Shenzhen University
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Optimization of Positive Linear Systems via Geometric Programming

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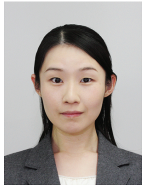


M. Ogura, M. Kishida, and J. Lam, "Geometric programming for optimal positive linear systems," *IEEE Transactions on Automatic Control (accepted for publication)*, 2020.

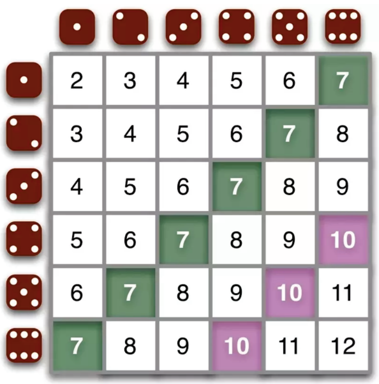
Outline

- Positive linear systems
- Parameter Tuning Problem
- Synthesis by geometric programming
- (Hidden convexity)
- Numerical example

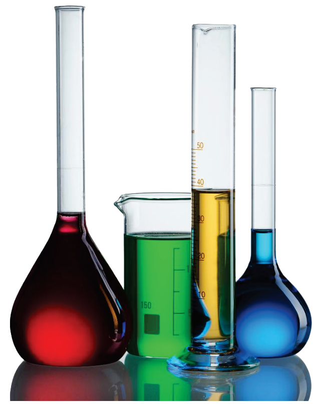
Positive linear systems

Nonnegative variables

Probability



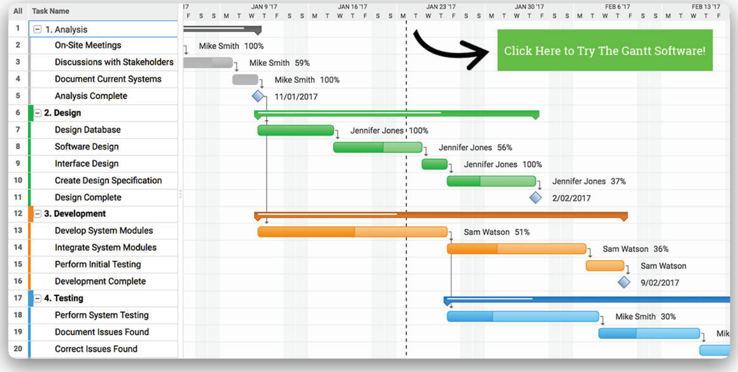
Chemistry



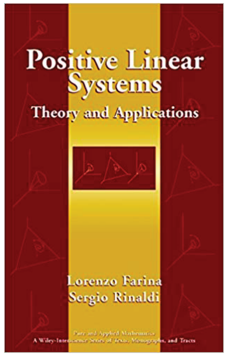
Economics

Math biology

Project management

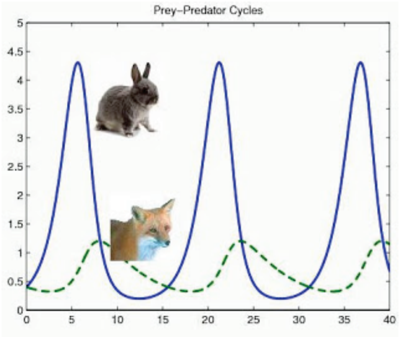


Dynamical systems with positive variables



- Lotka-Volterra equation

$$\frac{dx}{dt} = \alpha x - \beta xy,$$
$$\frac{dy}{dt} = \delta xy - \gamma y,$$



- Buffer network

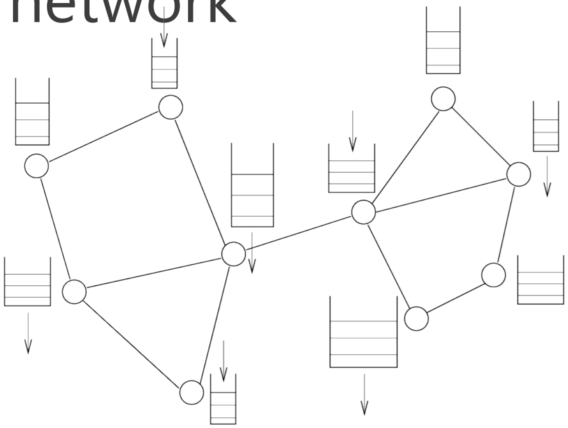


Fig. 1. Positive systems are commonly used to model dynamics of buffer networks (1). Each state represents the content of a buffer. Content can be transferred from one buffer to another via the network links. The content of a buffer can also change as a result of local production or consumption.

Rantzer, Valcher, "A tutorial on positive systems and large scale control," *2018 IEEE Conference on Decision and Control*, 2018.

Metzler matrix (nonnegative off-diagonals)

$$\frac{dx}{dt} = Ax + Bw$$

$$y = Cx$$

Nonnegative matrices

Positivity

- A Metzler, $t \geq 0 \Rightarrow \exp(At) \geq 0$ (nonnegative matrix)
- $x_0 \geq 0$ and $w(t) \geq 0 \Rightarrow x(t) \geq 0$
- $x_0 \geq 0$ and $w(t) \geq 0 \Rightarrow y(t) \geq 0$

Problem statement

Informal statement...

Positive linear system

$$\frac{dx}{dt} = Ax + Bw$$

$$y = Cx$$

Investment

$$\frac{dx}{dt} = \bar{A}x + \bar{B}w$$

$$y = \bar{C}x$$

Desirable control-theoretical properties

Problem statement

- Parametrized positive linear system

$$\Sigma_\theta : \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ y = C(\theta)x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

- Parameter tuning problem

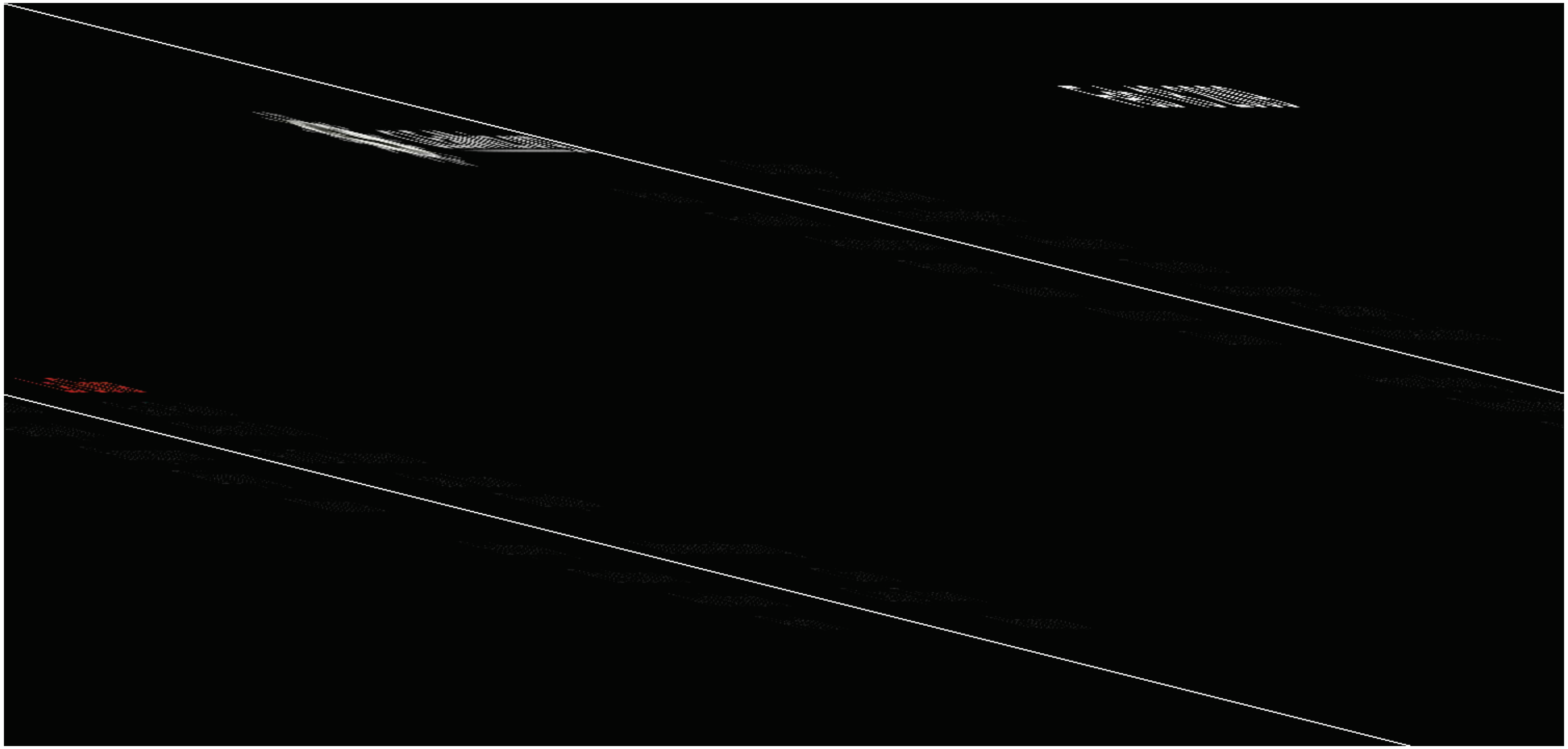
$$\begin{array}{ll} \underset{\theta \in \Theta}{\text{minimize}} & L(\theta) \\ \text{subject to} & \Sigma_\theta \text{ is internally stable} \\ & J(\Sigma_\theta) \le \gamma \end{array}$$

Parameter tuning cost

Performance functional
H^2 norm, H^∞ norm, Hankel norm, …

Networked epidemic processes

H1N1 Flu outbreak

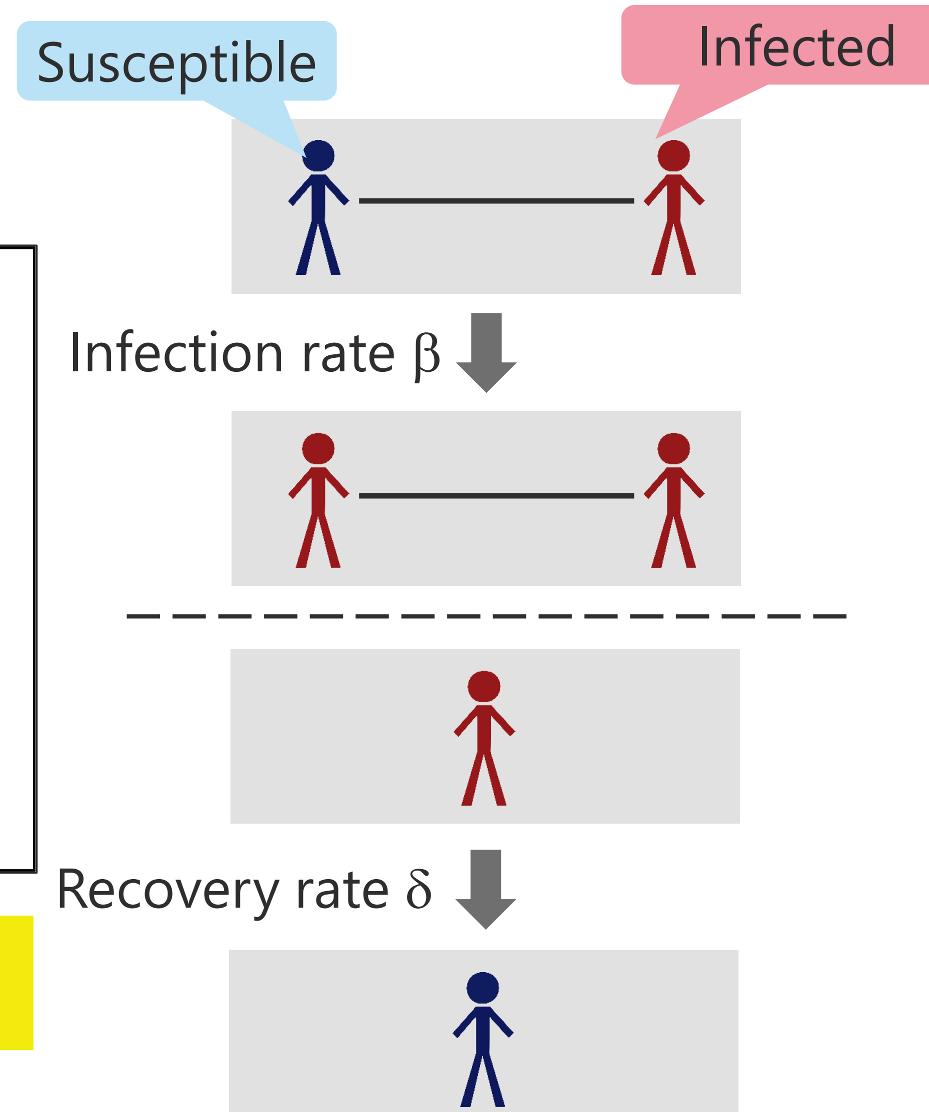


https://www.barabasilab.com/

Networked Susceptible-Infected-Susceptible model

Nodes = individuals

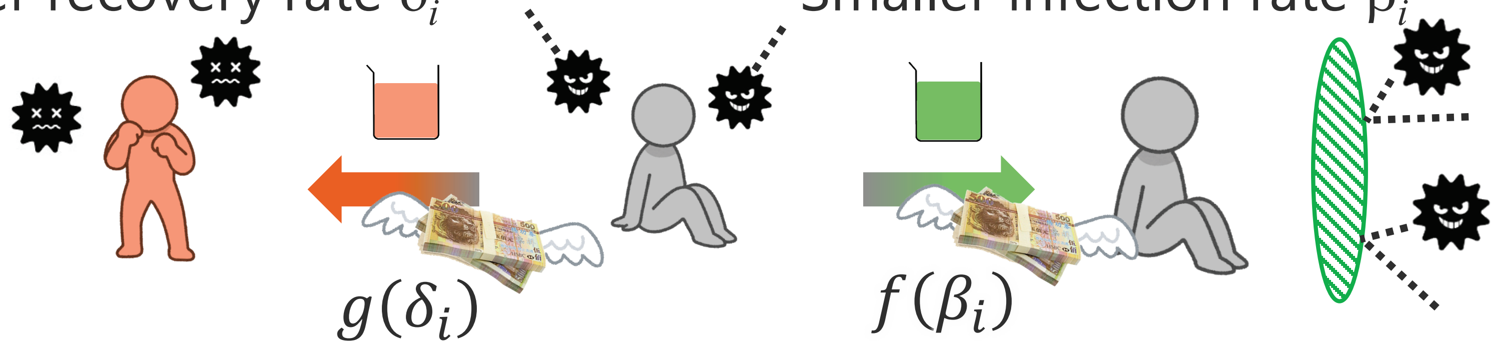
Edges = relationships



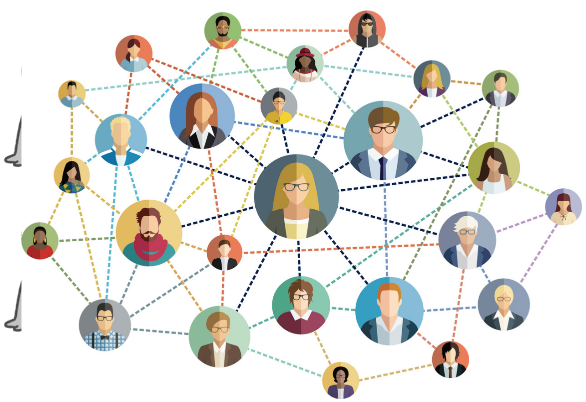
Infection probability ≥ 0

Epidemic containment problem [Preciado 2014, *IEEE TCNS*]

Larger recovery rate δ_i Smaller infection rate β_i

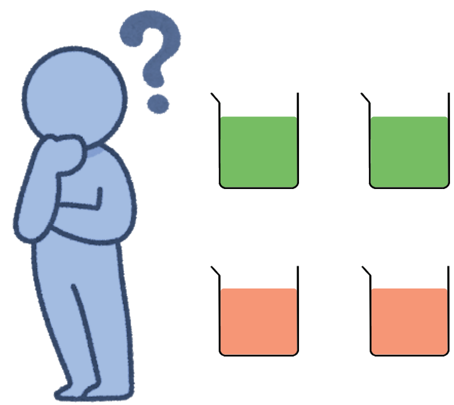


Finite, limited amount of medicine



Network (V, E)

$V = \{1, \dots, N\}$



Total cost

$$\sum_{i=1}^{N} (f(\beta_i) + g(\delta_i))$$

Example: Epidemic Containment

- Linearlized SIS model

Infection probability vector

Adjacency matrix

$$\Sigma_{\beta,\delta}: \begin{cases} \dfrac{dx}{dt} = \left(\mathrm{diag}(\beta_1, \dots, \beta_N) A_G - \mathrm{diag}(\delta_1, \dots, \delta_N)\right) x + Bw \\ y = Cx, \end{cases}$$

External disturbance

Performance output

- Cost function

$$L(\theta) = \sum_{i=1}^{N} \left(f(\beta_i) + g(\delta_i)\right)$$

Problem statement (p. 8)

- Parametrized positive linear system

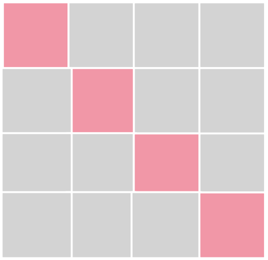
$$\Sigma_\theta : \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ y = C(\theta)x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

- Parameter tuning problem

$$\begin{array}{ll} \underset{\theta \in \Theta}{\text{minimize}} & L(\theta) \\ \text{subject to} & \Sigma_\theta \text{ is internally stable} \\ & J(\Sigma_\theta) \leq \gamma \end{array}$$

Related works

$$\Sigma_\theta: \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ y = C(\theta)x, \end{cases}$$



Convexity results

- Assumption: Off-diagonals of A fixed. B and C fixed.
- Convexity of $x(T)$ [Colaneri 2014, *Automatica*]
- Convexity of H^2 or H^∞ norm [Dhingra 2018, *IEEE TCNS*]
- Don’t apply to epidemic containment (and other problems)

Research question

Is it possible to optimally tune the off-diagonals of A as well as B and C matrices?

Geometric programming

A framework for optimizing "posynomial functions"

Posynomial

$$f(x_1, \dots, x_n) = \text{(sum of monomials)}$$

Monomial

$$g(x_1, \dots, x_n) = c\, {x_1}^{a_1} \cdots {x_n}^{a_n}$$

- Positive coefficient: $c > 0$
- Positive variables: $x_1, \dots, x_n > 0$
- Real exponents: $a_1, \dots, a_n \in \mathbb{R}$

Boyd, Kim, Vandenberghe, Hassibi, "A tutorial on geometric programming," Optimization and Engineering, vol. 8, no. 1, pp. 67–127, 2007.

Definition

Posynomial

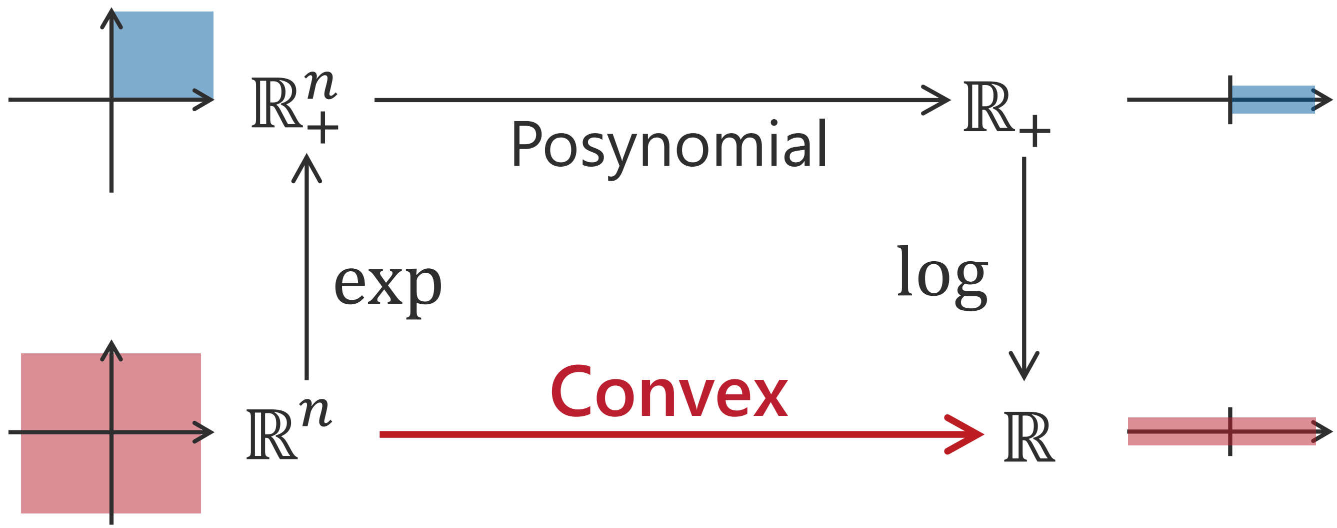
$$\begin{array}{ll} \underset{x_1,\dots,x_n>0}{\text{minimize}} & f(x_1,\dots,x_n) \\ \text{subject to} & f_i(x_1,\dots,x_n) \le 1 \ \ (i=1,\dots,p) \\ & g_j(x_1,\dots,x_n) = 1 \ \ (j=1,\dots,q) \end{array}$$

Monomial

- An example

$$\begin{array}{ll} \underset{x,y,z>0}{\text{minimize}} & x^{-1}y^{-1/2}z^{-1} + 2.3xz + 4xyz \\ \text{subject to} & (1/3)x^{-2}y^{-2} + (4/3)y^{1/2}z^{-1} \le 1, \\ & x + 2y + 3z \le 1, \\ & (1/2)xy = 1, \end{array}$$

Reduction to a convex optimization

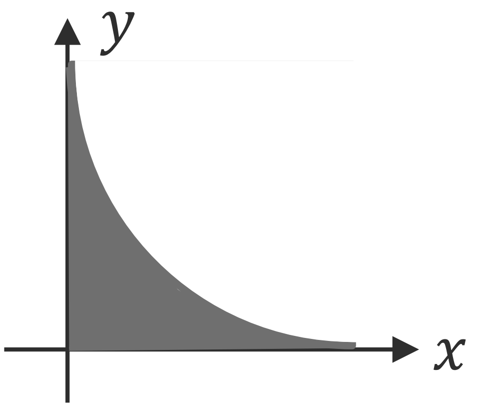


Log transformation

Geometric program → **Convex optimization**

- Geometric program

$$\begin{array}{ll} \text{minimize} & x^{-1/3}y^{-1/4} \\ \text{subject to} & x^{1/2}+y^{1/2}\le 1 \end{array}$$

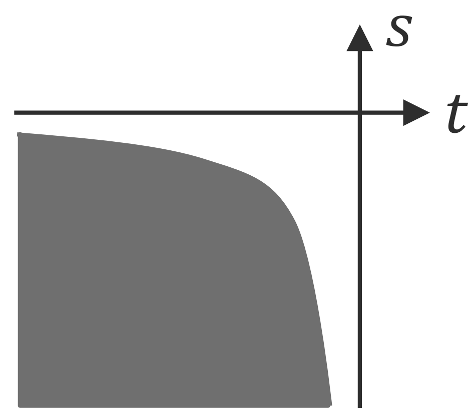


- Variable transformations: $x = e^t, y = e^s$

$$\begin{array}{ll} \text{minimize} & e^{\left(-\frac{t}{3}-\frac{s}{4}\right)} \\ \text{subject to} & e^{t/2}+e^{s/2}\le 1 \end{array}$$

- Equivalent convex optimization

$$\begin{array}{ll} \text{minimize} & -\frac{t}{3}-\frac{s}{4} \\ \text{subject to} & s\le 2\log(1-e^{t/2}) \end{array}$$



Applications

Circuit design

- Gate sizing (Chu and Wong 2001b; Cong and He 1998; Coudert et al. 1996; Kasamsetty et al. 2000; Marković et al. 2004; Matson and Glasser 1986; Pattanaik et al. 2003; Sancheti and Sapatnekar 1996; Sapatnekar et al. 1993; Shyu et al. 1988; Swahn and Hassoun 2006)
- Wire sizing (Chen and Wong 1999; Chen et al. 2004; Cong and He 1996; Cong and Koh 1994; Cong and Leung 1995; Cong and Pan 2002; Gao and Wong 1999; Kay and Pileggi 1998; Lee et al. 2002; Lin and Pileggi 2001; Sapatnekar 1996)
- Buffering and wire sizing (Chu and Wong 1999, 2001a)
- Simultaneous gate and wire sizing (Chen et al. 1999; Jiang et al. 2000)
- Sizing and placement (Chen et al. 2000; Lou et al. 1999)
- Routing (Borah et al. 1997)
- Design with multiple supply voltages (Krishnamurthy and Carley 1997)
- Yield maximization (Boyd et al. 2006; Kim et al. 2007; Patil et al. 2005)
- Power optimization (Bhardwaj et al. 2006; Bhardwaj and Vrudhula 2005; Horowitz et al. 2005; Satish et al. 2005)
- Parasitic reduction (Qin and Cheng 2003)
- Clock skew optimization (Sathyamurthy et al. 1998)

Other fields

- Chemical engineering (Clasen 1984; Salomone and Iribarren 1993; Salomone et al. 1994; Wall et al. 1986)
- Environment quality control (Greenberg 1995; Smeers and Tyteca 1984)
- Resource allocation in communication and network systems (Boche and Stańczak 2004; Chiang 2005a; Chiang and Boyd 2004; Chiang et al. 2002a, 2002b; Dutta and Rama 1992; Greenberg 1995; Julian et al. 2002; Kandukuri and Boyd 2002; Palomar et al. 2003)
- Information theory (Ben-Tal and Teboulle 1986; Ben-Tal et al. 1988; Chiang and Boyd 2004; Karlof and Chang 1997; Klafszky et al. 1992; Muqattash et al. 2006)
- Probability and statistics (Bricker et al. 1997; El Barmi and Dykstra 1994; Feigin and Passy 1981; Fuh and Hu 2000; Hu and Wei 1989; Mazumdar and Jefferson 1983; Stark and Machida 1993; Vardi 1985)
- Structural design (Adeli and Kamal 1986; Chan and Turlea 1978; Chen 1992; Dhillon and Kuo 1991; Hajela 1986; Vanderplaats 1984)
- Computer system architecture design (Trivedi and Sigmon 1981)
- Inventory control (Abou-El-Ata and Kotb 1997; Abuo-El-Ata et al. 2003; Cheng 1991; Hariri and Abou-El-Ata 1997; Jung and Klein 2001; Lee and Kim 1993; Scott et al. 2004)
- Production system optimization (Choi and Bricker 1996a)
- Mechanical engineering (Jha 1990; Sonmez et al. 1999)
- Transportation engineering (Wong 1981)
- Management science (Corstjens and Doyle 1979)
- Computational finance (Rajasekera and Yamada 2001)
- Geometric modeling (Cheng et al. 2002; Cheng et al. 2005a, 2005b)

Geometric programming to solve parameter tuning problem for positive linear systems?

Boyd, Kim, Vandenberghe, Hassibi, “A tutorial on geometric programming,” Optimization and Engineering, vol. 8, no. 1, pp. 67–127, 2007.

Solution by geometric programming

Problem statement (p. 8)

- Parametrized positive linear system

$$\Sigma_\theta\colon \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ y = C(\theta)x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

- Parameter tuning problem

$$\begin{aligned} \underset{\theta \in \Theta}{\text{minimize}} \quad & L(\theta) \\ \text{subject to} \quad & \Sigma_\theta \text{ is internally stable} \\ & J(\Sigma_\theta) \le \gamma \end{aligned}$$

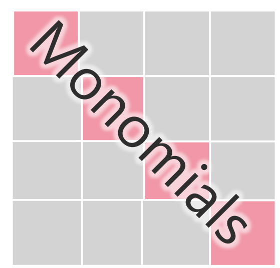
- Assumptions in literature: Off-diagonals of A fixed. B and C fixed.

Assumption: system matrices

- Parametrized positive linear system

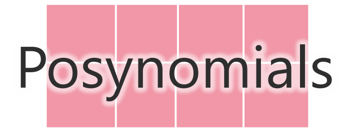
$$\Sigma_\theta : \begin{cases} \dfrac{dx}{dt} = \boxed{A(\theta)}x + \boxed{B(\theta)}w \\ y = \boxed{C(\theta)}x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

$$A(\theta) \quad = \quad \tilde{A}(\theta) \quad - \quad R(\theta)$$



$$B(\theta)$$

$$C(\theta)$$



Assumption: parameter set

- Parametrized positive linear system

$$\Sigma_\theta : \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ y = C(\theta)x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

$$\Theta = \{\theta \in \mathbb{R}^{n_\theta} \mid \theta > 0, f_1(\theta) \le 1, \dots, f_p(\theta) \le 1\}$$

Posynomials

Assumption: cost function

- Parametrized positive linear system

$$\Sigma_\theta\colon \begin{cases} \dfrac{dx}{dt} = A(\theta)x + B(\theta)w \\ \quad y = C(\theta)x, \end{cases} \qquad \theta \in \Theta \subset \mathbb{R}^{n_\theta}$$

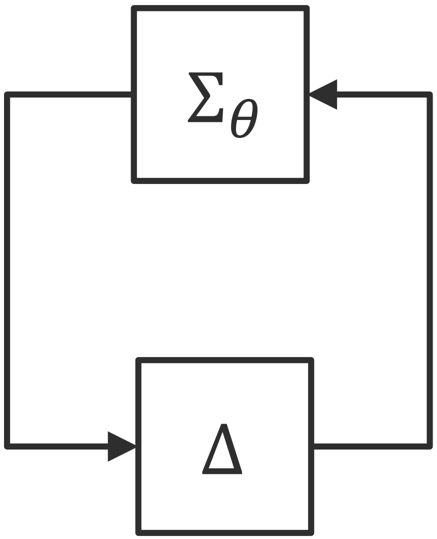
- Parameter tuning problem

$$\begin{array}{ll} \underset{\theta \in \Theta}{\text{minimize}} & L(\theta) \\ \text{subject to} & \Sigma_\theta \text{ is internally stable} \\ & J(\Sigma_\theta) \le \gamma \end{array}$$

Posynomial

If $J(\Sigma_\theta)$ is either

- H^2 norm;
- H^∞ norm;
- Worst-case growth rate under structural uncertainty;

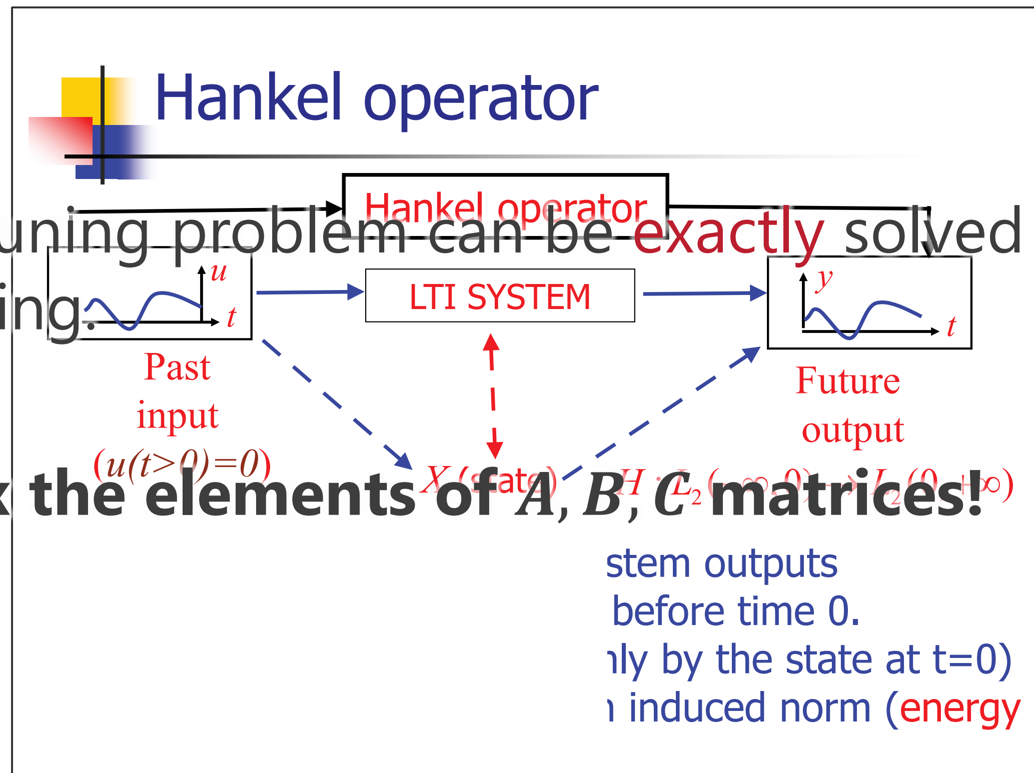


Main result: An overview

If $J(\Sigma_\theta)$ is either

- H^2 norm;
- H^∞ norm;
- Worst-case growth rate under structural uncertainty;
- Hankel norm;
- Schatten p-norm;

then, the parameter tuning problem can be exactly solved by geometric programming.



No need to fix the elements of A, B, C matrices!

https://www.slideserve.com/helene/introduction-to-the-hankel-based-model-order-reduction-for-linear-systems

Reduction to a geometric program

$$\begin{array}{ll} \underset{\theta \in \Theta}{\text{minimize}} & L(\theta) \\ \text{subject to} & \Sigma_\theta \text{ is internally stable} \\ & \boxed{J(\Sigma_\theta) \leq \gamma} \end{array}$$

How to rewrite into a constraint in terms of posynomials?

Strategy

Original $J(\Sigma_\theta)$ → Algebraic → Posynomial

- Linear systems theory

- Perron-Frobenius
- Matrix theory tools

Example: H^∞-norm

DC-dominance property
[Tanaka 2013, *SCL*]

$\|\Sigma_\theta\|_\infty \leq \gamma$ → $\|\hat{G}(0)\| \leq \gamma$ → Posynomial

$$\|\hat{G}(0)\| = \|C_\theta A_\theta^{-1} B_\theta\|$$

Max singular value lemma

Matrix inversion lemma

Lemmas

Maximum singular value lemma

Let M be a nonnegative matrix and γ be a positive number. Then, the following conditions are equivalent:

- $\|M\| < \gamma$
- $\exists$ positive vectors u and v such that $Mu < \gamma v$ and $Mv < \gamma u$

Matrix inversion lemma

Let F be a Metzler matrix, g be a nonnegative vector, and H be a nonnegative matrix. The following conditions are equivalent:

- F is Hurwitz stable and $-HF^{-1}g < v$
- $\exists$ positive vector χ such that $H\chi < v$ and $F\chi + g < 0$

Example: H^∞ norm

Theorem 4.1: The solution of the H^∞ norm-constrained optimization problem (13) is given by the solution of the following geometric program:

$$\begin{aligned}
\underset{\substack{\theta\in\mathbb{R}^{n_\theta}_{++},\\ u\in\mathbb{R}^{n_w}_{++},\, v\in\mathbb{R}^{n_y}_{++},\\ \xi,\zeta\in\mathbb{R}^{n_x}_{++}}}{\text{minimize}} \quad & \tilde{L}(\theta) && \text{(14a)}\\
\text{subject to} \quad & \gamma_\infty^{-1} D_v^{-1} C(\theta)\xi < \mathbb{1}, && \text{(14b)}\\
& D_\xi^{-1} R(\theta)^{-1}(\tilde{A}(\theta)\xi + B(\theta)u) < \mathbb{1}, && \text{(14c)}\\
& \gamma_\infty^{-1} D_u^{-1} B(\theta)^\top \zeta < \mathbb{1}, && \text{(14d)}\\
& D_\zeta^{-1} R(\theta)^{-1}(\tilde{A}(\theta)^\top \zeta + C(\theta)^\top v) < \mathbb{1}, && \text{(14e)}\\
& f_i(\theta) \le 1, \quad i = 1, \ldots, p. && \text{(14f)}
\end{aligned}$$

Theorem 5.1: Assume that there exist a monomial $r(\theta)$ and a diagonal matrix R_0 with positive diagonals such that the matrix $R(\theta)$ given in (3) satisfies (7). Then, the solution of the Hankel norm-constrained optimization problem (21) is given

Any fundamental reason for the effectiveness of geometric programming?

$$\min_{\substack{\theta \in \mathbb{R} \\ \chi_1 \in \mathbb{R}_{++}^{n_x^2 n_w},\ \chi_2 \in \mathbb{R}_{++}^{n_x n_y}}} \tag{24a}$$

$$\text{subject to} \quad \gamma^{-2} D_v^{-1} \bar{B}_1(\theta)\chi_1 < \mathbb{1}, \tag{24b}$$

$$D_{\chi_1}^{-1}(R_0 \oplus O_{n_w} \oplus R_0)^{-1} \frac{\bar{B}_2(\theta)\bar{C}_1(\theta)\chi_2 + (\tilde{A}(\theta) \oplus O_{n_w} \oplus \tilde{A}(\theta)^\top)\chi_1}{r(\theta)} < \mathbb{1}, \tag{24c}$$

$$D_{\chi_2}^{-1}(R_0 \oplus O_{n_y} \oplus R_0)^{-1} \frac{(\tilde{A}(\theta)^\top \oplus O_{n_y} \oplus \tilde{A}(\theta))\chi_2 + \bar{B}_2(\theta)v}{r(\theta)} < \mathbb{1}, \tag{24d}$$

$$f_i(\theta) \le 1, \quad i = 1, \ldots, p. \tag{24e}$$

Hidden convexity

Positive linear system (not parametrized)

$$\Sigma\colon \begin{cases} \dfrac{dx}{dt} = Ax + Bw \\ y = Cx, \end{cases}$$

- Impulse response function

$$\Phi(t) = C \exp(At)\, B$$

- Claim (informal)

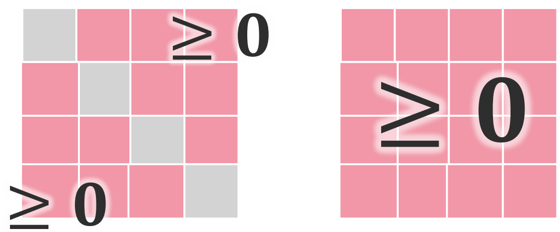
$\Phi(t)$ is a “posynomial” in the entries of A, B, C

$$\begin{tikzcd}
(A, B, C) \arrow[r, "\Phi(t)"] & \mathbb{R}_+ \arrow[d, "\log"'] \\
\text{“}(\log A, \log B, \log C)\text{”} \arrow[u, "\exp"'] \arrow[r, "\text{Convex}"] & \mathbb{R}
\end{tikzcd}$$

Diagonals of A matrix

Diagonal shift

$$A \quad = \quad \tilde{A} \quad - \quad rI$$



$$\Phi(t) = e^{-rt} C \exp\left(\tilde{A}t\right) B$$

Claim

Each entry of $\Phi(t)$ is the limit of a sequence of posynomials in the entries of $\tilde{A}, B, C$.

Proof

$$\Phi(t) = e^{-rt} C \exp\left(\tilde{A}t\right) B$$

$$= e^{-rt} C \sum_{k=0}^{\infty} \frac{\tilde{A}^k t^k}{k!} B$$

$$= \lim_{L\to\infty} e^{-rt} \sum_{k=0}^{L} \frac{t^k}{k} C \tilde{A}^k B$$

Posynomial in entries of $\tilde{A}, B, C$.

Corollary

The following quantities are log-convex in entries of $\tilde{A}$, B, C

Hidden convexity

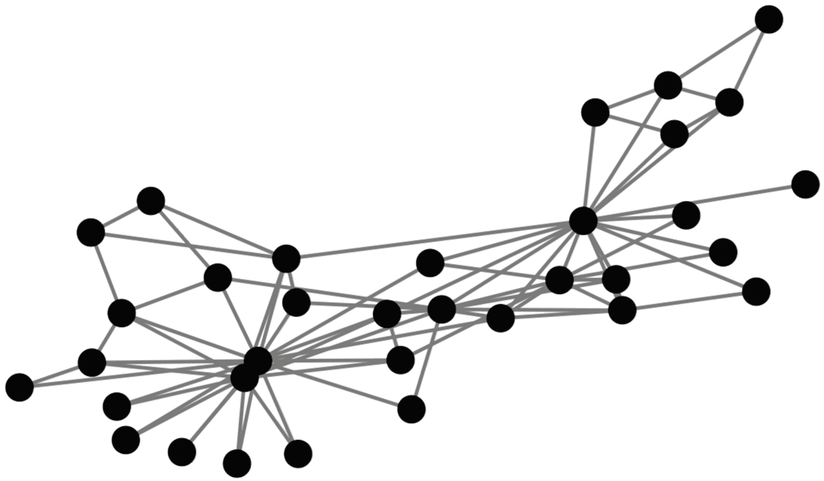
$$
\begin{tikzcd}
(\tilde{A}, B, C) \arrow[r, "L^p \text{ gain}"] & \mathbb{R}_+ \arrow[d, "\log"'] \\
\text{“}(\log \tilde{A}, \log B, \log C)\text{”} \arrow[u, "\exp"'] \arrow[r, "\text{Convex}"] & \mathbb{R}
\end{tikzcd}
$$

- Appearance of GP not necessarily surprising!

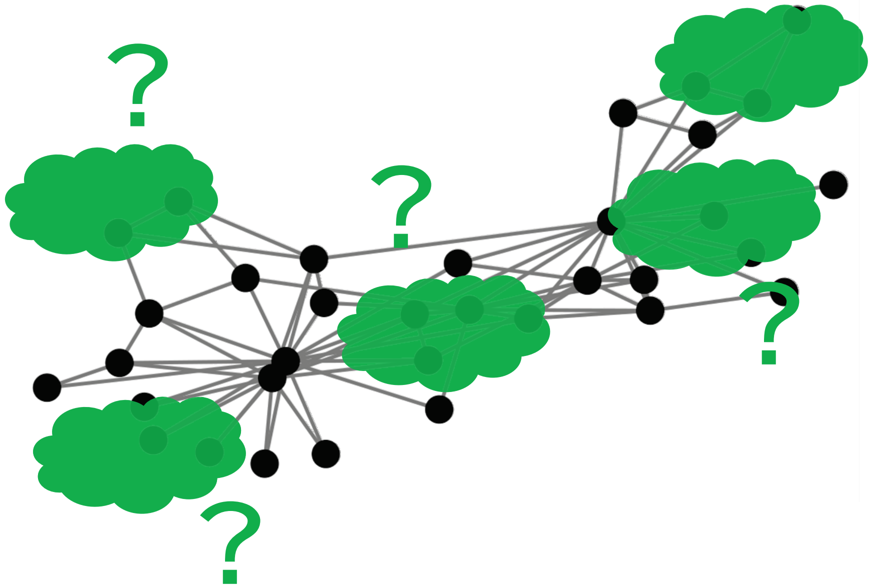
Example

Network uncertainty

Perfect information



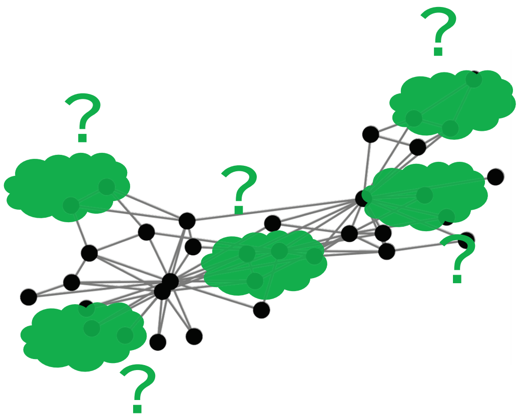
Uncertain network?



Optimal medical investment taking into account uncertainty in network structure?

- Privacy issue
- Information acquisition cost
- Data reliability

Additive uncertainty on adjacency matrix



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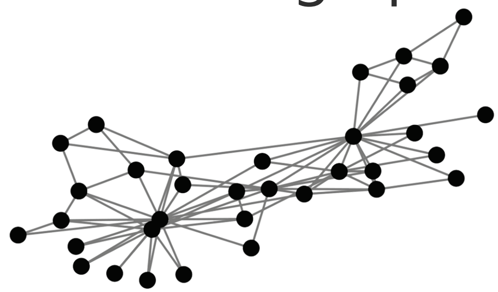
Nominal graph



+

Adjacency matrix $=$ A_G $+$ Δ

Fixed

Uncertain

Example: Epidemic Containment

- Linearlized SIS model w/ uncertainty

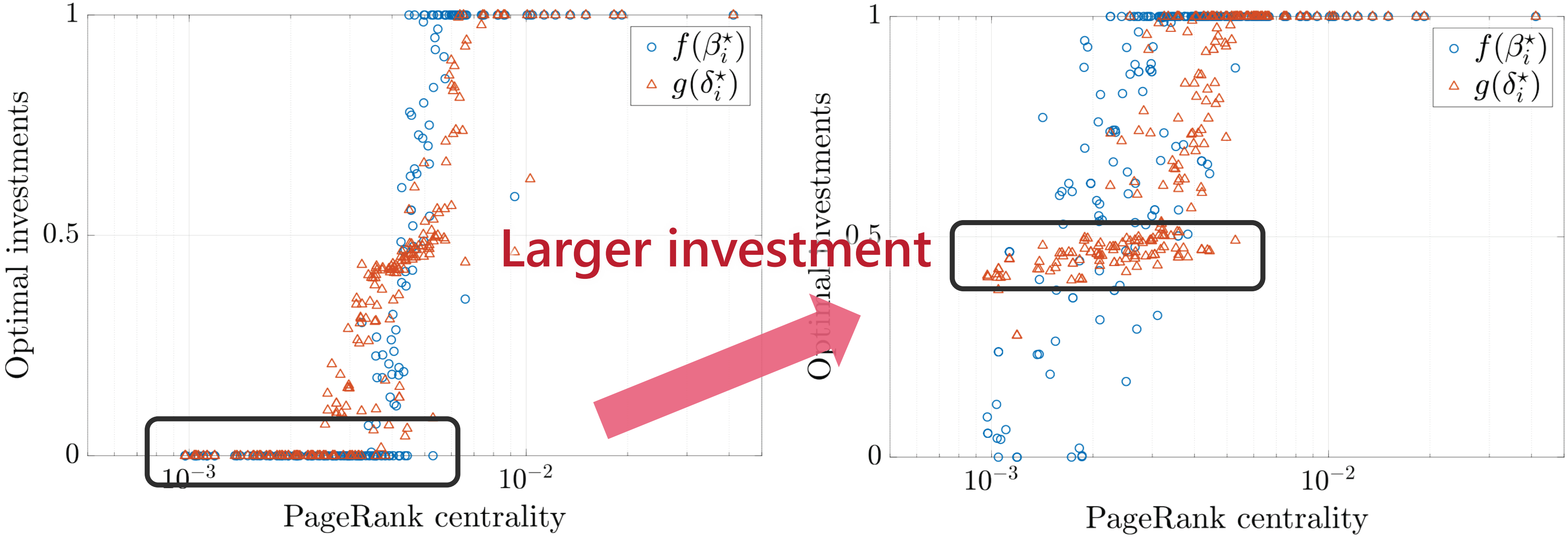
$$\Sigma_{\beta,\delta}: \begin{cases} \dfrac{dx}{dt} = \big(\mathrm{diag}(\beta_1, \dots, \beta_N)(\boldsymbol{A_G}+\boldsymbol{\Delta}) - \mathrm{diag}(\delta_1, \dots, \delta_N)\big)x + Bw \\ y = Cx, \end{cases}$$

- Cost function

$$L(\theta) = \sum_{i=1}^{N} (f(\beta_i) + g(\delta_i))$$

- Problem: maximize the worst-case decay rate of the size of infected population $\Rightarrow$ geometric programming

PageRank and optimal investments



Without uncertainty **With uncertainty**

- We need to invest more to prepare for worst case scenario
- GP allows us to find cost-efficient medical resource allocation

Conclusions

Conclusion

Parameter tuning of positive linear systems

- Geometric programs w/ finitely many variables
- Hidden convexity

Future research directions

- Switched linear systems
- Discrete-time systems
- Nonlinear systems

Research questions

- Can we combine convexity results and log-convexity results to better synthesize positive linear systems?