The 4th young researcher's workshop of the Extreme Universe Collaboration

Quantum magic and Conformal Field Theory

Masahiro Hoshino
Department of Physics, UTokyo

Masaki Oshikawa
ISSP, UTokyo

Yuto Ashida
Dept. Phys. & IPI, UTokyo

Quantum resource theory

Resource ≈ things that cannot be easily obtained

example: entanglement, nonstabilizerness (magic), non-Gaussianity, quantum coherence, …

Free states
restricted class of states (easily prepared)

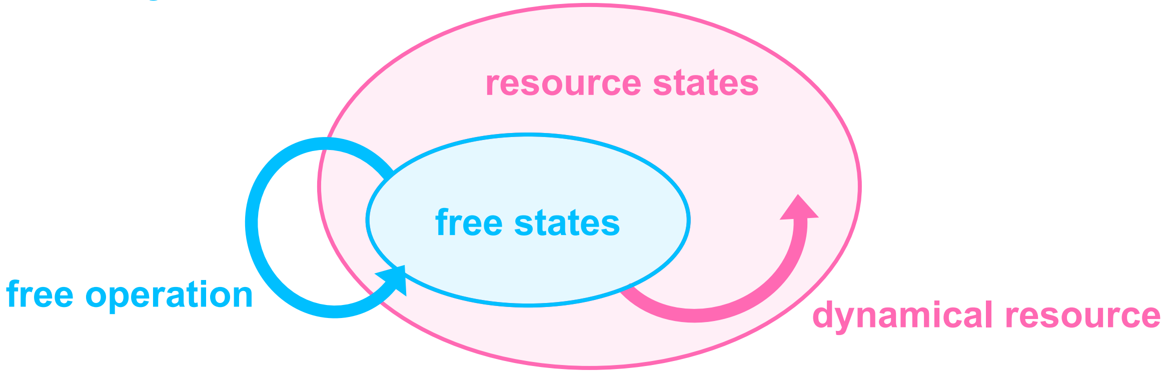
Free operations
operations acting invariantly on free states

easy

Resource states
states that are not free

Dynamical resources
operations that are not free

hard



E. Chitambar & G. Gour, Rev. Mod. Phys. (2019)

Entanglement: resource for quantum communication

Local operations & classical communication (LOCC)

$$|0\rangle|0\rangle \;\not\longrightarrow\; \frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}}$$

cannot create entanglement

Separable state

$$\rho = \sum_i p_i\, \rho_i^A \otimes \rho_i^B$$

LOCC

quantum instrument acting locally

$$\mathcal{E} = \mathcal{E}^A \otimes \mathcal{E}^B$$

Entangled state

$$\rho \neq \sum_i p_i\, \rho_i^A \otimes \rho_i^B$$

Entangling gates & measurements

ex)

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad |\mathrm{Bell}\rangle\langle\mathrm{Bell}|$$

Local measurements tend to destroy entanglement

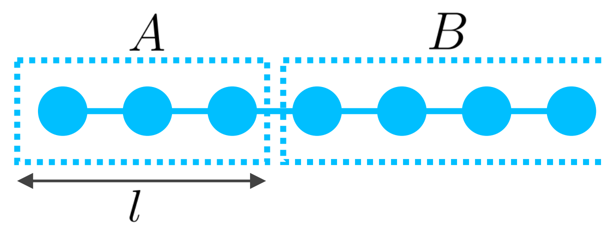
$$\frac{|0\rangle|0\rangle + |1\rangle|1\rangle}{\sqrt{2}} \longrightarrow |0\rangle|0\rangle,\ |1\rangle|1\rangle$$

measure the first qubit in $\{|0\rangle, |1\rangle\}$

R. Horodecki et al., Rev. Mod. Phys. (2009)

Entanglement in many-body systems

Scaling of EE in 1D pure states



- gapped ground states

$$S_A \sim \text{const.}$$

- critical ground states (CFT)

$$S_A \sim \frac{c}{3} \ln l$$

characterized by central charge c

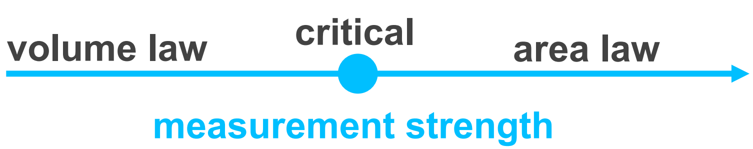
- thermal pure states

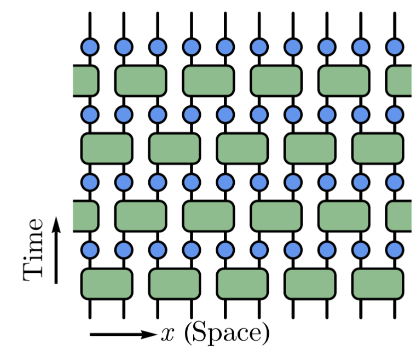
$$S_A \sim l$$

M B Hastings, J. Stat. Mech (2007)
P. Calabrese & J. Cardy, J. Stat. Mech. (2004,2009)

Many-body entanglement under measurements

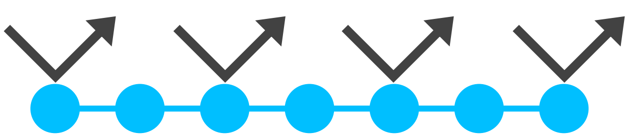
- measurement-induced transition





Jian et al., PRB (2020)

- critical states under local measurements



- described by boundary CFT
- tend to destroy entanglement

Nonstabilizerness: resource for universal quantum computation

Universal gate set = Clifford + T

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Clifford group

- easy to implement fault-tolerantly (transversal gates)
- classically simulable due to Gottesman-Knill theorem

T gate

Stabilizer states

(convex hull of) Clifford $|0\rangle^{\otimes L}$

Stabilizer protocol

Clifford + other easy operations

Non-stabilizer state (magic state)

$\neq$ Clifford $|0\rangle^{\otimes L}$

Non-Clifford gates

ex) T gate, CCZ gate

Gottesman-Knill theorem: stabilizer protocols can be simulated classically

Stabilizer formalism

$$\sigma^{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma^{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^{11} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

stabilizer state $|\psi\rangle$

stabilizer group $S = \{P \in \mathcal{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Pauli group: n-fold tensor products of Pauli matrices

generators $S = \langle g_1, g_2, \ldots, g_n \rangle$

stabilizer states are completely determined by their generators

1. **Clifford unitary: maps Pauli string to Pauli string**

$$H\sigma^{b_1 b_2} H^\dagger = \sigma^{b_2 b_1} \qquad S\sigma^{b_1 b_2} S^\dagger = \sigma^{b_1 (b_1 \oplus b_2)}$$

$$\mathrm{CNOT}\underbrace{\sigma^{b_1 b_2}}_{\text{control}}\underbrace{\sigma^{b'_1 b'_2}}_{\text{target}}\mathrm{CNOT}^\dagger = \sigma^{b_1 (b_2 \oplus b'_2)} \sigma^{(b_1 \oplus b'_1) b'_2} \qquad b_1 b_2 \in \{0,1\}^2$$

(with possible phase factor)

2. **measurement in the computational basis**

$$|0\rangle\langle 0| = \frac{I+Z}{2} \qquad |1\rangle\langle 1| = \frac{I-Z}{2} \qquad (Z = \sigma^{01})$$

measure the j-th qubit

- Z_j **commutes with** $g_1, g_2, \ldots, g_n$ $\rightarrow$ $\langle g_1, g_2, \ldots, g_n \rangle$
- Z_j **anti-commutes with** g_k $\rightarrow$ $\langle g_1, \ldots, \underset{Z_j}{\cancel{g_k}}, \ldots, g_n \rangle$

Gottesman-Knill theorem: Clifford can be simulated classically

Stabilizer formalism

$$\sigma^{00} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\ \sigma^{01} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\sigma^{10} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma^{11} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

stabilizer state $|\psi\rangle$

stabilizer group $S = \{P \in \mathcal{P}_n \mid P|\psi\rangle = |\psi\rangle\}$

Pauli group: n-fold tensor products of Pauli matrices

generators $S = \langle g_1, g_2, \ldots, g_n \rangle$

Example $|\mathrm{Bell}^{00}\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

stabilizer group $S = \langle X_1 X_2, Z_1 Z_2 \rangle$ $\because$

$$Z_1 Z_2 |\mathrm{Bell}^{00}\rangle = |\mathrm{Bell}^{00}\rangle$$

$$X_1 X_2 |\mathrm{Bell}^{00}\rangle = |\mathrm{Bell}^{00}\rangle$$

projection operator $|\mathrm{Bell}^{00}\rangle\langle\mathrm{Bell}^{00}| = \dfrac{I + X_1 X_2}{2} \dfrac{I + Z_1 Z_2}{2}$

measurement in the first qubit ($X_1 X_2$ anti-commutes with Z_1)

- **outcome +1** $\rightarrow \langle Z_1, Z_1 Z_2 \rangle$ **post-measurement state** $|00\rangle$
- **outcome -1** $\rightarrow \langle -Z_1, Z_1 Z_2 \rangle$ **post-measurement state** $|11\rangle$

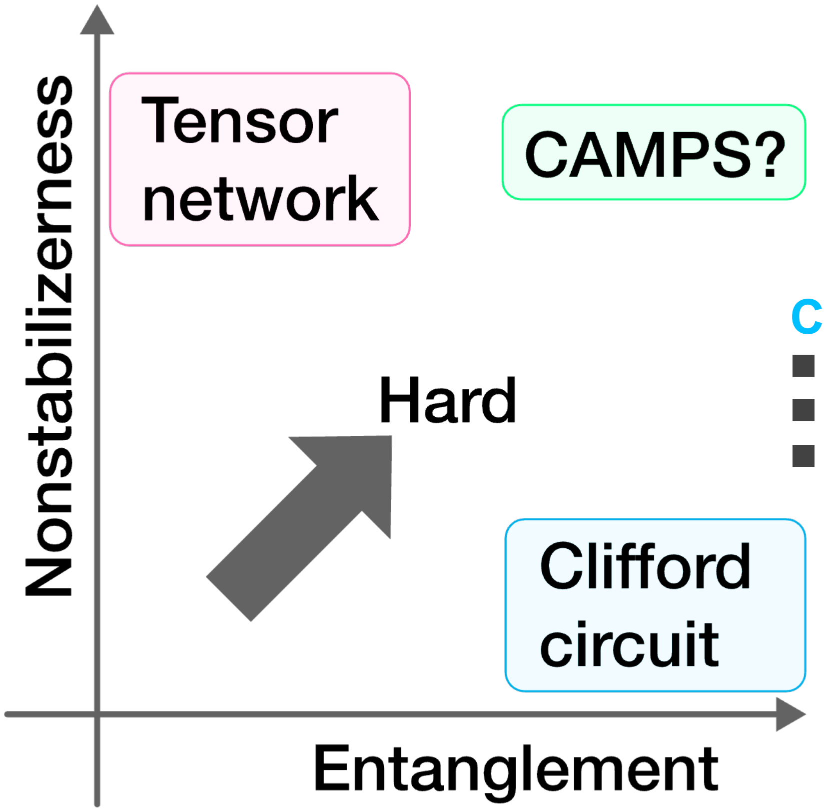
highly entangled states can still be simulated efficiently

Hardness of classical simulation

Entanglement alone does not indicate “quantumness”

Tensor network

- able to simulate low-entangled states
- cost $\approx$ bond dimension χ
- scales as $\chi \propto \exp(S_A)$



Clifford circuit

- able to simulate low-magic states
- cost $\approx$ number of T-gates
- scales as $\mathrm{time} \sim \mathrm{poly}(n, m) + \exp(T)$
  (n: # of qubits, m: # of Clifford gates, T: # of T-gates)

Bravyi & Gosset, PRL (2016) / arXiv:1601.07601

Quantifying magic

Amount of magic ≈ distance from stabilizer states

Stabilizer Rényi entropy (SRE): measure for pure states

$$M_\alpha(\psi) = \frac{1}{1-\alpha} \ln \sum_{\vec{m}} \frac{\mathrm{Tr}^{2\alpha}[\sigma^{\vec{m}}\psi]}{2^L}$$

$\psi = |\psi\rangle\langle\psi|$ **: L-qubit pure state**

$\sigma^{\vec{m}} = \sigma^{m_1}\sigma^{m_2}\cdots\sigma^{m_L}$ **: Pauli string**

L. Leone et al., PRL (2022)

Properties

1. **Faithfulness** : $M_\alpha(\psi) = 0 \iff$ stabilizer state $\quad \because \mathrm{Tr}[\sigma^{\vec{m}}\psi] = 1 \quad \text{if } \sigma^{\vec{m}} \in S \,, |S| = 2^L$
2. **Additivity** : $M_\alpha(\psi \otimes \phi) = M_\alpha(\psi) + M_\alpha(\phi)$
3. **Monotonicity : non-increasing under stabilizer protocol**
   only for $\alpha \geq 2$

Operational meaning

Stabilizer protocol : $|R_1\rangle^{\otimes M} \to |R_2\rangle^{\otimes N}$

conversion process

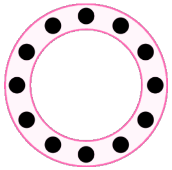
$$\frac{N}{M} \leq \frac{M_{\alpha\geq 2}(R_1)}{M_{\alpha\geq 2}(R_2)}$$

rate of conversion ≤ ratio of SRE

Quantum magic in many-body systems

Full state SRE

$$M_\alpha(\psi) = m_\alpha L - c_\alpha$$



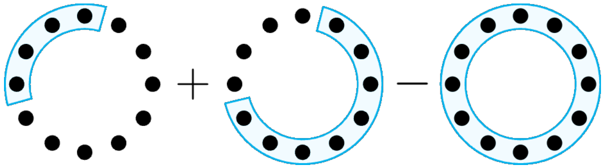
Turkeshi, et al., PRB 111, 054301, (2025)

- **Harr random:** $c_2 = 2,\ c_{>2} = 0$
- **typical state with time-reversal symmetry:** $c_2 = \ln 7,\ c_{>2} = 0$

for critical states described by CFT, c_α is a universal quantity determined by the g-factor

Mutual SRE

$$W_\alpha(A:B) = M_\alpha(\rho_A) + M_\alpha(\rho_B) - M_\alpha(\rho_{AB})$$



- **logarithmic scaling for CFT, coefficient determined by the scaling dimension of a BCCO**

$$W_\alpha(l) \sim \ln l \quad \left(l \to \frac{L}{\pi}\sin\frac{\pi l}{L}\right)$$ for periodic chains

Piemontese et al., PRB (2023)
Tarabunga et al., PRXQ (2023), Quantum (2024)
Falcao et al., PRB (2025)
Ding et al., arXiv:2501.12146
…

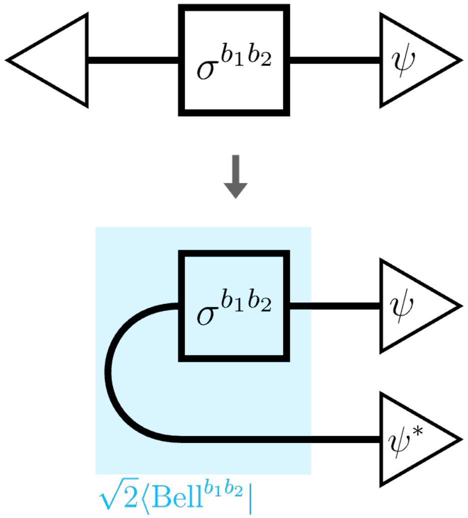
- **related to entanglement that can be reduced by Clifford circuits**

Frau, et al., arXiv:2411.11720

SRE as participation entropy in the Bell basis

Choi-Jamiolkowski isomorphism

$$\mathrm{Tr}^2[\sigma^{\vec{m}}\psi] = 2^L \mathrm{Tr}[P^{\vec{m}}(\psi \otimes \psi^*)]$$



Participation entropy

$$\mathrm{PE} = \frac{1}{1-\alpha} \ln \sum_i p_i^\alpha \quad (p_i = |\langle i|\Psi\rangle|^2)$$

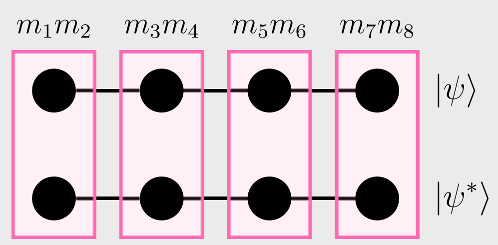
- **classical entropy in an orthonormal basis**
- **Well-explored for 1D CFT in computational basis**

Stephan et al., PRB (2009)
Ashida, Furukawa, Oshikawa, PRB (2024)
…

SRE = participation entropy in the Bell basis

$$M_\alpha(\psi) = \frac{1}{1-\alpha} \ln \sum_{\vec{m}} \underbrace{\mathrm{Tr}^\alpha[P^{\vec{m}}(\psi \otimes \psi^*)]}_{p_{\vec{m}}^\alpha} - (\ln 2)L$$

- **constant offset to ensure faithfulness**
- **PE with** $|i\rangle = \bigotimes_{j=1}^{L} |\mathrm{Bell}^{m_{2j-1}m_{2j}}\rangle$ **and** $|\Psi\rangle = |\psi\rangle|\psi^*\rangle$



see also: Haug and Kim, PRXQ (2023)

BCFT: boundary state formalism

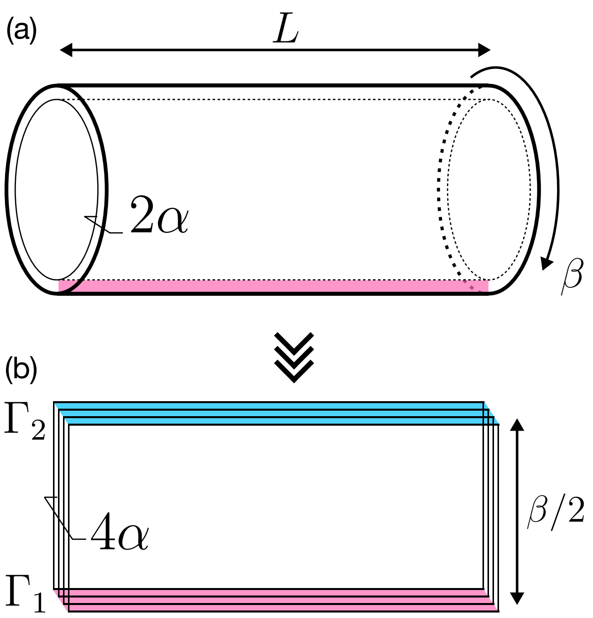
Replica trick

$$M_\alpha(\psi) = \frac{1}{1-\alpha} \ln \sum_{\vec{m}} \mathrm{Tr}^\alpha [P^{\vec{m}} (\psi \otimes \psi^*)] - (\ln 2) L$$

$$= \frac{1}{1-\alpha} \ln \mathrm{Tr} \left[\sum_{\vec{m}} (P^{\vec{m}})^{\otimes \alpha} (\psi \otimes \psi^*)^{\otimes \alpha} \right] - (\ln 2) L$$

projection operator

partition function of 2α-component CFT with a line defect



Folding trick

$$\frac{Z_{2\alpha}}{Z^{2\alpha}} = \mathrm{Tr} \left[\sum_{\vec{m}} (P^{\vec{m}})^{\otimes \alpha} (\psi \otimes \psi^*)^{\otimes \alpha} \right]$$

torus of size $L \times \beta$

$$Z_{2\alpha} = \langle \Gamma_2 | e^{-\frac{\beta}{2} H} | \Gamma_1 \rangle$$

cylinder of circumference L, length $\beta/2$
H: Hamiltonian of 4α-component CFT on a cylinder

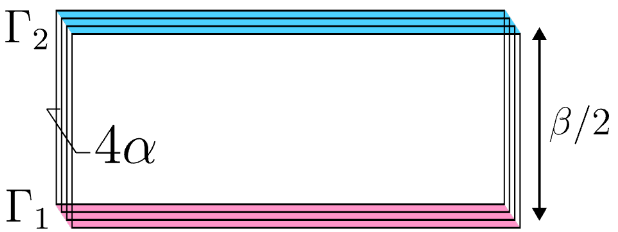
boundary state originating from the line defect

boundary state of the artificially created boundary

BCFT: g-factor

Noninteger ground-state degeneracy (g-factor)

$$\ln \frac{Z_{2\alpha}}{Z^{2\alpha}} = bL + \ln g + o(1)$$



1. non-universal line energy
2. universal constant term = g-factor $g = g_1 g_2$

$g_1 = \langle \mathrm{GS} | \Gamma_1 \rangle$ determined by the boundary condition imposed by Bell-state measurements

$g_2 = \langle \mathrm{GS} | \Gamma_2 \rangle$ trivial

$$Z_{2\alpha} \simeq \langle \Gamma_1 | \mathrm{GS} \rangle \langle \mathrm{GS} | \Gamma_2 \rangle e^{-\frac{\beta}{2} E_{\mathrm{GS}}(L)} \qquad E_{\mathrm{GS}}(L) = -\frac{\pi c}{6L}$$

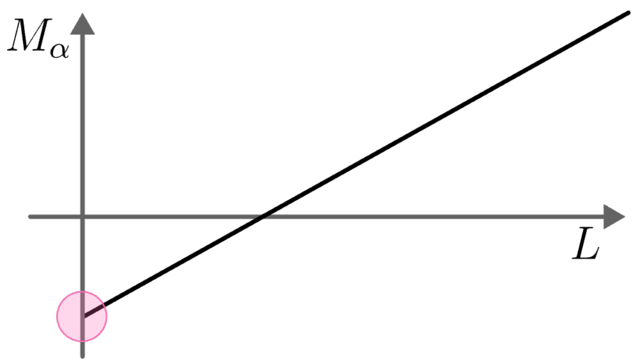
$(\beta \gg L)$

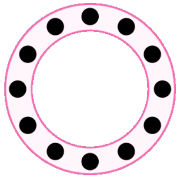
Universality class of the SRE is characterized by the g-factor

$$M_\alpha(\psi) = \frac{1}{1-\alpha} \ln \frac{Z_{2\alpha}}{Z^{2\alpha}} - (\ln 2) L = m_\alpha L - c_\alpha + o(1)$$

$$c_\alpha = \frac{\ln g_1}{\alpha - 1}$$

size-independent constant term derived from the g-factor of Γ_1

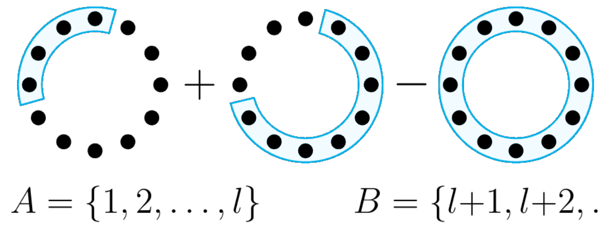




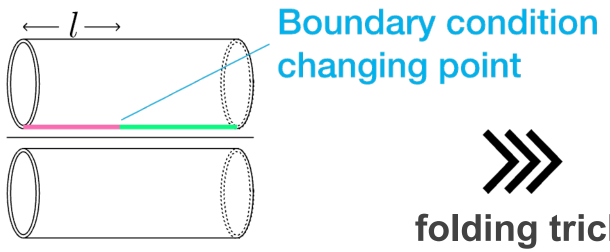
BCFT: boundary condition changing operator

Mutual SRE

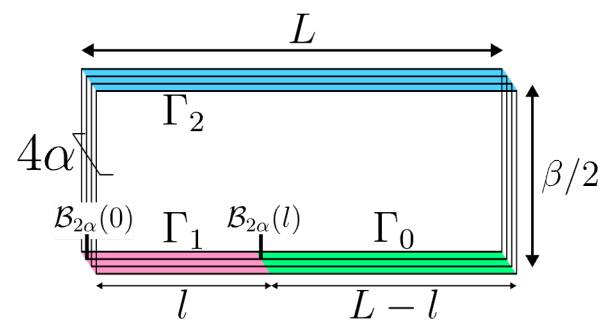
$$W_\alpha(A:B) = M_\alpha(\rho_A) + M_\alpha(\rho_B) - M_\alpha(\rho_{AB})$$



$$W_\alpha(l) = \frac{2}{1-\alpha} \ln$$



folding trick



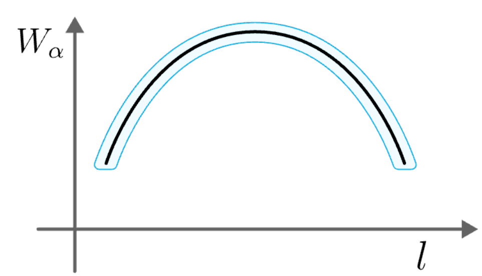
$$Z_{2\alpha}(A) = \langle \mathcal{B}_{2\alpha}(0)\mathcal{B}_{2\alpha}(l)\rangle \sim l^{-2\Delta_{2\alpha}}$$

two-point function of BCCO

Universal logarithmic scaling

$$W_\alpha(l) \sim \frac{4\Delta_{2\alpha}}{\alpha - 1} \ln l$$

scaling dimension of the BCCO
not solely determined by the central charge c



How to determine the boundary condition

participation entropy

$$\sum_{\vec{m}} p_{\vec{m}}^n = \mathrm{Tr}\left[\sum_{\vec{m}} (P^{\vec{m}})^{\otimes n} \rho^{\otimes n}\right] = \frac{1}{Z^n} \int \mathcal{D}\vec{\varphi}\, e^{-\sum_{j=1}^n \mathcal{S}[\varphi_j] - \delta\mathcal{S}_n[\vec{\varphi}]}$$

replicated theory

$$\vec{\varphi} = (\varphi_1, \varphi_2, \ldots, \varphi_n)$$

boundary perturbation

$$e^{-\delta\mathcal{S}_n[\vec{\varphi}]} = \sum_{\vec{m}} (P^{\vec{m}})^{\otimes n}$$

Example: Bell-state measurement, $n = 2$

$$P^{b_1 b_2} = |\mathrm{Bell}^{b_1 b_2}\rangle\langle\mathrm{Bell}^{b_1 b_2}| = \frac{I + (-1)^{b_1} ZZ}{2} \frac{I + (-1)^{b_2} XX}{2}$$

$$\Rightarrow \quad \sum_{b_1, b_2} (P^{b_1 b_2})^{\otimes 2} = \frac{I + ZZ^{(1)} ZZ^{(2)}}{2} \frac{I + XX^{(1)} XX^{(2)}}{2}$$

In general,

$$\sum_{b_1, b_2} (P^{b_1 b_2})^{\otimes \alpha} = \prod_{s \in \tilde{S}^{(\alpha)}} \frac{I + s}{2} \propto \lim_{\mu \to \infty} e^{\mu \sum s}$$

$$\tilde{S}^{(\alpha)} = \{\underbrace{XX^{(1)} XX^{(2)}}, \underbrace{ZZ^{(1)} ZZ^{(2)}}, \ldots, \underbrace{ZZ^{(\alpha-1)} ZZ^{(\alpha)}}\}$$

interlayer coupling

$$\Longrightarrow \quad \delta\mathcal{S}_{2\alpha} = -\mu \int d\tau\, \delta(\tau) \int dx \sum_{s \in \tilde{S}^{(\alpha)}} s$$

should be expressed by operators in the CFT (requires knowledge of lattice ↔ CFT correspondence)

Ising CFT: bosonization

Lattice ↔ CFT $Z \sim \sigma,\ X - \langle X \rangle \sim \varepsilon$

Bosonization

$$\sigma_1\sigma_2 = \cos\phi,$$
$$\varepsilon_1\varepsilon_2 = (\partial_\mu\phi)^2,$$
$$\varepsilon_1 + \varepsilon_2 = \cos 2\phi.$$

$$\mathcal{L} = \frac{1}{2\pi}(\partial_\mu\phi)^2, \quad \phi \sim \phi + 2\pi, \quad \phi \sim -\phi$$

free-boson CFT **compactification** **$\mathbb{Z}_2$ orbifold**

$$\begin{aligned} ZZ^{(i)}ZZ^{(i+1)} &\sim \cos\phi_i \cos\phi_{i+1} \\ &= \cos(\phi_i + \phi_{i+1}) + \cos(\phi_i - \phi_{i+1}) \\ &= 2\cos(\phi_i - \phi_{i+1}), \\ XX^{(i)}XX^{(i+1)} &\sim \langle X \rangle^2[(\partial_\mu\phi_i)^2 + (\partial_\mu\phi_{i+1})^2] \end{aligned}$$

$$\delta\mathcal{S}_{2\alpha} = -\mu \int dx \sum_{i=1}^{\alpha} \cos(\phi_i - \phi_{i+1})$$

boundary perturbation of α-component CFT (the most relevant terms)

Boundary condition (after the folding trick)

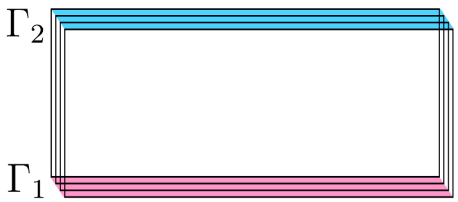
$$\Gamma_1 : \begin{cases} \sum_{i=1}^{2\alpha}\phi_i & \text{NBC} \\ \phi_1 - \phi_2 = 0 & \text{DBC} \\ \phi_2 - \phi_3 = 0 & \text{DBC} \\ \vdots & \\ \phi_{2\alpha-1} - \phi_{2\alpha} = 0 & \text{DBC} \end{cases}$$

Neumann of the “center of mass motion”
Dirichlet of the “relative motion”
+ symmetrization for the orbifold

Ising CFT: result

$$\Gamma_2 : \begin{cases} \phi_i - \phi_{i+\alpha} = 0 & \text{DBC} \\ \phi_i + \phi_{i+\alpha} & \text{NBC} \end{cases}$$

artificial boundary



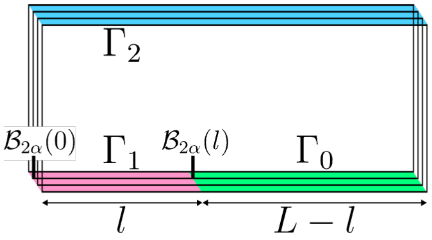
full state SRE

$$\Gamma_1 : \begin{cases} \sum_{i=1}^{2\alpha} \phi_i & \text{NBC} \\ \phi_1 - \phi_2 = 0 & \text{DBC} \\ \phi_2 - \phi_3 = 0 & \text{DBC} \\ \vdots & \\ \phi_{2\alpha-1} - \phi_{2\alpha} = 0 & \text{DBC} \end{cases}$$

boundary imposed by
Bell-state measurements

$$\Gamma_0 : \begin{cases} \phi_1 = \pi/2 & \text{DBC} \\ \phi_2 = \pi/2 & \text{DBC} \\ \vdots & \\ \phi_{2\alpha} = \pi/2 & \text{DBC} \end{cases}$$

boundary by $|\mathrm{Bell}^{00}\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}}$
(corresponds to partial trace)



mutual SRE

Constant term of the full state SRE

$$g_1 = \langle \mathrm{GS} | \Gamma_1 \rangle = \sqrt{\alpha} \quad \Longrightarrow \quad c_\alpha = \frac{\ln \sqrt{\alpha}}{\alpha - 1}$$

Coefficient of the log-scaling mutual SRE

$$\Delta_{2\alpha} = \frac{1}{16} \quad \Longrightarrow \quad W_\alpha(l) = \frac{4\Delta_{2\alpha}}{\alpha - 1} \ln l = \frac{1}{4(\alpha-1)} \ln l$$

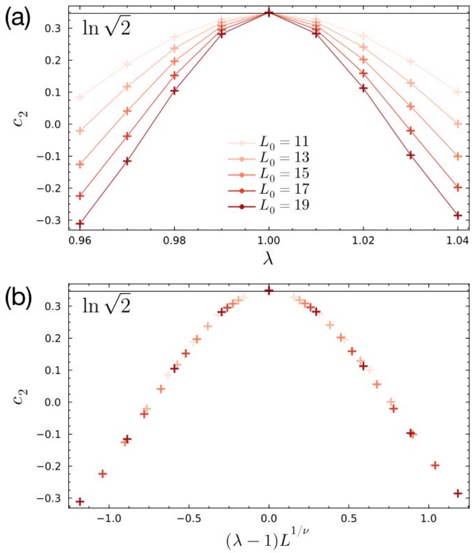
$\because$ lowest conformal weight included in the amplitude

$$\langle \Gamma_0 | q^{L_0 + \bar{L}_0 - c/12} | \Gamma_1 \rangle = \sum_h n^h_{\Gamma_0\Gamma_1} \chi_h(\tilde{q}) \quad (n^h_{\Gamma_0\Gamma_1} \in \mathbb{Z}_{\geq 0})$$

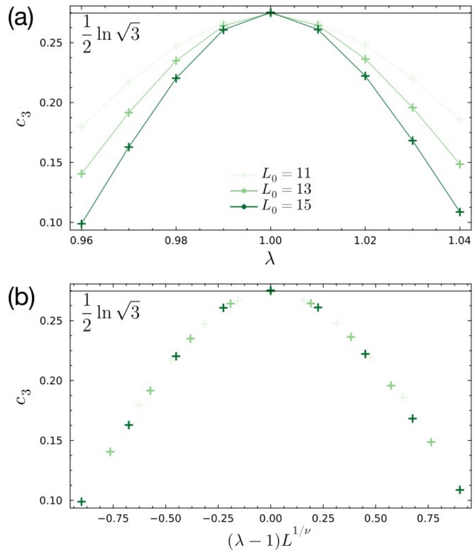
Ising CFT: numerical calculations

universal constant term of the full state SRE

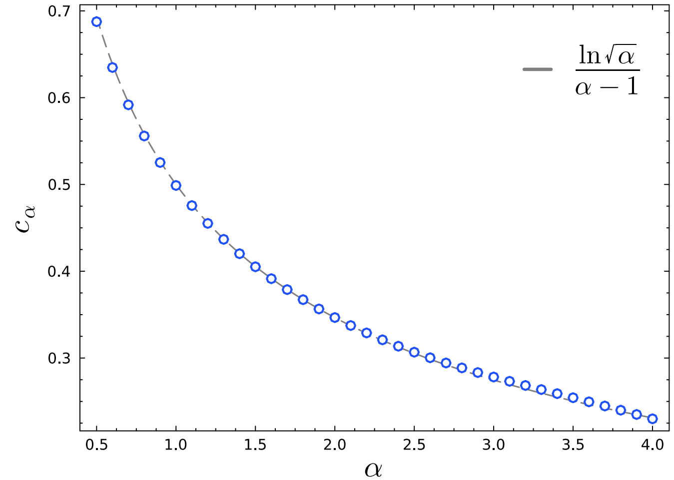
$$H_{\text{Ising}} = -\sum_{j=1}^{L}(Z_j Z_{j+1} + \lambda X_j)$$



$\alpha = 2$

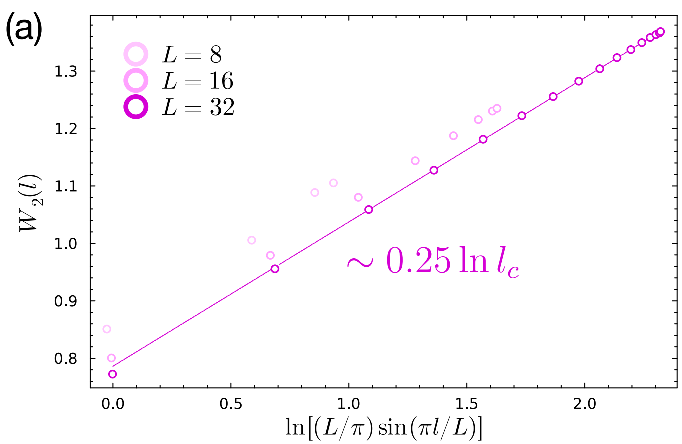


$\alpha = 3$



$\alpha \in [0.5, 4.0]$

universal logarithmic scaling of the mutual SRE



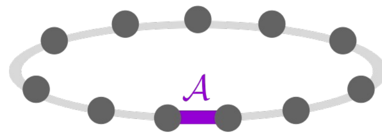
$\alpha = 2$

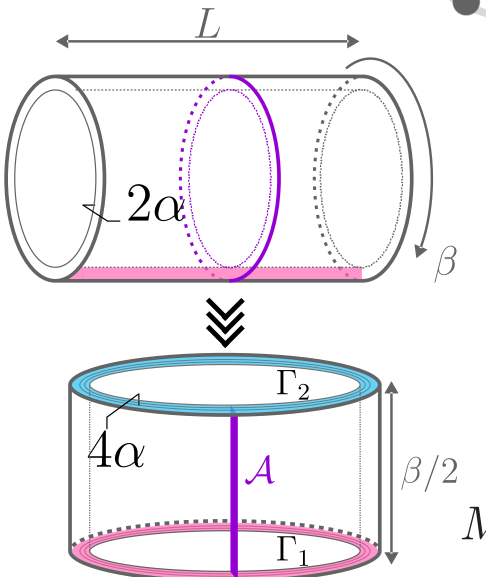
$$W_2(l) = \frac{1}{4}\ln l$$

MH, M. Oshikawa, Y. Ashida, arXiv:2503.13599

Stabilizer Rényi entropy with conformal defects

Topological defect (totally transmissive)

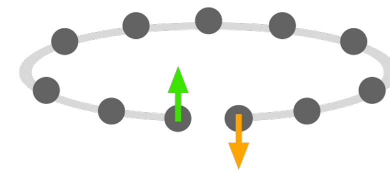


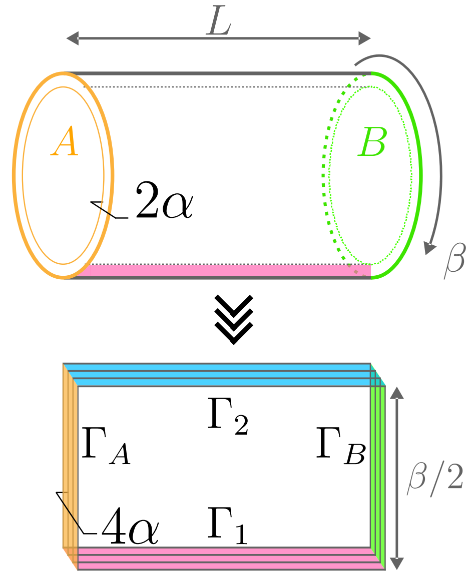


$$M_\alpha = m_\alpha L - c_\alpha^{\mathcal{A}}$$

different bulk sector
different constant term

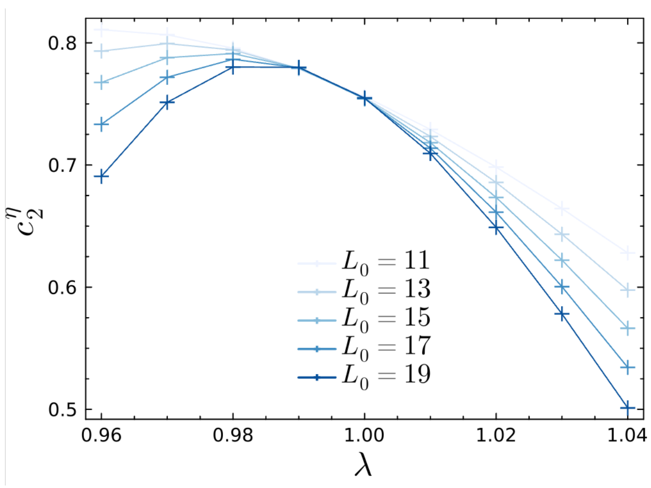
Open boundaries (totally reflective, factorising)

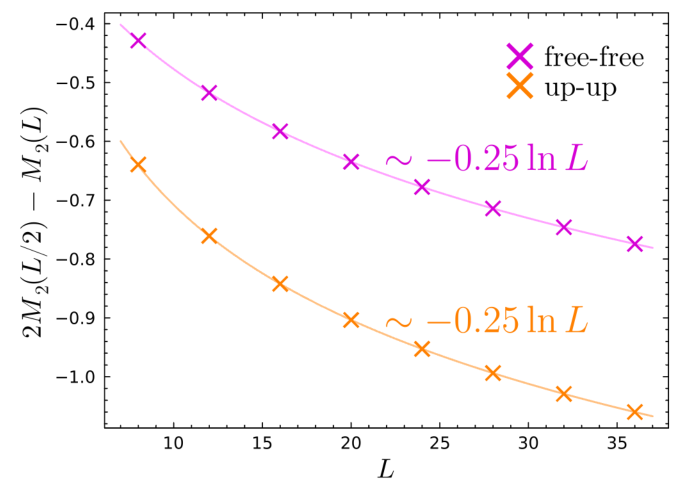




$$M_\alpha = m_\alpha L + \gamma_\alpha \ln L$$

logarithmic term due to the corners





Ising CFT: conformal defects move and fuse via Clifford unitaries

Cardy states $|f\rangle\,, \ |{\uparrow}\rangle\,, \ |{\downarrow}\rangle$

Verlinde lines $1,\ \eta,\ \mathcal{D}$

N. Seiberg, et al., SciPost (2024)

Fusion rules

$$\eta * |f\rangle = |f\rangle\,, \qquad \eta * |{\uparrow}\rangle = |{\downarrow}\rangle\,, \qquad \eta * |{\downarrow}\rangle = |{\uparrow}\rangle\,,$$

$$\mathcal{D} * |f\rangle = |{\uparrow}\rangle + |{\downarrow}\rangle\,, \qquad \mathcal{D} * |{\uparrow}\rangle = |f\rangle\,, \qquad \mathcal{D} * |{\downarrow}\rangle = |f\rangle\,.$$

$$\eta \otimes \eta = 1, \quad \eta \otimes \mathcal{D} = \mathcal{D} \otimes \eta = \mathcal{D},$$

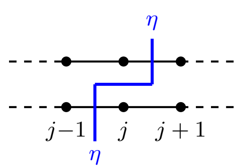
$$\mathcal{D} \otimes \mathcal{D} = \mathcal{T}^- \oplus \mathcal{T}^- \eta.$$

Lattice translation defect $\mathcal{T}^-$

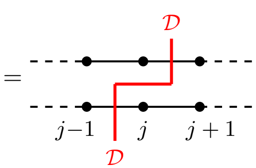
In the continuum: $\mathcal{N} \otimes \mathcal{N} = 1 \oplus \eta$

Movement operators

$U^j_\eta = X_j =$

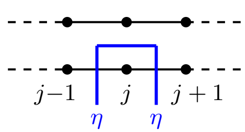


$U^j_{\mathcal{D}} = \mathsf{CZ}_{j,j+1}\mathsf{H}_j =$

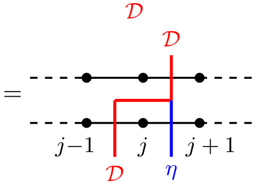


Fusion operators

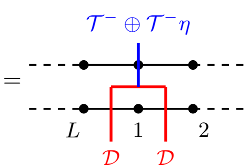
$\lambda^j_{\eta\otimes\eta} = X_j =$



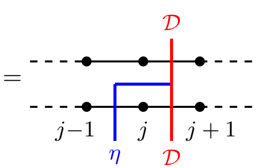
$\lambda^j_{\mathcal{D}\otimes\eta} = U^j_{\mathcal{D}} X_j =$



$\lambda^1_{\mathcal{D}\otimes\mathcal{D}} = \mathsf{H}_1\mathsf{CZ}_{1,2} =$



$\lambda^j_{\eta\otimes\mathcal{D}} = Z_j X_{j+1} =$



also for fusion with boundaries

SRE with multiple defects is determined by the fusion rule!

Summary

Nonlinear function of the density matrix (typical in informational quantity)

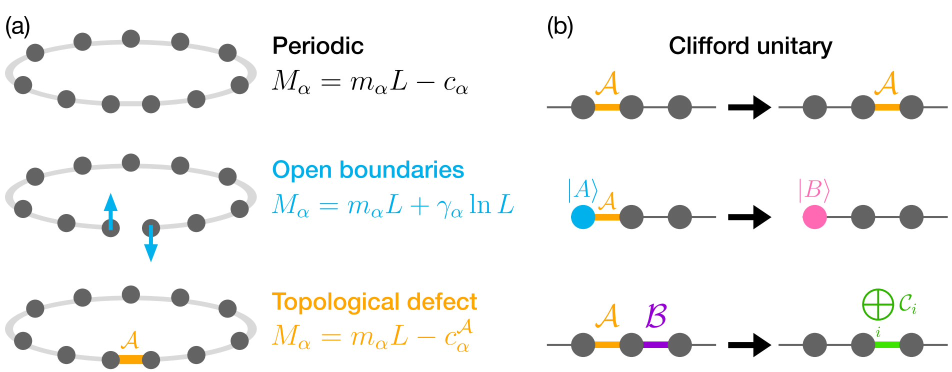
➢ **replica trick & folding trick: express the partition function as amplitudes between boundary states of the multicomponent theory**

Stabilizer Rényi Entropy and Conformal Field Theory

	Entanglement	Entanglement (long-range)	Nonstabilizerness	Nonstabilizerness (long-range)
measure	Rényi entropies	mutual information	SRE	mutual SRE
universal behavior	$S_\alpha = \frac{c}{6}\left(1+\frac{1}{\alpha}\right)\ln l$	$I_\alpha(l) = \frac{c}{3}\left(1+\frac{1}{\alpha}\right)\ln l$	$M_\alpha(\psi) = m_\alpha L - \boxed{c_\alpha}$	$W_\alpha(l) = \boxed{\frac{4\Delta_{2\alpha}}{\alpha-1}}\ln l$
characterized by	central charge c	central charge c	g-factor $g_1 = e^{(\alpha-1)c_\alpha}$	scaling dimension of BCCO $\Delta_{2\alpha}$

MH, M. Oshikawa, Y. Ashida, arXiv:2503.13599

SRE with open boundaries and topological defects



SRE reflects universal data of the defects and their fusion rules

MH, Y. Ashida, in preparation

Appendix

Measurements as boundary perturbation

Gibbs state

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr} e^{-\beta H}}$$

expectation value

$$\langle A \rangle = \mathrm{Tr}[A\rho] = \frac{1}{Z}\int \mathcal{D}\varphi\ A[\varphi]\, e^{-\mathcal{S}[\varphi]}$$

partition function

$$Z = \int \mathcal{D}\varphi\, e^{-\mathcal{S}[\varphi]}$$

projective measurement $\{P^{\vec{m}}\}_{\vec{m}}$

post-measurement state

$$\rho^{\vec{m}} = \frac{P^{\vec{m}} \rho P^{\vec{m}}}{p_{\vec{m}}}$$

Born probability

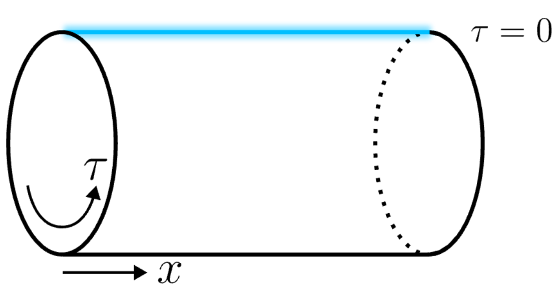
$$p_{\vec{m}} = \mathrm{Tr}[P^{\vec{m}}\rho] = \frac{1}{Z}\int \mathcal{D}\varphi\ P^{\vec{m}}[\varphi]\, e^{-\mathcal{S}[\varphi]}$$

$$= \frac{1}{Z}\int \mathcal{D}\varphi\ e^{-\mathcal{S}[\varphi]-\delta\mathcal{S}[\varphi]}$$

$$P^{\vec{m}}[\varphi] = e^{-\delta\mathcal{S}[\varphi]}$$

$$Z^{\vec{m}} = \int \mathcal{D}\varphi\, e^{-\mathcal{S}[\varphi]-\delta\mathcal{S}[\varphi]}$$

$=$



Boundary perturbation $\delta\mathcal{S}[\varphi]$

- flows to conformal boundary condition in the IR limit
- infinitely large coupling for projective measurements

S. J. Garratt et al., PRX (2023)

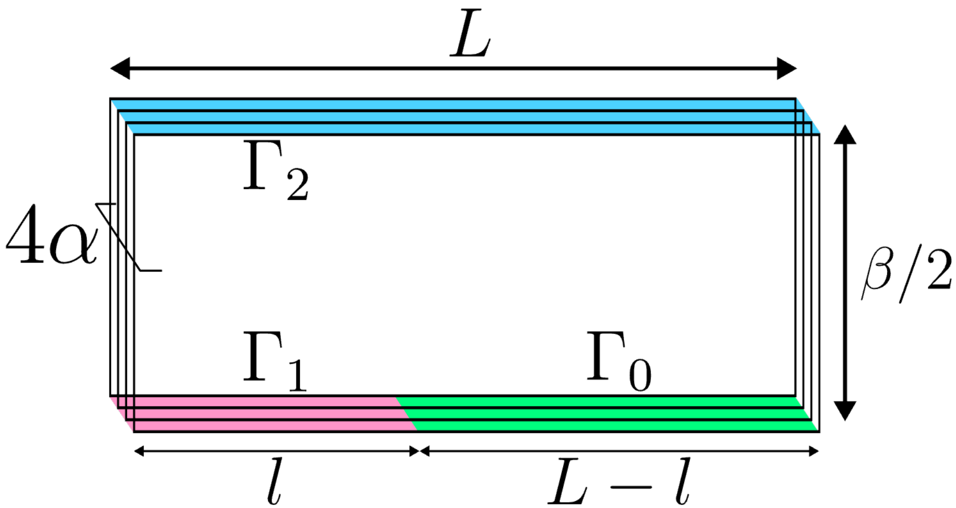
The boundary condition Γ_0

$$\mathrm{Tr}_A[\sigma^{\vec{m}_A}\rho_A] = \mathrm{Tr}\big[(\sigma^{\vec{m}_A}\otimes I_B)\psi\big] \qquad \vec{m}_A \in \{0,1\}^{2l}$$

$$\mathrm{Tr}_A^2[\sigma^{\vec{m}_A}\rho_A] = 2^L\,\mathrm{Tr}\big[(P^{\vec{m}_A}\otimes (P^{00})^{\otimes L-l})(\psi\otimes\psi^*)\big]$$

$$M_\alpha(\rho_A) = \frac{(\alpha L - l)\ln 2}{1-\alpha} + \frac{1}{1-\alpha}\ln \mathrm{Tr}\left[\sum_{\vec{m}_A}(\tilde{P}^{\vec{m}_A})^{\otimes\alpha}(\psi\otimes\psi^*)^{\otimes\alpha}\right]$$

$$\tilde{P}^{\vec{m}_A} = P^{\vec{m}_A}\otimes (P^{00})^{\otimes(L-l)}$$



Boundary condition imposed by uniform measurement of Bell00

For the Ising model, $|f\rangle \otimes |f\rangle = |D(\pi/2)\rangle_{\mathrm{orb}}$

$$\Longrightarrow \qquad |\Gamma_0\rangle = |D(\pi/2)\rangle_{\mathrm{orb}}^{\otimes 2\alpha}$$

Construction of the boundary states

Ishibashi condition

$$(L_n - \bar{L}_{-n})\,|\Gamma\rangle = 0$$

in terms of U(1) currents

$$(\vec{\alpha}^L_m - \mathcal{R}\vec{\alpha}^R_{-m})\,|\Gamma\rangle = 0$$

$\mathcal{R}$: **N×N orthogonal matrix**

Ishibashi state (coherent state)

$$S(\mathcal{R})\,|\vec{R},\vec{K}\rangle$$

$$S(\mathcal{R}) = \exp\left[-\sum_{n=1}^{\infty}(\vec{a}^L_n)^\dagger\cdot\mathcal{R}(\vec{a}^R_n)^\dagger\right] \qquad \vec{K}+\kappa\vec{R} = \mathcal{R}(\vec{K}-\kappa\vec{R})$$

Consistent boundary state of mixed Dirichlet-Neumann boundary condition

$$|\mathcal{R}(\vec{\phi}_D,\vec{\theta}_D)\rangle := g_{\mathcal{R}} \sum_{\vec{R}\in\Lambda_{\mathcal{R}}}\sum_{\vec{K}\in\Lambda^*_{\mathcal{R}}} e^{-i\vec{R}\cdot\vec{\theta}_D - i\vec{K}\cdot\vec{\phi}_D} S(\mathcal{R})\,|\vec{R},\vec{K}\rangle$$

g-factor

Boundary state of the orbifold theory (symmetrization)

$$a = \mathrm{diag}(\pm1,\pm1,\ldots,\pm1)\in G$$

$$|G| = 2^N$$

$$G_0 = G/\{\pm I\}$$

$$|\mathcal{R}(\vec{\phi}_D,\vec{\theta}_D)\rangle_{\mathrm{orb}} = \frac{1}{\sqrt{|G|}}\sum_{a\in G} D(a)\,|\mathcal{R}(\vec{\phi}_D,\vec{\theta}_D)\rangle$$

generic

$$|\mathcal{R}(\vec{\phi}_E,\vec{\theta}_E)\rangle_{\mathrm{orb}} = \sum_{b\in G_0} D(b)\left[\frac{1}{\sqrt{|G|}}\,|\mathcal{R}(\vec{\phi}_E,\vec{\theta}_E)\rangle \pm 2^{-N/4}\,|\mathcal{R}(\vec{\phi}_E,\vec{\theta}_E)\rangle_t\right]$$

endpoint