IIT ROORKEE

NPTEL ONLINE CERTIFICATION COURSE

MATHEMATICAL MODELING : ANALYSIS AND APPLICATIONS

Lecture 3.2 : Solution of First Order First Degree Differential Equations

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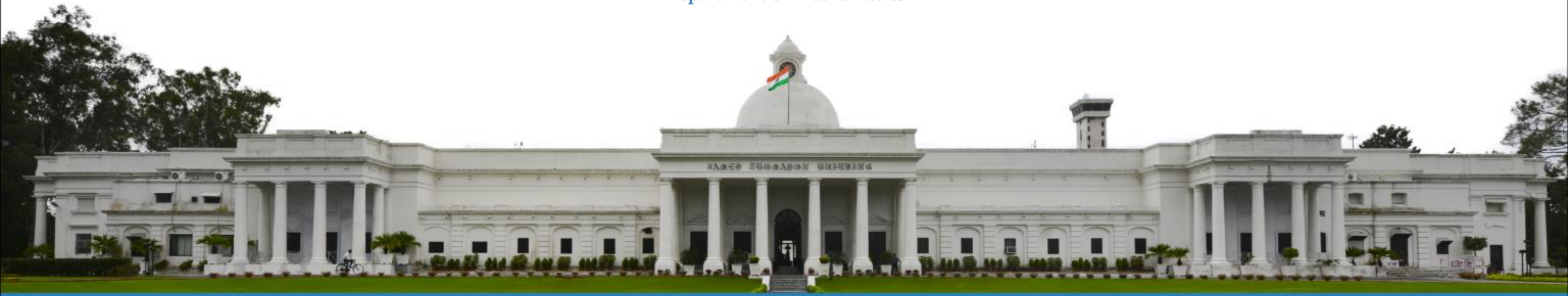

Contents:

- Methods to Solve First Order First Degree Differential Equation.
- Method of Separation of Variables.
- Homogenous Differential Equation.
- Linear Equations.

Methods to Solve First Order First Degree ODE:

- The general form of first order first degree ODE is:

$$\boxed{\frac{dy}{dx} = f(x,y)} \quad \text{.... 12.1}$$

- In general it's not easy to solve ODE as given in eq. 12.1. There are some methods which are applicable to some particular type of first order first degree equations. We will discuss some of them as listed below:

1. Method of Separation of Variables,
2. Method for Homogeneous Equations,
3. Bernoulli Equation (Method for Linear ODE's).

1. Method of Separation of Variables:

- If in the equation 12.1 it is possible to collect all the functions of x and dx on one side and the function of y and dy on the other side, then the variables are said to be separable.

- The general form of this equation is:

$$f(y)\,dy = g(x)\,dx \qquad \text{.... 12.2}$$

- By integrating both the side with respect to y and x respectively, the general solution of equation 12.2 can be obtained i.e.

$$\int f(y)dy = \int g(x)dx + constant \qquad \text{.... 12.3}$$

Example on Method of Separation of Variables:

Question: Consider a differential equation

$$\frac{dN}{dt} = \kappa N(C_0 - aN) \qquad \text{.... 12.4}$$

where all the parameters other than N are constant.

Solution:

$$\Rightarrow \int_{N_0}^{N} \frac{1}{N(C_0 - aN)} dN = \int_{t_0}^{t} \kappa dt$$

$$\Rightarrow \int_{N_0}^{N} \frac{C_0 - aN + aN}{NC_0(C_0 - aN)} dN = \int_{t_0}^{t} \kappa dt$$

$$\Rightarrow \frac{1}{C_0} \int_{N_0}^{N} \frac{1}{N} dN + \frac{a}{C_0} \int_{N_0}^{N} \frac{1}{C_0 - aN} dN = \int_{t_0}^{t} \kappa dt$$

$$\Rightarrow \frac{1}{C_0} log\left|\frac{N}{N_0}\right| - \frac{1}{C_0} log\left|\frac{C_0 - aN}{C_0 - aN_0}\right| = \kappa(t - t_0)$$

$$\Rightarrow \boxed{\frac{1}{C_0} log\left|\frac{N(C_0 - aN_0)}{N_0(C_0 - aN)}\right|} = \kappa(t - t_0)$$

where N_0 and t_0 are initial population and time respectively

Example on Method of Separation of Variables:

- For simplicity let's consider $\boxed{t_0 = 0}$ then

$$\Rightarrow \frac{N}{C_0 - aN} = \frac{N_0}{C_0 - aN_0} e^{kC_0 t}$$

$$\Rightarrow \frac{C_0 - aN}{N} = \frac{C_0 - aN_0}{N_0} e^{-kC_0 t}$$

$$\Rightarrow \frac{C_0}{N} - a = \left(\frac{C_0}{N_0} - a\right) e^{-kC_0 t}$$

$$\Rightarrow \frac{C_0}{N} = \frac{(C_0 - aN_0)e^{-kC_0 t} + aN_0}{N_0}$$

$$\Rightarrow \boxed{N = \frac{N_0 C}{(C_0 - aN_0)e^{-kC_0 t} + aN_0}}$$

- To reduce the number of parameters, assume $\boxed{A = C_0/a}$ and $\boxed{\gamma = \kappa C_0}$

$$N(t) = \frac{AN_0}{N_0 + (A - N_0)e^{-\gamma t}} \quad \text{…. 12.5}$$

$\Rightarrow N_0 + (A - N_0) e^{-\gamma t}$

$= A$

- Equation 12.5 represents a particular solution to 12.4 (because the curve of N(t) passes through point $(N_0, 0)$).

2. Homogeneous Differential Equation:

$f(xt, yt) = \boxed{t^n g(x/y)}$

$= t^n g(x/y, 1)$

- The general form of homogeneous differential equation is:

$$\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$$

where function f(x, y) and g(x, y) are homogeneous i.e. they have same degree with respect to all the variables x and y.

- The basic idea of solving this type of differential equation is to convert it into a simple form by substituting y/x = v or y = vx, then $\frac{dy}{dx} = \boxed{v + x\frac{dv}{dx}}$

- After this, use the method of separation of variables and proceed for solution.

- Sometimes, differential equations are not directly presented in homogeneous form. They need to be reduced into that one.

Example on Homogeneous Differential Equation:

Question: Solve $$\frac{dy}{dx} = \frac{y + x - 2}{y - x - 4}$$ $y = ux$

Answer: This equation is not homogeneous due to the presence of constants (-2 and -4). To make it homogeneous we will have to get rid off these constants by substituting $x = X + h$ and $y = Y + k$ which implies $dx = dX$ and $dy = dY$.

$$\frac{dY}{dX} = \frac{Y + X + h + k - 2}{Y - X - h + k - 4}$$

Now we want $h + k - 2 = 0$ and $-h + k - 4 = 0$. By solving these equations, we will get $h = -1$ and $k = 3$. Now the differential equation is reduced in form of:

$$\boxed{\frac{dY}{dX} = \frac{Y + X}{Y - X}}$$

Example on Homogeneous Differential Equation:

- To solve this homogeneous equation, substitute $\boxed{Y = vX}$ which leads to $\frac{dY}{dX} = v + X\frac{dv}{dX}$

$$\Rightarrow v + X\frac{dv}{dX} = \frac{v+1}{v-1}$$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+2v-v^2}{v-1}$$

$$\Rightarrow \int \frac{v-1}{1+2v-v^2} = \int \frac{1}{X} dX + c$$

$$\Rightarrow -\frac{1}{2}\int \frac{2-2v}{1+2v-v^2} = \int \frac{1}{X} dX + c$$

$$\Rightarrow -\frac{1}{2} log\left|1 + 2v - v^2\right| = log|X| + c$$

$$\Rightarrow log\left|1 + 2\frac{Y}{X} - \frac{Y^2}{X^2}\right| + log\left|X^2\right| = -2c$$

$$\Rightarrow \boxed{log\left|X^2 + 2XY - Y^2\right| = -2c}$$

Example on Homogeneous Differential Equation:

- Now substitute back X = x – h = x + 1 and Y = y – k = y – 3. The final solution will be: (Verify!!)

$$\boxed{x^2 + 2xy - y^2 - 4x + 8y - 14 = C_1} \quad \text{where } C_1 = e^{-2c}$$

3. Linear Equation:

- The standard form of a linear equation of the first order (also known as Leibnitz's Equation) is:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

- To solve this type of equation, multiply both the sides by Integrating factor (I.F.) $e^{\int P(x)dx}$

- Hence the final solution will be:

$$ye^{\int P(x)dx} = \int Q(x)\underbrace{e^{\int P(x)dx}}_{IF}dx + constant$$

3. Linear Equation:

- Some equations can be converted into standard linear equation form. The equation given below is called Bernoulli's equation:

$$\frac{dy}{dx}+P(x)y=Q(x)y^n$$

- This equation can be reduced into Leibnitz's equation by substituting $y^{1-n}=z$. The reduced linear equation will be:

$$\frac{1}{n-1}\frac{dz}{dx}+P(x)z=Q(x)$$

$$\frac{1}{y^n}\frac{dy}{dx}+\frac{P(x)}{y^{n-1}}=Q(x)$$

$$\Rightarrow y^{1-n}=z$$

$$\Rightarrow \frac{dy}{dx}=\frac{1}{1-n}\frac{dz}{y^{-n}dx},\quad \frac{dz}{dx}=(1-n)y^{1-n-1}\frac{dy}{dx}$$

Example on Linear Equation:

Question: Solve $\boxed{x\frac{dy}{dx}+y=x^3y^6}$

Solution: This equation is neither in the Leibnitz's form nor in the Bernoulli's form (standard). To convert this into the Bernoulli's form divide the whole equation by x and then to convert it into the Leibnitz's form substitute $z = y^{-5}$, which means

$$\boxed{\frac{dz}{dx}=-5y^{-6}\frac{dy}{dx}}$$

The simplified linear equation will be:

$$\frac{dz}{dx}-5\frac{z}{x}=-5x^2$$

$IF = e^{\int P\,dx} = e^{-\int \frac{5}{x}dx}$

Example on Linear Equation:

- The integrating factor (I.F.) of last equation is:

$$e^{\int -\frac{5}{x}dx} = e^{-5lnx} = \frac{1}{x^5}$$

- Hence the solution is given by:

$$z\left(\frac{1}{x^5}\right) = \int(-5x^2)\left(\frac{1}{x^5}\right)dx + c$$

$$\Rightarrow y^{-5}x^{-5} = -5\frac{x^{-2}}{-2} + c$$

Summary:

- Methods to solve first order first degree differential equation.
- Method of Separation of Variables.
- Homogeneous and reducible to homogeneous differential equation.
- Linear equation – Leibnitz's and Bernoulli's equation.


Thank you!
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