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Subjective Expected Utility with Non-additive Probabilities on Finite State Spaces

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Schmeidler and Gilboa's representation generalizes subjective expected utility to cope with the Ellsberg paradox, so that the probability measure over states of the world need not be additive. This paper examines a similar generalization in Savage's framework when the set of states is finite, while Savage's states are continuously divisible. Our axiomatization requires that the set X of consequences be infinite in contrast to Savage's arbitrary X. Three representational forms are axiomatized to give non-additivity, complementary additivity, and additivity of probability measures, respectively. *Journal of Economic Literature* Classification Numbers: 022, 026. © 1990 Academic Press, Inc.

1. INTRODUCTION

Theories of decision making under uncertainty provide subjective expected utility models that represent numerically the personal beliefs (subjective probabilities) and preferences of a decision maker. Under various frameworks numerous axiom systems and their numerical representations have been proposed (see the survey by Fishburn [8]). One of the most elegant and well-known axiomatizations of subjective probability and utility was given by Savage [17]. He developed an act-oriented theory which applies a preference relation over the set of acts. Acts are defined as functions from the states of the world into the consequences.

There is abundant evidence that people's carefully considered decisions often violate the assumptions of subjective expected utility theories (see Allais [1], Davidson, Suppes, and Siegel [6], Ellsberg [7], Slovic and Tversky [19], Kahneman and Tversky [13], Grether and Plott [12], and others.) Recently, Schmeidler [18] and Gilboa [10] generalized subjective expected utility theory to accommodate Ellsberg-type violations of additive probability measures. Schmeidler adopted the idea of lottery acts (functions from the states of the world into probability distributions on the consequence space) introduced by Anscombe and Aumann [2]. Gilboa used

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Savage's basic formulation, so that the set of states is required to be continuously divisible, but required at least three consequences. Moreover, his theory provides that utility must be bounded as in Savage's.

This paper axiomatizes a generalization of subjective expected utility to accommodate Ellsberg-type violation of additive probability in Savage's framework when the set of states is finite as in Wakker [22]. Wakker assumed that the set X of consequences is a connected separable topological space, and derived a non-additive probability measure over states and a continuous utility function on X. Also, Wakker [21] extended his result to arbitrary state spaces. Our approach is different in that we adopt Savage's framework and X is not necessarily a connected separable topological space,[1], and it then gives rise to a more general result.

The axiomatization requires that the set of consequences be infinite, while it is arbitrary in Savage's utility theory. The axioms are shown to give a utility function over the consequence space, and a non-additive probability measure over states. The expectation of a utility function with respect to non-additive probability measures is given as Choquet integration as in Schmeidler and Gilboa's representation, which is discussed in the next section. Section 3 states the axioms and the representation theorem that includes two special representations with complementary additivity and additivity of probability measures, respectively. Section 4 establishes a representation for all binary acts, then Section 5 completes the proof of the representation theorem in Section 3.

2. Representations without Additivity

Let S be the set of states. Subsets of S are called events. For events in $2^S = \{A : A \subseteq S\}$, denote the complement $S \backslash A$ of A by A^c. An act is a function from S into the set X of consequences. F is the set of all acts. Let $\precsim$ on F be the binary preference relation with $\sim$ and $\prec$ defined in the usual way: for $f, g \in F$, $f \sim g$ if $f \precsim g$ and $g \precsim f$; $f \prec g$ if $f \precsim g$ and not $(g \precsim f)$.

Schmeidler and Gilboa's representation has the following form: for all $f, g \in F$,

$$f \precsim g \qquad \text{iff} \quad \int_S u(f(s))\, d\pi(s) \leqq \int_S u(g(s))\, d\pi(s),$$

where u is a real valued function on X, and π is a monotonic (not

[1] Wakker [21, 22] noted that the topological approach needs topological separability only for ordered partitions ω of the set of states with exactly one ω-essential event in his terminology.

necessarily additive) measure on 2^S that satisfies $\pi(S)=1$ and $\pi(\text{empty})=0$. Monotonicity of π means that if $A \subseteq B$ then $\pi(A) \leqq \pi(B)$. Moreover, u is unique up to a positive linear transformation, and π is unique, so that u and $\pi(A)$ for $A \in 2^S$ can be interpreted as a utility function on X and a subjective probability for an event A, respectively.

The integration in the representation is defined in Choquet's sense (see Choquet [5]) to account for non-additive probability measures as follows. Since we are concerned with a finite S, assume that $S=\{s_1, ..., s_n\}$ and $u_1 \leqq \cdots \leqq u_n$, where $u_i = u(f(s_i))$ for $s_i \in S$ and all i. Let π_f denote the cumulative distribution function induced by π through f so that

$$\pi_f(t) = \pi(\{s : u(f(s)) \leqq t\})$$

with $\pi_f(t)=0$ for $t<u_1$ and $\pi_f(t)=1$ for $t \geqq u_n$. Then the integration is given by

$$\int_S u(f(s))\, d\pi(s) = \sum_{i=1}^{n-1} (u_i - u_{i+1})\, \pi_f(u_i) + u_n,$$

which is referred to as the lower Choquet integral in Gilboa [11]. Chew, Karni, and Safra [4] adopted this definition in risky situations. This means that expected utilities are calculated with respect to cumulative distributions.

On the other hand, the Choquet integration, referred to as the upper Choquet integral in Gilboa [11], is defined by the expectations with respect to the decumulative distribution functions induced by π through acts. Schmeidler and Gilboa's representation applies this definition. In risky situations, Quiggin [16] used it in the anticipated utility representation. Let π^* be the dual measure of π such that $\pi^*(A) = 1 - \pi(A^c)$ for all $A \in 2^S$. Gilboa [11] showed that the lower Choquet integral with respect to π is equivalent to the upper Choquet integral with respect to π^*. *We say that if* $\pi = \pi^*$, *then* π *is* complementarily additive. Throughout the paper, the integrations always stand for the lower Choquet integral.

3. Axioms and Theorem

Several additional notations and definitions will be useful in stating axioms. When F_1 and F_2 are subsets of F, $F_1 \precsim F_2$ means $f \precsim g$ for all $f \in F_1$ and all $g \in F_2$. We write $f \precsim g \precsim h$ when $f \precsim g$ and $g \precsim h$; $f \prec g \prec h$ when $f \prec g$ and $g \prec h$; $f \sim g \sim h$ when $f \sim g$ and $g \sim h$. For all $f, g \in F$, all $x \in X$, and all $A \in 2^S$, $f =_A g$ means that $f(s) = g(s)$ for every $s \in A$; $f =_A x$ means that $f(s) = x$ for every $s \in A$. Let xAf denote the act f' with $f' =_A x$ and $f' =_{A^c} f$. A binary act is defined as an act f with $f =_A x$ and $f =_{A^c} y$ for

$A \in 2^S$ and $x, y \in X$, denoted by xAy. A constant act is an act $f =_S x$ for some $x \in X$, so every $x \in X$ is identified with a constant act.

A partition of S is a sequence of non-empty events that are mutually disjoint and whose union equals S. For an n-partition $P = \{A_1, ..., A_n\}$, if $f =_{A_i} x_i$ for all i, then sometimes we write $f_P(x_1, ..., x_n)$ instead of f. Also, xAy is sometimes written by $f_A(x, y)$. Suppose that $\{x_1, ..., x_n\} = \{x_{i_1}, ..., x_{i_n}\}$, $P = \{A_1, ..., A_n\}$, $P' = \{A_{i_1}, ..., A_{i_n}\}$, and $P = P'$. Then by the definition of the acts, $f_P(x_1, ..., x_n) = f_{P'}(x_{i_1}, ..., x_{i_n})$. If $P'' = \{A_1 \cup A_2, A_3, ..., A_n\}$ and $x_1 = x_2$, then $f_P(x_1, ..., x_n) = f_{P''}(x_2, ..., x_n)$. In particular, $xAx = x$ and $xAy = yA^c x$.

A *null* event $A \in 2^S$ is an event for which for all $x, y, z \in X$, $xAz \sim yAz$ whenever $x \precsim y \precsim z$. A *universal* event $A \in 2^S$ is an event for which for all $x, y, z \in X$, $xAy \sim xAz$ whenever $x \precsim y \precsim z$. An empty event is null. If $x \prec y$ for some $x, y \in X$, then S is not null but universal. If $\prec$ is empty, then every $A \in 2^S$ is null and universal. Note that if there is an event which is neither null nor universal, then S includes at least two states, and $\prec$ is not empty, so that $x \prec y$ for some $x, y \in X$. We say that $\precsim$ is *bounded* if, for each $f \in F$, there are $x, y \in X$ such that $x \precsim f \precsim y$. Let N be any set of consecutive integers. Given an event $A \in 2^S$ which is neither null nor universal, we define a *standard sequence* as a set $\{a_i : a_i \in X, i \in N\}$ for which there exist $a, b \in X$ such that not$(a \sim b)$, and either $\{a, b\} \precsim \{a_i\}$ and $aAa_i \sim bAa_{i+1}$ for all $i, i+1 \in N$, or $\{a_i\} \precsim \{a, b\}$ and $a_iAa \sim a_{i+1}Ab$ for all $i, i+1 \in N$.

The following axioms apply to all $f \in F$, all $x, y, z \in X$, and all $A, B \in 2^S$:

A1. $\precsim$ on F is a bounded weak order.

A2. If $xAz \precsim f \precsim yAz$ then $f \sim aAz$ for some $a \in X$.

A3. If A is not null and $\{x, y\} \precsim z$, then $x \precsim y$ iff $xAz \precsim yAz$; if A is not universal and $z \precsim \{x, y\}$, then $x \precsim y$ iff $zAx \precsim zAy$.

A4. If $x \precsim y$ and $A \subseteq B$, then $xBy \precsim xAy$.

A5. Every strictly bounded standard sequence is finite.

A bounded weak order means by definition that $\precsim$ is bounded, complete, and transitive. A2 is a restricted solvability axiom. A3 and A4 are monotonicity axioms. A5 is an Archimedean axiom.

Since $\precsim$ is bounded, for each $f \in F$, there are $x, y \in X$ such that $x \precsim f \precsim y$. By the definition of acts, $x = xSz$ and $y = ySz$. Therefore, by A2, $f \sim aSz$ for some $a \in X$. Thus $f \sim a$. Let m be a mapping from F into X that assigns a constant act $m(f)$ for each $f \in F$ such that $f \sim m(f)$. Let M be the set of all such mappings, so that if $m, m' \in M$, then by A1, $m(f) \sim m'(f)$ for all $f \in F$. For an n-partition $P = \{A_1, ..., A_n\}$, if $f =_{A_i} x_i$ for all i, then $m(f)$ is denoted by $m_P(x_1, ..., x_n)$. When $P = \{A, A^c\}$ and $f = xAy$, we denote $m(f) = m_A(x, y)$. We say that $\prec$ is *dense* if, for each

$x, y \in X$ with $x \prec y$, there is an $a \in X$ such that $x \prec a \prec y$. If $x \prec y$ for $x, y \in X$, and $A \in 2^S$ is neither null nor universal, then by A1 and A3, $x \prec m_A(x, y) \prec y$, so that $\prec$ is dense.

Now we introduce the key axioms that apply to all $x, y, z, w, x_1, ..., x_n, y_1, ..., y_n \in X$ for $n \geqq 1$, all $f \in F$, all n-partitions, all $m \in M$, and all $A, B \in 2^S$:

A6. If $x_1 \precsim \cdots \precsim x_n$ and $y_1 \precsim \cdots \precsim y_n$ with $x_i \precsim y_i$ for all i, then $f_A(m_p(x_1, ..., x_n), m_p(y_1, ..., y_n)) \sim f_p(m_A(x_1, y_1), ..., m_A(x_n, y_n))$.

A7. $m_A(m_B(x, y), m_B(z, w)) \sim m_B(m_A(x, z), m_A(y, w))$.

A8. $f_A(m_p(x_1, ..., x_n), m_p(y_1, ..., y_n)) \sim f_p(m_A(x_1, y_1), ..., m_A(x_n, y_n))$.

A6 is a version under uncertainty of the condition under risk proposed in Fishburn [9, Chap. 3] that modifies the third axiom in Chew [3]. Chew's axiom was originally motivated by Quiggin [16]. When $P = \{B, B^c\}$, A7 is similar to the isometry condition if $A \neq B$, and the bisymmetry condition if $A = B$ in Pfanzagl [15]. A8 is a version of the isometry condition.

The purpose of the paper is to prove the following:

THEOREM 1. *Suppose that S is finite, and that there is an event which is neither null nor universal. If* A1–A6 *hold, then*

(1) *there are a monotonic measure π on 2^S and a real valued function u on X such that for all $f, g \in F$,*

$$f \precsim g \quad \textit{iff} \quad \int_S u(f(s))\, d\pi(S) \leqq \int_S u(g(s))\, d\pi(s);$$

(2) *π is unique, and u is unique up to a positive linear transformation;*

(3) *π is complementarily additive if* A7 *holds;*

(4) *π is additive if* A8 *holds instead of* A6.

The proof of the theorem will appear in Section 5. We note that the axioms except A1, A3, A4, and A5 are not necessary for the representation, as shown in the following example. Suppose that X is the set of all rational numbers, and that $u(x) = x$ for all $x \in X$. If π takes on values of irrational numbers, then A2 does not hold, so that M is empty. In this case, A2 requires that π take on values of rational numbers. Theorem 1(4) gives an expected utility representation with an additive subjective probability measure in Savage's framework when S is finite. Wakker [20] obtained a similar representation in the additive case when X is a connected separable topological space.

4. A Representation for Binary Acts

We assume throughout the rest of the paper that A1–A6 hold, and there is an event which is neither null nor universal. Note that S includes at least two states, and $\prec$ is not empty but dense. This section proves the following proposition, which implies that the representation of Theorem 1(1) holds for all binary acts.

Proposition 1. *There are two real valued functions, π on 2^S, and u on X, such that $\pi(empty) = 0$, $\pi(S) = 1$, and for all $x, y, z, w \in X$, and all $A, B \in 2^S$, if $x \precsim y$ and $z \precsim w$ then*

$$xAy \precsim zBw \quad \textit{iff} \quad \pi(A)\, u(x) + (1 - \pi(A))\, u(y) \leqq \pi(B)\, u(z) + (1 - \pi(B))\, u(w);$$

$$A \subseteq B \quad \textit{implies} \quad \pi(A) \leqq \pi(B).$$

Moreover, π is unique, and u is unique up to a positive linear transformation.

First we show three lemmas, and then give the proof of Proposition 1. For $a \in X$, let

$$X^* = \{x \in X : b \prec x \prec c \text{ for some } b, c \in X\},$$

$$X^a = \{x \in X : x \precsim a\},$$

$$X_a = \{x \in X : a \precsim x\},$$

$$X_{\max} = \{x \in X : x \prec b \text{ for no } b \in X\},$$

$$X_{\min} = \{x \in X : b \prec x \text{ for no } b \in X\}.$$

Given $a, a' \in X^*$ and an event $A \in 2^S$ which is neither null nor universal, we define a *standard sequence with respect to a* and a' as a set $\{a_i : a_i \in X, i \in N\}$ for which there exist $b, c \in X$ such that not$(b \sim c)$, either $\{b, c\} \precsim \{a, a'\} \precsim \{a_i\}$ and $bAa_i \sim cAa_{i+1}$ for all $i, i+1 \in N$, or $\{a_i\} \precsim \{a, a'\} \precsim \{b, c\}$ and $a_i Ab \sim a_{i+1}Ac$ for all $i, i+1 \in N$. Let $\precsim^A$ be a binary relation on $X^a \times X_{a'}$ induced by $\precsim$ as follows: for ordered pairs $xy, zw \in X^a \times X_{a'}$, $xy \precsim^A zw$ iff $xAy \precsim zAw$.

We say that the triple $\langle X^a, X_{a'}, \precsim^A \rangle$ is an *additive conjoint structure* (Krantz *et al.* [14, Chap. 6]) if for all $x, y, z \in X^a$ and all $x', y', z' \in X_{a'}$, the following five conditions hold for a weakly ordered $\precsim$:

1. *Independence*: If $xAb' \precsim yAb'$ for some $b' \in X_{a'}$, then $xAx' \precsim yAx'$; if $bAx' \precsim bAy'$ for some $b \in X^a$, then $xAx' \precsim xAy'$.

2. *Thomsen condition*: If $xAz' \sim zAy'$ and $zAx' \sim yAz'$, then $xAx' \sim yAy'$.

3. *Restricted solvability*: If $xAx' \precsim yAy' \precsim zAx'$, then $bAx' \sim yAy'$ for some $b \in X^a$; if $xAx' \precsim yAy' \precsim xAz'$, then $xAb' \sim yAy'$ for some $b' \in X_{a'}$.

4. *Archimedean*: Every strictly bounded standard sequence with respect to a and a' is finite.

5. *Essentiality*: not$(bAb' \sim cAb')$ for some $b, c \in X^a$ and some $b' \in X_{a'}$; not$(bAb' \sim bAc')$ for some $b \in X^a$ and some $b', c' \in X_{a'}$.

LEMMA 1. *If $A \in 2^S$ is neither null nor universal, then for $a, a' \in X^*$ with $a \precsim a'$, the triple $\langle X^a, X_{a'}, \precsim^A \rangle$ is an additive conjoint structure.*

LEMMA 2. *If $A \in 2^S$ is neither null nor universal, then there are two real valued functions, ϕ and ψ, on X such that for all $x, y, z, w \in X$ with $x \precsim y$ and $z \precsim w$,*

(1) *$xAy \precsim zAw$ iff $\phi(x) + \psi(y) \leqq \phi(z) + \psi(w)$;*

(2) *if ϕ' and ψ' satisfy (1) instead of ϕ and ψ, respectively, then there exist constants $\alpha > 0$, β_1, and β_2 such that*

$$\phi'(x) = \alpha\phi(x) + \beta_1 \qquad \textit{for all} \quad x \in X \backslash X_{\max},$$

$$\psi'(x) = \alpha\psi(x) + \beta_2 \qquad \textit{for all} \quad x \in X \backslash X_{\min}.$$

LEMMA 3. *If $A \in 2^S$ is neither null nor universal, then there is a real valued function u on X such that for some $0 < \alpha < 1$, and for all $x, y, z, w \in X$ with $x \precsim y$ and $z \precsim w$,*

$$xAy \precsim zAw \qquad \textit{iff} \qquad \alpha u(x) + (1-\alpha)\, u(y) \leqq \alpha u(z) + (1-\alpha)\, u(w).$$

Moreover, α is unique, and u is unique up to a positive linear transformation.

Proof of Lemma 1. Suppose that A is neither null nor universal. By A1, $\precsim$ is a weak order. It easily follows from A2, A3, and A5 that the independence, the restricted solvability, and Archimedean conditions of the additive conjoint structure hold. The essentiality condition follows from $a, a' \in X^*$ and the assumption that A is neither null nor universal. In what follows, we show the Thomsen condition.

Suppose that $xAz' \sim zAy'$ and $zAx' \sim yAz'$. Then by A1, $m_A(x, z') \sim m_A(z, y')$ and $m_A(z, x') \sim m_A(y, z')$. We are to show that $m_A(x, x') \sim m_A(y, y')$, so $xAx' \sim yAy'$. The following three cases cover all possibilities: $\{x', y'\} \precsim z'$; $\{x', z'\} \precsim y'$; $\{y', z'\} \precsim x'$.

Suppose first that $\{x', y'\} \precsim z'$. By A3, $m_A(x, z') \precsim m_A(x', z')$ and $m_A(y, z') \precsim m_A(y', z')$. Thus A3 and A6 imply

$$\begin{aligned} f_A(m_A(x, x'), m_A(z', z')) &\sim f_A(m_A(x, z'), m_A(x', z')) && \text{(by A6)} \\ &\sim f_A(m_A(z, y'), m_A(x', z')) && \text{(by A3)} \\ &\sim f_A(m_A(z, x'), m_A(y', z')) && \text{(by A6)} \\ &\sim f_A(m_A(y, z'), m_A(y', z')) && \text{(by A3)} \\ &\sim f_A(m_A(y, y'), m_A(z', z')), && \text{(by A6)} \end{aligned}$$

so by A1, $f_A(m_A(x, x'), m_A(z', z')) \sim f_A(m_A(y, y'), m_A(z', z'))$. By A1 and A3, $\{m_A(x, x'), m_A(y, y')\} \precsim m_A(z', z')$. Hence by A3, $m_A(x, x') \sim m_A(y, y')$.

Suppose next that $\{x', z'\} \precsim y'$. Then similarly we obtain $f_A(m_A(x, x'), m_A(z', y')) \sim f_A(m_A(y, y'), m_A(z', y'))$. By A1 and A3, $\{m_A(x, x'), m_A(y, y')\} \precsim m_A(z', y')$. Hence by A3, $m_A(x, x') \sim m_A(y, y')$. Suppose last that $\{y', z'\} \precsim x'$. Then similarly, $f_A(m_A(x, x'), m_A(z', x')) \sim f_A(m_A(y, y'), m_A(z', x'))$. By A1 and A3, $\{m_A(x, x'), m_A(y, y')\} \precsim m_A(z', x')$. Hence by A3, $m_A(x, x') \sim m_A(y, y')$. Q.E.D.

Proof of Lemma 2. Suppose that A is neither null nor universal. Lemma 1 implies that for $a \in X^*$, the triple $\langle X^a, X_a, \precsim^A \rangle$ is an additive conjoint structure. Thus by Theorem 2 of Chapter 6 in Krantz *et al.* [14], there exist two functions, ϕ_a on X^a and ψ_a on X_a, such that for all $x, y \in X^a$ and all $z, w \in X_a$,

$$xAz \precsim yAw \quad \text{iff} \quad \phi_a(x) + \psi_a(z) \leqq \phi_a(y) + \psi_a(w).$$

If $a \precsim b$ for $a, b \in X^*$, then by Lemma 1, the triple $\langle X^a, X_b, \precsim^A \rangle$ is an additive conjoint structure. Since $X^a \times X_b \subseteq X^a \times X_a$ and $X^a \times X_b \subseteq X^b \times X_b$, for all $x, y \in X^a$ and all $z, w \in X_b$, we have

$$\begin{aligned} xAz \precsim yAw \quad &\text{iff} \quad \phi_a(x) + \psi_a(z) \leqq \phi_a(y) + \psi_a(w) \\ &\text{iff} \quad \phi_b(x) + \psi_b(z) \leqq \phi_b(y) + \psi_b(w). \end{aligned}$$

By the uniqueness of the additive representations for the additive conjoint structure, there are real valued functions $k(a, b) > 0$, $k_1(a, b)$, and $k_2(a, b)$ for all $a, b \in X^*$ with $a \precsim b$ such that

$$\begin{aligned} \phi_a(x) &= k(a, b)\, \phi_b(x) + k_1(a, b) && \text{for all} \quad x \in X^a, \\ \psi_a(x) &= k(a, b)\, \psi_b(x) + k_2(a, b) && \text{for all} \quad x \in X_b. \end{aligned}$$

Given ϕ_a and ψ_a for a fixed $a \in X^*$, scale ϕ_b and ψ_b for $b \in X_a \backslash X_{\max}$, and ϕ_c and ψ_c for $c \in X^a \backslash X_{\min}$ such that $k(a, b) = k(c, a) = 1$ and $k_i(a, b) = k_i(c, a) = 0$ for $i = 1, 2$. Then it easily follows that if $b \precsim c$ and $b, c \in X^*$, then $\phi_b = \phi_c$ on X^b, and $\psi_b = \psi_c$ on X_c. Therefore, for all $x \in X$, define

$$\begin{aligned} \phi(x) &= \phi_x(x) && \text{if} \quad x \in X^*, \\ &= \phi_a(x) && \text{for some} \quad a \in X^* \quad \text{if} \quad x \in X_{\min}, \\ \psi(x) &= \psi_x(x) && \text{if} \quad x \in X^*, \\ &= \psi_a(x) && \text{for some} \quad a \in X^* \quad \text{if} \quad x \in X_{\max}, \end{aligned}$$

so ϕ on $X \backslash X_{\max}$ and ψ on $X \backslash X_{\min}$ are uniquely specified. If $x, y \in X \backslash X_{\max}$, then $\{x, y\} \precsim z$ for some $z \in X \backslash X_{\max}$, since $\prec$ is dense. Thus for all $x, y, z \in X \backslash X_{\max}$, if $\{x, y\} \precsim z$, then

$$\begin{aligned} x \precsim y \quad & \text{iff} \quad xAz \precsim yAz && \text{(by A3)} \\ & \text{iff} \quad \phi_z(x) + \psi_z(z) \leqq \phi_z(y) + \psi_z(z) && \text{(by Lemma 1)} \\ & \text{iff} \quad \phi_x(x) \leqq \phi_y(y) && \text{(by scaling)} \\ & \text{iff} \quad \phi(x) \leqq \phi(y). && \text{(by definition)} \end{aligned}$$

Similarly, for all $x, y \in X \backslash X_{\min}$, $x \precsim y$ iff $\psi(x) \leqq \psi(y)$. With ϕ and ψ thus obtained, let

$$\begin{aligned} \phi(x^*) &\geqq \sup\{\phi(x) : x \in X \backslash X_{\max}\} && \text{if} \quad x^* \in X_{\max}; \\ \psi(x_*) &\leqq \inf\{\psi(x) : x \in X \backslash X_{\min}\} && \text{if} \quad x_* \in X_{\min}. \end{aligned}$$

Since $\prec$ is dense, we obtain that $\phi(x) < \phi(x^*)$ for all $x \in X \backslash X_{\max}$ and all $x^* \in X_{\max}$; $\psi(x_*) < \psi(x)$ for all $x \in X \backslash X_{\min}$ and all $x_* \in X_{\min}$.

We are to show that ϕ and ψ satisfy (1) and (2). Suppose that $x \precsim y$ and $z \precsim w$. First, assume that $z \precsim y$ and $x \precsim w$. Then by A1, A2, and A3, $\{x, z\} \precsim a \precsim \{y, w\}$ for some $a \in X$. If $a \in X^*$, then by Lemma 1 and the constructions of ϕ and ψ,

$$\begin{aligned} xAy \precsim zAw \quad & \text{iff} \quad \phi_a(x) + \psi_a(y) \leqq \phi_a(z) + \psi_a(w) \\ & \text{iff} \quad \phi_x(x) + \psi_y(y) \leqq \phi_z(z) + \psi_w(w) \\ & \text{iff} \quad \phi(x) + \psi(y) \leqq \phi(z) + \psi(w). \end{aligned}$$

Note that $\phi(x_*)$ for $x_* \in X_{\min}$ is finite, otherwise, $x_* Ax \prec a$ for all $a, x \in X^*$. If $a \in X_{\min}$, then $a \sim \{x, z\}$, so we obtain

$$\begin{aligned} xAy \precsim zAw \quad & \text{iff} \quad aAy \precsim aAw && \text{(by A1 and A3)} \\ & \text{iff} \quad y \precsim w && \text{(by A3)} \\ & \text{iff} \quad \psi(y) \leqq \psi(w) \\ & \text{iff} \quad \phi(a) + \psi(y) \leqq \phi(a) + \psi(w) \\ & \text{iff} \quad \phi(x) + \psi(y) \leqq \phi(z) + \psi(w). \end{aligned}$$

If $a \in X_{\max}$, then (1) similarly follows. Next assume that either $y \prec z$ or $w \prec x$. Then (1) easily follows from A1, A3, and the constructions of ϕ and ψ. The constructions of ϕ and ψ immediately give (2). Q.E.D.

Proof of Lemma 3. Suppose that A is neither null nor universal. Let ϕ and ψ satisfy Lemma 2(1) with $\phi(x^*) \geqq \sup\{\phi(x): x \in X^*\}$ if $x^* \in X_{\max}$, and $\psi(x_*) \leqq \inf\{\psi(x): x \in X^*\}$ if $x_* \in X_{\min}$. Then we show that choosing the appropriate $\phi(x^*)$ and $\psi(x_*)$ if $x^* \in X_{\max}$ and $x_* \in X_{\min}$, there are constants, $\beta > 0$ and γ, such that $\psi(x) = \beta\phi(x) + \gamma$ for all $x \in X$. Define

$$\phi_{1a}(x) = \phi(m_A(a, x)), \quad \psi_{1a}(x) = \psi(m_A(a, x)) \quad \text{for} \quad x \in X_a \quad \text{if} \quad a \notin X_{\max};$$

$$\phi_{2a}(x) = \phi(m_A(x, a)), \quad \psi_{2a}(x) = \psi(m_A(x, a)) \quad \text{for} \quad x \in X^a \quad \text{if} \quad a \notin X_{\min}.$$

For $x, y, z, w \in X_a$ and $a \notin X_{\max}$, if $x \precsim y$ and $z \precsim w$, then by A1 and A3, $m_A(a, x) \precsim m_A(a, y)$ and $m_A(a, z) \precsim m_A(a, w)$, so

$$\begin{aligned} & \phi_{1a}(x) + \psi_{1a}(y) \leqq \phi_{1a}(z) + \psi_{1a}(w) \\ & \quad \text{iff} \quad \phi(m_A(a, x)) + \psi(m_A(a, y)) \leqq \phi(m_A(a, z)) + \psi(m_A(a, w)) \\ & \quad \text{iff} \quad f_A(m_A(a, x), m_A(a, y)) \precsim f_A(m_A(a, z), m_A(a, w)) \quad \text{(by Lemma 2(1))} \\ & \quad \text{iff} \quad f_A(m_A(a, a), m_A(x, y)) \precsim f_A(m_A(a, a), m_A(z, w)) \quad \text{(by A1 and A6)} \\ & \quad \text{iff} \quad m_A(x, y) \precsim m_A(z, w) \quad \text{(by A3)} \\ & \quad \text{iff} \quad xAy \precsim zAw. \end{aligned}$$

If $x \precsim y$ and $z \precsim w$ for $x, y, z, w \in X^a$ and $a \notin X_{\min}$, then similarly we get

$$xAy \precsim zAw \quad \text{iff} \quad \phi_{2a}(x) + \psi_{2a}(y) \leqq \phi_{2a}(z) + \psi_{2a}(w).$$

Thus ϕ_{ia} and ψ_{ia} satisfy Lemma 2(1) on X_a for $i = 1$, and on X^a for $i = 2$. Therefore, there are real valued functions, $k_i > 0$ and k_{1i}, on $X \backslash X_{\max}$ for $i = 1$ and $X \backslash X_{\min}$ for $i = 2$ such that for $a \notin X_{\max}$ and $b \notin X_{\min}$,

$$\phi(m_A(a, x)) = k_1(a)\,\phi(x) + k_{11}(a) \qquad \text{for} \quad x \in X_a \backslash X_{\max},$$

$$\psi(m_A(x, b)) = k_2(b)\,\psi(x) + k_{12}(b) \qquad \text{for} \quad x \in X^b \backslash X_{\min}.$$

Suppose that $x \precsim \{y, z\} \precsim w$ for $y, z \in X^*$. Then $x \notin X_{\max}$ and $w \notin X_{\min}$.

By A1 and A3, $m_A(x, y) \precsim m_A(z, w)$ and $m_A(x, z) \precsim m_A(y, w)$. It follows from A6, Lemma 2(1), and the preceding paragraph that

$$\begin{aligned}
&f_A(m_A(x, y), m_A(z, w)) \sim f_A(m_A(x, z), m_A(y, w)) \\
&\quad \text{iff} \quad \phi(m_A(x, y)) + \psi(m_A(z, w)) = \phi(m_A(x, z)) + \psi(m_A(y, w)) \\
&\quad \text{iff} \quad k_1(x)\,\phi(y) + k_{11}(x) + k_2(w)\,\psi(z) + k_{12}(w) \\
&\qquad\qquad = k_1(x)\,\phi(z) + k_{11}(x) + k_2(w)\,\psi(y) + k_{12}(w) \\
&\quad \text{iff} \quad k_1(x)(\phi(y) - \phi(z)) = k_2(w)(\psi(y) - \psi(z)).
\end{aligned}$$

Since the above equations are satisfied for all $x, y, z, w \in X$ with $y, z \in X^*$ and $x \precsim \{y, z\} \precsim w$, there are constants, λ and δ, such that for all $x \notin X_{\max}$ and all $w \notin X_{\min}$,

$$k_1(x) = \lambda > 0 \qquad \text{and} \qquad k_2(w) = \delta > 0.$$

Thus $\lambda(\phi(y) - \phi(z)) = \delta(\psi(y) - \psi(z))$, so $\delta\psi(y) - \lambda\phi(y) = \delta\psi(z) - \lambda\phi(z)$ for all $y, z \in X^*$. Therefore, there are constants, $\beta > 0$ and γ, such that $\psi(x) = \beta\phi(x) + \gamma$ for all $x \in X^*$.

Suppose that $X_{\max}$ is not empty. If $\sup \phi(x) = +\infty$, then by the preceding paragraph, $\psi(x^*) = +\infty$ for $x^* \in X_{\max}$. By Lemma 2(1), this implies that $a \prec xAx^*$ for all $a, x \in X^*$. Therefore, $\sup \phi(x)$ must be finite. The preceding paragraph implies that $\psi(x^*) \geqq \sup \psi(x) = \beta \sup \phi(x) + \gamma$ for $x^* \in X_{\max}$. Thus define $\phi(x^*) = (\psi(x^*) - \gamma)/\beta$, so that $\psi(x) = \beta\phi(x) + \gamma$ for $x \in X_{\max}$. Suppose next that $X_{\min}$ is not empty. Then similarly, we can define $\psi(x_*) = \beta\phi(x_*) + \gamma$ for $x_* \in X_{\min}$. Hence $\psi(x) = \beta\phi(x) + \gamma$ for all $x \in X$. With $\beta > 0$, $\phi(x^*)$ for $x^* \in X_{\max}$, and $\psi(x_*)$ for $x_* \in X_{\min}$ thus obtained, let $\alpha = 1/(1 + \beta)$. Then $0 < \alpha < 1$. Define $u(x) = \phi(x)/\alpha$ for all $x \in X$. Then the representation of the lemma easily follows.

To show the uniqueness of u, suppose that u' also satisfies the representation. By the uniqueness of ϕ and ψ, it easily follows that there are constants, $\beta > 0$ and γ, such that $u'(x) = \beta u(x) + \gamma$ for all $x \in X^*$. Suppose that $x_* Ay \sim z$ for $x_* \in X_{\min}$ and $y \in X^*$. Then $z \in X^*$. We obtain

$$\alpha u(x_*) + (1 - \alpha)\, u(y) = u(z),$$

$$\alpha u'(x_*) + (1 - \alpha)\, u'(y) = u'(z),$$

so $u'(x_*) = \beta u(x_*) + \gamma$. Similarly, $u'(x^*) = \beta u(x^*) + \gamma$ for $x^* \in X_{\max}$. Hence the uniqueness part of the lemma obtains. Q.E.D.

Proof of Proposition 1. If A is neither null nor universal, then Lemma 3 implies that there is a real valued function u_A on X such that for some $0 < \alpha_A < 1$ and for all $x, y, z, w \in X$ with $x \precsim y$ and $z \precsim w$,

$$xAy \precsim zAw \qquad \text{iff} \quad \alpha_A u_A(x) + (1 - \alpha_A)\, u_A(y) \leqq \alpha_A u_A(z) + (1 - \alpha_A)\, u_A(w).$$

Thus we define π on 2^S as follows:

$$\begin{aligned}\pi(A) &= \alpha_A && \text{if } A \text{ is neither null nor universal;}\\ &= 1 && \text{if } A \text{ is universal;}\\ &= 0 && \text{if } A \text{ is null.}\end{aligned}$$

Suppose that A and B are neither null nor universal. Then we show that $u_A = \alpha u_B + \beta$ for some constants, $\alpha > 0$ and β. Given $a \in X^*$, a binary relation $\precsim^a$ on $X^a \times X_a$ is defined as follows: for $xy, zw \in X^a \times X_a$,

$$xy \precsim^a zw \quad \text{iff} \quad f_A(m_B(x, a), m_B(a, y)) \precsim f_A(m_B(z, a), m_B(a, w)).$$

By A1 and A6,

$$xy \precsim^a zw \quad \text{iff} \quad f_B(m_A(x, a), m_A(a, y)) \precsim f_B(m_A(z, a), m_A(a, w)).$$

Since by A1 and A3, $m_C(x, a) \precsim m_C(a, y)$ and $m_C(z, a) \precsim m_C(a, w)$ for $C = A$ or B, Lemma 3 implies that

$$\begin{aligned} xy \precsim^a zw \quad &\text{iff} \quad \pi(A)\, u_A(m_B(x, a)) + (1 - \pi(A))\, u_A(m_B(a, y))\\ &\qquad \leqq \pi(A)\, u_A(m_B(z, a)) + (1 - \pi(A))\, u_A(m_B(a, w))\\ &\text{iff} \quad \pi(B)\, u_B(m_A(x, a)) + (1 - \pi(B))\, u_B(m_A(a, y))\\ &\qquad \leqq \pi(B)\, u_B(m_A(z, a)) + (1 - \pi(B))\, u_B(m_A(a, w)). \end{aligned}$$

By the uniqueness of additive representations, there are real valued functions, $\omega > 0$ and σ, on X^* such that for all $a \in X^*$ and all $x \in X^a$,

$$\pi(A)\, u_A(m_B(x, a)) = \omega(a)\, \pi(B)\, u_B(m_A(x, a)) + \sigma(a).$$

Since $m_A(x, y) \in X^a$ for $x, y \in X^a$, it follows from A1, A6, Lemma 3, and the above equation that for $x, y \in X^a$ with $x \precsim y$,

$$\begin{aligned} &\pi(A)\, u_A(m_B(m_A(x, y), m_A(a, a)))\\ &\quad = \pi(A)\, u_A(m_A(m_B(x, a), m_B(y, a)))\\ &\quad = \pi(A)^2\, u_A(m_B(x, a)) + (1 - \pi(A))\, \pi(A)\, u_A(m_B(y, a))\\ &\quad = \omega(a)\, \pi(B)\{\pi(A)\, u_B(m_A(x, a)) + (1 - \pi(A))\, u_B(m_A(y, a))\} + \sigma(a). \end{aligned}$$

On the other hand, by the preceding paragraph, Lemma 3, A1, A3, and A6,

$$\begin{aligned}
&\pi(A)\, u_A(m_B(m_A(x, y), m_A(a, a))) \\
&\quad = \pi(A)\, u_A(m_B(m_A(x, y), a)) \\
&\quad = \omega(a)\, \pi(B)\, u_B(m_A(m_A(x, y), a)) + \sigma(a) \\
&\quad = \omega(a)\, \pi(B)\, u_B(m_A(m_A(x, y), m_A(a, a))) + \sigma(a) \\
&\quad = \omega(a)\, \pi(B)\, u_B(m_A(m_A(x, a), m_A(y, a))) + \sigma(a).
\end{aligned}$$

Therefore, the above two equations give

$$\begin{aligned}
&u_B(m_A(m_A(x, a), m_A(y, a))) \\
&\quad = \pi(A)\, u_B(m_A(x, a)) + (1 - \pi(A))\, u_B(m_A(y, a)).
\end{aligned}$$

We show that $u_B(m_A(x, y)) = \pi(A)\, u_B(x) + (1 - \pi(A))\, u_B(y)$ for all $x, y \in X$ with $x \precsim y$. Thus by A1 and Lemma 3, we must have $u_A = \alpha u_B + \beta$ for some constants $\alpha > 0$ and β. Suppose first that $x \precsim y$ and $x, y \in X^*$. If $zAy \precsim x$ for some $z \in X$, then by A2, $wAy \sim x$ for some $w \in X$. Therefore, the desired result easily follows from the preceding paragraph. Thus assume that $x \prec zAy$ for all z with $z \prec x$. Then let $w = m_B(x, y)$. Noting by Lemma 3 that $u_B(w) = \pi(B)\, u_B(x) + (1 - \pi(B))\, u_B(y)$, it follows from Lemma 3 and the preceding paragraph that

$$u_B(m_B(m_A(x, y), m_A(y, y))) = \pi(B)\, u_B(m_A(x, y)) + (1 - \pi(B))\, u_B(y),$$

and

$$\begin{aligned}
u_B(m_A(w, y)) &= \pi(A)\, u_B(w) + (1 - \pi(A))\, u_B(y) \\
&= \pi(A)\, \pi(B)\, u_B(x) + (1 - \pi(A)\, \pi(B))\, u_B(y).
\end{aligned}$$

By A1, A3, and A6,

$$\begin{aligned}
m_A(w, y) &\sim m_A(m_B(x, y), m_B(y, y)) \\
&\sim m_B(m_A(x, y), m_A(y, y)),
\end{aligned}$$

so $u_B(m_A(w, y)) = u_B(m_B(m_A(x, y), m_A(y, y)))$. Therefore, $\pi(A)\, \pi(B)\, u_B(x) + (1 - \pi(A)\, \pi(B))\, u_B(y) = \pi(B)\, u_B(m_A(x, y)) + (1 - \pi(B))\, u_B(y)$. This is rearranged to give the desired result. Hence $u_B(m_A(x, y)) = \pi(A)\, u_B(x) + (1 - \pi(A))\, u_B(y)$ for all $x, y \in X^*$ with $x \precsim y$.

If either $x, y \in X_{\min}$ or $x, y \in X_{\max}$, then the desired result easily follows, so we show that it also holds when $x \in X_{\min}$ or $y \in X_{\max}$. Suppose that $x \in X_{\min}$ and $y \in X^*$. When $x \in X^*$ and $y \in X_{\max}$, the proof is similar. Then

$y \precsim z$ for some $z \in X^*$. Since $m_C(x, y)$ and $m_C(y, z)$ are in X^* for $C = A$ or B, Lemma 3 and the preceding paragraph imply that

$$\begin{aligned}
&u_B(m_B(m_A(x, y), m_A(y, z))) \\
&\quad = \pi(B)\, u_B(m_A(x, y)) + (1 - \pi(B))\, u_B(m_A(y, z)) \\
&\quad = \pi(B)\, u_B(m_A(x, y)) + (1 - \pi(B))\{\pi(A)\, u_B(y) + (1 - \pi(A))\, u_B(z)\},
\end{aligned}$$

$$\begin{aligned}
&u_B(m_A(m_B(x, y), m_B(y, z))) \\
&\quad = \pi(A)\, u_B(m_B(x, y)) + (1 - \pi(A))\, u_B(m_B(y, z)) \\
&\quad = \pi(A)\{\pi(B)\, u_B(x) + (1 - \pi(B))\, u_B(y)\} \\
&\qquad + (1 - \pi(A))\{\pi(A)\, u_B(y) + (1 - \pi(B))\, u_B(z)\}.
\end{aligned}$$

By A6, $m_B(m_A(x, y), m_A(y, z)) \sim m_A(m_B(x, y), m_B(y, z))$, so by Lemma 3, $u_B(m_B(m_A(x, y), m_A(y, z))) = u_B(m_A(m_B(x, y), m_B(y, z)))$. Thus $u_B(m_A(x, y)) = \pi(A)\, u_B(x) + (1 - \pi(A))\, u_B(y)$.

Suppose last that $x \in X_{\min}$ and $y \in X_{\max}$. Applying the results in the preceding paragraphs, the desired result similarly obtains. Hence $u_B(m_A(x, y)) = \pi(A)\, u_B(x) + (1 - \pi(A))\, u_B(y)$ for all $x, y \in X$ with $x \precsim y$.

Under appropriate positive linear transformations, there is a real valued function u on X such that $u = u_A$ for all $A \in 2^S$ which are neither null nor universal. Noting that if $x \prec y$ then $xAy \sim y$ when A is null, and $xAy \sim x$ when A is universal, it follows from the preceding paragraphs that for all $A \in 2^S$, and all $x, y \in X$ with $x \precsim y$,

$$u(m_A(x, y)) = \pi(A)\, u(x) + (1 - \pi(A))\, u(y).$$

Thus for all $A, B \in 2^S$ and all $x, y, z, w \in X$ with $x \precsim y$ and $z \precsim w$,

$$\begin{aligned}
xAy \precsim zBw \quad &\text{iff} \quad m_A(x, y) \precsim m_B(z, w) \\
&\text{iff} \quad u(m_A(x, y)) \leqq u(m_B(z, w)) \\
&\text{iff} \quad \pi(A)\, u(x) + (1 - \pi(A))\, u(y) \leqq \pi(B)\, u(z) + (1 - \pi(B))\, u(w),
\end{aligned}$$

so the representation of the proposition holds. It easily follows from A4 that $A \subseteq B$ implies $\pi(A) \leqq \pi(B)$. The uniqueness of π and u follows from the construction of u, and Lemma 3. Q.E.D.

5. Proof of the Theorem

Throughout the section, we shall assume that S is finite, and let u on X, and π on 2^S be real valued functions obtained in Proposition 1. We extend u on X as follows:

$$u(f)=u(x) \qquad \text{when} \quad f=_S x;$$

$$u(f)=u(m(f)) \qquad \text{for all} \quad f\in F.$$

Thus by A1, for all $f, g\in F$, $f\precsim g$ iff $u(f)\leqq u(g)$. We note by A1 and A6 that if $x_i\sim y_i$ for $i=1, ..., n$, then $f_p(x_1, ..., x_n)\sim f_p(y_1, ..., y_n)$ for all n-partitions.

Proof of Theorem 1. (1) We are to show that $u(f)=\int_S u(f(s))\,d\pi(s)$, so that the conclusion of (1) obtains. Let $P=\{A_1, ..., A_n\}$ be an n-partition, and

$$f_n=_{A_i} x_i \qquad \text{for} \quad i=1, ..., n,$$

where $x_1\precsim \cdots \precsim x_n$ and $x_i\in X$ for all i. Also let B_0 be empty, and $B_i=\sum_{j=1}^{i} A_j$, so $B_n=S$. For all $f_2\in F$, Proposition 1 gives the desired result. Thus we assume $n>2$. Suppose that the conclusion is true for all n-partitions with $n<k$. Then it suffices to show that for $n=k$,

$$u(f_k)=\sum_{i=1}^{k} (\pi(B_i)-\pi(B_{i-1}))\,u(x_i),$$

since this easily leads to the definition of the lower Choquet integral. If $x_i\sim x_j$ for some $1\leqq i<j\leqq k$, then the desired result easily follows from A1, A6, and the hypothesis of the induction. Thus we shall assume that $x_1\prec \cdots \prec x_k$.

We have the following three cases to examine:

CASE 1. $xAx_2\sim x_1$ and $x_{k-1}Ay\sim x_k$ for some $x, y\in X$ and some $A\in 2^S$.

CASE 2. either $xAx_2\sim x_1$ for some $x\in X$ and some nonuniversal $A\in 2^S$ or $x_{k-1}By\sim x_k$ for some $y\in X$ and some nonnull $B\in 2^S$.

CASE 3. not$(xAx_2\sim x_1)$ and not $(x_{k-1}By\sim x_k)$ for all $x, y\in X$, all nonuniversal $A\in 2^S$, and all nonnull $B\in 2^S$.

Let $P'=\{A_1, ..., A_{k-2}, A_{k-1}\cup A_k\}$ and $P''=\{A_1\cup A_2, A_3, ..., A_k\}$, so they are $(k-1)$-partitions.

Case 1. Since $xAx_2 \sim x_1$ and $x_{k-1}Ay \sim x_k$ for some $x, y \in X$ and some $A \in 2^S$, A3 implies that $x \prec x_1$ and $x_k \prec y$ It follows from A1 and A6 that

$$\begin{aligned} f_k &= f_P(x_1, ..., x_k) \\ &\sim f_P(m_A(x, x_2), m_A(x_2, x_2), ..., m_A(x_{k-1}, x_{k-1}), m_A(x_{k-1}, y)) \\ &\sim f_A(m_P(x, x_2, ..., x_{k-2}, x_{k-1}, x_{k-1}), m_P(x_2, x_2, x_3, ..., x_{k-1}, y)) \\ &\sim f_A(m_{P'}(x, x_2, ..., x_{k-1}), m_{P''}(x_2, ..., x_{k-1}, y)), \end{aligned}$$

so by A1, $f_k \sim m_A(m_{P'}(x, x_2, ..., x_{k-1}), m_{P''}(x_2, ..., x_{k-1}, y))$. Since $xAx_2 \sim x_1$ and $x_{k-1}Ay \sim x_k$, Proposition 1 implies that $u(x_1) = \pi(A)\,u(x) + (1-\pi(A))\,u(x_2)$ and $u(x_k) = \pi(A)\,u(x_{k-1}) + (1-\pi(A))\,u(y)$. Thus by the definition of $u(f_k)$, and the hypothesis of the induction, we get

$$\begin{aligned} u(f_k) &= u(m_A(m_{P'}(x, x_2, ..., x_{k-1}), m_{P''}(x_2, ..., x_{k-1}, y))) \\ &= \pi(A)\,u(m_{P'}(x, x_2, ..., x_{k-1})) + (1-\pi(A))\,u(m_{P''}(x_2, ..., x_{k-1}, y)) \\ &= \pi(A)\left\{\pi(B_1)\,u(x) + \sum_{i=2}^{k-2} (\pi(B_i) - \pi(B_{i-1}))\,u(x_i)\right. \\ &\qquad \left. + (1-\pi(B_{k-2}))\,u(x_{k-1})\right\} \\ &\qquad + (1-\pi(A))\left\{\pi(B_2)\,u(x_2) + \sum_{i=3}^{k-1} (\pi(B_i) - \pi(B_{i-1}))\,u(x_i)\right. \\ &\qquad \left. + (1-\pi(B_{k-1}))\,u(y)\right\} \\ &= \sum_{i=1}^{k} (\pi(B_i) - \pi(B_{i-1}))\,u(x_i). \end{aligned}$$

Case 2. Suppose that $x_{k-1}By \sim x_k$ for some $y \in X$ and some nonnull $B \in 2^S$. A similar proof applies when $xAx_2 \sim x_1$ for some $x \in X$ and some nonuniversal $A \in 2^S$. By the assumption and A3, B is neither null nor universal. Then let $z \sim x_1Bx_2$ for some $z \in X$, so $x_1 \prec z \prec x_2$. Thus by A1 and A6,

$$\begin{aligned} f_P(z, x_2, ..., x_k) &\sim f_P(m_B(x_1, x_2), m_B(x_2, x_2), ..., m_B(x_k, x_k)) \\ &\sim f_B(m_P(x_1, ..., x_k), m_P(x_2, x_2, x_3, ..., x_k)) \\ &\sim f_B(m_P(x_1, ..., x_k), m_{P''}(x_2, ..., x_k)). \end{aligned}$$

Since $f_k \sim m_P(x_1, ..., x_k)$, Proposition 1 implies

$$u(m_P(z, x_2, ..., x_k)) = \pi(B)\,u(f_k) + (1-\pi(B))\,u(m_{P''}(x_2, ..., x_k)).$$

Noting that $u(m_P(z, x_2, ..., x_k)) = \pi(B_1)\, u(z) + \sum_{i=2}^{k} (\pi(B_i) - \pi(B_{i-1}))\, u(x_i)$ by Case 1, and $u(z) = \pi(B)\, u(x_1) + (1 - \pi(B))\, u(x_2)$ by Proposition 1, the above equation is rearranged to give

$$\begin{aligned}\pi(B)\, u(f_k) &= \pi(B_1)\, u(z) + \sum_{i=2}^{k} (\pi(B_i) - \pi(B_{i-1}))\, u(x_i) \\ &\quad - (1 - \pi(B)) \left\{ \pi(B_2)\, u(x_2) + \sum_{i=3}^{k} (\pi(B_i) - \pi(B_{i-1}))\, u(x_i) \right\} \\ &= \pi(B) \left\{ \pi(B_1)\, u(x_1) + \sum_{i=2}^{k} (\pi(B_i) - \pi(B_{i-1}))\, u(x_i) \right\}.\end{aligned}$$

Hence the desired result obtains.

Case 3. Let $z \sim x_1 A x_2$ for some $A \in 2^S$ which is neither null nor universal. Then $x_1 \prec z \prec x_2$. Hence applying the result of Case 2, a similar analysis of Case 2 gives the desired result.

(2) This easily follows from Proposition 1.

(3) Suppose that A7 holds. Let u and π be real valued functions that satisfy (1). First we assume that $A \in 2^S$ is neither null nor universal. Since $\prec$ is not empty but dense, assume also that $x \prec y \prec z$. By A7 and Proposition 1,

$$u(m_A(m_A(y, x), m_A(y, z))) = u(m_A(m_A(y, y), m_A(x, z))).$$

Since by A1 and A2, $m_A(y, x) \prec m_A(y, z)$, (1) implies that the left hand side of the above equation is given as follows:

$$\begin{aligned}&u(m_A(m_A(y, x), m_A(y, z))) \\ &\quad = \pi(A)\, u(m_A(y, x)) + (1 - \pi(A))\, u(m_A(y, z)) \\ &\quad = \pi(A)\{\pi(A^c)\, u(x) + (1 - \pi(A^c))\, u(y)\} \\ &\qquad + (1 - \pi(A))\{\pi(A)\, u(y) + (1 - \pi(A))\, u(z)\} \\ &\quad = \pi(A)\, \pi(A^c)\, u(x) + \pi(A)(2 - \pi(A) - \pi(A^c))\, u(y) + (1 - \pi(A))^2\, u(z).\end{aligned}$$

We have two cases to examine: $m_A(y, y) \precsim m_A(x, z)$; $m_A(x, z) \prec m_A(y, y)$.

Case 1 $(m_A(y, y) \precsim m_A(x, z))$. It follows from (1) that

$$\begin{aligned}&u(m_A(m_A(y, y), m_A(x, z))) \\ &\quad = \pi(A)\, u(m_A(y, y)) + (1 - \pi(A))\, u(m_A(x, z)) \\ &\quad = \pi(A)\, u(y) + (1 - \pi(A))\{\pi(A)\, u(x) + (1 - \pi(A))\, u(z)\} \\ &\quad = \pi(A)(1 - \pi(A))\, u(x) + \pi(A)\, u(y) + (1 - \pi(A))^2\, u(z).\end{aligned}$$

Since $u(m_A(m_A(y, x), m_A(y, z))) = u(m_A(m_A(y, y), m_A(x, z)))$, we obtain

$$\begin{aligned}&\pi(A)\,\pi(A^c)\,u(x) + \pi(A)(2 - \pi(A) - \pi(A^c))\,u(y)\\ &\quad= \pi(A)(1 - \pi(A))\,u(x) + \pi(A)\,u(y),\end{aligned}$$

which is rearranged to give $\pi(A)(1 - \pi(A) - \pi(A^c))(u(x) - u(y)) = 0$. Since $u(x) < u(y)$ and A is not null, we must have $\pi(A) + \pi(A^c) = 1$.

Case 2 ($m_A(x, z) \prec m_A(y, y)$). It follows from (1) that

$$\begin{aligned}&u(m_A(m_A(y, y), m_A(x, z)))\\ &\quad= \pi(A^c)\,u(m_A(x, z)) + (1 - \pi(A^c))\,u(m_A(y, y))\\ &\quad= \pi(A^c)\{\pi(A)\,u(x) + (1 - \pi(A))\,u(z)\} + (1 - \pi(A^c))\,u(y)\\ &\quad= \pi(A)\,\pi(A^c)\,u(x) + (1 - \pi(A^c))\,u(y) + \pi(A^c)(1 - \pi(A))\,u(z).\end{aligned}$$

Since $u(m_A(m_A(y, x), m_A(y, z))) = u(m_A(m_A(y, y), m_A(x, z)))$, we obtain

$$\begin{aligned}&\pi(A)(2 - \pi(A) - \pi(A^c))\,u(y) + (1 - \pi(A))^2\,u(z)\\ &\quad= (1 - \pi(A^c))\,u(y) + \pi(A^c)(1 - \pi(A))\,u(z),\end{aligned}$$

which is rearranged to give $(1 - \pi(A))(1 - \pi(A) - \pi(A^c))(u(z) - u(y)) = 0$. Since $u(y) < u(z)$ and A is not universal, we must have $\pi(A) + \pi(A^c) = 1$.

Next we assume that A is null. When A is universal, the proof is similar. We are to show that A^c is universal, so that $\pi(A) + \pi(A^c) = 1$. If A^c is neither null nor universal, then the preceding paragraphs imply that A is neither null nor universal. This is a contradiction, so that A^c is either null or universal.

Assume that A^c is null. Also assume that $x \prec y \prec z$. By the hypothesis of the theorem, there is an event $B \in 2^S$ such that B is neither null nor universal. Since $m_A(y, x) = m_{A^c}(x, y) \sim y$, and $m_A(y, z) \sim z$ by the definition of null events, (1) implies that

$$u(m_B(m_A(y, x), m_A(y, z))) = u(m_B(y, z)).$$

On the other hand, (1) implies that

$$\begin{aligned}u(m_A(m_B(y, y), m_B(x, z))) &= u(m_B(x, z)) &&\text{if } m_B(y, y) \precsim m_B(x, z);\\ &= u(m_B(y, y)) &&\text{if } m_B(x, z) \prec m_B(y, y).\end{aligned}$$

Since B is neither null nor universal, A1 and A3 imply that $\{m_B(x, z), m_B(y, y)\} \prec m_B(y, z)$. Thus by (1), $m_A(m_B(y, y), m_B(x, z)) \prec m_B(m_A(y, x), m_A((y, z))$. This contradicts A7. Hence A^c must be universal.

(4) Suppose that A8 holds. Note that A8 implies A6 and A7. To show additivity of π, it suffices to prove that if $A \subset B$ then $\pi(B) = \pi(A) + \pi(B \backslash A)$. By the hypothesis of the theorem, suppose that $C \in 2^S$ is neither null nor universal, and $x \prec y$ for some $x, y \in X$. Let $z = m_C(x, y)$ and $w = m_C(z, x)$, so by A3, $x \prec w \prec z \prec y$. Also let $P = \{A, B \backslash A, B^c\}$ be a 3-partition of S. Then by A1 and A8,

$$\begin{aligned} f_P(x, w, z) &\sim f_P(m_C(x, x), m_C(z, x), m_C(x, y)) \\ &\sim f_C(m_P(x, z, x), m_P(x, x, y)) \\ &\sim f_C(m_{B \backslash A}(z, x), m_B(x, y)). \end{aligned}$$

Since $x \prec w \prec z$, (1) implies

$$u(f_P(x, w, z)) = \pi(A)\, u(x) + (\pi(B) - \pi(B \backslash A))\, u(w) + (1 - \pi(B))\, u(z).$$

Note by (1) and (3) that $u(z) = \pi(C)\, u(x) + (1 - \pi(C))\, u(y)$ and $u(w) = \pi(C)\, u(z) + (1 - \pi(C))\, u(x)$. Thus (1) and (3) imply

$$\begin{aligned} &u(f_C(m_{B \backslash A}(z, x), m_B(x, y)) \\ &\quad = \pi(C)\, u(m_{B \backslash A}(z, x)) + (1 - \pi(C))\, u(m_B(x, y)) \\ &\quad = \pi(C)\{\pi(B \backslash A)\, u(z) + (1 - \pi(B \backslash A))\, u(x)\} \\ &\qquad + (1 - \pi(C))\{\pi(B)\, u(x) + (1 - \pi(B))\, u(y)\} \\ &\quad = \pi(B \backslash A)\{\pi(C)\, u(z) + (1 - \pi(C))\, u(x)\} + (\pi(B) - \pi(B \backslash A))\, u(x) \\ &\qquad + (1 - \pi(B))\{\pi(C)\, u(x) + (1 - \pi(C))\, u(y)\} \\ &\quad = \pi(B \backslash A)\, u(w) + (\pi(B) - \pi(B \backslash A))\, u(x) + (1 - \pi(B))\, u(z). \end{aligned}$$

Since $u(f_P(x, w, z)) = u(f_C(m_{B \backslash A}(z, x), m_B(x, y)))$, we obtain

$$\pi(A)\, u(x) + (\pi(B) - \pi(B \backslash A))\, u(w) = \pi(B \backslash A)\, u(w) + (\pi(B) - \pi(B \backslash A))\, u(x),$$

which is rearranged to give $(\pi(A) - \pi(B) + \pi(B \backslash A))(u(x) - u(w)) = 0$. Hence, $\pi(B) = \pi(A) + \pi(B \backslash A)$. Q.E.D.

6. Conclusions

The purpose of this paper has been to axiomatize a generalization of the subjective expected utility in Savage's framework when the set of states is finite. Our axiomatization required that the set X of consequences be infinite, in contrast to Savage's arbitrary X. However, X is not necessarily a connected topological space. A generalized representational form yielded a utility function u on X and a monotonic probability measure over states, and adopted Choquet integration as in Schmeidler and Gilboa's representation to account for non-additive probability measures.

The six axioms were shown to be sufficient for the representation, where the sixth axiom was crucial in the finite state formulation. Also, we examined two additional axioms that gave complementarily additive or additive probability measures over states, respectively.

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