2.2 Separation of Horizontal and Vertical Structure

Our first step in solving the governing equations is to separate off the vertical structure. This is done by performing a vertical normal mode transform on equations (2.6). Appendix B details this process by following the treatment in Fulton and Schubert (1985), which results in the variables taking the following series forms,

$$\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ \phi(x,y,z,t) \end{pmatrix} = \sum_{\ell=1}^{\infty} \begin{pmatrix} u_\ell(x,y,t) \\ v_\ell(x,y,t) \\ \phi_\ell(x,y,t) \end{pmatrix} Z_\ell(z), \quad \begin{pmatrix} T(x,y,z,t) \\ w(x,y,z,t) \\ Q(x,y,z,t) \end{pmatrix} = \sum_{\ell=1}^{\infty} \begin{pmatrix} T_\ell(x,y,t) \\ w_\ell(x,y,t) \\ Q_\ell(x,y,t) \end{pmatrix} Z'_\ell(z), \tag{2.7}$$

where the orthogonal functions $Z_\ell(z)$ and their derivatives $Z'_\ell(z) = (d/dz)Z_\ell(z)$ compose the vertical structure. The specific form of the vertical modes can be found by solving the Sturm-Liouville problem

$$\left(\frac{d}{dz} - 1\right)\frac{d}{dz}Z_\ell(z) = -\left(\frac{\ell^2\pi^2}{z_T^2} + \frac{1}{4}\right)Z_\ell(z), \tag{2.8}$$

$$Z'_\ell(0) = Z'_\ell(z_T) = 0, \tag{2.9}$$

which arose from imposed conditions during the separation process. The separation constants $\bar{c}_\ell^2$ turn out to be the square of the internal pure gravity wave phase speeds corresponding to the vertical internal modes $\ell = 1, 2, ...$ and are defined as

$$\frac{R\Gamma}{\bar{c}_\ell^2} = \left(\frac{\ell^2\pi^2}{z_T^2} + \frac{1}{4}\right)$$

The first internal vertical mode $Z_1(z)$ and its derivative $Z'_1(z)$ can be written in normalized form as

$$Z_1(z) = \left(\frac{\pi^2}{z_T^2} + \frac{1}{4}\right)^{-\frac{1}{2}} e^{(z-z_m)/2}\left[\frac{z_T}{2\pi}\sin\left(\frac{\pi z}{z_T}\right) - \cos\left(\frac{\pi z}{z_T}\right)\right], \tag{2.10}$$

and

$$Z'_1(z) = \left(1 + \frac{z_T^2}{4\pi^2}\right)^{\frac{1}{2}} e^{(z-z_m)/2}\sin\left(\frac{\pi z}{z_T}\right), \tag{2.11}$$

where z_m, the level at which $Z'_1(z)$ reaches its maximum value, is given by $\pi z_m/z_T = \pi + \tan^{-1}(-2\pi/z_T)$, which, for $z_T \approx 1.619$, turns out to be $z_m \approx 0.5803z_T$. For later

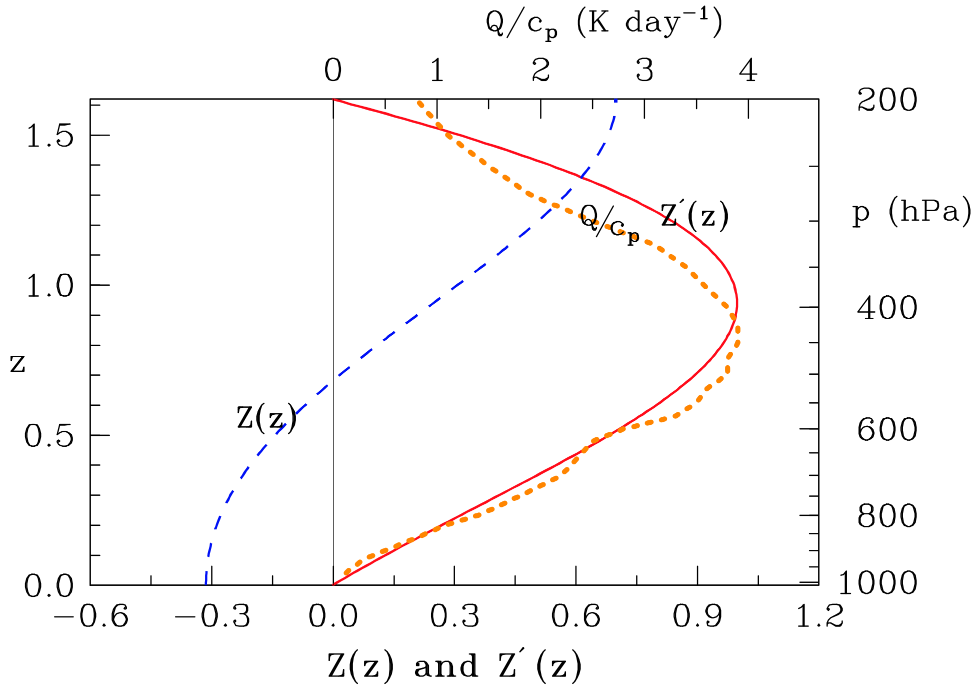

Figure 2.1: The curves labeled $Z(z)$ (blue,dashed) and $Z'(z)$ (red,solid)—interpreted using the lower scale—are the vertical structure functions defined by (2.10) and (2.11). The curve labeled Q/c_p (orange,dotted)—interpreted using the upper scale—is the 120-day mean vertical profile of heating rate for the western Pacific warm pool, as determined by Johnson and Ciesielski (2000). Note that $Z'(z)$ reaches its maximum at $p \approx 395$ hPa.

convenience, the normalization chosen in (2.10) and (2.11) yields $Z_1'(z_m) = 1$. Relabeling these functions to be $Z(z) \equiv Z_1(z)$ and $Z'(z) \equiv Z_1'(z)$, their plots are shown in Fig. 2.1. Also shown in Fig. 2.1 is the 120-day mean (November 1992 – February 1993) vertical profile of the heating rate for the western Pacific warm pool (Johnson and Ciesielski, 2000). The observed mean profile of Q/c_p has a peak value of approximately 4 K day^{-1}, and its shape is closely approximated by $Z'(z)$, the vertical structure of our constructed Q/c_p. Based on this good approximation we make the simplifying assumption that the heating excites only the first vertical internal mode. In doing so, the infinite series is truncated to this single mode. The variable expressions in (2.7) can then be rewritten as

$$\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ \phi(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \hat{u}(x,y,t) \\ \hat{v}(x,y,t) \\ \hat{\phi}(x,y,t) \end{pmatrix} Z(z), \qquad \begin{pmatrix} T(x,y,z,t) \\ w(x,y,z,t) \\ Q(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \hat{T}(x,y,t) \\ \hat{w}(x,y,t) \\ \hat{Q}(x,y,t) \end{pmatrix} Z'(z), \tag{2.12}$$

where the $(\hat{\ })$'s on the transformed variables denote their lack of z-dependence. The gravity wave speed that corresponds to the first internal vertical mode is

$$\bar{c} \equiv \bar{c}_1 = \frac{(R\Gamma)^{1/2}}{\left[\frac{\pi^2}{z_T^2} + \frac{1}{4}\right]^{1/2}} \approx 41.25 \text{ m s}^{-1}. \tag{2.13}$$

Over the 120-day observational period there were two MJO passages (Lin and Johnson, 1996; Yanai et al., 2000). During these two periods of enhanced convection the shape of the vertical profile of Q/c_p was very similar to the shape of the time mean profile shown in Fig. 2.1, but the peak values were considerably larger, 10 K day^{-1} for the first MJO and 16 K day^{-1} for the second. Thus, for the vertical profile of heating at the time of peak convective activity during the passage of an MJO, our model uses the $Z'(z)$ profile shown in Fig. 2.1, but scaled so the peak value is 12 K day^{-1} (chosen as an intermediate value between the two MJO peak values) rather than 4 K day^{-1}. Assuming now that u, v, ϕ, T, w, Q have the separable forms given in (2.12) we can convert (2.6) into the following system for the horizontal structure functions $\hat{u}, \hat{v}, \hat{\phi}, \hat{T}, \hat{w}$:

$$\begin{aligned} \frac{\partial \hat{u}}{\partial t} - \beta y \hat{v} + \frac{\partial \hat{\phi}}{\partial x} &= -\alpha \hat{u}, \\ \frac{\partial \hat{v}}{\partial t} + \beta y \hat{u} + \frac{\partial \hat{\phi}}{\partial y} &= -\alpha \hat{v}, \\ \hat{\phi} &= R\hat{T}, \\ \frac{\partial \hat{u}}{\partial x} + \frac{\partial \hat{v}}{\partial y} - \left(\frac{\pi^2}{z_T^2} + \frac{1}{4}\right)\hat{w} &= 0, \\ \frac{\partial \hat{T}}{\partial t} + \Gamma \hat{w} &= -\alpha \hat{T} + \frac{\hat{Q}}{c_p}. \end{aligned} \tag{2.14}$$

In section 2.4 we present an analytical solution of (2.14). The solution can be considered as the primitive equation generalization of the simplest MJO model involving the first baroclinic mode response to a moving planetary scale heat source under the long wave approximation (Chao, 1987).