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MAXMIN EXPECTED UTILITY AND WEIGHT OF EVIDENCE

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1. Introduction

THIS PAPER is concerned with decision-making when some or all of the relevant probabilities are not objectively known. The dominant theory of decision-making in such situations is subjective expected utility theory (SEU). According to SEU if a decision-maker follows certain axioms for rational choice, then he or she will have beliefs in the forms of a probability distribution and will maximise the expected values of a utility function with respect to this probability distribution (see Savage, 1954; Anscombe and Aumann, 1963). This kind of belief is called a subjective probability distribution. It has been observed that individuals deviate from SEU by showing a preference for situations in which uncertainty is resolved with probabilities, which are better known. Such behaviour has been called 'uncertainty aversion'. For evidence on this see Ellsberg (1961) and Anand (1990a). Ellsberg's evidence is discussed in Section 2 of the present paper.

In addition, it can be argued that SEU has some implausible implications. Consider the following example. Miss Julie is invited to bet on the outcomes of two tennis matches. In both she is indifferent between betting on either player to win. In match A this arises because she knows the players well and is firmly convinced that it will be an even match, while in match B she knows nothing about either of the players. (This is adapted from an example used by Gardenfors and Sahlin, 1982). Subjective expected utility theory would describe this situation by saying that she believes that either player would win with probability $\frac{1}{2}$ in both match A and match B. Further SEU would predict that she should be equally willing to bet on either match. Some people might find this counter-intuitive, and wish to argue that she would express a preference for betting on match A. In addition SEU may be criticised for providing an inadequate description of the situation. Miss Julie's beliefs about the outcomes of the two matches are different, yet SEU describes them both by the same subjective probability distribution. Thus, as well as possible empirical failings, SEU can be criticised since it may provide an inadequate description.

Hence it can be argued that a single probability distribution is insufficient to describe a decision-maker's beliefs in some situations of uncertainty. One possible criticism, is that in these circumstances decisions will be based not only upon the numerical value of a subjective probability but will also take account of the weight of evidence upon which this probability judgement is based. This is not a new idea. Discussion of it can be found in Keynes (1921). In recent years, there has been a revival of interest in the concept of weight of evidence; see Gardenfors and Sahlin (1982) and Anand (1991).

Consider the following example. Suppose an engineer states that the probability of a serious accident at a nuclear power station is about one in a million, but then adds that he is not sure of this. How should we interpret this statement? It cannot be the case that the engineer has a second-order probability distribution over probabilities, since if this were so we could obtain a single probability for the chances of an accident by using the rules for reducing compound lotteries to simple lotteries. One possible explanation is that the engineer is saying that the weight of evidence behind this judgement is low. A statement of this sort, can make sense in a theory which allows for the notion of weight of evidence. In Section 4 of the present paper we suggest a way in which weight of evidence can be represented, that does not collapse onto itself as second-order probabilities do.

Gilboa and Schmeidler (1989) have proposed an alternative to SEU which they call maxmin expected utility (MMEU). According to their theory the decision-maker has a convex set C of subjective probabilities. The intuition for this is that when information is limited the decision-maker may consider a range of possible probabilities. They show that the decision-maker has a von Neumann–Morgenstern utility function. Since the subjective probability is not unique there is an interval of possible expected utilities for any action. MMEU predicts that action a will be preferred to action b if and only if the minimum possible value of the expected utility of a is greater than the minimum expected utility of b.

MMEU is a compromise between SEU and maximin, both of which are special cases of it. If the set C consists only of a single probability distribution then MMEU will coincide with SEU. On the other hand if C consists of all possible probability distributions MMEU will coincide with maximin. Thus MMEU encompasses a continuum of decision rules with maximin and SEU as extreme cases. MMEU can be thought of as 'over-weighting' the worst outcomes of actions.

In Section 2 we derive MMEU from some basic assumptions about preferences over uncertain actions. Both out assumptions and conclusions are similar to those of Gilboa and Schmeidler (1989). However we assume the set of states is finite; consequently our proof is simpler. MMEU can allow dominated actions to be chosen. In Section 3 we present a modified version of MMEU, which does not have this disadvantage. In Section 4 we relate MMEU to the notion of weight of evidence. Section 5 considers normative properties of MMEU and argues that this theory is immune to the 'Dutch Book' argument. An application to welfare economics is considered in Section 6 and Section 7 concludes.

2. Maxmin expected utility

2.1. *Axioms*

Our framework for decision-making under uncertainty contains a finite set S of states of nature. *Ex post*, precisely one of these states will be realised. An action is a function from S to $\mathbb{R}$. $A(S)$ denotes the space of all actions. Thus an action a can be identified with the vector $\langle a_1, a_2, \ldots, a_n \rangle \in \mathbb{R}^n$, where n is the number of states. We shall interpret a_i as the utility the agent would receive in state s_i: we implicitly assume that the decision-maker can also take risks with objective probabilities and has formulated a strategy for purchase or sale of such risks. In the Appendix we show how the framework of this section may be derived from preferences over state-contingent lotteries (as in Anscombe and Aumann, 1963). In particular we derive the existence of a von Neumann–Morgenstern utility function.

We use $e_i \in A(S)$ to denote that action which gives utility 1 in state s_i, 0 otherwise. A constant action is an action which yields the same utility in all states. It is an element of $A(S)$ of the form αe where $\alpha \in \mathbb{R}$ and $e = \langle 1, 1, \ldots, 1 \rangle$ is the unit vector in $\mathbb{R}^n$. We assume that the decision maker's preferences can be represented by a function V, which satisfies the following properties.

Assumption 1 Quasi-Concavity: We assume that V is quasi-concave.

Assumption 2 Certainty-Independence: If $b \in A(S)$ and $\alpha e \in A(S)$ is a constant action and $0 < \lambda < 1$ then

$$V(\lambda b + (1 - \lambda)\alpha e) = \lambda V(b) + (1 - \lambda)V(\alpha e)$$

Assumption 3 Weak Dominance: V is increasing in all arguments.

The standard axioms for choice under uncertainty are transitivity, continuity, monotonicity and independence (see for instance Fishburn, 1970). Our assumptions are variants of these. Representing preferences by a real-valued function implies transitivity. Both quasi-concavity and certainty independence may be seen as weak versions of the independence axiom. Weak dominance is a variant of the monotonicity axiom.

One aim of this theory is to explain the Ellsberg paradox, the essence of which can be explained in the following problem. There are two urns, each containing 100 balls. Urn I contains a mixture of red and black balls in unknown proportions, while urn II contains precisely 50 red balls and 50 black balls. Let Red-I denote the gamble of receiving \$100 if a ball randomly drawn from urn I is red. Similar notation will be used for other gambles. Ellsberg reports that most individuals are indifferent between Red-I and Black-I and are also indifferent between Red-II and Black-II. However, the same individuals also express a strict preference for Red-II over Red-I and for Black-II over Black-I. It is easy to check that these preferences are not compatible with subjective expected utility or indeed any other plausible theory of decision-making under uncertainty which is based on conventional subjective probabilities.

The Ellsberg paradox could be described within the Anscombe–Aumann framework as follows. Suppose that there are two states s_R and s_B, where s_R (respectively s_B) is the state where a red (respectively black) ball is drawn from urn I. Consider actions a, b and c as described in the Table 1.

TABLE 1

	s_R	s_B
a	\$100	\$0
b	\$0	\$100
c	$\frac{1}{2}$ \$100 + $\frac{1}{2}$ \$0	$\frac{1}{2}$ \$100 + $\frac{1}{2}$ \$0

The choices Red-I and Black-I are represented by actions a and b respectively. We assume that the individual views the choices Red-II and Black-II as 50:50 randomisations between \$100 and \$0, hence both can be identified with action c. Since $c = \frac{1}{2}a + \frac{1}{2}b$, Ellsberg's evidence can be interpreted as implying that preferences are quasi-concave in probabilities. Thus a typical decision-maker will prefer a 50:50 randomisation between a and b to either of these actions for certain.

A possible motivation for this preference, is that the 50:50 randomisation involves 'hedging'. For a given state good and bad outcomes are averaged, producing a distribution which is preferred to either a or b. This motivates weakening the usual independence axiom, which would imply that if two actions are indifferent, then any mixture of them it itself indifferent to either of the original actions. Two of our axioms are weaker versions of independence. Quasi-concavity says that a mixture of two actions should be at least as good as the least preferred of the original actions. This allows for the possibility that 'hedging' might make the mixture more attractive, by reducing the dependence of the outcomes on the state-uncertainty. In the above example the dependence on the state-uncertainty was eliminated. However, in general, taking probability mixtures will only reduce this dependence. It is not possible to reduce the dependence on state-uncertainty by taking mixtures with a constant action. Certainty independence requires that the usual independence axiom apply to such mixtures. This assumption is made, since the above motivation for deviating from independence does not apply in this context.

2.2. *Algebraic derivation*

Normalisation We may normalise by setting $V(0) = 0$. Certainty independence implies that V is linear on the space of constant actions. By multiplying by a positive constant, if necessary, we may ensure that $V(\alpha e) = \alpha$ for all $\alpha \in \mathbb{R}$.

Lemma 1 V is homogeneous of degree 1.

Proof We need to show if $a, b \in A(S)$ and $\alpha \in \mathbb{R}$, $\alpha > 0$ then $a = \alpha b$ implies $V(a) = \alpha V(b)$. The case where $\alpha = 1$ is trivial. By interchanging a and b, if necessary, we may assume that $0 < \alpha < 1$.

Then

$$\begin{aligned} V(a) = V(\alpha b) = \; & V(\alpha b + (1 - \alpha)0) \\ & = \alpha V(b) + (1 + \alpha)V(0) \qquad \text{(by certainty independence)}, \\ & = \alpha V(b) \qquad \text{(by the normalisation of } V\text{)}. \end{aligned}$$ □

Since V is quasi-concave, using a separation theorem, we may show that for each $a \in A(S)$, there exists a non-zero vector p_a such that

$$V(b) \geqslant V(a) \Rightarrow p_a \cdot b \geqslant p_a \cdot a \tag{1}$$

Lemma 2 $p_a > 0$.

Proof By weak dominance $V(a + e_i) \geqslant V(a)$. Therefore $p_a \cdot (a + e_i) \geqslant p_a \cdot a$, which implies $p_a e_i \geqslant 0$ but $p_a \cdot e_i$ is the ith component of p_a. Since i was arbitrary this implies $p_a > 0$. □

Note that p_a is not unique since if $\beta > 0$ the vector βp_a will also satisfy equation (1). Thus, by multiplying by a positive constant, if necessary, we may ensure that $p_a \cdot e = \sum_{i=1}^{n} p_a \cdot e_i = 1$. With this normalisation the vector p_a has the formal properties of a probability distribution over S.

Lemma 3 $p_a \cdot a = V(a)$.

Proof $V(a)e$ is the constant action which is indifferent to a. If $\lambda > 0$ then $b = (1 - \lambda)V(a)e + \lambda a$ is a point on the half line starting at $V(a)e$ and passing through a. By certainty independence if $0 < \lambda < 1$,

$$V(b) = (1 - \lambda)V(a)V(e) + \lambda V(a) = (1 - \lambda)V(a) + \lambda V(a) = V(a)$$

If $\lambda > 1$, then $a = (1/\lambda)b + (1 - 1/\lambda)V(a)e$. Hence $V(a) = (1/\lambda)V(b) + (1 - 1/\lambda)V(a)$, by certainty independence, which implies $V(b) = V(a)$. By definition of p_a, $p_a \cdot ((1 - \lambda)V(a)e + \lambda a) \geqslant p_a \cdot a \Rightarrow (1 - \lambda)V(a) \geqslant (1 - \lambda)p_a \cdot a$ (since $p_a \cdot e = 1$). This implies $V(a) = p_a \cdot a$, since $1 - \lambda$ may be either negative or positive. □

Lemma 4 For all $a, b \in A(S)$, $V(a) \leqslant p_b \cdot a$.

Proof Suppose first that $V(a) = V(b)$. By quasi-concavity for $0 < \lambda < 1$, $V(\lambda a + (1 - \lambda)b) \geqslant V(b)$. Therefore by definition of p_b, $p_b \cdot (\lambda a + (1 - \lambda)b) \geqslant p_b \cdot b$. Expanding, $\lambda p_b \cdot a + (1 - \lambda)p_b \cdot b \geqslant p_b \cdot b$ hence $\lambda p_b \cdot a \geqslant \lambda p_b \cdot b = \lambda V(b) = \lambda V(a)$, using Lemma 3. Therefore $p_b \cdot a \geqslant V(a)$ in this case. Suppose now $V(b) > V(a)$. Let $\gamma = V(b) - V(a)$, then $V(a + \gamma e) = (1 + \gamma)V[(a/(1 + \gamma)] + [(\gamma/1 + \gamma)e]$ by linear homogeneity, $= (1 + \gamma)V(a/(1 + \gamma)) + (1 + \gamma)V(\gamma e/1 + \gamma))$ by certainty independence, $= V(a) + \gamma = V(b)$ by linear homogeneity. By the previous case, $V(a + \gamma e) \leqslant p_b \cdot (a + \gamma e)$, which implies, $V(a) + \gamma \leqslant p_b \cdot a + \gamma$. Therefore $V(a) \leqslant p_b \cdot a$. □

Theorem 1 There exists a closed convex set C of probability distributions on S such that $V(a) = \min_{p \in C} E_p a$ if and only if V satisfies, certainty independence, monotonicity and quasi-concavity (1)–(3).

Proof Suppose that V satisfies these axioms. Let C denote the closure of the convex hull of $\{p_a: a \in A(S)\}$. By Lemma 7, $V(a) \leqslant p_b \cdot a$ and by Lemma 6 there is equality if $p_b = p_a$. Hence $V(a) = \min_{p \in C} E_p a$. It is fairly easy to check that if V has this functional form then it satisfies our three axioms. □

Some intuition for this result can be obtained from the following argument. Quasi-concavity implies that indifference curves must be convex. As proved above, certainty independence implies linear homogeneity. Together these assumptions imply that the indifference surface through the origin is a convex cone, K. A closed convex cone has a dual cone C and $K = \{x \in \mathbb{R}^n: p \cdot x \geqslant 0, \forall p \in C\}$. This is equivalent to the formula for the indifference curve through the origin due to MMEU. By normalising the members of C we can interpret them as probability distributions. Finally certainty independence implies that all indifference surfaces have the same shape as the indifference surface through the origin.

2.3. *Diagrammatic exposition*

In this subsection we present a diagrammatic demonstration of the result for the case where there are two states. In Figs 1 and 2 the quantities on the axes are utility in state one, u_1 and utility in state two, u_2. When there are two states,

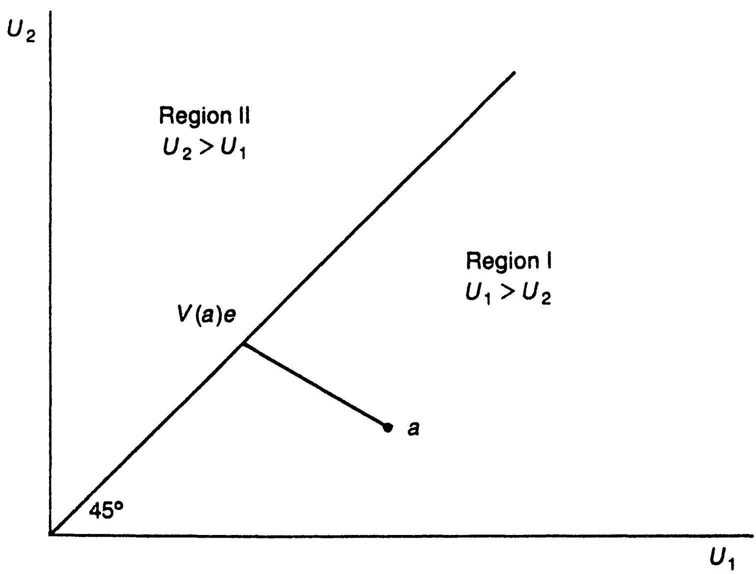

FIG. 1.

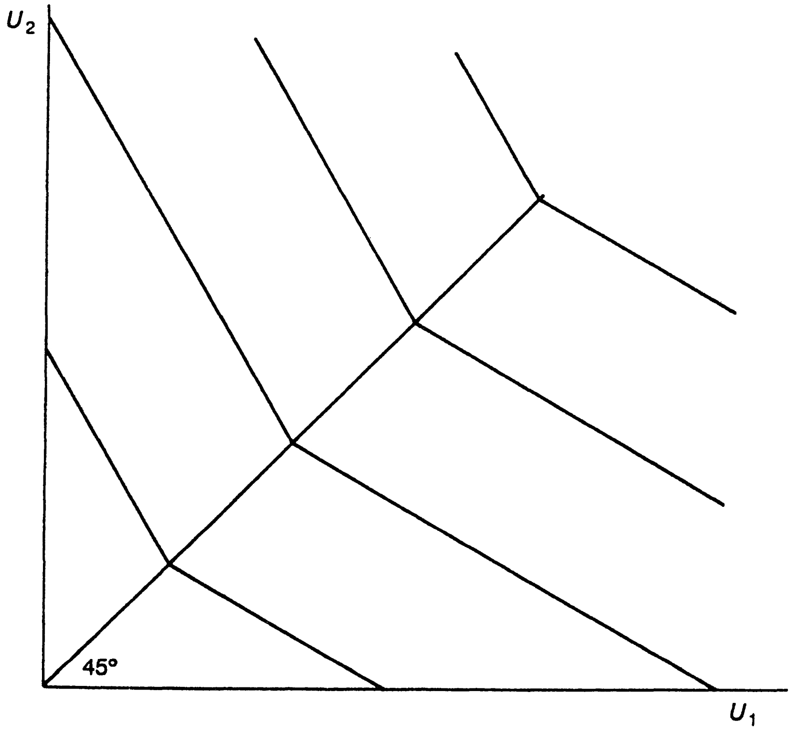

FIG. 2.

actions may be identified with points in these diagrams. As shown in Fig. 1 the positive quadrant of $\mathbb{R}^2$ can be divided into two regions. In region I, state one gives a better outcome than state two, while in region II, state two gives the better outcome.

Let a be an arbitrary point in region I. Then by certainty-independence any point of the form,

$$b = (1 - \lambda)V(a)e + \lambda a, \qquad \lambda \geqslant 0 \tag{2}$$

is indifferent to a. By monotonicity this is the equation of that section of the indifference curve through a which lies in region I. Now (2) is the equation of a straight line. By certainty independence all the indifference curves in region I must be straight lines with the same slope. By similar reasoning the indifference curves in region II are all straight lines with the same slope as one another. We are not, however, able to deduce that the indifference curves have the same slope in both regions. If this were the case then the decision-maker would be maximising subjective expected utility. Finally quasi-concavity implies that the indifference curves are kinked at the 45° line so that the upper contour set is convex as shown in Fig. 2. But these are precisely the indifference curves predicted by MMEU.

3. Lexicographic extension of MMEU

A possible criticism of MMEU is that it can allow a dominated alternative to be chosen.[1] Consider the following example.

Example 1 Suppose there are two states s_1 and s_2. The decision-maker has the following set of subjective probabilities $\{(\alpha, 1-\alpha): 0 \leqslant \alpha \leqslant \frac{1}{4}\}$. Actions a and b give payoffs as follows:

	s_1	s_2
a	15	10
b	16	10

Then $V(a) = V(b) = 10$. Hence MMEU would recommend that a could be chosen even though a is dominated.

While SEU can suffer from similar problems, it could be argued that violation of weak dominance poses a greater problem for MMEU. As with SEU, violation of weak dominance can occur when the decision-maker assigns probability zero to a state. (While in general it may be rational to ignore outcomes which only arise with zero probability, this is not always the case. In some games the outcome may depend crucially on what happens at nodes with zero probability, see Blume *et al.*, 1991.) In MMEU, where the decision-maker simultaneously holds a number of subjective probabilities, it is more likely that there will be states which are assigned probability zero in at least one of the subjective probability distributions.

Example 1 suggests that weak dominance should be strengthened. However, only preferences which are lexicographic can satisfy a stronger dominance relation. It is not possible to represent lexicographic preferences by a real-valued function hence we redefine our assumptions in terms of a binary relation R. We shall use P to denote the corresponding strict preference.

Assumption 4 Preferences are represented by a binary relation R on $A(S)$ which is complete, reflexive and transitive.

Assumption 5 Certainty Independence For all $a, b \in A(S)$, λ: $0 < \lambda < 1$ and $\alpha \in \mathbb{R}$, $aRb \Leftrightarrow (\lambda a + (1-\lambda)\alpha e)R(\lambda b + (1-\lambda)\alpha e)$.

Assumption 6 Quasi Concavity For all $a, b \in A(S)$ such that aRb and all λ: $0 < \lambda < 1$, $(\lambda a + (1-\lambda)b)Rb$.

Assumption 7 Strict Dominance For all $a, b \in A(S)$ if $a_i \geqslant b_i$ for $1 \leqslant i \leqslant n$ and there exists j such that $a_j > b_j$ then aPb.

Theorem 2 shows that if strict dominance is imposed, then the resulting

[1] I would like to thank Salvador Barbera for suggesting this argument.

preferences must be lexicographic, the first component of which is MMEU while the second is an extension of the dominance sub-relation. The second component generates corresponding orderings on all indifference surfaces of the first component. Apart from these two restrictions the second component is arbitrary.

Theorem 2 Let R be a set of preferences on $A(S)$ which satisfy strict dominance, quasi-concavity and certainty independence (Assumptions 4–7), then there exists a closed convex set C of probability distributions on S such that

$$\min_{p \in C} E_p a > \min_{p \in C} E_p b \Rightarrow aPb$$

Let $I = \{a \in A(S): \min_{p \in C} E_p a = 0\}$ then there is an ordering R^* on I such that, $aRb \Leftrightarrow [(\min_{p \in C} E_p a > \min_{p \in C} E_p b)$ or $(\min_{p \in C} E_p a = \min_{p \in C} E_p b)$ and there exists $\beta \in \mathbb{R}$ such that $(a + \beta e)R^*(b + \beta e)]$. Moreover the dominance partial order on I is a sub-relation of R^*.

Theorem 2 is proved after the following two lemmas. If $a \in A(S)$ define $J(a) = \inf\{\alpha: \alpha e P a\}$.

Lemma 5 J satisfies, quasi-concavity (Assumption 1), certainty independence (2) and weak dominance (3).

These properties of J follow from the corresponding properties of R, hence we omit the proof. It follows from Theorem 1 that there is a convex set C of probability distributions over S such that

$$J(a) = \min_{p \in C} E_p a \tag{3}$$

Lemma 6 Suppose $a, b \in A(S)$ and $J(a) > J(b)$ then aPb.

Proof Since $J(a) > J(b)$ there exists $\gamma \in \mathbb{R}$ such that $\gamma e P b$ but $\gamma < \inf\{\alpha: \alpha e P a\}$. This implies $aR\gamma e$, which with transivity of R implies aPb. □

Proof of Theorem 2 By certainty independence,

$$J(a + \alpha e) = J(b + \alpha e) \Leftrightarrow J(a) = (b)$$

which says that the indifference surfaces of J are a constant distance apart (as measured along the 45° line). Let I denote $\{a: J(a) = 0\}$ i.e. I is a surface on which J is constant. Let R^* denote the restriction of R to I. Suppose $J(a) = J(b) = \beta$, then $J(a - \beta e) = J(b - \beta e) = 0$, by eq. (3). It follows that $aRb \Leftrightarrow (a - \beta e)R^*(b - \beta e)$. Strict dominance implies that the dominance relation on I must be a sub-relation of R^*. This conpletes the proof of Theorem 2. □

Example 2 shows that there exist rules which satisfy the assumptions of this section.

Example 2 Let C be a convex set of subjective probabilities over S and let q be a probability distribution over S which assigns strictly positive probability to every state. Define a strict preference P by $aPb \Leftrightarrow \min_{p \in C} E_p a > \min_{p \in C} E_p b$ or ($\min_{p \in C} E_p a = \min_{p \in C} E_p b$ and $E_q a > E_q b$.) Since q does not assign zero probability to any state it follows that this binary relation must satisfy strict dominance. Suppose a dominates b then for all $p \in C$, $E_p a \geqslant E_p b$ and $E_q a > E_q b$ hence aPb. It is easy to see that this example satisfies our other axioms.

This example bears a resemblance to the decision theory proposed in Blume *et al.* (1991), in which the framework of Anscombe and Aumann (1963) was modified by adding an axiom similar to our strict dominance and weakening the continuity axiom. The motivation for this change is that, like MMEU, SEU can allow a strictly dominated action to be chosen. They conclude that the agent has a hierarchy of subjective probabilities. The decision-maker maximizes expected utility with respect to the first probability distribution. In the event of a tie he or she will maximise expected utility using the second probability distribution and so on. The authors apply this modified version of SEU to game theory. Example 2 could be extended in a similar fashion with a hierarchy of convex sets of probabilities. This literature should be distinguished from the kind of lexicographic preferences in Fishburn (1971). In the latter theory utilities are lexicographic but probabilities are not. This can be viewed as dual to the analysis of Blume *et al.*

4. Weight of evidence

Previous literature has used weight of evidence to describe situations, in which probabilities are ambiguous. In this section we shall relate this to MMEU and show that the Ellsberg paradox is compatible with MMEU.

In the introduction we discussed an example (concerning nuclear accidents), where the concept of weight of evidence seemed to be necessary. In addition it could be argued that weight of evidence is implicitly used in hypothesis testing. Statistical procedures such as significance tests depend on the number of observations. In particular, the outcomes of two sets of trials together may be considered significant even though neither is significant on its own. For instance 'critical n' seems to embody weight of evidence (see Anand, 1990b, p. 65). Critical n indicates the number of observations required to make a result statistically significant.

We should like to argue that MMEU contains a reasonable representation of weight of evidence. Recall that MMEU represents beliefs by sets of subjective probabilities. Suppose that C_1 and C_2 are two sets of probabilities. We would like to say that C_1 has greater weight than C_2 if C_1 is a subset of C_2. That is, weight of evidence can be represented by the partial order of set inclusion. Hence if a person has more weight of evidence, he or she will assign a smaller range of probabilities to any event. We may then extend our comparison of weight to probability distributions by saying that if $p_1 \in C_1$ and $p_2 \in C_2$ then there is greater weight behind p_1 than p_2 if $C_1 \sqsubseteq C_2$.

Previous theories of weight have used a numerical representation (see Gardenfors and Sahlin, 1982), which implies that weight must provide a complete ordering of beliefs. We shall argue to the contrary that weight should not be represented by a numerical scale. Evidence is inherently multi-dimensional. For instance, suppose an agent wishes to assess the probability that a particular coin will come up heads. We can distinguish two kinds of evidence which might be relevant. The agent could observe that results of a few tosses of the coin or alternatively he or she could perform a physical examination to check whether it is symmetric. It is not *a priori* clear that such different kinds of evidence could be combined. However they could certainly be used to generate a partial ordering. For instance, it is clear that there is greater weight of evidence when there is more of both kinds of information. At a theoretical level our representation of weight is superior since the sets C are derived within an axiomatic theory.

4.1. *The Ellsberg paradox*

Weight of evidence can be used to explain the Ellsberg Paradox, which was described in Section 2. Recall most subjects expressed a preference for a gamble on a ball drawn from the urn with known proportions of red and black balls. It could be argued that the subjects attach greater weight to the known probability distributions and this explains their choices.

A related explanation of the Ellsberg Paradox can be given using MMEU. We shall retain the notation of Section 2. Suppose the decision-maker has the following set of subjective probabilities; $\{p_R = \frac{1}{2} + \alpha, p_B = \frac{1}{2} - \alpha\colon -\varepsilon \leqslant \alpha \leqslant \varepsilon\}$, where $\varepsilon > 0$ is given, and p_R (respectively p_B) is the probability that a red (respectively black) ball is drawn from urn I. If we adopt the normalisation that $u(\$0) = 0$ then we obtain

$$V(\text{red-I}) = (\tfrac{1}{2} - \varepsilon)u(\$100) = V(\text{black-I})$$

$$V(\text{red-II}) = \tfrac{1}{2}u(\$100) = V(\text{black-II})$$

It is seen that for any positive ε these preferences are compatible with the typical response in the Ellsberg Paradox. This set of subjective probabilities can be given the interpreation, that the subject believes that a range of probabilities are possible. Hence it could be said that the reason that the subject prefers to bet on urn II is, that he or she is not certain whether the probability of getting a red ball from urn-I is one half.

5. Normative properties of MMEU

A common reaction to proposed alternatives to SEU is to argue that while they may provide a better description of behaviour they are not suitable as normative theories. There are two main arguments in favour of the superior normative performance of SEU.

5.1. *Axiomatic foundations*

The first argument says that it is desirable to follow SEU, because the latter theory has been axiomatically derived from a set of appealing axioms for decision-making under uncertainty. Two of the more prominent axiomatisations of SEU are Savage (1954) and Anscombe and Aumann (1963).

Anscombe and Aumann's framework is very similar to that of the present paper. In their model there are both known and unknown probabilities. The principal difference is that we have relaxed Anscombe and Aumann's independence axiom to certainty independence. Thus to establish that the axioms for MMEU are less rational than those for SEU it is necessary to establish that it is desirable to apply the independence axiom to all actions, not just to certain actions. We might not wish to apply independence to all actions since by taking mixtures of non-certain actions it is possible to reduce the dependence of the pay-offs on the unknown probabilities. For more details see Section 2. If a decision-maker displays uncertainty-aversion in the sense of Ellsberg (1961), then this could make a mixture of two actions more attractive than either of them on their own. Savage uses a framework without objective probabilities which cannot easily be compared to the present paper. However it would be possible to make similar criticisms of Savage's sure thing principle to the above criticism of the independence axiom.

5.2. *The Dutch Book argument*

The second normative argument in favour of SEU, is the Dutch Book argument, which says that if an individual does not follow SEU then there is a sequence of trades, each of which on its own appears acceptable, however the effect of the whole sequence is such that he or she will lose money for certain. Classic statements of this argument can be found in de Finetti (1975) and Freedman and Purves (1969). Both assume that the individual is prepared to bet both for and against any given event at the same odds. MMEU preferences do not have this property, hence these formulations of the Dutch Book argument do not apply to them.

For any given event, MMEU implies that the individual will have an interval of subjective probabilities. He or she will only accept bets that the event will occur at odds less than the minimum of this interval. Similarly the individual will only accept bets that the given event will not occur, at odds greater than the maximum of the interval. There are no odds at which the individual is prepared to bet both for and against the same event. Moreover for odds within the interval the individual is prepared to accept neither bet.

It is clear that MMEU is immune to Dutch Books of the conventional kind. Suppose that an individual obeys the axioms for MMEU and has a set C of subjective probabilities. Let $p \in C$. Then the individual will only accept a deal if it would be accepted by a decision-maker who followed SEU and had subjective probability p and will not accept all the deals the SEU decision-maker would accept. Since it is generally agreed that an SEU decision-maker will never accept a Dutch Book neither will the individual who follows MMEU.

Machina (1989) presents a second argument which can be used to show that MMEU preferences are not subject to the Dutch Book argument. If we view the same physical commodity in different states of nature as different goods then a formal equivalence can be established between a problem of decision-making under uncertainty and one without uncertainty. MMEU preferences are transitive and hence would correspond, under the above equivalence, to transitive preferences under certainty. If a Dutch Book were possible against an individual with MMEU preferences, this would induce a Dutch Book against an individual with transitive preferences in a situation of certainty. We know that the latter kind of Dutch Book is not possible since it would contradict standard consumer theory. Hence it is not possible to construct Dutch Books against individuals with MMEU preferences.

5.3. *Preference reversal incoherence*

Buehler (1976) has presented a variation on the Dutch Book argument which poses problems for MMEU. He defines a set of preferences to be preference reversal (PR)-incoherent if they contain a subset such that if all the preferences in the given subset are reversed then the decision-maker will be better off regardless of how uncertainty is resolved. Buehler shows the absence of PR-incoherence implies that preferences must obey the SEU axioms. MMEU preferences are potentially subject to PR-incoherence as can be seen from the following example.

Example 3 Suppose there are two states s_1, s_2. Let p_1 and p_2 denote the decision-maker's subjective probabilities, for s_1 and s_2 respectively. Assume that the decision-maker has the following set of subjective probabilities, $\{\langle p_1, p_2\rangle : p_1 = \alpha, p_2 = 1 - \alpha, 1/3 \leqslant \alpha \leqslant 2/3\}$. Assume, for simplicity, that utility is linear in money. Consider the following four actions with monetary outcomes as defined below.

	s_1	s_2
a	0	0
b	-9	12
c	12	-9
d	3	3

With the set of probabilities specified above we find $V(a) = 0$, $V(b) = -2$, $V(c) = -2$, $V(d) = 3$. If the decision-maker were offered a choice of a or b she would choose a. Similarly she would choose a rather than c. Buehler argues that these preferences are incoherent, since the decision-maker rejected b and c. Yet had she accepted both b and c would have gained \$3 for certain.

However Buehler's argument appears to be misspecified. In the Savage framework the actions were supposed to be mutually exclusive. If the decision-maker is being offered the choice of a, b, c or b and c together then this further option should be specified as a separate action. If this is done it would correspond to action d. This is clearly preferred to all the other actions. Thus a decision-making who used MMEU would not pass up a certain \$3, provided the choices are correctly specified.

One might attempt to reformulate Buehler's Dutch Book argument as follows. Suppose the decision-maker is offered the choice between a and b. He or she will choose a. Now suppose that unexpectedly he or she is offered c in addition to his or her previous choice. It is true that the decision-maker will then reject c and in the two choices will have forgone the opportunity to make money for certain.

I would like to argue that this is not really irrational. New information has come to light which means that *ex-post* the decision-maker may wish that he or she had not chosen a, but since this information was not available when the original decision was made it does not seem sensible to describe it as irrational. In Example 4 we shall indicate how a similar phenomena can arise without uncertainty if an individual has preferences over goods which are not additively separable.

Example 4 Suppose an individual prefers gin and tonic together to whisky, prefers whisky on its own to gin on its own, and least of all she likes the combination whisky and tonic. Such an individual if offered the choice between whisky and gin would choose whisky. If subsequently she was unexpectedly asked if she would like tonic with her drink she would refuse. One could argue that she is irrational, since if both her preferences were reversed she would have obtained a more preferred combination of drinks i.e. gin and tonic.

The origin of this 'inconsistency' is preferences which are not separable. Her preferences between whisky and gin are reversed if tonic is available. Somebody with additively separable preferences over drinks and mixers, would achieve their preferred combination whether or not the subsequent offer was anticipated. Most economists would not accept that the possibility of receiving a less preferred drink, if unexpectedly offered tonic, was a convincing argument for preferences which are separable between alcoholic drinks and mixers. If the analogy is accepted, we should not accept that the possibility of 'PR-incoherence' implies that preferences under uncertainty should be additively separable across states of nature or that MMEU preferences are irrational. (Machina, 1989, discusses the relation between non-separable preferences under certainty and non-expected utility preferences under uncertainty in greater detail.)

In summary, an individual with MMEU preferences will not express PR-incoherence provided that he or she fully anticipates all the options available. If new options unexpectedly become available it is possible that the decision-maker could have been better off had he or she made other choices in the past. However this is not restricted to MMEU but can occur with any set of preferences which are not additively separable, whether or not there is uncertainty. Viewed in this light it seems hard to judge a so-called PR incoherence as being irrational.

6. Application to welfare economics

In this section we consider the choice of a social welfare function by a group of people in the original position. These people understand what a human society is, but do not know which position they will occupy in society. It is argued that in such circumstances the social welfare function which would be picked would be 'fair' in the sense that it would not favour any particular group.

The type of social welfare function derived from this analysis depends crucially upon which decision theory the individuals use. Harsanyi (1953, 1955) assumed that individuals would use expected utility theory in the original position and deduced that they would choose utilitarianism. In contrast if it is assumed that individuals use maximin, then they will choose to maximise the utility of the worst-off individual in society. (This is a very simplified account of the difference principle of Rawls, 1972.) We shall refer to the latter rule as a 'Rawlsian' social welfare function.

Both the Rawlsian and utilitarian social welfare functions have some properties, which may be seen as being counterintuitive. Utilitarianism can be very non-egalitarian. For instance, consider a group of people with identical linear utility functions, who wish to divide a cake between them. Utilitarianism would say that any allocation, however unequal was optimal. On the other hand the Rawlsian social welfare function can be criticised for its extreme egalitarianism. Under the Rawlsian social welfare function there is no bound to the size of the sacrifices which better-off people can be required to make, in order to produce small gains for the worst-off individual. In addition there is no obligation to introduce policies which benefit the second worst-off individual, or indeed any individual, who is not actually the worst-off.

Suppose that individuals use MMEU in the original position. Then the set of probabilities C could be interpreted as a set of social welfare weights. Social welfare would be a weighted sum of utilities, where for any given distribution of utilities the weights which gave the lowest utility sum would be used. This social welfare function would give the worst-off individuals greater weight than the better-off but does not give them lexicographic priority. The extent to which worse-off individuals receive greater weight depends upon the size of C. The larger C the greater the weight placed on worse-off individuals. This will give rise to a class of social welfare functions depending upon the set C. If C consists of all possible probability distributions then the individuals would choose the Rawlsian social welfare function. On the other hand if C consists of the single probability distribution in which each individual believes he or she is equally likely to occupy any position within society, then the social welfare function will be utilitarian.

This overcomes the criticism of the utilitarian and Rawlsian social welfare functions discussed previously. Provided the set C is symmetric and does not consist of a single probability distribution, the resulting social welfare function will require equality in the cake division problem with linear utility. On the other hand this rule does not require the better-off individuals to make unboundedly large sacrifices to help the worst-off individual and it does give a role for redistribution towards people other than the worst-off individual. A similar social welfare function has been proposed by Ebert (1988).

7. Related literature

7.1. *Multiple subjective probabilities*

Decision theories similar to MMEU have been proposed in the philosophical literature by Levi (1980, 1986) and Gardenfors and Sahlin (1982).

Levi's theory is slightly different from MMEU since he allows both the utility function and the subjective probabilities to be non-unique. He defines an admissible action to be one which maximises expected utility with respect to some utility function and some probability distribution. He assumes that the decision-maker will choose that admissible action, which maximises the minimal value of expected utility. In other words the decision-maker uses MMEU subject to the additional constraint that only admissible actions be chosen.

Gardenfors and Sahlin (1982) propose a decision theory which is very similar to MMEU. In their theory the utility function is unique, there are multiple subjective probabilities and the decision-maker is assumed to maximise the minimal value of expected utility. There is one difference, they do not require the decision-maker's set of subjective probabilities to be convex. This is not important since MMEU preferences only depend on the extremal points of the set of subjective probabilities.

The first axiomatisation of MMEU was by Gilboa and Schmeidler (1989). Their axioms were similar to those used in Section 2 of the present paper. This chief difference is that they allow for infinite as well as finite state spaces. The main advantage of the present proof is its simplicity as it relies only on elementary properties of convex sets.

7.2. *Complete ignorance*

Arrow and Hurwicz (1972), Barbera and Jackson (1988), Barrett and Pattanaik (1993), and Cohen and Jaffray (1980) provide related theories which could also be used as models for Keynes–Knight uncertainty. In these papers, it is assumed that the decision-maker has no information about the process which determines the state. This is called 'complete ignorance' by Cohen and Jaffray.

All of these papers use a 'merger of states axiom' such as Arrow and Hurwicz's property C. This axiom can be illustrated by the two decision problems in the Tables 2 and 3. These tables represent two different decision problems, one defined over a two-member set of states, and the other defined over a three-member set of states. The merger of states axiom says that a should be chosen in the first decision if and only if a' is chosen in the second. Recall

TABLE 2

	s_1	s_2	s_3
a	x	y	y
b	z	w	w

TABLE 3

	s_1'	s_2'
a'	x	y
b'	z	w

that in these models the decision-maker is assumed to have no information about the process which determines the states. Thus the fact that the outcomes y and w can occur in two states in the first problem does not mean that they are more likely than in the second problem. Hence it can be argued that there is a certain symmetry between the two decisions and there should be a corresponding symmetry between the choices made in the two cases.

The other axioms typically used in these models are a symmetry axiom, which says that decisions should not be changed by permuting the outcomes between the states; a dominance axiom; and transitivity of strict preferences. All of these papers conclude that the decision-maker must follow a rule, such as maximin, which depends only on the extreme values of the actions.

We should like to argue that MMEU is superior to complete ignorance models, because it is a generalisation of SEU, while complete ignorance is not. When there is a unique probability distribution our theory coincides with expected utility maximisation. On the other hand when the set C is large the sort of decision rules our theory predicts are quite close to maximin. MMEU has the advantage that a decision-maker can have different degrees of ignorance measured by the size of the set C. In contrast in the complete ignorance models, ignorance is an all or nothing matter.

A second advantage, which MMEU has over the complete ignorance models is they assume that the decision-maker has no beliefs about the likelihood of states. While it is easy to find examples in which it is difficult to assign subjective probabilities, in most cases individuals have some beliefs about the relative likelihood of different events, even if they are incomplete and only qualitative. Without the assumption that the decision-maker has no beliefs, the merger of states axiom is difficult to motivate. This axiom is not crucial for the theory of complete ignorance. Kelsey (1993) shows that it is possible to prove similar results if the merger of states axiom is replaced with a variant of Savage's (1954) sure-thing principle. However the symmetry axiom would also be difficult to motivate if the assumption of no beliefs were dropped. The symmetry and merger of states axioms together are crucial for complete ignorance theory. In MMEU a decision-maker who does not have a unique subjective probability distribution will still have some beliefs over the states which are represented by the set of probabilities C.

A third advantage is that there appears to be no natural way to model learning in the complete ignorance models. Two ways to model learning within MMEU have been proposed. Jaffray (1990) has proposed that after observing a signal the decision-maker should revise each probability distribution in C according to Bayes' formula. Gilboa and Schmeidler (1993) have axiomatised a rule for updating MMEU beliefs, which they refer to as Classical Updating. According to this rule if the decision-maker's initial set of beliefs are represented by the set C of probabilities, then after the observation of a signal the decision-maker rejects those members of C which do not assign the highest probability to the signal and updates the remaining probabilities by Bayes' rule.

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APPENDIX

In this appendix we indicate how the framework of Section 2 can be derived from preferences over state-contingent routlette lotteries (i.e. horse lotteries). We derive the existence of a cardinal utility function and show that actions which yield the same expected utility in every state, are indifferent.

Let X denote the space of outcomes. We shall use $\Delta(X)$ to denote the space of probability distributions over X, with finite support. Let S denote the set of states. We shall assume that there are a finite number, n, of states. An action is defined to be a function $a: S \to \Delta(X)$. We shall use $A(S)$ to denote the space of all actions. Thus an action assigns a lottery with a finite number of outcomes and known probabilities to each state. Let $C(S) \subset A(S)$ denote the space of constant actions, i.e. if $c \in C(S)$ then $c(s) = c(s')$ for all $s, s' \in S$.

Axiom A.1 The decision-maker has a weak preference relation R which is complete reflexive and transitive.

Axiom A.2 Certainty Independence For all a, $b \in A(S)$, all $c \in C(S)$ and all $\lambda: 0 \leqslant \lambda \leqslant 1$, $aRb \Leftrightarrow \lambda a + (1 - \lambda)cR\lambda b + (1 - \lambda)c$.

Notation The certain actions $C(S)$ can be identified with the space of probability distributions on X, $\Delta(X)$ in a natural way. Let r denote the weak preference induced on $\Delta(X)$ by R on $C(S)$.

Axiom A.3 Weak Dominance Let $a, b \in A(S)$ if $a(s)rb(s)$ for $1 \leqslant s \leqslant n$ then aRb.

Axiom A.4 Quasi-Concavity For all $a, b \in A(S)$ and all λ: $0 \leqslant \lambda \leqslant 1$, $aRb \Rightarrow \lambda a + (1 - \lambda)bRb$.

Axiom A.5 Continuity For all a, b, $c \in A(S)$ with $aPbPc$ there exists a unique λ with $0 \leqslant \lambda \leqslant 1$, such that, $bI\lambda a + (1 - \lambda)c$.

Axioms A.2–A.4 are the analogues of Assumptions 1–3 in the main text. The chief difference is that A.2–A.4 apply to preference relations while 1–3 describe the properties of a numerical representation of a preference relation. Axiom A.5 is a fiarly standard continuity assumption, which ensures that preferences have a numerical representation.

Proposition A.1 Under axioms A.1, A.2 and A.5, there exists a utility function $u: X \to \mathbb{R}$ such that for all $a, b \in C(S)$, $aRb \Leftrightarrow E_a u \geqslant E_b u$. (If $a \in C(S)$ then a defines a unique probability distribution over X. $E_a u$ denotes the expected value of u with respect to this probability distribution.) Moreover u is unique up to a positive affine transformation.

Proof On the subspace $C(S)$, A.2 implies the independence axiom. With this observation it can be seen that A.1, A.2 and A.5 are a standard set of axioms for expected utility maximisation. The result can be proved as in Fishburn (1970, Ch. 8). □

Definition A.1 Define J: $A(S) \to \mathbb{R}$ by $J(a) = inf\{u(c): c \in C(S) \text{ and } cRa\}$.

Proposition A.2 Under axioms A.1, A.2 and A.5, $J(a) > J(b)$ implies aPb. If in addition dominance (A.3) is assumed we can conclude that $J(a) > J(b) \Leftrightarrow aPb$.

Proof Suppose $J(a) > J(b)$ then there exists $c \in C(S)$ such that $J(a) > u(c)$ and cRb. Now $J(a) > u(c)$ implies aPc. Transitivy implies aPb, which demonstrates the first part of the proposition. Now suppose aPb. Let m (respectively l) denote the most (respectively least) preferred outcome yielded by a and b. Then by weak dominance $\delta_m RaPbR\delta_1$, where for $x \in X$, δ_x denotes the element of $A(S)$, which gives outcome x with probability one in all states. By continuity there exist λ_1, $\lambda_2 \in [0, 1]$ such that $aI\lambda_1\delta_m + (1 - \lambda_l)\delta_l$, $bI\lambda_2\delta_m + (1 - \lambda_2)\delta_l$. Let $\lambda = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$. By weak dominance and the uniqueness part of the continuity axiom, $aP\lambda\delta_m + (1 - \lambda)\delta_l Pb$. Since $\lambda\delta_m + (1 - \lambda)\delta_j$ is a constant action it follows that $J(a) > J(b)$. □

Lemma A.1 If $a, b \in A(S)$ and $E_{a(s)}u = E_{b(s)}u$ for $1 \leqslant s \leqslant n$ then aIb.

Proof This follows directly from weak dominance. □

Remark We have established that actions are ranked solely by the expected utilities which they yield in each state. Let $a' \in \mathbb{R}^n$ and let $a \in A(S)$ such that $Eu(a(s)) = a'_s$ for $1 \leqslant s \leqslant S$. We may define V: $\mathbb{R}^n \to \mathbb{R}$ by $V(a') = J(a)$. Lemma A.1 shows that V is well defined.

Proposition A.3 V satisfies Assumptions 1, 2, and 3.

Each assumption follows directly from the corresponding property of preferences over $A(S)$, i.e. 1 follows from axiom A.4 etc. We have derived the function V used in Section 2 from an axiomatic theory of preferences over state-contingent lotteries. This justifies the assumptions used in that section.