both *S* and *R* (and they choose *S* over *R*).

Such quasi-concave preferences can be represented by model (4). For instance, if utility function *u*(.) is linear, ρ=0.8 and τ=0.5, the certainty equivalents of *S*0.95*R*, *S* and *R* are £10.02, £10 and £9.98 correspondingly. Thus, a decision maker would prefer mixture *S*0.95*R* over both *S* and *R*.

Bernasconi (1994) also found that the majority of subjects reveal quasi-convex preferences when λ=0.05: subjects choose both *S* and *R* over mixture *S*0.05*R* (and they choose *S* over *R*). If preferences are represented by (4) with a linear utility function *u*(.), ρ=0.8 and τ=0.5, then the certainty equivalents of *S*, *R* and *S*0.05*R* are £10, £9.98 and £9.97 correspondingly. Thus, a decision maker would prefer both *S* and *R* over mixture *S*0.05*R*.

2.5 Samuelson's example

Samuelson (1963) observed that some people are not willing to accept a lottery that yields a 0.5 chance of \$200 with a 0.5 chance of -\$100 but they are willing to play this lottery several times. Such switching behaviour is consistent with model (4). Let us consider for simplicity a decision maker with a linear utility function *u*(.). If Samuelson's lottery is played only once, its expected utility is 50, its expected utility deviation is 75 and its expected utility skewness is zero. Thus, a decision maker with ρ>2/3 is not willing to play such a lottery. Yet, if Samuelson's lottery is played twice, its expected utility becomes 100 but its expected utility deviation and skewness remain unchanged (75 and zero respectively). Thus, a decision maker with $\rho \in (2/3,1]$ is willing to play Samuelson's lottery twice even though he is not willing to play this lottery only once (*cf*. Blavatskyy, 2010, p. 2055).

[1] Systematic violations of betweenness only by a minority of subjects (*e.g.*, experiments 1 and 3 in Gigliotti and Sopher, 1993) can be an artifact of random errors (*cf*. Blavatskyy, 2006). .

Lottery	A	B	C	D
Description	0.25 chance of \$100 0.01 chance of \$55 0.74 chance of \$0	0.25 chance of \$100 0.02 chance of \$25 0.73 chance of \$0	0.26 chance of \$55 0.74 chance of \$0	0.25 chance of \$55 0.02 chance of \$25 0.73 chance of \$0
EU	25.55	25.5	14.3	14.25
EUD	18.907	18.625	10.582	10.4025
EUS	9.07536	8.95	5.07936	4.78515

Table 1 Lotteries used by Wu (1994) for testing ordinal independence and their expected utility (EU), expected utility deviation (EUD) and expected utility skewness (EUS) with a linear utility function

2.6 Violations of ordinal (or upper tail) independence

Consider four lotteries presented in table 1. Wu (1994, Table 2, p.45) found that 35% of subjects choose A over B and D—over C. This choice pattern violates ordinal independence and challenges Quiggin (1981) rank-dependent utility and Tversky and Kahneman (1992) cumulative prospect theory. Birnbaum (2005, Table 4, p. 1356) reported similar results as "violations of upper tail independence".

Model (4) can violate ordinal independence. For example, table 1 shows expected utility (EU), expected utility deviation (EUD) and expected utility skewness (EUS) of lotteries A-D in the simplest case when utility function *u*(.) is linear. In this case, a decision maker chooses A over B whenever

$$5.64\rho - 2.5072\tau < 1$$

and chooses D over C whenever

$$3.59\rho - 5.8842\tau > 1$$

These inequalities can hold at the same time (*e.g*. when ρ=0 and τ=-0.3). Thus, model (4), unlike rank-dependent utility or cumulative prospect theory, can accommodate violations of ordinal independence.

2.7 Violations of upper and lower cumulative independence

Consider four lotteries presented in table 2. Birnbaum and Navarrete (1998, p.64) found that 34 out of 100 subjects choose E over F as well as H—over G. Such a choice pattern violates upper cumulative independence and falsifies rank-dependent utility as well as cumulative prospect theory. Birnbaum *at al*. (1999) and Birnbaum (2004) also report violations of upper cumulative independence.

Model (4) can violate upper cumulative independence. Note that lotteries E and F yield the same expected utility, which we denote as *EU*, if *u*(\$98)+*u*(\$10)=*u*(\$44)+*u*(\$40). In addition, if *u*(\$110) is sufficiently high then *EU* is greater than both *u*(\$98) and *u*(\$44). Lotteries E and F then have the same expected utility deviation *EUD*. Moreover, if *u*(\$110) is sufficiently high then *EU-EUD* is greater then both *u*(\$98) and *u*(\$44). Lotteries E and F then also have the same expected utility skewness. In such a case, a decision maker who maximizes utility (4) is exactly indifferent between E and F.

Lottery	E	F	G	H
Description	0.8 chance of \$110 0.1 chance of \$98 0.1 chance of \$10	0.8 chance of \$110 0.1 chance of \$44 0.1 chance of \$40	0.9 chance of \$98 0.1 chance of \$10	0.8 chance of \$98 0.2 chance of \$40

Table 2 Lotteries used in Birnbaum and Navarrete (1998) for testing upper cumulative independence

Therefore, model (4) implies that a decision maker chooses lottery E over lottery F if *u*($110) is sufficiently high and the sum of utilities of $98 and $10 is greater than the sum of utilities of $44 and $40. The latter condition can be rearranged as inequality (13).

$$u(\$44) - u(\$10) < u(\$98) - u(\$40) \tag{13}$$

Lotteries G and H are binary lotteries, for which utility function (4) simplifies into formula (8). A decision maker who maximizes utility (8) chooses lottery H over G if and only if inequality (14) holds.

$$u(\$98) - u(\$40) < [u(\$40) - u(\$10)]\frac{1 + 0.9\rho + 0.72\tau}{1 + 0.7\rho + 0.24\tau} \tag{14}$$

When ρ=τ=0 the right-hand-side of inequality (14) simplifies to *u*($40)-*u*($10), which is always less then the left-hand-side of inequality (13). In this case inequalities (13) and (14) cannot hold at the same time. In other words, an expected utility maximizer cannot choose E over F and H—over G.

Yet, if ρ>-2.4τ then the fraction on the right-hand-side of inequality (14) is greater than one. In this case, the right-hand-side of inequality (14) can be greater then the left-hand-side of inequality (13). For instance, both inequalities (13) and (14) hold when *u*($10)=10, *u*($40)=50, *u*($44)=52, *u*($98)=98, ρ=0.8 and τ=0.5. Thus, when ρ>-2.4τ a decision maker, who behaves as if maximizing utility (4), can violate upper cumulative independence by choosing E over F and H—over G.

Consider now four lotteries presented in table 3. Birnbaum and Navarrete (1998, p.63) found that 38 out of 100 subjects choose K over L as well as N—over M. This choice pattern violates lower cumulative independence and falsifies rank-dependent utility as well as cumulative prospect theory. Birnbaum *at al*. (1999) and Birnbaum (2004) also found violations of lower cumulative independence.

Model (4) can violate lower cumulative independence. If *u*($52)+*u*($48)=*u*($98)+*u*($10) then K and L yield the same expected utility *EU*. If *u*($3) is sufficiently low then *EU* is greater than both *u*($48) and *u*($10). Lotteries K and L then have the same expected utility deviation *EUD*. Moreover, if *u*($3) is sufficiently low then *EU-EUD* is greater then both *u*($48) and *u*($10). Lotteries K and L then also have the same expected utility skewness. In such a case, a decision maker who maximizes utility (4) is exactly indifferent between K and L. Therefore, model (4) implies that a decision maker chooses lottery K over lottery L if *u*($3) is sufficiently low and the sum of utilities of $52 and $48 is greater than the sum of utilities of $98 and $10. The latter condition can be rearranged as inequality (15).

$$u(\$52) - u(\$10) > u(\$98) - u(\$48) \tag{15}$$

Lotteries M and N are binary lotteries, for which utility function (4) simplifies into formula (8). A decision maker who maximizes utility (8) chooses N over M if and only if inequality (16) holds.

Lottery	**K**	**L**	**M**	**N**
Description	0.1 chance of $52 0.1 chance of $48 0.8 chance of $3	0.1 chance of $98 0.1 chance of $10 0.8 chance of $3	0.2 chance of $52 0.8 chance of $10	0.1 chance of $98 0.9 chance of $10

Table 3 Lotteries used in Birnbaum and Navarrete (1998) for testing lower cumulative independence

$$(16) \qquad u(\$52) - u(\$10) < [u(\$98) - u(\$52)]\frac{1 - 0.9\rho + 0.72\tau}{1 - 0.7\rho + 0.24\tau}$$

When $\rho=\tau=0$ the right-hand-side of inequality (16) simplifies to $u(\$98)$-$u(\$52)$, which is always less then the right-hand-side of inequality (15). In this case inequalities (16) and (15) cannot hold at the same time. In other words, an expected utility maximizer cannot choose K over L and N—over M.

Yet, if $\rho<2.4\tau$ then the fraction on the right-hand-side of inequality (16) is greater than one. In this case, the right-hand-side of inequality (16) can be greater then the right-hand-side of inequality (15). For instance, both inequalities (15) and (16) hold when $u(\$10)=10$, $u(\$48)=54$, $u(\$52)=56$, $u(\$98)=98$, $\rho=0.8$ and $\tau=0.5$. Thus, when $\rho>2.4\tau$ a decision maker, who behaves as if maximizing utility (4), can violate lower cumulative independence by choosing K over L and N—over M.

2.8 Preference reversals between valuations elicited through the BDM mechanism and choice

Let us consider two binary lotteries. The \$-bet yields \$100 with probability 0.3 (and \$0 with probability 0.7). The P-bet yields \$32 with probability 0.9 (and \$0 with probability 0.1). A standard preference reversal is observed when a decision maker chooses the P-bet over the \$-bet in a direct binary choice but states a higher evaluation for the \$-bet than for the P-bet in the BDM mechanism (Becker *et al*., 1964). A non-standard preference reversal is observed when a decision maker chooses the \$-bet over the P-bet in a direct binary choice but states a higher evaluation for the P-bet. The preference reversal phenomenon refers to a frequent experimental finding that standard reversals significantly outnumber non-standard reversals (*e.g*., Grether and Plott, 1979, p. 632).

A decision maker who states evaluation $\$x$ for the \$-bet in the BDM mechanism receives outcome \$100 with probability $0.003x$, outcome \$0—with probability $0.007x$ and every outcome between $\$x$ and \$100—with equal probability.[2] A solid curve at the top of figure 4 shows utility (4) of this lottery for a decision maker with a linear utility function $u(.)$ and $\rho=\tau=0$. This decision maker obtains the highest utility from stating evaluation \$30, which is the expected value of the S-bet.

A decision maker who states evaluation $\$x$ for the P-bet in the BDM mechanism receives \$32 with probability $0.9x/32$, \$0—with probability $0.1x/32$ and every amount between $\$x$ and \$32—with equal probability. A dotted curve on figure 4 shows the utility of this lottery for a decision maker who maximizes expected value (function $u(.)$ is linear and $\rho=\tau=0$). This decision maker obtains the highest utility from stating evaluation \$28.8, which is the expected value of the P-bet. Therefore, in the BDM mechanism, the decision maker with a linear $u(.)$ and $\rho=\tau=0$ states a higher evaluation for the \$-bet than for the P-bet. In a direct binary choice, this decision maker also chooses the \$-bet over the P-bet. In sum, a decision maker who maximizes the expected value of lotteries (and any expected utility maximizer in general) cannot exhibit systematic standard or non-standard preference reversals.

---

[2] In other words, cumulative probability $1-0.01x$ is uniformly distributed over interval $[x,100]$.

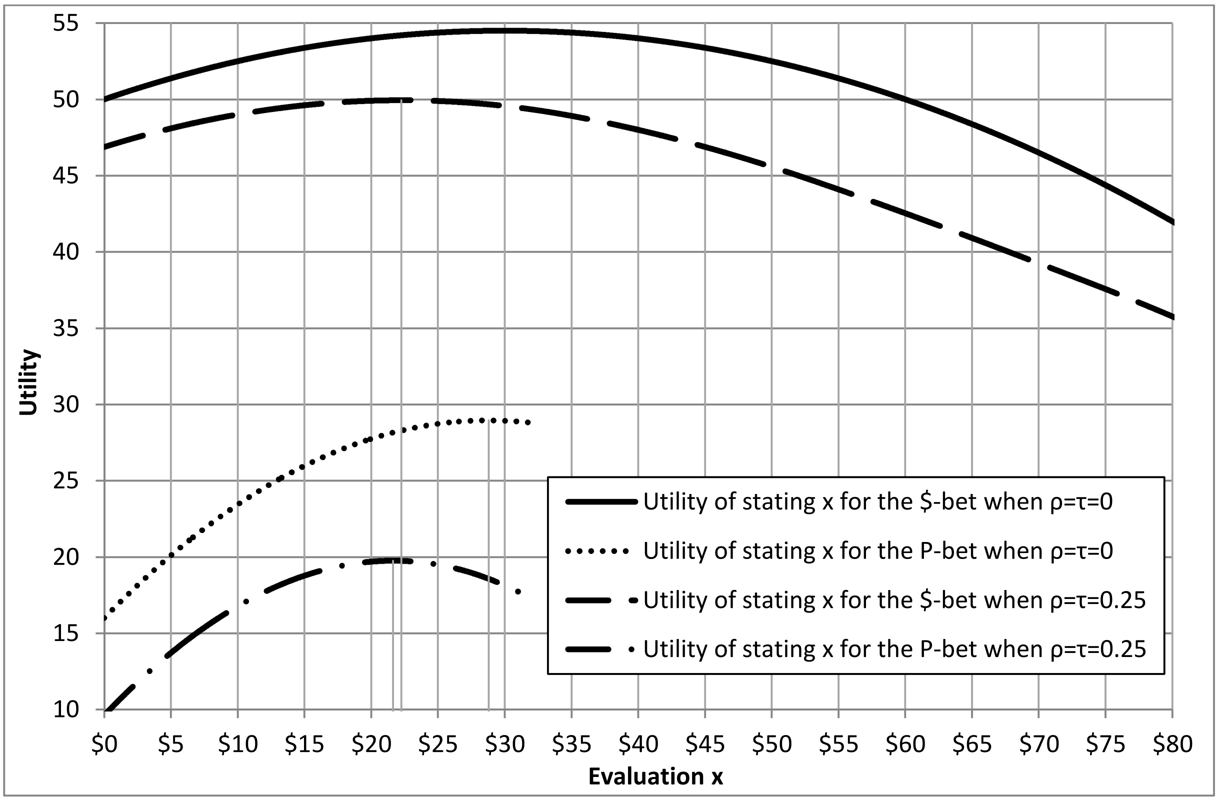


Figure 4 Utility of stating evaluation $x for the $-bet and the P-bet in the BDM mechanism

Let us now consider a decision maker with a linear utility function *u*(.) and *ρ*=*τ*=0.25. A dashed curve on figure 4 shows the utility from stating evaluation $*x* for the $-bet in the BDM mechanism. The decision maker obtains the highest utility from stating evaluation $22.26 for the $-bet. A dashed-dotted curve on figure 4 shows the utility from stating evaluation $*x* for the P-bet in the BDM mechanism. The highest utility is obtained by stating evaluation $21.64 for the P-bet. Thus, the decision maker states a higher evaluation for the $-bet than for the P-bet in the BDM mechanism.

In a direct binary choice, however, the decision maker with a linear *u*(.) and *ρ*=*τ*=0.25 prefers the P-bet over the S-bet (the certainty equivalent of the $-bet is $26.85 and the certainty equivalent of the P-bet is $27.50). In other words, the decision maker exhibits a standard preference reversal. This example shows that a decision maker who cares not only about expected utility but also—about utility deviation and skewness—may optimally state evaluations in the BDM mechanism that do not necessarily reflect his/her preference ordering of the $-bet and the P-bet (*cf*. Karni and Safra, 1987).

Our proposed theory can also rationalize the findings of Blavatskyy and Köhler (2009) who observed that restricting the range of possible evaluations in the BDM mechanism has a systematic effect on stated evaluations. Yet, our theory, like any theory representing a transitive preference order, cannot rationalize preference reversals when evaluations are inferred from binary choice between $-/P-bets and sure amounts. Blavatskyy (2009) showed that such preference reversals[3] may be an artifact of randomness in choice (as modelled in Blavatskyy, 2011; 2012a).

[3] As well as preference reversals with probability equivalents discovered in Butler and Loomes (2007).

3. Preference foundation (behavioral characterization of subjective parameters)

Behavioral characterization of model (4) is analogous to the axiomatization of expected utility theory by von Neumann and Morgenstern (1947). Specifically, model (4) represents a preference relation over lotteries that is a continuous weak order and satisfies the independence axiom, which is imposed on "transformed" lotteries rather than usual lotteries. For constructing transformed lotteries, we need only one additional assumption (axiom 4 below).

Let $\mathcal{L}$ denote the set of all lotteries. A decision maker has a *preference relation* $\succcurlyeq$ on $\mathcal{L}$. As usual, the symmetric part of $\succcurlyeq$ is denoted by $\sim$ and the asymmetric part of $\succcurlyeq$ is denoted by $\succ$. We assume that preference relation $\succcurlyeq$ is a weak order (axioms 1-2 below).

Axiom 1 (*Completeness*) For all $F, G \in \mathcal{L}$ either $F \succcurlyeq G$ or $G \succcurlyeq F$ (or both).

Axiom 2 (*Transitivity*) For all $F, G, H \in \mathcal{L}$ if $F \succcurlyeq G$ and $G \succcurlyeq H$ then $F \succcurlyeq H$.

For any two (simple) lotteries $H, G \in \mathcal{L}$ and probability $\alpha \in [0,1]$, let $H\alpha G \in \mathcal{L}$ denote a *compound* lottery that yields outcome $x \in X$ with probability $\alpha \cdot H(x)+(1-\alpha)\cdot G(x)$. With this notation, a standard continuity assumption can be formulated (axiom 3 below). Axioms 1-3 are conventional assumptions necessary for any real-valued representation of the preference relation $\succcurlyeq$ (Debreu, 1954), including model (4).

Axiom 3 (*Continuity*) For all $F, G, H \in \mathcal{L}$ the sets $\{\alpha \in [0,1]: F \succcurlyeq H\alpha G\}$ and $\{\alpha \in [0,1]: H\alpha G \succcurlyeq F\}$ are closed.

Let $\boldsymbol{x} \in \mathcal{L}$ denote a degenerate lottery that yields one outcome $x \in X$ with probability one. Thus, $\boldsymbol{x}\alpha\boldsymbol{y} \in \mathcal{L}$ denotes a binary lottery that yields outcome $x \in X$ with probability $\alpha \in [0,1]$ and outcome $y \in X$ with probability $1-\alpha$. Note that axioms 1-3 imply that for any triple of outcomes $x, y, z \in X$ such that $x \succ y \succ z$ there exists a probability $\alpha \in (0,1)$ such that $\boldsymbol{y} \sim \boldsymbol{x}\alpha\boldsymbol{z}$. This probability α is sometimes called the *probability equivalent* of $\boldsymbol{y}$. Given such α, axioms 1-3 also imply the existence of probability equivalents for lotteries $\boldsymbol{x}\alpha\boldsymbol{y}$, $\boldsymbol{y}\alpha\boldsymbol{z}$, $\boldsymbol{x}(1-\alpha)\boldsymbol{y}$ and $\boldsymbol{y}(1-\alpha)\boldsymbol{z}$. For constructing transformed lotteries we need to restrict these probability equivalents as described in axiom 4 below.

Axiom 4 For any three outcomes $x, y, z \in X$ such that $x \succ y \succ z$ and probabilities $\alpha, \beta, \gamma, \delta, \eta \in (0,1)$ such that $\boldsymbol{y} \sim \boldsymbol{x}\alpha\boldsymbol{z}$, $\boldsymbol{x}\alpha\boldsymbol{y} \sim \boldsymbol{x}\beta\boldsymbol{z}$, $\boldsymbol{y}\alpha\boldsymbol{z} \sim \boldsymbol{x}\gamma\boldsymbol{z}$, $\boldsymbol{x}(1-\alpha)\boldsymbol{y} \sim \boldsymbol{x}\delta\boldsymbol{z}$ and $\boldsymbol{y}(1-\alpha)\boldsymbol{z} \sim \boldsymbol{x}\eta\boldsymbol{z}$ the ratio

$$\rho \equiv \frac{\dfrac{\beta+\gamma-2\alpha}{\beta(1-\beta)(1-2\beta)+\gamma(1-\gamma)(1-2\gamma)-2\alpha(1-\alpha)(1-2\alpha)} - \dfrac{\delta+\eta-1}{\delta(1-\delta)(1-2\delta)+\eta(1-\eta)(1-2\eta)}}{\dfrac{\beta(1-\beta)+\gamma(1-\gamma)-2\alpha(1-\alpha)}{\beta(1-\beta)(1-2\beta)+\gamma(1-\gamma)(1-2\gamma)-2\alpha(1-\alpha)(1-2\alpha)} - \dfrac{\delta(1-\delta)+\eta(1-\eta)-2\alpha(1-\alpha)}{\delta(1-\delta)(1-2\delta)+\eta(1-\eta)(1-2\eta)}}$$

and the ratio

$$\tau \equiv \frac{\dfrac{\delta+\eta-1}{\delta(1-\delta)+\eta(1-\eta)-2\alpha(1-\alpha)} - \dfrac{\beta+\gamma-2\alpha}{\beta(1-\beta)+\gamma(1-\gamma)-2\alpha(1-\alpha)}}{\dfrac{\beta(1-\beta)(1-2\beta)+\gamma(1-\gamma)(1-2\gamma)-2\alpha(1-\alpha)(1-2\alpha)}{\beta(1-\beta)+\gamma(1-\gamma)-2\alpha(1-\alpha)} - \dfrac{\delta(1-\delta)(1-2\delta)+\eta(1-\eta)(1-2\eta)}{\delta(1-\delta)+\eta(1-\eta)-2\alpha(1-\alpha)}}$$

are constant.

Note that the independence axiom of expected utility theory implies $2\alpha = \beta + \gamma$ and $\eta = 1 - \delta$. In this case, both ratios in axiom 4 are equal to zero and axiom 4 holds trivially. Given axiom 4 we can introduce the concept of a *transformed lottery*. For a degenerate lottery $\boldsymbol{x}\in\mathcal{L}$ the transformed lottery is the same degenerate lottery that yields outcome $x\in X$ with probability one. For a binary lottery $B\in\mathcal{L}$ that yields a more desirable outcome $x\in X$ with probability $p\in[0,1]$ and a less desirable outcome $y\in X$ with probability $1-p$, the transformed lottery, denoted as B^T, yields outcome $x\in X$ with probability $p[1-(1-p)(\rho+\tau[2p-1])]$ and outcome $y\in X$ with probability $(1-p)[1+p(\rho+\tau[2p-1])]$.

The definition of transformed lottery L^T for a generic lottery $L\in\mathcal{L}$ requires additional notation. For simplicity, we assume the existence of the most preferred outcome $\overline{x}\in X$ (*i.e.*, $\overline{\boldsymbol{x}}\succcurlyeq\boldsymbol{y}$ for all $y\in X$) and the least preferred outcome $\underline{x}\in X$ (*i.e.*, $\boldsymbol{y}\succcurlyeq\underline{\boldsymbol{x}}$ for all $y\in X$).[4] The case when $\overline{\boldsymbol{x}}\sim\underline{\boldsymbol{x}}$ is trivial and we consider only the case when $\overline{\boldsymbol{x}}\succ\underline{\boldsymbol{x}}$. For any outcome $x\in X$ let $p_x\in[0,1]$ denote a probability such that $\boldsymbol{x}\sim\overline{\boldsymbol{x}}p_x\underline{\boldsymbol{x}}$ and let $v(x)\equiv p_x[1-(1-p_x)(\rho+\tau[2p_x-1])]$ denote the transformed probability p_x (*i.e.*, $v(x)$ is the probability of $\overline{x}$ in lottery $\left[\overline{\boldsymbol{x}}p_x\underline{\boldsymbol{x}}\right]^T$). Let $X_L^-\equiv\{x\in X|v(x)\leq\sum_{x\in X}L(x)\,v(x)\}$ denote the set of "disappointing" outcomes in lottery $L\in\mathcal{L}$ and let $Q_2(L)\equiv\sum_{x\in X_L^-}L(x)$ be the cumulative probability of all such "disappointing" outcomes in L. Let $X_L^+\equiv X\backslash X_L^-$ denote the set of "elating" outcomes. Furthermore, let $X_L^{--}\equiv\left\{x\in X|v(x)\leq[1-Q_2(L)]\sum_{x\in X}L(x)\,v(x)+\sum_{x\in X_L^-}L(x)\,v(x)\right\}$ denote the set of "misfortunate" outcomes in lottery L and let $Q_1(L)\equiv\sum_{x\in X_L^{--}}L(x)$ denote the cumulative probability of all such "misfortunate" outcomes. Finally, let $X_L^{++}\equiv\left\{x\in X|v(x)>[1+Q_2(L)]\sum_{x\in X}L(x)\,v(x)-\sum_{x\in X_L^-}L(x)\,v(x)\right\}$ denote the set of "euphoric" outcomes in lottery L and let $Q_3(L)\equiv\sum_{x\in X_L^{++}}L(x)$ denote the cumulative probability of all such "euphoric" outcomes in lottery L. The transformed lottery L^T of lottery $L\in\mathcal{L}$ is then defined as follows:

$$L^T(x)=\begin{cases}L(x)\cdot\left[1+\rho\left[1-Q_2(L)\right]-\tau\left(2Q_1(L)-1+Q_2(L)\left[Q_3(L)-Q_1(L)\right]\right)\right], & x\in X_L^{--}\\ L(x)\cdot\left[1+\rho\left[1-Q_2(L)\right]-\tau\left(2Q_1(L)+Q_2(L)\left[Q_3(L)-Q_1(L)\right]\right)\right], & x\in X_L^-\backslash X_L^{--}\\ L(x)\cdot\left[1-\rho Q_2(L)-\tau\left(Q_3(L)+Q_1(L)+Q_2(L)\left[Q_3(L)-Q_1(L)\right]\right)\right], & x\in X_L^+\backslash X_L^{++}\\ L(x)\cdot\left[1-\rho Q_2(L)-\tau\left(Q_3(L)+Q_1(L)-1+Q_2(L)\left[Q_3(L)-Q_1(L)\right]\right)\right], & x\in X_L^{++}\end{cases}$$

Note that, by construction, $\sum_{x\in X}L^T(x)=\sum_{x\in X}L(x)=1$. Thus, $L^T\in\mathcal{L}$ for all $L\in\mathcal{L}$ as long as $L^T(x)\geq0$ for all $x\in X$. This condition is satisfied when either $\tau\in[-1,0]$ and $|\rho|\leq1+\tau$, or $\tau\in[0,⅓]$ and $|\rho|\leq1$, or $\tau\in[⅓,8/9]$ and $(\rho\pm\tau)^2\leq8\tau(1-\tau)$.

Under expected utility theory, when $\rho=\tau=0$, the transformed lottery of any lottery is just the lottery itself. When $\rho>0$ ($\rho<0$) the transformation relatively increases (decreases) the probabilities of "disappointing" outcomes and relatively decreases (increases) the probabilities of "elating" outcomes (*cf*. Blavatskyy, 2010, p. 2052). When $\tau>0$ ($\tau<0$) the transformation relatively decreases (increases) the probabilities of outcomes that are neither "misfortunate" nor "euphoric" and increases (decreases) the probabilities of either "misfortunate" or "euphoric" outcomes (or both).

[4] Behavioral characterization of model (4) for an unbounded outcome set is analogous to that of standard additive utility (*cf*. working paper Blavatskyy, 2014). .

Let L^{-T} denote a lottery such that its transformed lottery is *L*. With this last notation we can formulate the independence axiom for transformed lotteries (axiom 5 below).

Axiom 5 (*Independence axiom for transformed lotteries*) For all *F*, *G*, $H \in \mathcal{L}$ and $\alpha \in [0,1]$ we have $F \succcurlyeq G$ if and only if $[F^T \alpha H^T]^{-T} \succcurlyeq [G^T \alpha H^T]^{-T}$.

Theorem 1 (*von Neumann and Morgenstern, 1947*) Preference relation $\succcurlyeq$ on $\mathcal{L}$ satisfies axioms 1-5 if and only if there exists utility function $u: X \to \mathbb{R}$, unique up to a positive affine transformation, such that for any *F*, $G \in \mathcal{L}$ we have $F \succcurlyeq G$ if and only if $\sum_{x \in X} F^T(x) \cdot u(x) \geq \sum_{x \in X} G^T(x) \cdot u(x)$.

Proof: The necessity of axioms 1-5 is straightforward. Their sufficiency follows from the fact that, given axiom 4, preference relation $\succcurlyeq$ on $\mathcal{L}$ induces a preference relation $\succcurlyeq^t$ on transformed lotteries, which is a continuous weak order and satisfies the independence axiom. Preference relation $\succcurlyeq^t$ then admits expected utility representation (von Neumann and Morgenstern, 1947). *Q.E.D.*

Theorem 1 effectively states that the preference relation $\succcurlyeq$ is represented by utility function $U(L) = \sum_{x \in X} L^T(x) \cdot u(x)$. This utility function can be rewritten as (17).

$$\begin{aligned}
U(L) = EU(L) - \rho \sum_{x \in X_L^-} L(x) \cdot [EU(L) - u(x)] \\
+ \tau \sum_{x \in X_L^{++}} L(x) \cdot \left[u(x) - EU(L) - \sum_{x \in X_L^-} L(x) \cdot [EU(L) - u(x)] \right] \\
- \tau \sum_{x \in X_L^{--}} L(x) \cdot \left[EU(L) - \sum_{x \in X_L^-} L(x) \cdot [EU(L) - u(x)] - u(x) \right]
\end{aligned} \tag{17}$$

If preferences are represented by utility function (17) then a decision maker is indifferent between a degenerate lottery $\boldsymbol{x}$ and a binary lottery $\overline{\boldsymbol{x}} p_x \underline{\boldsymbol{x}}$ if and only if equality (18) holds.

$$\frac{u(x) - u(\underline{x})}{u(\overline{x}) - u(\underline{x})} = p_x[1 - (1 - p_x)(\rho + \tau[2p_x - 1])] \tag{18}$$

Hence, the right-hand-side of (18), which we denoted by *v*(*x*), is nothing but the utility of outcome $x \in X$ (up to a positive affine transformation). This implies that set X_L^- is identical to the set $\{x \in X | u(x) \leq EU(L)\}$ and utility function (17) can be rewritten as (19).

$$\begin{aligned}
U(L) = EU(L) - \rho \cdot EUD(L) + \tau \sum_{x \in X_L^{++}} L(x) \cdot [u(x) - EU(L) - EUD(L)] \\
- \tau \sum_{x \in X_L^{--}} L(x) \cdot [EU(L) - EUD(L) - u(x)]
\end{aligned} \tag{19}$$

Finally, note that the set X_L^{--} is identical to the set $\{x \in X | u(x) \leq EU(L) - EUD(L)\}$ and the set X_L^{++} is identical to the set $\{x \in X | u(x) > EU(L) + EUD(L)\}$. Thus, utility function (19) can be rewritten as utility function (4).

4. An application: demand for insurance

Consider a decision maker who may lose D dollars with probability $p\in(0,1)$. The decision maker can purchase $a\in[0,D]$ units of insurance. One unit of insurance costs π dollars and pays off one dollar when the loss occurs. Let us investigate the optimality of full insurance (*i.e.*, $a=D$) when preferences of the decision maker are represented by utility function (4).

In case of full insurance, the decision maker loses πD dollars for sure. If only $a\in[0,D)$ units of insurance are purchased, the decision maker loses πa dollars with probability $1-p$ and loses $D + a(\pi - 1)$ dollars with probability p. If preferences over binary lotteries are represented by utility function (8), the full insurance is optimal if and only if for any $a\in[0,D)$ the following inequality holds:

$$u(-\pi D) > u\big(-D - a(\pi - 1)\big) + \big[u(-\pi a) - u\big(-D - a(\pi - 1)\big)\big](1 - p)[1 - \rho p - \tau p(1 - 2p)]$$

This inequality can be rearranged as inequality (20).

$$(20)\quad \frac{u(-\pi D)-u\big(-D-a(\pi-1)\big)}{(D-a)(1-\pi)}(1 - \pi) > \frac{u(-\pi a)-u\big(-D-a(\pi-1)\big)}{D-a}(1 - p)[1 - \rho p - \tau p(1 - 2p)]$$

The fraction on the left-hand side of inequality (20) is nothing but the slope of utility function $u(.)$ between points $-\pi D$ and $-D - a(\pi - 1)$. Similarly, the fraction on the right-hand side of inequality (20) is the slope of utility function $u(.)$ between points $-\pi a$ and $-D - a(\pi - 1)$. For a strictly concave utility function $u(.)$ the slope between points $-\pi D$ and $-D - a(\pi - 1)$ is always greater than the slope between points $-\pi a$ and $-D - a(\pi - 1)$. Thus, full insurance is optimal when

$$(21)\qquad 1 - \pi \geq (1 - p)[1 - \rho p - \tau p(1 - 2p)]$$

which can be further simplified into inequality (22).

$$(22)\qquad \pi \leq p + p(1 - p)[\rho + \tau(1 - 2p)]$$

Inequality (22) is not only sufficient but also necessary for full insurance. The necessity follows from the fact that it is always possible to choose $a\in[0,D)$ sufficiently close to D such that the fraction on the left-hand side of inequality (20) is arbitrary close to the fraction on the right-hand side of (20). Thus, if inequality (22) is violated then it is possible to choose $a\in[0,D)$ sufficiently close to D such that inequality (20) is violated as well.

When $\rho=\tau=0$ inequality (22) becomes $\pi \leq p$. This is a well-known result—an expected utility maximizer with a concave utility function never buys full insurance if the insurance premium is actuarially unfair. Yet, inequality (22) shows that a decision maker, who cares not only about expected utility but also—about expected utility deviation and skewness, may insure in full even when the insurance premium is actuarially unfair (*i.e.*, when $\pi > p$). In particular, full insurance with an actuarially unfair premium is possible when $\rho + \tau(1 - 2p) > 0$.

Let us now consider the case when insurance premium   is so high that the decision maker is exactly indifferent between full insurance and staying uninsured. Kahneman and Tversky (1979, p. 269) considered the possibility of *probabilistic* insurance that works as follows. The decision maker can pay only a fraction $\beta \in [0,1]$ of the insurance premium π. If the loss occurs, with probability β the decision maker pays the remaining fraction of the premium $(1-\beta)\pi$ to receive full indemnity of the loss; but with probability $1-\beta$ the insurance company does not honor the contract (it reimburses the paid fraction of the premium $\beta\pi$ and offers no indemnity). Kahneman and Tversky (1979, p. 270) find that most people are not willing to purchase probabilistic insurance even though such insurance is superior to normal insurance according to expected utility theory with a concave utility function.

A decision maker with probabilistic insurance policy faces a loss *D* with probability $(1-\beta)p$, a loss π—with probability βp and a loss $\beta\pi$—with probability $1-p$. A decision maker who is fully insured ($\beta=1$) faces a loss π with probability one. A decision maker who stays uninsured ($\beta=0$) faces a loss *D* with probability *p* and no loss with probability $1-p$. For simplicity, we use normalization $u(-D)=0$ and $u(0)=1$. If a decision maker is indifferent between full insurance and staying uninsured and preferences over binary lotteries are represented by (8) then equality (23) must hold.

$$u(-\pi)=(1-p)[1-\rho p-\tau p(1-2p)] \tag{23}$$

Using equality (23), the expected utility of probabilistic insurance can be written as formula (24).

$$EU(PI)=\beta p(1-p)[1-\rho p-\tau p(1-2p)]+u(-\beta\pi)(1-p) \tag{24}$$

We consider the case when the loss of π dollars is disappointing under probabilistic insurance, *i.e*. when $u(-\pi)\le EU(PI)$. Using (23) and (24) this condition can be rewritten as inequality (25).

$$u(-\beta\pi)\ge(1-\beta p)[1-\rho p-\tau p(1-2p)] \tag{25}$$

Using (23) and (24) the expected utility deviation of probabilistic insurance can be written as (26).

$$EUD(PI)=p(1-p)u(-\beta\pi)-\beta p(1-p)^2[1-\rho p-\tau p(1-2p)] \tag{26}$$

Two subcases are possible. First, when $u(-\pi)>EU(PI)-EUD(PI)$ the loss of π dollars is not misfortunate. In this case, if preferences are represented by utility (4), the decision maker strictly prefers full insurance (*i.e*., $\beta=1$) over probabilistic insurance with $\beta \in (0,1)$ if and only if inequality (27) holds.

$$u(-\beta\pi)<\frac{1-\beta p-\rho\beta p(1-p)+\tau\beta p^2(3-2\beta-2p+\beta p)}{1-\rho p-\tau p(1-\beta-2p+\beta p)}[1-\rho p-\tau p(1-2p)] \tag{27}$$

When $\rho=\tau=0$ the right-hand side of inequality (27) coincides with the right-hand side of inequality (25). Thus, an expected utility maximizer with a concave utility function, *i.e*., when $u(-\beta\pi)\ge 1-\beta p$, always prefers probabilistic insurance over full insurance (*cf*. Kahneman and Tversky, 1979, p. 270). Yet, a decision maker, who cares not only about expected utility but also—about expected utility deviation and skewness, may dislike probabilistic insurance. A necessary condition for such aversion to probabilistic insurance is that the right-hand side of inequality (25) is smaller than the right-hand side of inequality (27). This can be written as inequality (28).

$$\rho+\tau\big(1-p(2-\beta)\big)>0 \tag{28}$$

Interestingly, inequality (28) holds for all $\beta \in (0,1)$ when $\rho+\tau(1-2p)>0$, which we established previously to be a necessary condition for full insurance with an actuarially unfair insurance premium.

Finally, let us consider the second subcase. When $u(-\pi) \leq EU(PI) - EUD(PI)$ the loss of π dollars is misfortunate. Using (23), (24) and (26) this condition can be rewritten as inequality (29).

$$u(-\beta\pi) \geq \frac{1-\beta p(2-p)}{1-p}[1-\rho p-\tau p(1-2p)] \tag{29}$$

In this case, if preferences are represented by utility function (4), the decision maker strictly prefers full insurance (*i.e.*, $\beta = 1$) over probabilistic insurance with $\beta \in (0,1)$ if and only if inequality (30) holds.

$$u(-\beta\pi) < 1-\beta p-\beta p(1-p)[\rho+\tau(1-2p)] \tag{30}$$

When $\rho=\tau=0$ the right-hand side of inequality (30) is less then the right-hand side of inequality (29), *i.e*. expected utility maximizer cannot prefer full insurance over probabilistic insurance. Yet, when $\rho+\tau(1-2p) > 1$ the right-hand side of inequality (29) is less than the right-hand side of inequality (30), *i.e*. a decision maker, who cares not only about expected utility but also—about expected utility deviation and skewness, may prefer to avoid probabilistic insurance.

5. An application: optimal portfolio investment

Consider $N \in \mathbb{N}$ assets that were traded for $T \in \mathbb{N}$ time periods in the past. Assets are indexed with a subscript $i \in \{1,\ldots,N\}$. Time periods are indexed with a subscript $t \in \{1,\ldots,T\}$. Thus, $x_{it} \in \mathbb{R}$ denotes the return on asset $i \in \{1,\ldots,N\}$ in time period $t \in \{1,\ldots,T\}$. An investor can invest wealth $w \in \mathbb{R}$ into any of N available assets. Let $\alpha_i \in [0,1]$ denote the share of wealth w that is invested into asset $i \in \{1,\ldots,N\}$, $\sum_{i=1}^{N} \alpha_i = 1$. We assume that the frequency of assets' returns over the past T periods fully describes the distribution of future returns. An investor who maximizes expected utility solves problem (31).

$$\begin{gathered} \max_{\alpha_1,\alpha_2,\ldots,\alpha_N} \frac{1}{T}\sum_{t=1}^{T} u\left(w\sum_{i=1}^{N}\alpha_i x_{it}\right) \\ a_1, a_2, \ldots, a_N \geq 0 \\ a_1 + a_2 + \cdots + a_N = 1 \end{gathered} \tag{31}$$

For example, table 4 shows quarterly returns on seven most popular investment options of the Australian superannuation fund UniSuper over ten years (2004-2013). As per 31 January 2014, Unisuper clients—employees of Australia's higher education and research sector—invested AUD 1157 million (5.66% of all invested funds) into option "Cash" that yields the lowest expected return but also—the lowest variance of returns. "Balanced" was, by far, the most popular investment option with AUD 9 832.2 million (or 48.07%) invested into it.

Let us consider a constant relative risk aversion utility function (32) that has one subjective parameter—a coefficient of relative risk aversion r. The main advantage of this functional form is that the optimal portfolio becomes independent of the wealth level w.

$$u(x) = \begin{cases} \frac{x^{1-r}}{1-r}, & r \neq 1 \\ \ln x, & r = 1 \end{cases} \tag{32}$$

Option / Period	Balanced	Capital Stable	Conservative Balanced	Growth	High Growth	Cash	Socially Responsible High Growth
Q1 2004	1.0368	1.0247	1.0313	1.0415	1.0446	1.0126	1.0318
Q2 2004	1.0263	1.0155	1.0197	1.0325	1.0302	1.0122	1.0310
Q3 2004	1.0389	1.0300	1.0317	1.0432	1.0372	1.0122	1.0242
Q4 2004	1.0536	1.0293	1.0411	1.0621	1.0699	1.0121	1.0657
Q1 2005	1.0211	1.0081	1.0128	1.0279	1.0248	1.0126	1.0208
Q2 2005	1.0395	1.0315	1.0352	1.0419	1.0453	1.0132	1.0517
Q3 2005	1.0442	1.0205	1.0328	1.0543	1.0694	1.0128	1.0618
Q4 2005	1.0402	1.0266	1.0354	1.0457	1.059	1.0132	1.0515
Q1 2006	1.0466	1.0241	1.0368	1.0594	1.0714	1.0131	1.0766
Q2 2006	0.9875	0.9972	0.9890	0.9817	0.9639	1.0135	0.9780
Q3 2006	1.0406	1.0274	1.0330	1.0450	1.0446	1.0142	1.0379
Q4 2006	1.0492	1.0229	1.0372	1.0594	1.0793	1.0167	1.0713
Q1 2007	1.0319	1.0207	1.0265	1.0365	1.0485	1.0142	1.0416
Q2 2007	1.0315	1.0132	1.0222	1.0393	1.0497	1.015	1.0363
Q3 2007	1.0259	1.0229	1.0250	1.0261	1.0181	1.0115	1.0213
Q4 2007	0.9873	0.9978	0.9906	0.9845	0.9807	1.012	0.9566
Q1 2008	0.9443	0.9889	0.9624	0.9268	0.9105	1.0063	0.9036
Q2 2008	0.9830	0.9952	0.9880	0.9808	0.9774	1.0182	0.9784
Q3 2008	1.0014	1.0173	1.0053	0.9970	0.9907	1.0159	0.9566
Q4 2008	0.9074	0.9696	0.9261	0.8763	0.8612	1.0081	0.8423
Q1 2009	0.9845	0.9958	0.9921	0.9844	0.9812	1.0059	0.9749
Q2 2009	1.0140	1.0081	1.0239	1.0144	1.0217	1.0075	1.0342
Q3 2009	1.0955	1.0581	1.0852	1.1082	1.1192	1.0089	1.1362
Q4 2009	1.0291	1.0101	1.0203	1.0312	1.0369	1.0094	1.0402
Q1 2010	1.0164	1.0183	1.0182	1.0155	1.0127	1.0082	1.0119
Q2 2010	0.9635	1.0007	0.9775	0.9531	0.9451	1.0097	0.9140
Q3 2010	1.0493	1.0283	1.0394	1.0563	1.0619	1.0109	1.0543
Q4 2010	1.0261	1.0142	1.0202	1.0302	1.0330	1.0120	1.0270
Q1 2011	1.017	1.0154	1.0160	1.0183	1.0196	1.0105	1.0259
Q2 2011	0.9824	1.0039	0.9940	0.9740	0.9683	1.0110	0.9624
Q3 2011	0.9536	0.9962	0.9751	0.9396	0.9307	1.0112	0.9123
Q4 2011	1.0258	1.0230	1.0247	1.0263	1.0278	1.0116	1.0287
Q1 2012	1.0589	1.0294	1.0432	1.0688	1.0747	1.0095	1.0656
Q2 2012	0.9866	1.0195	1.0025	0.9758	0.9653	1.0098	0.9549
Q3 2012	1.0552	1.0252	1.0396	1.0638	1.0702	1.0085	1.0678
Q4 2012	1.0294	1.0167	1.0268	1.0506	1.0380	1.0087	1.0687
Q1 2013	1.0529	1.0239	1.0334	1.0485	1.0704	1.0066	1.0615
Q2 2013	1.0195	1.0044	1.0199	1.0216	1.0233	1.0067	1.0074
Q3 2013	1.0600	1.0356	1.0398	1.0739	1.0834	1.0066	1.1050
Q4 2013	1.0320	1.0155	1.0216	1.0395	1.0430	1.0066	1.0383

Table 4 Quarterly returns on seven Unisuper investment options over time period 2004-2013

An investor who maximizes expected utility, *i.e*. solves problem (31), with a constant relative risk aversion utility function (32) includes investment option “Cash” in his or her optimal portfolio only when his or her coefficient of relative risk aversion is greater than 19.36. In contrast, empirical studies typically elicit coefficients of relative risk aversion that are less than one (*e.g*., Blavatskyy and Pogrebna, 2010a, table III, p. 973). This puzzling discrepancy is known as the equity premium puzzle: only implausibly high levels of risk aversion could rationalize investing in bonds when stocks are available (*cf*. Mehra and Prescott, 1985; Blavatskyy and Pogrebna, 2010, p. 160).

Model (4) can rationalize the equity premium puzzle. Let us consider an example when an investor has a linear utility (*i.e*., utility function (32) with r=0), a positive ρ and τ=0. In this case, the expected utility of a portfolio becomes simply the portfolio’s expected return. Let $\mu_i = \frac{1}{T}\sum_{t=1}^{T} x_{it}$ denote the expected return on asset $i \in \{1,\ldots,N\}$. The expected return μ_α on one dollar invested in portfolio $\{\alpha_i\}_{i=1}^{N}$ is a weighted sum of expected returns on N assets that compose this portfolio:

(33)
$$\mu_\alpha = \sum_{i=1}^{N} \alpha_i \mu_i$$

Portfolio’s expected utility deviation σ_α can be written as (34).

(34)
$$\sigma_\alpha = \frac{1}{2T}\sum_{t=1}^{T}\left|\sum_{i=1}^{N} \alpha_i (x_{it} - \mu_i)\right|$$

An asset has a disappointing return in period t if the asset’s return in that period is less than or equal to the asset’s average return in all other periods. Two assets have co-disappointing returns if in every time period when a return on one asset is disappointing, so is the return on the other asset. If all available assets have co-disappointing returns, then portfolio’s expected utility deviation (34) is a weighted sum of expected utility deviations on N assets that compose this portfolio:

(35)
$$\sigma_{co-dis.} = \sum_{i=1}^{N} \alpha_i \left(\frac{1}{2T}\sum_{t=1}^{T} |x_{it} - \mu_i|\right)$$

Thus, when all available assets have co-disappointing returns, there is no additional benefit from diversification. An optimal portfolio is to invest all funds in one asset (which is not necessarily the asset with the highest expected return when parameter ρ is relatively high).

When not all available assets have co-disappointing returns, portfolio’s expected utility deviation (34) is less then the weighted sum (35) of expected utility deviations on N assets that compose this portfolio. In this case, diversification may reduce portfolio’s expected utility deviation more than proportionately to the reduction in the portfolio’s expected return.

In the financial literature, the problem of optimal portfolio investment is typically illustrated on a diagram that plots the expected return μ_α on the vertical axis and a measure of financial risk (in our case—expected utility deviation σ_α) on the horizontal axis. In such a diagram, the indifference curves representing the preferences of an investor with r=0 and τ=0 are straight lines with a slope ρ. Note that the investor does not violate the first-order stochastic dominance when the slope of indifference lines is between -1 and 1 (*i.e.*, $\rho \in [-1,1]$). Since indifference curves are straight lines, the optimal portfolio must be always located on the convex hull (envelope) over all feasible portfolios. Figure 5 shows the convex hull over all feasible portfolios with seven assets described in table 4 as well as the highest attainable indifference line with a slope ρ=0.8379. In this case, the optimal portfolio is such that the investor invests 5% of his or her investment funds into "Cash". This simple example illustrates that our proposed theory can rationalize the equity premium puzzle with a reasonably high parameter ρ (even in the extreme case when an investor has a linear utility function).

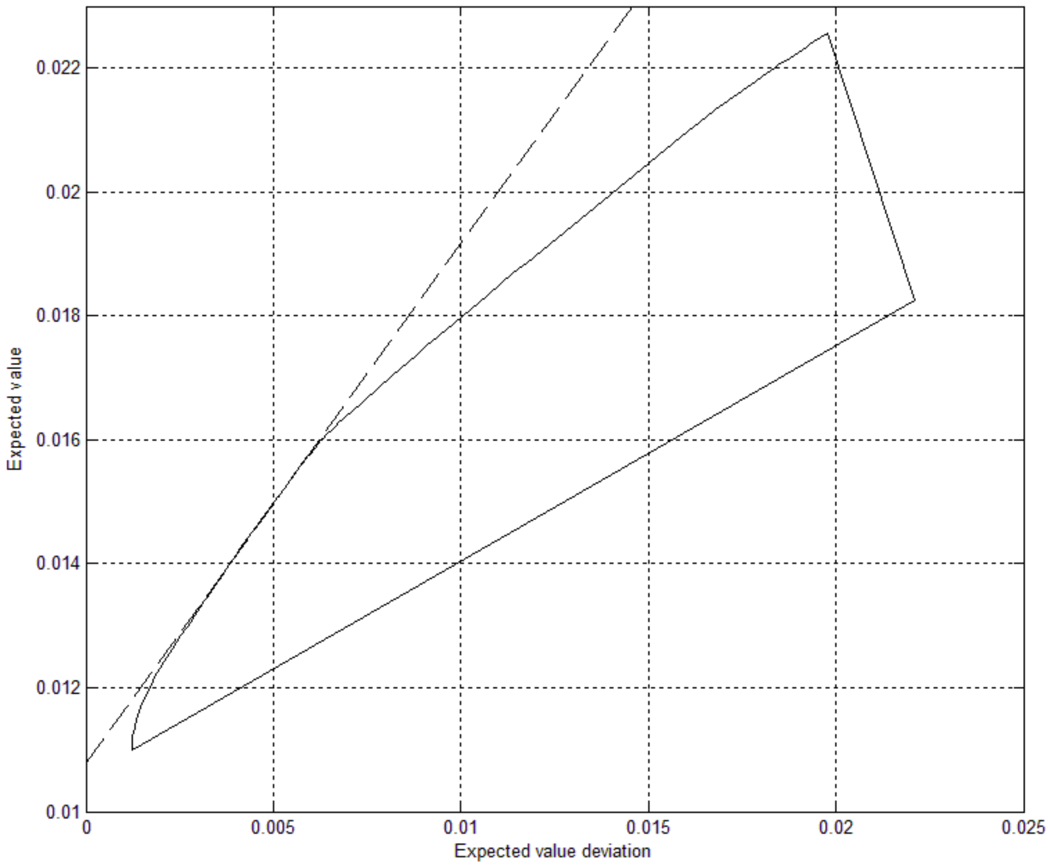


Figure 5 A convex hull over all available portfolios with seven assets described in table 4 and the highest attainable indifference line that represents investor's preferences when r=0, ρ=0.8379 and τ=0

6. Related literature

In the context of monetary lotteries, Markowitz (1952, p.91) considered utility function that depends on the expected value, the variance and the third central moment[5] of a lottery. Model (4) is based on a similar premise—a decision maker may care not only about expected utility but also about expected utility dispersion and skewness. Yet, model (4) avoids central moments for measuring dispersion or skewness. Consequently, model (4) satisfies the first-order stochastic dominance (when ρ and τ are small) whereas Markowitz utility function inevitably violates dominance (Borch, 1969).

According to Bell (1985, p.5), the expected value of a lottery may reflect the psychological expectation of the lottery's outcome by a decision maker with a linear utility function over money. Such a decision maker experiences disappointment (elation) when receiving a lower (higher) outcome than the expected value of the lottery. Bell (1985) considered only a binary lottery that yields a more desirable outcome *x* with a probability $p \in [0,1]$ and a less desirable outcome *y* with probability 1-*p*. Bell (1985, p.12) proposed to evaluate such a lottery with utility function $u(y) + \pi(p)[u(x) - u(y)]$ and considered the case when $\pi(p) = p[1 + (e - d)(1 - p)]$ for some constant *e*, $d \geq 0$. Bell's utility function is a special case of our utility function for binary lotteries (8) with τ=0 and $\rho = d - e$.

Loomes and Sugden (1986) argued that a decision maker may form a prior expectation, which they call *basic* expected utility of a lottery, and experiences disappointment (elation) when a realized *ex post* outcome of the lottery falls short of (exceeds) this expectation. Loomes and Sugden (1986) proposed to represent such preferences with utility function, which in our notation can be written as

$$U(L) = EU(L) + \sum_{x \in X} L(x)\, D\big(u(x) - EU(L)\big) \tag{36}$$

where *D*(.) is a differentiable non-decreasing real-valued function satisfying *D*(0)=0. Loomes and Sugden (1986, p. 276) apparently realized that such a function *D*(.) must be linear on the positive domain *and* linear on the negative domain if utility function *u*(.) is unique up to a positive affine transformation. Yet, such a piece-wise linear function *D*(.) is differentiable only when *D*(*t*)=*t* for all *t*, *i.e*. when the theory of Loomes and Sugden (1986) coincides with expected utility theory.

If we drop the assumption that function *D*(.) is differentiable but maintain the assumption that *u*(.) is a standard von Neumann-Morgenstern utility function, which is unique up to a positive affine transformation, then the theory of Loomes and Sugden (1986) is a special case of our proposed theory when τ=0. Note that if function *D*(.) is linear on the positive domain with a slope *e* and linear on the negative domain with a slope *d* then utility (36) becomes utility (4) with τ=0 and $\rho = d - e$. This special case is also analyzed in Blavatskyy (2010).

[5] Markowitz (1952, p.90) noted that the third moment "may be connected with a propensity to gamble".

Gul (1991) developed a theory of disappointment aversion that uses disappointment-elation decomposition of outcomes. In Gul's theory, a lottery's outcome is disappointing (elating) if the lottery is (not) strictly preferred over a degenerate lottery yielding this outcome with probability one. The theory proposed in this paper uses a different notion of disappointment/elation: a lottery's outcome is disappointing (elating) if its utility is less (greater) than the expected utility of the lottery, not the total utility (4) of the lottery. Thus, Gul's notion of disappointment/elation coincides with the one used in this paper only when a decision maker behaves as an expected utility maximizer. Gul's theory of disappointment aversion belongs to a larger class of models that satisfy the betweenness axiom, whereas our proposed model can violate betweenness (*cf*. subsection 2.4).

Conlisk (1993) proposed to model choice under risk by adding a small term, called utility of gambling, to the expected utility of a lottery. Conlisk (1993, p.263) assumed that utility of gambling is always strictly positive for non-degenerate lotteries. Similar to Conlisk (1993) model, utility function (4) adds two terms (expected utility deviation and skewness) to the expected utility of a lottery. Unlike in Conlisk (1993) model, the combined effect of these two terms is not necessarily strictly positive (*cf*. Figure 1 in subsection 2.1).

Jia et al. (2001) proposed to evaluate lotteries over monetary outcomes with a "generalized disappointment model"

$$U(L) = EV(L) - \frac{\lambda}{2}\sum_{x \in X} L(x)\,|x - EV(L)| \tag{37}$$

where $EV(L)$ denotes the expected value of lottery L and $\lambda \in [-1,1]$ is a constant. This model is a special case of our proposed model (4) with a linear utility function over money, τ=0 and $\rho = \lambda/2$.

Delquié and Cillo (2006, section 3, p. 203) propose a model of "disappointment without a prior expectation" that is equivalent to Quiggin (1981) rank-dependent utility with a quadratic weighting function. For binary lotteries, this model is a special case of model (8) when τ=0. More generally, for binary lotteries our proposed theory is equivalent to Quiggin (1981) rank-dependent utility with a cubic probability weighting function,[6] *cf*. equation (8). This equivalence, however, does not hold for lotteries with three or more outcomes. Rank-dependent utility with a cubic weighting function can be written as a weighted sum of 1) expected utility, 2) one half of Gini (1912) mean difference of utilities of lottery's outcomes, and 3) expected difference between the mean and the median utility across all triples of outcomes, which is known as the third L-moment or L-skewness (Hosking, 1990). Thus, rank-dependent utility with a cubic weighting function has a similar structure as model (4) but employs different measures of utility dispersion and skewness. In particular, one half of Gini (1912) mean difference of utilities of lottery's outcomes generally exceeds the expected utility deviation (2).

[6] This cubic probability weighting function $w(p) = p - \rho \cdot p(1-p) + \tau \cdot p(1-p)(1-2p)$ is analysed in Blavatskyy (2014a).

Lottery P	90% white $0	6% red win $45	1% green win $30	1% blue lose $15	2% yellow lose $15
Lottery Q	90% white $0	6% red win $45	1% green win $45	1% blue lose $10	2% yellow lose $15

Table 5 Lotteries used by Tversky and Kahneman (1986, p. 263) for testing the first-order stochastic dominance: description with a transparent dominance relation

7. Limitations and possible extensions of the proposed theory

Examples from section 2 show that model (4) can accommodate numerous behavioral patterns. Yet, our theory cannot rationalize some empirical findings such as a violation of the first-order stochastic dominance. Consider the following example from Tversky and Kahneman (1986, p. 263). Lotteries P and Q are presented to subjects as shown in table 5. Probability information is described by the percentage of marbles of different colors (one of which is drawn at random from a box). Note that Q first-order stochastically dominates P. Moreover, the dominance relation is transparent in the description shown in table 5: for every color the outcome in Q is at least as good as the outcome in P. Tversky and Kahneman (1986, p. 263) found that every subject chooses Q over P. Similarly, Carbone and Hey (1995), Loomes and Sugden (1998, table 2, p. 591) and Hey (2001, table 2, p.14) found that the first-order stochastic dominance is violated in 1 out of 320 cases (0.3%), 13 out of 920 cases (1.4%) and 24 out of 1590 cases (1.5%) respectively, when the dominance relation is transparent.

Yet, Tversky and Kahneman (1986, p. 264) also found that 72 out of 124 subjects (58%) choose a dominated lottery P when lotteries P and Q are presented as shown in table 6 below. Stochastic dominance is not transparent in this presentation. In fact, for every color, the outcome in P is at least as good as the outcome in Q but the composition of colors now differs for P and Q (*i.e*., marbles are drawn from different, not the same box). Birnbaum and Navarrete (1998, p. 61) as well as Birnbaum (2005, p.1356) also found that the majority of subjects violate the first-order stochastic dominance when choice alternatives are presented in such a way that the dominance relation is not transparent.

The theory proposed in this paper can be extended to an arguably more descriptive model that can accommodate violations of the first-order stochastic dominance if parameters ρ and τ are unrestricted, *i.e*. they can be arbitrary large in the absolute value (*cf*. section 1). Yet, such parameterization is not considered in this paper because model (4) then does not have a clear preference foundation (we cannot define transformed lotteries in section 3). One could also argue that the theory loses much of its normative appeal when parameters ρ and τ are left unrestricted.

Lottery P	90% white $0	6% red win $45	1% green win $30	3% yellow lose $15
Lottery Q	90% white $0	7% red win $45	1% green lose $10	2% yellow lose $15

Table 6 Lotteries used by Tversky and Kahneman (1986, p. 264) for testing the first-order stochastic dominance: description with a non-transparent dominance relation

Tversky (1969) found systematic intransitive cycles, which challenge any model representing a transitive preference relation over lotteries, including model (4). Birnbaum and Gutierrez (2007) replicated Tversky (1969) experiment only to discover that “the vast majority of participants … were transitive”. Similarly, Birnbaum and Schmidt (2010) found “no evidence to reject the hypothesis that everyone had a transitive preference order”. Loomes et al. (1989, 1991) as well as Starmer and Sugden (1998) found that, even though the majority of subjects reveal transitive preferences, most of those few, who do not, exhibit an intransitive choice cycle that is consistent with regret theory.

Consider the following example from Day and Loomes (2010). Lottery U yields a 0.6 chance of £25 (and a 0.4 chance of £0), lottery V yields a 0.8 chance of £15 (and a 0.2 chance of £0) and lottery Z yields £10 for sure. Day and Loomes (2010, table 4, p. 238) found that 13 out of 100 subjects choose Z over V, V—over U and U—over Z. This intransitive choice pattern is consistent with regret theory (Loomes and Sugden, 1982). Only one out of 100 subjects chooses V over Z, U—over V and Z—over U. This intransitive choice pattern is consistent with similarity theory (Tversky, 1969; Rubinstein, 1988).

Both “regret” and “similarity” cycles falsify model (4). Our proposed theory, however, can be extended into a model of probabilistic choice.[7] Let us consider a simple illustration. A decision maker has a linear utility function *u*(.), ρ=0.9, τ=0 and makes constant errors by choosing his/her preferred lottery with probability $1-\varepsilon$, and his/her non-preferred lottery—with a (relatively small) probability $\varepsilon < 0.5$ that can be interpreted as the probability of a tremble (Harless and Camerer, 1994, p. 1260-62). With probability $\varepsilon(1-\varepsilon)^2$ this decision maker chooses Z over V, V—over U and U—over Z. With probability $\varepsilon^2(1-\varepsilon)$ the decision maker chooses V over Z, U—over V and Z—over U. Thus, “regret” cycles outnumber “similarity” cycles in proportion $1-\varepsilon$ to ε. Obviously, if probability ε is small then “regret” cycles can significantly outnumber “similarity” cycles (*cf*. Sopher and Gigliotti, 1993).

Day and Loomes (2010, table 4, p. 238) also found that behavior changes when probabilities of all non-zero outcomes in lotteries U, V and Z are divided by 4. In this “scaled-down” version, 22 out of 100 subjects exhibit the “similarity” cycle and only 8 out of 100 subjects exhibit the “regret” cycle. With the “scaled-down” lotteries, the decision maker with a linear utility function *u*(.), ρ=0.9 and τ=0 reveals the “similarity” cycle with probability $\varepsilon(1-\varepsilon)^2$ and the “regret” cycle—with probability $\varepsilon^2(1-\varepsilon)$. Therefore, if probability ε is small, “similarity” cycles can now significantly outnumber “regret” cycles. This example illustrates that even the simplest possible probabilistic extension of our proposed theory (a so-called constant error or tremble model) can rationalize a higher incidence of “regret” (“similarity”) cycles with the “scaled-up” (“scaled-down”) triple of lotteries U, V and Z.

[7] Many studies find that decision making under risk is a probabilistic rather than a deterministic process. For example, Camerer (1989, p. 81), Hey and Orme (1994, p. 1296) as well as Ballinger and Wilcox (1997, p. 1100) found that 31.6%, 25% and 20.8% of subjects respectively reverse their choice when presented with the same decision problem within a short period of time.

Starmer and Sugden (1993), Humphrey (1995) as well as Birnbaum (2004; 2005, p. 1356) found event-splitting effects—choice patterns may differ when identical lotteries are presented in a coalesced and split form. Model (4), like any model based on the consequentialist premise, cannot rationalize event-splitting (and framing) effects. Blavatskyy (2012) reports violations of the hexagon condition (Blaschke and Bol, 1938, p. 10), which is also known as the Thomsen-Blaschke condition (*e.g.*, Wakker, 1984, p.112). This finding falsifies any model with a separable utility function for binary lotteries, including a family of rank-dependent models as well as their special case—model (8).

8. Conclusion

The theory presented in this paper is a parsimonious two-parameter generalization of the classical economic theory (expected utility). It also extends the standard financial theory along three dimensions: 1) a risk measure (mean absolute deviation) that does not violate stochastic dominance; 2) a non-linear utility function over money; and 3) a preference for skewness. Last but not least, our theory extends an intuitive psychological notion of disappointment (elation) to misfortune (euphoria). Arguably, synergy between economics, finance and psychology gives our theory a normative appeal.

Our proposed theory also appears to be descriptively accurate. It compares well with other existing theories in terms of goodness of fit to experiment data. For example, for each individual choice pattern in the data set collected by Blavatskyy (2013a), we estimated our proposed theory as well as nine other models: a simple heuristic[8], expected utility theory, Yaari (1987) dual model, Chew (1983) weighted utility, which coincides with Loomes and Sugden (1982) regret theory, Quiggin (1981) rank-dependent utility, Gul (1991) disappointment aversion theory, Chew *et al.* (1991) quadratic utility and Viscusi (1989) prospective reference theory. Details of the estimation procedure are described in Blavatskyy (2013a) and "raw" experimental data are available in Blavatskyy (2014a). Figure 6 shows the percentage of choice patterns for which various theories provide the best goodness of fit.[9] Clearly, our proposed theory compares quite favorably with all other theories.

The new theory appears to be useful in common applications. Many people prefer to purchase full insurance coverage even though most, if not all, insurance companies offer an actuarially unfair insurance premium. Such demand for insurance cannot be modeled with classical expected utility theory but it can be rationalized within the framework of our proposed theory. Some investors invest into bonds even though available shares often yield significantly higher returns. Expected utility theory can hardly rationalize such financial portfolio as optimal (the problem known as "the equity premium puzzle"). Our proposed theory can be used for modelling such investment decisions.

[8] The rule of thumb is the following: 1) choose a lottery with the lowest probability of the worst possible outcome; 2) if both lotteries yield the same probability of the worst possible outcome then choose the lottery with the highest probability of the best possible outcome.

[9] If two or more theories are not significantly different in terms of goodness of fit, then only a theory with fewer subjective parameters is shown on figure 6.

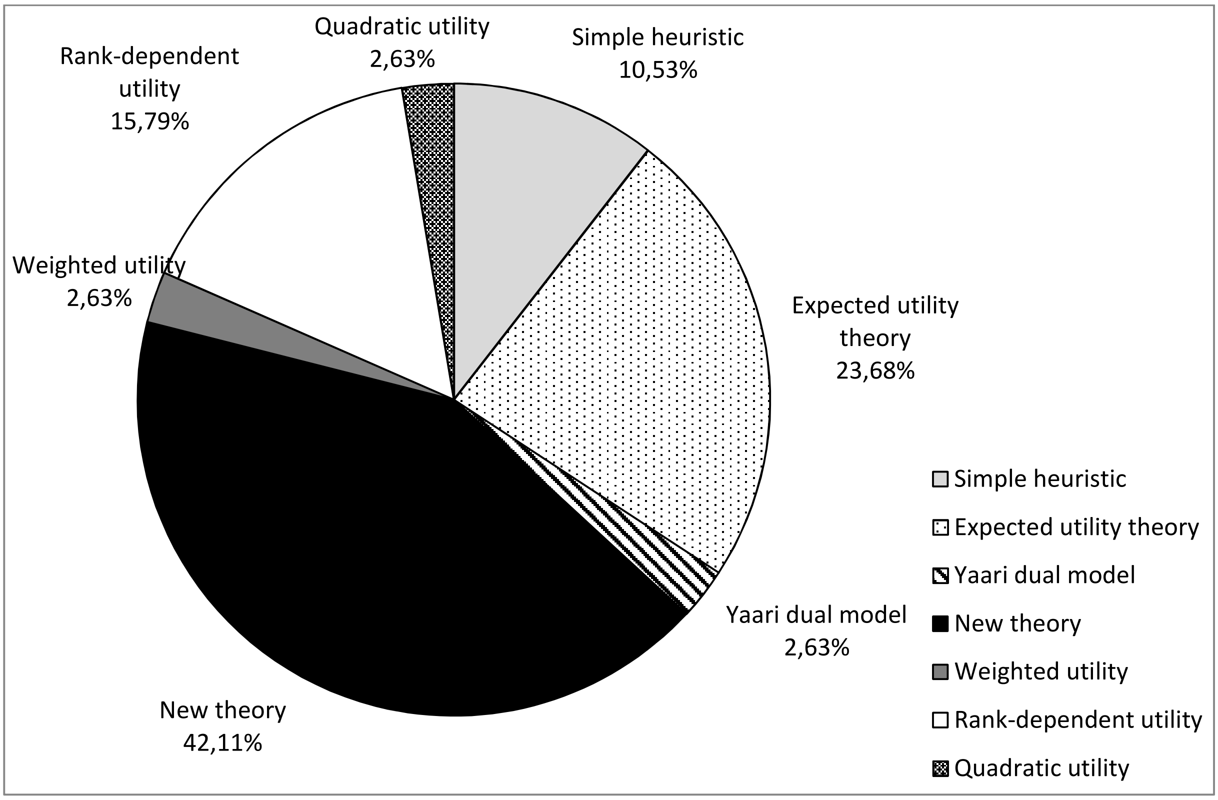


Figure 6 Percentage of subjects in Blavatskyy (2013a) data set whose choice patterns are best described by various theories

References

Allais, Maurice (1953) “Le Comportement de l’Homme Rationnel devant le Risque: Critique des Postulates et Axiomes de l’Ecole Américaine” *Econometrica* **21**, 503-546

Ballinger, Parker and Nathaniel Wilcox (1997) “Decisions, error and heterogeneity” *Economic Journal* **107**, 1090-1105

Becker, Gordon M., Morris H. DeGroot, and Jacob Marschak (1964) “Measuring utility by a single-response sequential method” *Behavioral Science* **9**, 226-232

Bell, David (1985) “Disappointment in Decision Making under Uncertainty” *Operations Research* **33**,1–27

Bernasconi, Michele (1994) “Nonlineal preference and two-stage lotteries: theories and evidence” *Economic Journal* **104**, 54-70

Bernoulli, D. (1738) “Specimen theoriae novae de mensura sortis” Commentarii Academiae Scientiarum Imperialis Petropolitanae; translated in Bernoulli, D., (1954) “Exposition of a new theory on the measurement of risk” *Econometrica* **22**, 23–36

Birnbaum, Michael (2005) “Three New Tests of Independence That Differentiate Models of Risky Decision Making” *Management Science* **51** (9), 1346–1358

Birnbaum, Michael (2004) “Tests of rank-dependent utility and cumulative prospect theory in gambles represented by natural frequencies: Effects of format, event framing, and branch splitting”

Organizational Behavior and Human Decision Processes **95**, 40–65

Birnbaum, Michael and R. Gutierrez (2007) "Testing for intransitivity of preferences predicted by a lexicographic semi-order" *Organizational Behavior and Human Decision Processes* **104**, 96–112

Birnbaum, Michael and Juan Navarrete (1998) "Testing descriptive utility theories: Violations of stochastic dominance and cumulative independence" *Journal of Risk Uncertainty* **17**, 49–78

Birnbaum, Michael, Jamie Patton and Melissa Lott (1999) "Evidence against rank-dependent utility theories: Violations of cumulative independence, interval independence, stochastic, dominance, and transitivity" *Organizational Behavior and Human Decision Processes* **77**, 44–83

Birnbaum, Michael and Ulrich Schmidt (2010) "Testing transitivity in choice under risk" *Theory and Decision* **69**, 599-614

Blaschke, W. and Bol G. (1938) "Geometrie der Gewebe: Topologische Fragen der Differentialgeometrie" Berlin: Springer

Blavatskyy, P. (2014) "Geometric utility theory" working paper ssrn.com/abstract=2305138

Blavatskyy, P. (2014a) "A Probability Weighting Function for Cumulative Prospect Theory and Mean-Gini Approach to Optimal Portfolio Investment" working paper ssrn.com/abstract=2380484

Blavatskyy, Pavlo (2013) "Reverse Allais Paradox" *Economics Letters* **119(1)**, 60-64

Blavatskyy, Pavlo (2013a) "Which Decision Theory?" *Economics Letters* **120 (1)**, 40-44

Blavatskyy, Pavlo (2012) "Troika Paradox" *Economics Letters* **115** (**2**), 236-239

Blavatskyy, Pavlo (2012a) "Probabilistic Choice and Stochastic Dominance" *Economic Theory* **50** (**1**), 59-83

Blavatskyy, Pavlo (2011) "A Model of Probabilistic Choice Satisfying First-Order Stochastic Dominance" *Management Science* **57 (3)**, 542-548

Blavatskyy, Pavlo (2010) "Modifying the Mean-Variance Approach to Avoid Violations of Stochastic Dominance" *Management Science* **56 (11)**, 2050-2057

Blavatskyy, P. (2010a) "Reverse common ratio effect" *Journal of Risk and Uncertainty* **40**:219-241

Blavatskyy, Pavlo (2009) "Preference Reversals and Probabilistic Choice" *Journal of Risk and Uncertainty* **39 (3)**, 237-250

Blavatskyy, Pavlo (2006) "Violations of Betweenness or Random Errors?" *Economics Letters* **91 (1)**, 34-38

Blavatskyy, Pavlo and Wolfgang Köhler (2009) "Range Effects and Lottery Pricing" *Experimental Economics* **12 (3)**, 332-349

Blavatskyy, Pavlo and Ganna Pogrebna (2010) "Reevaluating Evidence on Myopic Loss Aversion: Aggregate Patterns Versus Individual Choices" *Theory and Decision* **68**, 159-171

Blavatskyy, P. and G. Pogrebna (2010a) "Models of Stochastic Choice and Decision Theories: Why Both are Important for Analyzing Decisions" *Journal of Applied Econometrics* **25 (6)**, 963-986

Borch, K. (1969) "A Note on Uncertainty and Indifference Curves" *The Review of Economic Studies*, Vol. 36, No. 1, pp. 1-4

Butler, J. David and Graham C. Loomes (2007) “Imprecision as an Account of the Preference Reversal Phenomenon” *American Economic Review* **97(1):** 277-297

Camerer, Colin (1989) “An experimental test of several generalized utility theories.” *Journal of Risk and Uncertainty* **2**, 61-104

Camerer, Colin and Ho, T. (1994) “Violations of the Betweenness Axiom and Nonlinearity in Probability” *Journal of Risk and Uncertainty* **8**, 167-196

Carbone, Enrica and John Hey (1995) “A comparison of the estimates of EU and non-EU preference functionals using data from pairwise choice and complete ranking experiments” *Geneva Papers on Risk and Insurance Theory* **20**, 111-133

Chew, S. (1983) "A Generalization of the Quasilinear Mean with Applications to the Measurement of Income Inequality and Decision Theory Resolving the Allais Paradox" *Econometrica* **51**, 1065-1092

Chew, S., L. Epstein, U. Segal (1991) "Mixture Symmetry and Quadratic Utility" *Econometrica* **59**, 139-163

Conlisk, John (1989) “Three Variants on the Allais Example” *American Economic Review* **79 (3)**, 392-407

Conlisk, John (1993) “The Utility of Gambling” *Journal of Risk and Uncertainty* **6**: 255-275

Day, Brett and Graham Loomes (2010) “Conflicting violations of transitivity and where they may lead us” *Theory and Decision* **68**, 233-242

Debreu, G. (1954) “Representation of a Preference Ordering by a Numerical Function” in R. M. Thrall, C. H. Coombs, and R. L. Davis, eds., “Decision Processes” NY: John Wiley and Sons, 159-165

Dekel, E. (1986) “An axiomatic characterization of preferences under uncertainty” *Journal of Economic Theory* **40**, 304-318

Delquié Philippe and Alessandra Cillo (2006) “Disappointment without prior expectation: a unifying perspective on decision under risk” *Journal of Risk and Uncertainty* **33**, 197-215

Friedman, M. and Savage, L. (1948) “The utility analysis of choices involving risk” *Journal of Political Economy* **56**, 279-304

Gigliotti, G. and Sopher, B. (1993). “A Test of Generalized Expected Utility Theory.” *Theory and Decision* **35**, 75-106

Gini, Corrado (1912) “Variabilità e Mutabilità, contributo allo studio delle distribuzioni e delle relazione statistiche” Studi economici-giuridici dela Regia Università di Cagliari **3**, 3-159

Grether, David and Charles Plott (1979) “Economic Thoery of Choice and the Preference Reversal Phenomenon” *American Economic Review* **69** (4), 623-638

Gul, Faruk (1991) "A Theory of Disappointment Aversion" *Econometrica* **59**, 667-686

Harless, D. and C. Camerer (1994) The predictive utility of generalized expected utility theories, *Econometrica* **62**, 1251-1289

Hey, John (2001) “Does repetition improve consistency?” *Experimental Economics* **4**, 5-54

Hey, John and Chris Orme (1994) Investigating generalisations of expected utility theory using experimental data, *Econometrica* **62**, 1291-1326

Hosking, J.R.M. (1990) "L-moments: analysis and estimation of distributions using linear combinations of order statistics" *Journal of the Royal Statistical Society, Series B* **52**: 105–124

Humphrey, Steven (1995) “Regret aversion or event-splitting effects? More evidence under risk and uncertainty” *Journal of Risk and Uncertainty* **11** 263–274

Humphrey, Steven and Arjan Verschoor (2004) “The probability weighting function: experimental evidence from Uganda, India and Ethiopia” *Economics Letters* **84**, 419-425

Huygens, Christiaan (1657) “De Ratiociniis in Ludo Aleae” in Schooten, Frans van *ed*. “Exercitationum mathematicarum” Leiden: Elzevir

Jia, Jianmin, Dyer, James S., and John C. Butler (2001) “Generalized Disappointment Models” *Journal of Risk and Uncertainty* **22** (1), 59–78

Kahneman, Daniel, and Amos Tversky (1979) "Prospect Theory: An Analysis of Decision Under Risk" *Econometrica* **XLVII**, 263-291

Karni, Edi and Zvi Safra (1987) “"Preference Reversal" and the Observability of Preferences by Experimental Methods” *Econometrica* **55** (3), 675-685

Loomes, Graham, Chris Starmer and Robert Sugden (1989) “Preference reversal: information-processing effect or rational non-transitive choice?” *Economic Journal* **99**, 140-151

Loomes, Graham and Robert Sugden (1989) “Testing different stochastic specifications of risky choice” *Economica* **65**, 581–598

Loomes, Graham and Robert Sugden (1991) “Observing violations of transitivity by experimental methods” *Econometrica* **59**, 425-440

Loomes, Graham and Robert Sugden (1986) “Disappointment and dynamic consistency in choice under uncertainty” *Review of Economic Studies* **53**, 271–282

Loomes, Graham and Robert Sugden (1982) “Regret theory: An alternative theory of rational choice under uncertainty” *Economic Journal* **92**, 805–824

Markowitz, Harry M. (1952) "Portfolio Selection" *The Journal of Finance* **7** (1), pp. 77–91

Mehra, R., and Prescott, E. (1985) “The equity premium: A puzzle” *Journal of Monetary Economics*, **15** (2), 341–350

Prelec, D. (1990) “A 'pseudo-endowment' effect, and its implications for some recent nonexpected utility models” *Journal of Risk and Uncertainty* **3**, 247-259

Quiggin, John (1981) “Risk perception and risk aversion among Australian farmers” *Australian Journal of Agricultural Recourse Economics* **25**, 160-169

Rubinstein, Ariel (1988) “Similarity and decision making under risk: is there a utility theory resolution to the Allais paradox?” *Journal of Economic Theory* **46**, 145-153

Samuelson, Paul (1963) "Risk and Uncertainty: A Fallacy of Large Numbers" *Scientia*, Vol. 98, pp. 108-113

Sopher, Barry and Gary Gigliotti (1993) “Intransitive Cycles: Rational Choice or Random Errors? An Answer Based on Estimation of Error Rates with Experimental Data” *Theory and Decision* **35**, 311-336

Starmer, Chris (1992) “Testing New Theories of Choice under Uncertainty Using the Common Consequence Effect” *Review of Economic Studies* **59**, 813–830

Starmer, Chris (2000) “Developments in non-expected utility theory: the hunt for a descriptive theory of choice under risk" *Journal of Economic Literature* **38**, 332-382

Starmer, Chris and Robert Sugden (1991) “Does the random-lottery incentive system elicit true preferences? An experimental investigation” *American Economic Review,* **81**, 971–978

Starmer, Chris and Robert Sugden (1993) “Testing for juxtaposition and event splitting effects” *Journal of Risk and Uncertainty* **6** 235–254

Starmer, Chris and Robert Sugden (1998) “Testing alternative explanations of cyclical choices” *Economica* **65**, 347-361

Tversky, Amos (1969) “Intransitivity of preferences” Psychological Review **76**, 31-48

Tversky, Amos and Daniel Kahneman (1986) “Rational choice and the framing of decisions” *Journal of Business* **59** (**4**), 251-278

Tversky, Amos and Daniel Kahneman (1992) “Advances in prospect theory: Cumulative representation of uncertainty” *Journal of Risk and Uncertainty* **5**, 297-323

Viscusi, Kip (1989) “Prospective Reference Theory: Toward an Explanation of the Paradoxes” *Journal of Risk and Uncertainty* **2**, 235-264

von Neumann, John and Oscar Morgenstern (1947) “Theory of Games and Economic Behavior” Second edition, Princeton: Princeton University Press

Wakker, Peter P. (1984) “Cardinal coordinate independence for expected utility” *Journal of Mathematical Psychology* 28, 110-117

Wu, George (1994) “An empirical test of ordinal independence” *Journal of Risk and Uncertainty* **9**, 39–60

Yaari, Menahem (1987) “The Dual Theory of Choice under Risk” *Econometrica* **55**, 95-115