convective regions, while it is small, and the corresponding PV anomaly is small, for weakly raining, zonally narrow, fast moving convective regions. For the case shown in the bottom panel of Fig. 3.2, $a_0 = 1250$ km and $c = 5$ m s^{-1} so that $\tau_p = 69.4$ hr, while $\bar{c} = 41.25$ m s^{-1} and $Q_0/c_p = 12$ K day^{-1} so that $\tau_c = 11.9$ hr, resulting in $\tau_p/\tau_c = 5.8$.

Finally it is worth noting that the solution (3.10) can be used to obtain a rough indication of when nonlinear effects are expected to become important. For example, at 850 hPa and $x = -a_0$, and for $\alpha\tau_p \approx 0.72$, (3.10) becomes $q \approx -0.15(\tau_p/\tau_c)\beta y \exp[-(y - y_0)^2/b_0^2]$. As τ_p/τ_c becomes larger we expect that the potential vorticity anomaly q (and hence the relative vorticity $\partial v/\partial x - \partial u/\partial y$) will eventually become larger than βy, in which case the factor βy in the last term of (3.3) and (3.5) should be replaced by the total potential vorticity and the factor βy in the divergence term of (3.1) should be replaced by the absolute vorticity (a nonlinear effect). Although the inclusion of nonlinear terms in the case $\tau_p/\tau_c = 5.8$ should not lead to qualitative changes from the linear solution, larger values of τ_p/τ_c that cause intense westerly wind bursts (~ 15 m s^{-1}) should be expected to have significant nonlinear effects.

3.3 Invertibility Principle

We now show that the wind and mass fields in the wake of a convective envelope can be approximately recovered from the PV through a simple invertibility principle. The argument begins by returning to the PV anomaly expression, (3.4), written in the form

$$\nabla^2\psi + \frac{\beta y}{R\Gamma}\left(\frac{\partial}{\partial z} - 1\right)\frac{\partial \phi}{\partial z} = q, \tag{3.12}$$

where ψ is the streamfunction for the rotational part of the flow and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the horizontal Laplacian operator. No approximation has been made in writing the relative vorticity anomaly as $\nabla^2\psi$. Equation (3.12) can be converted into an invertibility principle by introducing an approximate balance relation between the wind and mass fields, i.e., a relation between ψ and ϕ. To derive this relationship start by differentiating the horizontal momentum equations, (2.6), along their respective orientations to get the divergence equation,

$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)-\frac{\partial}{\partial x}(\beta y v)+\frac{\partial}{\partial y}(\beta y u)+\nabla^2\phi=-\alpha\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right). \tag{3.13}$$

In nearly balanced flows the divergence would be expected to be small. Furthermore, the magnitude of the dissipation and time rate of change of the divergence should be even smaller. Based on this the divergence terms that appear in (3.13) will be neglected. By the same rationale, it seems natural to approximate the total wind field by its rotational component. Using ψ, which was introduced above, this is done by writing

$$u \approx u_\psi \equiv -\frac{\partial \psi}{\partial y}, \quad v \approx v_\psi \equiv \frac{\partial \psi}{\partial x}. \tag{3.14}$$

Use of these two approximations in (3.13) reduces it to

$$-\frac{\partial}{\partial x}\left(\beta y\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(\beta y\frac{\partial \psi}{\partial y}\right)+\nabla^2\phi=0, \tag{3.15}$$

which is known as the linear balance relation and can be written neatly as,

$$\nabla\cdot(\beta y\nabla\psi)=\nabla^2\phi. \tag{3.16}$$

If we make the additional assumption that βy is slowly varying in space compared to ψ, (3.16) becomes

$$\nabla^2(\beta y\psi)=\nabla^2\phi. \tag{3.17}$$

Imposing the condition $\phi-\beta y\psi\rightarrow 0$ as $y\rightarrow\pm\infty$ on (3.17), it immediately follows that

$$\phi=\beta y\psi. \tag{3.18}$$

The validity of this last step is demonstrated in Appendix H. Returning to (3.12) and using the balance relation (3.18) to express ϕ in terms of ψ we arrive at

$$\nabla^2\psi+\frac{\beta^2y^2}{R\Gamma}\left(\frac{\partial}{\partial z}-1\right)\frac{\partial\psi}{\partial z}=q. \tag{3.19}$$

Concerning the boundary conditions for (3.19), we here use the simple conditions $\psi\rightarrow 0$ as $y\rightarrow\pm\infty$ and $\partial\psi/\partial z=0$ at $z=0$, z_T, noting that the lower condition could easily be generalized to allow temperature variations at $z = 0$. Equation (3.19), together with its boundary conditions, constitute an invertibility principle. From this principle we can recover ψ, and then compute the rotational wind field from (3.14) and the mass field from (3.18). Although the invertibility principle (3.19) can be used in conjunction with an approximate PV equation to form a closed balanced theory (see section 3.4), our discussion in this section is limited to the use of (3.19) as a diagnostic aid for interpretation of the primitive equation model results.

To solve (3.19) we follow the arguments of sections 2.2 and 2.4 by separating off the vertical dependence, assuming the solution is steady in a reference frame moving eastward at speed c, and then taking the Fourier transform in the zonal direction. We find that (3.19) reduces to

$$\epsilon^{\frac{1}{2}}\left(\frac{d^2}{d\hat{y}^2} - \hat{y}^2\right)\hat{\psi}_m - m^2\hat{\psi}_m = a^2\hat{q}_m, \tag{3.20}$$

where, as before, $\hat{y} = \epsilon^{\frac{1}{4}}(y/a)$ is the dimensionless meridional coordinate. We now solve (3.20) by transforming in $\hat{y}$. The mathematical apparatus is simpler than in section 2.4 where we introduced the vector inner product (2.25), and the vector transform pair (2.36)–(2.37). To solve the scalar equation (3.20) we can use the scalar transform pair

$$\hat{\psi}_{mn} = \int_{-\infty}^{\infty}\hat{\psi}_m(\hat{y})\mathcal{H}_n(\hat{y})d\hat{y}, \tag{3.21}$$

$$\hat{\psi}_m(\hat{y}) = \sum_{n=0}^{\infty}\hat{\psi}_{mn}\mathcal{H}_n(\hat{y}). \tag{3.22}$$

A transform pair similar to (3.21) and (3.22) also exists for $\hat{q}_{mn}$, $\hat{q}_m(\hat{y})$. Note that (3.21) can be obtained through multiplication of (3.22) by $\mathcal{H}_{n'}(\hat{y})$, followed by integration over $\hat{y}$ and use of the orthogonality relation (2.35). Multiplying (3.20) by $\mathcal{H}_n(\hat{y})$ and integrating over $\hat{y}$ (i.e., taking the Hermite transform of (3.20)) we obtain

$$\hat{\psi}_{mn} = -\frac{a^2\hat{q}_{mn}}{m^2 + \epsilon^{\frac{1}{2}}(2n+1)}. \tag{3.23}$$

In the derivation of (3.23) we have used two integrations by parts (with vanishing boundary terms) and the fact that $\mathcal{H}_n(\hat{y})$ is an eigenfunction of the operator $(d^2/d\hat{y}^2 - \hat{y}^2)$, i.e.,

$(d^2/d\hat{y}^2 - \hat{y}^2)\mathcal{H}_n(\hat{y}) = -(2n+1)\mathcal{H}_n(\hat{y})$. The simplicity of the spectral form of the invertibility principle (3.23) is in fact due to this eigenfunction property of $\mathcal{H}_n(\hat{y})$.

Combining the inverse Hermite transform (3.22), inverse Fourier transform defined in (2.19), and the assumed vertical structure, we obtain

$$\psi(\xi, y, z) = Z(z) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \hat{\psi}_{mn} \mathcal{H}_n(\hat{y}) e^{im\xi/a}, \tag{3.24}$$

so that the physical space streamfunction field can be plotted by substituting (3.23) in (3.24) and then numerically evaluating the sums over m and n. However, we would like to examine more than just the $\psi(\xi, y, z)$ field, in particular the $u_\psi(\xi, y, z)$, $v_\psi(\xi, y, z)$, and $\phi(\xi, y, z)$ fields, all of which can be determined from the ψ field. Recalling that $u_\psi = -\partial\psi/\partial y$, $v_\psi = \partial\psi/\partial\xi$, and $\phi = \beta y \psi$, and using the recurrence and derivative relations (2.30) and (2.31), we have

$$\begin{pmatrix} u_\psi(\xi, y, z) \\ v_\psi(\xi, y, z) \\ \phi(\xi, y, z) \end{pmatrix} = Z(z) \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \frac{\hat{\psi}_{mn}}{a} \begin{pmatrix} U_{mn}(\xi, \hat{y}) \\ V_{mn}(\xi, \hat{y}) \\ \Phi_{mn}(\xi, \hat{y}) \end{pmatrix} \tag{3.25}$$

where

$$\begin{pmatrix} U_{mn}(\xi, \hat{y}) \\ V_{mn}(\xi, \hat{y}) \\ \Phi_{mn}(\xi, \hat{y}) \end{pmatrix} = \begin{pmatrix} \epsilon^{\frac{1}{4}} \left[\left(\frac{n+1}{2}\right)^{\frac{1}{2}} \mathcal{H}_{n+1}(\hat{y}) - \left(\frac{n}{2}\right)^{\frac{1}{2}} \mathcal{H}_{n-1}(\hat{y}) \right] \\ im\mathcal{H}_n(\hat{y}) \\ \bar{c}\epsilon^{\frac{1}{4}} \left[\left(\frac{n+1}{2}\right)^{\frac{1}{2}} \mathcal{H}_{n+1}(\hat{y}) + \left(\frac{n}{2}\right)^{\frac{1}{2}} \mathcal{H}_{n-1}(\hat{y}) \right] \end{pmatrix} e^{im\xi/a}, \tag{3.26}$$

which are readily seen to be the Rossby wave approximations of (2.28). To summarize, once $\hat{q}_{mn}$ is known, we can compute $\hat{\psi}_{mn}$ from (3.23), and then $u_\psi(\xi, y, z)$, $v_\psi(\xi, y, z)$, $\phi(\xi, y, z)$ via numerical evaluation of (3.25). For $\hat{q}_{mn}$ in (3.23) we use the Rossby wave contribution to the total $q(\xi, y, z)$ field given in (3.7), i.e., we set $\hat{q}_{mn} = \hat{q}_{mn0}$, the latter of which is defined in (3.8). This is a reasonable approximation since the total $q(\xi, y, z)$ field is dominated by the Rossby wave contribution, as argued in section 3.1. For Fig. 3.3 the center of convective forcing is set to $y_0 = 0$ km. The top panel displays the Rossby PV field for reference. The second panel contains the flow field, $u_\psi(\xi, y, z)$, $v_\psi(\xi, y, z)$, and $\phi(\xi, y, z)$, attained from inverting the Rossby PV with (3.19). Lastly, the primitive

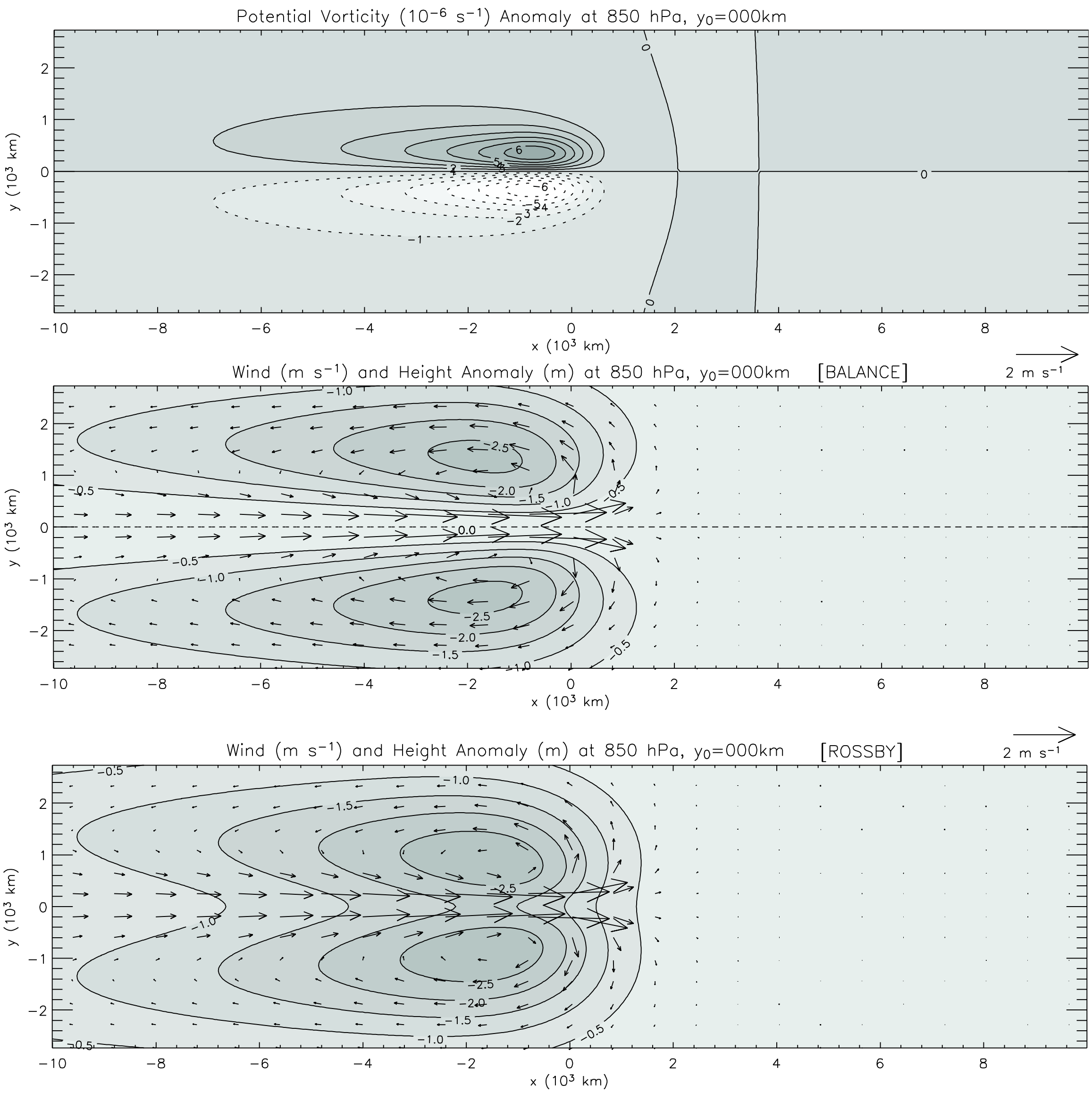

Figure 3.3: The top panel shows the Rossby wave contribution to the $q(\xi, y, z)$ field, while the second panel shows the $u_\psi(\xi, y, z)$, $v_\psi(\xi, y, z)$, $\phi(\xi, y, z)$ fields resulting from the solution of the invertibility principle (3.19) when its right hand side is given by the PV field shown in the top panel. The bottom panel is the Rossby contribution to the primitive equation circulation, included here for comparison with the middle panel. This figure is for $y_0 = 0$ km.

equation Rossby component flow field is shown for comparison with the flow field recovered from the invertibility principle. The same fields are shown for $y_0 = 450$ km in Fig. 3.4. A comparison of the circulation fields for each figure shows that the balanced model mass field and rotational wind field are fairly good approximations of the Rossby wave contributions to the primitive equation results, the most apparent difference being that the zonal pressure

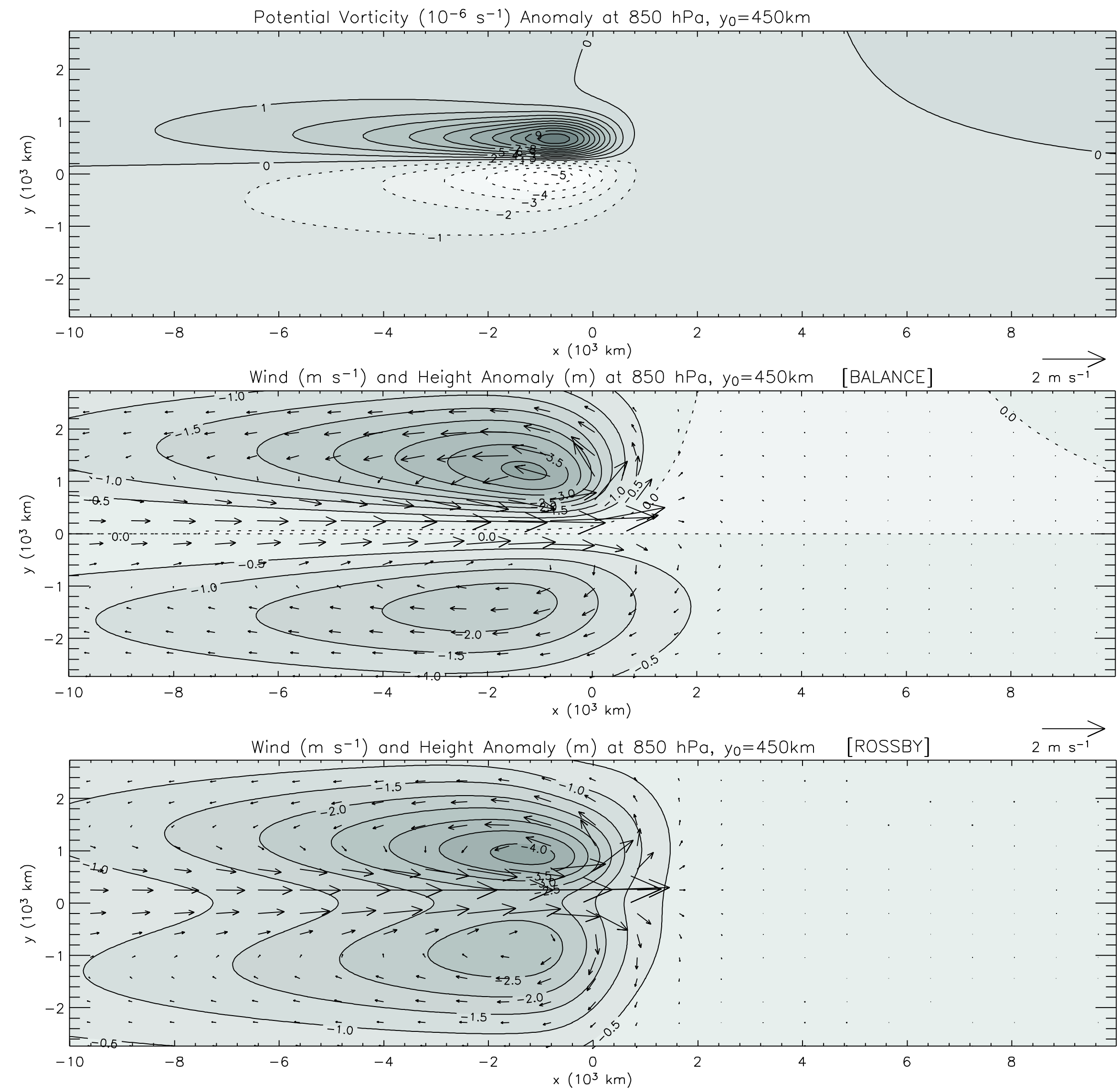

Figure 3.4: Same as Fig. 3.3 but for $y_0 = 450$ km.

gradient force at the equator in the primitive equation model is not reproduced in the solutions of the invertibility principle. Mathematically this is expected from the balance relationship, (3.18), which dictates that $\phi = 0$ at the equator due to the βy factor. However, an advantage of this balance relation should be noted as well. Compared to geostrophic balance, (3.18) is preferred because it is valid over the entire equatorial domain including the equator. In other words, the streamfunction derivative relationships for the rotational wind components, (3.14), can be evaluated at the equator, while the geostrophic wind relations have a singularity there due to vanishing of the Coriolis force. It's worthwhile to point out the distinction between the wind fields computed in Quasi-Geostrophic (QG) theory and those computed under our equatorial β-plane approximate balance theory. In QG theory the horizontal wind is considered to be composed of geostrophic and ageostrophic components. After determining the geopotential by inverting the PV, the geostrophic component of the wind field is recovered from the geopotential. Under our equatorial balance theory we consider the wind to be composed of rotational and divergent components. In this case the PV is inverted to determine the streamfunction, which is then used to recover the rotational wind components. Having pointed out the difference, some similarity can also be seen. Differentiating (3.18) independently in the zonal direction and in the meridional direction, then using (3.14) yields

$$\beta y v_\psi = \frac{\partial \phi}{\partial x}, \quad \beta y u_\psi = -\frac{\partial \phi}{\partial y} + \beta \psi. \tag{3.27}$$

The relationship involving the meridional rotational wind and zonal pressure gradient is evocative of geostrophic balance, though the zonal rotational wind and meridional pressure gradient expression involves an additional term.

As a side note, the invertibility principle could have been expressed in terms of ϕ instead of ψ using (3.18), though inversion would probably be more difficult if possible. Interestingly, it has been suggested that, for some flow patterns, inversion of PV for ψ may give more accurate forecast results than inverting PV for ϕ (Phillips, 2000).

3.4 Proposed Prognostic Equation

It was noted that the invertibility principle (3.19) could be used as part of a complete balanced theory, i.e., it can be used in conjunction with an approximate PV equation. For example, in the context of inviscid, adiabatic flow we could consider the system

$$\frac{\partial q}{\partial t} + \beta \frac{\partial \psi}{\partial x} = 0, \tag{3.28}$$

$$\nabla^2 \psi + \frac{\beta^2 y^2}{R\Gamma} \left(\frac{\partial}{\partial z} - 1 \right) \frac{\partial \psi}{\partial z} = q, \tag{3.29}$$