UTokyo

Quantum Magic and Conformal Field Theory

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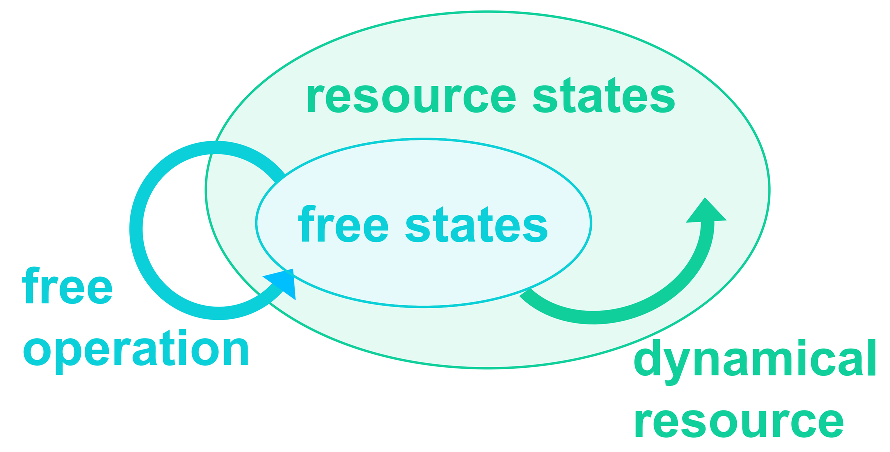
Magic state resource theory

■ **Information processing consists of EASY and HARD operations**



■ **Quantum Computation**

Free states: Stabilizer states — Free operation: Clifford gate,…

Resource states: Magic states — Dynamical resource: T gate,…

■ **Clifford + T gate set**

$$\mathrm{CNOT}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad H=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\1&-1\end{pmatrix}\quad S=\begin{pmatrix}1&0\\0&i\end{pmatrix}\qquad T=\begin{pmatrix}1&0\\0&e^{i\pi/4}\end{pmatrix}$$

Clifford group

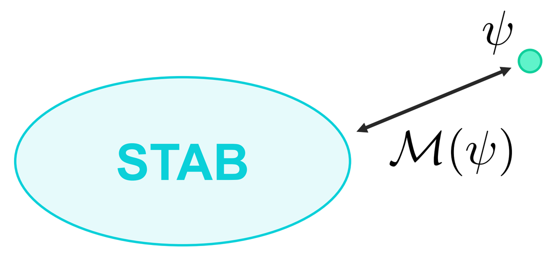
◆ Transversal: easy to implement fault-tolerantly

◆ Gottesman-Knill theorem: classically simulable

T gate

◆ generate magic states

■ **Quantifying magic ≈ distance from stabilizer states**



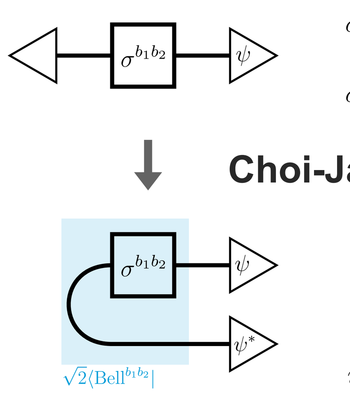
1. **Faithfulness:** $\mathcal{M}(\psi)=0$ iff $\psi\in\mathrm{STAB}$
2. **Monotonicity:** $\mathcal{M}(\mathcal{E}(\psi))\le\mathcal{M}(\psi),\quad\forall\mathcal{E}\in\mathcal{S}$
3. **Additivity:** $\mathcal{M}(\psi\otimes\phi)=\mathcal{M}(\psi)+\mathcal{M}(\phi)$

Stabilizer Rényi entropy in critical quantum spin chains

■ **Stabilizer Rényi entropy (SRE)**

$$M_\alpha(\psi)=\frac{1}{1-\alpha}\ln\sum_{\vec m}\frac{\mathrm{Tr}^{2\alpha}[\sigma^{\vec m}\psi]}{2^L}\qquad \psi=|\psi\rangle\langle\psi|\ \text{ L-qubit pure state}\qquad \sigma^{\vec m}=\sigma^{m_1}\sigma^{m_2}\cdots\sigma^{m_L}\ \text{ Pauli string}$$

■ **Participation entropy in the Bell basis**



$$\sigma^{00}=\begin{pmatrix}1&0\\0&1\end{pmatrix},\quad \sigma^{10}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma^{01}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma^{11}=\begin{pmatrix}0&-i\\i&0\end{pmatrix}.$$

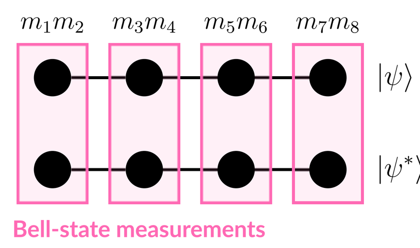
↓ Choi-Jamiolkowski isomorphism

$$|\sigma^{00}\rangle=\sqrt{2}|\mathrm{Bell}^{00}\rangle=|00\rangle+|11\rangle,\quad |\sigma^{10}\rangle=\sqrt{2}|\mathrm{Bell}^{10}\rangle=|01\rangle+|10\rangle,$$
$$|\sigma^{01}\rangle=\sqrt{2}|\mathrm{Bell}^{01}\rangle=|00\rangle-|11\rangle,\quad i|\sigma^{11}\rangle=\sqrt{2}|\mathrm{Bell}^{11}\rangle=|01\rangle-|10\rangle.$$

$$\mathrm{Tr}^2[\sigma^{\vec m}\psi]=2^L\mathrm{Tr}[P^{\vec m}(\psi\otimes\psi^*)]$$

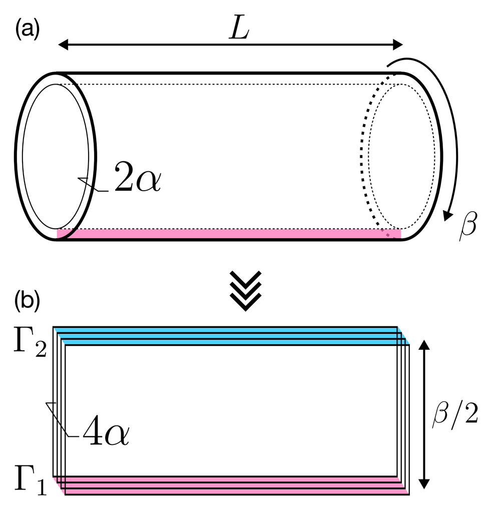
$$M_\alpha(\psi)=\frac{1}{1-\alpha}\ln\sum_{\vec m}\underbrace{\mathrm{Tr}^\alpha[P^{\vec m}(\psi\otimes\psi^*)]}_{p^\alpha_{\vec m}}-(\ln 2)L$$

Classical entropy of the participation in the Bell basis



■ **Replica trick & Folding trick**

$$\begin{aligned}M_\alpha(\psi)&=\frac{1}{1-\alpha}\ln\sum_{\vec m}\mathrm{Tr}^\alpha[P^{\vec m}(\psi\otimes\psi^*)]-(\ln 2)L\\&=\frac{1}{1-\alpha}\ln\mathrm{Tr}\Big[\sum_{\vec m}(P^{\vec m})^{\otimes\alpha}(\psi\otimes\psi^*)^{\otimes\alpha}\Big]-(\ln 2)L\\&=\frac{1}{1-\alpha}\ln\frac{Z_{2\alpha}}{Z^{2\alpha}}-(\ln 2)L\end{aligned}$$



(a) 2α-component CFT on a torus with a line defect

(b) 4α-component CFT on a cylinder with boundaries

General behavior

$$Z_{2\alpha}=\langle\Gamma_2|e^{-\frac{\beta}{2}H}|\Gamma_1\rangle\sim\underbrace{\langle\Gamma_2|\mathrm{GS}\rangle}_{g_2}\underbrace{\langle\mathrm{GS}|\Gamma_1\rangle}_{g_1}e^{-\frac{\beta}{2}E_{\mathrm{GS}}}\quad(\beta\gg L)$$

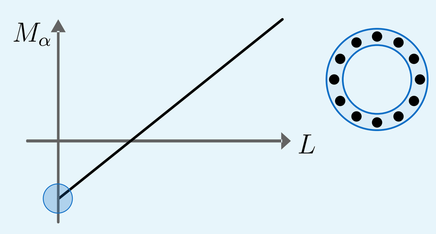
$$-\ln\frac{Z_{2\alpha}}{Z^{2\alpha}}=b_\alpha L-\ln g_1g_2+o(1)$$

⇓

Universal scaling of the SRE

$$M_\alpha(\psi)=m_\alpha L-c_\alpha+o(1)$$

$$c_\alpha=\frac{\ln g_1}{\alpha-1}$$



Example: Ising CFT

■ **Lattice ↔ CFT** $\quad Z\sim\sigma,\ X-\langle X\rangle\sim\varepsilon$

■ **Bosonization**

$$\sigma_1\sigma_2=\cos\phi,\qquad \varepsilon_1\varepsilon_2=(\partial_\mu\phi)^2,\qquad \varepsilon_1+\varepsilon_2=\cos 2\phi.$$

$$\mathcal{L}=\frac{1}{2\pi}(\partial_\mu\phi)^2,\quad \phi\sim\phi+2\pi,\ \phi\sim-\phi$$

$S^1/\mathbb{Z}_2$ free-boson CFT

■ **Boundary perturbation**

$$\sum_{b_1,b_2}(P^{b_1b_2})^{\otimes\alpha}=\prod_{s\in\tilde{S}^{(\alpha)}}\frac{I+s}{2}\propto\lim_{\mu\to\infty}e^{\mu\sum s}$$

$$\tilde{S}^{(\alpha)}=\{\underbrace{XX^{(1)}XX^{(2)}},\underbrace{ZZ^{(1)}ZZ^{(2)}},\dots,\underbrace{ZZ^{(\alpha-1)}ZZ^{(\alpha)}}\}$$

Interchain coupling

$$\sum_{\vec m}(P^{\vec m})^{\otimes\alpha}=e^{-\delta S_{2\alpha}}\qquad \delta S_{2\alpha}=-\mu\int d\tau\,\delta(\tau)\int dx\sum_{s\in\tilde{S}^{(\alpha)}}s$$

$$Z_{2\alpha}=\int\mathcal{D}\vec\phi\exp\left(-\mathcal{S}[\vec\phi]-\delta S_{2\alpha}\right)$$

Line defect: configuration minimizing the perturbation

■ **Boundary condition (after folding)**

$$\begin{aligned}ZZ^{(i)}ZZ^{(i+1)}&\sim\cos\phi_i\cos\phi_{i+1}\\&=\cos(\phi_i+\phi_{i+1})+\cos(\phi_i-\phi_{i+1})\\&=2\cos(\phi_i-\phi_{i+1}),\\XX^{(i)}XX^{(i+1)}&\sim\langle X\rangle^2[(\partial_\mu\phi_i)^2+(\partial_\mu\phi_{i+1})^2]\end{aligned}$$

$$\delta S_{2\alpha}=-\mu\int dx\sum_{i=1}^{\alpha}\cos(\phi_i-\phi_{i+1})$$

Boundary perturbation (most relevant terms)

$$\Longrightarrow\ \Gamma_1:\begin{cases}\sum_{i=1}^{2\alpha}\phi_i & \mathrm{NBC}\\ \phi_1-\phi_2=0 & \mathrm{DBC}\\ \phi_2-\phi_3=0 & \mathrm{DBC}\\ \vdots & \\ \phi_{2\alpha-1}-\phi_{2\alpha}=0 & \mathrm{DBC}\end{cases}$$

2α-component CFT

Neumann for "center of mass motion"

Dirichlet for "relative motion"

■ **Symmetrization for the orbifold boundary states**

Ishibashi condition $\quad(L_n-\bar L_{-n})|\Gamma\rangle=0$

in terms of U(1) current $\quad(\vec\alpha^L_m-\mathcal{R}\vec\alpha^R_{-m})|\Gamma\rangle=0$

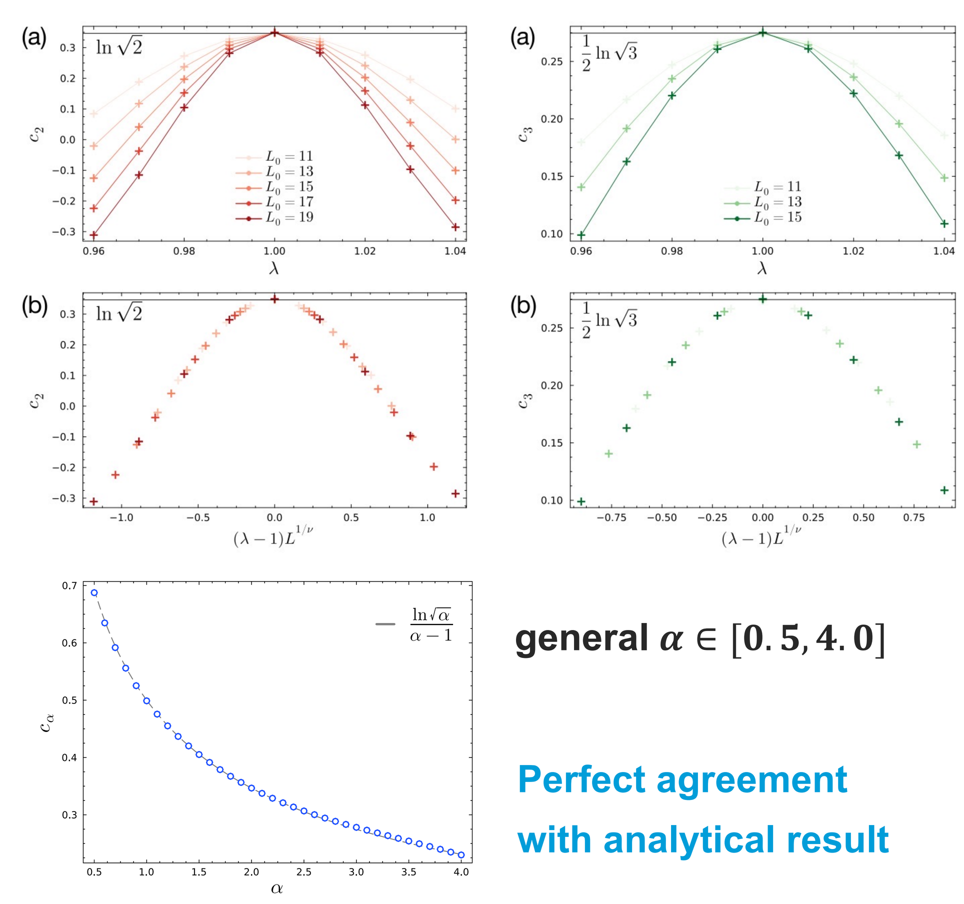
Generic $\quad|\mathcal{R}(\vec\phi_D,\vec\theta_D)\rangle\rangle_{\mathrm{orb}}=\frac{1}{\sqrt{|G|}}\sum_{a\in G}D(a)|\mathcal{R}(\vec\phi_D,\vec\theta_D)\rangle\rangle$

$a=\mathrm{diag}(\pm1,\pm1,\dots,\pm1)\in G$, $|G|=2^N$, $G_0=G/\{\pm I\}$

Endpoint $\quad|\mathcal{R}(\vec\phi_E,\vec\theta_E)\rangle\rangle_{\mathrm{orb}}=\sum_{b\in G_0}D(b)\left[\frac{1}{\sqrt{|G|}}|\mathcal{R}(\vec\phi_E,\vec\theta_E)\rangle\rangle\pm2^{-N/4}|\mathcal{R}(\vec\phi_E,\vec\theta_E)\rangle\rangle_t\right]$

■ **Results**

$$g_1=\langle\mathrm{GS}|\Gamma_1\rangle=\sqrt{\alpha}\ \Longrightarrow\ c_\alpha=\frac{\ln\sqrt{\alpha}}{\alpha-1}$$

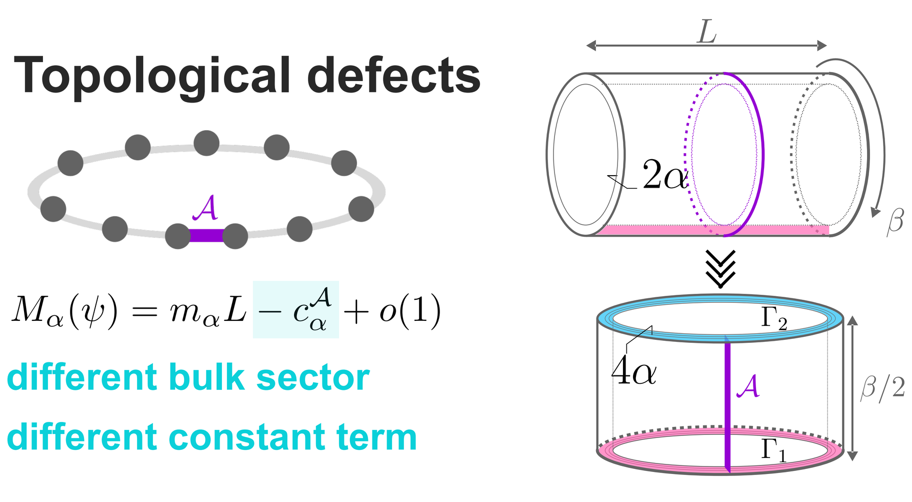


general $\alpha\in[0.5,4.0]$

Perfect agreement with analytical result

SRE encodes fusion rules of topological defects and boundaries

■ **Topological defects**

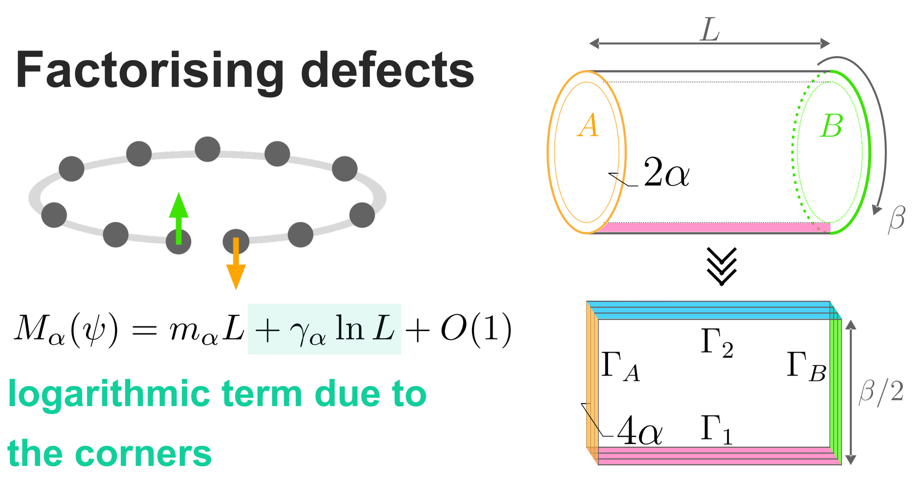


$$M_\alpha(\psi)=m_\alpha L-c^{\mathcal{A}}_\alpha+o(1)$$

different bulk sector

different constant term

■ **Factorising defects**



$$M_\alpha(\psi)=m_\alpha L+\gamma_\alpha\ln L+O(1)$$

logarithmic term due to the corners

■ **Ising CFT**

Cardy states $\quad|f\rangle,\ |\uparrow\rangle,\ |\downarrow\rangle$

Verlinde lines $\quad 1,\ \eta,\ \mathcal{D}$

■ **Fusion rules**

$$\eta*|f\rangle=|f\rangle,\qquad \eta*|\uparrow\rangle=|\downarrow\rangle,\qquad \eta*|\downarrow\rangle=|\uparrow\rangle,$$
$$\mathcal{D}*|f\rangle=|\uparrow\rangle+|\downarrow\rangle,\qquad \mathcal{D}*|\uparrow\rangle=|f\rangle,\qquad \mathcal{D}*|\downarrow\rangle=|f\rangle.$$

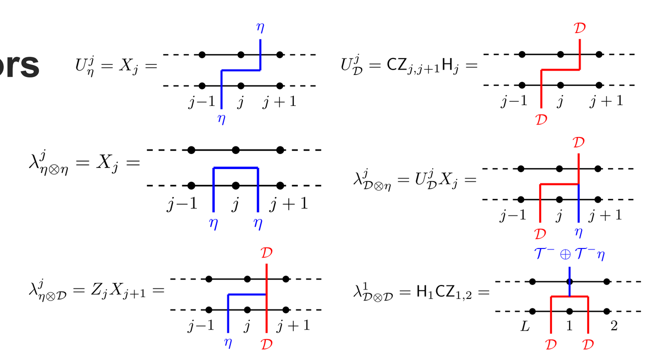
$$\eta\otimes\eta=1,\quad \eta\otimes\mathcal{D}=\mathcal{D}\otimes\eta=\mathcal{D},\quad \mathcal{D}\otimes\mathcal{D}=\mathcal{T}^-\oplus\mathcal{T}^-\eta$$

Lattice translation defect $\mathcal{T}^-$

(in the continuum $\mathcal{N}\otimes\mathcal{N}=1\oplus\eta$)

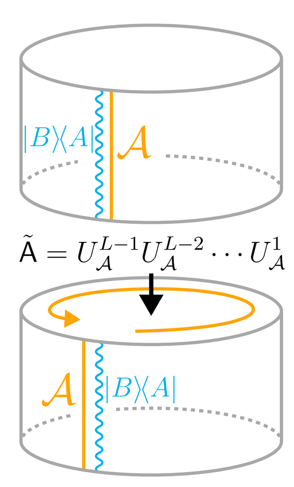
Movement operators

Fusion operators



■ **SRE encodes fusion rules**

◆ equivalence between two different pairs of boundaries $\quad(\mathcal{A}*|A\rangle,|B\rangle)\quad(|A\rangle,\mathcal{A}*|B\rangle)$



◆ fusion rule determines the constant term

$$c^{\mathcal{A}\otimes\mathcal{B}}_\alpha=c^{\mathcal{C}_0}_\alpha$$

$\mathcal{C}_0$: lowest energy sector in the fusion $\mathcal{A}\otimes\mathcal{B}=\bigoplus_i\mathcal{C}_i$

Full paper available!

◆ SRE and CFT (2503.13599)

◆ SRE with defects (2507.10656)