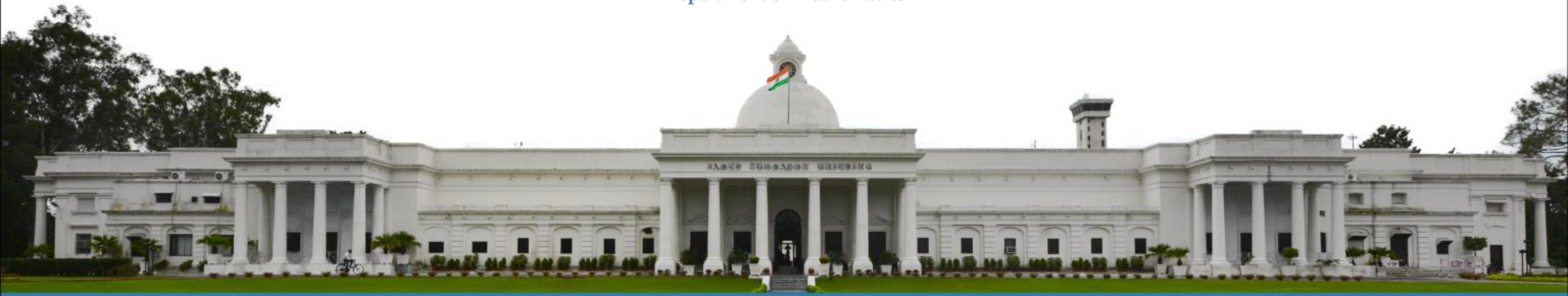
IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELS : ANALYSIS AND APPLICATIONS

Lecture 5 : Numerical Methods to Compute the Eigen Values

Dr. Ameeya Kumar Nayak
Department of Mathematics



Contents:

- Physical Significance of Eigen Values.
- Why Numerical Methods?
- LR Method.

Physical Significance of Eigen Values:

- In the previous lectures, we have talked a lot on eigen values, characteristic equations and stability.
- By the time, you might have realized the importance of eigen values in finding the stability of solution and hence analyzing the behavior of the same.
- Do these eigen values describe something? Or are they mere some numbers?
- Eigen values (and eigen vectors) are very well related with the transformation of a system.
- The need of transformation of system is to make it easier for study and analysis. When you transform any matrix, you need two things:

1. By what amount are you going to transform your system?
2. In which direction are you transforming it?

Physical Significance of Eigen Values:

- The answer to your first question is given by eigen values and to second is given by eigen vectors.

- Suppose you have a 2 x 2 matrix and it's eigen values are λ_1, λ_2 and corresponding eigen vectors are v_1, v_2 respectively.

  not unique

- λ_1 gives the amount by which one is transforming matrix in direction v_1. Similar is the case with λ_1 in direction of v_2.

  If $0 < \lambda < 1$, means system is contracted in the same direction as of v.
  If $\boxed{\lambda > 1}$, means system is expanded in the same direction as of v.
  If $-1 < \lambda < 0$, means system is contracted in the opposite direction as of v.
  If $\lambda < -1$, means system is expanded in the opposite direction of v.
  If $\lambda = 0$, means system is not at all transformed in the direction of v.

Need of Numerical Methods:

- Now we have seen the physical and mathematical importance of eigen values (and eigen vectors). We have also seen that how tedious job is to calculate eigen values when matrix is of high dimension.

- Also, when degree of characteristic equation is large (more than or equal to 3), it's not even easy to find eigen values.

  Jury's test only help us to know whether all the eigen values lie in a unit disk or not. But to know the exact solution, we need eigen values.

- When to find the solution of a system or an equation becomes difficult task with analytical methods, then we go with numerical methods.

- Numerical methods are the computer based approach which try to converge near the exact solution as far as possible.

Numerical Methods to find Eigen Values:

- There are many numerical algorithms to compute the eigen values. Some of the widely used algorithms are:

1. Power Method
2. LR Method
3. QR Method
4. Householder Method etc.

- Each algorithm has some limitations and some advantages.

- In this lecture we will study only LR Method which can be used to any arbitrary matrix to find all the eigen values.

LR Method:

- The LR Method was suggested by a Swiss Mathematician H. Rutishauser in 1958.

- This method can be used to any arbitrary matrix to compute all the eigen values. This method generally takes two steps:

$$\boxed{A_k = L_k R_k,} \qquad (A) = (L\,U)$$

$$A_{k+1} = R_k\, L_k. \qquad \ldots\ 5.1$$

mn, $m \times n$, $n \times n$, $n \times n$, n^2

where k = 1, 2, 3, … and $A_1 = A$. L_k is unit lower triangular matrix while R_k is upper triangular matrix.

- For large k, matrix A_{k+1} generally approaches to an upper triangular matrix. (**Note:** The determinant of an upper or a lower triangular matrix is simply the product of its diagonal entries.) So the eigen values are given by the diagonal elements.

$$\lambda_1 \lambda_2 \cdots \lambda_n = a_{11} a_{22} \cdots a_{nn}$$

Example on LR Method:

Question: Find the eigen values of matrix [A] given as: $A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$

Solution: $A = A_1 = L_1R_1$. Assume $L_1 = \begin{bmatrix} 1 & 0 \\ \alpha_1 & 1 \end{bmatrix}$ and $R_1 = \begin{bmatrix} \alpha_2 & \alpha_3 \\ 0 & \alpha_4 \end{bmatrix}$

$L_1R_1 = \begin{bmatrix} \alpha_2 & \alpha_3 \\ \alpha_1\alpha_2 & \alpha_1\alpha_3+\alpha_4 \end{bmatrix}$

From first iteration, one can find the values of α's as:

$\alpha_2 = 3,\ \alpha_3 = 2,\ \alpha_1 = 1/3$ and $\alpha_4 = 10/3$.

L_1R_1

$R_1L_1 = [\alpha_2 + \alpha_3\alpha_1\ \ldots$

Now $A_2 = R_1L_1 = \begin{bmatrix} \frac{11}{3} & 2 \\ \frac{10}{9} & \frac{10}{3} \end{bmatrix}$. Again find the L_2, R_2 by the same method as followed for L_2, R_2. This process will be repeated until we get same matrix [A] in successive iterations. After performing this iteration 10 times you will get,

$$A_9 = \begin{bmatrix} 4.9952 & 2 \\ 0 & 2.0023 \end{bmatrix} \qquad A_{10} = \begin{bmatrix} 4.9976 & 2 \\ 0.0024 & 2.0023 \end{bmatrix}$$

You can observe that A_9 and A_{10} are almost identical and diagonal matrices. Hence the eigen values are 5 and 2.

Summary:

- Physical significance of eigen values.
- Need of numerical methods.
- Numerical methods for computing eigen values.
- LR Method.


Thank you!
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