Appendix B

VERTICAL NORMAL MODE TRANSFORM

Separation of the horizontal and vertical structure in (2.6) is accomplished by designing an integral transform that will eliminate the vertical derivatives (Fulton and Schubert, 1985). The kernel ($\Psi_\ell(z)$) of this transform is initially unspecified and is determined by solving the eigenvalue-eigenfunction problem it will be required to satisfy. Define $\phi_\ell(x, y, t)$, the vertical integral transform of $\phi(x, y, z, t)$ by

$$\phi_\ell(x, y, t) = \int_0^{z_T} \phi(x, y, z, t)\Psi_\ell(z)\, e^{-z/2}\, dz, \tag{B.1}$$

where $e^{-z/2}$ is the weight.

When our continuously stratified, compressible atmosphere is in the basic resting state it is completely described by the hydrostatic equation, continuity equation, and unforced thermodynamic equation,

$$\frac{\partial \phi}{\partial z} = RT, \tag{B.2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} - w = 0, \tag{B.3}$$

$$\frac{\partial T}{\partial t} + \Gamma w = 0. \tag{B.4}$$

These can be combined into a single equation that contains all the vertical derivatives associated with (2.6),

$$e^z \frac{\partial}{\partial z}\left[\frac{e^{-z}}{R\Gamma}\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right] - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0. \tag{B.5}$$

To find $\Psi_\ell(z)$, the intrinsic vertical modes, apply the constructed transform to (B.5),

$$\int_0^{z_T} e^z \frac{\partial}{\partial z}\left[\frac{e^{-z}}{R\Gamma}\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right]\Psi_\ell(z)\, e^{-z/2}\, dz - \int_0^{z_T}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\Psi_\ell(z)\, e^{-z/2}\, dz = 0. \tag{B.6}$$

Assuming transforms $u_\ell(x,y,t)$ and $v_\ell(x,y,t)$ similar to (B.1) exist for $u(x,y,z,t)$ and $v(x,y,z,t)$, then (B.6) becomes

$$\underbrace{\int_0^{z_T} e^z \frac{\partial}{\partial z}\left[\frac{e^{-z}}{R\Gamma}\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right]\Psi_\ell(z)\, e^{-z/2}\, dz}_{I_1} - \left(\frac{\partial u_\ell}{\partial x} + \frac{\partial v_\ell}{\partial y}\right) \;=\; 0. \tag{B.7}$$

The horizontal divergence has been transformed so only the first term (denoted by I_1 for convenience) needs attention. Using integration by parts on I_1 twice gives

$$I_1 = \int_0^{z_T} \frac{\partial \phi}{\partial t}\frac{\partial}{\partial z}\left[\frac{e^{-z}}{R\Gamma}\frac{\partial}{\partial z}\left(\Psi_\ell(z)e^{z/2}\right)\right]dz - \left[\frac{e^{-z}}{R\Gamma}\left\{\frac{\partial}{\partial z}\left(\Psi_\ell(z)e^{z/2}\right)\frac{\partial \phi}{\partial t} - \Psi_\ell(z)e^{z/2}\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right)\right\}\right]_{z=0}^{z=z_T}. \tag{B.8}$$

Section 2.1 gave upper and lower model boundary conditions $w = 0$ at $z = z_T, 0$. Enforcing these for (B.4) together with (B.2) implies that $\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t}\right) = 0$ at $z = z_T, 0$, which eliminates the last term in (B.8),

$$I_1 = \int_0^{z_T} \frac{\partial \phi}{\partial t}\left\{e^{z/2}\frac{\partial}{\partial z}\left[\frac{e^{-z}}{R\Gamma}\frac{\partial}{\partial z}\left(\Psi_\ell(z)e^{z/2}\right)\right]\right\}e^{-z/2}dz - \left[\frac{e^{-z}}{R\Gamma}\frac{\partial \phi}{\partial t}\left\{\frac{\partial}{\partial z}\left(\Psi_\ell(z)e^{z/2}\right)\right\}\right]_{z=0}^{z=z_T}. \tag{B.9}$$

Now impose conditions on the two terms contained in $\{\}$ (which we are allowed to do as $\Psi_\ell(z)$ is yet unspecified) so that I_1 will yield the transform of $\partial\phi/\partial t$, as desired. This will in fact occur if the entire second term disappears (require second term in $\{\}$ to vanish at boundaries), and if the entire first term is proportional to $\int_0^{z_T}(\partial\phi/\partial t)\Psi_\ell(z)e^{-z/2}dz$ (insist first term in $\{\}$ is proportional to $\Psi_\ell(z)$). These conditions,

$$e^{z/2}\frac{d}{dz}\left[\frac{e^{-z}}{R\Gamma}\frac{d}{dz}\left(\Psi_\ell(z)e^{z/2}\right)\right] \;=\; -\frac{1}{\bar{c}_\ell^2}\Psi_\ell(z), \tag{B.10}$$

$$\left.\frac{d}{dz}\left(\Psi_\ell(z)e^{z/2}\right)\right|_{z=0,z_T} \;=\; 0, \tag{B.11}$$

form a Sturm-Liouville eigenproblem. The set of eigenfunction solutions ($\Psi_\ell(z)$) are known to be orthogonal and complete, and can thus be used as a basis for the inverse transform. An example of the inverse transform is

$$\phi(x,y,z,t) = \sum_{\ell=1}^{\infty}\phi_\ell(x,y,t)\Psi_\ell(z)e^{z/2}, \tag{B.12}$$

which forms a transform pair with (B.1). Define the total vertical structure in (B.12) as

$$Z_\ell(z) = \Psi_\ell(z)e^{z/2}, \tag{B.13}$$

and use it to rewrite (B.10) as

$$\left(\frac{d}{dz}-1\right)\frac{d}{dz}Z_\ell(z) = -\frac{R\Gamma}{\bar{c}_\ell^2}Z_\ell(z). \tag{B.14}$$

Substituting the known general solution for the internal modes ($Z_\ell(z) = e^{z/2}[A\cos(\gamma_\ell z) + B\sin(\gamma_\ell z)]$, where $\gamma_\ell^2 = [R\Gamma/\bar{c}_\ell^2 - 1/4]$ and $\ell = 1, 2, 3, ...$) in (B.14) and applying the boundary conditions determines the eigenvalues,

$$\frac{1}{\bar{c}_\ell^2} = \frac{1}{R\Gamma}\left(\frac{\ell^2\pi^2}{z_T^2}+\frac{1}{4}\right). \tag{B.15}$$

Using (B.15) in (B.14) and (B.13) in (B.11) the Sturm-Liouville problem takes the form seen in section 2.2,

$$\left(\frac{d}{dz}-1\right)\frac{d}{dz}Z_\ell(z) = -\left(\frac{\ell^2\pi^2}{z_T^2}+\frac{1}{4}\right)Z_\ell(z), \tag{B.16}$$

$$\left.\frac{d}{dz}Z_\ell(z)\right|_{z=0} = \left.\frac{d}{dz}Z_\ell(z)\right|_{z=z_T} = 0. \tag{B.17}$$

From here $Z_\ell(z)$ for any $\ell = 1, 2, 3, ...$ can easily be found from (B.16) and (B.17), though as discussed in section 2.2, only $Z(z) \equiv Z_1(z)$ is necessary for this study. As a result the inverse transform for the variables $u(x, y, z, t)$, $v(x, y, z, t)$, and $\phi(x, y, z, t)$ reduces to

$$\begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ \phi(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \hat{u}(x,y,t) \\ \hat{v}(x,y,t) \\ \hat{\phi}(x,y,t) \end{pmatrix} Z(z), \tag{B.18}$$

where the $(\hat{\ })$'s denote the lack of vertical dependence. Use of (B.18) in the hydrostatic equation for $\phi(x, y, z, t)$ shows that to be consistent, T must have vertical structure given by $Z'(z) \equiv (d/dz)Z(z)$. Substituting this structure for T in the forced thermodynamic equation shows w and Q must have this structure as well. In summary,

$$\begin{pmatrix} T(x,y,z,t) \\ w(x,y,z,t) \\ Q(x,y,z,t) \end{pmatrix} = \begin{pmatrix} \hat{T}(x,y,t) \\ \hat{w}(x,y,t) \\ \hat{Q}(x,y,t) \end{pmatrix} Z'(z), \tag{B.19}$$

The expressions given in (B.18) and (B.19) give us the ability to separate the vertical and horizontal structure in equations (2.6).