Journal of Mathematical Economics 23 (1994) 451–473. North-Holland

A theory of expected utility with nonadditive probability

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Submitted March 1991, accepted February 1993

In this paper, a new concept of probability (*I*-nonadditive probability), which is partially additive and partially nonadditive, is introduced. By using this concept of probability and the former results of Savage's and Schmeidler's, a model of expected utility with nonadditive probability is formulated.

Key words: Nonadditive probability; Expected utility

1. Introduction

Expected utility with additive probability (EUAP) theories, e.g. Savage's (1954) and Anscombe and Aumann's (1963) are known as standard formulations of decision under uncertainty. But these EUAP theories cannot explain some phenomena in uncertain situations. The Ellsberg paradox is one example of such phenomena. This paradox is caused by the conflict between observations and the axioms under EUAP theories, especially substitution axiom. Let's take an example to illustrate this problem.

Example 1 (*the Ellsberg paradox*). Suppose there are two urns. Each urn contains four balls. Urn 1 contains two black balls and two red ones. Urn 2 contains four balls, either red or black, but there is no information about the ratio of the number of black balls to the number of red ones. In this case, there exist four acts.

1_B: betting on a black ball drawn from urn 1.
1_R: betting on a red ball drawn from urn 1.
2_B: betting on a black ball drawn from urn 2.
2_R: betting on a red ball drawn from urn 2.

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*I am grateful to Toshihide Mizuno and Mototsugu Fukushige for helpful discussions. I owe special thanks to Eric S. Maskin for helpful comments. The former version of this paper was presented at Far Eastern Meeting of Econometric Society at Seoul National University, 1991. Part of the research was done during a stay at Harvard University, Department of Economics.

SSDI 0304-4068(93)E0046-X

Table 1

	s_{BB}	s_{BR}	s_{RB}	s_{RR}
1_B	100	100	0	0
1_R	0	0	100	100
2_B	100	0	100	0
2_R	0	100	0	100

Table 2

Probabilities	Objective–subjective	Subjective–subjective	Subjective
Additive	Anscombe–Aumann		Savage
I-nonadditive		This paper	
Nonadditive	Schmeidler		Gilboa

If you draw a ball the color of which you have chosen, you will get $100. Otherwise you will get nothing.

There exist four states.

s_{BB}: a ball drawn from either urn is black.
s_{BR}: a ball drawn from urn 1 is black, while a ball drawn from urn 2 is red.
s_{RB}: a ball drawn from urn 1 is red, while a ball drawn from urn 2 is black.
s_{RR}: a ball drawn from either urn is red.

The relations between states and payoffs are shown in table 1. Empirical studies suggest that the following relation holds:

$$1_B \sim 1_R \succ 2_B \sim 2_R,$$

where $\succ$ and $\sim$ denote 'is preferred to', and 'is indifferent to' respectively. This result contradicts the following substitution axiom.

Substitution Axiom. Let S be the set of all states and X be the set of all consequences. And let F be the set of all functions from S to X. For all $f, g \in F$ and $A \subset S$ [$f = g$ and $f' = g'$ if A happens, and $f = f'$ and $g = g'$ if A^c happens] $\rightarrow$ ($f \succ g$ if and only if $f' \succ g'$).

If $1_B \succ 2_B$, then $2_R \succ 1_R$ is followed by the substitution axiom. But this result contradicts the former empirical one. Expected utility with nonadditive probability (EUNAP) theories, e.g. Schmeidler's (1984, 1989) and Gilboa's (1987) are able to resolve this paradox. These models use weakenings of the substitution axiom. Our model is one variation of these EUNAP theories. This model is classified as an intermediate one among four different models. (This is shown in table 2.)

Schmeidler formulated his model in the framework of Anscombe and Aumann (1963). His EUNAP model is a two stage model and uses objective–subjective probability. Gilboa (1987) formulated his EUNAP model in the framework of Savage (1954). His model uses purely subjective probability. Wakker (1989) and Nakamura (1990) also formulated different EUNAP models. Wakker studied an EUNAP model in the case of additive continuous utility, and Nakamura studied an EUNAP model in Savage's framework when the set of states is finite.

The purpose of this paper is to formulate a model, which is synthesizing Savage's model and Schmeidler's. The explanatory power of our EUNAP theory is not so much different from the explanatory power of other EUNAP theories. Because we use Schmeidler's theorem in one part of the proof of our theorem, our representation for acts is similar to his representation. The relation between Schmeidler's model and ours may be akin to the relation between von Neumann and Morgenstern's model and Savage's one. By using Savage's method of assigning the value of probability, we replace objective probabilities in Schmeidler's model by subjective ones. Gilboa's model is also using subjective probabilities, but Gilboa's representation is slightly different from ours because of the difference between his axioms and ours. So the implications of our model are different from those of Gilboa's in some cases. (See appendix).

In section 2, we define a new concept of probability (I-nonadditive probability), which is the amalgam of additive and nonadditive probability. And we provide the conditions which the binary relation should satisfy, in order that it has I-nonadditive probability representation. (Theorem 1). In section 3, we provide an EUNAP model by using Theorem 1, Savage's and Schmeidler's work (Theorem 2).

2. *I*-nonadditive probability

Let S be a (nonempty) *set of all states*, and $I=\{1,\dots,m\}$ an *index set*. Tentatively, we assume that m is finite. And let Σ be the set of all subsets of S (Σ is an algebra[1] on S), and A, an element of Σ, is called an *event*. Suppose $m \geq 1$ and there exists a *partition*, $\{S_1, S_2, \dots, S_m\}$ of S, with respect to I. A partition of S is a set of nonempty events that are mutually exclusive and whose union equals S. The partition with respect to I is called a *structural partition*, and each S_i is called a *structure*.

Example 2. In the former example of the Ellsberg paradox, it is said that there is no information about the ratio of the number of black balls to the number of red ones in urn 2. Although there is no information about the

[1]Algebra means Σ is closed under complementation and finite unions.

ratio, it is possible to get some information from this situation. We know that urn 2 contains four balls, either red or black. Therefore, only five cases are possible. The contents of urn 2 are as follows:

1. all balls are black.
2. three black balls and one red one.
3. two black balls and two red ones.
4. one black ball and three red ones.
5. all balls are red.

Let the set I, $I=\{1,2,3,4,5\}$, be an index set. Each index, an element of I, corresponds to a 'structure'. In this case, there exist twenty (5 times 4) states. In addition, we have no information about what structure is the true one. Therefore the decision maker confronts uncertainty about the true structure.[2]

The states, in which the color of the ball drawn from urn 1 is k, and the color of the ball drawn from urn 2 is l, are denoted by s_{ikl} for $i \in I$ and $k, l \in \{\mathrm{B}, \mathrm{R}\}$.

Let a binary relation $\succ^*$ be defined on Σ, and Q a function from Σ to $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. $A \succ^* B$ denotes that A is more probable than B, $\sim^*$ is defined by not $(A \succ^* B)$ and not $(B \succ^* A)$, and $A \succsim^* B$ is defined by $A \succ^* B$ and $A \sim^* B$. Furthermore, A and B are called *mutually I-monotonic* (*events*) if and only if, for all $i \in I$, either $A \cap S_i \succsim^* B \cap S_i$ or $B \cap S_i \succsim^* A \cap S_i$. (It is also said that A, B and C are mutually I-monotonic (events). This means A, B and C are pairwisely mutually I-monotonic.)

Definition 1. The function Q is called *nonadditive probability measure* if the following conditions hold:

1. $Q(\varnothing)=0$.
2. $Q(S)=1$.
3. $\forall A, B \in \Sigma: B \supset A \rightarrow Q(B) \geq Q(A)$.

Moreover, if the following condition holds, Q is called *I-nonadditive probability measure.*[3]

4. $\forall A, B \in \Sigma$ and $A \cap B = \varnothing$: if A and B are mutually I-monotonic, then $Q(A \cup B) = Q(A) + Q(B)$.

[2]This type of uncertainty is sometimes called 'ambiguity'. Strictly speaking, the uncertainty about the probabilities of structures is different from other 'heavier' types of uncertainty, e.g., uncertainty about the number of possible structures and perfect ignorance about the contents of the urn.

[3]Here I-nonadditive means 'not additive with respect to the index set I'.

If we replace this fourth condition by the following condition, then Q is called *additive probability measure.*

5. $\forall A, B \in \Sigma$ and $A \cap B = \varnothing$: $Q(A \cup B) = Q(A) + Q(B)$.

Remark 1. Condition 3 of Definition 1 can be replaced by the following condition in case of additive probability.

3'. $\forall A \in \Sigma: Q(A) \geq 0$.

Condition 3 implies Condition 3' since $A, \varnothing \in \Sigma$ and $A \supset \varnothing$. Also, in case of additive probability, Condition 1 and 2 are not independent. (If we suppose one of these conditions, another can be derived by the remaining conditions.)

As for the axioms for an additive probability representation, the following theorem is known as a standard one.

Theorem A (*Savage*). *Suppose a binary relation* $\succ^*$ *on* Σ *satisfies the following,* $\forall A, B, C \in \Sigma$:

A1. $S \succ^* \varnothing$.

A2. $A \succsim^* \varnothing$.

A3. $\succ^*$ *on* Σ *is a weak order.*[4]

A4. *If* $A \cup B$ *and* C *are disjoint, then*

$$A \succ^* B \quad \textit{if and only if} \quad A \cup C \succ^* B \cup C.$$

A5. $A \succ^* B \rightarrow$ *there exist a finite partition,* $\{D_1, D_2, \ldots, D_n\}$, *of* S *and* $A \succ^* B \cup D_j$ *for every* $j = 1, 2, \ldots, n$.

Then the binary relation $\succ^*$ *has a unique* (*additive*) *probability representation, i.e.,* $A \succ^* B$ *if and only if* $P(A) > P(B)$, *where* P *is an additive probability measure defined on* Σ.

Proof. This proof is given in Fishburn (1970, pp. 194–199).

Next we consider the axioms for an I-nonadditive probability representation. A partition $\{E_1, E_2, \ldots, E_n\}$ is called a *uniform partition* (of S) when $E_1 \sim^* E_2 \sim^* \cdots \sim^* E_n$ (and $\bigcup_{i=1}^n E_i = S$). A partition $\{E_1, E_2, \ldots, E_n\}$ is called a *basic partition* of S when $E_1, E_2, \ldots, E_n$ are mutually I-monotonic (i.e., pairwisely I-monotonic) and $\{E_1 \cap S_i, E_2 \cap S_i, \ldots, E_n \cap S_i\}$ is also a partition

[4] A relation which is asymmetric and negatively transitive, is called a weak order. In this case, if $A \succ^* B$, then not $(B \succ^* A)$, and if not $(A \succ^* B)$ and not $(B \succ^* C)$, then not $(A \succ^* C)$.

of S_i for all $i \in I$. The basic partition which is also a uniform partition is called a *basic uniform partition.* By using this concept, let's define the following sets.

Define $M_b(n) = \{E \in \Sigma \mid E$ is a member of the n-part basic uniform partition of S and $M_b^k = \bigcup_{n=1}^{k} M_b(n)$ for some positive integer k.

Note that $M_b(n)$ is not uniquely determined because of the multiplicity of the n-part basic uniform partition. Only we need in this paper the existence of one $M_b(n)$ and one M_b^k.

Theorem 1. Suppose a binary relation $\succ^*$ on Σ satisfies the following, $\forall A, B, C \in \Sigma$:

P1. $S \succ^* \varnothing$.

P2. $A \succeq^* \varnothing$.

P3. *$\succ^*$ on Σ is a weak order.*

P4. *If $A \cup B$ and C are disjoint, and A and B are mutually I-monotonic, then*

$$A \succ^* B \quad \text{if and only if} \quad A \cup C \succ^* B \cup C.$$

P5. *(1) For all nonempty $A \in \Sigma$, there exists $E \in M_b^k$ for some finite k such that A and E are mutually I-monotonic.*
(2) $A \succ^* B \rightarrow$ there is a finite basic partition of S, and $A \succ^* B \cup E$ for every member E of the partition.

Then $\succ^*$ has a unique I-nonadditive probability representation, i.e.,

$$A \succ^* B \quad \text{if and only if} \quad Q(A) > Q(B),$$

Remark 2. If $I = \{1\}$, then Theorem 1 coincides with Theorem A, because any A and $B \in \Sigma$ are mutually I-monotonic, and any partition of S is a basic partition of S.

Next consider the meaning of the axioms in Theorem 1. P1, P2, P3 are the same axioms as corresponding axioms in Savage's subjective probability theorem. But, P4 and P5 are slightly different from the usual axioms. So let's consider the meaning of these axioms.

P4 is a weakening of the usual independence axiom. This axiom makes our probability I-nonadditive. Suppose A denotes 'It (the weather at some date) is fine', while B denotes 'It is rainy'. Furthermore, suppose that each structure $i \in I$ denotes a climatic condition. Thus $A \cap S_i \succ^* B \cap S_i$ means that 'fine' is more probable than 'rainy' when the climatic condition is i. In this case, I-monotonicity means that 'fine' is more probable than 'rainy' whichever climatic condition happens. Suppose 'fine' is more probable than 'rainy' whichever climatic condition happens. Then it is natural to think that 'fine' is more probable than 'rainy' even if we don't know which climatic condition (structure) happens. Thus we assume that if A and B are mutually I-monotonic events, i.e., we can arrange these two events by 'structure dominance', then the usual substitution axiom holds for these events.

P5 is our Archimedian axiom. This axiom makes it possible to use Savage's method of assigning the value of probability. A sufficiently fine basic uniform partition of S gives us something like 'a unit for measure'. Gilboa (1987) used another method of assigning the value of probability. His method is original but more complex.

Formally, I-nonadditive probability is a special case of nonadditive probability, and a general case of additive probability. Why do we introduce this intermediate concept? One merit of this concept is its applicability. There exist some cases where the (simple) independence axiom doesn't hold. Axioms under I-nonadditive probability representation don't exclude some cases that violate the independence axiom. Another merit of this concept is its clarity. Nonadditive probability is a more general concept than the I-nonadditive one, but one demerit of nonadditive probability is that it is ambiguous whether the probabilities, which correspond to some specific events, are additive or not.

Normally the probabilities of the consequences of rolling a 'fair' dice, are considered as additive ones. The events, which are made by rolling a 'fair' dice, can be considered as mutually I-monotonic, so the system of axioms under I-nonadditive probability make the probabilities on such events additive.

Before proving Theorem 1, we will show the following lemmas.

Lemma 1.1. Suppose P1–P5 hold for $\succ^*$, then the following statements hold.

(1) *Suppose A and $B \in \Sigma$ are mutually I-monotonic. Then, if $A \succ^* B$, $A \cap S_i \succ^* B \cap S_i$ $\forall i \in I$.*

(2) *For A, B, C and $D \in \Sigma$, suppose A and B, and C and D are mutually I-monotonic respectively, and $A \cup B$ and C, and $C \cup D$ and B are disjoint respectively. Then, if $A \succ^* B$ and $C \succ^* D$, then $A \cup C \succ^* B \cup D$.*

(3) *For A, B, C and $D \in \Sigma$, suppose A and B, and C and D are mutually I-monotonic respectively, and $A \cup B$ and C, and $C \cup D$ and B are disjoint respectively. Then, if $A \sim^* B$ and $C \sim^* D$, then $A \cup C \sim^* B \cup D$.*

(4) *If $A \cap S_i \succ^* \varnothing$, then for any positive integer n there is a n-part uniform partition of $A \cap S_i$ $\forall i \in I$.*

Proof. (1) Suppose not $(A \cap S_i \succ^* B \cap S_i \;\; \forall i \in I)$, then $B \cap S_i \succeq^* A \cap S_i$ $\forall i \in I$ because A and B are mutually I-monotonic. Define $C^0 = A$, $C^i = (B \cap S_i) \cup C^{i-1}$, $(C^m = B)$ for $i = 1, 2, \ldots, m$ successively. For all i, $A \cap S_i$

and $B \cap S_i$ are mutually I-monotonic because $(B \cap S_i) \cap S_i \succeq^* (A \cap S_i) \cap S_i$ and $(B \cap S_j) \cap S_i \sim^* (A \cap S_j) \cap S_i$ for $j \neq i$. So, by P4, $B \cap S_i \succeq^* A \cap S_i$ iff $C^i = (B \cap S_i) \cup (C^{i-1} \backslash S_i) \succeq^* (A \cap S_i) \cup (C^{i-1} \backslash S_i) = C^{i-1}$ for all i. Thus $B = C^m \succeq^* C^0 = A$ by P3, a contradiction.
(2) Since $A \succ^* B$ are mutually I-monotonic and $A \cup B$ and C are disjoint, $A \succ^* B$ iff $A \cup C \succ^* B \cup C$ by P4. Similarly, since $C \cup D$ and B are disjoint, $C \succ^* D$ iff $C \cup B \succ^* D \cup B$. Thus, by P3, $A \cup C \succ^* B \cup D$.
(3) By the definition of $\sim^*$ and Lemma 1.1(2), this result will be obtained easily.
(4) This proof is basically the same which is given, for example, in Fishburn (1970, pp. 195–198). Q.E.D.

By using Lemma 1.1, several results about $M_b(n)$ and M_b^k are obtained.

Lemma 1.2. Suppose a binary relation $\succ^*$ on Σ satisfies P1–P5. Then the following statements hold:

(1) *Both $M_b(n)$ and M_b^k (for finite k) are nonempty.*
(2) *If $A, B \in M_b^k$, then A and B are mutually I-monotonic.*
(3) *$\forall A \in M_b(k')$ and $\forall B \in M_b(k'')$: if $k'' > k'$, then $A \succ^* B$.*

Proof. (1) By Lemma 1.1(4), $A \cap S_i$ has its n-part uniform partition, $\{A_i^1, A_i^2, \ldots, A_i^n\}$ $\forall i \in I$. Let $E^j = \bigcup_{i \in I} A_i^j$ $\forall j \in I$. Then $\{E^1, E^2, \ldots, E^m\}$ is a basic uniform partition of S, because $E^1 \sim^* E^2 \sim^* \cdots \sim^* E^m$ by Lemma 1.1(3) and $\bigcup_{j \in I} E^j = S$. Thus $M_b(n)$ is nonempty and $M_b(n)$ is a set of all elements of one of the partition. M_b^k is also nonempty by the definition.
(2) By P3, $A \cap S_1 \succ^* B \cap S_1$ or $B \cap S_i \succ^* A \cap S_1$ or $A \cap S_1 \sim^* B \cap S_1$ holds. Then by the definition of basic uniform partition and P3, if $A \cap S_1 \succ^* B \cap S_1$ [or $B \cap S_1 \succeq^* A \cap S_1$], then $A \cap S_i \succeq^* B \cap S_i$ [or $B \cap S_i \succeq^* A \cap S_i$] for all $i \in I$. Hence, A and B are mutually I-monotonic.
(3) If $A \in M_b(n)$ for finite n, then $A \succ^* \varnothing$. Because if A is empty, then $S \sim^* \varnothing$ by P3 and P4. This contradicts P1. Let $A^1, A^2, \ldots, A^k \in M_b(k')$ and $B^1, B^2, \ldots, B^{k'} \in M_b(k'')$. Suppose $A^1 \succ^* B^1$, then $A^i \succ^* B^j$ $\forall i, j$ by the definition of basic uniform partition and P3. Suppose not ($A^1 \succ^* B^1$), then $B^j \succeq^* A^i$ $\forall i, j$. By using Lemma 1.1(2),(3) repeatedly, $\bigcup_{j=1}^k B^j \succeq^* \bigcup_{i=1}^k A^i = S$ is obtained. Remind that every $B^1, B^2, \ldots, B^{k'}$ is nonempty. So by using P4, $\bigcup_{j=1}^k B^j = (\bigcup_{j=1}^{k'} B^j) \cup (\bigcup_{j=k+1}^{k'} B^j) \succ^* \bigcup_{j=1}^k B^j$ is derived. Thus $S = \bigcup_{j=1}^{k'} B^j \succ^* \bigcup_{j=1}^k B^j \succeq^* \bigcup_{i=1}^k A^i = S$, a contradiction. Hence, if $k'' > k'$, then $A \succ^* B$. Q.E.D.

Let $N(E)$ be a set such that, if $\forall A, B \in N(E)$, then A and B and E are mutually I-monotonic events (i.e., A and B and E are pairwisely I-monotonic events) for an $E \in M_b^k$ for some k. And let $\Sigma'(E, k)$ be $M_b^k \cup N(E)$.

Lemma 1.3. Suppose $\succ^*$ on Σ satisfies P1–P5.

(1) *If A and B are mutually I-monotonic events, then there exists $\Sigma'(E,k) \subset \Sigma$ for some $E \in M_b^k$ and for some k, such that $A, B \in \Sigma'(E,k)$.*
(2) *If $\succ^*$ is defined on $\Sigma'(E,k)$ for some $E \in M_b^k$, then $\succ^*$ satisfies A1–A5, thus it has an additive probability representation.*

Proof. (1) By P5, if A and $B \in \Sigma$, there exist E and $E' \in M_b^k$ for some k such that $A \succ E$ and A and E are mutually I-monotonic, and $B \succ E'$ and B and E' are mutually I-monotonic. Because E and E' are in M_b^k, E and E' are mutually I-monotonic events by Lemma 1.2(2). Suppose $E' \succsim E$ without loss of generality. Then B and E are mutually I-monotonic events because $B \cap S_i \succ^* E' \cap S_i \succsim^* E \cap S_i$ $\forall i \in I$. Hence, A and B are in $\Sigma'(E,k)$ for some finite k.
(2) In this case, it is apparent P1–P5 means A1–A5. Hence, by Theorem A, $\succ^*$ has an additive probability representation. Q.E.D.

Proof of Theorem 1. Let $C_b(r, 2^n) = \{A \mid$ the union of r-members of $M_b(2^n)\}$. This set is nonempty because $M_b(2^n)$ is nonempty by Lemma 1.2(1). Let $r(n)$ be the largest number of r such that $A \succsim_* B_n$ where $B_n \in C_b(r, 2^n)$. Define $Q(A) = \sup_n r(n)/(2^n)$ $(n = 1, 2, \ldots)$. By the definition of Q, $Q(\emptyset) = 0$ and $Q(S) = 1$.

(1) If $A \in \Sigma'(E,k)$ for some $E \in M_b$ and for some k, then $Q(A) = P(A)$, by Lemma 1.3(2). Hence, if A and B are mutually I-monotonic and disjoint, then $Q(A \cup B) = Q(A) + Q(B)$. Moreover, if $A \in C_b(r, 2^n)$, then $Q(A) = P(A) = r(n)/2^n$, by the definition of Q and $C_b(r, 2^n)$. If $A, B \in \Sigma$ and $B \supset A$, then A and B are mutually I-monotonic events. Thus $Q(B) \geq Q(A)$.
(2) In general, if $A \succsim^* B$, then $Q(A) \geq Q(B)$ by the definition of Q. If $A \succ^* \emptyset$, then, by P5, there exists $E \in M_b(n)$ for some n such that $A \succ^* E$. Thus $Q(A) > 0$. [Because $Q(A) \geq Q(E) > 0$ for some E. If not ($Q(E) > 0$), then $Q(S) = 0$, a contradiction]. If $A \succ^* B$, then, by P5, there exists $E \in M_b$ such that E and B are mutually I-monotonic and $A \succ^* B \cup E$. Then $Q(A) \geq Q(B + E) = Q(B) + Q(E)$ and $Q(E) > 0$ hold. Thus $Q(A) > Q(B)$. Remind that if $A \succsim^* B$, then $Q(A) \geq Q(B)$. $A \succ^* B$ iff $Q(A) > Q(B)$ is obtained. Uniqueness of Q is apparent by the definition of Q. Q.E.D.

Next let's consider the following variations. First, structure-dependent (additive) probability measures are introduced.

Definition 2. Let Q_i be a function from Σ to $\mathbb{R}$ for every $i \in I$. Q_i is called *structure-dependent* (*additive*) *probability measure* if the following conditions hold. Let $A_i = A \cap S_i$, and $B_i = B \cap S_i$ $\forall A, B \in \Sigma$.

1. $\forall A \in \Sigma: Q_i(A) = Q_i(A_i)$.

2. $Q_i(S_i)=1$.
3. $\forall A_i \in \Sigma: Q_i(A_i) \geq 0$.
4. $\forall A_i, B_i \in \Sigma,\ A_i \cap B_i = \varnothing: Q_i(A_i \cup B_i) = Q_i(A_i) + Q_i(B_i)$.

In what follows, we sometimes use $Q_i(A_i)$ and $Q_i(A)$ exchangeably because these two values are equivalent.

Corollary 1. Suppose $\succ_i^*$ is defined as follows. $\forall A, B \in \Sigma: A \succ_i^* B$ if $A \cap S_i \succ^* B \cap S_i$ for every $i \in I$. If $\succ^*$ satisfies P1–P5, then $\succ_i^*$ has a unique (additive) probability representation for every $i \in I$.

Proof. First we will show the following statement. 'If $\succ^*$ satisfies P1–P5, then A1–A5 hold for $\succ_i^*$ for every $i \in I$'. If this statement holds, then $\succ^*$ has a unique (additive) probability representation by Theorem A.

A1. By the definition of structural partition, if $i \in I$ then S_i is nonempty.

A2. If $A \succeq^* \varnothing$, then $A \cap S_i \succ^* \varnothing$. Hence $A \succ_i^* \varnothing$.

A3. This is easily derived by P3.

A4. Because $A \cap S_i$ and $B \cap S_i$ are mutually I-monotonic, and $(A \cup B) \cap S_i$ and $C \cap S_i$ are disjoint, then $A \cap S_i \succ^* B \cap S_i$ if and only if $(A \cup C) \cap S_i \succ^* (B \cup C) \cap S_i$ by P4.

A5. By P5, if $A \succ^* B$, then there is a basic partition of S such that $A \succ^* B \cup E$ for every member E of the partition. Put $D = E \cap S_i$. Then D is considered as an element of a partition of S_i. And $A \succ_i^* B \cup D$ for all $i \in I$. Q.E.D.

Remark 3. By the definition of I-nonadditive probability, Q is additive for every pair of mutually I-monotonic events. Therefore, we can consider $Q_i(A)$ as $Q(A \cap S_i)/Q(S_i)$.

Proof. By Theorem A, Q_i is a unique representation for $\succ_i^*$. So if another expression exists, it is equal to Q_i.

Every S_i is not empty by our assumption. Then $Q(S_i) > 0$. So $Q(S_i \cap S_i)/Q(S_i)$ is well defined. $Q(\cdot \cap S_i)/(Q(S_i)$ is also a structure dependent (additive) probability measure. The following four conditions are satisfied for all $i \in I$.

1. $Q(A \cap S_i)/Q(S_i) = Q(A_i \cap S_i)/Q(S_i)$.
2. $Q(S_i \cap S_i)/Q(S_i) = 1$.
3. $Q(A \cap S_i)/Q(S_i) \geqq 0$.

4. Because A_i, B_i are disjoint and mutually I-monotonic, $Q((A_i \cup B_i) \cap S_i)/Q(S_i) = Q(A_i)/Q(S_i) + Q(B_i)/Q(S_i)$.

Furthermore, for all $A, B \in \Sigma$, $A \succ_i^* B$ iff $A \cap S_i \succ^* B \cap S_i$. Thus $Q_i(A) > Q_i(B)$ iff $Q(A \cap S_i) > Q(B \cap S_i)$ iff $Q(A \cap S_i)/Q(S_i) > Q(B \cap S_i)/Q(S_i)$. So we can consider $Q(\cdot \cap S_i)/Q(S_i)$ and Q_i are equivalent. Q.E.D.

Remark 4. Let $\sigma(J)$ be $Q(\bigcup_{j \in J} S_j)$ for all $J \subset I$. Then σ is a nonadditive probability. Because $\sigma(I) = Q(\bigcup_{j \in I} S_j) = 1$ and $\sigma(\emptyset) = Q(\emptyset) = 0$, and if $J \subset K \subset I$, then $\sigma(K) = Q(\bigcup_{j \in K} S_j) \geq Q(\bigcup_{j \in J} S_j) = \sigma(J)$.

The following example is an application of Theorem 1. The results of this example will be used in Example 4.

Example 3. Suppose the decision maker in the former example have the following relations.

$$\{s_{i\mathrm{BB}}, s_{i\mathrm{BR}}\} \sim^* \{s_{i\mathrm{RB}}, s_{i\mathrm{RR}}\} \sim^* \{s_{j\mathrm{BB}}, s_{j\mathrm{BR}}\},$$

$$\{s_{i\mathrm{BB}}, s_{i\mathrm{RB}}\} \sim^* \{s_{i\mathrm{BR}}, s_{i\mathrm{RR}}\}, \quad \text{and}$$

$$\{s_{i\mathrm{BB}}, s_{i\mathrm{BR}}\} \succ^* \{s_{i\mathrm{BB}}, s_{i\mathrm{RB}}\} \quad \text{for} \quad \text{all } i, j \in I.$$

Denote $\bigcup_{i=1}^{5} \{s_{ikl}\}$ as $\{s_{kl}\}$ for $i \in I$ and $k, l \in \{\mathrm{B}, \mathrm{R}\}$.
Then, the following results are obtained by using Theorem 1.

(1) $Q(\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\}) = Q(\{s_{\mathrm{RB}}, s_{\mathrm{BR}}\}) = 1/2$.
(2) $Q_i(\{s_{kl}\}) = Q(\{s_{kl}\})/Q(S_i) = 1/4Q(S_i)$ for $\forall k, l \in \{\mathrm{B}, \mathrm{R}\}$.
(3) $\sigma(J) \leq \sigma(J')$ if $J \subset J'$ for $\forall J, J' \subset I$.

Proof. (1) From $\{s_{i\mathrm{BB}}, s_{i\mathrm{BR}}\} \sim^* \{s_{i\mathrm{RB}}, s_{i\mathrm{RR}}\}$ and Lemma 1.1(1), $\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\} \sim^* \{s_{\mathrm{RB}}, s_{\mathrm{RR}}\}$. Since $\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\}$ and $\{s_{\mathrm{RB}}, s_{\mathrm{RR}}\}$ are mutually I-monotonic, then, by Theorem 1, $Q(\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\}) = Q(\{s_{\mathrm{RB}}, s_{\mathrm{RR}}\}) = 1/2$.
(2) Because $s_{i\mathrm{BB}}$, $s_{i\mathrm{BR}}$, $s_{i\mathrm{RB}}$, $s_{i\mathrm{RR}}$ are mutually I-monotonic, then

$$Q(\{s_{i\mathrm{BB}}\}) = Q(\{s_{i\mathrm{BR}}\}) = Q(\{s_{i\mathrm{RB}}\}) = Q(\{s_{i\mathrm{RR}}\}) = Q(S_i)/4.$$

Hence, $Q_i(\{s_{kl}\}) = Q(\{s_{kl}\}/Q(S_i) = 1/4Q(S_i)$ $\forall k, l \in \{\mathrm{B}, \mathrm{R}\}$.
(3) This relation is diretcly derived from the definition of σ, i.e., $\sigma(J) = Q(\bigcup_{j \in J} S_j)$ for all $J \subset I$.

Similarly, we will show the following relation: $0 = \sigma(\emptyset) \leq \sigma(\{1\}) \leq \sigma(\{1, 2\}) \leq \sigma(\{1, 2, 3\}) \leq \sigma(\{1, 2, 3, 4\}) \leq \sigma(I) = 1$. Q.E.D.

3. An EUNAP model

First we provide Schmeidler's EUNAP model. Let X be the *set of consequences* and x, an element of X, is called a *consequence*. And let Y be the *set of all additive probability measures* on X. Let Ω be the set of all subsets of I.[5] A *lottery act*, a, is a Ω-measurable function from I to Y and let L be the *set of all lottery acts* in Y^I. Y^I is the linear space of all functions from I to Y. Y is treated as a subset of L because $y \in Y$ can be considered as $a(i)=u$ for all $i \in I$.

Suppose the binary relation $\succ^{**}$ is defined on L, and $\succeq^{**}$, and $\sim^{**}$ are defined in the usual way. Also suppose, for $y, z \in Y$ and $a, b \in L$, $y \succ^{*} z$ if and only if $a \succ^{**} b$, where $a(i)=y$ and $b(i)=z$ for all $i \in I$. Therefore, we will sometimes use the same notation $\succ^{**}$ for the binary relation on L and on Y. (Because $\succ^{**}$ coincides with $\succ^{*}$ on Y).

Additionally, the following definition is needed for Schmeidler's Theorem. a and b are called *mutually comonotonic* if there are no $i, j \in I$ such that $a(i) \succ^{**} a(j)$ and $b(j) \succ^{**} b(i)$.

Theorem B (*Schmeidler*). *Suppose a binary relation $\succ^{**}$ on L is a weak order. For $a, b, c \in L$:*

B1. *$\succ^{**}$ on L is a weak order.*

B2. *[a, b and c are mutually comonotonic, $0<\lambda<1$] $a \succ^{**} b \rightarrow \lambda a+(1-\lambda)c \succ^{**} \lambda b+(1-\lambda)c$.*

B3. *[For some $\alpha, \beta \in (0,1)$, where (,) means open interval] $\alpha \succ^{**} b$ and $b \succ^{**} c \rightarrow \alpha a+(1-\alpha)c \succ^{**} \beta a+(1-\beta)c$.*

B.4 *$a \succ^{**} b$ for some $a, b \in L$.*

B5. *$a(i) \succ^{**} b(i)$ for all $i \in I \rightarrow a \succeq^{**} b$.*

B6. *a is bounded, i.e., $\exists y, z \in Y$, $y \succeq^{**} a(i)$ and $a(i) \succeq^{**} z$ for all $i \in I$ and all $a \in L$.*

Then there is a unique nonadditive probability measure σ on Ω and a unique linear functional v on Y, unique up to positive affine transformation, for all lottery acts a, b,

$$a \succ^{**} b \quad \text{if and only if} \quad \int_I v(a(i))\,\mathrm{d}\sigma > \int_I v(b(i))\,\mathrm{d}\sigma.$$

Proof. See Schmeidler (1984, 1986, 1989).

[5] Ω is supposed to be a σ-algebra. σ-algebra means closed under complementation and infinite unions.

Remark 5. In Theorem B, we use *Choquet integral* for integral representation. Choquet integral is defined as follows: given a function w on I and a nonadditive probability measure σ on Ω,

$$\int_I w(i)\,\mathrm{d}\sigma = \int_{c=0}^{\infty} \sigma(\{i\in I: w(i)\geq c\})\,\mathrm{d}c - \int_{c=-\infty}^{0} [1-\sigma(\{i\in I: w(i)\geq c\})\,\mathrm{d}c.$$

The integrals on the right-hand side are Riemann integrals. [For details, see Choquet (1955) and Schmeidler (1986).] When w is constant on each member of the finite partition $\{J_1, J_2, \ldots, J_n\}$ with $w(i)=c_j$ for all $i\in J_j$ and $c_1>c_2> \cdots > c_n$. Then the definition is as follows:

$$\int_I w(i)\,\mathrm{d}\sigma(i) = \sum_{j=1}^{n-1} (c_j - c_{j+1})\sigma\left(\bigcup_{i=1}^{j} J_i\right) + c_n.$$

Secondly, we provide a modified Savage's EUAP model. Let F be the set of all functions from Σ to X. Every element $f\in F$ is called an *act* and F is called the *set of all acts.*

A binary relation $\succ$ is defined on F. And $\succeq$ and $\sim$ are defined as usual. Define $f=xAy$ as $f(s)=x$ $\forall s\in A$ and $f(s)=y$ $\forall s\in A^c$ for $x,y\in X$. Denote $f=x$ on A if $f(s)=x$ $\forall s\in A$, and $f=g$ on A if $f(s)=g(s)$ $\forall s\in A$. $f=g$ in A, and h on B means $f=g$ on A, and $f=h$ on B. Also denote $f\succ g$ on A if $fAh\succ gAh$ $\forall h\in F$.

Definition 3. Binary relations $\succ^*$, $\succ_i^*$ and $\succ_i$ (for all $i\in I$) are defined as follows, $\forall A,B\in\Sigma$ and $\forall f,g\in F$:

(1) $A\succ^* B$ if $xAy\succ xBy$ for all $x,y\in X$ for which $x\succ y$.
(2) $A\succ_i^* B$ if $x(A\cap S_i)y\succ x(B\cap S_i)y$ on S_i for all $x,y\in X$ for which $x\succ y$.
(3) $f\succ_i g$ if $fS_ih\succ gS_ih$ for all $h\in F$.

Theorem C (a modified version of Savage). Suppose a binary relation $\succ_i$ is defined on F for every $i\in I$, and satisfies the following, for all $f,g\in F$ and $x,y,x',y'\in X$:

C1. *$\succ_i$ on F is a weak order.*

C2. *$[f=f',\ g=g'$ on $A\cap S_i$, $f'-g'$, $f'=g'$ on $A^c\cap S_i]\to(f\succ_i g\Leftrightarrow f'\succ_i g')$.*

C3. *$[A\cap S_i$ is not null: $f=x$, $g=y$ on $A\cap S_i]\to(f\succ_i g$ on $A\cap S_i\Leftrightarrow x\succ_i y)$.*

C4. *$[x\succ_i y, x'\succ_i y']\to(x(A\cap S_i)y\succ_i x(B\cap S_i)y$ and $x'(A\cap S_i)y'\succ_i x'(B\cap S_i)y'$.*

C5. *$x\succ_i y$ for some $x,y\in X$.*

C6. *$f \succ_i g \rightarrow$ given x, there is a finite partition of S^i, $\{D_1, D_2, \dots, D_n\}$,* such that
$[f' = x$ on D_j, and $f' = f$ on $D_j^c \cap S_i] \rightarrow f' \succ_i g$,
$[g' = x$ on D_j, and $g' = g$ on $D_j^c \cap S_i] \rightarrow f \succ_i g'$.

C7. *$[f \succ_i g(s)$ on $A \cap S_i$ for all $s \in A \cap S_i] \rightarrow f \succeq_i g$ on $A \cap S_i$.*
$[f(s) \succ_i g$ on $A \cap S_i$ for all $s \in A \cap S_i] \rightarrow f \succeq_i g$ on $A \cap S_i$.

Then $\succ_i^*$ has a unique additive probability representation and there is a bounded functional u on X such that

$$f \succ_i g \quad \text{if and only if} \quad \int_{S_i} u(f)\,\mathrm{d}Q_i > \int_{S_i} u(g)\,\mathrm{d}Q_i.$$

Moreover, u is unique up to positive affine transformation for all $i \in I$.

Proof. This proof is basically the same as in Fishburn (1970, ch. 14).

Finally we will provide our EUNAP model by using the former theorems. We call f and g *mutually I-monotonic (acts)* if, $\forall i \in I$ and $\forall h \in F$, either $fS_ih \succeq gS_ih$ or $gS_ih \succeq fS_ih$, and *mutually I-comonotonic (acts)* if there are no $i, j \in I$ both $fS_ih \succ fS_jh$ and $gS_jh \succ gS_ih$ for all $h \in F$. (It is sometimes called that f, g and h are mutually I-comonotonic if f, g and h are pairwisely mutually I-comonotonic.)

The following representation theorem holds even if the index set I is not finite. From now on, we think that I is not necessarily finite.

Theorem 2. Suppose that the following conditions (Q1–Q7) hold for $\succ$ on F. For all $f, g, f', g' \in F$ and all $x, y, x', y' \in X$:

Q1. *$\succ$ on F is a weak order.*
Q2. *If f and g are mutually I-monotonic and/or mutually I-comonotonic, then $[f = f', g = g'$ on A, and $f = g$, $f' = g'$ on $A^c] \rightarrow (f \succ g \Leftrightarrow f' \succ g')$.*

Q3. *[A is not null: $f = x$, $g = y$ on $A] \rightarrow (x \succ y \rightarrow f \succ g$ on $A)$.*

Q4. *$[x \succ y$ and $x' \succ y'] \rightarrow (xAy \succ xBy$ and $x'Ay' \succ x'By')$.*

Q5. *$x \succ y$ for some $x, y \in X$.*

Q6. *Define f' and g' as follows: $f' = x$ on E, $f' = f$ on E^c and $g' = x$ on E, $g' = g$ on E^c for an element E of the partition of S.*
(1) $\forall f \in F$, there exists an appropriate finite basic uniform partition such that f and f' are mutually I-monotonic (acts) for a member E of the partition.
(2) $[f \succ g] \rightarrow$ given x, there is a finite basic partition of S, such that $f' \succ g$, $f \succ g'$ for every member E of the partition.

Q7. $f \succ g(s)$ *on* A *for all* $s \in A \to f \succeq g$ *on* A. $f(s) \succ g$ *on* A *for all* $s \in A \to f \succeq g$ *on* A.

Then every $\succ_i^*$ *has a unique additive probability representation for every* $i \in I$, *and there exists a nonadditive probability measure* σ *on* Ω, *and there is a bounded functional* u *on* X *such that*

$$f \succ g \quad \text{if and only if} \quad \int_I \int_{S_i} u(f)\,\mathrm{d}Q_i\,\mathrm{d}\sigma > \int_I \int_{S_i} u(g)\,\mathrm{d}Q_i\,\mathrm{d}\sigma.$$

Moreover u *is unique up to positive affine transformation.*

The relation between Savage's system of axioms and ours is considered in the following Remark 6 and Lemma 2.1.

Remark 6. If $I = \{1\}$, Q1–Q7 means C1–C7. So Theorem 2 can be seen as a generalization of Theorem C.

Proof. Q1, Q2, Q4–Q7 coincide with C1, C2, C4–C7 respectively if $I = \{1\}$. C3 is derived from Q2 and Q3 if $S = \{1\}$. $x \succ y \to f \succ g$ on A is directly derived by Q3. If $S = \{1\}$, then x and y are mutually I-monotonic. So, if $f = x$, $g = y$ on A and $f \succ g$, then $xAh \succ yAh$ for all $h \in F$ by Q2. Thus, $x \succ y$. [Because, if not $(x \succ y)$, then $yAh \succeq xAh$ $\forall h \in F$ by Q3, a contradiction.] Hence $f \succ g \to x \succ y$. Q.E.D.

Now we examine the meaning of our axioms respectively. Q1, Q4, Q5 and Q7 are the same axioms as the corresponding Savage's axioms. But Q2 and Q3 are weakenings of the corresponding Savage's ones. Q2 is a weakening of the usual substitution axiom. Q3 is the same axiom as Gilboa's P3*, which is a weakening of C3. But Q2 is different from his corresponding axiom P2*.[6]

I-monotonicity and I-comonotonicity can be considered as regularity conditions. If acts are not mutually I-monotonic and/or I-comonotonic, then there exist many cases that don't satisfy substitution axioms. The main cause of the violation of substitution axioms is the lack of transparency. We can think that the acts which are mutually I-monotonic and/or I-comonotonic are relatively easy to compare to each other. But I-comonotonicity may have less intuitive appeal than I-monotonicity. Technically, I-comonotonic additivity is needed for integral representation.

Q6 is our Archimedian axiom which is slightly different from the corresponding Savage's axiom.

Let's illustrate our Theorem 2 by using the former example.

[6]As for Gilboa's P2*, see appendix.

Example 4. Suppose u is a strictly increasing function with $u(0)=0$. By using Theorem 2 and the results of Example 3, we will get the EUNAP representations for the acts in the former example. They are as follows.

$$1_{\mathrm{B}}\colon u(100)Q(\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\})+u(0)Q(\{s_{\mathrm{RB}}, s_{\mathrm{RR}}\})=(1/2)u(100),$$

$$1_{\mathrm{R}}\colon u(100)Q(\{s_{\mathrm{RB}}, s_{\mathrm{RR}}\})+u(0)Q(\{s_{\mathrm{BB}}, s_{\mathrm{BR}}\})=(1/2)u(100),$$

$$\begin{aligned}2_{\mathrm{B}}\colon &\{u(100)-(3/4)u(100)\}\sigma(\{1\})+\{(3/4)u(100)-(1/2)u(100)\}\sigma(\{1,2\})\\ &+\{(1/2)u(100)-(1/4)u(100)\}\sigma(\{1,2,3\})\\ &+(1/4)u(100)\sigma(\{1,2,3,4\})\\ &=(1/4)u(100)[\sigma(\{1\})+\sigma(\{1,2\})+\sigma(\{1,2,3\})+\sigma(\{1,2,3,4\})],\end{aligned}$$

$$\begin{aligned}2_{\mathrm{R}}\colon &\{u(100)-(3/4)u(100)\}\sigma(\{5\})+\{(3/4)u(100)-(1/2)u(100)\}\sigma(\{5,4\})\\ &+\{(1/2)u(100)-(1/4)u(100)\}\sigma(\{5,4,3\})\\ &+(1/4)u(100)\sigma(\{5,4,3,2\})\\ &=(1/4)u(100)[\sigma(\{5\})+\sigma(\{5,4\})+\sigma(\{5,4,3\})+\sigma(\{5,4,3,2\})].\end{aligned}$$

Because of $2_{\mathrm{B}}\sim 2_{\mathrm{R}}$,

$$\begin{aligned}&[\sigma(\{1\})+\sigma(\{1,2,\})+\sigma(\{1,2,3\})+\sigma(\{1,2,3,4\})]\\ &=[\sigma(\{5\})+\sigma(\{5,4\})+\sigma(\{5,4,3\})+\sigma(\{5,4,3,2\})].\end{aligned}$$

Assume *uncertainty-aversion*, i.e., $\sigma(A\cup B)>\sigma(A)+\sigma(B)$.[7] Then

$$\begin{aligned}&[\sigma(\{1\})+\sigma(\{1,2\})+\sigma(\{1,2,3\})+\sigma(\{1,2,3,4\})]\\ &=1/2([\sigma(\{1\})+\sigma(\{1,2\})+\sigma(\{1,2,3\})+\sigma(\{1,2,3,4\})]\\ &\quad+[\sigma(\{5\})+\sigma(\{5,4\})+\sigma(\{5,4,3\})+\sigma(\{5,4,3,2\})])<2.\end{aligned}$$

Hence

$$(1/2)u(100)>(1/4)u(100)[\sigma(\{1\})+\sigma(\{1,2\})+\sigma(\{1,2,3\})+\sigma(\{1,2,3,4\})].$$

Therefore, we have the desired result,

$$1_{\mathrm{B}}\sim 1_{\mathrm{R}}\succ 2_{\mathrm{B}}\sim 2_{\mathrm{R}}.$$

[7]The meaning of this assumption is explained in Schmeidler (1984, 1989) and Wakker (1990).

Theorem 2 states that if $\succ$ on F satisfies Q1–Q7, then there exists a function Φ from F to $\mathbb{R}$ such that $\Phi(f) > \Phi(g)$ iff $f \succ g$ $\forall f, g \in F$. Let a function V from L to $\mathbb{R}$ be as follows:

$$V(a) = \int_I v(a(i))\, \mathrm{d}\sigma.$$

This function represents the binary relation $\succ^{**}$ on L (Theorem B). Technically, our destination is to get a function Φ, which represents $\succ$ on F. So the remaining problem is to find an appropriate function θ from F to L. If we find such a function, then we will get the function Φ through $\Phi = V(\theta)$. (Precisely, L should be restricted to appropriate L'. Because θ should be a mapping onto $L' \subset L$, in order to compose the functions.)

Lemma 2.1. If $\succ$ satisfies Q1–Q7 and $\succ_i$ is defined as Definition 3 (for every $i \in I$), then every $\succ_i$ on F satisfies C1–C7. Therefore $\succ_i^*$ has a unique additive probability representation and there is a bounded functional u on X such that

$$f \succ_i g \quad \textit{if and only if} \quad \int_{S_i} u(f)\, \mathrm{d}Q_i > \int_{S_i} u(g)\, \mathrm{d}Q_i.$$

Moreover u is unique up to positive affine transformation for every $i \in I$.

Proof. C1, C4, C5 and C7. These are easily derived by the respective conditions, Q1, Q4, Q5 and Q7.
C2. Since fS_ih and gS_ih are mutually I-monotonic, then C2 is directly derived by Q2.
C3. $x \succ_i y \to f \succ_i g$ on A_i is directly derived by Q3. If $f = x$, $g = y$ on A_i and $f \succ g$, then $xA_ih \succ yA_ih$ for all $h \in F$, by Q2. Thus, $x \succ_i y$. [Because, if not $(x \succ_i y)$, then $yA_ih \succeq xA_ih$ for some $h \in F$ by Q3, a contradiction.] Hence $f \succeq_i g$ on $A_i \to x \succeq_i y$.
C6. An appropriate finite basic partition of S, $\{E_1, E_2, \ldots, E_n\}$, corresponds to $\{D_1, D_2, \ldots, D_n\}$ where $D_j = E_j \cap S_i$ for $j = 1, 2, \ldots, n$. Then, by using Q6, C6 is obtained.

Hence the desired result is obtained by Theorem C. Q.E.D.

Lemma 2.2. Suppose $\succ$ on F satisfies Q1–Q6 and $\succ^*$ is defined as in Definition 3, then $\succ^*$ on Σ satisfies P1–P5. Then $\succ^*$ has a unique I-nonadditive probability representation.

Proof. P3. There are some $x, y \in X$ such that $x \succ y$ by Q5. Then $\succ^*$ is well-defined. $\succ^*$ is a weak order because $\succ$ is a weak order and the definition of $\succ^*$.

P1. Suppose $x \succ y$. Then $xSh \succ ySh$ $\forall h \in F$ by Q3. Thus $xSy \succ x\varnothing y$. Hence $S \succ^* \varnothing$.
P2. If A is null, then $xAy \sim x\varnothing y$. If A is not null, and if $x \succ y$, then $xAy \succ x\varnothing y$ on A by Q3 and $xAy = x\varnothing y$ on A^c. Then $xAy \succeq x\varnothing y$. Hence $A \succeq^* \varnothing$.
P4. Suppose $x \succ y$, and let A and B are mutually I-monotonic (events) such that $(A \cup B) \cap C = \varnothing$. Then xAy and xBy are mutually I-monotonic (acts). If $A \succ B$, then $xAy \succ xBy$ by Q4. Let $f = xAy$, $g = xBy$, $f' = fCy$ and $g' = gCy$. Then $f = xAy \succ xBy = g$ iff $f' = x(A \cup C)y \succ x(B \cup C)y = g'$ by Q2. Hence $A \succ^* B$ iff $A \cup C \succ^* B \cup C$.
P5. Given $A \succ^* B$. Suppose $x \succ y$, then $xAy \succ xBy$ by Q4. Define $f = xAy$ and $g = xBy$. Then there is an appropriate basic partition, $\{E_1, E_2, \ldots, E_n\}$, of S such that $g' = xE_ig$ and $f \succ g'$ by Q6. Hence $A \succ^* B \cup E_i$ for every $i \in \{1, 2, \ldots, n\}$.
Thus, by Theorem 1, $\succ^*$ has a unique I-nonadditive probability representation. Q.E.D.

Now define Q^f on X as follows: $Q^f(Z) = Q(\{s \in S \mid f(s) \in Z, Z \subset X\})$. Define $a_i^f(Z) = Q_i(\{s \in S_i \mid f(s) \in Z, Z \subset X\})$. $a_i^f(Z)$ also equals $Q(\{s \in S_i \mid f(s) \in Z, Z \subset X\})/Q(\{s \in S_i\})$ for all $i \in I$. $a_i^f(X)$ is simply denoted by a_i^f. Let $Y'(i) = \{a_i^f \in Y \mid \forall f \in F\}$. For convenience, we sometimes denote $Y'(i)$ by Y'. So the meaning of Y' is slightly different by context.
Then, we get a function ρ from $F \times I$ to Y', i.e., $\rho(f, i) = a_i^f \in Y$ for all $i \in I$. a_i^f is also denoted by $a(i)$, if $a(i) = a_i^f$ $\forall i \in I$. Furthermore, a_i^x denotes $Q_i(\{s \in S_i \mid f(s) = x \text{ for all } s \in S_i\})$.

Define $a^f = (a_1^f, a_2^f, \ldots, a_i^f, \ldots)$, where the ith element of a^f is a_i^f $\forall i \in I$. Let $L' = \{a^f \in L \mid \forall f \in F\}$. This gives us a function θ from F to L', i.e., $\theta(f) = a^f$.

Remark 7. Let $U(f, i)$ be as follows:

$$U(f, i) = \int_{S_i} u(f)\, \mathrm{d}Q_i.$$

$U(f, i)$ is a function from $F \times I$ to $\mathbb{R}$. Remind that ρ is a function from $F \times I$ to Y' such that $\rho(f, i) = a_i^f$, and v a function from $Y' \subset Y$ to $\mathbb{R}$, respectively. By their construction, $U(f, i) = v(\rho(f, i)) = v(a_i^f)$ holds.

Definition 4. A binary relation $\succ^{**}$ on L' is defined as follows.

$$\forall a, b \in L' : a \succ^{**} b \quad \text{if} \quad f \succ g \text{ whenever } a = a^f \text{ and } b = a^g.$$

To use Theorem B, we must show if $\succ$ on F satisfies Q1–Q7, then $\succ^{**}$ on L' satisfies B1–B6. If this statement holds, then we will obtain the function Φ by $\Phi = V(\theta)$. To consider the relation between these two binary relations, the following two lemmas are provided.

Lemma 2.3. [*Q1, Q2,* $\{A_1, A_2, \ldots\}$ *is a partition of A where* $A_i = A \cap S_i$ *for* $i \in I$]$\to$($f \succeq g$ *on* A_i *for all* $i \in I \to f \succeq g$ *on A; if in addition* $f \succ g$ *on* A_i *for some* $i \in I$, *then* $f \succ g$ *on A*).

Proof. Firstly, we will show this lemma in case the set I is finite, $I = \{1, 2, \ldots, m\}$. For f, g and $h \in F$, note that every k_i and k_j are mutually I-monotonic acts. By using

$$k_0 = f, \quad k_i = f \text{ on } \bigcup_{n=1}^{i} A_n,$$

$$= g \text{ on } \bigcup_{n=i+1}^{m} A_n,$$

$$= k \text{ on } A^c \quad (i = 1, 2, \ldots, m).$$

Q2 repeatedly, the following relation is obtained: $f = k_0 \succeq k_i \succeq k_{m-1} \succeq k_m = g$. Hence, $f \succeq g$.

Secondly, consider the set I is not necessarily finite. Let $J \subset I$ be nonempty and finite ($J \cup J^c = I$). Denote $\bigcup_{i \in J} A_i$ by A_j. Suppose not ($f \succeq g$), i.e., $g \succ f$. Let $f' = f$ on A_J, and h on A_J^c, and $g' = g$ on A_J, and h on A_J^c. Thus, by Q2, $g \succ f$ iff $g' \succ f'$. $g' \succ f'$ means $g \succ f$ on A_J. But, if $f \succeq g$ for on A_J, then $f \succ g$ on A_J by the former result, a contradiction.

The additional phrase in Lemma 2.3 is easily derived similarly. Q.E.D.

Remark 8. In Lemma 2.3, A is an arbitrary set in S. So we can take $A = S$ and consider the structural partition of S. Thus, the following relation is obtained:

$$f \succeq g \text{ on } S_i \quad \text{for} \quad \text{all } i \in I, \quad \text{then} \quad f \succeq g \text{ on } S.$$

Lemma 2.4. Suppose Q1–Q7, then the function v, which is defined on Y′ is bounded for all $i \in I$. *Here the function v is as follows:*

$$v(a_i^f) = \int_{S_i} u(f)\,\mathrm{d}Q_i.$$

Proof. With respect to the boundedness of u, it is shown in Fishburn (1970, pp. 206–207). By using this result, the boundedness of v is derived in Fishburn (1970, pp. 207–210). Q.E.D.

Lemma 2.5. If $\succ$ on F satisfies Q1–Q7, then $\succ^{**}$ on L′ satisfies B1–B6.

Proof. The key relation for this proof is that

$$f \succ g \quad \text{if and only if} \quad a^f \succ^{**} a^g.$$

By Definition 4, $a^f \succ^{**} a^g$ if $f \succ g$. Suppose if not ($f \succ g$), i.e., $g \succeq f$, then $a^g \succeq^{**} a^f$, i.e., not ($a^f \succ^{**} a^g$). Then $f \succ g$ if $a^f \succ^{**} a^g$. Hence, this key relation holds.

B1. This is easily derived by Q1 and the definition of $\succ^{**}$.
B2. If $f, g, h \in F$ are mutually I-comonotonic (acts), then $a^f, a^g, a^b \in L$ are mutually comonotonic. Let $f' = f$ on E, and h on E^c for $E, E^c \in M_b^k$ for some k. By Lemma 2.2, there exists and I-nonadditive probability representation Q. So, let $Q(E) = \lambda$. [Hence, $Q(E^c) = 1 - \lambda$]. $0 < \lambda < 1$ because E and E^c are nonempty. By the key relation, $f \succ g$ iff $a^f \succ^{**} a^g$, and $f' \succ g'$ iff $\lambda a^f + (1-\lambda) a^h \succ^{**} \lambda a^g + (1-\lambda) a^h$. And, by Q2, $f \succ g$ iff $f' \succ g'$. Therefore $a^f \succ a^g$ iff $\lambda a^f + (1-\lambda) a^h \succ^{**} \lambda a^g + (1-\lambda) a^h$.
B3. Let $a^f = a$, $a^g = b$ and $a^h = c$. So, $a \succ^{**} b$ means that $f \succ g$. Therefore, by Q6, if $f \succ g$, then there exists a basic partition $\{E_1, E_2, \ldots, E_n\}$ such that $f' = x$ on E_i, and f on E_i^c for given $x \in X$, and $f' \succ g$. Here f and f' are mutually I-monotonic (acts). Suppose Q is an I-nonadditive probability which represents $\succ^*$ induced by $\succ$. Then $Q(E_i) + Q(E_i^c) = 1$ and $0 < Q(E_i) < 1$ if both E_i and E_i^c are not null events. Let $\alpha = Q(E_i^c)$. Then $a^{f'} \succ^{**} \alpha a^f + (1-\alpha) a^x \succ^{**} a^g$.

Since x is arbitrary, we can take x to hold $a^b(i) \succeq^{**} a^x$ for all $i \in I$. Thus, $\alpha a^f + (1-\alpha) a^h \succeq^{**} a^g$. Thus $a \succ^{**} b$ and $b \succ^{**} c \to \alpha a + (1-\alpha) c \succ^{**} b$ for some $\alpha \in (0,1)$. Similarly, we can get $a \succ^{**} b$ and $b \succ^{**} c \to b \succ^{**} \beta a + (1-\beta) c$ for some $\beta \in (0,1)$.
B4. By Q5, there are some $x, y \in X$ such that $x \succ y$. Then we can define $a, b \in Y'$ as follows: $a = a^x$, $b = a^y$. So $a \succ^{**} b$ by the definition of $\succ^{**}$.
B5. Let $a^f = a$ and $a^g = b$. If $a(i) \succ^{**} b(i)$ for all $i \in I$, then $f \succ_i g$ for all $i \in I$. This means $f \succeq g$ by using Remark 8. Hence, $a \succ^{**} b$ by the definition of a and b.
B6. By Lemma 2.4, there is no $f \in F$ such that, for all $y' \in Y'$, $a^f(i) \succ^{**} y'$ or $y' \succ^{**} a^f(i)$ or both, for some $i \in I$. Because if there exists such f, then v is not bounded. Hence there exist $y, z \in Y'$ such that $y \succeq^{**} a^f(i) \succeq^{**} z$ for all $i \in I$. Q.E.D.

Proof of Theorem 2. If $\succ$ on F satisfies Q1–Q7, then by Lemma 2.1, $\succ_i$ satisfies C1–C7 for every $i \in I$ and every $\succ_i^*$ has an additive probability representation such that

$$A \succ_i^* B \quad \text{if and only if} \quad Q_i(A) > Q_i(B) \quad \text{for every } i \in I,$$

and there is a bounded functional u on X such that

$$f \succ_i g \quad \text{if and only if} \quad \int_{S_i} u(f)\,\mathrm{d}Q_i > \int_{S_i} u(g)\,\mathrm{d}Q_i.$$

Moreover, u is unique up to positive affine transformation for all $i \in I$.

Next we define $a_i^f = Q_i(\{s \in S_i \mid f(s) \in X\})$ and $a^f = (a_1^f, a_2^f, \ldots) \in L$ for all $f \in F$. These are well defined by Lemma 2.1 and Lemma 2.2. Q_i and Q are consistent by Remark 3.

If $\succ$ on F satisfies Q1–Q7, then $\succ^{**}$ satisfies B1–B6 by Lemma 2.5. Then, by Theorem B, there is a nonadditive probability measure σ on Ω and there is a unique linear functional v on Y' such that

$$f \succ g \quad \text{if and only if} \quad \int_I v(a(i))\,\mathrm{d}\sigma > \int_I v(b(i))\,\mathrm{d}\sigma \quad \text{for} \quad \text{all } i \in I.$$

Because of Lemma 2.1 and Remark 7, for every $i \in I$,

$$v(a(i)) = \int_{S_i} u(f)\,\mathrm{d}Q_i, \quad v(b(i)) = \int_{S_i} u(g)\,\mathrm{d}Q_i.$$

Consequently,

$$f \succ g \quad \text{if and only if} \quad \int_I \int_{S_i} u(f)\,\mathrm{d}Q_i\,\mathrm{d}\sigma > \int_I \int_{S_i} u(g)\,\mathrm{d}Q_i\,\mathrm{d}\sigma. \qquad \text{Q.E.D.}$$

Appendix: A note on the difference between Gilboa's Axiom (P2*) and Q2

Gilboa (1987) axiomizes his EUNAP model in a very general setting. So, his model may explain the phenomena that our model cannot explain. But his model doesn't include our model. To show this, let's consider the following example.

Gilboa's P2* is the following axiom.

P2*. For all $f_1, f_2, g_1, g_2 \in F$, all A, B and all $x_1, x_2, y_1, y_2 \in X$ such that $y_1 \succ x_1$ and $y_2 \succ x_2$, if

(1) $x_1 A f_1$, $y_1 A f_1$, $x_2 A g_1$, $y_2 A g_1$ are pairwise comonotonic (i.e., mutually comonotonic) and so are $x_1 B f_2$, $y_1 B f_2$, $x_2 B g_2$, $y_2 B g_2$, and
(2) $x_1 A f_1 \sim x_1 B f_2$, $x_2 A g_1 \sim x_2 B g_2$, $y_1 A f_1 \succeq y_1 B f_2$, then $y_2 A g_1 \geqq y_2 B g_2$.

Example. Suppose $A \cup B \subset S$ and $A \cap B = \varnothing$. Let $x_1, x_2, y_1, y_2 \in X (= \mathbb{R})$ be as follows: $x_1 = 0$, $x_2 = 1$, $y_1 = 1$, $y_2 = 2$. And,

$$f_1 = 1 \text{ on } S, \quad f_2 = 3 \text{ on } A, \quad 0 \text{ on } A^c,$$

$$g_1 = 2 \text{ on } S, \quad g_2 = 6 \text{ on } A, \quad 1 \text{ on } A^c.$$

Table A.1

	A	B	$A^c \cap B^c$
$x_1 A f_1$	0	1	1
$x_1 B f_2$	3	0	0
$y_1 A f_1$	1	1	1
$y_1 B f_2$	3	1	0

Table A.2

	A	B	$A^c \cap B^c$
$x_2 A g_1$	1	2	2
$x_2 B g_2$	6	1	1
$y_2 A g_1$	2	2	2
$y_2 B g_2$	6	2	1

Then, x_1Af_1, y_1Af_1, x_2Ag_1, y_2Ag_1 are pairwise comonotonic and so are x_1Bf_2, y_1Bf_2, x_2Bg_2, y_2Bg_2.

Assume $x_1Af_1 \sim x_1Bf_2$, $x_2Ag_1 \sim x_2Bg_2$, and $y_1Af_1 \succsim y_2Bf_2$, then Gilboa's P2* requires $y_2Ag_1 \succsim y_2Bg_2$.

But, we cannot exclude the possibility that $y_2Bg_2 \succ y_2Ag_1$. The reason is as follows.

Suppose the decision maker prefers the larger numbers, i.e., $x \succ y$ if $x > y$. In this example, the following relations hold:

$$y_1Bf_2 \succ y_1Af_1 \text{ and } y_2Bg_2 \succ y_2Ag_1 \text{ on } A,$$

$$y_1Bf_2 \sim y_1Af_1 \text{ and } y_2Bg_2 \sim y_2Ag_1 \text{ on } B,$$

$$y_1Af_1 \succ y_1Bf_2 \text{ and } y_2Ag_1 \succ y_2Bg_2 \text{ on } A^c \cap B^c.$$

But the difference of y_2Bg_2 and y_2Ag_1 is greater than that of y_1Bf_2 and y_1Af_1. Thus, it is possible that $y_2Bg_2 \succ y_2Ag_1$.

Axiom Q2 doesn't necessarily exclude this case. Suppose $A = S_1$, $B = S_2$, and $A^c \cap B^c = S_3$, i.e., $I = \{1, 2, 3\}$. If x_2Ag_1, x_2Bg_2, y_2Ag_1, and y_2Bg_2 are mutually I-monotonic and/or I-comonotonic, then $y_2Ag_1 \sim y_2Bg_2$ if $x_2Ag_1 \sim x_2Ag_1 \sim x_2Bg_2$. But x_2Ag_1 and y_2Bg_2 are neither mutually I-comonotonic nor I-monotonic. Therefore, Q2 doesn't require $y_2Ag_1 \geqq y_2Bg_2$.

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