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Generic Utility Theory: Measurement Foundations and Applications in Multiattribute Utility Theory

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A utility representation is formulated that is generic in the sense that it is implied by many stronger utility theories, yet it does not make assumptions peculiar to particular utility theories. The generic utility representation postulates that the preference order on a subset of two-outcome gambles with fixed probability is represented by an additive combination of the utility of the outcomes. Although many utility theories imply that this representation should be satisfied, standard additive conjoint formalizations do not provide an axiomatization of the generic utility representation because the preference ordering is only defined on a proper subset of a Cartesian product. (The precise specification of the subset is stated in the paper.) An axiomatization of the generic utility representation is presented here, along with proofs of the existence and interval scale uniqueness of the representation. The representation and uniqueness theorems extend additive conjoint measurement to a structure in which the empirical ordering is only defined on a proper subset of a Cartesian product. The generic utility theory is important because formalizations and experimental tests carried out within its framework will be meaningful from the standpoint of any stronger theory that implies the generic utility theory. Expected utility theory, subjective expected utility theory, Kahneman and Tversky's prospect theory ((1979). *Econometrica*, **47**, 263–291), and Luce and Narens' dual bilinear model ((1985). *Journal of Mathematical Psychology*, **29**, 1–72) all imply that selected subsets of gambles satisfy generic utility representations. Thus, utility investigations carried out within the generic utility theory are interpretable from the standpoint of these stronger theories. Formalizations of two-factor additive and multiplicative utility models and of parametric utility models are presented within the generic utility framework. The formalizations illustrate how to develop multiattribute and parametric utility models in the generic utility framework and, hence, within the framework of stronger theories that imply it. In particular, the formalizations show how to develop multiattribute and parametric utility models within prospect theory and the dual bilinear model. © 1988 Academic Press, Inc.

This essay investigates the measurement foundations of a generic utility theory. The theory is generic in the sense that its validity is implied by many stronger utility theories, yet it does not make assumptions peculiar to particular utility theories. Although the assumptions of the generic utility theory are weak, they are strong enough to provide a framework for the axiomatization of the basic utility models of multiattribute utility theory. Here we find the principal advantage of the generic utility theory. It has just enough structure to allow the formalization and

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testing of utility models without commitment to the full strength of a theory like expected utility (EU) or subjective expected utility (SEU) theory, Kahneman and Tversky's (1979) prospect theory, or Luce and Narens' (1985) dual bilinear model. Formalizations and experimental tests that are carried out within the generic utility framework will be meaningful from the standpoint of stronger theories that imply the generic theory. Thus, the generic utility theory provides a convenient framework in which to investigate utility models without being forced to address the complex issues pertaining to the validity of stronger theories.

The structure of this essay is as follows. First, I will formulate the utility representation of the generic theory and show how it is implied by EU and SEU theory, prospect theory, and the dual bilinear model. Second, the generic utility theory will be axiomatized and the existence and uniqueness of the representation will be proved. The representation theorem extends additive conjoint measurement to a structure in which the empirical ordering is defined only on a proper subset of a Cartesian product of the form $A \times A$. The precise definition of the subset will be given below. Third, it will be demonstrated that basic utility models can be axiomatized within the generic utility framework. In particular, the two-factor additive and multiplicative utility representations will be axiomatized. Axiomatizations will also be presented for the log/power family of utility functions, and for the linear/exponential family of utility functions. These axiomatizations are essentially standard EU axiomatizations translated into the generic utility framework. The present analysis emphasizes that each of these utility models can be derived from functional equations, that there are strong analogies between the functional equations underlying different utility models, and that qualitative axiomatizations of the functional equations can be formulated within the generic utility theory.

Finally, I will argue that the generic utility theory provides a useful framework for the investigation of utility models. Theoretical and empirical results established within the generic utility framework will be intelligible within any theory that implies it. In particular, the axiomatizations developed here in the generic utility framework constitute axiomatizations of these models in prospect theory or the dual bilinear model, for these theories imply the generic utility theory. These results are themselves of some importance, for it has not been previously shown how to introduce additive and multiplicative utility models, and parametric utility models into prospect theory or the dual bilinear model. The generic utility theory allows one to study the utility component of the theory of preference under risk while minimizing issues and problems concerning the mental representation of probability that are addressed in stronger theories.

I should also mention a secondary benefit of the generic utility theory, namely, that it provides a representation theorem that may be useful in the axiomatization of the dual bilinear model of Luce and Narens (1985).

The Generic Utility Theory

Let C denote a set of outcomes, let $[0, 1]$ denote the closed unit interval, and let $(0, 1) = [0, 1] - \{0, 1\}$. For any $p \in (0, 1)$ and $x, y \in C$, let $(x\,p\,y)$ denote a gamble with a p chance of receiving x and a $1 - p$ chance of receiving y. Let $T^*(p)$ denote the set of all such gambles, i.e.,

$$T^*(p) = \{(x\,p\,y): x, y \in C\}. \tag{1}$$

Finally, let $\geqslant_p$ denote a weak ordering of $T^*(p)$. The utility representation for the structure $(C, p, T^*(p), \geqslant_p)$ is well known, but it will be stated here for reference in later discussions. (In the present notation, "iff" stands for "if and only if" and "Re" stands for the real numbers.)

Definition 1. Let C be a nonempty set of consequences, $p \in (0, 1)$, $T^*(p) = \{(x\;p\;y): x, y \in C\}$, and $\geqslant_p$ be a weak ordering (transitive, connected relation) of $T^*(p)$. The structure $(C, p, T^*(p), \geqslant_p)$ is said to have an *identical components* (IC) *additive representation* iff there exists a scale $U: C \to \mathrm{Re}$ and real constants $\alpha > 0$ and $\beta > 0$ such that

$$(w\,p\,x) \geqslant_p (y\,p\,z) \qquad \text{iff} \qquad \alpha U(w) + \beta U(x) \geqslant \alpha U(y) + \beta U(z) \tag{2}$$

for every $w, x, y, z \in C$.

The representation (2) is called an identical components (IC) representation because the same scale U applies to the first and second outcomes of a gamble. If the scales for the two outcomes were not linearly related to each other, then (2) would be a two-factor additive representation, but not an IC additive representation. It is not assumed that α and β satisfy $\alpha + \beta = 1$. Although this constraint is consistent with the analysis developed here, it will not be required. The assumption that $\alpha > 0$ and $\beta > 0$ is made because it is natural under the interpretation that $\geqslant_p$ is a preference ordering of two-outcome gambles. In a more general context, one might also allow negative values of α and β, but this generalization will not be discussed here. The formal work developed below is based on the axiomatization of (2) in Krantz *et al.* (1971, pp. 245–259, 303–305).

The generic utility representation is a weakening of the IC additive representation, defined as follows. A riskless consequence x can be interpreted as a gamble with constant outcomes of the form $(x\,p\,x)$. Define a riskless preference relation R by the condition

$$xRy \qquad \text{iff} \qquad (x\,p\,x) \geqslant_p (y\,p\,y) \tag{3}$$

for any $x, y \in C$. The only reason for adopting this definition of R is that it reduces the number of primitives needed to state the generic utility theory. One could as easily think of R as having an independent definition, and (3) as a consistency condition whose satisfaction is required by the theory.

Define $T^+(p)$ and $T^-(p)$ by the conditions

$$T^-(p) = \{(x\,p\,y) \in T^*(p) \text{ such that } xRy\} \tag{4}$$

$$T^+(p) = \{(x\,p\,y) \in T^*(p) \text{ such that } yRx\}. \tag{5}$$

Note that $T^-(p) \cup T^+(p) = T^*(p)$. The sets $T^-(p)$ and $T^+(p)$ will be called triangular sets of gambles; $T^-(p)$ will be called a *lower triangular set* and $T^+(p)$ will be called an *upper triangular set*. The rationale for this terminology is as follows. If C were a set of monetary payoffs, and the gambles in $T^*(p)$ were arranged on the $X \times Y$ plane, with (x, y) being the coordinates of $(x\,p\,y)$, then the condition $x = y$ would determine a diagonal in this plane. Assuming that xRy whenever $x \geqslant y$, $T^-(p)$ is the set of gambles that are on or below the diagonal, and $T^+(p)$ is the set of gambles that are on or above the diagonal. Of course, the generic utility theory is not limited to this interpretation. Outcomes need not be monetary, and if they are, preference need not be increasing with money.

For any $p \in (0, 1)$, let $\geqslant_{p^-}$ and $\geqslant_{p^+}$ denote the restrictions of $\geqslant_p$ to $T^-(p)$ and $T^+(p)$, respectively. The generic utility representation asserts that the structures $(C, p, T^-(p), \geqslant_{p^-})$ and $(C, p, T^+(p), \geqslant_{p^+})$ possess utility representations.

DEFINITION 2. Let C be a nonempty set of consequences, $p \in (0, 1)$, let $T^-(p)$ be a lower triangular set of gambles, and let $\geqslant_{p^-}$ be a weak ordering of $T^-(p)$. *The structure* $(C, p, T^-(p), \geqslant_{p^-})$ *is said to have a generic utility representation* iff there exists a function $U: C \to \mathrm{Re}$, and real constants $\alpha > 0$ and $\beta > 0$ such that

$$(w\,p\,x) \geqslant_{p^-} (y\,p\,z) \quad \text{iff} \quad \alpha U(w) + \beta U(x) \geqslant \alpha U(y) + \beta U(z) \tag{6}$$

for every $(w\,p\,x)$, $(y\,p\,z) \in T^-(p)$. Similarly, let $T^+(p)$ be an upper triangular set of gambles, and let $\geqslant_{p^+}$ be a weak ordering of $T^+(p)$. Then $(C, p, T^+(p), \geqslant_{p^+})$ *is said to have a generic utility representation* iff there exists $U': C \to \mathrm{Re}$, and real constants $\alpha' > 0$ and $\beta' > 0$ such that

$$(w\,p\,x) \geqslant_{p^+} (y\,p\,z) \quad \text{iff} \quad \alpha' U'(w) + \beta' U'(x) \geqslant \alpha' U'(y) + \beta' U'(z) \tag{7}$$

for every $(w\,p\,x)$, $(y\,p\,z) \in T^+(p)$.

The IC additive representation is the special case of the generic utility representation where $\alpha = \alpha'$ and $\beta = \beta'$. I will show below that prospect theory and the dual bilinear model both posit cases where $\alpha \neq \alpha'$ and $\beta \neq \beta'$.

The noteworthy feature of the generic utility representation is that representations (6) and (7) are restricted to preferences for gambles in a triangular set. Standard axiomatizations of additive representations do not apply to the generic utility representation because they assume that the representation is satisfied by the preference ordering over the full set $T^*(p)$ of gambles. According to prospect theory and the dual bilinear model, it could be the case that there does not exist an IC additive representation for $(C, p, T^*(p), \geqslant_p)$ even though there exists a generic utility representation for $(C, p, T^-(p), \geqslant_{p^-})$ or $(C, p, T^+(p), \geqslant_{p^+})$. To axiomatize the generic utility representation, we require a generalization of additive conjoint measurement that is based on an ordering of a triangular set of gambles. The axiomatization of the generic utility representation developed here consists of minor modifications of standard additive conjoint axioms together with an innocuous regularity condition. From the measurement standpoint, what is interesting is not the choice of axioms but rather the demonstration that these axioms are sufficient to imply the existence of the additive representation (6) even when the empirical ordering is defined only on $T^-(p)$ or $T^+(p)$. Before developing the axiomatization, it will be instructive to see that the generic utility representation is implied by many theories of preference under risk.

EU, SEU, ASEU, NASEU, *and* SWU *Theories*

Obviously, if a theory implies that the utility of fixed-probability, two-outcome gambles satisfies an IC additive representation, then it also implies the validity of the generic utility representation. Thus, EU theory implies (6) and (7) with $\alpha = \alpha' = p$ and $\beta = \beta' = 1 - p$ (Luce & Raiffa, 1957). If SEU theory is construed as a theory in which the stated probability p is transformed to a subjective probability $s(p)$ prior to evaluating the utility of a gamble, then SEU theory implies (6) and (7) with $\alpha = \alpha' = s(p)$ and $\beta = \beta' = 1 - s(p)$ (Tversky, 1967). Edwards (1962) formulated an additive subjective expected utility (ASEU) model and a nonadditive subjective expected utility (NASEU) model. According to these models, (6) and (7) are satisfied, but it is not assumed that $\alpha + \beta = 1$, or $\alpha' + \beta' = 1$. The difference between the ASEU and the NASEU models is that the ASEU model implies that the weights assigned to mutually exclusive and exhaustive events sum to the same value for any choice of events, whereas the NASEU model allows for the possibility that the weights sum to different values for different choices of mutually exclusive and exhaustive events. Karmarkar (1978) proposed a subjectively weighted utility (SWU) model in which (6) and (7) are satisfied with $\alpha = \alpha' = p^{2\theta}/(p^{2\theta} + q^{2\theta})$ and $\beta = \beta' = q^{2\theta}/(p^{2\theta} + q^{2\theta})$, where $q = 1 - p$ and $\theta > 0$ is a parameter.

Prospect Theory

Kahneman and Tversky (1979) have proposed a theory of preference under risk, called prospect theory, that attempts to account for empirical violations of EU and SEU theory. I will follow Kahneman and Tversky in presenting prospect theory as a theory of preference between monetary gambles. It is clear from their discussion that a generalization to nonmonetary gambles is consistent with their approach.

Prospect theory proposes that outcomes are perceived as gains or losses relative to a neutral reference level. Let c_0 denote the reference level. The outcome x is a gain if $x > c_0$ and a loss if $x < c_0$. Kahneman and Tversky assume that $c_0 = 0$ in many choice problems; i.e., x is a gain if one receives money, and x is a loss if one pays out money. They note, however, that if one expects to receive a large sum of money, then it might be perceived as a loss to receive a lesser sum, and if one expects to lose a large sum of money, it might be perceived as a gain to suffer a smaller loss. The value of c_0 can vary depending on the context or framing of a choice problem (Kahneman & Tversky, 1979; Tversky & Kahneman, 1981).

By definition, a prospect is a three-outcome gamble $(x, p; y, q)$ in which one receives x with probability p, y with probability q, and c_0 with probability $1-p-q$. It is assumed that p and q are probabilities satisfying $p+q \leqslant 1$. Prospect theory postulates the existence of a scale V defined on prospects such that V preserves the preference order over prospects. The value $V(x, p; y, q)$ is a function of the subjective values of x and y, and a subjective weighting of p and q, but the combination rule determining $V(x, p; y, q)$ is different depending on whether x and y are greater or less than c_0. If $p+q<1$, or $x \geqslant c_0 \geqslant y$, or $x \leqslant c_0 \leqslant y$, then $(x, p; y, q)$ is said to be a *regular prospect*, and

$$V(x, p; y, q)=\pi(p)v(x)+\pi(q)v(y), \tag{8}$$

where π is a function mapping probabilities to subjective weigths in the unit interval, and v is a function mapping outcomes to real numbers. It is assumed that $\pi(0)=0$, $\pi(1)=1$, and $v(c_0)=0$.

If $p+q=1$ and either $x \geqslant y > c_0$ or $x \leqslant y < c_0$, then $(x, p; y, q)$ is said to be an *irregular prospect*, and

$$V(x, p; y, q)=\pi(p)v(x)+(1-\pi(p))v(y). \tag{9}$$

Equation (9) and the plausible assumption that $V(x, p; y, q)=V(y, q; x, p)$ imply that if $p+q=1$ and either $y \geqslant x > c_0$ or $y \leqslant x < c_0$, then

$$\begin{aligned} V(x, p; y, q) &= V(y, q; x, p) \\ &= (1-\pi(1-p))v(x)+\pi(1-p)v(y). \end{aligned} \tag{10}$$

I will call the gambles that occur in (9) *irregular prospects of the first kind*, and the gambles that occur in (10) *irregular prospects of the second kind.* Prospect theory must distinguish among these three classes of gambles in order to account for the Allais paradox, and assumptions of symmetry and betweenness that are discussed in Miyamoto (1987).

Note that if $y=c_0$, then $v(y)=0$, so

$$\begin{aligned} V(x, p; y, q) &= \pi(p)v(x)+\pi(q)v(y) \\ &= \pi(p)v(x)+[1-\pi(p)]v(y). \end{aligned}$$

Therefore the combination rule in (9) yields the value of $V(x, p; y, q)$ when $y=c_0$. Similarly, the combination rule in (10) yields the value of $V(x, p; y, q)$ when $x=c_0$. Thus, when $x=c_0$ or $y=c_0$, the value of $V(x, p; y, q)$ remains the same regardless of whether $V(x, p; y, q)$ is computed by Eq. (8), the equation for regular prospects, or Eq. (9) or (10), the equations for irregular prospects. In subsequent discussions, I will include prospects of the form $(x, p; y, 1-p)$ where $x=c_0$ or $y=c_0$ among the irregular prospects. Although Kahneman and Tversky's (1979) definition would classify this type of prospect as a regular prospect and not an irregular prospect, calling it an irregular prospect is perfectly consistent with prospect theory.

Prospect theory implies that the generic utility representation, but not the IC additive representation, is satisfied. To see this, let C be the set of outcomes that are nonlosses, i.e., C is the set of x such that $x \geqslant c_0$. Choose any $p \in (0, 1)$ and define $T^*(p)$, $T^-(p)$, and $T^+(p)$ by

$$T^*(p) = \{(x\,p\,y): x, y \in C\} \tag{11}$$

$$T^-(p) = \{(x\,p\,y): x, y \in C \text{ and } xRy\} \tag{12}$$

$$T^+(p) = \{(x\,p\,y): x, y \in C \text{ and } yRx\}. \tag{13}$$

$T^-(p)$ and $T^+(p)$ are triangular subsets of gambles. Let $\geqslant_{p^-}$ and $\geqslant_{p^+}$ be the restriction of $\geqslant_p$ to $T^-(p)$ and $T^+(p)$, respectively. Equation (9) implies that $(C, p, T^-(p), \geqslant_{p^-})$ has a generic utility representation, and (10) implies that $(C, p, T^+(p), \geqslant_{p^+})$ has a generic utility representation. Kahneman and Tversky's (1979) analysis of the Allais paradox implies that there exist p such that $\pi(p) \neq 1 - \pi(1-p)$. If $\pi(p) \neq 1 - \pi(1-p)$, then $(C, p, T^*(p), \geqslant_p)$ does not have an IC additive representation, because $V(x, p; y, 1-p)$ is given by (9) or (10) depending on whether $(x\,p\,y) \in T^-(p)$ or $(x\,p\,y) \in T^+(p)$, and $\alpha = \pi(p) \neq 1 - \pi(1-p) = \alpha'$. Similar arguments show that prospect theory implies the existence of generic utility representations for triangular subsets of irregular prospects whose outcomes are less than c_0. Thus, if one assumes prospect theory and Kahneman and Tversky's analysis of the Allais paradox, there are numerous sets of prospects of the form $T^-(p)$ or $T^+(p)$ that have generic utility representations, even when the full set $T^*(p)$ does not have an IC additive representation.

An axiomatization of the generic utility representation will facilitate the study of the v function of prospect theory. To study the v function on nonlosses, one would consider the preference order on triangular subsets of gambles with nonloss outcomes. To study the v function on nongains, one would consider the preference order over triangular subsets of gambles with nongain outcomes. Later it will be demonstrated that multiattribute utility models can be formalized in the generic utility framework. The formalizations permit the generalization of prospect theory to multiattribute utility representations, using the fact that triangular subsets of irregular prospects satisfy generic utility representations.

The Dual Bilinear Model

Luce and Narens (1985) have proposed a utility theory, called the dual bilinear model, that is intended to account for many of the same violations of SEU theory that motivated prospect theory (see also Narens & Luce, 1986). Their theory applies to more general classes of gambles than prospect theory. The notation used here is slightly different from that of Luce and Narens (1985).

For any event A, let $(x\,A\,y)$ denote a gamble in which one receives x if A occurs and y if A does not occur. The outcomes of gambles may themselves be gambles.

For example, $((w\,B\,z)\,Ay)$ denotes a gamble in which one receives w if A and B occur, z if A but not B occurs, and y if neither A nor B occurs. The different events in a gamble are regarded as independent; thus $((x\,A\,y)\,Az)$ denotes a gamble in which the two occurrences of A stand for independent experiments whose outcomes are A or not A. Let Ω be a nonempty set of events. Let X be a nonempty set consisting of pure outcomes and gambles. X is assumed to be closed under formation of gambles in the sense that $(x\,A\,y) \in X$ whenever, $x, y \in X$ and $A \in \Omega$. Let $\geqslant_G$ denote a binary relation on X. The structure $(X, \Omega, \geqslant_G)$ is said to have a *dual bilinear utility representation* iff there exists an interval scale $U: X \to \mathrm{Re}$ and scales $S^+: \Omega \to [0, 1]$ and $S^-: \Omega \to [0, 1]$ such that for any $x, y \in X$ and $A, B \in \Omega$,

$$(w\,A\,x) \geqslant_G (y\,B\,z) \quad \text{iff} \quad U(w\,A\,x) \geqslant U(y\,B\,z) \tag{14}$$

$$U(x\,A\,y) = S^+(A)\,U(x) + (1 - S^+(A))\,U(y) \quad \text{if} \quad x >_G y \tag{15}$$

$$U(x\,A\,y) = U(x) \quad \text{if} \quad x \sim_G y \tag{16}$$

$$U(x\,A\,y) = S^-(A)\,U(x) + (1 - S^-(A))\,U(y) \quad \text{if} \quad y >_G x. \tag{17}$$

No assumptions are made concerning the relationship between the two probability weighting functions, S^+ and S^-. If S^+ and S^- are identical, then (14)–(17) are called the bisymmetric utility representation, and if, in addition, S^+ is a probability measure, then (14)–(17) is the SEU theory (Luce & Narens, 1985).

The prospect theoretic representation for nonlosses is a special case of the dual bilinear model. Suppose that $p + q = 1$, and x and y are monetary gains that are equal to or more preferred than c_0. Then, prospect theory asserts that $V(x, p; y, q)$ satisfies (9) or (10) depending on whether $x \geqslant y$ or $y \geqslant x$. If $V(x, p; y, q) = U(x\,p\,y)$, where U is the dual bilinear utility function, then $V(x, p; y, q)$ also satisfies (15), (16), or (17) depending on whether $x > y$, $x = y$, or $x < y$. The case where $x > y$ implies that $S^+(p) = \pi(p)$, and the case where $x < y$ implies that $S^-(p) = 1 - \pi(1 - p) = 1 - S^+(1 - p)$. Thus, the prospect theoretic representation for nonlosses implies that the dual bilinear representation is satisfied subject to the constraint that $S^+(p) + S^-(1 - p) = 1$. The prospect theoretic representation for nongains implies the same constraint. The representation of gambles in prospect theory is a special case of the dual bilinear representation within the domain of nonlosses or, separately, within the domain of nongains. If $S^+(p) + S^-(1 - p) \neq 1$, then preferences for nonlosses or nongains satisfy the dual bilinear model, but not prospect theory. If $x \geqslant c_0 \geqslant y$, then $(x, p; y, 1 - p)$ and $(y, 1 - p; x, p)$ are regular prospects, and $V(x, p; y, 1 - p) = \pi(p)v(x) + \pi(1 - p)v(y) = \pi(1 - p)v(y) + \pi(p)v(x) = V(y, 1 - p; x, p)$. If regular prospects also satisfy the dual bilinear model, then $\pi(p) = S^+(p)$, $\pi(1 - p) = 1 - S^+(p)$, $\pi(1 - p) = S^-(1 - p)$, and $\pi(p) = 1 - S^-(1 - p)$. With respect to regular prospects, neither prospect theory nor the dual bilinear model is a special case of the other theory. If regular prospects satisfy both theories, then $\pi(p) + \pi(1 - p) = 1$, which is not assumed by prospect theory, and $S^+(p) + S^-(1 - p) = 1$, which is not assumed by the dual bilinear model. Furthermore, if regular prospects satisfy both theories, $S^+(p) = \pi(p) = S^-(p)$ so that the utility of regular prospects is given by an IC additive representation, which is not assumed by either theory in its general form.

The present discussion has compared prospect theory and the dual bilinear model in terms of the constraints that each theory imposes on the numerical representation of subjective probability. An analysis of the qualitative properties of preference that distinguish prospect theory from the dual bilinear model would be of considerable interest, but it is beyond the scope of the present discussion.

It should be clear that if there exists a dual bilinear utility representation of $(X, \Omega, \geqslant_G)$, then $(X, \Omega, \geqslant_G)$ contains many substructures that satisfy a generic utility representation. Let C denote the pure outcomes in X, i.e., $C = \{x \in X$: if $x = (y\,A\,z)$ for any $A \in \Omega$ and $y, z \in X$, then $y = x = z\}$. Define sets $T^*(A)$, $T^+(A)$, and $T^-(A)$ by

$$\begin{aligned} T^*(A) &= \{(x\,A\,y) \text{ such that } x, y \in C\}, \\ T^+(A) &= \{(x\,A\,y) \text{ such that } x, y \in C \text{ and } x \geqslant_G y\} \\ T^-(A) &= \{(x\,A\,y) \text{ such that } x, y \in C \text{ and } y \geqslant_G x\}. \end{aligned} \tag{18}$$

Let $\geqslant_A$ denote the restriction of $\geqslant_G$ to $T^*(A)$, let $\geqslant_{A^+}$ denote the restriction of $\geqslant_G$ to $T^+(A)$, and let $\geqslant_{A^-}$ denote the restriction of $\geqslant_G$ to $T^-(A)$. Obviously (15) and (16) imply that $(C, A, T^+(A), \geqslant_{A^+})$ has a generic utility representation with $\alpha = S^+(A)$ and $\beta = (1 - S^+(A))$. Similarly, (17) and (16) imply that $(C, A, T^-(A), \geqslant_{A^-})$ has a generic utility representation with $\alpha = S^-(A)$ and $\beta = (1 - S^-(A))$. But the combined structure $(C, A, T^*(A), \geqslant_A)$ does not have an IC additive representation if $S^+(A) \neq S^-(A)$, because in this case the coefficients of $U(x)$ and $U(y)$ depend on the relative preference for x and y.

A conjoint measurement analysis of the generic utility representation contributes to the analysis of the dual bilinear model in two ways. First, an axiomatization of the generic utility representation provides a basis for multiattribute and parametric generalizations of the dual bilinear model. These generalizations will be described below. Second, standard formalizations of additive representations do not apply in a straightforward way to the axiomatization of the dual bilinear model because they are formulated in terms of the elements of a Cartesian product, while in general, the dual bilinear representation is only additive on triangular subsets of gambles. Given an axiomatization of the generic utility representation, one can seek additional consistency conditions that allow the many substructures of the form $(C, A, T^+(A), \geqslant_{A_+})$ and $(C, A, T^-(A), \geqslant_{A_-})$ to fit together into a dual bilinear representation. I hope to explore this approach to the axiomatization of the dual bilinear representation in a later work. I should mention that Luce (1986) has described an alternative approach to the axiomatization of the dual bilinear model, in which a relational structure of the form $(C, A, T^+(A), \geqslant_{A_+})$ is extended formally to a virtual structure on the full set $T^*(A)$. I refer to the latter as a virtual structure because the relations on $T^*(A)$ are created by definition and construction from the primitive ordering $\geqslant_{A_+}$ defined on $T^+(A)$. A comparative analysis of these two approaches to the axiomatic analysis of the dual bilinear model cannot be given here.

Measurement Foundations for Generic Utility Theory

A measurement axiomatization of the generic utility representation (Definition 2) will be presented in this section that generalizes the axiomatization of the IC additive representation to an ordering that is only defined on a lower triangular set of gambles. Obviously, the representation theory for an ordering of an upper triangular set is analogous, so it will be omitted. The formalization will be developed in terms of abstract sets and relations. The primitives need not be interpreted as a preference ordering over two-outcome gambles, but it will be obvious that the primitives can be interpreted in this way. Once the main representation theorem has been stated in abstract terms, the application of this theorem to the existence and uniqueness of a generic utility representation will be formulated as a corollary of the theorem.

Representation Theory for a Lower Triangular Additive (LTA) *Representation*

Let A be an arbitrary nonempty set. Pairs in $A \times A$ will be denoted by juxtaposed symbols; e.g., if $x, y \in A$, then $xy \in A \times A$. Let $\geqslant_g$ denote a binary relation on $A \times A$. Although $\geqslant_g$ is assumed to be transitive, it is not assumed to be connected on $A \times A$. We need to formulate properties of $\geqslant_g$ that distinguish the elements on which $\geqslant_g$ is defined.

In the context of preference under risk, a riskless consequence can be interpreted as a gamble with constant outcomes of the form $(x\,p\,x)$. Hence a riskless preference of x over y can be viewed as a preference for $(x\,p\,x)$ over $(y\,p\,y)$, In the abstract formalism, this relationship is represented by the condition $xx \geqslant_g yy$. In addition, we want to axiomatize a utility representation using a preference relation that is only defined on gambles $(w\,p\,x)$ and $(y\,p\,z)$ where w is preferred to x and y is preferred to z. Thus the abstract relation $\geqslant_g$ should be defined on precisely the pairs wx and yz such that $ww \geqslant_g xx$ and $yy \geqslant_g zz$. These considerations motivate the following definition.

Definition 3. Let A be an arbitrary nonempty set. A binary relation $\geqslant_g$ is said to be a *lower triangular relation on* $A \times A$ iff the following conditions hold:

(i) For any $x, y \in A$, $xx \geqslant_g yy$ or $yy \geqslant_g xx$.

(ii) For any $w, x, y, z \in A$, $ww \geqslant_g xx$ and $yy \geqslant_g zz$ iff either $wx \geqslant_g yz$ or $yz \geqslant_g wx$.

Interpreted as statements about gambles, condition (i) asserts that the ordering over certain outcomes is connected, and condition (ii) asserts that $\geqslant_g$ is only defined on pairs wx and yz such that w is preferred to x and y is preferred to z. Define $P \subseteq A \times A$ by the condition $xy \in P$ iff $xx \geqslant_g yy$. Conditions (i) and (ii) imply that $\geqslant_g$ is a connected relation on P. The representation problem is formulated using the concept of a lower triangular relation.

DEFINITION 4. Let A be a nonempty set, let $\geqslant_g$ be a lower triangular relation on $A \times A$. The pair $(A, \geqslant_g)$ is said to have a *lower triangular additive* (LTA) *representation* iff there exists a function $\phi: A \to \mathrm{Re}$ and a constant $\lambda > 0$ such that

$$ax \geqslant_g bz \quad \text{iff} \quad \phi(a) + \lambda\phi(x) \geqslant \phi(b) + \lambda\phi(z) \tag{19}$$

whenever $a, b, x, z \in A$ and $ax, bz \in P$.

The axiomatization of (19) is similar to that of the IC additive representation (Krantz *et al.*, 1971, Sects. 6.1, 6.2, and 6.11), except that the axioms are formulated in terms of elements of P, rather than elements of $A \times A$. Conditions (i) and (ii) of Definition 3 are Axioms 1 and 2 of the LTA axiomatization. Axiom 3 is the assumption that $\geqslant_g$ is transitive. These three axioms imply that $\geqslant_g$ is a weak ordering of P.

We will need an independence assumption. Conditions (20) and (21) below are the standard independence assumptions restricted to pairs in P:

$$aw \geqslant_g bw \quad \text{iff} \quad ax \geqslant_g bx \tag{20}$$

$$ya \geqslant_g yb \quad \text{iff} \quad za \geqslant_g zb \tag{21}$$

whenever $aw, bw, ax, bx, ya, yb, za, zb \in P$. We will actually adopt a slightly stronger independence assumption. The assumption, stated in the following definition, implies that the orderings induced on the first and second components of $A \times A$ are the same.

DEFINITION 5. Let A be a nonempty set, let $\geqslant_g$ be a lower triangular relation on $A \times A$, and let $P \subseteq A \times A$ be defined as above. The relation $\geqslant_g$ is said to be *independent* provided that

$$ax \geqslant_g bx \quad \text{iff} \quad ya \geqslant_g yb \tag{22}$$

whenever $a, b, x, y \in A$, and $ax, bx, ya, yb \in P$.

Clearly (22) is implied by the LTA representation. It will be proved in Lemma 1 that (22) implies (20) and (21). Condition (22) is Axiom 4 of the LTA representation.

Axioms 5–7 of the LTA representation are taken from Krantz *et al.*'s (1971) axiomatization of additive conjoint measurement. Each axiom restricts an axiom of Krantz *et al.* to pairs in P. Axiom 5 is the Thomsen condition. Axiom 6 is a restricted solvability condition. Axiom 7 is an Archimedean condition. To state the Archimedean condition, we need to reformulate the definition of a standard sequence in terms of elements of P.

DEFINITION 6. Let A be a nonempty set, and $\geqslant_g$ a triangular relation on $A \times A$. Define P as above. Then, for any set N of consecutive integers (positive or negative, finite or infinite), a set $\{a_i \in A: i \in N\}$ is said to be *a standard sequence on the first component* iff for some $b, x, y \in A$, the following holds: $bx, by \in P$, $bx \not\sim_g by$, $a_i x, a_{i+1} y \in P$, and $a_i x \sim_g a_{i+1} y$ for every i, $i+1 \in N$. The sequence is said to be bounded iff there exist $u, v, z \in A$ such that $uz, vz, a_i z \in P$ and $uz \geqslant_g a_i z \geqslant_g vz$ for every $i \in N$. The definitions of *a standard sequence on the second component* and of *a bounded sequence on the second component* are analogous, with elements of the form xa_i substituted for elements of the form $a_i x$.

Axioms 8 and 9 of the LTA representation are technical axioms. Axiom 8 asserts that there exist $ax, bx, ay, by \in P$ such that $ax >_g bx$ and $ax >_g ay$. In Krantz *et al.* (1971), this axiom is called the assumption that each component is essential. It merely excludes the trivial case where all elements in $A \times A$ are equivalent. Axiom 9 asserts that if $ax, bx \in P$ and $ax >_g bx$, then there exists c such that $cx \in P$ and $ax >_g cx >_g bx$; similarly, if $ax, ay \in P$ and $ax >_g ay$, then there exists z such that $az \in P$ and $ax >_g az >_g ay$. When combined with the remaining LTA axioms, Axiom 9 implies that the ordering induced by $\geqslant_g$ on each component is like the rational numbers in the sense that it is free from empty gaps. Once the function ϕ has been constructed, it can be shown that $\phi(A)$ is dense in an interval of real numbers in the sense that for any α and β in the interval, if $\alpha > \beta$, then $\alpha > \phi(x) > \beta$ for some $x \in A$.

The last axiom, Axiom 10, is also taken from Krantz *et al.*'s formalization of the IC additive representation. Whereas the previous axioms pertain to an additive representation on a product of the form $A \times B$, an additional assumption is required if $B = A$ and the scale on the second component of $A \times A$ is linear with respect to the scale on the first component. Axiom 10 implies that if $a_0, a_1, a_2, \ldots$ is a standard sequence on the first component and if the spacing between each a_i and a_{i+1} is sufficiently fine, then $a_0, a_1, a_2, \ldots$ is also a standard sequence on the second component. The requirement that the spacing must be sufficiently fine merely reflects the fact that if the spacing is too large, there may not exist elements on the other component whose differences can be equated to the differences between the a_i and a_{i+1}. Axiom 9 is used to establish the existence of standard sequences whose spacing is arbitrarily fine. The linearity of the scale on the second component with respect to the scale on the first component follows from the fact that standard sequences on either component are also standard sequences on the other component (provided that they are sufficiently fine-grained).

The preceding ten axioms are sufficient to imply the LTA representation. It is of interest that only Axiom 9 does not correspond to an axiom in Krantz *et al.*'s axiomatization of the IC additive representation. The remaining axioms merely restrict axioms taken from Krantz *et al.* (1971) to elements of $P \subseteq A \times A$. Axioms 1–3 assert that $\geqslant_g$ is a weak ordering of P. Axiom 4 asserts that $\geqslant_g$ satisfies independence. Axiom 5 asserts that $\geqslant_g$ satisfies the Thomson condition. Axiom 6 is a solvability condition. Axiom 7 is an Archimedean condition. Axiom 8 asserts that each component is essential, and Axiom 10 asserts that elements that are equally spaced on one component cannot be unequally spaced when they are on the other component.

The following definition summarizes the properties of a lower triangular additive (LTA) structure. The theorem which follows asserts that any LTA structure possesses an LTA representation.

DEFINITION 7. Let A be any nonempty set, and let $\geqslant_g$ be a binary relation on $A \times A$. Let $P \subseteq A \times A$ be defined by $xy \in P$ iff $xx \geqslant_g yy$. The pair $(A, \geqslant_g)$ will be called a lower triangular additive (LTA) structure iff Axioms 1–10 holds:

(1) For any $x, y \in A$, $xx \geqslant_g yy$ or $yy \geqslant_g xx$.

(2) For any $w, x, y, z \in A$, $ww \geqslant_g xx$ and $yy \geqslant_g zz$ iff either $wx \geqslant_g yz$ or $yz \geqslant_g wx$.

(3) $\geqslant_g$ is transitive.

(4) For every $a, b, x, y \in A$, if $ax, bx, ya, yb \in P$, then $ax \geqslant_g bx$ iff $ya \geqslant_g yb$.

(5) For any $a, b, c, x, y, z \in A$, such that $ax, by, bz, cx, az, cy \in P$, if $ax \sim_g by$, and $bz \sim_g cx$, then $az \sim_g cy$.

(6) For any $a, b, c, x, y \in A$, if $ax, by, cx \in P$ and $ax \geqslant_g by \geqslant_g cx$, then there exists $d \in A$ such that $dx \sim_g by$. For any $a, b, x, y, z \in A$, if $ax, by, az \in P$ and $ax \geqslant_g by \geqslant_g az$, then there exists $w \in A$ such that $aw \sim_g by$.

(7) Every bounded standard sequence is finite (see Definition 6).

(8) There exist $a, b, y \in A$ such that $ax, bx, ay, by \in P$, $ax >_g bx$ and $ax >_g ay$.

(9) If $ax, ay \in P$ and $ax >_g ay$, then there exists $z \in A$ such that $ax >_g az >_g ay$. If $ax, bx \in P$ and $ax >_g bx$, then there exists $c \in A$ such that $ax >_g cx >_g bx$.

(10) Suppose that $a, b, c, x, y \in A$ and $aw, bx, bw, cx, ya, zb, yb, zc \in P$. If $aw \sim_g bx$, $bw \sim_g cx$, and $ya \sim_g zb$, then $yb \sim_g zc$. If $ya \sim_g zb$, $yb \sim_g zc$, and $aw \sim_g bx$, then $bw \sim_g cx$.

THEOREM 1. *Let $(A, \geqslant_g)$ be an* LTA *structure. Then there exists a function $\phi: A \to \mathrm{Re}$ and a constant $\lambda > 0$ such that*

$$ax \geqslant_g by \quad \textit{iff} \quad \phi(a) + \lambda\phi(x) \geqslant \phi(b) + \lambda\phi(y) \tag{23}$$

for every $a, b, x, y \in A$ such that $ax, by \in P$. Furthermore, if $\phi': A \to \mathrm{Re}$ and $\lambda' > 0$ are any other function and constant satisfying (23), *then $\phi' = \alpha\phi + \beta$ and $\lambda' = \lambda$, for some constants $\alpha > 0$ and β.*

I will briefly sketch the strategy of proof. The complete proof is given in Appendix I. For any $x \in A$, define U_x and D_x by

$$a \in U_x \quad \text{iff} \quad ax \in P$$

$$b \in D_x \quad \text{iff} \quad xb \in P.$$

Mnemonically, U_x and D_x can be thought of as Up-x and Down-x, i.e., the set of elements that are above x and below x, respectively. If $ax \in P$, let $\geqslant_{ax}$ denote the restriction of $\geqslant_g$ to $U_a \times D_x$. Lemma 3 establishes a crucial fact about substructures of the form $(U_a \times D_x, \geqslant_{ax})$: if U_a and D_x are essential in the sense that there exist $b, c \in U_a$ such that $bx \not\sim_g cx$ and $v, w \in D_x$ such that $av \not\sim_g aw$, then $(U_a \times D_x, \geqslant_{ax})$ satisfies the additive conjoint axioms of Krantz *et al.* (1971). Consequently, $(U_a \times D_a, \geqslant_{aa})$ is an additive conjoint structure whenever U_a and D_a are both essential.

Let K denote the set of all $a \in A$ such that U_a and D_a are both essential. Axioms 8 and 9 guarantee that K is nonempty, and indeed, K contains infinitely many elements. Furthermore, if $a \in K$, let $S(a)$ denote the set of all pairs of scales that represent the additive structure of $(U_a \times D_a, \geqslant_{aa})$. In other words, if $(\chi, \tau) \in S(a)$, then χ: $U_a \to \text{Re}$, τ: $D_a \to \text{Re}$, and $\chi + \tau$ preserves the $\geqslant_{aa}$ ordering of $U_a \times D_a$. The next set of results strengthens the relationships between the additive substructures of $(A, \geqslant_g)$.

Suppose that $a \in K$ and $(\chi, \tau) \in S(a)$ are chosen arbitrarily, and consider any other $b \in K$. It is easy to show from the definition of K that $U_a \cap U_b \neq \varnothing$ and $D_a \cap D_b \neq \varnothing$. Lemma 4 establishes that we can always find scales $(\chi', \tau') \in S(b)$ such that $\chi(x) = \chi'(x)$ for every $x \in U_a \cap U_b$, and $\tau(y) = \tau'(y)$ for every $y \in D_a \cap D_b$. If f and g are any functions such that $f(x) = g(x)$ for every x in the intersection of their domains, then I will write $f \cong g$. In this notation, Lemma 4 asserts that for any $a \in K$, $(\chi, \tau) \in S(a)$ and $b \in K$, there exist $(\chi', \tau') \in S(b)$ such that $\chi' \cong \chi$ and $\tau' \cong \tau$.

After showing that the sets $\chi(U_a)$ and $\tau(D_a)$ are dense in intervals of real numbers for any $a \in K$ and $(\chi, \tau) \in S(a)$, I prove the central result in the proof of Theorem 1. To formulate the result, consider first an analogous issue in the theory of the IC additive representation. To derive the IC additive representation, one first shows that there exists an additive representation for the ordering of $A \times A$, but then one must show that the scales on the first and second components are linear with respect to each other. In the present case, we do not have an additive representation defined on every pair in $A \times A$, but we can still establish the linear relationship between the scales on the first and second components by considering relations between additive substructures of the form $(U_a \times D_a, \geqslant_{aa})$. Suppose that $a, b \in K$, $bb >_g aa$, $(\chi, \tau) \in S(a)$, and $(\chi', \tau') \in S(b)$. Note that U_a is the domain of χ, that D_b is the domain of τ', and $U_a \cap D_b \neq \varnothing$. Lemma 7 establishes that whenever these conditions hold, the scale χ is linear with respect to τ' on $U_a \cap D_b$. Loosely speaking, the "upward" and "downward" scales of different additive substructures are linear with respect to each other whenever the intersection of their domains is nonempty. The lemma actually establishes a stronger result. For any $a \in K$ and $(\chi, \tau) \in S(a)$, there exist constants $\lambda > 0$ and v such that for any $b \in K$, if $(\chi', \tau') \in S(b)$, $\chi \cong \chi'$, and $\tau \cong \tau'$, then $\tau'(x) = \lambda\chi(x) + v$ for every $x \in U_a \cap D_b$, and $\tau(y) = \lambda\chi'(y) + v$ for every $y \in D_a \cap U_b$. What is important is that a single choice of λ and v determines the linear relationship between (χ, τ) and (χ', τ') when $\chi \cong \chi'$ and $\tau \cong \tau'$.

This last result can be further strengthened. It is possible to choose $(\chi, \tau) \in S(a)$ such that $v = 0$, for the constants λ and v that relate (χ, τ) to (χ', τ'). Thus, for any

$a \in K$, there exist $(\chi, \tau) \in S(a)$ and a constant $\lambda > 0$, such that for any $b \in K$, if $(\chi', \tau') \in S(b)$, $\chi \cong \chi'$, and $\tau \cong \tau'$, then $\tau'(x) = \lambda\chi(x)$ for every $x \in U_a \cap D_b$, and $\tau(y) = \lambda\chi'(y)$ for every $y \in D_a \cap U_b$ (Lemma 8). With this result it is possible to define the scale ϕ of the LTA representation.

Choose any $a \in K$, and choose $(\chi, \tau) \in S(a)$ and $\lambda > 0$ that satisfy this last relationship. Define ϕ: $A \to \mathrm{Re}$ by

$$\phi(x) = \begin{cases} \chi(x) & \text{if } xx \geqslant_g aa \\ \tau(x)/\lambda & \text{if } aa >_g xx. \end{cases}$$

It can be shown (Lemma 9) that if $c \in K$, $(\chi', \tau') \in S(c)$, $\chi \cong \chi'$, and $\tau \cong \tau'$, then $\chi'(x) = \phi(x)$ for every $x \in U_c$ and $\tau'(y) = \lambda\phi(y)$ for every $y \in D_c$.

With these lemmas in hand, it is not difficult to show that ϕ and λ satisfy the LTA representation. The proof requires that one consider various cases. I will describe only the most important of these cases here. Suppose that $wx \geqslant_g yz$. If there exists $b \in K$ such that $w, y \in U_b$ and $x, z \in D_b$, then choose $(\chi', \tau') \in S(b)$ such that $\chi' \cong \chi$ and $\tau' \cong \tau$, where χ and τ were used to define ϕ. We then have $wx \geqslant_g yz$ iff $wx \geqslant_{bb} yz$ iff $\chi'(w) + \tau'(x) \geqslant \chi'(y) + \tau'(z)$ iff $\phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z)$. Of course there may not exist $b \in K$ such that $w, y \in U_b$ and $x, z \in D_b$. For example, it could be that $ww \geqslant_g xx >_g yy \geqslant_g zz$. But in this case it is easy to show that $\phi(w) \geqslant \phi(x) > \phi(y) \geqslant \phi(z)$, and hence, $\phi(w) + \lambda\phi(x) > \phi(y) + \lambda\phi(z)$. There are a number of other cases to consider, but they all yield to similar kinds of argument.

Existence and Uniqueness of the Generic Utility Representation

Evidently the LTA structure constitutes an axiomatization of the generic utility representation when the primitives of the LTA structure are interpreted as a preference ordering on a triangular set of gambles. Here I will define a generic utility structure that is isomorphic to an LTA structure, and then state the representation theorem for the generic utility representation.

DEFINITION 8. Let C be a nonempty set of consequences, let $T^-(p)$ be a set of gambles of the form $(x\,p\,y)$ where $x, y \in C$, and let $\geqslant_{p^-}$ be a binary relation on $T^-(p)$. Define a binary relation $\geqslant_g$ on $C \times C$ by the condition

$$ax \geqslant_g by \qquad \text{iff} \quad (a\,p\,x) \geqslant_{p^-} (b\,p\,y) \tag{24}$$

for any $a, b, x, y \in C$. Then, $(C, p, T^-(p), \geqslant_{p^-})$ will be called *a generic utility structure* iff $(C, \geqslant_g)$ is a LTA structure in the sense of Definition 7.

Definition 8 defines the obvious translation from the generic utility primitives to the LTA primitives. It would be routine to restate the axioms of the LTA structure directly in terms of $(C, p, T^-(p), \geqslant_{p^-})$, but this formalization will be omitted for the sake of brevity. In subsequent discussions, the issue will arise whether a structure of form $(C, p, T^-(p), \geqslant_{p^-})$ is a generic utility structure. In these discussions, I will say that $(C, p, T^-(p), \geqslant_{p^-})$ satisfies or fail to satisfy axioms of the LTA structure, rather than to speak more precisely of the question whether the associated structure $(C, \geqslant_g)$ satisfies these axioms. This imprecision should cause no confusion.

The following corollary to Theorem 1 establishes that any generic utility structure has a generic utility representation.

COROLLARY 1. *Let* $(C, p, T^-(p), \geqslant_{p^-})$ *be a generic utility structure. Then there exist a function* $U: C \to \mathrm{Re}$ *and constants* $\alpha > 0$ *and* $\beta > 0$ *such that*

$$(w\,p\,x) \geqslant_{p^-} (y\,p\,z) \quad \textit{iff} \quad \alpha U(w) + \beta U(x) \geqslant \alpha U(y) + \beta U(z) \tag{25}$$

whenever $w, x, y, z \in C$ *and* $(w\,p\,x), (y\,p\,z) \in T^-(p)$. *Moreover if* $U': C \to \mathrm{Re}$, α' *and* β' *are any other function and constants that satisfy* (25), *then* $\alpha' = \tau\alpha$, $\beta' = \tau\beta$, *and* $U' = \eta U + \gamma$ *for some* $\tau, \eta > 0$ *and some* γ.

The proof of Corollary 1 is immediate from Theorem 1. Since $(C, \geqslant_g)$ is a LTA structure, there exist $\phi: C \to \mathrm{Re}$ and $\lambda > 0$ that satisfy the LTA representation (23). Define $U = \phi$, $\alpha = 1/(1+\lambda)$, and $\beta = \lambda/(1+\lambda)$. Then, for any $(w\,p\,x), (y\,p\,z) \in T^-(p)$,

$$\begin{aligned}(w\,p\,x) \geqslant_{p^-} (y\,p\,z) \quad &\text{iff} \quad wx \geqslant_g yz \\ &\text{iff} \quad \phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z) \\ &\text{iff} \quad \alpha U(w) + \beta U(x) \geqslant \alpha U(y) + \beta U(z),\end{aligned}$$

so (25) is satisfied. If $U': C \to \mathrm{Re}$, α', and β' are any other function and constants that satisfy (25), then define $\phi' = U'$, $\lambda' = \beta'/\alpha'$. We have

$$\begin{aligned}wx \geqslant_g yz \quad &\text{iff} \quad (w\,p\,x) \geqslant_{p^-} (y\,p\,z) \\ &\text{iff} \quad \alpha' U'(w) + \beta' U'(x) \geqslant \alpha' U'(y) + \beta' U'(z) \\ &\text{iff} \quad \phi'(w) + \lambda'\phi'(x) \geqslant \phi'(y) + \lambda'\phi'(z)\end{aligned}$$

for every $w, x, y, z \in C$. Therefore by Theorem 1, $\phi' = \eta\phi + \gamma$ for some $\eta > 0$ and some γ. Thus $U' = \eta U + \gamma$ by definition of ϕ'. Furthermore by Theorem 1, $\lambda' = \lambda$, so $(\beta'/\alpha') = (\beta/\alpha)$. Let $\tau = \beta'/\beta$. Then $(\beta'/\alpha') = (\tau\beta/\tau\alpha)$, so $\alpha' = \tau\alpha$. Thus Corollary 1 is proved.

MULTIATTRIBUTE UTILITY THEORY IN THE GENERIC UTILITY FRAMEWORK

Multiattribute utility theory, as developed by Fishburn, Keeney, Raiffa, and others, provides formalizations of additive and multiplicative utility models, and also parametric utility models. The formalizations of these models are based on functional equations that are derived from postulated isomorphisms between substructures of a utility structure, and from the fact that the restrictions of the utility scale to such substructures are interval scales. The generic utility theory provides a framework for formalizing these utility models because it is possible to axiomatize the relevant classes of isomorphisms within the generic utility framework. I will present formalizations of the two-factor additive and multiplicative utility models, of the log/power family of utility models, and of the linear/exponential family of utility models. Emphasis will be placed on the derivation of functional equations from isomorphisms between utility substructures, and on the role of the interval scale uniqueness of the utility scale. The purpose of the analysis is to show that the generic utility theory permits the formalization of basic utility models, and also to reveal why the structure of the generic utility theory is suited to the construction and analysis of utility models.

Additive and Multiplicative Utility Models

Let $C = C_1 \times C_2$ be a Cartesian product of nonempty sets. Pairs in C will be denoted by juxtaposed symbols; thus, if $a \in C_1$ and $x \in C_2$, then $ax \in C$. *A function* $U: C \to \mathrm{Re}$ *will be said to be additive* iff there exist functions $\phi_1: C_1 \to \mathrm{Re}$ and $\phi_2: C_2 \to \mathrm{Re}$ and a constant k such that

$$U(ax) = \phi_1(a) + \phi_2(x) + k \tag{26}$$

for any $ax \in C$. *A function* $U: C \to \mathrm{Re}$ *will be said to be multiplicative* iff there exist functions $\theta_1: C_1 \to \mathrm{Re}$ and $\theta_2: C_2 \to \mathrm{Re}$ and a constant k such that

$$U(ax) = \theta_1(a)\theta_2(x) + k \tag{27}$$

for any $ax \in C$. If U is an interval scale, then the constant k can be eliminated from (26) and (27) by an admissible change of scale. It will be more convenient, however, to retain k in the definition of the additive and multiplicative representations.

To formalize the additive and mutliplicative representations, we first need to define utility substructures of a generic utility structure. It will simplify the discussion to develop all formalizations with respect to a lower triangular set of gambles, even though an upper triangular set would serve equally as well. The structure $(C, p, T^-(p), \geqslant_{p^-})$ will be denoted by the simpler expression $(C, T^-, \geqslant_p)$.

DEFINITION 9. Let $(C, T^-, \geqslant_p)$ be a generic utility structure. Then $(D, Q^-, \geqslant_q)$ is said to be *a substructure of* $(C, T^-, \geqslant_p)$ iff $D \subseteq C$, $Q^- = \{(a\,p\,x) \in T^-: a, x \in D\}$, and $\geqslant_q$ is the restriction of $\geqslant_p$ to Q^-. A substructure $(D, Q^-, \geqslant_q)$ is said to be *a utility substructure* iff $(D, Q^-, \geqslant_q)$ is a generic utility structure in the sense of Definition 8.

It will simplify the discussion of utility substructures if we adopt a convention regarding the gambles in T^- and Q^-. We have no need to discuss gambles that are not elements of T^-. Therefore reference to a gamble will include an implicit stipulation that it is in T^-. For example, the expression $(w\,p\,x) \geqslant_p (y\,p\,z)$ is equivalent to the assertion that $(w\,p\,x)$, $(y\,p\,z) \in T^-$ and $(w\,p\,x) \geqslant_p (y\,p\,z)$. Similarly, if $x, y \in D$, then reference to $(x\,p\,y)$ will implicitly include the stipulation that $(x\,p\,y) \in Q^-$. This convention avoids the need to state repeatedly that the gambles under discussion are elements of T^- or Q^-. In stating definitions, lemmas, and theorems, however, I will explicitly state that the gambles in question are elements of T^- or Q^-.

It is easy to see that any substructure of a generic utility structure satisfies the universal axioms of the LTA structure. An axiom is universal if it states a condition that must hold of every element of C. For example, Axiom 1 of Definition 7 is a universal axiom because it asserts that $(x\,p\,x) \geqslant_p (y\,p\,y)$ or $(y\,p\,y) \geqslant_p (x\,p\,x)$ for every $x, y \in C$. If $x, y \in D$ and $(D, Q^-, \geqslant_q)$ is a substructure of $(C, T^-, \geqslant_p)$, then $(x\,p\,x) \geqslant_q (y\,p\,y)$ or $(y\,p\,y) \geqslant_q (x\,p\,x)$ because $\geqslant_q$ is the restriction of $\geqslant_p$ to Q^-. Hence $(D, Q^-, \geqslant_q)$ also satisfies Axiom 1.

Since a substructure of a generic utility structure necessarily satisfies the universal axioms of a generic utility structure, only the existential axioms (axioms that assert the existence of elements with specific properties) need to be checked to determine whether the substructure is a utility substructure. The existential axioms of the LTA structure are restricted solvability (Axiom 6), the assumption that each component is essential (Axiom 8), and the assumption that there always exist elements that are strictly between nonequivalent elements (Axiom 9). Therefore a substructure $(D, Q^-, \geqslant_q)$ is a utility structure if and only if it satisfies Axioms 6, 8, and 9 of the LTA structure. The following lemma states this result formally.

LEMMA 10. *Suppose that $(C, T^-, \geqslant_p)$ is a generic utility structure, and that $(D, Q^-, \geqslant_q)$ is a substructure of $(C, T^-, \geqslant_p)$. Then $(D, Q^-, \geqslant_q)$ is a utility substructure iff it satisfies Axioms* 6, 8, *and* 9 *of the* LTA *structure.*

The proof of Lemma 10 was sketched above, so it will be omitted here.

The multiattribute utility analysis of additive and multiplicative models is based upon classes of isomorphisms between utility substructures, defined as follows. Let $(C, T^-, \geqslant_p)$ be a generic utility structure, where $C = C_1 \times C_2$ is a Cartesian product of nonempty sets. For any $a \in C_1$ and $x \in C_2$, define substructures $(a \times C_2, a \times T^-, \geqslant_a)$ and $(C_1 \times x, T^- \times x, \geqslant_x)$ by the conditions

$$a \times C_2 = \{ay \in C\colon y \in C_2\},$$

$$a \times T^- = \{(ay\,p\,az) \in T^-\colon y, z \in C_2\},$$

$$\geqslant_a \text{ is the restriction of } \geqslant_p \text{ to } a \times T^-,$$

$$C_1 \times x = \{bx \in C\colon b \in C_1\},$$

$$T^- \times x = \{(bx\,p\,cx) \in T^-\colon b, c \in C_1\},$$

and

$$\geqslant_x \text{ is the restriction of } \geqslant_p \text{ to } T^- \times x.$$

Thus $(a \times C_2, a \times T^-, \geqslant_a)$ is the substructure based upon elements whose first component is constantly equal to a. Similarly $(C_1 \times x, T^- \times x, \geqslant_x)$ is the substructure based upon elements whose second component is constantly equal to x. The sets $a \times C_2$ and $C_1 \times x$ can be thought of as one dimensional subsets of $C_1 \times C_2$. I will say that $(C, T^-, \geqslant_p)$ *is rich in one dimensional utility substructures* iff $(a \times C_2, a \times T^-, \geqslant_a)$ and $(C_1 \times x, T^- \times x, \geqslant_x)$ are utility substructures for any choice of $a \in C_1$ and $x \in C_2$.

The assumption that $(C, T^-, \geqslant_p)$ is rich in one dimensional utility substructures is rather strong. It rules out the possibility of a multiplicative representation in which some elements are mapped to 0. For example, if the utility function is multiplicative and $c \in C_1$ is mapped to zero, then $(cw\ p\ cx) \sim_p (cy\ p\ cz)$ for any choice of $w, x, y, z \in C_2$. For such an element, $(c \times C_2, c \times T^-, \geqslant_c)$ violates the assumption that each component of the LTA representation is essential (Axiom 8). Although the present discussion could be generalized to take zero and negative elements into account, the generalization will not be developed here (cf. Fishburn & Keeney, 1974, 1975; Krantz *et al.*, 1971, Chap. 7).

Keeney and Raiffa introduced a property called utility independence that characterizes utility functions that are either additive or multiplicative (Keeney, 1967, 1971; Keeney & Raiffa, 1976; Raiffa, 1969). Although their definition of utility independence assumes that preferences under risk satisfies EU or SEU assumptions, it is straightforward to reformulate utility independence in the generic utility theory.

DEFINITION 10. Let $C = C_1 \times C_2$ be a Cartesian product of nonempty sets, and let $(C, T^-, \geqslant_p)$ be a generic utility structure. *Then C_1 is said to be utility independent of C_2* iff for any $x, y \in C_2$ and $a, b, c, d \in C_1$, the following condition holds:

$$\begin{aligned} &(ax\ p\ bx), (cx\ p\ dx) \in T^- \quad \text{and} \quad (ax\ p\ bx) \geqslant_p (cx\ p\ dx) \\ \text{iff}\ &(ay\ p\ by), (cy\ p\ dy) \in T^- \quad \text{and} \quad (ay\ p\ by) \geqslant_p (cy\ p\ dy). \end{aligned} \tag{28}$$

Conversely, C_2 *is said to be utility independent of* C_1 iff for any $a, b \in C_1$ and $w, x, y, z \in C_2$, the following condition holds:

$$\begin{aligned} &(aw\ p\ ax), (ay\ p\ az) \in T^- \quad \text{and} \quad (aw\ p\ ax) \geqslant_p (ay\ p\ az) \\ \text{iff}\ &(bw\ p\ bx), (by\ p\ bz) \in T^- \quad \text{and} \quad (bw\ p\ bx) \geqslant_p (by\ p\ bz). \end{aligned} \tag{29}$$

C_1 and C_2 are said to be mutually utility independent iff each is utility independent of the other.

The equivalences (28) and (29) imply the existence of families of isomorphisms between utility substructures. For instance, define a function $h_{ab}: (a \times C_2) \to (b \times C_2)$ by $h_{ab}(ax) = bx$ for $x \in C_2$. Then (29) implies that

$$(aw\ p\ ax) \geqslant_a (ay\ p\ az) \qquad \text{iff} \quad (h_{ab}(aw)\ p\ h_{ab}(ax)) \geqslant_a (h_{ab}(ay)\ p\ h_{ab}(az)).$$

Hence h_{ab} is an isomorphism of $(a \times C_2, a \times T^-, \geqslant_a)$ onto $(b \times C_2, b \times T^-, \geqslant_b)$ for any $a, b \in C_1$. Similarly, the function $j_{wz}: (C_1 \times w) \to (C_1 \times z)$ defined by $j_{wz}(cw) = cz$

for $c \in C_1$ is an isomorphism of $(C_1 \times w, T^- \times w, \geqslant_w)$ onto $(C_1 \times z, T^- \times z, \geqslant_z)$ for any $w, z \in C_2$. The following lemma states the principal implication of utility independence.

LEMMA 11. *Suppose that $(C_1 \times C_2, T^-, \geqslant_p)$ is a generic utility structure that is rich in one dimensional utility substructures. Let $U: C_1 \times C_2 \to \mathrm{Re}$, α, and β be a utility function and coefficients for a generic utility representation of $(C_1 \times C_2, T^-, \geqslant_p)$. Then the following two conditions hold.*

(i) *If C_1 is utility independent of C_2, then there exist functions $\phi_1: C_1 \to \mathrm{Re}$, $F_1: C_2 \to \mathrm{Re}$, and $G_1: C_2 \to \mathrm{Re}$ such that*

$$U(ax) = F_1(x)\,\phi_1(a) + G_1(x) \tag{30}$$

for every $a \in C_1$ and $x \in C_2$.

(ii) *If C_2 is utility independent of C_1, then there exist functions $\phi_2: C_2 \to \mathrm{Re}$, $F_2: C_1 \to \mathrm{Re}$, and $G_2: C_1 \to \mathrm{Re}$ such that*

$$U(by) = F_2(b)\,\phi_2(y) + G_2(b) \tag{31}$$

for every $b \in C_1$ and $y \in C_2$.

Equations (30) and (31) exemplify a basic approach to model construction in multiattribute utility theory. The equations are a numerical representation of the underlying isomorphisms determined by mutual utility independence of C_1 and C_2. These isomorphisms will be discussed after other analogous utility constructions have been presented.

I will sketch a proof of 30 to show the relationship between utility independence and the interval scale uniqueness of generic utility representations. The method of proof was developed by Keeney and Raiffa (Keeney, 1967; Keeney & Raiffa, 1976; Raiffa, 1969) and was analyzed in detail by Miyamoto (1983). If U, α, and β are the scale and coefficients of a generic utility representation for $(C, T^-, \geqslant_p)$, it must be that for any $x \in C_2$ and $a, b, c, d \in C_1$, we have

$$\begin{aligned} &(ax\,p\,bx) \geqslant_x (cx\,p\,dx) \\ \text{iff}\quad &(ax\,p\,bx) \geqslant_p (cx\,p\,dx) \\ \text{iff}\quad &\alpha U(ax) + \beta U(bx) \geqslant \alpha U(cx) + \beta U(dx). \end{aligned}$$

Therefore, for any $x \in C_2$, the restriction of U to $C_1 \times x$ is a generic utility representation of $(C_1 \times x, T^- \times x, \geqslant_x)$.

Choose an arbitrary fixed $y \in C_2$, and define functions $\phi_1: C_1 \to \mathrm{Re}$ and $\pi: C_1 \times y \to C_1$ by $\phi_1(a) = U(ay)$ and $\pi(ay) = a$ for every $a \in C_1$. Thus $\phi_1 \cdot \pi(ay) = U(ay)$ for every $ay \in C_1 \times y$. It must be that $\phi_1 \cdot \pi$, α, and β are the scale and coefficients of a generic utility representation for $(C_1 \times y, T^- \times y, \geqslant_y)$ because

$\phi_1 \cdot \pi$ is equivalent to the restriction of U to $C_1 \times y$. For any $x \in C_2$, the function j_{yx} defined by $j_{yx}(ay) = ax$ for every $a \in C_1$ is an isomorphism of $(C_1 \times y, T^- \times y, \geqslant_y)$ onto $(C_1 \times x, T^- \times x, \geqslant_x)$. Therefore $U \cdot j_{yx}$ satisfies

$$\begin{aligned} &(ay\, p\, by) \geqslant_y (cy\, p\, dy) \\ \text{iff} \quad &(j_{yx}(ay)\, p\, j_{yx}(by)) \geqslant_x (j_{yx}(cy)\, p\, j_{yx}(dy)) \\ \text{iff} \quad &\alpha U(j_{yx}(ay)) + \beta U(j_{yx}(by)) \geqslant \alpha U(j_{yx}(cy)) + \beta U(j_{yx}(dy)). \end{aligned}$$

Therefore $U \cdot j_{yx}$ is another utility scale for $(C_1 \times y, T^- \times y, \geqslant_y)$. By the uniqueness of the generic utility representation, there must exist $\eta > 0$ and ζ such that

$$U \cdot j_{yx}(ay) = \eta \phi_1 \cdot \pi(ay) + \zeta \tag{32}$$

for every $ay \in C_1 \times y$. But $U(ax) = U \cdot j_{yx}(ay)$ and $\pi(ay) = a$ for every $a \in C_1$. Hence,

$$U(ax) = \eta \phi_1(a) + \zeta \tag{33}$$

for every $a \in C_1$.

The values of η and ζ that satisfy (33) are unique because there exist $a, b \in C_1$ such that $\phi_1(a) \neq \phi_1(b)$ (see Axiom 8 of the LTA structure). Since the values of η and ζ depend on the choice of $x \in C_2$, we can define $F_1 : C_2 \to \mathrm{Re}$ and $G_1 : C_2 \to \mathrm{Re}$ by the conditions $F_1(x) = \eta$ and $G_1(x) = \zeta$ for the η and ζ that satisfy (33). Then,

$$U(ax) = F_1(x)\, \phi_1(a) + G_1(x)$$

for every $a \in C_1$ and $x \in C_2$. This proves part (i) of Lemma 11. The proof of part (ii) is similar.

It can be proved that if U satisfies (30) and (31), then there exist constants ρ, τ, σ, and ω such that

$$U(ax) = \rho \phi_1(a)\, \phi_2(x) + \tau \phi_1(a) + \sigma \phi_2(x) + \omega \tag{34}$$

for every $a \in C_1$ and $x \in C_2$ (Keeney, 1967; Keeney & Raiffa, 1976; Miyamoto, 1983; Raiffa, 1969). The proof that (30) and (31) imply (34) requires only that U be a nonconstant function of its arguments in the sense that there exist $a, b, c \in C_1$ and $x, y, z \in C_2$ such that $U(az) \neq U(bz)$ and $U(cx) \neq U(cy)$. No other assumptions of EU, SEU, or the generic utility theory are required to prove this implication (Miyamoto, 1983). The proof that (30) and (31) imply (34) will omitted for the sake of brevity.

Equation (34) implies that U is either additive or multiplicative. If $\rho = 0$, then clearly U is additive. If $\rho \neq 0$, then we can define θ_1, θ_2, and k by the conditions $\theta_1(a) = \rho \phi_1(a) + \sigma$, $\theta_2(x) = \phi_2(x) + \tau/\rho$, and $k = (\rho\omega - \tau\sigma)/\rho$. Substituting these values in (34) yields that $U(ax) = \theta_1(a)\, \theta_2(x) + k$ for every $a \in C_1$ and $x \in C_2$. Therefore U is either additive or multiplicative. It is not hard to show that the converse also holds: if U is either additive or multiplicative, then C_1 and C_2 must be mutually utility independent (Miyamoto, 1983). These results are stated in the following theorem.

THEOREM 2. *Let $C = C_1 \times C_2$ be a Cartesian product of nonempty sets, let $(C, T^-, \geqslant_p)$ be a generic utility structure that is rich in one dimensional utility substructures, and let U: $C \rightarrow \mathrm{Re}$, α, and β be the scale and constants of a generic utility representation for $(C, T^-, \geqslant_p)$. Then C_1 and C_2 are mutually utility independent iff U is either additive or multiplicative.*

Theorem 2 established conditions under which F is either additive or multiplicative, but it does not distinguish between the two representations. Fishburn (1965) pointed out within the EU framework that utility functions on a Cartesian product are additive iff gambles with identical marginal probability distributions over attributes are always equally preferred. This condition, called *marginality* or *additive independence*, cannot be formulated in the generic utility framework because the generic utility representation is restricted to a triangular set of two-outcome gambles with fixed probability, and marginality is stated in terms of gambles with more outcomes and probabilities. Furthermore, the application of this diagnostic assumes that one knows the numerical values of the subjective weights associated with the probabilities. Although these subjective weights might be estimated by a scaling procedure, they would not be easy to estimate at the level of precision needed to determine whether gambles have the same marginal subjective probability distributions over attributes.

A possible diagnostic for the multiplicative utility model is a property called sign dependence (Krantz & Tversky, 1971; Krantz *et al.*, Roskies, 1965). A general definition of sign dependence will not be given here (Krantz *et al.*, 1971), but the essential notion is that if attributes combine multiplicatively and some attributes take on negative or zero values as well as positive values, then the ordering over the remaining attributes will sometimes be inverted or made degenerate. If the attributes combine additively, however, the ordering over any attribute will be the same regardless of the chosen level of other attributes. It is possible to generalize utility independence to take account of the reversals or degeneracy of order that characterize sign dependence (Fishburn & Keeney, 1974). If a preference order is sign dependent and also satisfies such a generalized utility independence, then it must be multiplicative (Fishburn & Keeney, 1974, 1975). The converse is not true, however, for a function can be multiplicative without any components ever taking negative or zero values. Thus sign dependence is sufficient but not necessary for a multiplicative representation.

Next I will formulate a condition that is necessary and sufficient for the additive representation under the assumption that C_1 and C_2 are mutually utility independent. Suppose that

$$(aw\, p\, bx) \geqslant_p (cw\, p\, dx) \tag{35}$$

for some $a, b, c, d \in C_1$ and $w, x \in C_2$. If U is additive, then (35) implies that

$$\begin{aligned}\alpha U(aw) + \beta U(bx) &= \alpha(\phi_1(a) + \phi_2(w) + k) + \beta(\phi_1(b) + \phi_2(x) + k) \\ &\geqslant \alpha(\phi_1(c) + \phi_2(w) + k) + \beta(\phi_1(d) + \phi_2(x) + k) \\ &= \alpha U(cw) + \beta U(dx). \end{aligned} \tag{36}$$

Therefore $\alpha\phi_1(a) + \beta\phi_1(b) > \alpha\phi_1(c) + \beta\phi_1(d)$. But then we must have $\alpha U(ay) + \beta U(bz) > \alpha U(cy) + \beta U(dz)$ for any other $y, z \in C_2$. If $(ay\, p\, bz)$, $(cy\, p\, dz) \in T^-$, then

$$(ay\, p\, bz) \geqslant_p (cy\, p\, dz). \tag{37}$$

Therefore, the additivity of U implies that if $a, b, c, d \in C_1$, $w, x, y, z \in C_2$, and $(aw\, p\, bx)$, $(cw\, p\, dx)$, $(ay\, p\, bz)$, $(cy\, p\, dz) \in T^-$, then (35) implies (37).

This implication is also sufficient for the additive representation, provided that it holds true for every choice of gambles in T^-. To see this, note that mutual utility independence implies that U is either multiplicative or additive. I will show that if U is multiplicative rather than additive, then there must exist cases where (35) is satisfied and (37) is violated.

Assume that U is multiplicative. For any $a \in C_1$ and $x \in C_2$, the restriction of U to $a \times C_2$ is the scale of a generic utility representation for $(a \times C_2, a \times T^-, \geqslant_a)$, and the restriction of U to $C_1 \times x$ is the scale of a generic utility representation of $(C_1 \times x, T^- \times x, \geqslant_x)$. By Lemma 6 of the LTA representation theorem, $U(a \times C_2)$ and $U(C_1 \times x)$ must be dense in intervals of real numbers. This shows that if U is multiplicative as in (27), then $\theta_1(C_1)$ and $\theta_2(C_2)$ must both be dense in intervals of real numbers. Therefore we can choose $a, b, c, d \in C_1$ and $w, x, y, z \in C_2$ such that

$$\theta_1(a) > \theta_1(c) > \theta_1(d) > \theta_1(b) > 0, \qquad \theta_2(w) > \theta_2(x) > \theta_2(y) > \theta_2(z) > 0,$$

and

$$\frac{\alpha\theta_2(w)}{\beta\theta_2(x)} > \frac{\theta_1(d) - \theta_1(b)}{\theta_1(a) - \theta_1(c)} > \frac{\alpha\theta_2(y)}{\beta\theta_2(z)}.$$

Algebraic manipulation yields

$$\begin{aligned}\alpha U(aw) + \beta U(bx) &= \alpha(\theta_1(a)\,\theta_2(w) + k) + \beta(\theta_1(b)\,\theta_2(x) + k) \\ &> \alpha(\theta_1(c)\,\theta_2(w) + k) + \beta(\theta_1(d)\,\theta_2(x) + k) \\ &= \alpha U(cw) + \beta U(dx)\end{aligned}$$

and (38)

$$\begin{aligned}\alpha U(ay) + \beta U(bz) &= \alpha(\theta_1(a)\,\theta_2(y) + k) + \beta(\theta_1(b)\,\theta_2(z) + k) \\ &< \alpha(\theta_1(c)\,\theta_2(y) + k) + \beta(\theta_1(d)\,\theta_2(z) + k) \\ &= \alpha U(cy) + \beta U(dz).\end{aligned}$$

Furthermore, a, b, c, and d, and w, x, y, and z were chosen to satisfy the inequalities $\theta_1(a)\,\theta_2(w) > \theta_1(b)\,\theta_2(x)$, $\theta_1(c)\,\theta_2(w) > \theta_1(d)\,\theta_2(x)$, $\theta_1(a)\,\theta_2(y) > \theta_1(b)\,\theta_2(z)$ and $\theta_1(c)\,\theta_2(y) > \theta_1(d)\,\theta_2(z)$. Therefore $(aw\,p\,bx)$, $(cw\,p\,dx)$, $(ay\,p\,bz)$, $(cy\,p\,dz) \in T^-$, and $(aw\,p\,bx) >_p (cw\,p\,dx)$ and $(ay\,p\,bz) <_p (cy\,p\,dz)$. Hence (35) is satisfied and (37) is violated. Therefore, the condition that (35) implies (37) is both necessary and sufficient for the additive representation, under the assumption of mutual utility independence and the generic utility representation.

An analogous argument shows that U is additive iff (39) implies (40) whenever the gambles are in T^-:

$$(aw\,p\,bx) >_p (ay\,p\,bz), \tag{39}$$

$$(cw\,p\,dx) >_p (cy\,p\,dz). \tag{40}$$

If U is multiplicative, there exist gambles in T^- such that (39) is satisfied and (40) is violated.

We thus have the following corollary to Theorem 2.

COROLLARY 2. *Let* $C = C_1 \times C_2$ *be a Cartesian product of nonempty sets, let* $(C, T^-, \geqslant_p)$ *be a generic utility structure that is rich in one dimensional utility substructures, let* $U\colon C \to \mathrm{Re}$, α, *and* β *be the scale and constants of a generic utility representation for* $(C, T^-, \geqslant_p)$, *and suppose that* C_1 *and* C_2 *are mutually utility independent. Then the following conditions distinguish the additive and multiplicative representations.*

(i) U *is additive iff* (35) *implies* (37) *for every* $a, b, c, d \in C_1$ *and every* $w, x, y, z \in C_2$ *such that* $(aw\,p\,bx)$, $(cw\,p\,dx)$, $(ay\,p\,bz)$, $(cy\,p\,dz) \in T^-$. U *is multiplicative iff there exist* $a, b, c, d \in C_1$ *and* $w, x, y, z \in C_2$ *such that* (35) *is satisfied and* (37) *is violated.*

(ii) U *is additive iff* (39) *implies* (40) *for every* $a, b, c, d \in C_1$ *and every* $w, x, y, z \in C_2$ *such that* $(aw\,p\,bx)$, $(ay\,p\,bz)$, $(cw\,p\,dx)$, $(cy\,p\,dz) \in T^-$. U *is multiplicative iff there exist* $a, b, c, d \in C_1$ *and* $w, x, y, z \in C_2$ *such that* (39) *is satisfied and* (40) *is violated.*

Theorem 2 and Corollary 2 provide a diagnostic procedure for additive and multiplicative utility representations. If C_1 and C_2 are not mutually utility independent, neither the additive nor the multiplicative utility representation is valid. If they are mutually utility independent, then U is either additive or multiplicative, but it is not yet determined which representation obtains. If C_1 and C_2 are mutually utility independent, then either condition (i) or (ii) of Corollary 2 can be tested to determine whether U is additive or multiplicative. As far as I know, conditions (i) and (ii) of Corollary 2 have not previously been formulated as diagnostics for additivity.

In their discussion of polynomial conjoint measurement, Krantz *et al.* have said that the additive and multiplicative representations are notational variants when the attributes are not sign dependent (Krantz *et al.*, 1971; Krantz & Tversky, 1971).

This statement would appear to contradict the present results, where the additive and multiplicative representations are distinguished even when sign dependence is not present. This contradiction, however, is only apparent.

The generic utility representation implies an ordering of utility differences. Thus, if $\alpha U(au)+\beta U(bv) \geqslant \alpha U(cw)+\beta U(dx)$ and $\alpha U(ey)+\beta U(dx) \geqslant \alpha U(fz)+\beta U(bv)$, then

$$\alpha(U(au)-U(cw)) \geqslant \beta(U(dx)-U(bv)) \geqslant \alpha(U(fz)-U(ey)).$$

The difference $U(au)-U(cw)$ can thus be ordered relative to $U(fz)-U(ey)$. If $U=\phi_1+\phi_2+k$, then standard sequences on ϕ_1 or ϕ_2 are also equally spaced with respect to the intervals in U. If $U=\theta_1\theta_2+k$, then standard sequences on θ_1 or θ_2 correspond to sequences of *equal ratios* in $U-k$. Of course, a sequence of equal ratios in $U-k$ will be a sequence of unequal intervals in $U-k$. The generic utility representation allows one to distinguish whether the utility of attributes combines additively or multiplicatively, because one can compare the ordering of utility differences to the ordering of subjective differences or ratios on single attributes.

The polynomial conjoint measurement analysis of additive and multiplicative models is based on an ordering of multiattribute outcomes, and not on an ordering of gambles for such outcomes (Krantz *et al.*, 1971; Krantz & Tversky, 1971). Since only the ordering of outcomes is under consideration, the theory does not determine an ordering of utility differences independently from the ordering of subjective intervals or ratios on single attributes. In the absence of sign dependence, the additive and multiplicative representations are notational variants when the ordering of outcomes is the only available empirical property and it satisfies independence. If one observes an ordering of gambles as well as an ordering of outcomes, however, the additive and multiplicative representations are distinguishable even when sign dependence is not present.

Power and Exponential Utility Models

Let C be a one dimensional physical continuum like money or survival duration. Elements of C are denoted by their physical measure, and I will assume that C is a nonnegative real interval that is possibly bounded above. In other words, C is isomorphic to a set of real numbers of the form $\{x \in \mathrm{Re}: \varepsilon \leqslant x \leqslant \omega\}$ where $0 \leqslant \varepsilon < \omega$, and ω is finite or infinite. C is defined in terms of bounds ε and ω to allow for cases where the utility model is only posited of outcomes in an intermediate range of values.

A utility function $U: C \rightarrow \mathrm{Re}$ will be said to be a member of *the family of log/power utility functions* iff either

$$U(x)=\rho x^\gamma+\tau \qquad \text{for some} \quad \rho \neq 0,\ \gamma \neq 0, \text{ and some real } \tau \tag{41}$$

or

$$U(x)=\eta(\log x)+\xi \qquad \text{for} \quad \eta \neq 0 \text{ and some } \xi. \tag{42}$$

If (41) holds, then U is monotonic increasing iff ρ and γ are both positive or both negative; U is monotonic decreasing iff ρ and γ are of opposite sign. If (42) holds, then U is monotonic increasing or decreasing depending on whether η is positive or negative. The logarithmic function is included in the same family as the power functions because the preference ordering determined by the utility function $U(x) = x^\gamma$ approaches the preference ordering of a logarithmic utility function as γ approaches zero.

A utility function U: $C \to \text{Re}$ will be said to be a member of *the linear/exponential family of utility functions* iff

$$U(x) = ve^{\theta x} + \omega \qquad \text{for} \quad v \neq 0 \text{ and } \theta \neq 0, \tag{43}$$

or

$$U(x) = \lambda x + v \qquad \text{for} \quad \lambda \neq 0 \text{ and some } v. \tag{44}$$

If (43) holds, then U is monotonic increasing if v and θ are both positive or both negative, and it is monotonic decreasing if v and θ have opposite sign. If (44) holds, then U is monotonic increasing or decreasing depending on whether λ is positive or negative. The linear utility function is included in the family of exponential functions because the preference ordering determined by the utility function $U(x) = e^{\theta x}$ approaches the preference ordering of a linear utility function as θ approaches zero.

I will first formalize the class of power utility models. Working in the EU framework, Pratt (1964) showed that a utility function U is in the log/power family iff U is twice differentiable, and $-U''(x)/U'(x) = cx$ where c is a constant. It can be shown that this condition is equivalent to asserting that U is twice differentiable,

$$\lim(-xU''(x)/U'(x)) \text{ exists as } x \to 0,$$

and

$$(w\,p\,x) \geqslant_p (y\,p\,z) \qquad \text{iff} \quad (sw\,p\,sx) \geqslant_p (sy\,p\,sz) \tag{45}$$

for any $s \in \text{Re}^+$ and $w, x, y, z \in C$ such that $sw, sx, sy, sz \in C$ (Keeney & Raiffa, 1976). (Re^+ denotes the strictly positive real numbers). Condition (45) asserts that preference between gambles is unaffected by a scalar multiplication of outcomes. This condition has been interpreted to mean that the optimal gambling strategy is determined only by the proportion of the total assets that the gamble outcomes represent (Keeney & Raiffa, 1976; Pratt, 1964).

Here I will show that (45) also characterizes the log/power family of utility functions within the generic utility theory. The analysis is based on the solution of a functional equation investigated by Luce (1959). The use of functional equations to formalize the log/power utility functions has two advantages. First, one can replace the assumptions that U is twice differentiable and that the limit of $-xU''(x)/U'(x)$ exists at 0 by the weaker assumption that U is continuous. Indeed, if one assumes the validity of the generic utility theory, then one need only assume that U is strictly monotonic. Second, the functional equations approach emphasizes the role of interval scale uniqueness in the formalization of the log/power utility functions. As I will argue below, the utility models discussed in this section are all formalized in terms of isomorphisms that preserve interval scales.

Assume that $(C, T^-, \geqslant_p)$ is a generic utility structure, and let $U: C \to \text{Re}$, α, and β be the scale and coefficients of a generic utility representation. I will continue to observe the convention that reference to a gamble $(x\,p\,y)$ includes the assertion that $(x\,p\,y) \in T^-$. For any $s \in \text{Re}^+$, let C^*s denote the set of $x \in C$ such that $sx \in C$. Let T^*s denote the subset of T^- consisting of gambles with outcomes in C^*s. Finally, let $\geqslant_{*_s}$ denote the restriction of $\geqslant_p$ to T^*s. Translated into the generic utility framework, condition (45) can be stated as follows:

DEFINITION 11. Let C be a nonempty interval of nonnegative real numbers. Let T^- be a lower triangular set of gambles determined by the choice of some $p \in (0, 1)$, and let $\geqslant_p$ be a binary relation on T^-. Then *the ordering $\geqslant_p$ is said to be preserved under scalar multiplication* iff for any $s \in \text{Re}^+$ and $w, x, y, z \in C^*s$, if $(w\,p\,x)$, $(y\,p\,z) \in T^-$, then $(sw\,p\,sx)$, $(sy\,p\,sz) \in T^-$ and

$$(w\,p\,x) \geqslant_p (y\,p\,z) \quad \text{iff} \quad (sw\,p\,sx) \geqslant_p (sy\,p\,sz). \tag{46}$$

Condition (46) has essentially the same interpretation in the generic utility theory as in SEU theory.

Let $B = \{s \in \text{Re}: C^*s \text{ is a nonempty interval}\}$. B must be a nonempty interval because C is a nonempty interval. If $s \in B$, then by Lemma 10, the substructure $(C^*s, T^*s, \geqslant_{*_s})$ is a utility substructure of $(C, T^-, \geqslant_p)$ iff Axioms 6, 8, and 9 of the LTA structure are satisfied by $(C^*s, T^*s, \geqslant_{*_s})$. These axioms are obviously true of $(C^*s, T^*s, \geqslant_{*_s})$ if one assumes that U is continuous and strictly monotonic with respect to the $\geqslant$ ordering of C. Later I will show that if $(C, T^-, \geqslant_p)$ is a generic utility structure, then the assumption that U is strictly monotonic implies that U is continuous. For the moment, let us assume that U is continuous and strictly monotonic. Hence $(C^*s, T^*s, \geqslant_{*_s})$ is a utility substructure for any $s \in B$.

Note that if $s \in B$, then $s^{-1} \in B$. For any $s \in B$, let $t = s^{-1}$, and define a function $h_s: C^*s \to C^*t$ by $h_s(x) = sx$ for any $x \in C^*s$. If the ordering $\geqslant_p$ is preserved under scalar multiplication of outcomes, then h_s is an isomorphism of $(C^*s, T^*s, \geqslant_{*_s})$ onto $(C^*t, T^*t, \geqslant_{*_t})$. Clearly for any $s \in B$, the restriction of U to C^* is a generic utility representation for $(C^*s, T^*s, \geqslant_{*_s})$, because

$$\begin{aligned}(w\,p\,x) \geqslant_{*_s} (y\,p\,z) \quad &\text{iff} \quad (w\,p\,x) \geqslant_p (y\,p\,z)\\ &\text{iff} \quad \alpha U(w) + \beta U(x) \geqslant \alpha U(y) + \beta U(z).\end{aligned}$$

But $U \cdot h_s$ must also be a generic utility representaion for $(C^*s, T^*s, \geqslant_{*s})$, because

$$\begin{aligned}(w\,p\,x) \geqslant_{*s} (y\,p\,z) \quad &\text{iff} \quad (w\,p\,x) \geqslant_p (y\,p\,z)\\ &\text{iff} \quad (sw\,p\,sx) \geqslant_p (sy\,p\,sz)\\ &\text{iff} \quad \alpha U(sw) + \beta U(sx) \geqslant \alpha U(sy) + \beta U(sz)\\ &\text{iff} \quad \alpha U \cdot h_s(w) + \beta U \cdot h_s(x) \geqslant \alpha U \cdot h_s(y) + \beta U \cdot h_s(z).\end{aligned}$$

Since the restriction of U to C^*s is an interval scale representation of $(C^*s, T^*s, \geqslant_{*s})$, there must exist constants $\eta > 0$ and ζ such that $U \cdot h_s(x) = \eta U(x) + \zeta$ for every $x \in C^*s$. But $U \cdot h_s(x) = U(sx)$, so

$$U(sx) = \eta U(x) + \zeta \tag{47}$$

for every $x \in C^*s$. For any $s \in B$, the η and ζ in (47) must be unique because U is nonconstant. Since the particular η and ζ in (47) are determined by the choice of $s \in B$, we may define functions F: $B \to \mathrm{Re}$ and G: $B \to \mathrm{Re}$ by the conditions $F(s) = \eta$ and $G(s) = \zeta$ for the η and ζ in (47). Hence,

$$U(sx) = F(s)\,U(x) + G(s) \tag{48}$$

for any $s \in B$ and $x \in C$ such that $sx \in C$.

Luce (1959, Theorem 2) showed that the only continuous solutions to (48) are the logarithmic function or power functions. In the context of the generic utility theory, it suffices to assume that U is strictly monotonic because the LTA axioms imply that $U(C)$ is dense in an interval of real numbers. If U is strictly monotonic and $U(C)$ is dense in an interval of real numbers, then U is continuous. The following definition states a qualitative property that is equivalent to the assertion that U is strictly monotonic.

DEFINITION 12. Let C be a nonempty interval of nonnegative real numbers. Let T^- be a lower triangular set of gambles determined by the choice of some $p \in (0, 1)$, and let $\geqslant_p$ be a binary relation on T^-. Then *the relation $\geqslant_p$ is said to be strictly monotonic* iff for every $x, y \in C$, if $x > y$, then $(x\,p\,y) \in T^-$, and $(x\,p\,x) >_p (x\,p\,y) >_p (y\,p\,y)$.

It is routine to show that if U is the utility function of a generic utility representation and $\geqslant_p$ is strictly monotonic, then U is a strictly monotonic function with respect to the $\geqslant$ ordering of C. We can now state a representation theorem for power or logarithmic utility functions. The proof of the theorem is based on Eq. (48) and Luce's (1959) Theorem 2.

THEOREM 3. *Suppose that $(C, T^-, \geqslant_p)$ is a generic utility structure, where C is a nonempty interval of nonnegative real numbers and $\geqslant_p$ is strictly monotonic. Let*

$U: C \to \mathrm{Re}$, α, and β be the scale and coefficients of a generic utility representation. Then, $\geqslant_p$ is preserved under scalar multiplication iff either $U(x) = \eta(\log x) + \xi$ for some $\eta \neq 0$ and some ξ, or else $U(x) = \rho x^\gamma + \tau$ for some $\rho \neq 0$, $\gamma \neq 0$, and τ.

Proof. Since $\geqslant_p$ is strictly monotonic, U must be strictly monotonic. By Lemma 6, the scale of a LTA representation is onto a dense subset of a real interval. Hence $U(C)$ must be dense in a real interval. Since U is strictly monotonic, it must be continuous. Define sets B and C^*s as in the preceding text. It was demonstrated in the text that condition (46) implies that there exist functions $F: B \to \mathrm{Re}$, $G: B \to \mathrm{Re}$ such that $U(sx) = F(s)U(x) + G(s)$ for every $s \in B$ and $x \in C^*s$. By Luce's (1959) Theorem 2, U is either logarithmic or a power function, i.e., either (41) or (42) holds. The converse claim that log or power functions imply that $\geqslant_p$ is preserved under scalar multiplication is routine to prove. ∎

Theorem 3 characterizes the class of utility functions that are either power or logarithmic, but it does not distinguish between positive powers, negative powers, or logarithmic functions. Fagot (1963) has formulated diagnostic properties of these three classes of representations using the notion of a subjective midpoint. To apply Fagot's criteria, one must note that the generic utility representation determines utility midpoints as follows. Suppose $a, b, c, y, z \in C$ satisfy

$$(a\,p\,y) \sim_p (b\,p\,z), \qquad (b\,p\,y) \sim_p (c\,p\,z), \qquad (y\,p\,z) >_p (z\,p\,z). \tag{49}$$

Although a, b, and c satisfying (49) might not exist for arbitrarily chosen y and z, the LTA assumptions imply that they must exist for some choice of y and z, because a, b, and c are the first three elements in a standard sequence, and the LTA representation theory implies that standard sequences exist. Then (49) implies that $\alpha U(a) + \beta U(y) = \alpha U(b) + \beta U(z)$, $\alpha U(b) + \beta U(y) = \alpha U(c) + \beta U(z)$, and $\alpha U(y) + \beta U(z) > \alpha U(z) + \beta U(z)$. Therefore $U(a) - U(b) = U(b) - U(c) = (\beta/\alpha)(U(z) - U(y)) \neq 0$. Hence $U(a) \neq U(b) \neq U(c)$, and

$$U(b) = (U(a) + U(c))/2. \tag{50}$$

Equation (50) asserts that b is the midpoint in utility between a and c. Fagot (1963) pointed out that if a, b, and c are any elements that satisfy (50), and U is either power or logarithmic, then $U(x)$ is a positive power of x iff $b > \sqrt{ac}$, $U(x)$ is a negative power of x iff $b < \sqrt{ac}$, and $U(x)$ is a logarithmic function of x iff $b = \sqrt{ac}$, Thus, positive and negative power functions and logarithmic functions are distinguished by the relative magnitude of utility midpoints and geometric means of stimulus values. Thus, Theorem 3 provides a representation theorem for generic utility functions that are either power or logarithmic, and Fagot's conditions diagnose which form of power or logarithmic functions is the case.

The axiomatization of the linear/exponential family of utility function is similar to that of the log/power family, so the formalization will only be sketched. Pfanzagle formulated what he called a consistency axiom that is sufficient in combination with other preference axioms to imply that utility is in the linear/exponential family. Pfanzagl's consistency axiom can be reformulated in the generic utility framework as follows. Let $(C, T^-, \geqslant_p)$ be a generic utility structure, and let U, α, and β be the scale and coefficients of a generic utility representation. For any $s \in \mathrm{Re}$, let $C+s$ denote the set of $x \in C$ such that $x+s \in C$. Let $T+s$ denote the subset of T^- with outcomes in $C+s$. Let $\geqslant_{+s}$ denote the restriction of $\geqslant_p$ to $T+s$. Let V denote the set of $s \in \mathrm{Re}$ such that $C+s$ is a nonempty interval. It is clear that V is nonempty because C is a nonempty interval. Furthermore $s \in V$ implies that $-s \in V$. Pfanzagl's consistency axiom can be reformulated in the generic utility framework as follows:

Definition 13. Let C be a nonempty interval of nonnegative real numbers. Let T^- be a lower triangular set of gambles determined by the choice of some $p \in (0, 1)$, and let $\geqslant_p$ be a binary relation on T^-. Then $\geqslant_p$ *is said to be preserved under translation* iff for any $s \in \mathrm{Re}$ and $w, c, y, z \in C+s$, if $(w\,p\,x), (y\,p\,z) \in T^-$, then $(w+s\,p\,x+s), (y+s\,p\,z+s) \in T^-$ and

$$(w\,p\,x) \geqslant_p (y\,p\,z) \quad \text{iff} \quad (w+s\,p\,x+s) \geqslant_p (y+s\,p\,z+s). \tag{51}$$

I will sketch a proof that (51) implies that U is linear or exponential. The proof is based on a functional equation studied by Aczel (1966). Pfanzagl based his proof on a different functional equation from the one developed here.

Suppose that $\geqslant_p$ is strictly monotonic, and hence, that U is continuous. Then $(C+s, T+s, \geqslant_{+s})$ is a utility substructure for any $s \in V$. For any $s \in V$, let $t = -s$ and define $h_s\colon C+s \to C+t$ by $h_s(x) = x+s$. By (51), h_s is an isomorphism of $(C+s, T+s, \geqslant_{+s})$ onto $(C+t, T+t, \geqslant_{+t})$. Clearly the restriction of U to $C+s$ is the scale of a generic utility representation for $(C+s, T+s, \geqslant_{+s})$. But $U \cdot h_s$ is also a generic utility representation for $(C+s, T+s, \geqslant_{+s})$ because

$$\begin{aligned}(w\,p\,x) \geqslant_p (y\,p\,z) \quad &\text{iff} \quad (h_s(w)\,p\,h_s(x)) \geqslant_p (h_s(y)\,p\,h_s(z)) \\ &\text{iff} \quad \alpha U(h_s(w)) + \beta U(h_s(x)) \geqslant \alpha U(h_s(y)) + \beta U(h_s(z))\end{aligned}$$

for any $w, x, y, z \in C+s$. But the generic utility representation is an interval scale, so

$$U \cdot h_s(x) = \eta U(x) + \tau, \tag{52}$$

for some $\eta > 0$, some τ, and every $x \in C+s$. The constants η and τ are unique because $C+s$ is a nonempty interval. Therefore we can define functions $F\colon V \to \mathrm{Re}$, $G\colon V \to \mathrm{Re}$ by $F(s) = \eta$ and $G(s) = \tau$ for the η and τ in (52). Since $x+s = h_s(x)$, we have

$$U(x+s) = F(s)\,U(x) + G(s) \tag{53}$$

for every $s \in V$ and every $x \in C^*s$. Aczel (1966, Theorem 1, Sec. 3.13) showed that the only solutions to (53) that are continuous on an interval are linear or exponential functions. Therefore we have the following theorem.

THEOREM 4. *Suppose that* $(C, T^-, \geqslant_p)$ *is a generic utility structure, where* C *is a nonempty interval of nonnegative real numbers and* $\geqslant_p$ *is strictly monotonic. Let* $U\colon C \to \mathrm{Re}$, α, *and* β *be the scale and coefficients of a generic utility representation. Then,* $\geqslant_p$ *is preserved under translation iff either* $U(x) = ve^{\theta x} + \omega$ *for some* $v \neq 0$, $\theta \neq 0$, *and some* ω, *or else* $U(x) = \lambda x + v$ *for* $\lambda \neq 0$, *and some* v.

Characterization of Utility Models by Isomorphisms of Utility Substructures

In the preceding sections, formalizations of additive and multiplicative utility models and of log/power and linear/exponential utility models were presented. These formalizations all share a common abstract structure. To understand this structure, I will first describe the formalizations in abstract, set theoretic terms, and then point out how the abstraction applies to the utility models presented in the previous sections.

Suppose that $(C, T^-, \geqslant_p)$ is a generic utility structure, and that U, α, and β are the scale and coefficients of a generic utility representation for $(C, T^-, \geqslant_p)$. Let Γ denote a nonempty set of indices, and suppose that for each $\omega \in \Gamma$, there exist utility substructures $(D_\omega, R_\omega, \geqslant_r)$ and $(E_\omega, S_\omega, \geqslant_s)$ and an isomorphism f_ω of $(D_\omega, R_\omega, \geqslant_r)$ onto $(E_\omega, S_\omega, \geqslant_s)$. The restriction of U to D_ω is a generic utility function for $(D_\omega, R_\omega, \geqslant_r)$, but so is $U \cdot f_\omega\colon D_\omega \to \mathrm{Re}$ because f_ω is an isomorphism. Since generic utility representations are interval scales, there exist $\eta \neq 0$ and τ such that

$$U \cdot f_\omega(x) = \eta U(x) + \tau \tag{54}$$

for every $x \in D_\omega$. For any $\omega \in \Gamma$, the constants η and τ that satisfy (54) are unique because $U(D_\omega)$ is onto a dense subset of a real interval. Since η and τ depend on the choice of $\omega \in \Gamma$, we can define functions $F\colon \Gamma \to \mathrm{Re}$ and $G\colon \Gamma \to \mathrm{Re}$ by $F(\omega) = \eta$ and $G(\omega) = \tau$ for the η and τ in (54). Then,

$$U \cdot f_\omega(x) = F(\omega)\, U(x) + G(\omega) \tag{55}$$

for every $\omega \in \Gamma$ and every $x \in D_\omega$.

A functional equation of the form (55) played a central role in the formalization of each of the classes of utility models described above. For example, in the formalization of additive and multiplicative utility representations, let $\Gamma = C_2 \times C_2$, and define $j_{yx}\colon (C_1 \times y) \to (C_1 \times x)$ by $j_{yx}(ay) = ax$ for every $a \in C_1$. The assumption that C_1 is utility independent from C_2 implies that j_{yx} is an isomorphism of $(C_1 \times y, T^- \times y, \geqslant_y)$ onto $(C_1 \times x, T^- \times x, \geqslant_x)$. Choose any $y \in C_2$, and define ϕ_1 by $\phi_1(a) = U(ay)$. Then,

$$U(ax) = U \cdot j_{yx}(ay) \tag{56}$$

$$= F(y, x)\, U(ay) + G(y, x) \tag{57}$$

$$= F_1(x)\, \phi_1(a) + G_1(x), \tag{58}$$

where $F_1(x) = F(y, x)$ and $G_1(x) = G(y, x)$ for every x. The equivalence of (56) and (58) simply restates Eq. (30), the central result of Lemma 11 (i). The equivalence of (56) and (57) follows from the abstract argument underlying Eq. (55), and the equivalence of (57) and (58) is simply a definition of notation. A similar analysis shows that Eq. (31), the central result of Lemma 11 (ii), is also derived by the argument underlying Eq. (55).

In the formalization of the parametric utility models, the assumption that the preference ordering $\geqslant_p$ is preserved under scalar multiplication implies that for any s and $t = s^{-1}$, the structures of the forms $(C^*s, T^*s, \geqslant_{*s})$ and $(C^*t, T^*t, \geqslant_{*t})$ are isomorphic under the transformation $f_s(x) = sx$. Similarly, the assumption that the preference ordering $\geqslant_p$ is preserved under translation implies that for any s and $t = -s$, the structures $(C+s, T+s, \geqslant_{+s})$ and $(C+t, T+t, \geqslant_{+t})$ are isomorphic under the transformation $g_s(x) = x + s$. These isomorphisms lead to functional equations, (48) and (53), whose proofs are also based on the abstract argument underlying Eq. (55).

Each of the utility models analyzed here was axiomatized in terms of an empirical property (utility independence, invariance of preference under scalar multiplication or translation) that implied isomorphisms between utility substructures. Functional equations like (55) were inferred from the existence of the isomorphisms. It was then found that the utility models were the unique solutions of these functional equations, (30), (31), (48), and (53). It is important to note that the solutions of these equations depend only on the form of the equations, and not on any utility theoretic assumptions from which the equations were derived. Therefore to formalize these models, it was only necessary to provide a basis for the derivation of Eqs. (30), (31), (48), and (53). This basis consisted of (i) a specification of the relevant utility substructures, (ii) an axiomatization of the appropriate isomorphisms between substructures, and (iii) the existence of an interval scale of utility whose restrictions to such substructures are also interval scales. The generic utility theory provides a weak, but sufficient basis for axiomatizing these utility models because it fulfills these three requirements.

Empirical Applications of the Generic Utility Theory

Miyamoto and Eraker (1988) tested a utility independence assumption in the generic utility framework. They studied a two-attribute utility problem, where the attributes were duration of survival and health quality during survival. Medical patients were asked to judge durations of certain survival that were equivalent in preference to an even-chance gamble for survival duration. For example, a subject might be asked to state a duration X such that surviving X years for certain would be equal in value to owning the gamble (20 years .5 2 years). Subjects were instructed to assume that all of the survivals in the choice would be accompanied by the same specified health state, and to assume that health state was approximately constant during the period of survival. The duration X satisfying this preference equivalence will be called *the certainty match* of (20 years .5 2 years).

The utility independence of survival duration from health quality was tested as follows. Under one condition, subjects were instructed to assume that the certainty matches and gamble outcomes were associated with excellent health. Under a second condition, the same subjects were instructed to assume that the certainty matches and gamble outcomes were associated with "poor health," where a precise definition of poor health was given in terms of health problems that the subjects (medical patients) had been experiencing during the period prior to the experiment. The stimulus gambles had the form $(X\,.5\,Y)$ where $X >_p Y$. Thus the stimulus gambles were elements of a lower triangular set. It was found that, with the exception of a small minority of subjects, the certainty matches produced by subjects were independent of assumed health state. This result supports the utility independence of survival duration from health state.

Miyamoto and Eraker (1988) did not test the utility independence of health state from survival duration. Instead, they noted that for most individuals, survival duration and health state are sign dependent attributes. To see this, note that preference for survival duration is an increasing function of duration when desirable health states are assumed, but when the assumed health state is extremely undesirable, less duration can be preferable to more; i.e., some health states are regarded as worse than death. Furthermore, when survival duration is zero, the preference ordering over health state degenerates—all health states are equally desirable or undesirable when immediate death (duration = 0) is the associated duration. These patterns of preference are characteristic of sign dependence (Krantz & Tversky, 1971). It is possible to show that if survival duration and health quality are sign dependent, and if survival duration is utility independent of health quality, then the utility function must be multiplicative (Miyamoto, 1985). Empirical support for the utility independence of survival duration from health quality thus supports the hypothesis that the utility of survival duration and health quality is multiplicative.

Miyamoto and Eraker (in press) tested axioms for the log/power and linear/exponential families of utility functions. As in the previous study, they had subjects judge the certainty matches of even-chance gambles for survival duration of the form $(X\,.5\,Y)$, where $X >_p Y$. Subjects were instructed to assume that health state was approximately constant and very good. In terms of certainty matching, the assumption that the preference ordering is preserved under scalar multiplication asserts that X is the certainty match of $(Y\,.5\,Z)$ iff sX is the certainty match of $(sY\,.5\,sZ)$. Similarly, the assumption that the preference ordering is preserved under translation asserts that X is the certainty match of $(Y\,.5\,Z)$ iff $X+s$ is the certainty match of $(Y+s\,.5\,Z+s)$. Of 38 subjects, 29 (76%) violated the assumption that certainty matches are preserved under scalar multiplication and 22 (58%) violated the assumption that they are preserved under translation. Eighteen subjects (47%) violated both assumptions. It is clear that neither parametric class of models is generally valid for the utility of survival duration.

An interesting feature of this experiment was that an attempt was made to determine each subject's reference level for survival. Recall that prospect theory postulates that outcomes are coded as gains or losses. By definition, the reference level is the boundary between gains and losses. Miyamoto and Eraker (in press) asked subjects to state a duration c_0 such that any survival greater than c_0 would be regarded as a gain, while any survival less than c_0 would be regarded as loss. Subjects appeared to find the question meaningful after some explanation. Determination of a subject's reference level was important for the analysis in terms of prospect theory because the reference level determines which gambles in the stimulus set are evaluated as regular or irregular prospects.

A subset of subjects gave reference levels that were either greater than the outcomes of every gamble in the stimulus set or else less then the outcome of very gamble in the stimulus set. According to prospect theory, these subjects should evaluate every stimulus gamble by the same combination rule, because the stimulus gambles had the form $(X .5\ Y)$ where $X >_p Y \geqslant_p c_0$ for every gamble , or $c_0 \geqslant_p X >_p Y$ for every gamble. As demonstrated in the Introduction, prospect theory implies that the preferences of such subjects should satisfy a generic utility representation. There were 31 such subjects, of whom 24 (77%) violated the assumption that certainty matches are preserved under scalar multiplication, 21 (68%) violated the assumption that certainty matches are preserved under translation, and 17 (55%) violated both assumptions. Therefore the parametric utility models are also generally rejected for the subset of subjects whose data can be interpreted from the standpoint of prospect theory.

The last result is interesting for it shows how one can test parametric utility hypotheses in prospect theory. Tests of parametric utility hypotheses in the generic utility framework can be interpreted as tests within prospect theory for any subject whose reference level is outside the range of the gamble outcomes. Miyamoto and Eraker (1988) also attempted to determine individual subject reference levels in the test of the utility independence of survival duration from health quality. They distinguished a subset of subjects for whom every stimulus gamble was an irregular gamble. The majority of such subjects (11 of 17) satisfied the utility independence of survival duration from health quality.

The examples illustrate the fact that if a utility model is tested within the generic utility framework, the test is interpretable from the standpoint of any theory that implies the validity of the generic utility representation. Thus, the tests of utility models cited above are interpretable in terms of EU, SEU, SWU, NASEU, and ASEU theories, and the dual bilinear model. Furthermore, the results for subjects whose reference levels were outside the range of stimulus gamble outcomes were also interpretable from the standpoint of prospect theory.

Conclusions

There are three major problems in the theory of preference under risk. The first is to analyze the mental or subjective representation of probabilities and uncertainties. The second is to investigate the subjective value of outcomes or consequences. The third is to describe how these two components are integrated in judging the worth of gambles or in choosing between gambles. The main purpose of the generic utility theory is to provide a framework within which the second question, the subjective value of outcomes, can be addressed. The generic utility theory was formulated to avoid rather than to solve problems arising in the first and third domains. Of course, one would rather have a valid theory that accounted for preference in a great variety of gambling situations, but empirical psychological studies have revealed patterns of preference behavior that are difficult to incorporate into existing theories (Grether & Plott, 1979; Kahneman & Tversky, 1979, 1981; Slovic & Lichtenstein, 1983). The primary virtue of the generic utility theory is that it is weak enough to be implied by many theories of preference under risk, yet it is strong enough to formalize important utility models. Theoretical and empirical results developed within the generic utility framework are interpretable from the standpoint of stronger theories without being limited to the assumptions of these theories. The generic utility theory thus helps to disentangle questions of utility from issues concerning the integration of subjective probability and utility. It is hoped that the formulation of a generic utility theory will encourage theoretical and applied investigations of utility by providing testable formulations of utility models that are neutral (or as neutral as possible) with respect to the representation of subjective probability.

I have demonstrated that additive and multiplicative representations and parametric representations can be axiomatized in the generic utility framework. Although the additive and multiplicative formalizations were limited to the two-attribute case, the generalization to arbitrarily many attributes is straightforward. Furthermore, when more than two attributes are considered, a new kind of representation called a multilinear utility function can also be derived from utility independence assumptions (Keeney & Raiffa, 1976). The formalization of additive, multiplicative, and multilinear utility functions of arbitrarily many attributes is like the two-attribute case in that the different representations are characterized by classes of isomorphisms between utility substructures (Miyamoto, 1983). The generic utility theory provides a framework for these axiomatizations because the relevant utility substructures and isomorphisms are definable within the generic utility theory and because the utility measure of the generic utility representation is an interval scale.

The last point deserves some elaboration. Clearly, utility independence assumptions and the axioms for the parametric utility models could be introduced into prospect theory or the dual bilinear model without developing the present axiomatization of the generic utility theory. In the absence of the generic utility theory, however, it would not be apparent whether these assumptions imply the corresponding utility models. Each of these assumptions implies the existence of a family of isomorphisms between substructures of an overall preference structure, where the basic set of these substructures is a triangular set of gambles. *If the restriction of the utility function to these substructures is an interval scale*, then compositions of isomorphisms with the utility function satisfy functional equations that characterize utility models (see Eqs. (54) and (55)). Here we see the critical importance of the interval scale uniqueness of the generic utility representation. Without the generic utility theory, one cannot establish that the restriction of a utility measure to a triangular set of gambles is itself an interval scale, and without this condition, one cannot develop the functional equations that characterize the utility models formalized here. The uniqueness theorem of the generic utility theory is central to the introduction of multiattribute and parametric utility models into prospect theory and the dual bilinear, at least insofar as the formalization of these models is based on functional equations of the form (55), because the derivation of these equations requires that restrictions of the utility scale to triangular subsets of gambles be interval scales. It has not previously been shown how to formalize additive and multiplicative utility models and parametric utility models in prospect theory or the dual bilinear model. As demonstrated here, the formalizations are routine if we consider generic utility representations embedded within the stronger theories.

Finally, I should mention the possibility that the generic utility theory may provide a useful tool in the axiomatic analysis of the dual bilinear model. The standard additive conjoint measurement theory (Krantz, 1964; Krantz *et al.*, 1971; Luce & Tukey, 1964) does not apply in a simple way to this problem, because it formalizes an additive representation on a Cartesian product $X \times X$, while the dual bilinear model postulates different additive representations for the lower triangle and upper triangle of a Cartesian product. The generic utility theory provides a step toward an axiomatization of the dual bilinear model. It axiomatizes the special case of the dual bilinear model where the uncertain event is held fixed, and the preference order is restricted to a lower or upper triangle of two-outcome gambles for simple outcomes (i.e., not gambles for gambles). There are at least two directions in which this case must be generalized in the study of the dual bilinear model. First, if uncertain events are allowed to vary, it is necessary to formulate conditions that imply the consistency of the utility functions and subjective probability representations across different choices of events. Second, it is necessary to represent the utility of gambles whose outcomes are themselves gambles. No attempt will be made to develop the requisite theory here (cf. Luce, 1986), but I would suggest that the generic utility theory appears to be suited to the task of axiomatizing the dual bilinear model because the dual bilinear representation can be viewed as a system of interlocking generic utility representations.

APPENDIX I: Proof of Theorem 1

An informal sketch of the proof was presented earlier in the paper. The formal details are given here.

LEMMA 1. *Let* $(A, \succcurlyeq_g)$ *be an* LTA *structure, and let* P *be defined as in Definition* 7. *Then,* $\succcurlyeq_g$ *is a weak ordering of* P.

Proof. Choose any $ax, by \in P$. By definition of P, $aa \geqslant_g xx$ and $bb \geqslant_g yy$. Therefore either $ax \geqslant_g by$ or $by \geqslant_g ax$, according to Axiom 2. Hence $\geqslant_g$ is connected on P. Note that if $ax \geqslant_g by$ for any $a, b, x, y \in A$, then $aa \geqslant_g xx$ and $bb \geqslant_g yy$ from Axiom 2. Hence $ax, by \in P$. So $\geqslant_g$ is defined precisely on the pairs in P. By Axiom 3, $\geqslant_g$ is transitive, so $\geqslant_g$ is a weak ordering of P. ∎

LEMMA 2. *If $(A, \geqslant_g)$ is an* LTA *structure, then the following five conditions hold*:

(i) *For any $a, b, x, y \in A$, if $ax, ay, bx, by \in P$, then $ax \geqslant_g bx$ iff $ay \geqslant_g by$.*

(ii) *For any $a, b, x, y \in A$, if $ax, ay, bx, by \in P$, then $ax \geqslant_g ay$ iff $bx \geqslant_g by$.*

(iii) *For any $a, b, x \in A$, if $ax, bx \in P$, then $ax \geqslant_g bx$ iff $aa \geqslant_g bb$ iff $ab \in P$.*

(iv) *For any $a, b, y \in A$, if $ya, yb \in P$, then $ya \geqslant_g yb$ iff $aa \geqslant_g bb$ iff $ab \in P$.*

(v) *For any $a, b, c \in A$, if $ab, bc \in P$, then $ac \in P$.*

Proof. To prove (i), choose any $a, b, x, y \in A$ such that $ax, ay, bx, by \in P$. Let $w = a$ if $ab \in P$, and $w = b$ otherwise. In either case $wa, wb \in P$. By Axiom 4, $ax \geqslant_g bx$ iff $wa \geqslant_g wb$ iff $ay \geqslant_g by$. To prove (ii), choose any $a, b, x, y \in A$ such that $ax, ay, bx, by \in P$, and let $c = y$ if $xy \in P$, and $c = x$ otherwise. Then, $xc, yc \in P$, and by Axiom 4, $ax \geqslant_g ay$ iff $xc \geqslant_g yc$ iff $bx \geqslant_g by$.

To prove (iii), first suppose that $ax, bx \in P$ and $ax \geqslant_g bx$. I want to show that $aa \geqslant_g bb$ and $ab \in P$. Suppose $bb >_g aa$. Then $ba \in P$. Applying Axiom 4 to $ax \geqslant_g bx$ yields $ba \geqslant_g bb$ and $aa \geqslant_g ba$. Thus $aa \geqslant_g bb$, contradicting $bb \geqslant_g aa$. Thus $aa \geqslant_g bb$. Also, $ab \in P$ by definition of P. Hence one direction of the implication is proved. Now suppose that $ax, bx, ab \in P$. Then $aa \geqslant_g bb$ by definition of P. If $bx >_g ax$, then $ab >_g aa$ by Axiom 4, and $bb >_g ab$ by part (i), so $bb >_g aa$, contradicting $aa \geqslant_g bb$. Hence $ax \geqslant_g bx$, and (iii) is proved. The proof of part (iv) is similar. To prove part (v), if $ab, bc \in P$, then $aa \geqslant_g bb \geqslant_g cc$ by definition of P. Hence $ac \in P$ by transitivity of $\geqslant_g$ (Axiom 3). ∎

For any $x \in A$, define subsets U_x and D_x of A by the conditions

$$b \in U_x \quad \text{iff} \quad bx \in P$$

$$y \in D_x \quad \text{iff} \quad xy \in P.$$

Mnemonically, U_x and D_x can be thought of as Up-x and Down-x, respectively, i.e., the elements above x and below x in the ordering on A induced by $\geqslant_g$. Define a subset K of A by the condition $x \in K$ iff there exist $b, c \in U_x$ and $y, z \in D_x$ such that $bx \not\sim_g cx$ and $xy \not\sim_g xz$. K is the set of $x \in A$ such that neither U_x nor D_x is degenerate in the ordering induced by $\geqslant_g$. If $ax \in P$, then let $\geqslant_{ax}$ denote the restriction of $\geqslant_g$ to $U_a \times D_x$. The following lemma shows why the set K is important.

LEMMA 3. *The set K defined above is nonempty. Moreover, if $a, x \in K$ and $ax \in P$, then the structure $(U_a \times D_x, \geqslant_{ax})$ is an additive conjoint structure in the sense of Definition* 6.7 *of Krantz et al.* (1971, p. 256).

Proof. To show that $K \neq \varnothing$, Axiom 8 implies that there exist $a, b, x, y \in A$ such that $ax, bx, ay \in P$, $ax >_g ay$, and $ax >_g bx$. Then Lemma 2 (iv) implies that $xy \in P$, so Lemma 2 (v) implies that $by \in P$. Hence Lemma 2 (ii) implies that $bx >_g by$. Also, $ax >_g bx$ implies $ab >_g bb$ by Axiom 4 and Lemma 2 (i). Therefore $b \in K$ by definition of K, so $K \neq \varnothing$.

Let (F1)–(F6) denote the six axioms of Definition 6.7 of Krantz *et al.* (1971). I first show that Axiom F1 is satisfied. If $bz \in U_a \times D_x$, then $ba, xz \in P$. Hence $ax, bx, bz \in P$ by Lemma 2 (v). Thus $U_a \times D_x \subseteq P$. Hence $\geqslant_{ax}$ is a weak ordering of $U_a \times D_x$ because $\geqslant_g$ is a weak ordering of P. Thus F1 is satisfied. The $\geqslant_{ax}$ relation satisfies independence (F2) and the Thomsen condition (F3) because Lemma 2 and Axiom 5 assert that these conditions are satisfied by pairs in P, and $U_a \times D_x \subseteq P$. To establish solvability (F4), suppose that $bq, cp, dq \in U_a \times D_x$ and $bq \geqslant_{ax} cp \geqslant_{ax} dq$. Then $bq \geqslant_g cp \geqslant_g dq$ by definition of $\geqslant_{ax}$. By Axiom 6 there exists $e \in A$ such that $eq \sim_g cp \geqslant_g dq$. Hence $ee \geqslant_g dd$ by Lemma 2 (iii). Since $d \in U_a$, $dd \geqslant_g aa$. Thus $e \in U_a$, and $eq \in U_a \times D_x$. An analogous proof shows that if $bp, cq, br \in U_a \times D_x$ and $bp \geqslant_{ax} cq \geqslant_{ax} br$, then there exists $s \in D_x$ such that $bs \sim_{ax} cq$ and $bs \in U_a \times D_x$. Thus Axiom F4 is satisfied. The Archimedean property (F5) follows from Axiom 7 of Definition 6 and the definition of $\geqslant_{ax}$. Axiom F6 (each component is essential) follows from the fact that $a, x \in K$. ∎

As a special case of Lemma 3, if $a \in K$, then $(U_a \times D_a, \geqslant_{aa})$ is an additive conjoint structure. It will be useful to adopt a special notation for the additive scales for such a structure. If $a \in K$, let $S(a)$ denote the set of all pairs of functions that constitute additive scales for $(U_a \times D_a, \geqslant_{aa})$. In other words, $(\chi, \tau) \in S(a)$ iff $\chi: U_a \to \mathrm{Re}$, $\tau: D_a \to \mathrm{Re}$, and

$$bx \geqslant_{aa} cy \qquad \text{iff} \quad \chi(b) + \tau(x) \geqslant \chi(c) + \tau(y)$$

for every $bx, cy \in U_a \times D_a$. Suppose that we choose an arbitrary $a \in K$ and $(\chi, \tau) \in S(a)$. The next lemma establishes that for any $b \in K$, there exist scales $(\chi', \tau') \in S(b)$ such that $\chi(x) = \chi'(x)$ for every x in the domains of both χ and χ', and $\tau(y) = \tau'(y)$ for every y in the domains of both τ and τ'.

LEMMA 4. *Let $a \in K$ and scales $(\chi, \tau) \in S(a)$ be chosen arbitrarily. Then for any $b \in K$, there exist scales $(\chi', \tau') \in S(b)$ such that $\chi(x) = \chi'(x)$ for every $x \in U_a \cap U_b$, and $\tau(y) = \tau'(y)$ for every $y \in D_a \cap D_b$.*

Proof. Choose any $b \in K$ and $(\chi', \tau') \in S(b)$. Consider first the case where $ba \in P$. Note that $U_b \subseteq U_a$, and $D_b \supseteq D_a$. Therefore $wx \geqslant_{ba} yz$ iff $wx \geqslant_{bb} yz$ iff $wx \geqslant_{aa} yz$ for any $w, y \in U_b$ and $x, z \in D_a$. Therefore

$$\begin{aligned} wx \geqslant_{ba} yz \qquad &\text{iff} \quad \chi'(w) + \tau'(x) \geqslant \chi'(y) + \tau'(z) \\ &\text{iff} \quad \chi(w) + \tau(x) \geqslant \chi(y) + \tau(z) \end{aligned}$$

for any $wx, yz \in U_b \times D_a$. Then $(\chi \mid U_b, \tau)$ and $(\chi', \tau' \mid D_a)$ are two pairs of additive scales for $(U_b \times D_a, \geqslant_{ba})$. By the uniqueness of the additive conjoint representation, there exist constants $\alpha > 0$, β_1, and β_2 such that $\chi \mid U_b = \alpha\chi' + \beta_1$ and $\tau = \alpha\tau' + \beta_2$. Let $\chi'' = \alpha\chi' + \beta_1$ and $\tau'' = \alpha\tau' + \beta'$. Then $(\chi'', \tau'') \in S(b)$, and $\chi''(x) = \chi(x)$ for every $x \in U_a \cap U_b$, and $\tau''(y) = \tau(y)$ for every $y \in D_a \cap D_b$. This proves the lemma for the case where $ba \in P$. The proof for the case where $ab \in P$ is similar and will be omitted. ▌

It is useful to have a special notation for functions that are related as in Lemma 4. If f and g are any functions, then $f \cong g$ will denote that $f(x) = g(x)$ for every x in the intersection of their domains. In this notation, Lemma 4 asserts that for any $a \in K$, $(\chi, \tau) \in S(a)$, and $b \in K$, there exist $(\chi', \tau') \in S(b)$ such that $\chi \cong \chi'$, and $\tau \cong \tau'$.

The next two lemmas show that if $a \in K$ and $(\chi, \tau) \in S(a)$, then $\chi(U_a)$ and $\tau(D_a)$ are dense in intervals of real numbers. (A set X is dense in the interval J iff for every $y, z \in J$, if $y > z$, then there exists $x \in X$ such that $y > x > z$.)

LEMMA 5. *Let $a \in K$ and $(\chi, \tau) \in S(a)$. Then for any $\varepsilon > 0$ there exist $b, c \in U_a$ and $x, y \in D_a$ such that $\varepsilon > |\chi(b) - \chi(c)| > 0$, and $\varepsilon > |\tau(x) - \tau(y)| > 0$.*

Proof. I will only give the proof for χ. The proof for τ is similar. By definition of K, there exist $b_1, c \in U_a$ such that $ca >_g b_1 a$. Using Axiom 9, we can choose a sequence $b_2, b_3, b_4, \ldots$ satisfying $ca >_g b_{i+1} a >_g b_i a$ for every integer $i > 0$. Since χ is a scale for $(U_a \times D_a, \geqslant_{aa})$, we have $\chi(c) > \chi(b_{i+1}) > \chi(b_i)$ for every $i > 0$. But then the sequence $(\chi(b_i))_{i>0}$ is strictly increasing and bounded above by $\chi(c)$. Let d denote its least upper bound. Then for any $\varepsilon > 0$, there exists an integer k such that $\varepsilon > d - \chi(b_n) > 0$ for every $n > k$. Hence $\varepsilon > \chi(b_{n+1}) - \chi(b_n) > 0$ for every $n > k$. ▌

LEMMA 6. *Let $a \in K$ and let $(\chi, \tau) \in S(a)$.* (i) *If X is the smallest interval containing $\chi(U_a)$, then $\chi(U_a)$ is dense in X.* (ii) *If Y is the smallest interval containing $\tau(D_a)$, then $\tau(D_a)$ is dense in Y.*

Proof. I will prove part (i). The proof of part (ii) is similar. Choose any $u, v \in X$ such that $u > v$. I need to show that there exists $\chi(b) \in [v, u]$ for some $b \in U_a$. Choose $c, d_1 \in U_a$ such that $\chi(c) \geqslant u$ and $v \geqslant \chi(d_1)$. Such c and d_1 must exist because otherwise X would not be the least interval containing $\chi(U_a)$. Let $\varepsilon = u - v$. By Lemma 4, we can choose $w, x \in D_a$ such that $\varepsilon > \tau(w) - \tau(x) > 0$. Then $\chi(c) - \chi(d_1) > \tau(w) - \tau(x) > \chi(d_1) - \chi(d_1)$, so $\chi(c) + \tau(x) > \chi(d_1) + \tau(w) > \chi(d_1) + \tau(x)$. Hence, $cx >_{aa} d_1 w >_{aa} d_1 x$, so $cx >_g d_1 w >_g d_1 x$. By Axiom 6, there exists d_2 such that $d_2 x \sim_g d_1 w$. Therefore $d_2 x \sim_{aa} d_1 w$, so $\chi(d_2) - \chi(d_1) = \tau(w) - \tau(x)$. If $\chi(d_2) < v$, we can repeat the process to find d_3 such $\chi(d_3) - \chi(d_2) = \tau(w) - \tau(x)$. Let $(d_i)_{i>0}$ be a sequence constructed as above such that $\chi(d_{i+1}) - \chi(d_i) = \tau(w) - \tau(x)$ for every i. By Axiom 7 the sequence $(d_i)_{i>0}$ is finite. Let d_n be its last member. We must have $\chi(c) - \chi(d_n) < \tau(w) - \tau(x)$, for otherwise we could find d_{n+1} to continue the sequence. Therefore $\varepsilon > \chi(c) - \chi(d_n) \geqslant 0$. Since $\chi(c) > u$ and $u - v = \varepsilon$, we must have

$\chi(d_n) > v$. If $u \geq \chi(d_n)$, we are done. If $\chi(d_n) > u$, there must exist some d_k such that $u \geq d_k \geq v$, because $\varepsilon > \chi(d_{i+1}) - \chi(d_i) > 0$ for every i. This completes the proof of part (i). ▮

The next lemma is the central result in the proof of the Theorem 1. The lemma establishes that the "upward" and "downward" scales of different additive representations are linear with respect to each other on the overlap of their domains. Stated symbolically, if $a, b \in K$, $bb >_g aa$, $(\chi, \tau) \in S(a)$, and $(\chi', \tau') \in S(b)$, then χ is linear with respect to τ' on $U_a \cap D_b$. The lemma actually establishes a stronger result—for any $a \in K$ and $(\chi, \tau) \in S(a)$ there exist $\lambda > 0$ and v such that if $b \in K$, $(\chi', \tau') \in S(b)$, $\chi' \cong \chi$, $\tau' \cong \chi$, then $\tau'(x) = \lambda\chi(x) + v$ for every $x \in U_a \cap D_b$, and $\tau(y) = \lambda\chi'(y) + v$ for every $y \in D_a \cap U_b$. What is important about this result is that a single choice of λ and v determines a linear relationship between χ and τ', and between χ' and τ, for every $b \in K$ and $(\chi', \tau') \in S(b)$ such that $\chi' \cong \chi$ and $\tau' \cong \chi$. The method of proof borrows from Krantz *et al.*'s (1971) proof of the IC additive representation.

LEMMA 7. *For any $a \in K$ and $(\chi, \tau) \in S(a)$ there exist constants $\lambda > 0$ and v such that if $c \in K$, $(\chi', \tau') \in S(c)$, $\chi' \cong \chi$, and $\tau' \cong \chi$, then $\tau'(x) = \lambda\chi(x) + v$ for every $x \in U_a \cap D_c$, and $\tau(y) = \lambda\chi'(y) + v$ for every $y \in D_a \cap U_c$.*

Proof. I will first show that there exist λ and v that satisfy the lemma with respect to every $c \in K \cap U_a$. I will then show that there exist constants λ' and v' that satisfy the lemma with respect to every $c \in K \cap D_a$, and finally I will show that $\lambda = \lambda'$, and $v = v'$.

Choose any $a \in K$ and $(\chi, \tau) \in S(a)$. To derive values for λ and v, choose any $b \in K \cap U_a$ such that $da \not\sim_g aa$ for some $d \in U_a \cap D_b$. The element b must exist because $a \in K$. By Lemma 4 we can choose $(\chi', \tau') \in S(b)$ such that $\chi' \cong \chi$ and $\tau' \cong \tau$. Furthermore $\chi(U_a) \cap \tau'(D_b)$ must be dense in an interval of reals, because $\chi(U_a)$ and $\tau'(D_b)$ are both dense in intervals (by Lemma 6) and their intersection is nonempty. Now choose any $y, z \in U_a \cap D_b$ such that $\chi(y) > \chi(a)$, $\chi(z) > \chi(a)$, and $\chi(z) \neq \chi(y)$. I want to develop expressions for

$$\frac{\chi(y) - \chi(a)}{\chi(z) - \chi(a)} \quad \text{and} \quad \frac{\chi'(y) - \chi'(a)}{\chi'(z) - \chi'(a)}.$$

For any $\varepsilon > 0$, there exist $r, s \in D_a$ such that $\varepsilon > \tau(r) - \tau(s) > 0$. Let $\omega = \tau(r) - \tau(s)$. By Lemma 6, we can choose r and s to make ω arbitrarily small. Using r and s, we can construct a sequence $e_0, e_1, ..., e_n$ of elements of U_a such that $e_0 = a$, the sequence $\chi(e_0), \chi(e_1), ..., \chi(e_n)$ is strictly monotonic increasing, $\omega = \chi(e_{i+1}) - \chi(e_i)$ for every i $(1 \geq i \geq n)$, and $\omega > \chi(y) - \chi(e_n) \geq 0$. The method for constructing the sequence is the basic method for constructing standard sequences in an additive conjoint structure (Krantz *et al.*, 1971, Chap. 6), so it will not be described here.

Note that $\chi(y) \geq \chi(e_i)$ for every i. Therefore $\chi(y) + \tau(a) \geq \chi(e_i) + \tau(a)$ and $ya \geq_g e_i a$ and $yy \geq_g e_i e_i$ for every i. But $y \in D_b$, so $bb \geq_g yy$ Hence $bb \geq_g e_i e_i$ and

$e_i \in D_b$ for every i. The fact that $e_i \in D_b$ will be used later when I show that the sequence $\tau'(e_0), \tau'(e_1), ..., \tau'(e_n)$ is also equally spaced.

Since the number n of elements in the standard sequence is a function of the size ω of the spacing between adjacent elements, it will be convenient to denote $n = n(\omega)$. Let $\delta(\omega) = \chi(y) - \chi(e_{n(\omega)})$, and recall that $\omega = \chi(e_{i+1}) - \chi(e_i)$ for every i $(1 \leqslant i < n)$. In this notation, we have $\chi(y) - \chi(a) = n(\omega)\omega + \delta(\omega)$. Construct a second sequence $f_0, f_1, ..., f_m$ of elements of U_a such that $f_0 = a$, the sequence $\chi(f_0), \chi(f_1), ..., \chi(f_m)$ is strictly monotonic increasing, $\omega = \chi(f_{i+1}) - \chi(f_i)$ for every i $(1 \leqslant i < m)$, and $\omega > \chi(z) - \chi(f_m) \geqslant 0$. Since m is also a function of ω, let $m = m(\omega)$. Let $\theta(\omega) = \chi(z) - \chi(f_m)$. We have $\omega > \theta(\omega) \geqslant 0$, and $\chi(z) - \chi(a) = m(\omega)\omega + \theta(\omega)$. Therefore,

$$\frac{\chi(y) - \chi(a)}{\chi(z) - \chi(a)} = \frac{n(\omega)\omega + \delta(\omega)}{m(\omega)\omega + \theta(\omega)}$$

$$= \frac{n(\omega) + \delta(\omega)/\omega}{m(\omega) + \theta(\omega)/\omega}.$$

Note that as ω approaches 0, $n(\omega)$ and $m(\omega)$ approach infinitely while $\delta(\omega)/\omega$ and $\theta(\omega)/\omega$ are bounded by 0 and 1. The proofs of the additive conjoint representation theorem and Holder's theorem (Krantz *et al.*, 1971) show that the following limits exist and satisfy the relations,

$$\frac{\chi(y) - \chi(a)}{\chi(z) - \chi(a)} = \operatorname*{Lim}_{\omega \to 0} \frac{n(\omega) + \delta(\omega)/\omega}{m(\omega) + \theta(\omega)/\omega} \tag{59}$$

$$= \operatorname*{Lim}_{\omega \to 0} \frac{n(\omega)}{m(\omega)}. \tag{60}$$

Now we must derive an analogous expression for $(\tau'(y) - \tau'(a))/(\tau'(z) - \tau'(a))$. To do this, we must show that the sequence $\tau'(e_0), \tau'(e_1), ..., \tau'(e_{n(\omega)})$ determines a sequence of equal intervals. Hence, we need to find $r', s' \in U_b$ such that $r'e_{i+1} \sim_{bb} s'e_i$ for every i $(1 \leqslant i \leqslant n(\omega) - 1)$. The method for finding r' and s' is slightly indirect. The sequence $\chi(e_0), \chi(e_1), \chi(e_2), ..., \chi(e_{n(\omega)})$ is strictly monotonic increasing by hypothesis. Hence $ya >_g e_{i+1}a >_g e_ia$ for every i. By Axiom 4, by $>_g be_{i+1} >_g be_i$ for every i. Thus, the sequence of $\tau'(e_i)$ is strictly monotonic increasing, bounded by $\tau'(e_0)$ and $\tau'(y)$, and $n(\omega) \to \infty$ as $\omega \to 0$. Choose any $w, v \in U_b$ such that $wb >_g vb$. Let $\varepsilon = \chi'(w) - \chi'(v) > 0$. If $n(\omega)$ is sufficiently large, there must be some e_k and e_{k+1} in the sequence such that $\tau'(e_{k+1}) - \tau'(e_k) < \varepsilon$. Hence

$$\chi'(w) - \chi'(v) > \tau'(e_{k+1}) - \tau'(e_k) > \chi'(v) - \chi'(v) = 0.$$

Therefore $\chi'(w) + \tau'(e_k) > \chi'(v) + \tau'(e_{k+1}) > \chi'(v) + \tau'(e_k)$. Thus, $we_k >_g ve_{k+1} >_g ve_k$. By Axiom 6, there exists $s' \in A$ such that $s'e_k \sim_g ve_{k+1}$. Note that $s' \in U_b$ because $v \in U_b$ and $s'e_k \geqslant_g ve_k \geqslant_g be_k$. Let $r' = v$. Then $r'e_{k+1} \sim_g s'e_k$.

By Axiom 10, we must have $r'e_{i+1} \sim_g s'e_i$ for every i such that $1 \leqslant i \leqslant n(\omega) - 1$. Let $\sigma(\omega) = \tau'(e_{i+1}) - \tau'(e_i) > 0$ for any i. Then we must have $\sigma(\omega) > \tau'(y) - \tau'(e_{n(\omega)}) \geqslant 0$, for otherwise we could use restricted solvability (Axiom 6) to find $e_{n(\omega)+1}$ such that $\tau'(e_{n(\omega)+1})$ is between $\tau'(y)$ and $\tau'(e_{n(\omega)})$ and $\sigma(\omega) = \tau'(e_{n(\omega)+1}) - \tau'(e_{n(\omega)})$. But then Axioms 4 and 10 would imply that $\chi(e_{n(\omega)+1})$ is between $\chi'(y)$ and $\chi'(e_{n(\omega)})$ and $\omega = \chi'(e_{n(\omega)+1}) - \chi'(e_{n(\omega)})$, contrary to the definition of $e_{n(\omega)}$.

A similar argument shows that $f_0, f_1, ..., f_{m(\omega)}$ is a standard sequence on D_b such that $\tau'(f_{i+1}) - \tau'(f_i) = \sigma(\omega)$, and $\sigma(\omega) > \tau'(z) - \tau'(f_{m(\omega)}) \geqslant 0$.

Let $\gamma(\omega) = \tau'(y) - \tau'(e_{n(\omega)})$. Let $\pi(\omega) = \tau'(z) - \tau'(f_{m(\omega)})$. Now the same argument used to derive (59) and (60) can be used to show that

$$\frac{\tau'(y) - \tau'(a)}{\tau'(z) - \tau'(a)} = \frac{n(\omega)\sigma(\omega) + \gamma(\omega)}{m(\omega)\sigma(\omega) + \pi(\omega)} \tag{61}$$

$$= \frac{n(\omega) + \gamma(\omega)/\sigma(\omega)}{m(\omega) + \pi(\omega)/\sigma(\omega)}. \tag{62}$$

A previously noted, $n(\omega)$ and $m(\omega)$ approach infinity as ω approaches zero, while $\gamma(\omega)/\sigma(\omega)$ and $\pi(\omega)/\sigma(\omega)$ are bounded by 0 and 1. Therefore,

$$\frac{\tau'(y) - \tau'(a)}{\tau'(z) - \tau'(a)} = \operatorname*{Lim}_{\omega \to 0} \frac{n(\omega) + \gamma(\omega)/\sigma(\omega)}{m(\omega) + \pi(\omega)/\sigma(\omega)} \tag{63}$$

$$= \operatorname*{Lim}_{\omega \to 0} \frac{n(\omega)}{m(\omega)}, \tag{64}$$

where the limits exist by the additive conjoint representation theorem and Holder's theorem. Since the limits in (60) and (64) are the same, we must have

$$\frac{\chi(y) - \chi(a)}{\chi(z) - \chi(a)} = \frac{\tau'(y) - \tau'(a)}{\tau'(z) - \tau'(a)}. \tag{65}$$

Define constants λ_{bz} and v_{bz} by

$$\lambda_{bz} = \frac{\tau'(z) - \tau'(a)}{\chi(z) - \chi(a)}$$

and

$$v_{bz} = -\lambda_{bz}\chi(a) + \tau'(a).$$

Note that $\lambda_{bz} > 0$ by Axiom 4. The values of λ_{bz} and v_{bz} depend possibly on the choice of b and z (and on the original choice of a), but not on the value of y. But then (65) implies that

$$\tau'(y) = \lambda_{bz}\chi(y) + v_{bz} \tag{66}$$

for any choice of $y \in U_a \cap D_b$ such that $\chi(y) > \chi(a)$. It is routine to verify that if $\chi(y) = \chi(a)$, then $\tau'(y) = \tau'(a)$ and thus Eq. (65) holds when $\chi(y) = \chi(a)$. Therefore (66) holds for every $y \in U_a \cap D_b$.

Let $\lambda = \lambda_{bz}$ and $\upsilon = \upsilon_{bz}$. Next we must show that for any $x \in U_a \cap K$ and any scales $(\chi'', \tau'') \in S(x)$, if $\chi'' \cong \chi$ and $\tau'' \cong \tau$, then

$$\tau''(y) = \lambda\chi(y) + \upsilon \tag{67}$$

for every $y \in U_a \cap D_x$. To prove this, choose any $x \in U_a \cap K$ and $(\chi'', \tau'') \in S(x)$ such that $\chi'' \cong \chi$ and $\tau'' \cong \tau$.

I note first that $\chi'' \cong \chi'$ and $\tau'' \cong \tau'$. (Recall that $(\chi', \tau') \in S(b)$.) To see this, choose $(\kappa, \rho) \in S(x)$ such that $\kappa \cong \chi'$ and $\rho \cong \tau'$. Since also $(\chi'', \tau'') \in S(x)$, there must exist $\alpha > 0$, β_1, and β_2 such that $\chi'' = \alpha\kappa + \beta_1$ and $\tau'' = \alpha\rho + \beta_2$. We must have $\chi''(z) = \kappa(z)$ for every $z \in U_b \cap U_x$ because $\chi'' \cong \chi$, $\chi \cong \chi'$, and $\chi' \cong \kappa$ and $U_b \cap U_x \subseteq U_a \cap U_b \cap U_x$. Furthermore there must exist $z, z' \in U_b \cap U_x$ such that $\chi''(z) \neq \chi''(z')$ because $U_b \subseteq U_b \cap U_x$ or $U_x \subseteq U_b \cap U_x$, and $b, x \in K$. Since $\alpha\kappa(z) + \beta_1 = \chi''(z) = \kappa(z)$ for all $z \in U_b \cap U_x$, we must have $\alpha = 1$ and $\beta_1 = 0$. Similarly, we must have $\tau''(w) = \rho(w)$ for every $w \in D_a$ because $\tau'' \cong \tau$, $\tau \cong \tau'$, and $\tau' \cong \rho$ and $D_a \subseteq D_b \cap D_x$. Since $\alpha = 1$, we have $\rho(w) = \tau''(w) = \rho(w) + \beta_2$, so $\beta_2 = 0$. Therefore $\chi'' = \kappa$, $\tau'' = \rho$, and we have $\chi'' \cong \chi'$ and $\tau'' \cong \tau'$ by the choice of κ and ρ.

To show that (67) holds, there are two cases to consider. Case 1: if $bb \geqslant_g xx$, then $U_a \cap D_x \subseteq U_a \cap D_b$. Since $\tau'' \cong \tau'$, we have $\tau''(y) = \tau'(y) = \lambda\chi(y) + \upsilon$ for every $y \in U_a \cap D_x$. Case 2: now suppose that $xx >_g bb$. By hypothesis, there exists $d \in U_a \cap D_b$ such that $da \not\sim_g aa$. So $d \in U_a \cap D_x$. We can therefore repeat the construction of λ_{bz} and υ_{bz} to find constants λ_{cv} and υ_{cv}, which depend possibly on a choice of $c, v \in U_a \cap D_x$, such that $\tau''(y) = \lambda_{cv}\chi(y) + \upsilon_{cv}$ for every $y \in U_a \cap D_x$. But then $\lambda_{cv}\chi(y) + \upsilon_{cv} = \tau''(y) = \tau'(y) = \lambda\chi(y) + \upsilon$ for every $y \in U_a \cap D_b$, because $\tau'' \cong \tau'$. Therefore $\lambda_{cv} = \lambda$ and $\upsilon_{cv} = \upsilon$, and $\tau''(y) = \lambda\chi(y) + \upsilon$ for every $y \in U_a \cap D_x$. Thus (67) is established. We may note in passing that we have also shown that the constants λ_{bz} and υ_{bz} in Eq. (66) do not actually depend on the choice of b and z.

Next we need the analogous result for elements $c \in K \cap D_a$. An argument that is entirely parallel to the preceding argument shows that there exist constants λ'' and υ'' such that if $c \in K \cap D_a$, and if $(\chi'', \tau'') \in S(c)$, $\chi'' \cong \chi$, and $\tau'' \cong \tau$, then $\tau(y) = \lambda''\chi''(y) + \upsilon''$ for every $y \in U_c \cap D_a$. The proof will be omitted here.

Finally, we need to show that $\lambda = \lambda''$ and $\upsilon = \upsilon''$. Choose $b \in K \cap U_a$ such that $ba \not\sim_g aa$. Choose $c \in K \cap D_a$ such that $ac \not\sim_g aa$. Choose $(\chi', \tau') \in S(b)$ and $(\chi'', \tau') \in S(b)$ and $(\chi'', \tau'') \in S(c)$ such that $\chi' \cong \chi \cong \chi''$ and $\tau' \cong \tau \cong \tau''$. We have already established that $\tau'(y) = \lambda\chi(y) + \upsilon$ for every $y \in U_a \cap D_b$, and $\tau(z) = \lambda''\chi''(z) + \upsilon''$ for every $z \in U_c \cap D_a$. We also have $\chi' \cong \chi''$ and $\tau' \cong \tau''$ because $U_b \subseteq U_a$ and $D_c \subseteq D_a$. Therefore if we repeat the preceding argument with b and c playing the roles previously played by b and a, we can show that there exist constants ρ and γ such that $\tau'(y) = \rho\chi''(y) + \gamma$ for every $y \in U_c \cap D_b$. But $\rho\chi''(y) + \gamma = \tau'(y) = \tau(y) = \lambda''\chi''(y) + \upsilon''$ for every $y \in U_c \cap D_a$ because $\tau' \cong \tau$. Since

$ac \not\succ_g aa$, we must have $\rho = \lambda''$ and $\gamma = \upsilon''$. Thus $\tau'(y) = \lambda''\chi''(y) + \upsilon''$ for every $y \in U_c \cap D_b$.

But now $\lambda''(\chi''(b) - \chi''(a)) = \tau'(b) - \tau'(a) = \lambda(\chi(b) - \chi(a))$. We have $\chi''(b) - \chi''(a) \neq 0$ by the choice of b. Also $\chi''(b) - \chi''(a) = \chi(b) - \chi(a)$ because $\chi'' \cong \chi$. Therefore $\lambda'' = \lambda$. Also, $\lambda''\chi''(b) + \upsilon'' = \tau'(b) = \lambda\chi(b) + \upsilon$, so $\upsilon'' = \upsilon$. Therefore λ and υ satisfy the assertion of the lemma. ∎

The next lemma is a minor strengthening of Lemma 6. It shows that the scales $(\chi, \tau) \in S(a)$ can be chosen such that $\upsilon = 0$ for the constant υ of Lemma 6.

LEMMA 8. *For any $a \in K$, there exist $(\chi, \tau) \in S(a)$ and a real constant $\lambda > 0$ such that if $c \in K$, and if $(\chi', \tau') \in S(c)$, $\chi' \cong \chi$, and $\tau' = \tau$, then $\tau'(y) = \lambda\chi(y)$ for every $y \in U_a \cap D_c$, and $\tau(z) = \lambda\chi'(z)$ for every $z \in U_c \cap D_a$.*

Proof. Choose any $(\chi'', \tau'') \in S(a)$, and let λ and υ be the constants that satisfy Lemma 7 with respect to χ'' and τ''. Define $\chi: U_a \to \text{Re}$ and $\tau: D_a \to \text{Re}$ by $\chi = \chi''$ and $\tau = \tau'' - \upsilon$. Then $(\chi, \tau) \in S(a)$ by the uniqueness theorem for the additive conjoint representation. To show that (χ, τ) satisfies the lemma, choose any $c \in K$, and $(\chi', \tau') \in S(c)$, such that $\chi' \cong \chi$ and $\tau' \cong \tau$. Note that $\chi' \cong \chi \cong \chi''$ and $\tau' + \upsilon \cong \tau + \upsilon \cong \tau''$ by definition of χ and τ. If $y \in U_a \cap D_c$, then $\tau'(y) + \upsilon = \lambda\chi''(y) + \upsilon$, and thus, $\tau'(y) = \lambda\chi(y)$. If $z \in U_c \cap D_a$, then $\tau(z) + \upsilon = \tau''(z) = \lambda\chi'(z) + \upsilon$. Therefore $\tau(z) = \lambda\chi'(z)$. Hence (χ, τ) satisfies the lemma. ∎

Lemma 8 lets us construct a function that satisfies the LTA representation. The next lemma shows how to define this function, and relates the function to the additive scales of substructures of the form $(U_b \times D_b, \succcurlyeq_{bb})$.

LEMMA 9. *Let $a \in K$ be chosen arbitrarily. Choose $(\chi, \tau) \in S(a)$ such that Lemma 8 is satisfied with respect to the constant $\lambda > 0$. Define a functions $\phi: A \to \text{Re}$ by*

$$\phi(x) = \begin{cases} \chi(x) & \text{if } xx \succcurlyeq_g aa \\ \tau(x)/\lambda & \text{if } aa \succ_g xx. \end{cases}$$

The function ϕ has the following property: For any $c \in K$, if $(\chi', \tau') \in S(c)$, $\chi' \cong \chi$, and $\tau' \cong \tau$, then $\chi'(x) = \phi(x)$ for every $x \in U_c$ and $\tau'(y) = \lambda\phi(y)$ for every $y \in D_c$.

Proof. Choose any $c \in K$. If $x \in U_a \cap U_c$, then $\chi'(x) = \chi(x) = \phi(x)$ because $\chi' \cong \chi$ and the definition of ϕ. If $x \in U_c \cap D_a$, then $\chi'(x) = \tau(x)/\lambda = \phi(x)$ by Lemma 8 and the definition of ϕ. If $y \in U_a \cap D_c$, then $\tau'(y) = \lambda\chi(y) = \lambda\phi(y)$ by Lemma 8 and the definition of ϕ. If $y \in D_a \cap D_c$, then $\tau'(y) = \tau(y) = \lambda\phi(y)$ because $\tau' \cong \tau$, and the definition of ϕ. Thus the lemma holds for every choice of $y \in A$. ∎

We are now in a position to prove Theorem 1. The main tools for the proof are Lemmas 7, 8, and 9.

Proof of Theorem 1. Choose any $a \in K$, and let $(\chi, \tau) \in S(a)$ satisfy Lemma 8 with respect to the constant $\lambda > 0$. Define $\phi: A \to \text{Re}$ as in Lemma 9. I claim that

$$wx \geqslant_g yz \quad \text{iff} \quad \phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z) \tag{68}$$

for any $w, x, y, z \in A$ such that $wx, yz \in P$. To establish (68), suppose that $wx, yz \in P$. Then $ww \geqslant_g xx$ and $yy \geqslant_g zz$ by definition of P. Cases 1–6 below are the only orderings of ww, xx, yy, and zz that are compatible with these orderings. (Cases 1–6 are not mutually exclusive, but that will not impair the argument.)

Case 1: $ww \geqslant_g xx \geqslant_g yy \geqslant_g zz$ *Case* 4: $yy \geqslant_g ww \geqslant_g xx \geqslant_g zz$

Case 2: $ww \geqslant_g yy \geqslant_g xx \geqslant_g zz$ *Case* 5: $yy \geqslant_g ww \geqslant_g zz \geqslant_g xx$

Case 3: $ww \geqslant_g yy \geqslant_g zz \geqslant_g xx$ *Case* 6: $yy \geqslant_g zz \geqslant_g ww \geqslant_g xx.$

Condition (68) holds iff it holds for these six cases. Fortunately, these cases do not all require separate proofs.

First note that for any $b, c \in A$, $bb \geqslant_g cc$, iff $\phi(b) \geqslant \phi(c)$. To see this, let $w = c$ if $bb \geqslant_g cc$, and $w = b$ if $cc >_g bb$. Choose $(\chi', \tau') \in S(w)$ such that $\chi' \cong \chi$ and $\tau' = \tau$. By Lemma 9, $\chi'(y) = \phi(y)$ for any $y \in U_w$. But now we have $bb \geqslant_g cc$ iff $bw \geqslant_g cw$ iff $\chi'(b) + \tau'(w) \geqslant \chi'(c) + \tau'(w)$ iff $\phi(b) \geqslant \phi(c)$.

Now I will prove that (68) holds in Cases 1 and 6. In Case 1, we must have $\phi(w) \geqslant \phi(x) \geqslant \phi(y) \geqslant \phi(z)$ by what we have just proved. Hence $\phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z)$. Furthermore $wx \geqslant_g wz \geqslant_g yz$ by Lemma 2. Therefore the only possibility that is consistent with Case 1 is that $wx \geqslant_g yz$, and $\phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z)$. Hence (68) holds in Case 1. In Case 6, we must have $\phi(y) \geqslant \phi(z) \geqslant \phi(w) \geqslant \phi(x)$. Hence $\phi(y) + \lambda\phi(z) \geqslant \phi(w) + \lambda\phi(x)$. Furthermore $yz \geqslant_g yx \geqslant_g wx$ by Lemma 2. Therefore the only possibility that is consistent with Case 6 is that $yz \geqslant_g wx$, and $\phi(y) + \lambda\phi(z) \geqslant \phi(w) + \lambda\phi(x)$. Hence (68) holds in Case 6.

Cases 2–5 can be treated together, so suppose that any of these cases holds. It is clear from inspection of Cases 2–5 that there must exist $b \in A$ such that both

$$yy \geqslant_g bb \geqslant_g xx \quad \text{and} \quad ww \geqslant_g bb \geqslant_g zz. \tag{69}$$

It may or may not be the case that there exists $b \in K$ that satisfies (69).

If there exists $b \in K$ that satisfies (69), then choose $(\chi', \tau') \in S(b)$ such that $\chi' \cong \chi$ and $\tau' \cong \tau$. (Recall that ϕ was defined in terms of $(\chi, \tau) \in S(a)$). By (69), $wx, yz \in U_b \times D_b$. Hence

$$\begin{aligned} wx \geqslant_g yz \quad &\text{iff} \quad wx \geqslant_{bb} yz \\ &\text{iff} \quad \chi'(w) + \tau'(x) \geqslant \chi'(y) + \tau'(z) \\ &\text{iff} \quad \phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z), \end{aligned}$$

where the second line follows by Lemma 3, and the third line follows by Lemma 9. Therefore (68) holds when there exists $b \in K$ that satisfies (69).

Finally suppose that no such $b \in K$ exists. Choose any $b \in A$ that satisfies (69). Since not $b \in K$, either $cb \sim_g db$ for every $c, d \in U_b$, or else $bc \sim_g bd$ for every $c, d \in D_b$. Since $w, y \in U_b$ and $x, z \in D_b$, we must have $wb \sim_g yb$, or $bx \sim_g bz$. By Lemma 2, either $ww \sim_g yy$, or $xx \sim_g zz$, or both. Hence $\phi(w) = \phi(y)$ or $\phi(x) = \phi(z)$. Suppose that $ww \sim_g yy$ and $\phi(w) = \phi(y)$. Then

$$\begin{aligned} wx \geqslant_g yz \quad \text{iff} \quad wx \geqslant_g wz \quad &\text{iff} \quad xx \geqslant_g zz \quad \text{iff} \quad \phi(x) \geqslant \phi(z) \\ &\text{iff} \quad \phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z). \end{aligned}$$

Alternatively, suppose that $xx \sim_g zz$ and $\phi(x) = \phi(z)$. Then

$$\begin{aligned} wx \geqslant_g yz \quad &\text{iff} \quad wz \geqslant_g yz \quad \text{iff} \quad ww \geqslant_g yy \quad \text{iff} \quad \phi(w) \geqslant \phi(y) \\ &\text{iff} \quad \phi(w) + \lambda\phi(x) \geqslant \phi(y) + \lambda\phi(z). \end{aligned}$$

Therefore (68) holds in Cases 2–5 whether or not there exist $b \in K$ such that (69) holds.

This proves the existence of an LTA representation. Finally, to establish the uniqueness of ϕ and λ, suppose that ϕ' and λ' are any other function and constant that satisfy the representation (17) (restated in (68)). Recall that ϕ and λ were defined in terms of an $a \in K$, and $(\chi, \tau) \in S(a)$. Choose any $b, c \in K$ such that $bb >_g cc >_g aa$. Choose scales $(\chi_b, \tau_b) \in S(b)$ and $(\chi_c, \tau_c) \in S(c)$ such that $\chi_b \cong \chi$, $\tau_b \cong \tau$, $\chi_c \cong \chi$, and $\tau_c \cong \tau$. Let $\phi \mid U_b$ and $\lambda\phi \mid D_b$ denote the restrictions of ϕ and $\lambda\phi$ to U_b and D_b, respectively, and define $\phi \mid U_c$, $\lambda\phi \mid D_c$, $\phi' \mid U_b$, $\lambda'\phi' \mid D_b$, $\phi' \mid U_c$, and $\lambda'\phi' \mid D_c$ similarly. By hypothesis, ϕ and ϕ' satisfy $(\phi \mid U_b, \lambda\phi \mid D_b) \in S(b)$, $(\phi' \mid U_b, \lambda'\phi' \mid D_b) \in S(b)$, $(\phi \mid U_c, \lambda\phi \mid D_c) \in S(c)$, and $(\phi' \mid U_c, \lambda'\phi' \mid D_c) \in S(c)$. By the uniqueness of the additive conjoint representation, there exist constants $\alpha, \beta, \gamma, \alpha', \beta'$, and γ' such that

$$\phi' \mid U_b = \alpha\phi \mid U_b + \beta, \qquad \lambda'\phi' \mid D_b = \alpha\lambda\phi \mid D_b + \gamma, \tag{70}$$

$$\phi' \mid U_c = \alpha'\phi \mid U_c + \beta', \qquad \lambda'\phi' \mid D_c = \alpha'\lambda\phi \mid D_c + \gamma'. \tag{71}$$

But $bb >_g cc$, so $U_b \subseteq U_c$. Hence $\alpha = \alpha'$, and $\beta = \beta'$. Similarly, $D_c \subseteq D_b$, so $\gamma = \gamma'$.

Since $bb >_g cc$, we must have that $\phi(U_c \cap D_b)$ and $\phi'(U_c \cap D_b)$ are both dense subsets of real intervals. Choose any $x \in U_c \cap D_b$. Then $\phi'(x) = \alpha\phi(x) + \beta$ by (67), and $\lambda'\phi'(x) = \alpha\lambda\phi(x) + \gamma$ by (71). Hence

$$\begin{aligned} \lambda'\phi'(x) &= \alpha\lambda\phi(x) + \lambda\beta - \lambda\beta + \gamma = \lambda(\alpha\phi(x) + \beta) - \lambda\beta + \gamma \\ &= \lambda\phi'(x) - \lambda\beta + \gamma. \end{aligned} \tag{72}$$

Since (72) holds for every $x \in U_c \cap D_b$, we must have $\lambda' = \lambda$ and $\gamma = \lambda\beta$. But then from (67), we have $\lambda\phi'(x) = \lambda(\alpha\phi(x) + \beta)$ for every $x \in D_b$, and from (71), we have $\phi'(y) = \alpha\phi(y) + \beta$ for every $y \in U_c$. But $A = U_c \cup D_b$, so $\phi'(x) = \alpha\phi(x) + \beta$ for every $x \in A$. This proves that $\phi' = \alpha\phi + \beta$ and that $\lambda' = \lambda$. This completes the proof of Theorem 1. ▮

REFERENCES

ACZEL, J. (1966). *Lectures on functional equations and their applications.* New York: Academic Press.

EDWARDS, W. (1962). Subjective probabilities inferred from decisions. *Psychological Review*, **69**, 109–135.

FAGOT, R. F. (1963). On the psychological law and estimation procedures in psychophysical scaling. *Psychometrika*, **28**, 145–160.

FISHBURN, P. C. (1965). Independence in utility theory with whole product sets. *Operations Research*, **13**, 28–45.

FISHBURN, P. C., & KEENEY, R. L. (1974). Seven independence concepts and continuous multiattribute utility functions. *Journal of Mathematical Psychology*, **11**, 194–327.

FISHBURN, P. C., & KEENEY, R. L. (1975). Generalized utility independence and some implications. *Operations Research*, **23**, 928–940.

GRETHER, D. M., & PLOTT, C. R. (1979). Economic theory of choice and the preference reversal phenomenon. *American Economic Review*, **69**, 623–638.

KAHNEMAN, D., & TVERSKY, A. (1979). Prospect theory: An analysis of decision under risk. *Econometrica*, **47**, 263–291.

KARMARKAR, U. S. (1978). Subjectively weighted utility: A descriptive extension of the expected utility model. *Organization Behavior and Human Performance*, **21**, 61–72.

KEENEY, R. L. (1968). Quasi-separable utility functions. *Naval Research Logistics Quarterly*, **15**, 551–565.

KEENEY, R. L. (1971). Utility independence and preferences for multiattributed consequences. *Operations Research*, **19**, 875–893.

KEENEY, R. L., & RAIFFA, H. (1976). *Decisions with multiple objectives: Preferences and value tradeoffs.* New York: Wiley.

KRANTZ, D. H. (1964). Conjoint measurement: The Luce–Tukey axiomatization and some extensions. *Journal of Mathematical Psychology*, **1**, 248–277.

KRANTZ, D. H., LUCE, R. D., SUPPES, P., & TVERSKY, A. (1971). *Foundations of measurement* (Vol 1). New York: Academic Press.

KRANTZ, D. H. & TVERSKY, A. (1971). Conjoint-measurement analysis of composition rules in psychology. *Psychological Review*, **78**, 151–169.

LUCE, R. D. (1959). On the possible psychological laws. *Psychological Review*, **66**, 81–95.

LUCE, R. D. (1986). Uniqueness and homogeneity of ordered relational structures. *Journal of Mathematical Psychology*, **30**, 391–415.

LUCE, R. D., & NARENS, L. (1985). Classification of concatenation measurement structures according to scale type. *Journal of Mathematical Psychology*, **29**, 1–72.

LUCE, R. D., & RAIFFA, H. (1957). *Games and decisions: Introduction and critical survey.* New York: Wiley.

LUCE, R. D., & TUKEY, J. W. (1964). Simultaneous conjoint measurement: A new type of fundamental measurement. *Journal of Mathematical Psychology*, **1**, 1–27.

MIYAMOTO, J. M. (1983). Measurement foundations for multiattribute psychophysical theories based on first order polynomials. *Journal of Mathematical Psychology*, **27**, 152–182.

MIYAMOTO, J. M. (1985). *The utility of survival duration and health quality: A conjoint measurement analysis.* Ann Arbor, MI: University Microfilms, Inc., Publication Number 8512473.

MIYAMOTO, J. M. (1987). Constraints on the representation of gambles in prospect theory. *Journal of Mathematical Psychology*, **31**, 410–418.

MIYAMOTO, J. M., & ERAKER, S. A. (1988). A multiplicative model of the utility of survival duration and health quality. *Journal of Experimental Psychology: General*, **117**, 3–20.

MIYAMOTO, J. M., & ERAKER, S. A. (in press). Parametric models of the utility of survival duration: Tests of axioms in a generic utility framework. *Organizational Behavior and Human Decision Processes.*

NARENS, L., & LUCE, R. D. (1986). Measurement: The theory of numerical assignments. *Psychological Bulletin*, **99**, 166–180.

PFANZAGL, J. (1959). A general theory of measurement: Applications to utility. *Naval Research Logistic Quaterly*, **6**, 283–294.

PRATT, J. W. (1964). Risk aversion in the small and in the large. *Econometrica*, **32**, 122–136.

RAIFFA, H. (1969). Preferences for multi-attributed alternatives. Santa Monica, C.A: The Rand Corp. RM–5868–DOT/RC.

ROSKIES, R. (1965). A measurement axiomatization for an essentially multiplicative representation of two factors. *Journal of Mathematical Psychology*, **2**, 266–276.

SLOVIC, P., & LICHTENSTEIN, S. (1983). Preference reversals: A broader perspective. *American Economic Review*, **73**, 569–605.

TVERSKY, A. (1967). Additivity, utility and subjective probability. *Journal of Mathematical Psychology*, **4**, 175–202.

TVERSKY, A., & KAHNEMAN, D. (1981). The framing of decisions and the psychology of choice. *Science*, **211**, 453–458.

VON NEUMANN, J., & MORGENSTERN, O. (1944, 1947, 1953). *Theory of games and economic behavior*. Princeton: Princeton Univ. Press.

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