Efficiently computable lower bounds for the p-radius of switching linear systems

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Stability of switched linear systems

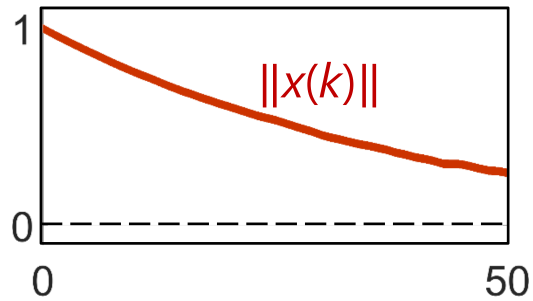
$$x(k+1) = \begin{cases} A_1 x(k) & \text{prob.} = 1/2 \\ A_2 x(k) & \text{prob.} = 1/2 \end{cases}$$

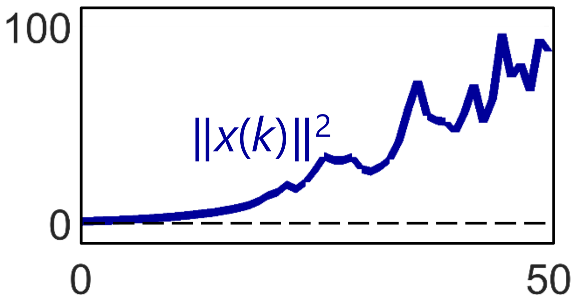
- Stability: When $x(k) \to 0$?
- Theorem [Kalman '57]
  - $E[\|x(k)\|^2] \to 0 \quad \Leftrightarrow \quad \rho\left(\dfrac{A_1 \otimes A_1 + A_2 \otimes A_2}{2}\right) < 1$
  - Mean square stability
- The most used stability notion
  - Is it the best?

Example

$$x(k+1) = \begin{cases} 1.4\,x(k) & \text{prob.} = 1/2 \\ 0.55\,x(k) & \text{prob.} = 1/2 \end{cases}$$

- Sample averages





- In general:

Mean stability $E[\|x(k)\|] \to 0$	$\Leftarrow$ $\not\Rightarrow$	Mean square stability $E[\|x(k)\|^2] \to 0$

1-radius

$$x(k+1) = \begin{cases} A_1 x(k) & \text{prob.} = 1/2 \\ A_2 x(k) & \text{prob.} = 1/2 \end{cases}$$

- 1-radius [Jia91, Wang92]

$$\rho_1(A_1, A_2) = \lim_{k\to\infty} \left(\frac{\sum_{i_1,\dots,i_k \in \{1,2\}} \|A_{i_k} \cdots A_{i_1}\|}{2^k} \right)^{1/k}$$

  - Averaged growth rate of $\|x(k)\|$

- Mean stability $\Leftrightarrow$ $\rho_1 < 1$
- NP-hard to approximately compute

  [JungersProtasov '11]

Bounds

$$\rho_1(A_1, A_2) = \lim_{k\to\infty} \left(\frac{\sum_{i_1,\ldots,i_k \in \{1,2\}} \|A_{i_k} \cdots A_{i_1}\|}{2^k} \right)^{1/k}$$

- Upper bound
  - The sequence is decreasing.
  - High computational cost
- Lower bound [Zhou '98]

$$\rho_1(A_1, A_2) \geq \rho\left(\frac{A_1 + A_2}{2}\right)$$

  - $\rho_1(+1, +1) = 1 =$ (lower bound).
  - $\rho_1(+1, -1) = 1$, but (lower bound) $= 0$.

An improvement

- Theorem [BarthélemyMarx '13]: If

$$w_1, w_2 \in [-1, +1],$$

  then

$$\rho_1(A_1, A_2) \geq \rho\left(\frac{w_1 A_1 + w_2 A_2}{2}\right)$$

- Examples
  - $\rho_1(+1, -1) = 1 = \rho\left(\frac{(+1)(+1) + (-1)(-1)}{2}\right)$
  - $\rho_1(R^0, R^1, R^2, R^3) = 1$

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

- Better lower bound?

Main result

- Theorem: Let W_1 and W_2 be square matrices. If

  $$\|\text{any product from } W_1, W_2\| \leq \text{constant}$$

  then

  $$\rho_1(A_1, A_2) \geq \rho\left(\frac{W_1 \otimes A_1 + W_2 \otimes A_2}{2}\right)$$

  - Extends many lower bounds in the literature
    - Including the one in [BarthelemyMarx '13]
  - Equivalent condition

    $$(\text{Joint spectral radius of } W_1, W_2) \leq 1$$

    - Measure of stability for arbitrary switching
    - The JSR toolbox [Jungers]

Why Kronecker products?

- Need to show: The transform

$$x(k+1) = \begin{cases} A_1\, x(k) \\ A_2\, x(k) \end{cases} \longrightarrow x(k+1) = \begin{cases} (W_1 \otimes A_1)\, x(k) \\ (W_2 \otimes A_2)\, x(k) \end{cases}$$

does not increase 1-radius.

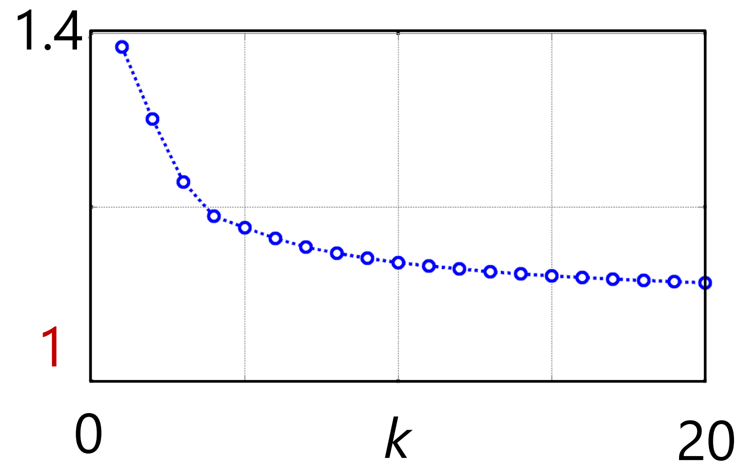
  - Compare k-products:

$$\|(W_{i_k} \otimes A_{i_k})\,(W_{i_{k-1}} \otimes A_{i_{k-1}}) \cdots (W_{i_1} \otimes A_{i_1})\|$$
$$= \|(W_{i_k}\, W_{i_{k-1}} \cdots W_{i_1}) \otimes (A_{i_k}\, A_{i_{k-1}} \cdots A_{i_1})\|$$
$$= \|W_{i_k}\, W_{i_{k-1}} \cdots W_{i_1}\|\; \|A_{i_k}\, A_{i_{k-1}} \cdots A_{i_1}\|$$
$$\leq C\, \|A_{i_k}\, A_{i_{k-1}} \cdots A_{i_1}\|$$

  - Usual products don't make them commute!

Example

- $A_1 = \begin{bmatrix} -0.87 & -0.77 \\ 1.17 & -1.09 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0.14 & 0.40 \\ 0.89 & -0.73 \end{bmatrix}$



- The decreasing sequence converging to 1-radius
  - Conjecture: $\rho_1 > 1$
  - Recall: Mean stability $\Leftrightarrow \rho_1 < 1$
- BarthélemyMarx '13 $\simeq 0.73$
- Our lower bound $\simeq 1.02$
  - Proves the conjecture
  - Weighting matrices $W_1 = \begin{bmatrix} -0.64 & -0.57 \\ 0.86 & -0.80 \end{bmatrix}$, $W_2 = \begin{bmatrix} 0.14 & 0.39 \\ 0.86 & -0.71 \end{bmatrix}$

Extension to p-radius

- *p*-radius

$$\rho_p = \left(\frac{\sum_{i_1,\ldots,i_k \in \{1,\ldots,N\}} \|A_{i_k} \cdots A_{i_1}\|^p}{2^k} \right)^{1/kp}$$

  - Characterizes *p*th mean stability

$$E[\|x(k)\|^p] \to 0 \quad \Leftrightarrow \quad \rho_p < 1$$

  - Fact: If *p* is an integer then

$$\rho_p(A_1, A_2) = \rho_1(\overbrace{A_1 \otimes \cdots \otimes A_1}^{p \text{ copies}}, A_2 \otimes \cdots \otimes A_2)^{1/p}$$

    - Reduces to 1-radius
    - The obtained lower bound applies

Conclusion

- A novel lower bound of p-radius
  - Weighting by matrices via Kronecker products
  - Extends the lower bounds in the literature
- Open problems
  - How can one choose good (or best) weights?
  - Does the lower bound attain the exact value of p-radius?
    - Sufficient condition based on sparsity (in the proceeding)