THE PROBABILITY WEIGHTING FUNCTION

By Drazen Prelec[1]

A *probability weighting function* $w(p)$ is a prominent feature of several non-expected utility theories, including prospect theory and rank-dependent models. Empirical estimates indicate that $w(p)$ is regressive (first $w(p) > p$, then $w(p) < p$), s-shaped (first concave, then convex), and asymmetrical (intersecting the diagonal at about $1/3$). The paper states axioms for several $w(p)$ forms, including the compound invariant, $w(p) = \exp\{-\{-\ln p\}^{\alpha}\}$, $0 < \alpha < 1$, which is regressive, s-shaped, and with an invariant fixed point and inflection point at $1/e = .37$.

Keywords: Expected utility theory, non-expected utility theory, prospect theory, Allais paradox.

1. INTRODUCTION

THE COMMON-RATIO EFFECT (Allais (1953)) refers to the observation that the more risky of two simple prospects becomes relatively more attractive when the probability of winning is reduced by equal proportion in both prospects. Thus a person who prefers a sure gain of $100,000 over a coin toss for $300,000 or nothing, might also prefer a one-in-a-million lottery ticket for $300,000 over a two-in-a-million lottery ticket for $100,000. This contradicts expected utility but not common sense: There is a world of difference between certainty and a 50-50 shot; the difference between one or two chances in a million is negligible. The example has all the force and simplicity of classical arguments for nonlinear utility, where the utility interval from, say $10 and $20, is taken as self-evidently greater than the utility interval from $1,000,010 and $1,000,020. Both arguments appeal directly to our intuitions about numbers. A major point of difference is that the money argument involves a constant money interval and the probability argument a constant probability ratio. A minor difference is that in the money domain a constant interval has less impact as the numbers get *larger*, while in the probability domain a constant ratio has less impact as the numbers get *smaller*.

Putting the two demonstrations side by side like this raises an interesting question, namely, whether an account of nonlinearity-in-probabilities can be somehow adapted from classical utility theory (modulo a log transformation, converting statements about probability ratios into statements about utility intervals). Indeed, in their seminal paper on prospect theory, Kahneman and Tversky (1979) explained the common-ratio effect by means of a nonlinear transformation of probabilities into "decision weights," $p \rightarrow w(p)$, with $\log(w)$
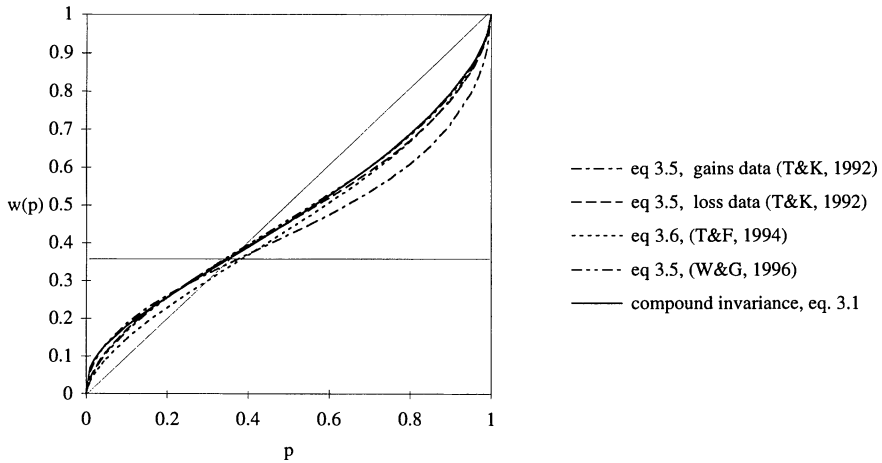

FIGURE 1.—The compound invariant form (solid line) and several empirical probability weighting functions. Estimates of the one-parameter equation (3.5) are taken from Tversky and Kahneman (1992) and Wu and Gonzalez (1996a); estimates of the two-parameter equation (3.6) are taken from Tversky and Fox (1994).

convex in $\log(p)$. They called such functions *subproportional*. However, they also listed other compelling expected utility violations that required additional, apparently unrelated restrictions on $w(p)$.

My objective in this paper is to derive the observed properties of probability weighting from preference axioms, relying on the common-ratio effect as the basic building block. A fair summary of current knowledge about $w(p)$ is displayed in Figure 1, where the dotted and dashed lines reproduce several recent empirical estimates. Unlike utility functions, which are characterized by a single property—concavity, here we encounter functions that are:

regressive—intersecting the diagonal from above,

asymmetric—with fixed point at about $1/3$,

s-shaped—concave on an initial interval and convex beyond that,

reflective—assigning equal weight to a given loss-probability as to a given gain-probability.

Each of these four properties has a distinctive impact on risk attitudes, and, to complicate matters, each one is independent of subproportionality. The first property, regressiveness, generates the important "four-fold pattern of risk attitudes, which is risk-seeking for small-probability gains and large-probability losses, and risk-aversion for small-probability losses and large-probability gains (Tversky and Kahneman (1992)). This is because the overweighting of small probabilities, below the fixed point of the regressive function, enhances the attraction of small-p gains (e.g., lottery tickets) and the aversion to small-p losses (e.g., rare accidents), while the underweighting of larger probabilities, above the fixed point, diminishes the attraction of larger-p gains and the

aversion to larger-p losses. The asymmetrical depression of the fixed point below $p = .5$ further reduces the weight of uncertain relative to certain outcomes, and so tends to increase risk-aversion for gains and (by reflection) risk-seeking for losses. The s-shape property "fits naturally" with regressiveness, but is logically independent—one could have a function that is regressive but not s-shaped, and vice versa. An s-shape implies that changes in probability have less impact as one moves away from the boundary of the probability interval, making, for instance, an initial sweepstakes ticket seem more valuable than any additional ones.[2]

Proposition 1, given in Section 3 of this paper, axiomatizes a subproportional function, $w(p) = \exp(-(-\ln p)^\alpha)$, $0 < \alpha < 1$, that satisfies all four target properties, and that has an invariant fixed point and inflection point at $p = 1/e = .37$. This form, shown by the solid line in Figure 1, is the weighting function counterpart to the power utility function, $u(x) = x^\alpha$. I derive also a more general form, $w(p) = \exp(-\beta(-\ln p)^\alpha)$, which is not constrained to the l/e fixed point value. Such "compound invariant" functions are simple to estimate, and provide a consistent ordering of individuals across several categories of expected-utility violations (because a smaller α-parameter value in: $\exp(-\beta(-\ln p)^\alpha)$, implies a more subproportional, more regressive, and a more s-shaped function). In Section 4.1, I show how Pratt-Arrow analysis can be used to generate other functional families. Section 5 discusses the complementary work by Tversky and Wakker (1995) and Wu and Gonzalez (1996b) on common-consequence related properties of the weighting function.

2. THEORETICAL FRAMEWORK

2.1. *Cumulated Prospect Theory*: *Notation and Assumptions*

The results to follow are developed in context of a sign- and rank-dependent representation of preferences (Starmer and Sugden (1989), Luce (1991), Luce and Fishburn (1991), Tversky and Kahneman (1992), Wakker and Tversky (1993)), defined over lotteries on a real interval. Specifically, I follow the notation and assumptions of Cumulated Prospect Theory or CPT, specialized for the domain of risk (Tversky and Kahneman (1991), Wakker and Tversky (1993)). Outcomes are elements, $x, y, z, \ldots$, of a real interval, $X = [x^-, x^+]$, containing positive and negative values, $x^- < 0 < x^+$. The preference relation $\succeq$ is defined over the set, $\mathscr{P}$, of probability distributions, $P, Q, \ldots$, on X. $\mathscr{P}^+$ and $\mathscr{P}^-$ are the sets of distributions concentrated on the nonnegative and nonpositive parts of X. $V(P)$ *represents* $\succeq$ if: $P \succeq Q \Leftrightarrow V(P) \geq V(Q)$, for any P, Q in $\mathscr{P}$. *Prospects* are distributions with finite support. CPT preferences are represented by a sign- and rank-dependent functional, $V(P)$, with a *value function* $v(x)$ for money

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[2] The four-fold pattern of risk attitudes was described and documented by Tversky and Kahneman (1992). Preference conditions equivalent to concavity/convexity have been axiomatized by Wakker (1994), Prelec (1995), and Wu and Gonzalez (1996b), and those equivalent to reflection by Tversky and Wakker (1995).

outcomes, and two *probability weighting functions*, $w^+(p)$ and $w^-(p)$, for gains and losses, respectively (equation (2.0), Appendix 1). Throughout the paper we assume the following axioms, taken with minor modifications from Wakker (1994):

ASSUMPTION A0: $\succsim$ *satisfies axioms $W1$–$W6$ in Appendix 1, which support a sign- and rank-dependent representation (equation (2.0)) with a ratio scale $v(x)$, continuous and strictly increasing, and unique $w^-(p), w^+(p)$, continuous on $(0,1)$, strictly increasing, and satisfying $w^+(0) = w^-(0) = 0$, $w^+(1) = w^-(1) = 1$.*

Only three types of prospects matter in this paper: Certain prospects, $(x, 1)$, abbreviated as (x); *simple prospects* with one nonzero outcome $(0, 1 - p; x, p)$, abbreviated as (x, p); *binary prospects* with two nonzero outcomes of the same sign $(0, 1 - p - q; x, p; y, q)$, $0 < |x| < |y|$, abbreviated as $(x, p; y, q)$. Hence the only parts of the representation that we need to be concerned with are

$$(2.1) \qquad V((x, p)) = \begin{cases} w^+(p)v(x), & x > 0, \\ w^-(p)v(x), & x < 0, \end{cases}$$

$$(2.2) \qquad V((x, p; y, q)) = \begin{cases} w^+(p + q)v(x) + w^+(q)(v(y) - v(x)), \\ \qquad 0 < x < y, \\ w^-(p + 1)v(x) + w^-(q)(v(y) - v(x)), \\ \qquad y < x < 0. \end{cases}$$

Equation (2.2) shows the accounting assumptions of sign- and rank-dependence. The binary prospect $(x, p; y, q)$ may be interpreted as "a '$p + q$' chance of gaining [losing] at least $v(x)$ and a 'q' chance of gaining [losing] an extra $v(y) - v(x)$." The extension of this idea to more complex prospects is then straightforward (equation (2.0) in Appendix 1). A pure rank-dependent model arises if: $w^-(p) = 1 - w^+(1 - p)$.

Because the separable equation for simple prospects (2.1) is common to several models of risky choice (e.g., Kahneman and Tversky (1979), Quiggin (1982), Yaari (1987), Rubinstein (1988), Luce (1988, 1991)), those results that involve only simple prospects will be highlighted by the following assumption:

ASSUMPTION A1: *The restriction of $\succsim$ to simple prospects has a separable representation (2.1) with $v(x)$, $w^-(p)$, and $w^+(p)$ satisfying the conditions in Assumption A0.*

I will omit the superscripts where the distinction between losses and gains is not at issue.

2.2. *Supplementary Axioms*

The mandatory axioms in A0 will be supplemented at various points with one or more optional conditions, which can be selectively "turned on" to further constrain a particular model. The first of these conditions is boundary continuity:

BOUNDARY CONTINUITY: *For any $y > x > 0$ or $0 > x > y$ there exist $p, q \in (0, 1)$ such that $(x) > (y, p)$ and $(x) < (y, q)$.*

This axiom, adapted from Wakker (1994), combines four conditions that ensure continuity at $p = 0$ and $p = 1$ for gains and losses.

The next condition states that common-ratio violations will be observed at all probabilities.

SUBPROPORTIONALITY: *For any $0 < \lambda < 1$, $p \neq q$, $(x, p) \sim (y, q)$ implies: $(y, \lambda q) > (x, \lambda p)$ if $y > x > 0$, and $(y, \lambda q) < (x, \lambda p)$ if $0 > x > y$.*

The term "subproportionality" abbreviates the more correct "strict subproportionality in probabilities on every nondegenerate interval."[3] *Weak* subproportionality holds if the preferences in the definitions are weak. The function $w(p)$ is [weakly] subproportional if $p > q$, $0 < \lambda < 1$, implies: $w(p)w(\lambda q) > [\geq]w(q)w(\lambda p)$.

The third and final optional axiom is new to the literature, and has the effect of rendering a subproportional $w(p)$ *diagonally concave*, which is to say, concave where it is overweighting and convex where it is underweighting. Empirically estimated weighting functions have approximately satisfied this requirement.

The relevant preference axiom involves mixtures of a binary prospect $(x, p; y, q)$ with the sure-thing middle outcome (x), in cases where this outcome also happens to be the certainty equivalent for the binary prospect, $(x) \sim (x, p; y, q)$. Informally, diagonal concavity of the weighting function precludes preferences where a person might like (i.e., dislike) mixtures of $(x, p; y, q)$ with (x), but at the same time dislike (i.e., like) local mixtures of $(x, p; y, q)$ with other binary prospects near $(x, p; y, q)$.

To state the axiom requires some more notation. Let $\Delta^+[s, r] = \{(x, p; y, q)|0 < x < y, s \leq q, p + q \leq r\}$ and $\Delta^-[s, r] = \{(x, p; y, q)|y < x < 0, s \leq q, p + q \leq r\}$, i.e., the sets of binary prospects where the probability of the extreme outcome is at least s and the probability of the zero outcome at least $1 - r$. Preferences are *quasiconvex* [resp. *quasiconcave*] on $\Delta^+[s, r]$ if $P \sim Q$ implies: $Q \gtrsim \mu P + (1 - \mu)Q$ [resp. $Q \lesssim \mu P + (1 - \mu)Q$] for any three P, Q, and $\mu P + (1 - \mu)Q$ in $\Delta^+[s, r]$, and similarly for $\Delta^-[s, r]$. Preferences are *certainty-equivalent* (*or CE*)-*quasiconvex* [resp. *CE-quasiconcave*] on $\Delta^+[s, r]$ if $P \sim Q$ implies: $Q \gtrsim \mu P$

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[3] From a formal standpoint, this definition is unnecessarily strong. Most results go through with weak subproportionality and at least one strict common ratio violation.

$+ (1 - \mu)Q$ [resp. $Q \lesssim \mu P + (1 - \mu)Q$] for any certain outcome Q, and for any P and $\mu P + (1 - \mu)Q$ in $\Delta^+[s, r]$, and similarly for $\Delta^-[s, r]$. (Note that $Q \notin \Delta^+[s, r]$, $\Delta^-[s, r]$, unless $s = 0$, $r = 1$. In that case, however, CE-quasiconvexity is subsumed by quasiconvexity. *Strict* means strict preference unless $P = Q$.

Proposition 6, proved in Appendix 3, states that the following preference axiom is equivalent to the diagonal concavity of the probability weighting function:

DIAGONAL CONCAVITY: *There is no nondegenerate interval $[s, r]$ such that $\succeq$ is quasiconvex and strictly CE-quasiconcave on $\Delta^+[s, r]$ or $\Delta^-[s, r]$, nor quasiconcave and strictly CE-quasiconvex on $\Delta^+[s, r]$ or $\Delta^-[s, r]$.*

What might constitute a violation of the diagonal concavity axiom? Here is an example. Consider a prospect $(10p, \$100K)$ consisting of 10 lottery tickets (p-chances) for a $100K sweepstakes, with each ticket tradable for a 10% chance at a "consolation prize" of $50. Imagine a person who is:

(i) not willing to exchange *one* ticket for a 10% chance of $50.

(ii) willing to exchange *two* tickets for a 20% chance of $50.

(iii) not willing to exchange *all* 10 tickets for $50.

(i) and (ii) indicate quasiconvexity, or probabilistic risk-aversion, as $(.2, \$50; 8p, \$100K) > (10p, \$100K) > (.1, \$50; 9p, \$100K)$. (ii) and (iii) indicate CE-quasiconcavity, as (iii) implies that $CE > \$50$, and so from (ii) we have: $(.2, CE; 8p, \$100K) > (.2, 50; 8p, \$100K) > (10p, \$1000K) > (CE)$. Although intuitions about probabilistic risk may not be robust, nevertheless the preference pattern (i)–(iii) is faintly implausible. A person who is willing to trade some—but not all—of the lottery tickets for a better chance at a "consolation prize," should, in some sense, derive a greater marginal benefit from trading the first rather than the second ticket.

3. COMPOUND INVARIANCE

3.1. *Compound Invariance*

Expressed in the indifference mode, an example of a common-ratio violation is a pair of judgments like the following:

(Schema 1)   ($10,000 for sure) ~ (1/2 chance of $30,000),
              (1/2 chance of $10,000) ~ (1/6 chance of $30,000),

which show that certainty is to a coin toss as a coin toss is to a die roll, or, $w(1)/w(1/2) = w(1/2)/w(1/6)$. What might this pattern indicate about other judgments?

In this section I consider the implications of a particularly simple extrapolation of Schema 1. The idea is to require that common ratio violations be preserved under probability compounding, as might take place in situations

where ultimate success requires one or more independent successes in a row. Suppose that, starting with Schema 1, we could find a dollar value x that generates a second schema:

(Schema 2)   (x for sure) $\sim$ (1/4 chance of \$30,000),

(1/4 chance of x) $\sim$ (1/36 chance of \$30,000).

The person in Schema 1 is indifferent between \$10,00 and tossing a coin for \$30,000, and is also indifferent between tossing a coin for \$10,000 and rolling a die for \$30,000. The assumption states that requiring two (or N) successive wins, either on coin toss or die roll, will preserve the common ratio violation structure, after adjusting the dollar outcomes. (This would be true for expected utility preferences, except that the die in the example would have to have *four* sides, rather than six). More generally, we have the following definition.

DEFINITION 1: $\geq$ exhibits *compound invariance* if for any outcomes $x, y, x', y' \in X$, probabilities $q, p, r, s \in [0,1]$, and compounding integer $N \geq 1$:
If $(x, p) \sim (y, q)$ and $(x, r) \sim (y, s)$, then $(x', p^N) \sim (y', q^N)$ implies $(x', r^N) \sim (y', s^N)$.

In combination with the assumptions of Section 2, compound invariance yields a weighting function that meets the requirements stated in the introduction to the paper.

PROPOSITION 1: (A) *Let* $\geq$ *be a preference relation on* $\mathscr{P}$ *satisfying A0, diagonal concavity, subproportionality and compound invariance. Then the weighting of probabilities in the representation of* $\geq$ *is characterized by a unique value* $\alpha, 0 < \alpha < 1$, *such that for all* $p, 0 < p < 1$:

(3.1) $w^+(p) = w^-(p) = \exp\{-(-\ln p)^\alpha\}$.

(B) *If* $\geq$ *satisfies A0 or A1, subproportionality and compound invariance, then the weighting functions are characterized by* $\alpha, 0 < \alpha < 1$, *and* $\beta^+, \beta^- > 0$:

(3.2) $w^+(p) = \exp\{-\beta^+(-\ln p)^\alpha\}$,

$w^-(p) = \exp\{-\beta^-(-\ln p)^\alpha\}$.

(C) *If* $\geq$ *satisfies A0 or A1, and compound invariance, then the weighting functions are as in (3.2) with* $\alpha, \beta^+, \beta^- > 0$. *The special case,* $\alpha = 1$, *yields the power functions*:

(3.3) $w^+(p) = p^{\beta^+}$, $w^-(p) = p^{\beta^-}$.

(D) *If* $\geq$ *satisfies A0 or A1, and if compound invariance holds for probabilities in the open interval* $(0,1)$, *then the weighting functions for* $p < 1$ *are characterized*

by: $\alpha, \beta^+, \beta^- > 0,$ *and* $0 < \gamma^+, \gamma^- \leq 1$:

$$
\begin{aligned}
(3.4) \qquad w^+(p) &= \gamma^+ \exp\{-\beta^+(-\ln p)^\alpha\}, \\
w^-(p) &= \gamma^- \exp\{-\beta^-(-\ln p)^\alpha\}.
\end{aligned}
$$

I will refer to (3.1)–(3.4) as the *compound invariant or CI-family* of functions. Because the limit of any CI form as $p \to 0$, is zero, the functions (3.1)–(3.4) are all continuous on $[0, 1)$, and continuously differentiable on $(0, 1)$. If compound invariance holds for all probabilities, as it does on (3.1)–(3.3), the continuity extends to the entire interval $[0, 1]$. Boundary continuity is not invoked in the proposition; it is rather a consequence of compound invariance. Note that (3.1) requires a full sign- and rank-dependent representation (A0); the other forms only presuppose a separable representation for simple prospects (A1).

3.2. *Expected Utility Plus One Parameter*

The axioms behind Proposition 1A define a highly constrained theory, adding one degree of freedom to EU. For any allowed value of the α-parameter, probability weighting is regressive and s-shaped. The common loss-gain weighting function is intrinsically asymmetric, with a fixed point and inflection point at $p = 1/e = .37$. This particular "signature property" of (3.1) is displayed in the left panel of Figure 2, which traces (3.1) for different α. The derivative of $w(p)$ at $1/e$ equals α, hence the α-value is also the numerical lower bound for $w'(p)$. This also means that we can visually infer α by the slope of the function at the inflection point. As $\alpha \to 1$, probability weighting approximates the linear, expected utility case. As $\alpha \to 0$, $w(p)$ approximates a step function, flat everywhere except near the endpoints of the probability interval.

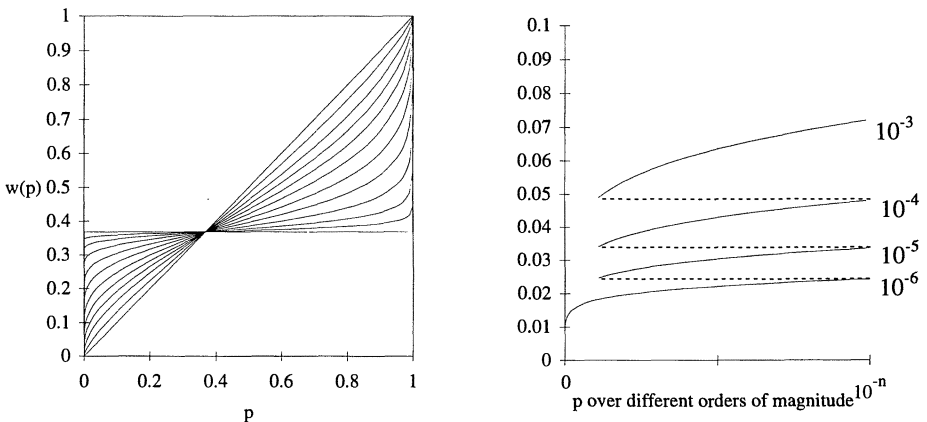FIGURE 2.—The function: $w(p) = \exp(-(-\ln p)^\alpha)$, plotted for different α (left panel) and for small probabilities with $\alpha = .65$ (right panel).

The α-parameter remains a comprehensive index of nonlinear weighting in the more general CI form, equation (3.2). If two functions in (3.2) are parameterized by (α_i, β_i) and if $\alpha_1 < \alpha_2$, it then follows that w_1 is: (a) more subproportional, (b) more s-shaped, and (c) more regressive than w_2, in the usual sense that the composition, $w_1\{w_2^{-1}\}$, is itself a subproportional, s-shaped, and regressive function. These comparative statements derive from the fact that $w_1\{w_2^{-1}\}$ is also CI with $\alpha = \alpha_1/\alpha_2$. The β-parameter in (3.2) is a "net" index of convexity in that increasing β increases the convexity of the function without affecting subproportionality. The γ-parameter, which appears only in (3.4), is an index of the weight of sure outcomes ($p = 1$) relative to uncertain ones ($p < 1$), holding constant subproportionality and convexity on $[0, 1)$.

The behavior of a CI function near zero is perhaps its most interesting feature, both theoretically and for practical applications. The right panel of Figure 2 traces out the CI form ($\alpha = .65$, $\beta = 1$) over four orders of magnitude, ranging from one chance in a thousand (top segment) to one chance in a million (bottom segment). The vertical axis in the figure is probability weight; the horizontal axis is the probability interval for one order of magnitude, $[0, 10^{-n}]$, with $n = 3$, 4, 5, or 6.

In the picture we see how the CI accommodates three intuitions about the decision impact of small chances. First, the weight of ever smaller chances tends to zero (continuity). Second, the slope tends to infinity at zero, giving a qualitative character to the transition from impossibility to possibility. Third, the graph becomes *relatively* flatter at smaller probabilities, capturing the feeling that, e.g., two chances in a million are really no different from one chance in a million. Formally, one can show that for any p, q, the limit as $\lambda \to 0$ of $w(\lambda p)/w(\lambda q)$ equals 1, and that probability weights approximate a step function at zero. At the level of preferences, this means that for very small chances the money dimension overrides the probability dimension: for any (x, p) and (y, q) where $x < y$, there exists a $\lambda > 0$ such that the prospect with the higher-prize is preferred, $(y, \lambda q) > (x, \lambda p)$.

The picture at the other endpoint is almost, but not quite the same. $w(p)$ is unboundedly convex at $p = 1$, by the Pratt-Arrow measure (w''/w'), and the slope, dw/dp, again tends to infinity. However, probabilities near $p = 1$ are not relatively indistinguishable in the same sense as are probabilities near $p = 0$, because the limit as $\lambda \to 0$ of: $(1 - w(\lambda p))/(1 - w(\lambda q))$, $p \neq q$, does not approach 1.

3.3. *Evidence on the Location of the Fixed and Inflection Points*

The main theoretical advantage of the one-parameter form (3.1) over the two-parameter form (3.2) is the stability of the fixed and inflection point across different levels of nonlinearity. The exact location of this point is perhaps less important, provided the $1/e$ value is not too far off the mark. We can make a

rough assessment of the $1/e$ prediction using published estimates of functional forms that do not have an invariant fixed point. Something like this is attempted in Figure 1, where the solid line traces equation (3.1) ($\alpha = .65$), against estimates of

$$(3.5) \qquad w(p) = \frac{p^\delta}{\left(p^\delta + (1-p)^\delta\right)^{1/\delta}}, \qquad \delta > 0,$$

$$(3.6) \qquad w(p) = \frac{ap^\delta}{ap^\delta + (1-p)^\delta}, \qquad a, \delta > 0.$$

The four functions in the figure are remarkably close.[4] The fixed points inferred from Tversky and Kahneman's (1992) estimates of (3.5) are $p° = .34$ for gains and $p° = .38$ for losses. The estimate for equation (3.6) is $p° = .30$ (Tversky and Fox (1994)). Wu and Gonzalez (1996a, p. 1686) report $p° = .39$ as the pooled fixed point estimate, with equation (3.5). Camerer and Ho (1994) conducted a maximum likelihood estimate of (3.5) for nine previously published data sets. The mean fixed point value using their nine separate estimates is $p° = .35$, with standard error of .04, which is again close to the conjectured value of .37.

These parametric estimates are not conclusive, relying as they do on possibly incorrect functional forms for $w(p)$ and for money value. Fully nonparametric evidence on the location of the *inflection* point (rather than the fixed point) has been collected in a recent study by Wu and Gonzalez (1996a). They constructed p-indexed ladders of "common-consequence" choice pairs, such that preference for the less risky prospect in a pair increases with the local slope of $w(p)$ near p, and is therefore minimized near the inflection point of $w(p)$. The estimate of the inflection point is valid independent of assumptions about money value. Across the five ladders in the study, the point of minimal preference occurred at p-values of $p = .37$, $p = .40$, $p = .32$, $p = .15$, and $p = .35$, leading them to conclude that the weighting function, "is concave up to $p \sim .40$ and convex beyond that probability."

3.4. *Application to the Four-fold Pattern of Risk Attitudes*

A concise, global characterization of subproportional CI functions is that they are regressive not only with respect to the diagonal line, but also with respect to the entire family of increasing power functions. In other words, the subproportional CI intersects from above every increasing power function, p^β, $\beta > 0$. I will call such functions *strongly regressive*.[5]

This bears directly on risk-attitudes, at least in the elementary setting where the choice is between a risky simple prospect and a sure thing. Here one

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[4] Equations (3.5) and (3.6) are not subproportional at small probabilities, however.
[5] Formally, $w(p)$ is strongly regressive if: $\forall c > 0$, $\exists p_c^* \in (0, 1)$ such that $p \in (0, p_c^*) \Rightarrow w(p) > p^c$, and $p \in (p_c^*, 1) \Rightarrow w(p) < p^c$.

typically observes a risk-seeking for small-p gains and larger-p losses, and a risk-aversion for small-p losses and larger-p gains (i.e., the "four-fold pattern" of Tversky and Kahneman (1992)). In a sign- and rank-dependent model with a *linear* value function, regressive weighting alone would be a sufficient explanation of the pattern. However, if value is not linear, then probability nonlinearity must somehow dominate money nonlinearity for the pattern to hold. For example, to explain risk-seeking behavior with respect to longshot lottery tickets, the overweighting of the small chance of winning would have to dominate the concavity of the value function, which works against risk-seeking.

Such dominance arises in representations that combine CI probability weighting with power functions for money value:

$$v(x) = \begin{cases} x^{\sigma+} & \text{for } x > 0, \\ -(-x)^{\sigma-} & \text{for } x < 0. \end{cases}$$

The relative certainty equivalent for a simple prospect is then itself a CI function of probability, as $(c) \sim (x, p)$ implies $c^{\sigma+} = x^{\sigma+} \exp\{-\beta^+(-\ln p)^\alpha\}$, or[6]

$$(3.7) \qquad \frac{c}{x} = \exp\{-(\beta^+/\sigma^+)(-\ln p)^\alpha\}.$$

Risk attitude is established by comparing the certainty equivalent and the expected value of the prospect, px. Substituting from (3.7), we find that the risky prospect (x, p) will be preferred to the EV whenever

$$(3.8) \qquad \begin{aligned} &(\text{for gains, } x > 0) \text{: } c > px \Leftrightarrow \exp\{-(\beta^+/\sigma^+)(-\ln p)^\alpha\} > p, \\ &(\text{for losses, } x < 0) \text{: } c > px \Leftrightarrow \exp\{-(\beta^-/\sigma^-)(-\ln p)^\alpha\} < p. \end{aligned}$$

For gains, the risky prospect will be preferred at probabilities where the CI expression in (3.8) is overweighting, while for losses it will be preferred where the CI is underweighting. Because any CI form with $\alpha < 1$ is regressive, overweighting small probabilities and underweighting larger ones, this confirms the four-fold pattern. Notably, it confirms the pattern independent of the exponent of the value function, i.e., independent of whether the functions are concave or convex; the value exponents only affect the sizes of the risk-seeking regions. The nonlinearity of weighting functions overrides the nonlinearity of the value functions, in this particular case.

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[6] The combination of equation (3.1) and a power value function can be estimated with OLS, because the plot of *relative* certainty equivalents, c/x, against p is linear in double-log coordinates. From (3.7), with $\beta = 1$, we have

$$-\ln(-\ln(c/x)) = \ln \sigma + \alpha(-\ln(-\ln p)).$$

3.5. *The Compounding Factor and ε-subproportionality*

Having considered one axiom, compound invariance, one might ask whether this is an anomalous special case or whether there are less restrictive axioms, conveying similar properties on $w(p)$ without forcing a particular parametric form. A possibility that immediately comes to mind—the class of all subproportional preferences—is clearly too broad, containing functions that are neither regressive nor *s*-shaped.

A more productive generalization takes its cue from a specific numerical implication of CI. By computation with equation (3.2), one can show that there exists a compounding factor, ρ, such that for any probability p,

$$(x) \sim (y, p) \qquad \text{implies} \qquad (x, p) \sim (y, p^\rho), \qquad p = 2^{1/\alpha},$$

i.e., $w(p)^2 = w(p^\rho)$ for $\rho = 2^{1/\alpha}$. The CI function in Figure 1, with $\alpha = .65$, yields a compounding factor value of $\rho = 2.9$. In this case, it would take (approximately) a cubing of probabilities to produce a squaring of probability weights. Of course $\rho = 2$ is the only value consistent with expected utility preferences. For subproportional preferences, $\rho > 2$, but the exact value of ρ will depend on the probability in the definitional schema, $(x) \sim (y, p)$ & $(x, p) \sim (y, p^\rho)$.

The condition I now state requires that there exist a minimal value $\varepsilon < 0$ such that at any base probability level, the compounding factor differs from 2 by at least ε:

DEFINITION 2: We say that $\geq$ is *ε-subproportional* if subproportional and if there exists an $\varepsilon > 0$ such that for all $0 < x < y$, and probabilities, $0 < p < 1$, $(x) \sim (y, p)$ implies $(x, p) < (y, p^{2+\varepsilon})$, and for all $y < x < 0$, $(x) \sim (y, p)$ implies $(x, p) > (y, p^{2+\varepsilon})$.

Compound invariant preferences satisfy the definition for any $2 < 2 + \varepsilon < 2^{1/\alpha}$, provided $\alpha < 1$.

PROPOSITION 2: *Let $\geq$ be a preference relation on $\mathscr{P}$ satisfying A0 or A1, and ε-subproportionality. Then the weighting function of probabilities in the representation of $\geq$ is strongly regressive.*

Although Definition 2 makes much weaker demands on the preference relation than Definition 1, it still implies a weighting function whose global shape and boundary nonlinearities are qualitatively similar to the CI form. In particular, the approximation to a step function at zero, shown in Figure 2, would also be a property of any strongly regressive weighting function.

4. PRATT-ARROW CLASSIFICATION OF SUBPROPORTIONAL FUNCTIONS

In utility theory, the basic functional forms are introduced as hypotheses about how risk aversion depends on wealth. Thus the exponential and power

functions for $u(x)$ define the hypotheses that absolute or proportional risk-aversion is invariant with respect to absolute or proportional changes in wealth. Stating these hypotheses requires, in turn, a precise definition of greater or lesser risk aversion, one that permits comparison of two utility functions, or of the same function before and after a given transformation (Pratt (1964)).

The situation with probability weight is similar. A general theory of functional forms for $w(p)$ should begin with a definition of greater or lesser divergence from the linear norm. In Section 5, I will discuss proposals by Tversky and Wakker (1995) and Wu and Gonzalez (1996b) that take the Allais common-consequence effect as the effective measure of nonlinearity. In this section, however, I look at comparative nonlinearity in the context of common-ratio violations, i.e., of subproportional weighting functions. For simplicity, I consider only positive outcome prospects.

4.1. *Comparative Subproportionality*

Comparing two preference relations, we use the following definition.

DEFINITION 3: $\succsim_1$ is *at least as subproportional* as $\succsim_2$ on $\mathscr{P}^+$ if for any probabilities $p > r \geq q > s$, and outcomes $0 < x < y$, $0 < x' < y'$, $(x, p) \sim_2 (y, q)$, $(x, r) \sim_2 (y, s)$, and $(x', p) \sim_1 (y', q)$ implies $(x', r) \precsim_1 (y', s)$.

Kahneman and Tversky (1979) pointed out that subproportionality implies convexity of the weighting function in log-log coordinates, which is to say, convexity of the ω-function,[7]

$$(4.1) \qquad \omega(\ln p) = \ln(w(p)).$$

Indeed, the ordering of preference relations by subproportionality is the same as the ordering of the ω-functions by convexity:

PROPOSITION 3: *Assume that* $\succsim_1$ *and* $\succsim_2$ *are two preference relations on* $\mathscr{P}^+$ *satisfying A0 or A1 and boundary continuity, with weighting functions,* $w_1(p)$ *and* $w_2(p)$, *representing* $\succsim_1$ *and* $\succsim_2$, *respectively. Then, conditions A, B, and C are equivalent:*
  (A) $\succsim_1$ *is at least as subproportional as* $\succsim_2$;
  (B) $w_1(w_2^{-1}\{z\})$ *is a weakly subproportional function of* z;
  (C) $\omega_1(\omega_2^{-1}\{z\})$ *is a convex function of* z, *with* $\omega_i(\ln p) = \ln w_i(p)$.

Proposition 3 offers a general recipe for generating subproportional weighting functions. Taking any increasing, strictly concave "utility function," $u(t)$, $t \geq 0$, we can reflect $u(t)$ into the negative domain, $\omega(t) = -u(-t)$, and create a weighting function, $w(p) = \exp(\omega(\ln p))$, that is subproportional and continuous

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[7] Tversky and Wakker (1995) derive a result similar to 3AB but for subadditivity, rather than subproportionality.

on $(0, 1)$.[8] As a practical matter, if we were only interested in empirical estimation we could borrow such forms and compare $w(p) = \exp(-u(-\ln p))$ against data. However, in utility theory the forms are justified by intuitions about how the risk-premium might depend on wealth. In the probability domain, the formal counterpart to a risk-premium has little significance. To engage our intuition we have to go directly to the axioms that support particular functional forms. I illustrate this with three examples.

4.2. *Compound Invariance as the "Constant-Relative" Benchmark*

The first example, we have already encountered in Section 3. The ω-function for the CI form, equation (3.2), is $\omega(\ln p) = -\beta(-\ln p)^{\alpha}$, which has constant-relative Pratt-Arrow convexity, $(-\ln p)\omega''/\omega' = \alpha$. I now restate the compound invariance axiom as follows. For any preference relation, $\gtrsim$, define the compounded preferences, $\gtrsim^N$, as

$$(x, p) \gtrsim^N (y, q) \quad \text{iff} \quad (x, p^N) \gtrsim (y, q^N).$$

Compound invariance is just the claim that $\gtrsim^N$ and $\gtrsim$ are equally subproportional, in the sense of Definition 3. Actual preferences might systematically deviate from invariance in one of two directions. If $\gtrsim^N$ is observed to be more subproportional than $\gtrsim$, i.e., if common-ratio violations are more common after compounding, then preferences exhibit *increasing-relative* subproportionality (or *IRS*). The other possibility would correspond to the *decreasing-relative* or *DRS* class.

4.3. *Conditional Invariance*

A second Pratt-Arrow benchmark, "constant-absolute" subproportionality, also flows from a simple property of preference. Let us return to the schema at the start of Section 3:

(Schema 1)   ($10,000 for sure) $\sim$ (1/2 chance of $30,000),
                (1/2 chance of $10,000) $\sim$ (1/6 chance of $30,000),

and require now that common ratio violations be preserved under the proportional reduction of probabilities, as might correspond to a scenario where ultimate success is made conditional on some other independent event. The idea is that starting with Schema 1 and a conditional probability (e.g., 1/2) we can find a dollar value x that generates a second schema:

(Schema 3)   (1/2 chance of x) $\sim$ (1/4 chance of $30,000),
                (1/4 chance of x) $\sim$ (1/12 chance of $30,000).

More generally, we have the following definition.

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[8] New functional forms may be generated by taking advantage of the fact that if $w_1(p)$ and $w_2(p)$ are subproportional, then so are the functions $w_1(p)w_2(p)$, $w_1(w_2(p))$, and $\max\{w_1(p), w_2(p)\}$.

DEFINITION 4: $\geq$ exhibits *conditional invariance* if for any outcomes x, y, x', $y' \in X$, probabilities $q, p, r, s \in [0, 1]$, and "conditional probability" λ, $0 < \lambda < 1$:
If $(x, p) \sim (y, q)$ and $(x, r) \sim (y, s)$, then $(x', \lambda p) \sim (y', \lambda q)$ implies $(x', \lambda r) \sim (y', \lambda s)$.

Actual preferences can systematically diverge from the axiom in two directions. Let $\geq_\lambda$ denote the "conditional" preferences, $(x, p) \geq_\lambda (y, q)$ if $(x, \lambda p) \geq (y, \lambda q)$. Then $\geq$ exhibits *increasing-absolute subproportionality* or IAS if $\geq_\lambda$ is as subproportional as $\geq$, and *decreasing-absolute subproportionality* or DAS if $\geq$ is as subproportional as $\geq_\lambda$. The functions implied by the invariant, constant-absolute case are given below:

PROPOSITION 4: (A) *Let* $\geq$ *be a preference relation on* $\mathscr{P}^+$ *satisfying A0 or A1 and conditional invariance. Then the weighting function for* $p > 0$ *in the representation of* $\geq$ *is either an exponential-power function,*

$$(4.2) \qquad w(p) = \exp\left\{ -\frac{\beta}{\alpha}(1 - p^\alpha) \right\}, \qquad \alpha \neq 0, \beta > 0,$$

or a power function,

$$(4.3) \qquad w(p) = p^\beta, \qquad \beta > 0.$$

(B) *If* $\geq$ *satisfies A0 or A1, and conditional invariance holds for probabilities in the open interval* $(0, 1)$, *then the weighting function for* $0 < p < 1$ *is* $\gamma w(p)$, *where* $0 < \gamma \leq 1$ *and* $w(p)$ *is either* (4.2) *or* (4.3).

(C) *If* $\geq$ *satisfies A0 or A1, and conditional invariance, then the axioms*: *subproportionality and boundary continuity, are inconsistent.*

The exponential-power function is subproportional for $\alpha > 0$. However, a sufficiently small β can render it concave and overweighting on $(0, 1)$, hence regressiveness and s-shape are not generic properties in the sense that they are with CI. A more important difference is noted in part C of the proposition. Equation (4.2) has a positive limit as $p \to 0$, which creates discontinuous weighting at zero (the other possibility, (4.3), is not subproportional). Going beyond this specific example, any $w(p)$ function whose ω-function, equation (4.1), has a finite lower bound as $\ln p \to -\infty$ will be discontinuous at zero. In particular, ω-functions that exhibit increasing absolute convexity in the direction of smaller probabilities will be bounded.[9] Therefore, of the three possibilities: decreasing-, increasing-, or constant-absolute, only the decreasing case can accommodate subproportionality and continuity at $p = 0$.

There is no clear argument in favor of increasing vs. decreasing relative subproportionality as a more likely departure from the constant case—at least nothing like the continuity argument favoring the decreasing absolute case. It is

[9] This follows from standard utility theory results (Pratt (1964, Theorems 1 and 8)). The CI function is decreasing-absolute, constant-relative.

perhaps interesting that among functions that are diagonally concave, continuous on $[0, 1]$, and twice differentiable on $(0, 1)$, there exists a connection between the location of the fixed point and the direction of relative subproportionality. Specifically, a diagonally concave IRS function cannot have an interior fixed point above $1/e$, and a diagonally concave DRS function cannot have one below $1/e$ (Prelec (1995)).

4.4. *Projection Invariance*

Our final example illustrates a DAS-IRS function. Let us return again to Schema 1 and observe that (r, s) are related to the first pair (p, q) by the factors $1/2$ and $1/3$: $r = (1/2)p$, $s = (1/3)q$. In other words, the smaller value in the initial pair $q = 1/2$, has to be reduced by a greater factor than the larger value, $p = 1$. Suppose that starting with Schema 1, we could then generate a new indifference judgment by multiplying the left and right side chances by $1/2$ and $1/3$:

$$(\$10,000) \sim (1/2, \$30,000),$$

$$(1/2, \$10,000) \sim (1/6, \$30,000),$$

$$(1/4, \$10,000) \sim (1/18, \$30,000).$$

In the general case, we have the following definition.

DEFINITION 5: $\succeq$ exhibits *projection invariance* if for any outcomes $x, y \in X$, and probabilities $q, p, r, s \in [0, 1]$: $(x, p) \sim (y, q)$ and $(x, rp) \sim (y, sq)$ imply $(x, r^2 p) \sim (y, s^2 q)$.

The axiom is the simplest of the three invariances because it involves two outcomes, x and y, and three preference judgments, instead of the four outcomes and four preference judgments required to disconfirm Definitions 1 or 4.

PROPOSITION 5: (A) *Let* $\succeq$ *be a preference relation on* $\mathscr{P}^+$ *satisfying* $A0$ *or* $A1$ *and projection invariance. Then* $\succeq$ *is weakly subproportional, and* $w(p)$ *is either a power function*, (3.3), *or a hyperbolic logarithm,*

$$(4.4) \qquad w(p) = (1 - \alpha \ln p)^{-\beta/\alpha}, \qquad \alpha, \beta > 0.$$

(B) *If* $\succeq$ *satisfies* $A0$ *or* $A1$, *and projection invariance holds for probabilities in the open interval* $(0, 1)$, *then the weighting function for* $p < 1$ *is* $\gamma w(p)$, *where* $0 < \gamma \leq 1$ *and* $w(p)$ *is either* (4.4) *or* (4.3).

Equation (4.4) is regressive, diagonally concave if: $(\alpha + \beta) - \beta \ln(\alpha + \beta) = 1$, $\alpha + \beta \geq 1$. Under this constraint, the fixed point falls within the interval $(1/e^2, 1/e)$. This is proved in Appendix 2, after the derivation of (4.4).

As noted in 5A, weak subproportionality is an implication of projection invariance. This is a bit surprising, because, on the face of it, Definition 5 does not preclude a *supra*proportional common-ratio violation, which would be diagnosed by the pattern $p > q$ and $r < s$. However, the axiom has no solutions that might accommodate such a combination of probabilities. Regarding the other "essential" properties, the hyperbolic-log is typically regressive and *s*-shaped, but, just like the exponential power function, equation (4.2), it becomes concave and overweighting when β is very small. This failure of generic regressiveness and *s*-shape again highlights the special qualities of the CI form.

5. COMPOUND INVARIANCE AND THE ALLAIS COMMON-CONSEQUENCE EFFECT

In addition to the common-ratio effect, the 1953 article by Allais contained a second class of counterexamples of which the following "certainty effect" is perhaps the most famous:

(Schema 4)
$$(11\% \text{ chance of } \$1,000,000) < (10\% \text{ chance of } \$5,000,000),$$
$$(\$1,000,000 \text{ for sure}) > \left(\begin{array}{l} 10\% \text{ chance of } \$5,000,000 \\ 89\% \text{ chance of } \$1,000,000 \end{array} \right).$$

The second pair of prospects are derived from the first pair by adding a "common-consequence" to both sides, in this case a .89 probability of $1M. In a rank-dependent representation, Schema 4 leads to the inequality: $w(1) - w(.99) > w(.11) - w(.10)$, showing that, in the domain of gains, the marginal impact of the worst centile is greater than that of the 11th best centile. Many such common-consequence examples have been studied. For instance, the following schema (Prelec (1990)):

(Schema 5)
$$(2\% \text{ chance of } \$20,000) > (1\% \text{ chance of } \$30,000),$$
$$(34\% \text{ chance of } \$20,000) < \left(\begin{array}{l} 1\% \text{ chance of } \$30,000 \\ 32\% \text{ chance of } \$20,000 \end{array} \right),$$

generates an inequality in the interior of the probability interval: $w(.02) - w(.01) > w(.34) - w(.33)$. The generic difference between common-consequence and common-ratio schemas is that the former tell us about the ordering of intervals, $w(p + \Delta) - w(p)$ versus $w(q + \Delta) - w(q)$, while the latter tell us about the ordering of ratios, $w(\Delta p)/w(p)$ versus $w(\Delta q)/w(q)$.

Is the ordering of intervals, as related by common-consequence violations, consistent with the ordering of ratios, as revealed by common-ratio violations of expected utility? Fortunately, the answer to this question appears to be *Yes*. The implications of common-consequence violations for $w(p)$ have been explained in recent papers by Tversky and Wakker (1995) and Wu and Gonzalez (1996a, b). Tversky and Wakker show that the generalization of the classic Schema 4, and a "dual" common-consequence effect, is equivalent to *subadditivity* of the weighting function, which holds if there exist boundary constants ε and $\varepsilon' \geq 0$ such

that

(Subadditivity) $w(q) \geq w(p+q) - w(p), \qquad p+q \leq 1 - \varepsilon \leq 1,$
                   $1 - w(1-q) \geq w(p+q) - w(p), \qquad p \geq \varepsilon' \geq 0.$

Tversky and Wakker observe that "violations of subadditivity are rare." The results of Wu and Gonzalez (1996b) are in the same spirit. They show that more general classes of common-consequence schemas are equivalent to $w(p)$ being concave or convex on an interval. Schema 5, for instance, supports concavity on the interior interval $(.01, .34)$. Both papers provide comparative propositions showing that greater subadditivity (Tversky and Wakker (1995)) or concavity/convexity (Wu and Gonzalez (1996b)) is equivalent to more common-consequence violations "of the right kind."

A subproportional CI function is subadditive, as is any strongly regressive function.[10] The compound invariance assumption is thus compatible with most, if not all, current evidence on common-consequence violations of expected utility. Furthermore, within the family of compound invariant preference relations, the ordering of relations by more common-ratio violations (in the sense of Definition 3) coincides with their ordering by more common-consequence violations (in the sense of conditions (6.3) and (6.4) in Tversky and Wakker (1995)). This is because a CI form with a smaller α is both more subproportional and more subadditive.

The coherence of the two Allais paradoxes within the CI framework illustrates a more general proposition. If common-ratio violations exceed some ε-minimum (as in Definition 2), then the slope of $w(p)$ at zero and one is unbounded (Proposition 2). These endpoint nonlinearities are independently confirmed by the certainty effect and by the four-fold pattern of risk attitudes. This does not mean that one behavioral paradox is primary and the others derivative. What it suggests, instead, is that the form of the probability weighting function is overdetermined by the empirical evidence, with more than one road leading to the same end.

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[10] To prove that any strongly regressive function, including the CI, is subadditive, let p° equal the fixed point of the function, $p^\circ = w(p^\circ)$, and pick $p+q < p^\circ$. Then $w(p+q) > p+q$, and $w(p+q) = (p+q)^c$, for some $c < 1$. It follows from $q < p+q$ and strong regressivity that $w(q) > q^c = (q/(p+q))^c(p+q)^c = (q/(p+q))^c w(p+q)$, as $w(p+q) = (p+q)^c$. But $c < 1$ then indicates that $(q/(p+q))^c > q/(p+q)$, or $w(q) > (q/(p+q))w(p+q)$, and, by the same token, that $w(p) > (p/(p+q))w(p+q)$, proving: $w(p) + w(q) > w(p+q)$, for boundary constant p°. A similar argument shows that the second line in the definition of subadditivity holds as well. This proves the formal claim. Tversky and Wakker also indicate that the boundary constants $\varepsilon, \varepsilon'$ ought to be small, which cannot be derived from strong regressiveness alone. However, the special CI form, equation (3.1), does imply small $\varepsilon, \varepsilon'$.

APPENDIX 1: REPRESENTATIONAL ASSUMPTIONS

This appendix collects axioms from Wakker (1994), which together with the results in Wakker and Tversky (1993) are sufficient for the sign- and rank-dependent representation. Let $\geq$ denote a preference relation on the set, $\mathscr{P}$, of probability distributions, $P, Q, \ldots$, on $X = [x^-, x^+]$, with $x^- < 0 < x^+$. Prospects are distributions with finite support. The following five axioms are assumed to hold on $\mathscr{P}$ without restriction:

W1. WEAK ORDERING: $\geq$ is complete and transitive.

W2. STRICT STOCHASTIC DOMINANCE: $P > Q$ if $P \neq Q$ and P stochastically dominates Q.

W3. CERTAINTY-EQUIVALENT CONDITION: For all P there exists an x such that $(x) \sim P$.

W4. CONTINUITY IN PROBABILITIES: If $(y, p) > (x)$, $0 < p < 1$, then there exist q, r such that $q < p < r$, $(y, q) > (x)$, and $(y, r) > (x)$. If $(y, p) < (x)$, $0 < p < 1$, then there exist q, r such that $q < p < r$, $(y, q) < (x)$ and $(y, r) < (x)$.

Now define $S(k, n)$, $0 \leq k \leq n$, as the set of all k nonpositive and $(n-k)$ nonnegative rank-ordered n-tuples from X, $S(k, n) = \{(x_1, \ldots, x_n) \in X^n : x_1 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n\}$.

W5. SIMPLE-CONTINUITY: For any probability vector $(p_1, \ldots, p_n)$ the preference relation induced on each $S(k, n)$ is continuous.

The central (and final) axiom is trade-off consistency, restricted to prospects that are sign- and rank-order compatible. Let $(x, p_i; \mathbf{x}_{-i}, \mathbf{p}_{-i})$ indicate a prospect with outcome x of rank "i" singled out, and let $\mathscr{P}(k, n, p)$ denote the set of all sign- and rank-order compatible prospects that have a p-chance of yielding a negative outcome: $\mathscr{P}(k, n, p) = \{(x_1, p_1; \ldots; x_n, p_n) : (x_1, \ldots, x_n) \in S(k, n)$ and: $p_1 + \cdots + p_k = p\}$.

W6. TRADEOFF CONSISTENCY: There do not exist eight prospects, $(x, p_i; \mathbf{a}_{-i}, \mathbf{p}_{-i})$, $(x, p_i; \mathbf{b}_{-i}, \mathbf{p}_{-i})$, $(x', p_i; \mathbf{a}_{-i}, \mathbf{p}_{-i})$, $(y', p_i; \mathbf{b}_{-i}, \mathbf{p}_{-i})$, $(x', q_j; \mathbf{c}_{-j}, \mathbf{q}_{-j})$, $(y', q_j; \mathbf{d}_{-j}, \mathbf{q}_{-j})$, $(x, q_j; \mathbf{c}_{-j}, \mathbf{q}_{-j})$, and $(y, q_j; \mathbf{d}_{-j}, \mathbf{q}_{-j})$, such that the first four and the second four belong to the same sign- and rank-order compatible set, and that:

$$(x, p_i; \mathbf{a}_{-i}, \mathbf{p}_{-i}) \geq (y, p_i; \mathbf{a}_{-i}, \mathbf{p}_{-i}),$$

$$(x', p_i; \mathbf{a}_{-i}, \mathbf{p}_{-i}) \leq (y', p_i; \mathbf{b}_{-i}, \mathbf{p}_{-i}),$$

$$(x', q_j; \mathbf{c}_{-j}, \mathbf{q}_{-j}) \geq (y', q_j; \mathbf{d}_{-j}, \mathbf{q}_{-j}),$$

$$(x, q_j; \mathbf{c}_{-j}, \mathbf{q}_{-j}) < (y, q_j; \mathbf{d}_{-j}, \mathbf{q}_{-j}).$$

The necessity of this condition is straightforward to check. Because the first four prospects are sign- and rank-order compatible, the terms involving $p_i, \mathbf{a}_{-i}, \mathbf{b}_{-i}$, and $\mathbf{p}_{-i}$ will cancel out after applying (2.1), yielding: $v(x) - v(y) \geq v(x') - v(y')$. Because the second four prospects are also sign- and rank-order compatible, the terms involving q_j, $\mathbf{c}_{-j}, \mathbf{d}_{-j}$, and $\mathbf{q}_{-j}$ will also cancel out, yielding the contradictory ordering, $v(x) - v(y) < v(x') - v(y')$.

In view of Observations 8.1 and 8.4 of Wakker and Tversky (1993), assumptions W1–W6 conform to the requirements of their Theorem 6.3, which ensures a CPT representation with unique, nondecreasing weighting functions, satisfying $w(0) = 0$, $w(1) = 1$, and a ratio scale value function. (Theorem 6.3 of Wakker and Tversky (1993) gives a representation for uncertainty; in the presence of W2 it extends to the case of risk, as shown by Wakker (1990)). A prospect $P = (x_1, p_1; \ldots; x_n, p_n)$,

$x_1 \leq \cdots \leq x_k \leq 0 \leq x_{k+1} \leq \cdots \leq x_n$, would be evaluated as:

$$(2.0) \qquad V(P) = \sum_{i=1}^{k} \left(w^- \left(\sum_{j=1}^{i} p_j \right) - w^- \left(\sum_{j=1}^{i-1} p_j \right) \right) v(x_j)$$

$$+ \sum_{i=k+1}^{n} \left(w^+ \left(\sum_{j=1}^{n} p_j \right) - w^+ \left(\sum_{j=i+1}^{n} p_j \right) \right) v(x_j).$$

Distributions are handled by the natural continuous generalization of (2.0). Considering only the restriction of $\geq$ to nonnegative and nonpositive prospects, Theorem 12 of Wakker (1994) proves that W1–W6 imply, first, that $w^+(p)$ $w^-(p)$ are strictly increasing on $[0,1]$ and continuous on $(0,1)$, and second, that if W4 is augmented by boundary continuity then they are also continuous at $p = 0$ or $p = 1$, respectively.

APPENDIX 2: PROOFS OF PROPOSITIONS 1–5

PROOF OF PROPOSITION 1: I prove the four parts of Proposition 1 in reverse order, D, C, B, and A. Proposition 6, which is invoked in the proof of part A, is proved in Appendix 3.

PROOF OF PROPOSITION 1D (Gains): The proof is based on a functional equation in Aczel (1966, Theorem 2, p. 153), which gives the only solution to

$$(A1.1) \qquad h^{-1}\{\mu h(vt_1) + (1-\mu)h(vt_2)\} = vh^{-1}\{\mu h(t_1) + (1-\mu)h(t_2)\},$$

for μ constant, $0 < \mu < 1$, and v, t_1, t_2 positive, as

$$(A1.2) \qquad \begin{aligned} &h(t) = A\ln(t) + B, \\ &h(t) = At^C + B, \end{aligned}$$

with $A \neq 0$, $C \neq 0$.

To prove Proposition 1, I show that compound invariance gives rise to (A1.1) with $h(t) = \ln(w(\exp(-t)))$. Once (A1.1) is established, $w(p)$ is derived by inserting the solutions in (A.1.2), solving for w, and refining the constants so that the representational assumptions in A0 are satisfied. Specifically, these assumptions would be violated if $w(p)$ is nonincreasing at some p, or if $w(p) = 0$ for $p \neq 0$, or if $w(p)$ is unbounded on $(0,1)$.

PROOF: I first show that compound invariance holds for any rational exponent, not just integers. Define $\zeta(p, s, q, r) = w(p)w(s) - w(q)w(r)$. Pick any p, q, r, s such that $\zeta(p, s, q, r) = 0$, and choose positive outcomes x, y such that, $v(x)/v(y) = w(q)/w(p) = w(s)/w(r)$ (such x, y exist because v is a continuous ratio scale). From $w(p)v(x) = w(q)v(y)$ and $w(r)v(x) = w(s)v(y)$, we have $(x, p) \sim (y, q)$ and $(x, r) \sim (y, s)$. For any integer N, pick outcomes x', y', such that $v(x')/v(y') = w(q^N)/w(p^N)$, or $(x, p^N) \sim (y, q^N)$. Compound invariance then implies that $(x', r^N) \sim (y', S^N)$, or $w(r^N)v(x') = w(s^N)v(y')$. Cancelling $v(x')$ and $v(y')$ from $w(p^N)v(x') = w(q^N)v(y')$ and $w(r^N)v(x') = w(s^N)v(y')$, yields $w(p^N)w(s^N) - w(q^N)w(r^N)$ or $\zeta(p^N, s^N, q^N, r^N) = 0$. Hence, for any probabilities, p, q, r, s, and any integer N, $\zeta(p, s, q, r) = 0$ implies $\zeta(p^N, s^N, q^N, r^N) = 0$. Because ζ is strictly monotonic in each argument it follows, also, that $\zeta(p, s, q, r) > 0$ implies $\zeta(p^N, s^N, q^N, r^N) > 0$ (if not, and $\zeta(p^N, s^N, q^N, r^N) \leq 0$, then by decreasing the first two arguments and/or increasing the second two one could bring about $\zeta(p', s', q', r') = 0$ while $\zeta(p'^N, s'^N, q'^N, r'^N) < \zeta(p^N, s^N, q^N, r^N) \leq 0$). Likewise $\zeta(p, s, q, r) < 0$ implies $\zeta(p^N, s^N, q^N, r^N) < 0$. Consequently, $\zeta(p, s, q, r) = 0$ if and only if $\zeta(p^N, s^N, q^N, r^N) = 0$. Therefore, $\zeta(p, s, q, r) = 0$ also implies $\zeta(p^{1/N}, s^{1/N}, q^{1/N}, r^{1/N}) = 0$, as well as $\zeta(p^{M/N}, s^{M/N}, q^{M/N}, r^{M/N}) = 0$, for any positive integers, M, N. This shows that for any positive rational, $\lambda = N/M$, $(x, p) \sim (y, q)$, $(x, r) \sim (y, s)$, and $(x', p^\lambda) \sim (y', q^\lambda)$ imply $(x', r^\lambda) \sim (y', s^\lambda)$.

Now pick any pair of positive reals t_1, t_2, and any rational $\lambda > 0$. Define probabilities p, q, r, and outcomes x, y, x', y', such that

$$p = \exp(-t_1),$$

$$r = \exp(-t_2),$$

$$q = w^{-1}\{\sqrt{w(p)w(r)}\}, \qquad \text{implying} \qquad w(q)^2 = w(p)w(r),$$

$$\frac{v(x)}{v(y)} = \frac{w(q)}{w(p)},$$

$$\frac{v(x')}{v(y')} = \frac{w(q^\lambda)}{w(p^\lambda)}.$$

Correct q exists because w is continuous, strictly increasing on $(0,1)$. Correct x, y, x', y' exist by continuity of $v(x)$ and the fact that $v(0) = 0$. Jointly $w(p)w(r) = w(q)^2$ and $v(x)/v(y) = w(q)/w(p)$ imply $(x, p) \sim (y, q)$, and $(x, q) \sim (y, r)$, while $v(x')/v(y') = w(q^\lambda)/w(p^\lambda)$ implies $(x', p^\lambda) \sim (y', q^\lambda)$. Invoking compound invariance we extend to $(x', q^\lambda) \sim (y', r^\lambda)$, i.e., $v(x')/v(y') = w(q^\lambda)/w(p^\lambda)$, which combines with $v(x')/v(y') = w(q^\lambda)/w(p^\lambda)$ to show that $w(p^\lambda)w(r^\lambda) = w(q^\lambda)^2$. This proves that $w(p)w(r) = w(q)^2$ implies $w(p^\lambda)w(r^\lambda) = w(q^\lambda)^2$, or, letting $h(t) = \ln w(\exp(-t))$, i.e., $w(p) = \exp(h(-\ln p))$, that $h(-\ln p) + h(-\ln r) = 2h(-\ln q)$ implies $h(-\lambda \ln p) + h(-\lambda \ln r) = 2h(-\lambda \ln q)$. Solving for $-\ln q$ and $-\lambda \ln q$,

(A1.3)
$$-\ln q = h^{-1}\{.5h(-\ln p) + .5h(-\ln r)\},$$
$$-\lambda \ln q = h^{-1}\{.5h(-\lambda \ln p) + .5h(-\lambda \ln r)\}.$$

From the first line of (A1.3), we have: $-\lambda \ln q = \lambda h^{-1}\{.5h(-\ln p) + .5h(-\ln r)\}$, which we may set equal to the bottom line of (A1.3) to produce

$$h^{-1}\{.5h(-\lambda \ln p) + .5h(-\lambda \ln r)\} = \lambda h^{-1}\{.5h(-\ln p) + .5h(-\ln r)\},$$

or

$$h^{-1}\{.5h(\lambda t_1) + .5h(\lambda t_2)\} = \lambda h^{-1}\{.5h(t_1) + .5h(t_2)\},$$

after substituting for t_1, t_2. Both sides of the equality are continuous in λ, as $h(t) = \ln w(\exp(-t))$, with $w(p)$ continuous, strictly increasing on $(0,1)$, and λ, t_1, and t_2 strictly positive. Because the equality holds for every positive rational λ, it follows, by taking limits of both sides, that it holds also for any positive real λ. Hence $h(t) = \ln w(e^{-t})$ satisfies the functional equation (A1.1). The first solution $h(t) = At^C + B$ yields $w(p) = \exp(A(-\ln p)^C + B)$, $A, C \neq 0$. Monotonicity implies that $AC < 0$. The case $A > 0$, $C < 0$ is unbounded. The case $A < 0$, $C > 0$ is given in (3.4), with $\alpha = C$, $\beta = -A$, $\gamma = \exp(B)$. The normalization $w(1) = 1$, constrains: $\gamma \leq 1$. The second solution, $h(p) = A \ln t + B$, yields $w(p) = e^B(-\ln p)^A$, which is unbounded for $A < 0$ and decreasing for $A > 0$.

PROOF OF PROPOSITION 1D (Gains and Losses): To prove that $w^+(p)$ and $w^-(p)$ have the same value of the α-parameter, note that compound invariance allows us to reflect a common-ratio schema from the positive into the negative domain, and vice versa. Therefore, $(x) \sim (y, p)$ and $(x, p) \sim (y, q)$ also implies $(z) \sim (w, p)$ iff $(z, p) \sim (w, q)$, even if the sign of $z\&w$ differs from the sign of $x\&y$. Because the probabilities in each schema uniquely determine the value of the α parameter—and these probabilities are the same—the value of α must be the same in both $w^+(p)$ and $w^-(p)$. The remaining parameters, β, γ are not determined by preferences over simple prospects; hence they may be distinct in the gains and loss part of the representation.

PROOF OF PROPOSITIONS 1C, 1B, 1A: Direct computation with (3.4) shows that compound invariance with $p = 1$ is only possible for $\gamma = 1$, proving 1C (take $(x) \sim (y, q)$, $(x, q) \sim (y, s)$, $(x') \sim (y', q^2)$, $x \neq y$, and show that $(x', q^2) \sim (x', s^2)$ fails unless $\gamma = 1$). Subproportionality implies that $\omega(\ln p) = \ln w(p)$ is strictly convex (viz. Proposition 3) which can only be satisfied for $\alpha < 1$,

proving 1B. By Proposition 6, diagonal concavity precludes $w(p)$ strictly overweighting and convex on any nondegenerate interval. For the twice differentiable equation (3.2), this means that the fixed and inflection points coincide. Computation with (3.2) shows that this forces $\beta = 1$ in equation (3.2), proving 1A.

PROOF OF PROPOSITION 2: I only prove the proposition for the case of gains. Let w_ε denote an ε-subproportional weighting function and w_β a power function, $w_\beta = p^\beta$. The proof involves establishing two claims. The first claim is that w_ε can cross w_β *at most* once; the second claim is that w_ε must cross w_β *at least* once.

Consider the functions ω and ψ, defined by

(A2.1)
$$\omega(\ln p) \equiv \ln w(p),$$
$$\psi(-\ln(-\ln p)) \equiv -\ln(-\ln w(p)) = -\ln(-\omega(\ln p)).$$

ω is the graph of $w(p)$ in log-coordinates, and ψ the graph in double-log coordinates. Because $w(p)$ is continuous, strictly increasing on $(0, 1)$, it follows that $\omega(\ln p)$ is continuous, strictly increasing on the negative reals, and that $\psi(-\ln(-\ln p))$ is continuous, strictly increasing on the entire real line. If p° is a fixed point of w, then $\ln p^\circ$ is a fixed point of ω, and $-\ln(-\ln p^\circ)$ is a fixed point of ψ. Note also that the ψ-graph of a power function is a linear function with slope equal to one:

(A2.2) $w_\beta(p) = p^\beta \Rightarrow \psi_\beta(-\ln(-\ln p)) = -\ln\beta + (-\ln(-\ln p)).$

To prove the first claim, of at most one intersection, I show that subproportional preferences imply that the slope of ψ_ε on any nondegenerate interval is strictly less than one. By Proposition 3, subproportional preferences indicate that the ω-function is convex, i.e., that for any probabilities $q < p$,

$$\frac{\omega_\varepsilon(\ln p)}{\omega_\varepsilon(\ln q)} > \frac{\ln p}{\ln q}.$$

(Recall that subproportionality is defined as "strict on any nondegenerate subinterval," hence ω is strictly convex on any subinterval.) Because $\omega_\varepsilon(\ln p) = -\exp(-\psi_\varepsilon(-\ln(-\ln p)))$, from (A2.1), this means that

$$\frac{\exp(-\psi_\varepsilon(-\ln(-\ln p)))}{\exp(-\psi_\varepsilon(-\ln(-\ln q)))} > \frac{\ln p}{\ln q} = \frac{-\ln p}{-\ln q},$$

or that $-\psi_\varepsilon(-\ln(-\ln p)) + \psi_\varepsilon(-\ln(-\ln q)) > \ln(-\ln p) - \ln(-\ln q)$, i.e., $\psi_\varepsilon(-\ln(-\ln p)) - \psi_\varepsilon(-\ln(-\ln q)) < -\ln(-\ln p) - (-\ln(-\ln q))$, or

$$\frac{\psi_\varepsilon(-\ln(-\ln p)) - \psi_3(-\ln(-\ln q))}{-\ln(-\ln p) - (-\ln(-\ln q))} < 1,$$

for any $-\ln(-\ln p) > -\ln(-\ln q)$. Therefore, the slope of ψ_ε on any nondegenerate interval is strictly less than one. Because ψ_β has slope exactly equal to one (A2.2), the two graphs can intersect at most once, with ψ_ε intersection ψ_β from above.

To prove the second claim, of at least one intersection, I define,

(A2.3) $\psi^N \equiv \psi(N\ln(2 + \varepsilon)),$

where the value of ε is taken from ψ_ε, and N is any integer, positive or negative. I then show that the difference, $\psi_\varepsilon^N - \psi_\beta^N$, has no upper bound as $N \to -\infty$, nor lower bound as $N \to +\infty$.

The preference conditions given by ε-subproportionality, $(x) \sim (y, p) \Rightarrow (x, p) < (y, p^{2+\varepsilon})$, indicate that $v(x) = w(p)v(y) \Rightarrow w(p)v(x) < w(p^{2+\varepsilon})v(y)$, i.e.,

$$w_\varepsilon(p)^2 < w_\varepsilon(p^{2+\varepsilon}),$$

$$-2\ln w_\varepsilon(p) > -\ln w_\varepsilon(p^{2+\varepsilon}),$$

$$\ln 2 + \ln(-\ln w_\varepsilon(p)) > \ln(-\ln w_\varepsilon(p^{2+\varepsilon})),$$

$$\ln 2 - \psi_\varepsilon(-\ln(-\ln p)) > -\psi_\varepsilon(-\ln(-\ln p^{2+\varepsilon})) \qquad \text{(from A4.1)},$$

$$= -\psi_\varepsilon(-\ln(-\ln p) - \ln(2+\varepsilon)).$$

In particular, for $-\ln(-\ln p) = N\ln(2+\varepsilon)$, this implies

$$\psi_\varepsilon(N\ln(2+\varepsilon)) - \psi_\varepsilon((N-1)\ln(2+\varepsilon)) = \psi_\varepsilon^N - \psi_\varepsilon^{N-1} < \ln 2,$$

for any integer N. By induction, we have,

(A2.4)
$$\psi_\varepsilon^N - \psi_\varepsilon^0 < N\ln 2, \qquad \text{for } N \geq 1,$$

$$\psi_\varepsilon^N - \psi_\varepsilon^0 > N\ln 2, \qquad \text{for } N \leq -1.$$

(The second line can also be read as $\psi_\varepsilon^0 - \psi_\varepsilon^N < (-N)\ln 2$, for $N \leq -1$.)

The difference between ψ_ε and ψ_β can now be expressed as the sum of three terms:

$$\psi_\varepsilon^N - \psi_\beta^N = (\psi_\varepsilon^N - \psi_\varepsilon^0) + (\psi_\beta^0 - \psi_\beta^N) + (\psi_\varepsilon^0 - \psi_\beta^0)$$

$$= (\psi_\varepsilon^N - \psi_\varepsilon^0) - N\ln(2+\varepsilon) + (\psi_\varepsilon^0 - \psi_\beta^0) \qquad \text{(from A2.2)}.$$

For positive N, we substitute for $\psi_\varepsilon^N - \psi_\varepsilon^0$ from the top line of (A2.4) to obtain the inequality

$$\psi_\varepsilon^N - \psi_\beta^N < N\ln 2 - N\ln(2+\varepsilon) + (\psi_\varepsilon^0 - \psi_\beta^0) \qquad (N \geq 1).$$

Because $\varepsilon > 0$ and $\psi_\varepsilon^0 - \psi_\beta^0$ is a constant, the difference, $\psi_\varepsilon^N - \psi_\beta^N$, must be negative for sufficiently large N. For negative N, we substitute for $\psi_\varepsilon^N - \psi_\varepsilon^0$ from the bottom line of (A2.4) to obtain the inequality,

$$\psi_\varepsilon^N - \psi_\beta^N > N\ln 2 - N\ln(2+\varepsilon) + (\psi_\varepsilon^0 - \omega_\beta^0) \qquad (N \leq -1),$$

showing that the difference, $\psi_\varepsilon^N - \psi_\beta^N$, must be positive for sufficiently large N. Consequently, the two graphs have at least one intersection. As we have previously shown that ψ_ϵ and ψ_β can intersect at most once, this proves that there is a unique intersection point, with the ε-subproportional function intersecting the power function from above.

PROOF OF PROPOSITION 3—(C) $\Rightarrow$ (A): Assume that (A) is false, and that we have probabilities $p > r \geq q > s$, and outcomes $x < y$, $x' < y'$, such that $(x, p) \sim_2 (y, q)$, $(x, r) \sim_2 (y, s)$, and $(x', p) \sim_1 (y', q)$, but $(x', r) >_1 (y', s)$. Applying the separable formula to each relation and cancelling $v(x)$ and $v(y)$, we arrive at $w_2(p)/w_2(r) = w_2(q)/w_2(s)$, and $w_1(p)/w_1(r) < w_1(q)/w_1(s)$. In terms of ω-functions, $\psi(\ln p) = \ln w(p)$, this implies $\exp(\omega_2(\ln p))/\exp(\omega_2(\ln r)) = \exp(\omega_2(\ln q))/\exp(\omega_2(\ln s))$, and $\exp(\omega_1(\ln p))/\exp(\omega_1(\ln r)) < \exp(\omega_1(\ln q))/\exp(\omega_1(\ln s))$, or $\omega_2(\ln p) - \omega_2(\ln r) = \omega_2(\ln q) - \omega_2(\ln s)$, and $\omega_1(\ln p) - \omega_1(\ln r) < \omega_1(\ln q) - \omega_1(\ln s)$. Consequently,

$$\frac{\omega_1(\ln p) - \omega_1(\ln r)}{\omega_1(\ln q) - \omega_1(\ln s)} < 1 = \frac{\omega_2(\ln p) - \omega_2(\ln r)}{\omega_2(\ln q) - \omega_2(\ln s)},$$

which, in view of $\ln p > \ln r \geq \ln q > \ln s$, contradicts the claim (B) that ω_1 is a convex transformation of ω_2.

To prove, (A) $\Rightarrow$ (C), pick any $0 < s < r \leq q < p \leq 1$, such that

(A3.1) $\omega_2(\ln p) - \omega_2(\ln q) = \omega_2(\ln r) - \omega_2(\ln s)$.

Substituting $\omega(\ln p) = \ln w(p)$, we can rewrite this as

(A3.2) $w_2(p)/w_2(q) = w_2(r)/w_2(s)$.

Because each $v_i(x)$ is a continuous ratio scale with $v(0) = 0$, we can also choose $x < y$, $x' < y'$, such that

(A3.3) $w_2(p)/w_2(q) = v_2(y)/v_2(x)$,

(A3.4) $w_1(p)/w_1(q) = v_1(y')/v_1(x')$.

(A3.3) implies $w_2(p)v_2(x) = w_2(q)v_2(y)$ or $(x, p) \sim_2 (y, q)$, (A3.4) implies $w_1(p)v_1(x') = w_1(q)v_1(y')$ or $(x', p) \sim_1 (y', q)$, while (A3.2) and (A3.3) jointly imply $w_2(r)v_2(x) = w_2(s)v_2(y)$ or $(y, s) \sim_2 (x, r)$. Because $\geq_1$ exhibits more common-ration violations than $\geq_2$, it follows that $(y', s) \geq_1 (x', r)$, i.e., $w_1(s)v_1(y') \geq w_1(r)v_1(x')$, or $v_1(y')/v_1(x') \geq w_1(r)/w_1(s)$. Combining this inequality with (A3.4) yields $w_1(p)/w_1(q) \geq w_1(r)/w_1(s)$, or $\exp \omega_1(\ln p)/\exp \omega_1(\ln q) \geq \exp \omega_1(\ln r)/\exp \omega_1(\ln s)$, or $\omega_1(\ln p) - \omega_1(\ln q) \geq \omega_1(\ln r) - \omega_1(\ln s)$. In view of (A3.1), this shows that for any nonpositive $\ln p > \ln q \geq \ln r > \ln s$, the equality $\omega_2(\ln p) - \omega_2(\ln q) = \omega_2(\ln r) - \omega_2(\ln s)$ implies $\omega_1(\ln p) - \omega_1(\ln q) \geq \omega_1(\ln r) - \omega_1(\ln s)$, which proves that ω_1 is a convex transform of ω_2 on $t \leq 0$.

The equivalence of (B) and (C) follows from the definitions. Weak subproportionality, $w(p)w(\lambda q) \geq w(q)w(\lambda p)$, for $p > q, 1 > \lambda > 0$, may be rendered as $w(p)/w(\lambda p) \geq w(q)/w(\lambda q)$, i.e., $\omega(\ln p) - \omega(\ln p + \ln \lambda) \geq \omega(\ln q) - \psi(\ln q + \ln \lambda)$, for $0 > \ln p > \ln q$, $0 > \ln \lambda$. Hence $\omega(\ln p)$ is convex iff $w(p)$ is weakly subproportional.

PROOF OF PROPOSITIONS 4 AND 5: The proofs are based on a second functional equation in Aczel (1966, Theorem 2, p. 153), which gives the only solutions to

(A4.1) $g^{-1}\{\mu g(v + t_1) + (1 - \mu)g(v + t_2)\} = v + g^{-1}\{\mu g(t_1) + (1 - \mu)g(t_2)\}$,

for μ constant, $0 < \mu < 1$, and v, t_1, t_2 positive as

(A4.2)
$$g(t) = At + B,$$
$$g(t) = Ae^{Ct} + B,$$

with $A \neq 0$, $C \neq 0$. To prove Propositions 4 and 5, one shows that both preference patterns give rise to one of these forms, specifically:
   (i) $g(t) = \ln(w(\exp(-t)))$, for conditional invariance,
   (ii) $g(t) = \ln(w^{-1}\{\exp(-t)\})$, for projection invariance.
Once (A4.1) is established, the corresponding $w(p)$ forms are derived by inserting the solutions in (A4.2), solving for w, and refining the constants to meet whatever other axioms apply in a particular case.

PROOF OF 4B: As with Proposition 1, I first prove the more general case (4B), where conditional invariance is assumed to hold only for probabilities in the open interval $(0, 1)$. For any positive

v, t_1, t_2, define probabilities λ, p, q, r and outcomes x, y, x', y', such that

$$\lambda = \exp(-v),$$

$$p = \exp(-t_1),$$

$$r = \exp(-t_2),$$

$$q = w^{-1}\{\sqrt{w(p)w(r)}\}, \qquad \text{implying} \qquad w(q)^2 = w(p)w(r),$$

$$\frac{v(x)}{v(y)} = \frac{w(q)}{w(p)},$$

$$\frac{v(x')}{v(y')} = \frac{2(\lambda q)}{w(\lambda p)}.$$

Correct q exists because w is continuous, strictly increasing on $(0,1)$. Correct x, y, x', y' exist by continuity of $v(x)$ and the fact that $v(0) = 0$.

Now, $w(p)w(r) = w(q)^2$ and $v(x)/v(y) = w(q)/w(p)$ jointly imply $(x, p) \sim (y, q)$, and $(x, q) \sim (y, r)$, while $v(x')/v(y') = w(\lambda q)/w(\lambda p)$ implies $(x', \lambda p) \sim (y', \lambda q)$. Invoking conditional invariance (Definition 4) we can extend this to $(x', \lambda q) \sim (y', \lambda r)$, i.e., $v(x')/v(y') = w(\lambda r)/w(\lambda q)$, which combines with $v(x')/v(y') = w(\lambda q)/w(\lambda p)$ to show that $w(\lambda p)w(\lambda r) = w(\lambda q)^2$. This proves that $w(p)w(r) = w(q)^2$ implies $w(\lambda p)w(\lambda r) = w(\lambda q)^2$, or letting $h(p) = \ln w(p)$, that $h(p) + h(r) = 2h(q)$ implies $h(\lambda p) + h(\lambda r) = 2h(\lambda q)$. Solving for q and λq,

(A4.3)
$$q = h^{-1}\{.5h(p) + .5h(r)\},$$
$$\lambda q = h^{-1}\{.5h(\lambda p) + .5h(\lambda r)\}.$$

From the first line of (A4.3), we obtain $\lambda q = \lambda h^{-1}\{.5h(p) + .5h(r)\}$, which we may set equal to the bottom line of (A4.3) to produce

$$h^{-1}\{.5h(\lambda p) + .5h(\lambda r)\} = \lambda h^{-1}\{.5h(p) + .5h(r)\},$$

or,

$$g^{-1}\{.5g(v + t_1) + .5g(v + t_2)\} = v + g^{-1}\{.5g(t_1) + .5g(t_2)\},$$

with $g(t) = h(\exp(-t))$, $h^{-1}\{t\} = \exp(-g^{-1}\{t\})$. Hence $g(t) = \ln w(\exp(-t))$, or $g(-\ln p) = \ln w(p)$. The solution, $g(-\ln p) = \ln w(p) = -A \ln p + B$, yields $w(p) = e^B p^{-A}$, while $g(-\ln p) = \ln w(p) = Ae^{-C \ln p} + B = Ap^{-C} + B$ yields $w(p) = \exp(Ap^{-C} + B)$. The first solution is the power function in (3.2). For the second solution, monotonicity implies $AC < 0$. This corresponds to $\gamma w(p)$ where $w(p)$ is given in equation (4.2) in the text, with $\alpha = -C$, $\beta = -AC$, $\gamma = \exp(A + B)$.

PROOF OF 4A: Direct computation with $\gamma w(p)$, $w(p)$ either (4.2) or (4.3), shows that conditional invariance with $p = 1$ is only possible for $\gamma = 1$ (take $(x) \sim (x, q) \sim (y, s)$, $(x') \sim (y', .5q)$, $x \neq y$, and show that $(x', .5q) \sim (x', .5s)$ fails unless $\gamma = 1$).

PROOF OF 4C: By 4A, conditional invariance implies $w(p)$ of the form (4.2) or (4.3). Subproportionality is inconsistent with (4.3). The ω-function for (4.2) is $\omega(\ln p) = (-\beta/\alpha)(-p^\alpha)$, or $\omega(t) = (-\beta/\alpha)(1 - \exp(\alpha t))$, which is convex for $\alpha > 0$. Hence subproportionality implies $\alpha > 0$. The limit as $p \to 0$ of equation (4.2) is then $\exp(-\beta/\alpha) > 0$, which violates boundary continuity.

PROOF OF 5B: I take the two parts in the order 5B, 5A. First I will slightly rephrase the statement of the projection invariance axiom. Note that in the context of our separable representation with continuous, strictly increasing $w(p)$, the axiom $(x, p) \sim (y, q)$ and $(x, rp) \sim (y, sq) \Rightarrow (x, r^2 p) \sim (y, s^2 q)$, also implies $(x, p) \sim (y, q)$ and $(x, r^2 p) \sim (y, s^2 q) \Rightarrow (x, rp) \sim (y, sq)$. Letting $r' = pr^2$, $q' =$

sq^2, this indicates that, $(x, p) \sim (y, q)$ and $(x, r') \sim (y, s') \Rightarrow (x,(r'/p)^{.5}p) \sim (y,(s'/q)^{.5}q)$, i.e.,

$$(x, p) \sim (y, q) \text{ and } (x, r') \sim (y, s') \Rightarrow (x, p^{.5}r'^{.5}) \sim (y, q^{.5}s'^{.5}).$$

Now for any positive v, t_1, t_2, define probabilities p, q, r, s and outcomes x, y, such that

$$p = w^{-1}\{w_1 \exp(-t_1)\},$$

$$r = w^{-1}\{w_1 \exp(-t_2)\},$$

(A5.1) $$q = w^{-1}\{w_1 \exp(-(v + t_1))\},$$

$$s = w^{-1}\{w_1 \exp(-(v + t_2))\},$$

$$\frac{v(x)}{v(y)} = \exp(-v),$$

where $w_1 = \lim\{w(p)\}$ as $p \to 1$. Correct x, y exist by continuity of $v(x)$ and the fact that $v(0) = 0$. Correct p, q, r, s exist because w is a continuous, strictly increasing map of $(0, 1)$ onto $(0, w_1)$. To show that $w(p)$ is indeed continuous at zero, observe that projection invariance implies, in particular, that $(x) \sim (y, q)$ and $(x, q) \sim (y, sq) \Rightarrow (x, q^2) \sim (y, s^2q)$, and by induction, for any integer $N \geq 1$, that $(x, q^N) \sim (y, s^Nq)$. Hence for any N, $w(q^N)v(x) = w(s^Nq)v(y)$, or $v(x)/v(y) = w(s^Nq)/w(q^N)$, which indicates that the limit as $N \to \infty$ of $w(q^N)$ is zero (otherwise the limit of the ratio, $w(s^Nq)/w(q^N)$, would equal one).

Combining lines one, three, and five in (A5.1), yields: $w(p)v(x) = w(q)v(y)$, i.e., $(x, p) \sim (y, q)$; combining lines two, four, and five yields: $w(r)v(x) = w(s)v(y)$, i.e, $(x, r) \sim (y, s)$. By projection invariance, we may infer that $(x, p^{.5}r^{.5}) \sim (y, q^{.5}s^{.5})$, or

$$v(x)w(p^{.5}r^{.5}) = v(y)w(q^{.4}sq^{.5}).$$

Substituting for p, q, r, s, x, y, from (A5.1),

$$\exp(-v)w(w^{-1}\{w_1 \exp(t_{-1})\}^{.5}w^{-1}\{w_1 \exp(-t_2)\}^{.5})$$

$$= w(w^{-1}\{w_1 \exp(-v - t_1)\}^{.5}w^{-1}\{w_1 \exp(-v - t_2)\}^{.5}),$$

and writing

(A5.2) $$\exp(g(t)) = w^{-1}\{w_1 \exp(-t)\}, \qquad \text{i.e., } \ln w(p) = g^{-1}\{\ln p\} + \ln w_1,$$

we substitute first for $w^{-1}\{w_1 \exp(-t)\}$:

$$\exp(-v)w(\exp(.5g(t_1))\exp(.5g(t_2))) = w(\exp(.5g(v + t_1))\exp(.5g(v + t_2))).$$

Then, taking logs, we substitute for $\ln w(p)$ from (A5.2), to obtain the basic functional equation (A4.1):

$$v + g^{-1}\{.5g(t_1) + .5g(t_2)\} = g^{-1}\{.5g(v + t_1) + .5g(v + t_2)\}.$$

As $g(t) = \ln w^{-1}\{w_1 \exp(-t)\}$, the first solution, $g(t) = At + B$, yields the power function: $w(p) = w_1 \exp(B/A)p^{-1/A}$. The second solution, $g(t) = Ae^{Ct} + B$, yields the log-hyperbola: $w(p) = w_1((\ln p - B)/A)^{+1/C}$. The case, $B \leq 0$ is inconsistent with the representational assumptions. Hence $B > 0$ and $A < 0$. By monotonicity, $C < 0$, yielding,

$$w(p) = \gamma(1 - \alpha \ln p)^{-\beta/\alpha}, \qquad \alpha, \beta, \gamma > 0,$$

with $\alpha = 1/B$, $\beta = BC$, $\gamma = w_1(-A/B)^C$. The condition $w(1) = 1$ constrains $\gamma \leq 1$.

PROOF OF 5A: From 5B we know that $\alpha > 0$ in (4.4). The ω-function for (4.4) is $\omega(\ln p) = (-\beta/\alpha)\ln(1 - \alpha \ln p)$, or $\omega(t) = (-\beta/\alpha)\ln(1 - \alpha t)$, which is convex for $\alpha > 0$. Hence (4.4) is

subproportional. The other solution, (4.3), is weakly subproportional. Hence preferences must be weakly subproportional. To show that $\gamma = 1$, take $(x) \sim (y, q)$, $(x, q) \sim (y, sq)$, $x \neq y$, and show by direct computation that for either (4.4) or (4.3), the implication $(x, q^2) \sim (y, s^2 q)$ fails unless $\gamma = 1$.

To assess the implications of diagonal concavity, I first define the inflection point in terms of the ω-function, $\omega(\ln p) = \ln w(p)$, which we can rewrite as $\exp(\omega(t)) = w(\exp(t))$, for $\exp(t) = p$, $t = \ln p$. Differentiating once yields $\omega'(t)\exp(\omega(t)) = \exp(t)w'(\exp(t))$, and twice yields $\omega'(t)^2 \exp(\omega(t)) + \omega''(t)\exp(\omega(t)) = \exp(t)w'(\exp(t)) + \exp(2t)w''(\exp(t))$. Dividing the left and right terms in the second-order equality by the left and right terms in the first-order equality, gives

$$\omega_1'(t) + \frac{\omega_1''(t)}{\omega_1'(t)} = 1 + \exp(t)\frac{w_1''(\exp(t))}{w_1'(\exp(t))},$$

$$= 1 \qquad \text{if } t \text{ is an inflection point.}$$

The ω-function associated with (4.4) is $\omega(t) = -(\beta/\alpha)\ln(1 + \alpha t)$. If t° is a fixed point, then $\alpha t^\circ = -\beta \ln(1 - \alpha t^\circ)$. If t° is also an inflection point, it follows from the above expression that $\alpha + \beta = 1 - \alpha t^\circ$. Eliminating αt° yields the constraint $(\alpha + \beta) - \beta \ln(\alpha + \beta) = 1$. Letting $a = \alpha + \beta$, we may write $\alpha = (a \ln(a) - (a - 1))/\ln(a)$, and $\beta = (a - 1)/\ln(a)$, with $\alpha, \beta > 0$ implying $a > 1$. To compute the lower and upper fixed point bounds, we express t° as

$$t^\circ = -\frac{\beta}{\alpha}\ln(1 - \alpha t^\circ) = -\frac{(a - 1)\ln(a)}{a \ln(a) - (a - 1)} = \frac{f(a)}{g(a)}.$$

Applying l'Hopital's rule twice yields $f'(a)/g''(a) = -(1 + a^{-1})$, which, in the two limits, at $a = 1$, and $a \to +\infty$, equals -2 and -1, respectively. Therefore, the fixed point probability lies between $e^{-2} (= .14)$ and $e^{-1} (= .37)$.

APPENDIX 3: PROOF OF PROPOSITION 6, ON DIAGONAL CONCAVITY

I derive here a general result on diagonally concave representations, which is needed in the proof of Proposition 1. As in the text, let $\Delta^+[s, r] = \{(x, p; y, q) | 0 < x < y, s \leq q, p + q \leq r\}$ and $\Delta^-[s, r] = \{(x, p; y, q) | y < x < 0, s \leq q, p + q \leq r\}$, i.e., the sets of binary prospects where the probability of the extreme outcome is at least s and the probability of the zero outcome at least $1 - r$. I first state and prove the proposition for gains, and then briefly discuss the proposition for losses.

PROPOSITION 6 (Gains): *Let $\geq$ be a subproportional preference relation on $\mathscr{P}^+$ satisfying A0. Then for any interval $[s, r]$, $0 < s < r < 1$, the following two conditions are equivalent.*
   (i) *$\geq$ is quasiconvex and strictly CE-quasiconcave on $\Delta^+[s, r]$ [respectively, $\geq$ is quasiconcave and strictly CE-quasiconvex on $\Delta^+[s, r]$];*
   (ii) *$w^+(p)$ is convex and strictly overweighting on $[s, r]$ [respectively, concave and strictly underweighting on $[s, r]$].*

The restriction of $[s, r]$ to the interior of the probability interval takes account of the possible discontinuities of $w^+(p)$ at the endpoints. If $w^+(p)$ is continuous on $[0, 1]$ then the statement of Proposition 6 can include $s = 0$, $r = 0$.

The proof requires two separate steps. The first step, in Lemma 1, involves translating concavity or convexity of the weighting function into the language of preferences. Here I apply Wakker's (1994) full-interval argument to the subinterval $[s, r]$. The second step involves translating over- and under-weighting into the language of preferences. This is done in Lemmas 2 and 3. The key point is the use of CE-quasiconvexity to derive an inequality, (A6.3), that supports conclusions about the sign of $w(p) - p$.

LEMMA 1: *Under the assumptions in Proposition 6, $\geq$ is quasiconvex [quasiconcave] on $\Delta^+[s, r]$ iff $w^+(p)$ is convex [concave] on $[s, r]$.*

PROOF: If $w^+(p)$ is convex on $[s, r]$ then quasiconvexity follows by direct application of the rank-dependent formula. Wakker (1994, Theorem 24) proves the converse for the special case of full probability interval, $s = 0$, $r = 1$, and the main idea in his proof can be applied to any $[s, r]$. Suppose that $\geq$ is quasiconvex on $\Delta^+[s, r]$ but that $w^+(p)$ is not convex. In that case, for some $s \leq p < q \leq r$, and $0 < \mu < 1$,

$$(A6.1) \qquad \mu w^+(p) + (1 - \mu)w^+(q) < w^+(\mu p + (1 - \mu)q).$$

If $q < r$, then because w^+ is strictly increasing there is a $p^* \in (q, r)$ at which w^+ has finite right derivative. Define prospects $P = (x, q^* - p; y, p)$, $Q = (x, p^* - q; y, q)$, for $q < p^* < q^* < r$, with $v(x) = v^\circ < 1 = v(y)$ adjusted so that $P \sim Q$, i.e.,

$$(A6.2) \qquad (1 - v^\circ)(w^+(q) - w^+(p)) = v^\circ(w^+(q^*) - w^+(p^*)).$$

By quasiconvexity, $P, Q \geq \mu P + (1 - \mu)Q$ so that $\mu V(P) + (1 - \mu)V(Q) \geq V(\mu P + (1 - \mu)Q)$, or

$$\mu((1 - v^\circ)w^+(p) + v^\circ w^+(q^*)) + (1 - \mu)((1 - v^\circ)w^+(q) + v^\circ w^+(p^*))$$
$$\geq (1 - v^\circ)w^+(\mu p + (1 - \mu)q) + v^\circ w^+(\mu q^* + (1 - \mu)p^*),$$

which yields

$$(1 - v^\circ)(\mu w^+(p) + (1 - \mu)w^+(q) - w^+(\mu p + (1 - \mu)q))$$
$$\geq v^\circ(w^+(\mu q^* + (1 - \mu)p^*) - (\mu w^+(q^*) + (1 - \mu)w^+(p^*))).$$

Dividing by the left and right sides of (A6.2) produces a sum of two fractions:

$$\frac{(\mu w^+(p) + (1 - \mu)w^+(q)) - w^+(\mu p + (1 - \mu)q)}{w^+(q) - w^+(p)}$$
$$+ \frac{(\mu w^+(p^*) + (1 - \mu)w^+(q^*)) - w^+(\mu p^* + (1 - \mu)q^*)}{w^+(q^*) - w^+(p^*)} \geq 0,$$

that must be nonnegative for all $s < p < q < p^* < q^* < r$, if indeed preferences are quasiconvex on $\Delta^+[s, r]$. By (A6.1) the first fraction is strictly negative, which means that the second fraction must be positive. However, as $w^+(p^*)$ has finite right derivative, the second fraction can be brought down to zero by drawing q^* down to p^*. Consequently, whenever (A6.1) involves a strict subinterval of $[p, q]$ we can construct a contradiction of quasiconvexity. But, if (A6.1) holds for $p = s$, $q = r$, it will also hold for some subinterval, in which case we are again able to create a quasiconvexity violation. Hence w^+ is necessarily convex. This proves Lemma 1.

LEMMA 2: *Under the assumptions of Proposition 6, if $w^+(p)$ is concave and underweighting on $[s, r]$, then $\geq$ is CE-quasiconvex on $\Delta^+[s, r]$. If $w^+(p)$ is convex and overweighting on $[s, r]$, then $\geq$ is CE-quasiconcave on $\Delta^+[s, r]$. If either concavity/convexity or under/overweighting is strict, then the relevant preference property is strict.*

PROOF: Consider a sure outcome $Q = (x)$, and prospects $P = (x, q - p; y, p)$ and $\mu Q + (1 - \mu)P = (x, \mu + (1 - \mu)(q - p); y, (1 - \mu)p)$. Preferences will be CE-quasiconvex with respect to these three prospects if $P \sim Q$ implies $Q \geq \mu P + (1 - \mu)Q$, which is to say, $\mu V(P) + (1 - \mu)V(Q) \geq V(\mu P + (1 - \mu)Q)$. Let $v(y) = 1 > v^\circ = v(x)$ (note that v is a ratio scale). Applying the rank-dependent formula for binary prospects, equation (2.2), we have:

$$Q \sim P \text{ implies } v^\circ = (1 - v^\circ)w^+(p) + v^\circ w^+(q),$$
$$P \geq Q + (1 - \mu)P \text{ implies } (1 - v^\circ)w^+(p) + v^\circ w^+(q)$$
$$\geq (1 - v^\circ)w^+(p - \mu p) + v^\circ w^+(q + \mu(1 - q)).$$

Cancelling the $(1 - v°)/v°$ ratio from both equations yields

$$\frac{1 - w^+(q)}{w^+(p)} \geq \frac{w^+(q + \mu(1 - q)) - w^+(q)}{w^+(p) - w^+(p - \mu p)},$$

or

$$(A6.3) \qquad \frac{1 - w^+(q)}{1 - q} \frac{p}{w^+(p)} \geq \frac{\dfrac{w^+(q + \mu(1 - q)) - w^-(q)}{\mu(1 - q)}}{\dfrac{w^+(p) - w^+(p - \mu p)}{\mu p}}.$$

Lemma 2 now follows immediately. Assume $w^+(p)$ is concave and underweighting on $[s, r]$. By assumption, P and $\mu P + (1 - \mu)Q$ are in $\Delta^+[s, r]$, hence $s \leq p - \mu p \leq q + \mu(1 - q) \leq r$. Underweighting implies $w^+(p) \leq p$ and $1 - w^+(q) \geq 1 - q$, so the left side in (A6.3) is not less than 1. Concavity and $p \leq q$ imply that

$$\frac{w^+(1 + \mu(1 - q)) - w^+(q)}{\mu(1 - q)} \leq \frac{w^+(p) - w^+(p - \mu p)}{\mu p},$$

which proves the right side in (A6.3) is not greater than 1. Therefore, the inequality in (A6.3) holds, and preferences are CE-quasiconvex. Strict CE-quasiconvexity arises if the left side in (A6.3) is strictly less than one, and/or if the right side is strictly greater than one, i.e., for strict underweighting and/or strict concavity of $w^+(p)$ on $[s, r]$.

LEMMA 3: *Under the assumptions of Proposition 6, $w^+(p) > p$ $[< p]$ implies that $\succeq$ is not CE-quasiconvex [not CE-quasiconcave] on any $\Delta^+[s, r]$ with $s < p < r$.*

PROOF: Note first that under the assumptions of Proposition 6, $w^+(p)$ is absolutely continuous because subproportionality implies that the function, $\omega(\ln p) = \ln w^+(p)$ is convex, and therefore absolutely continuous. Because w^+ is also strictly increasing, $w^+(p) > p$ implies that there exists a p^* near p, such that $s < p^* < r$, $w^+(p^*) > p^*$, and w^+ has a finite, nonzero derivative at p^*. Now letting $p = q = p^*$ in (A6.3), CE-quasiconvexity on $\Delta^+[s, r]$ implies that for all μ such that: $s \leq p - \mu p$, $q + \mu(1 - q) \leq r$, we will have

$$\frac{1 - w^+(p^*)}{1 - p^*} \frac{p^*}{w^+(p^*)} \geq \frac{\dfrac{w^+(p^* + \mu(1 - p^*)) - w^+(p^*)}{\mu(1 - p^*)}}{\dfrac{w^+(p^*) - w^+(p^* - \mu p^*)}{\mu p^*}}.$$

Differentiability at p^* with nonzero derivative implies that the right side approaches one as $\mu \to 0$; the left side, however, is a constant strictly less than one, as $w^+(p^*) > p^*$. Therefore the inequality reverses for sufficiently small μ, and $\succeq$ is not CE-quasiconvex on any $\Delta^+[s, r]$ such that $s < p < r$.

To prove Proposition 6 it is sufficient to show that for any nondegenerate interval $[s, r]$, quasiconvexity and strict CE-quasiconcavity of preference on $\Delta^+[s, r]$ is equivalent to the statement that $w^+(p)$ is convex and strictly overweighting on $[s, r]$. A similar argument would show that quasiconcavity and strict CE-quasiconvexity of preference on $\Delta^+[s, r]$ is equivalent to the statement that $w^+(p)$ is concave and strictly underweighting on $[s, r]$.

PROOF OF PROPOSITION 6($\Rightarrow$): Quasiconvexity and strict CE-quasiconcavity imply that $w^+(p)$ is convex and strictly overweighting on $[s, r]$. We take the two implications in turn:

(i) By Lemma 1, quasiconvexity implies that $w^+(p)$ is convex.

(ii) It remains to be shown that $w^+(p)$ is strictly overweighting on $[s, r]$. I prove the contrapositive, i.e., if $w^+(p)$ is not strictly overweighting, then either quasiconvexity or strict CE-quasiconcavity must fail. On the assumption that $w^+(p)$ is not strictly overweighting, either $w^+(p) = p$ on all of

$[s, r]$ or $w^+(p) < p$ for some $p \in (s, r)$. If $w^+(p) = p$, then it is also concave and underweighting, which, by Lemma 2, implies CE-quasiconvexity, hence failure of strict CE-quasiconcavity on $\Delta^+[s, r]$. If, however, $w^+(p) < p$ for some p, then by Lemma 3, preferences are not CE-quasiconcave on $\Delta^+[s, r]$. Therefore, $w^+(p)$ must be strictly overweighting on $[s, r]$.

PROOF ($\Leftarrow$): $w^+(p)$ convex and strictly overweighting on $[s, r]$ implies quasiconvexity and strict CE-quasiconcavity on $\Delta^+[s, r]$. We take the two implications in turn:
  (i) By Lemma 1, $w^+(p)$ convex implies quasiconvexity on $\Delta^+[s, r]$.
  (ii) By Lemma 2, $w^+(p)$ convex and strictly overweighting implies strict CE-quasiconcavity on $\Delta^+[s, r]$.

The loss-version of Proposition 6 has a parallel statement and proof, but with reversal of quasiconcavity and quasiconvexity, and of CE-quasiconcavity and CE-quasiconvexity.

PROPOSITION 6 (Losses): *Let $\geq$ be a subproportional preference relation on $\mathscr{P}$ satisfying A0. Then for any interval $[s, r]$, $0 < s < r < 1$, the following two conditions are equivalent*:
  (i) *$\geq$ is quasiconcave and strictly CE-quasiconvex on $\Delta^-[s, r]$, [respectively, $\geq$ is quasiconvex and strictly CE-quasiconcave on $\Delta^-[s, r]$]*;
  (ii) *$w^-(p)$ is convex and strictly overweighting on $[s, r]$ [respectively, concave and strictly underweighting on $[s, r]$]*.

COMMENTS ON THE PROOF: The proof follows the same sequence in Lemmas 1–3, with parallel prospect-constructions. In particular, Lemma 1 for losses asserts that $\geq$ is quasiconcave [quasiconvex] on $\Delta^-(s, r)$ iff $w^-(p)$ is convex [concave] on $[s, r]$; Lemma 2 for losses asserts that $w^-(p)$ is concave and underweighting on $[s, r]$ implies $\geq$ is CE-quasiconcave on $\Delta^-[s, r]$, and $w^-(p)$ convex and overweighting on $[s, r]$ implies $\geq$ is CE-quasiconvex on $\Delta^-[s, r]$; Lemma 3 for losses asserts that $w^-(p) > p$ ($< p$) implies that $\geq$ is not CE-quasiconcave [not CE-quasiconvex] on any $\Delta^-[s, r]$, $s < p < r$. In terms of constructions, the generic positive binary prospect, $P^+ = (x, q - p; y, p)$, with $0 < v^\circ \equiv v(x) < 1 \equiv v(y)$, evaluating as $V(P^+) = (1 - v^\circ)w^+(p) + v^\circ w^+(q)$, is replaced by a "reflected" counterpart, $P^- = (x, q - p; y, p)$, with $0 > -v^\circ \equiv v(x) > -1 \equiv v(y)$, evaluating as $V(P^-) = -((1 - v^\circ)w^-(p) + v^\circ w^-(q))$. In the proofs, statements about probabilistic risk-attitude for losses lead to opposite inequalities at the level of the weighting function, e.g., as in the central (A6.3). The flow of the argument is otherwise the same.

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