# Mean stability of continuous-time semi-Markov jump linear positive systems

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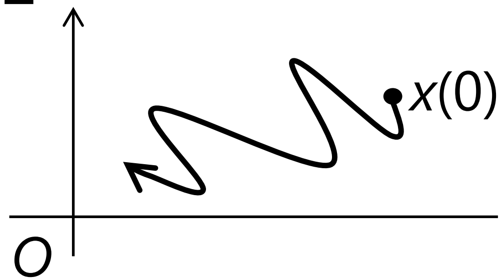
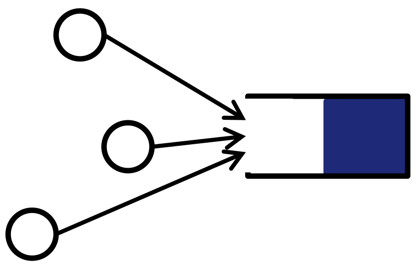
# Switching positive linear system

$$dx/dt = A_{\sigma(t)}\, x(t), \quad \sigma(t) = 1 \text{ or } 2$$




- Positivity $x(0) \geq 0 \Rightarrow x(t) \geq 0$ (entry-wise)
  - Metzler matrices $a_{ij} \geq 0$ if $i \neq j$
  - $t \geq 0 \Rightarrow e^{At} \geq 0$
- Examples
  - Infection treatment [Hernandez-Vargas et al. '10]
    - Switching between multiple drugs to avoid resistance
    - # of viruses $\geq 0$
  - Network congestion [Shorten et al. '06]
    - Unsynchronized networks
    - TCP window size (# of packets sent) $\geq 0$



# Mean stability of positive systems

$$dx/dt = A_{\sigma(t)}\, x(t), \ \text{positive}$$

- σ: stochastic process
- Stochastic stability notions

| Mean square stability $E[\|x(t)\|^2] \to 0$ | $\Rightarrow$ | Mean stability $E[\|x(t)\|] \to 0$ | $\Rightarrow$ | Almost sure stability $\|x(t)\| \to 0$ w.p.1 |
|---|---|---|---|---|

Compatible with positivity

$$\|x(t)\|_1 = |x_1(t)| + |x_2(t)| = x_1(t) + x_2(t) = [1 \ \ 1]\, x(t)$$

# Stability analysis | Markovian case

$$dx/dt = A_{\sigma(t)}\, x(t), \text{ positive}$$

- Time-homogeneous Markov process

$$\mathrm{Prob}(\,\sigma(t+h) = j \mid \sigma(t) = i\,) \approx q_{ij}\, h \quad (h \text{ small}, i \neq j)$$

- Theorem Let $M := Q^{\mathsf{T}} \otimes \mathrm{Id} + \begin{bmatrix} A_1 & \\ & A_2 \end{bmatrix}$.

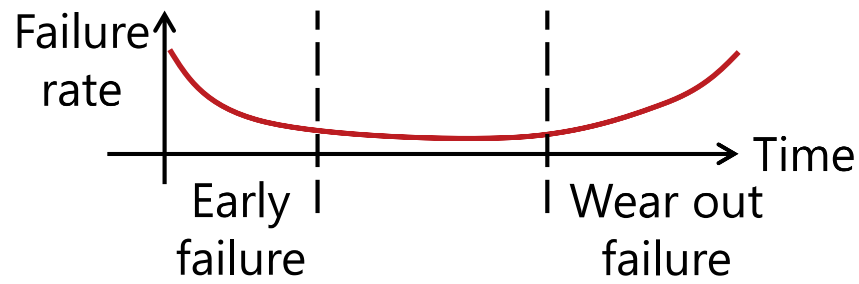
$$\text{Mean stability} \iff M \text{ is Hurwitz stable}$$

[Bolzern et al. '14, OguraMartin '14]

- Question: How general are Markov processes?
  - Dwell times must follow exponential distributions

# Non-exponential distributions

- Time-to-failure
  - Bathtub curve




  - Weibull distribution

$$f(x) = \alpha\, x^{k-1} \exp(-\beta x^{k})$$

- Time-to-response on social media
  - Lognormal distribution

$$f(x) = \exp(-(\log x - \mu)^2/2\sigma^2)/(2\pi)^{1/2}\sigma x$$

    - Verified for Twitter and Digg [Doerr et al. '13]

# Semi-Markovian case | main result

- Time-homogeneous semi-Markov process

$$\mathrm{Prob}(\,\sigma(t+h)=j \mid \sigma(t)=i\,) \approx q_{ij}(h)\,h \quad (h \text{ small}, i \neq j)$$

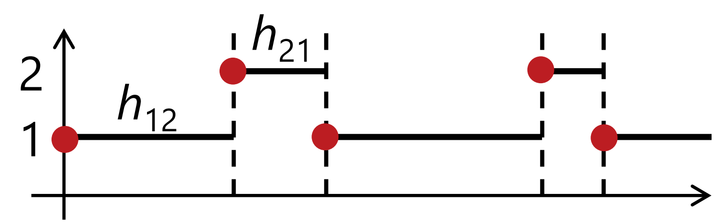
  - Dwell times can follow any distribution

- Theorem If dwell times are essentially bounded then

$$\text{Mean stability} \iff \rho(M) < 1$$

where $M = \begin{bmatrix} p_{11}E[\exp(A_1 h_{11})] & p_{21}E[\exp(A_2 h_{21})] \\ p_{12}E[\exp(A_1 h_{12})] & p_{22}E[\exp(A_2 h_{22})] \end{bmatrix}$

  - $p_{ij}$ = Transition prob. of underlying Markov chain $\{\sigma(t_k)\}$
  - $h_{ij}$ = Dwell times (random variable)



# Difficulty

$$dx/dt = A_{\sigma(t)}\, x(t), \ \text{positive}$$

- Extended state variable: $\tilde{x}(t) = \begin{cases} \begin{bmatrix} x(t) \\ 0 \end{bmatrix} & \sigma(t) = 1 \\ \begin{bmatrix} 0 \\ x(t) \end{bmatrix} & \sigma(t) = 2 \end{cases}$
  - Markovian case

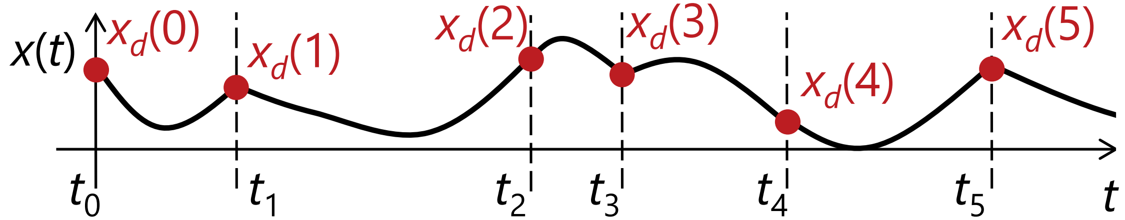
$$\frac{d}{dt} E[\tilde{x}] = M E[\tilde{x}]$$

  - Semi-Markovian case

$$E[\tilde{x}] = \Theta(E[\tilde{x}]) + h \qquad \Theta: \text{ integral operator}$$

    - Investigated in [Antunes et al. '13] for mean square stability analysis
    - They directly studied the integral equation

# Idea | discretization




$$E[\tilde{x}_d(k+1)] = ME[\tilde{x}_d(k)]$$

$$\rho(M) < 1$$

$$\Updownarrow$$

Mean stability
of discretized system

$\Updownarrow$ ?

1. Positivity
2. Linearity
3. Dwell times (sampling periods) $< T < \infty$

Mean stability

# Conclusion

- Stability of positive switched linear system
  - Semi-Markov switching signal
  - Discretization approach
  - Reference
    - M. Ogura and C. F. Martin, “Stability analysis of positive semi-Markovian jump linear systems with state resets,” *SIAM Journal on Control and Optimization*, vol. 52, no. 3, pp. 1809–1831, May 2014

Thank you!