Chapter 3

POTENTIAL VORTICITY ASPECTS

3.1 Primitive Equation Potential Vorticity

To derive the potential vorticity principle associated with (2.6) we first cross-differentiate the horizontal momentum equations to obtain the vorticity equation

$$\left(\frac{\partial}{\partial t}+\alpha\right)\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+\beta y\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\beta v=0, \tag{3.1}$$

and then combine the hydrostatic, continuity, and thermodynamic equations in such a way as to eliminate T and w, which results in

$$\left(\frac{\partial}{\partial t}+\alpha\right)\left(\frac{\partial}{\partial z}-1\right)\frac{\partial \phi}{\partial z}-R\Gamma\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=\kappa\left(\frac{\partial}{\partial z}-1\right)Q. \tag{3.2}$$

Then, eliminating the horizontal divergence between (3.1) and (3.2) we obtain the potential vorticity principle

$$\frac{\partial q}{\partial t}+\beta v=-\alpha q+\frac{\beta y}{c_p\Gamma}\left(\frac{\partial}{\partial z}-1\right)Q, \tag{3.3}$$

where

$$q=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}+\frac{\beta y}{R\Gamma}\left(\frac{\partial}{\partial z}-1\right)\frac{\partial \phi}{\partial z} \tag{3.4}$$

is the potential vorticity anomaly. From here it is an easy step to obtain the PV principle that is associated with the shallow water system (2.14). Refer back to the separable form (2.12) used in solving the primitive equation system in chapter 2. Using the variable expressions to separate off the vertical structure from (3.3) and (3.4), we arrive at the shallow water PV principle,

$$\frac{\partial \hat{q}}{\partial t}+\beta\hat{v}=-\alpha\hat{q}-\frac{\beta y}{\bar{c}^2}\kappa\hat{Q}, \tag{3.5}$$

where

$$\hat{q} = \frac{\partial \hat{v}}{\partial x} - \frac{\partial \hat{u}}{\partial y} - \frac{\beta y}{\bar{c}^2}\hat{\phi} \tag{3.6}$$

is the shallow water PV anomaly. Following the general scheme used to solve the primitive equations, a spectral space series expression for PV can be derived. First perform a zonal Fourier transform (2.19) on (3.6), next use (2.37) to expand the variables in terms of meridional normal modes. The physical space PV can then be computed from its spectral components by

$$q(\xi, y, z) = Z(z) \sum_{m=-\infty}^{\infty} \sum_{n=-1}^{\infty} \sum_{r} \hat{q}_{mnr} \mathcal{H}_n(\hat{y}) e^{im\xi/a}, \tag{3.7}$$

where

$$\hat{q}_{mnr} = A_{mnr} \left(\frac{m^2 - \epsilon \hat{\nu}_{mnr}^2}{a \hat{\nu}_{mnr}} \right) \hat{\eta}_{mnr}. \tag{3.8}$$

According to (3.7) and (3.8), the physical space potential vorticity field is a superposition of the potential vorticity associated with all wave types except for the Kelvin wave. To see this recall that the Kelvin wave dispersion relation is $\epsilon^{\frac{1}{2}} \hat{\nu} = m$. Using this to evaluate the numerator in (3.8) we see that it is identically zero for all m, so the Kelvin wave contributes nothing to the sum. Although equatorial β-plane dynamics differ from midlatitude f-plane dynamics in the sense that inertia-gravity waves have zero potential vorticity on the midlatitude f-plane but not on the equatorial β-plane, the contribution of the equatorial β-plane inertia-gravity waves to the potential vorticity tends to be quite small. This is especially true for the higher zonal wavenumbers because for inertia-gravity waves $m^2 - \epsilon \hat{\nu}_{mnr}^2$ decreases and $|\hat{\nu}_{mnr}|$ increases as m increases. Recall that for $y_0 = 0$, the mixed Rossby-gravity wave is not excited. Even when it is excited ($y_0 \neq 0$), its contribution to the total circulation, and furthermore the PV, is small. It should be evident then that the total PV field is almost entirely due to the Rossby response, which is what we will use in what follows. It is interesting to interpret the flow patterns associated with the moving heat source in terms of a PV wake. For this interpretation the y-factor in the last term of (3.3) or (3.5) plays a crucial role. It causes the source term Q to be ineffective at generating a PV anomaly at the equator but maximizes the PV response near the poleward edges of the

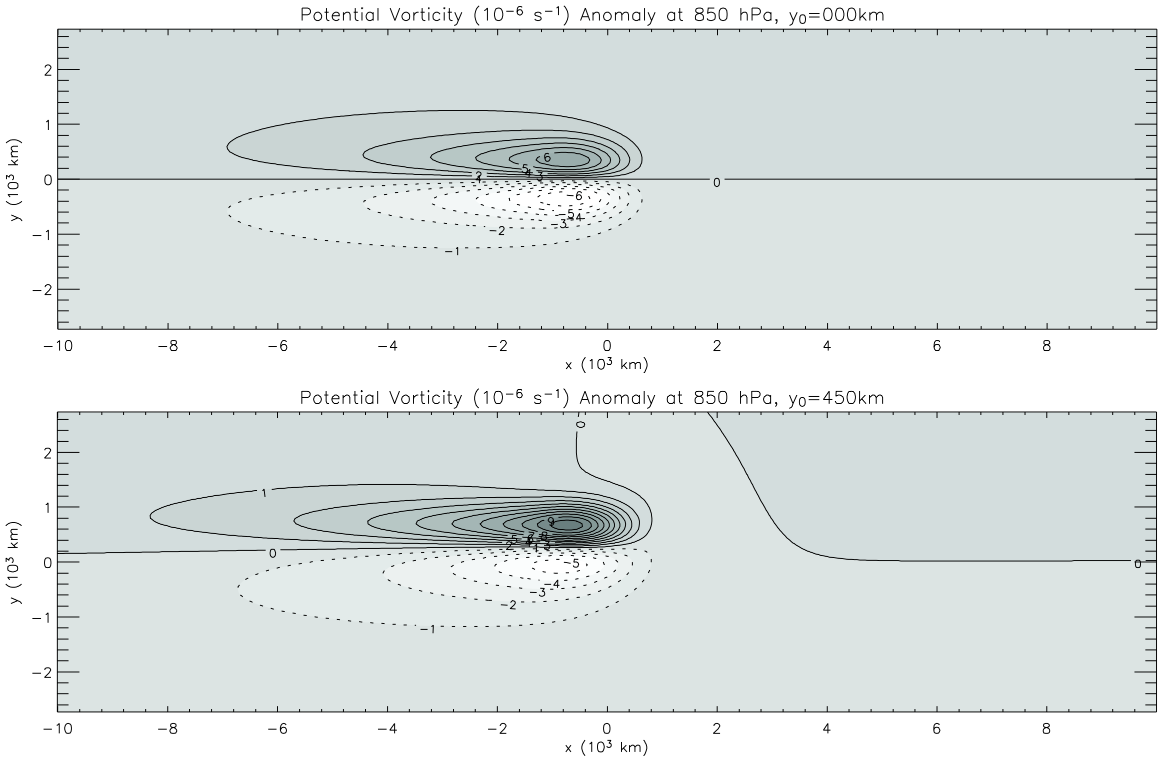

Figure 3.1: Both panels show the 850 hPa Rossby wave potential vorticity anomaly q, with a contour interval of 1×10^{-6} s^{-1}. Top panel $y_0 = 0$ km, bottom panel $y_0 = 450$ km.

heat source. In this manner a moving equatorial heat source can produce two ribbons of lower tropospheric PV anomaly, a positive one off the equator in the northern hemisphere and a negative one off the equator in the southern hemisphere, with oppositely signed PV anomalies in the upper troposphere.

Examining the top panel of Fig. 3.1 and the second panel of Fig. 2.5, the PV anomaly patterns and the associated equatorial trough shear zones are zonally elongated because of the eastward movement of the convective envelope and the equatorward advection of the basic state PV in the wake of the convective envelope. As mentioned in section 2.5, shifting the center of convection north of the equator produces very little change in the response east of the forcing (compare last panel Figs. 2.5 and 2.7), yet the response west of the forcing (second panel Figs. 2.5, 2.7) becomes biased to the northern hemisphere. The behavior to the west of the forcing can be seen from the perspective of PV in Fig. 3.1. This plot shows the PV anomaly is nearly twice as strong in the northern hemisphere when $y_0 = 450$ km.