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NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 4 : Discrete Time Linear Age – Structured Models

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Contents:

- Bernoulli – Lewis – Leslie (BLL) Model.
- Projection Matrix.
- Leslie Matrix.
- Jury's Stability Test.

Introduction to Age-Structured Models :

- If the species involved in modeling are studied after categorizing them in several age-classes, then the model is termed as age-structured.

- Till so far, we have studied 3 models out of which 2 models, linear cell division and linear prey-predator models are non age structured because we haven't consider the ages of species.

- In some species, the amount of reproduction varies greatly with age of individuals. Hence, they can not be formulated using non-age structured models.

- Fibonacci's Rabbit model may be considered as an age structured model because we included the age factor in formulating the model.

- In age structured models, first the whole population is divided in to n age groups. For modeling, separate birth and death rates are considered for each age groups.

Modeling Human Population :

- The human reproduction rate is proportional to the age of the individual so, we will be requiring an age structured model.

- Let's divide human population in 5 age classes.

$x_1(n)$ = Number of individuals from age 0 to 19 at time n.
$x_2(n)$ = Number of individuals from age 20 to 39 at time n.
$x_3(n)$ = Number of individuals from age 40 to 59 at time n.
$x_4(n)$ = Number of individuals from age 60 to 79 at time n.
$x_5(n)$ = Number of individuals from age 80 to 99 at time n.

- We are assuming that no one survives after age 100 ! This assumption could of-course be remedied by considering additional age groups.

Modeling Human Population :

- The mathematical model of the human population with this age class is given as:

$$
\begin{aligned}
x_1(n+1) &= f_1\, x_1(n) + f_2\, x_2(n) + f_3\, x_3(n) + f_4\, x_4(n) + f_5\, x_5(n) \\
x_2(n+1) &= g_{1,2}\, x_1(n) \\
x_3(n+1) &= g_{2,3}\, x_2(n) \\
x_4(n+1) &= g_{3,4}\, x_3(n) \\
x_5(n+1) &= g_{4,5}\, x_4(n) \qquad \ldots.\ 4.1
\end{aligned}
$$

here, f_i denotes the birth rate (over a period of 20 years) for parents in the i^{th} age class and $g_{i,i+1}$ denotes the survival rate for those in the i^{th} age class passing into $(i+1)^{th}$.

- A single set of parents may be in different age groups, we should attribute half of their offspring to each in choosing values for f_i.

Modeling Human Population :

- In matrix notation, the previous model can be written as:

$$x(n+1) = P\,x(n). \qquad \text{.... } 4.2$$

where, vector $x = \{x_1, x_2, x_3, x_4, x_5\}$ and matrix [P] is termed as projection matrix.

$$P = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 \\ g_{1,2} & 0 & 0 & 0 & 0 \\ 0 & g_{2,3} & 0 & 0 & 0 \\ 0 & 0 & g_{3,4} & 0 & 0 \\ 0 & 0 & 0 & g_{4,5} & 0 \end{bmatrix}$$

Analysis on Projection Matrix :

- The numbers written in matrix [P] will be obtained from data collection. Do these numbers have some associated meaning? Or are they just some arbitrary numbers?!

- Well, to answer this we need to perform a proper analysis on the data.

- Among all f_i ‘s, which one will have largest magnitude and smallest magnitude? Can f_i have zero value also ? and the similar questions for $g_{i,i+1}$.

- We might expect f_1 to be smaller than f_2 to because parents of age group 0-19 years may have less reproduction rate than parents belong to 20-39 years group (Think!).

- Some very elder parents might not be capable for reproduction and this results in zero value for some of the f_i ‘s.

Analysis on Projection Matrix :

- We had assumed 5 different age groups to model the human population and hence we obtained a 5 X 5 projection matrix [P].

- If we increase the number of classes, the size of population matrix will be increased proportionally.

  For example, if we consider the length of an age group as 5 years and the last age as 100 years then, size of [P] will be 20 X 20.

- Whatever be the size of the projection matrix, it will have the same structure. The top row deduce the fecundity information and sub-diagonal shows the rate of survival. The rest of the entries will be 0.

- This kind of projection matrix is coined as Leslie Matrix.

Analysis on Projection or Leslie Matrix :

- So stability of the human population model will entirely depend on the Leslie matrix. In the last lecture we have learnt about the stability of system by finding the eigen values of coefficient matrix (here it's [P]).

- Again, the size of Leslie matrix [P] will decide the degree of characteristic polynomial to solve. In general, there is no generalized method to solve the polynomial of degree higher than 2.

- Then, what to do now? How can we solve it further? Because without finding the eigen values one can not tell about the stability of system. Is there any other method to analyze the polynomial of higher degree?

- An American electrical engineer Eliahu Ibraham Jury gave a concrete method to check the stability of linear discrete time system which is known as **Jury's stability test**.

Jury's Stability Test :

- Suppose a given discrete time linear system is

$$Z_{n+1} = [M]\, Z_n \, .$$

where [M] is n x n matrix.

- To find the eigen values, we need to solve for $\det(M - \lambda I) = 0$. This will give a polynomial of degree n. Suppose the polynomial in simple form is represented as:

$$Q(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots + a_{n-1}\lambda + a_n = 0, \text{ where } a_0 \neq 0.$$

- To apply the Jury's test, we need to form a table. To create the same, the procedure is given as below:

1. In the first row, write all the coefficients of polynomial starting from highest power of λ.
2. In the second row, write these coefficients in reverse order i.e. starting from lowest power.

Jury's Stability Test :

$$\begin{array}{cccccc} a_0 & a_1 & a_2 & \dots & a_{n-1} & a_n \\ a_n & a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{array}$$

3. For the third row, multiply second row by coefficient a_n/a_0 and then subtract the same from first row. The last term will be 0.
4. For the fourth row, write the first n coefficient of the third row in reverse order.

$$\begin{array}{cccccc} a_0 - a_n^2/a_0 & a_1 - a_{n-1}\,a_n/a_0 & a_2 - a_{n-2}\,a_n/a_0 & \dots & a_{n-1} - a_1\,a_n/a_0 & 0 \\ a_{n-1} - a_1\,a_n/a_0 & a_{n-2} - a_2\,a_n/a_0 & a_{n-3} - a_3\,a_n/a_0 & \dots & a_0 - a_n^2/a_0 & \end{array}$$

5. Repeat the same procedure till a row contains only one non zero element.

- For $a_0 > 0$ if all the coefficients in first column of odd numbered rows are positive (if any one of them is 0, then system is **asymptotically stable** because of unit magnitude eigen value), $Q(1) > 0$ and $(-1)^n Q(-1) > 0$ then all the roots of polynomial lie inside an unit circle with center at origin, which is the required condition on eigen values for **stable** system . If $a_0 < 0$, then multiply whole polynomial by -1 to make it positive. If any one of the 3 conditions not satisfied, system is **unstable**.

Jury's Stability Condition for Quadratic Equation:

- Let's consider a quadratic equation $Q(\lambda)=b_0\lambda^2 + b_1\lambda + b_2 = 0$, where $b_0 > 0$.
- Jury's stability table will look as:

1. b_0	b_1	b_2
2. b_2	b_1	b_0
3. $c_0 = b_0 - b_2^2/b_0$	$c_1 = b_1 - b_1 b_2/b_0$	0
4. $c_1 = b_1 - b_1 b_2/b_0$	$c_0 = b_0 - b_2^2/b_0$	
5. $d_0 = c_0 - c_1^2/c_0$		

- So in the 5th step only, we got a single non-zero number d_0. Since $b_0 > 0$ the condition for stability is $c_0 > 0$ and $d_0 > 0$.
- Without loss of generality, say $b_0 = 1$ (for simplicity). Then the conditions for stability turned out are (Exercise!):

$$|b_2| < 1 \text{ and } |b_1| < |1 + b_2| \qquad \text{.... 4.3}$$

Jury's Stability Condition for Cubic Equation:

- Let's consider a cubic equation $Q(\lambda) = \lambda^3 + p_1\lambda^2 + p_2\lambda + p_3 = 0.$
- Jury's stability table will look as:

1. 1	p_1	p_2	p_3
2. p_3	p_2	p_1	1
3. $q_1 = 1 - p_3^2$	$q_2 = p_1 - p_1p_2$	$q_3 = p_2 - p_1p_3$	0
4. $q_3 = p_2 - p_1p_3$	$q_2 = p_1 - p_1p_2$	$q_1 = 1 - p_3^2$	
5. $r_2 = q_1 - q_3^2/q_1$	$r_3 = q_2 - q_2q_3/q_1$	0	
6. $r_3 = q_2 - q_2q_3/q_1$	$r_2 = q_1 - q_3^2/q_1$		
7. $s_3 = r_2 - r_3^2/r_2$			

- So in the 7th step only, we got a single non-zero number s_3. The condition for stability is $Q(1) > 0$, $Q(-1) < 0$, $q_1 > 0$, $r_2 > 0$ and $s_3 > 0$. After solving for same we will get simplified form as:

$|p_3| < 1, |p_1 + p_3| < p_2 + 1, |1 + p_2|$ and $|p_2 - p_3p_1| < |1 - p_3^2|$ …. 4.4

Summary:

- Introduction to age structured models.
- Bernoulli – Lewis – Leslie (BLL) model.
- Model for human population.
- Analysis on projection matrix and Leslie matrix.
- Jury's stability criterion for discrete linear system


Thank you!
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