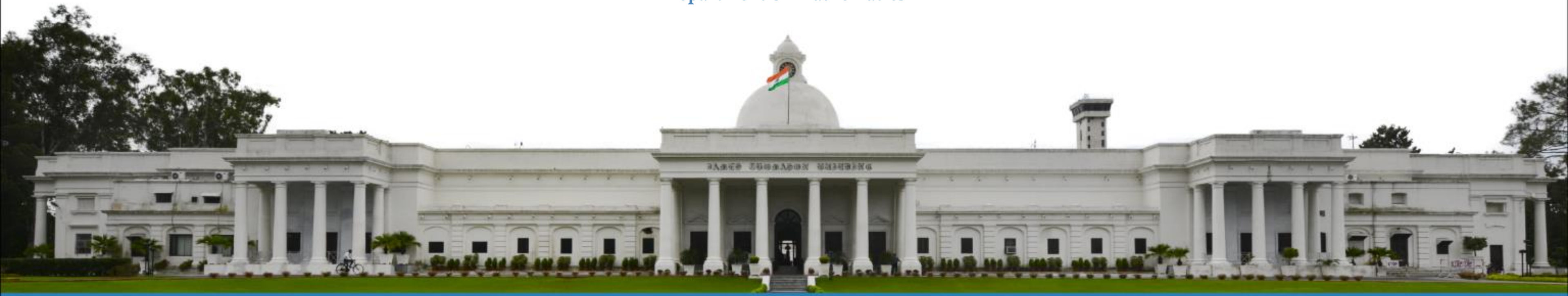
IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 3.1 : Introduction to Continuous Time Models

Dr. Ameeya Kumar Nayak
Department of Mathematics



Contents:

- Introduction to Continuous Time Modeling.
- Introduction to Differential Equations.
- Solution of a Differential Equation.
- Geometrical Meaning of a Differential Equation.

Introduction to Continuous Time Modeling:

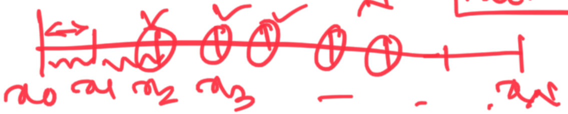
- In previous lectures, we have studied and analyzed the discrete time models.
- If you observe the natural phenomena's, all of them are depending on the time continuously. Then, why did we use the discrete time models?
- There is a very close link between discrete time and continuous time models. If numerical methods (finite difference, finite element etc.) are applied on continuous time models then we get discrete time models.
- As, you are aware of using the numerical methods in the cases (especially) where finding the analytical solution is a cumbersome or impossible task.
- But this is not the only case where discrete time models are being formulated. If the data are given at discrete points (e.g. population data given for every 10 years interval), then also discrete time modeling is done.

Introduction to Continuous Time Modeling:

- Thus there are two important reasons to use the discrete time models.

  1. when data points are at finite distances,
  2. to solve the differential equations .

$\frac{x_N - x_0}{N}$ $\boxed{x_i = x_0 + ih}$



- Are discrete time models not able to show the actual behavior of system? No!. Discrete time models have their own limitations.

- The major disadvantage of using the difference equations is the dilemma of choosing the type/order of difference equation. It's the matter of **ACCURACY** !! When any differential equation is discretized to form a difference equation, then we must care about accuracy. The accuracy of solution generally increases with the increase in the order of difference equation. But we must take care of **COMPUTATIONAL COST** also. (trade off between accuracy and number of computations.)

Introduction to Continuous Time Modeling:

- To overcome the limitations of discrete time models, we need an evolution in formulation of a model !

- Our earlier approach was to make a suitable model which satisfy the data and the law governing this data. After solving this model, we got an expression representing all the data in sequential manner.

- To find a model in a proper equation form representing all the datum in continuous mode, we require continuous time models !

→ fixed → time dependent

- With classical approach, deterministic continuous time dynamic systems are formulated by using ordinary differential equation(s) or partial differential equation(s).

Introduction to the Differential Equations:

- An equation is called as differential equation if it involves the differential operators e.g. D = d/dt. $Cf(x)$

  operator

- A differential equation is termed as an ordinary differential equation **(ODE)** if there is only one independent variable and rest all are dependent variables.

- If there are more than one independent variables involved in differential equation, it is called as partial differential equation **(PDE)**.

  Example: i) $dx/dt + dy/dt = 1$ has only one independent variable 't' while $x = x(t)$ and $y = y(t)$ are dependent. Hence it is ordinary differential equation.

  ii) $\partial y/\partial t + \partial y/\partial x = 1$ has two independent variables t and x while only one dependent variable $y = y(t, x)$. Hence it is partial differential equation.

Introduction to the Differential Equations:

- The order of a differential equation is the order of highest derivative involved in it. The degree of a differential equation is the degree of the highest derivative appearing in it, (degree should be considered for highest order differential operator only) after the equation has been expressed in a form free from radicals and fractions as far as possible.

  Note: Degree is always a whole number. Order may be fractional (Differential equations with fraction order are termed as fractional order differential equation).

  Fractional Calculus.

  Example: i) $\boxed{x\frac{dx}{dt} = \left(\frac{dx}{dt}\right)^2}$ It is of first order and second degree ODE.

  $\left[1 + \left(\frac{dx}{dt}\right)^2\right]^3 = \left(\frac{d^2x}{dt^2}\right)^2$

  ii) $\frac{\left[1+\left(\frac{dx}{dt}\right)^2\right]^{3/2}}{\frac{d^2x}{dt^2}} = c$ It is of second order and second degree ODE.

  (Exercise ! Remove the fraction and radical to verify the result.)

Introduction to the Differential Equations:

- A differential equation is termed as linear if the degree of each differential operator is 1 and no two operators are multiplied together or with dependent variable .

- If differential equation is not linear, then it is called non-linear differential equation.

Example: i) $\boxed{x\frac{dx}{dt} = \left(\frac{dx}{dt}\right)^2}$ It's non linear (because x is multiplied with dx/dt and also the degree of right hand side term is 2).

ii) $\boxed{dx/dt = x}$ It's linear.

iii) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y = 0$ It is non-linear (because the degree of dy/dx is 3).

Solution of a Differential Equation:

- A solution of a differential equation is a relation between all the dependent and independent variables involved in the differential equation.

- A general solution of a differential equation is that in which the number of arbitrary constants is equal to the order of the differential equation.

$y_G = C_1 x + C_2 x + C_3 e^{3x}$ $\quad \frac{d^3y}{dx^3}$

- A particular solution is one which can be obtained from the general solution by giving particular values to arbitrary constants.

- A differential equation may sometimes have an additional solution which can not be obtained from general solution by assigning particular values to arbitrary constants. This solution is called as singular solution.

Example:

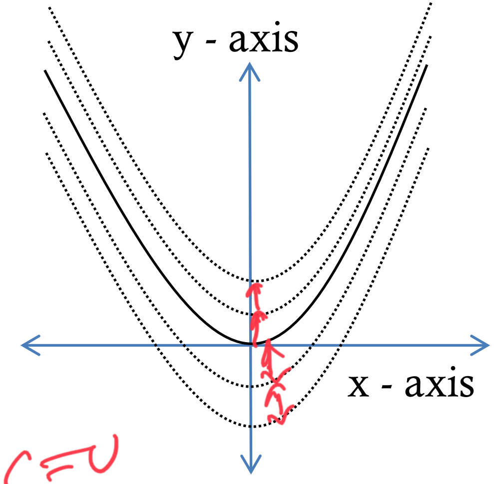
Question: Solve the differential equation $\boxed{dy/dx = x}$.

Solution: $\int dy = \int x\, dx$.

- By integrating both the sides, $\boxed{y = x^2/2 + \text{constant}}$. This curve represents the **general solution**.

- A general solution will always have as many constants as the order of differential equation. The given differential equation is of first order, so it has only one constant. All the general solution differ only by constant terms.

- If the curve (solution) $y = x^2/2 + \text{constant}$, is passing through a particular point say (0, 0) then the value of constant is turned out to be 0. Hence $\boxed{y = x^2/2}$ is the **particular solution** of differential equation.



C=0

Fig. 11.1: Graph of $y = x^2/2 +$ constant for different constant values

Geometrical Meaning of a Differential Equation:

Consider any differential equation of first order and first degree:

$$\frac{dy}{dx} = f(x,y) \quad \text{…. 11.1}$$

- Suppose the solution of this equation 11.1 is given as g(x, y) =0.
- If we take any arbitrary point P =P(x_0 ,y_0) satisfying g(x_0 , y_0) = 0 then this point will also satisfy the corresponding differential equation and vice-versa.
- The term dy/dx in differential equation, will given the slope of the curve g(x, y) =0 at point P.
- All such curves which satisfy g(x, y) =0 also referred as family of curves, represent general solution.
- The curve g(x_0 , y_0) = 0 is particular solution to the differential equation.

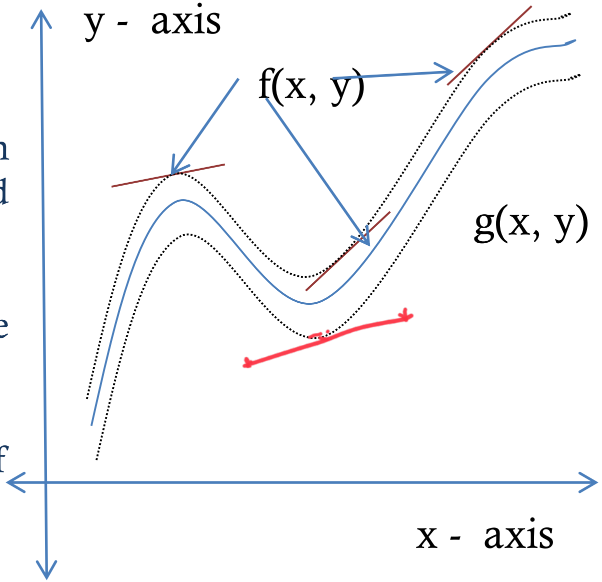


Fig. 11.2: Geometrical meaning of differential equation

Summary:

- Advantages and limitations of discrete time models.
- Need for continuous time models.
- Conversion of differential to difference equation – Accuracy and Computational cost.
- Introduction to differential equation – ODE and PDE.
- Type of solutions of ODE.
- Geometrical meaning associated with differential equation.


Thank you!
STABILO