Efficiency in Job-Ladder Models

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Abstract

This paper examines the efficiency of a decentralized equilibrium in a broad class of random-search job-ladder models. We decompose the source of inefficiency into two margins: (i) the investment margin, that is, the difference between the private and social benefit of job creation given the surplus of a match, and (ii) the valuation margin, that is, the difference between the private valuation and the social valuation of a match surplus. In the presence of on-the-job searches, the well-known Hosios condition no longer guarantees the market equilibrium aligns with the efficient allocation along both margins. On-the-job searches contribute to the overvaluation of the match surplus in market equilibrium, especially at the top of the job ladder. Consequently, the decentralized equilibrium with the Hosios condition features excess creation of vacancies in the steady state. On-the-job searches also lead to excess volatility in unemployment in response to aggregate productivity shocks. Quantitatively, we find a significant difference between the equilibrium outcome and the efficient allocation under standard calibration. We also consider several decentralizations of the efficient allocation to shed light on the optimal policies under the frictional labor market.

Keywords: search and matching, job ladder, on-the-job search, Hosios condition

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1 Introduction

In designing and evaluating macroeconomic policies under frictional labor markets, understanding the normative properties of the equilibrium outcome is essential. In a frictional environment, assessing the efficiency of decentralized equilibrium is a nontrivial task. For example, since matching workers and firms is costly, the mere presence of unemployment is not necessarily a sign of inefficiency.

Many recent studies emphasize the significance of job-to-job transitions in understanding the positive aspects of the labor market, both at the micro and macro levels. The commonly employed approach in modeling the job-to-job transition considers the *job ladder*. That is, workers move up to better and better jobs by conducting on-the-job searches and accepting better job offers. At the micro level, job ladders are important determinants of earnings growth and workers' career development.[1] At the macro level, job ladders shape the dynamics of aggregate productivity, wages, inflation, and unemployment.[2]

Despite their popularity, the normative properties of the job-ladder models are not well understood. In this paper, we examine the efficiency of a broad class of job-ladder models with random search. As in the standard Diamond-Mortensen-Pissarides (DMP) model, firms match with workers by posting vacancies. The probability of a match is governed by the matching function, which produces matches from the inputs of vacancies and searching workers. Jobs differ in match quality, and workers can engage in on-the-job searches. Consequently, employed workers gradually find a better job and move up the job ladder.

Our starting point is the well-known result by Hosios (1990). Hosios shows that, in the standard DMP model without on-the-job search, an efficient outcome is achieved by the market equilibrium with Nash bargaining if the elasticity of the matching function coincides with the worker's bargaining power (*the Hosios condition*). As Hosios points out, two inefficiencies, congestion externality and hold-up problems due to wage bargaining, exist in the DMP-style search-and-matching model. The Hosios condition ensures that these inefficiencies cancel out to achieve efficiency.

There are several advantages to using Hosios' result as the measuring stick in our paper. First, our model is a direct extension of the DMP model, and examining inefficiency under the Hosios condition highlights the additional sources of inefficiency resulting from on-the-job searches. Second, the Hosios condition is frequently used for calibration in the quantitative macroeconomic

[1] For example, Topel and Ward (1992) attribute about one-third of earnings growth for young workers to job-to-job transitions. More recently, Hahn et al. (2021) show workers (on average) gain both wages and hours with job-to-job transitions.

[2] For example, reallocating workers across jobs and improving allocation (Barlevy, 2002), generating wage dispersion (Postel-Vinay and Robin, 2002; Hornstein et al., 2011), inflation (Moscarini and Postel-Vinay, 2023; Faccini and Melosi, 2023; Birinci et al., 2024), wage rigidity (Fukui, 2020), unemployment dynamics (Moscarini and Postel-Vinay, 2018; Faberman et al., 2022), and aggregate productivity dynamics (Mukoyama, 2013).

literature (e.g., Shimer, 2005; Faberman et al., 2022), and clarifying the efficiency properties under the Hosios condition would shed light on the efficiency implications of that calibration strategy. Even when the elasticity of the matching function and the worker's bargaining power are separately calibrated (e.g. Lise et al., 2016; Bilal et al., 2022), the Hosios condition provides straightforward and useful guidance in deducing the efficiency properties of the equilibrium. Third, various papers that extend the standard DMP model in different directions also use Hosios' result as the point of comparison (e.g., Acemoglu, 2001; Davis, 2001; Mukoyama, 2019). Following a similar strategy helps to sort out the differences in this literature.

Building on the insight of Hosios (1990), we categorize these inefficiencies into two margins, which we refer to as the *investment margin* and the *valuation margin*. The DMP model treats the job creation as a firm's investment. Firms invest in vacancies by paying the vacancy-posting cost and receive returns in the form of profit. On the investment margin, the question is whether the firm's incentive to invest, given the reward upon matching, is aligned with the social benefit of the investment activity. A separate question is whether the firm's reward upon matching is aligned with the social value of creating a match, that is, the value of moving a worker from unemployment to employment. We refer to this margin as the valuation margin. In the standard DMP model without on-the-job search (Pissarides, 1985), the Hosios condition (coincidentally) ensures efficiency in both margins—killing two birds with one stone.

We show the Hosios condition does not guarantee efficiency once on-the-job search is taken into account. On the investment margin, the planner takes into account how much the additional vacancy creation congests the market. In contrast, in the decentralized equilibrium, the firm is concerned with the extent to which the additional vacancy creation is rewarded. When a worker is poached, the new match receives all surplus, including the part created by previous employers. When a worker is poached, existing matches are destroyed, which may not be fully internalized by the poachers. This outcome leads to a ″worker-stealing″ externality, similar to the business-stealing externality in the economic growth literature. We demonstrate that imposing the sequential-auction wage-setting protocol of Dey and Flinn (2005) and Cahuc et al. (2006), together with the Hosios condition, ensures that the increased wage payment due to the auction coincides with the worker-stealing externality, thereby achieving efficiency along the investment margin.

The same condition, however, does not ensure efficiency on the valuation margin. The social value of moving a worker from unemployment to employment includes the change in congestion externality of switching from an off-the-job search to an on-the-job search. By contrast, the private value of a match includes the loss of opportunity to conduct an off-the-job search and the gain of opportunity to conduct an on-the-job search. Once again, the presence of wage bargaining implies that a part of the surplus from both searching on the job and off the job is not taken into account when the private surplus is computed. We demonstrate that the presence of on-the-job searches leads to the overvaluation of matches in our model.

The intuition is clearest in the case where on-the-job search efficiency is the same as off-the-job search efficiency. In this case, there is no change in congestion externality from forming a match, as an employed worker and an unemployed worker equally congest the labor market. However, privately, the undercounting of the off-the-job search opportunity cost is larger than the undercounting of the gain from on-the-job search opportunities. This result follows because the total private gain from an on-the-job search is smaller than the private gain from an off-the-job search and the Nash bargaining implies that the constant share of private gains is not taken into account. Consequently, the undercounting of the off-the-job search opportunity cost dominates, and the market equilibrium ends up overvaluing the surplus from match formation.

We build on these observations to formally demonstrate a general result that, under the Hosios condition, the decentralized equilibrium always exhibits excessive vacancy creation in the steady state. The excess creation of vacancies implies workers climb the job ladders too quickly, and the decentralized equilibrium has *too many* good matches compared with the efficient allocation. This finding is in contrast to a model with ex-ante job heterogeneity á la Acemoglu (2001) that has *too few* good jobs in the decentralized equilibrium.

We consider two decentralizations of efficient allocation. The first is a simple implementation based on two fiscal instruments: unemployment insurance and tax on vacancy creation. We show that, under the Hosios condition, both the efficient unemployment insurance and the vacancy tax are positive. These instruments, while having the advantage of their simplicity, have shortcomings in that they do not directly map into two wedges (investment and valuation margins) we identified, and they are not robust to various extensions of the model with endogenous efforts. For this reason, we also consider another implementation where the government imposes taxes on vacancy creation, search on the job, and search off the job. Under the Hosios condition, the optimal taxes have the following features. First, off-the-job search should be subsidized. Second, the tax on on-the-job search increases as one ascends the job ladder, starting with a subsidy at the bottom and transitioning into taxes at the top. Third, the tax on vacancy creation is weakly positive and is zero whenever the sequential auction wage-setting protocol is in place.

In the next part of the paper, we quantify the discrepancy between the decentralized equilibrium and the efficient allocation. We find that the discrepancy between the market equilibrium and the efficient allocation is quantitatively significant under the standard calibration in the literature, assuming the Hosios condition and sequential auction. The decentralized equilibrium allocation has an unemployment rate that is about 1 percentage point lower than the efficient allocation.

We then show that, under the same calibration, the discrepancy between the decentralized equilibrium and the efficient allocation is further exacerbated in response to positive productivity shocks. The decentralized equilibrium exhibits excessive volatility in unemployment compared to the efficient allocation. The reason is that underaccounting of the net opportunity cost of forming a match is larger when the match surplus is high. As a result, the equilibrium overvaluation of matches is magnified when the productivity is high.

These results are important for applied work because when a job-ladder model is calibrated to satisfy the Hosios condition, which is a common practice in the literature, the model automatically favors a policy that suppresses vacancy creation in the steady state, such as income taxes and unemployment insurance, and stabilizes labor market fluctuations, such as monetary and fiscal policies, even without any other frictions. Even if one moves away from the Hosios condition, our result serves as a reminder that the efficiency property cannot be assessed solely by the Hosios condition in the presence of an on-the-job search.

Related literature

Our paper is related to several strands of the literature. The most directly related literature includes the papers that examine efficiency in the models with on-the-job search. An earlier paper by Gautier et al. (2010) analyzes the efficiency of matching models with on-the-job searches and costly vacancy creation. They use a model with a structure (a "circular" heterogeneity) that is not typically used in macroeconomics. They focus on a particular case where on-the-job search and off-the-job search have the same efficiency, and the discount rate is equal to the population growth rate. We do not impose such restrictions. Our model has a structure that is more commonly found in the macroeconomic literature, and we consider wage-setting protocols that are widely used in the applied quantitative literature.[3] In addition to the difference in setting, our important innovation is the explicit analysis of the investment and valuation margins. This approach enables us to identify which inefficiency matters in each margin. For example, we show that a market arrangement with the Hosios condition and a sequential-auction wage setting mechanism (Dey and Flinn, 2005; Cahuc et al., 2006) delivers efficiency in the investment margin, whereas it fails to achieve efficiency in the valuation margin. Previous papers have no explicit discussion of the valuation margin.[4]

The second strand of related literature is the analysis of the DMP model with heterogeneous jobs. Papers such as Acemoglu (2001), Davis (2001), and Mukoyama (2019) explicitly consider efficiency in DMP models with heterogeneous jobs and no on-the-job search. These papers do not deal with job-to-job transitions.

[3]Gautier et al. (2010) mainly consider two wage-setting protocols, and both are different from ours. Both protocols assume the wages are set once and for all, either by negotiation upon the match or commitment via wage posting. We allow wages to change at any point during the match, as in most models in the quantitative macroeconomic literature.

[4]In a recent paper, Cai (2020) considers a discrete-time model where multiple firms can match with a worker within a period. Similarly to Gautier et al. (2010), his setting is restricted to the case where the discount rate is the same as the population growth rate and does not separate the investment and valuation margins. The wage determination in his model is restricted to proportional sharing of the current surplus. In contrast, our model is a direct extension of the standard DMP model.

Finally, previous studies have examined the efficiency of equilibrium in directed-search settings. For example, Menzio and Shi (2011) establish the efficiency of directed-search equilibrium in a model with on-the-job search. Our paper considers random-search models, where different types of workers interact in the same labor market.

This paper is organized as follows. Section 2 sets up the terminology we use in our analysis by reviewing the intuition of the Hosios (1990) condition in a model without on-the-job searches. Section 3 sets up our model and compares the efficient allocation and the equilibrium outcome. In Section 4, we derive some analytical results. Section 5 examines the model quantitatively. Section 6 analyzes how efficient allocation can be implemented as a market equilibrium with taxes. Section 7 concludes.

2 Revisiting the basic intuition of the Hosios condition

Before considering our baseline model, let us set up the model without an on-the-job search to develop basic intuition. Consider the textbook Pissarides (1985) model in continuous time. A continuum of infinitely lived workers with a population of 1 exists in the economy. The workers are risk-neutral, and the discount rate is $r > 0$. The workers are either employed or unemployed. For a worker to be employed, she must be matched with a firm. Production takes place with a one-worker, one-firm match. The match produces $z > 0$ units of the consumption good. An unemployed worker receives the flow value of $h \in [0, z)$ from home production. Firms post vacancies to fill their positions. The flow vacancy cost is $\kappa > 0$. We assume free entry: any firm can post a vacancy and begin producing once it is matched with a worker.

In this section, on-the-job searches do not exist; only unemployed workers look for jobs. We assume the matching is random, and therefore, all vacancies have an equal chance to match with a worker, and all unemployed workers have an equal chance to match with a vacancy. The matching process is governed by the matching function: $M(u, v)$ represents the number of matches created when v vacancies and u unemployed workers exist. The matching function satisfies the following conditions: (i) $M(u, v)$ is strictly increasing and strictly concave in each of u and v and satisfies Inada conditions; (ii) $M(u, v)$ exhibits constant-returns to scale; and (iii) $M(u, v) \le \min\{u, v\}$. We assume the separation is random with the Poisson probability $\sigma > 0$. In market equilibrium, wages are determined according to the generalized Nash bargaining solution. Nash bargaining is conducted on the expected present value of surpluses, and the worker's bargaining power is set at $\gamma \in [0, 1)$. Because of the random-matching assumption, the Poisson rate at which an unemployed worker finds a job and a vacancy finds a worker can be written as

$$p(\theta) \equiv M(1, \theta)$$

and

$$q(\theta) \equiv M\left(\frac{1}{\theta}, 1\right),$$

where $\theta \equiv v/u$ is the market tightness. Note $p(\theta) = \theta q(\theta)$ holds.

The details of the social planner's problem and the market equilibrium are analyzed in Appendix A. It shows the Hosios condition (Hosios, 1990)

$$\gamma = \eta(\theta), \tag{1}$$

where $\eta(\theta) \equiv -\theta q'(\theta)/q(\theta)$ is the elasticity of the matching function, ensures the constrained efficiency of the market outcome. That is, under condition (1), the solution to the social planner's problem coincides with the market equilibrium. Intuitively, the Hosios condition can be understood as the condition where two inefficiencies exactly cancel each other out.

Let us review the intuition more closely. For the ease of exposition, we focus on the steady state. First, consider the optimization problem for vacancy posting. This exercise amounts to comparing between

$$\kappa = (1 - \eta(\theta))q(\theta)\mu \tag{2}$$

for the social planner's problem, where μ is the social value of moving one additional unemployed worker into employment, and

$$\kappa = (1 - \gamma)q(\theta)S, \tag{3}$$

where S is the surplus of a match between a worker and a firm. Here, suppose $\mu = S$ holds. Even in that situation, two inefficiencies arise in comparing equations (2) and (3). We refer to this margin as the "investment margin" because it relates to the firm's vacancy creation as an investment. The first inefficiency is the hold-up problem. Matches are formed because of the firms' active investment (vacancy posting), and workers incur no costs. Thus, all returns from the match should be paid to the firms to ensure an efficient level of investment. However, at the time the worker and the firm engage in Nash bargaining, the investment costs are already sunk. As a result, the firm can collect only $(1 - \gamma)$ share of the surplus (the hold-up problem) on the right-hand side of (3). This inefficiency, resulting from firms' imperfect appropriation of surplus, leads to too few vacancies compared to the socially desirable level. When γ (the worker's bargaining power) is large, the inefficiency is large.

The second inefficiency is the matching externality. In general, a firm posting a vacancy generates externalities to both workers and firms. On the worker side, an increased vacancy raises the probability of an unemployed worker finding a job. This externality does not lead to inefficient allocation here because the workers do not make a decision. On the firm side, the increase in vacancy by one firm makes the matching of the other firms difficult due to congestion. The firm does not account for this congestion externality, and thus, the outcome is an excessive number of vacancies. To see why this externality is related to $\eta(\theta)$ in equation (2), consider the effect of a marginal increase in vacancies by one firm. From this firm's (private) perspective, the expected number of matches increases by $q(\theta) = M(u, v)/v$. From the social perspective, however, the increase in the match is $M_2(u, v)$, which is lower than $M(u, v)/v$ (recall that M is concave in each term). The difference $M_2(u, v) - M(u, v)/v$ represents the externality. It is straightforward to show that $M_2(u, v) - M(u, v)/v = -q(\theta)\eta(\theta)$, and thus, the term $-q(\theta)\eta(\theta)$ represents the (negative) externality each vacancy creation generates.

Now, consider how the inefficiency shows up in the calculation of the social and private values of the match. We refer to this margin as the "valuation margin." This exercise compares

$$(r + \sigma)\mu = z - h - (p(\theta)\mu - \kappa\theta) \tag{4}$$

and

$$(r + \sigma)S = z - h - p(\theta)\gamma S. \tag{5}$$

In equation (4), the (flow) social value of the match is the sum of the current surplus $z - h$ and the opportunity cost of keeping a worker employed. The opportunity cost is that, by keeping the worker unemployed, she could have generated a new match with probability $p(\theta)$. However, this calculation does not take into account that increasing u changes θ if v is kept constant. To keep θ constant, v has to be increased by θ units (because $(v + \theta)/(u + 1) = v/u$). These additional vacancies would cost $\kappa\theta$, which is subtracted from the gain from matching. Another way of thinking about the second term is the match generated by the marginal (unemployed) worker, $M_1(u, v)$, times the value of the match, μ. This fact can be seen from $p(\theta)\mu - \kappa\theta = p(\theta)\mu - \theta q(\theta)(1 - \eta(\theta))\mu = \theta q(\theta)\eta(\theta)\mu$, which is equal to $M_1(u, v)\mu$. In the market equilibrium equation (5), the second term on the right-hand side, $p(\theta)\gamma S$, represents the worker's opportunity cost from working (not searching). The full opportunity cost is $p(\theta)S$, but here, the fraction $p(\theta)(1 - \gamma)S$ is unaccounted for because the worker only receives a γ fraction of the surplus.

A similar logic as above holds: the worker's private value looks at $p(\theta) = M(u, v)/u$, whereas the social value is $M_1(u, v)$—the private value does not take into account the negative externality imposed on the other workers. The private value corresponds to $p(\theta)\mu$, and the externality term is $\kappa\theta$ (it is the amount of resources required to "undo" the externality). Under the Hosios condition, this inefficiency is offset by the inefficiency that the worker only recognizes the γ fraction of the opportunity cost (the surplus it could have created by being unemployed).

As we can see, the Hosios condition achieves "offsetting one inefficiency by another inefficiency" on the firm side as well as the worker side. It can "kill two birds with one stone" because both inefficiencies are symmetric: the firms and the unemployed workers create similar externalities, and they are on the opposite sides of the bargaining. Moreover, the direction of the total inefficiencies is aligned when the Hosios conditions do not hold: when γ is too small, S is too large in the valuation margin. These two imply $(1 - \gamma)S$ is too large in the investment margin. Thus, θ is too large in equilibrium.

It is useful to note that, in both margins, the nature of the inefficiencies are different even though they can exactly offset each other via the Hosios condition. In particular, the search externality is imposed on *other firms*, whereas the imperfect appropriation is about the *own match*.[5]

3 The model with on-the-job search

In this section, we present our main model. The model now features on-the-job search.

3.1 Market equilibrium

The economy is populated by a unit mass of workers. All workers have the following linear preferences:

$$\int_0^\infty e^{-rt} c_t dt,$$

where $r > 0$ is, once again, the discount rate, and c_t is the consumption at t. Workers receive wages (and consume them) while working and enjoy h units of consumption (home production) while unemployed.

Here, the new assumption is that workers search both on and off the job. That is, matches are heterogeneous, and an employed worker may meet with a new job (new match). When the new job is better, the worker moves. Thus, over time, a worker may climb up the job ladder.

Firms create vacancies at per-vacancy cost $\kappa > 0$. When firms meet with workers, they draw permanent match quality, $z \in [0, \infty)$, with pdf $g(z)$ and cdf $G(z)$. We assume the match quality has a finite mean. Firms with match quality $A_t z$ produce z units of output in each period, where A_t is the aggregate productivity.

The economy is subject to matching friction. We normalize the search efficiency of unemployed workers to one. Employed workers have a search intensity of $\zeta \in [0, 1]$. The total efficiency of the search on the worker side is

$$x_t \equiv u_t + \zeta(1 - u_t).$$

We assume the matching is random. Let

$$f_t^u \equiv \frac{u_t}{x_t}, \quad f_t(z) \equiv \frac{\zeta n_t(z)}{x_t} \tag{6}$$

be the probability (conditional on the meeting) of a vacancy encountering unemployed workers and the probability density (conditional on the meeting) of a vacancy encountering employed workers with match quality z, respectively. Here, $n_t(z)$ denotes the measure of workers employed at z.

[5] Also note that the valuation margin does not show up in a two-period model where the first-period search creates the second-period match, because the inefficiency stems from (i) the externality imposed on the unemployed workers' search in the second period and (ii) the opportunity cost of the second-period search is not properly taken into account.

Given the number of vacancies created (v_t), the number of matches in the economy is given by the matching function $M(x_t, v_t)$, where we assume this function is increasing in both terms and exhibits constant returns to scale. We denote

$$\theta \equiv \frac{v_t}{x_t}$$

as the labor market tightness. The Poisson rate that an unemployed worker meeting a vacancy and a vacancy meeting a worker is $p(\theta_t) \equiv M(1, \theta_t)$ and $q(\theta_t) \equiv M(1/\theta_t, 1)$, respectively. The meeting rate of workers employed in z is $\zeta p(\theta_t)$. The match separates at rate $\sigma > 0$.

We let U_t denote a worker's value of being unemployed, $E_t(z, \bar{O})$ denote the value of an employed worker with match quality z and the worker's outside option $\bar{O}$, $J_t(z, \bar{O})$ denote the value of a firm with match quality z and the worker's outside option $\bar{O}$, and V denote the value of a vacancy. Define the joint match surplus to be

$$S_t(z) \equiv W_t(z, \bar{O}) + J_t(z, \bar{O}) - U_t - V_t, \tag{7}$$

where $S_t(z)$ is independent of the worker's outside option and increasing in z, which we confirm below.

The wages are determined by Nash bargaining, where the worker's outside option may depend on the history of past meetings. When unemployed workers decide to form a match, they engage in Nash bargaining with the worker's bargaining weight $\gamma \in [0, 1]$. The outside option for unemployed workers is the value of unemployment, U_t.

When employed workers at firm z with outside option $\bar{O}$ meet with poachers z', the following takes place. When $S_t(z) \geq S_t(z')$, workers stay and bargain with outside option $\max\{\bar{O}, U_t + \omega S_t(z')\}$, where $\omega \in [0, 1]$ is a parameter governing the degree of offer-matching and $S_t(z')$ is the joint match surplus of job z', which we define below. When $S_t(z) < S_t(z')$, workers are poached, and the outside option of workers is given by $U_t + \omega S_t(z)$. The offer-matching parameter ω is sufficiently flexible to nest many existing wage-setting protocols. Two important special cases are the following:

1. Nash bargaining with no commitment (e.g., McCrary, 2022; Borovičková and Shimer, 2024): $\omega = 0$;

2. Sequential auction (e.g., Dey and Flinn, 2005; Cahuc et al., 2006): $\omega = 1$.[6]

We will highlight these two cases later on.

The value function of the unemployed is

$$rU_t = h + p(\theta_t) \int g(z) \max\{W_t(z, U_t) - U_t, 0\} dz + \dot{U}_t, \tag{8}$$

[6] A subtle point is that here we assume the wage renegotiation occurs continuously, as in the standard DMP model, whereas in Dey and Flinn (2005) and Cahuc et al. (2006), a negotiation occurs only when a new meeting happens. This difference does not result in different outcomes in the steady state.

where the "dot notation" represents the time derivative: $\dot{U}_t \equiv \partial U_t / \partial t$. Value function of workers employed in firm z with an outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned} rW_t(z, \bar{O}) = & \, w_t(z, \bar{O}) \\ & + \zeta p(\theta_t) \int_z^\infty g(z') \left(W_t(z', U_t + \omega S_t(z)) - W_t(z, \bar{O}) \right) dz' \\ & + \zeta p(\theta_t) \int_0^z g(z') \left(W(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - W(z, \bar{O}) \right) dz' \\ & + \sigma(U_t - W_t(z, \bar{O})) + \dot{W}_t(z, \bar{O}). \end{aligned} \tag{9}$$

The value of the filled job with productivity z and outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned} rJ_t(z, \bar{O}) = & \, A_t z - w_t(z, \bar{O}) \\ & + \zeta p(\theta_t) \int_z^\infty g(z') \left(V_t - J_t(z', \bar{O}) \right) dz' \\ & + \zeta p(\theta_t) \int_0^z g(z') \left(J_t(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - J_t(z, \bar{O}) \right) dz' \\ & + \sigma \left(V_t - J_t(z, \bar{O}) \right) + \dot{J}_t(z, \bar{O}). \end{aligned} \tag{10}$$

Given the above bargaining protocol, the outside option $\bar{O}$ cannot exceed $U + \omega S(z)$ for a match with productivity z along the equilibrium path. Thus, we ignore such a possibility. The value of vacancy is given by

$$\begin{aligned} V_t = & -\kappa \\ & + f_t^u q(\theta_t) \int_0^\infty g(z) \max\{J_t(z, U) - V_t, 0\} dz \\ & + q(\theta_t) \int_0^\infty g(z) \int_z^\infty f(z') \max\{J_t(z, U + \omega S_t(z')) - V_t, 0\} dz' dz + \dot{V}_t, \end{aligned} \tag{11}$$

where $\kappa > 0$ is the vacancy-posting cost. We assume free entry into vacancy posting. This assumption implies zero value from vacancy posting:

$$V_t = 0. \tag{12}$$

The Nash bargaining with worker's outside option $\bar{O}$ solves

$$\max_{w_t(z, \bar{O})} \left(W_t(z, \bar{O}) - \bar{O} \right)^\gamma \left(J_t(z, \bar{O}) - V_t \right)^{1-\gamma}.$$

After imposing the free-entry condition (12), the solution to the bargaining problem yields

$$W_t(z, \bar{O}) = \bar{O} + \gamma(S_t(z) - (\bar{O} - U_t)) \tag{13}$$

and

$$J_t(z, \bar{O}) = (1 - \gamma)(S_t(z) - (\bar{O} - U_t)). \tag{14}$$

Plugging (8), (9), and (10) into (7) and using (13), we obtain the recursive expression for the job-match surplus:

$$
\begin{aligned}
(r+\sigma)S_t(z) = {} & A_t z - h \\
& + \zeta p(\theta) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z) \right] - S_t(z) \right) dz' \\
& - p(\theta_t) \int_{\underline{z}_t}^\infty g(z') \gamma S_t(z') dz' + \dot{S}_t(z).
\end{aligned} \tag{15}
$$

The reservation match quality $\underline{z}_t$, above which the match is formed, satisfies

$$S_t(\underline{z}_t) = 0. \tag{16}$$

The expression (15) confirms our original presumption that the joint match surplus is independent of the worker's outside option $\bar{O}$.

After imposing free-entry condition (12) and also (14) in (11), we obtain

$$\kappa = (1-\gamma) q(\theta_t) \left[f_t^u \int_{\underline{z}} g(z) S_t(z) dz + \int_0^\infty \int_z^\infty f_t(z) g(z') \left[S_t(z') - \omega S_t(z) \right] dz' dz \right]. \tag{17}$$

We denote the mass of workers employed at match quality below z as $N_t(z)$. Its law of motion satisfies

$$\dot{N}_t(z) = \left(G(z) - G(\underline{z}_t) \right) p(\theta_t) u_t - N_t(z)(1 - G(z)) \zeta p(\theta_t) - \sigma N_t(z), \tag{18}$$

with a boundary condition $N_t(\underline{z}_t) = 0$. By its definition,

$$\partial_z N_t(z) = n_t(z), \tag{19}$$

where $\partial_z N_t(z) \equiv \partial N_t(z)/\partial z$. The law of motion of unemployment over the time interval dt is

$$du_t = \left[-p(\theta_t)(1 - G(\underline{z}_t)) + \sigma(1 - u_t) \right] dt + dN_t(\underline{z}_t), \tag{20}$$

where $dN_t(\underline{z}_t)$ is the mass of workers endogenously separating over the time interval dt.

We are now ready to define equilibrium.

Definition 1 *Given $\{N_0(z), u_0\}$, the equilibrium consists of a sequence of the joint match surplus $\{S_t(z)\}$, the employment distribution across the job-ladder, $\{N_t(z), n_t(z), f_t(z), f_t^u\}$, and the unemployment rate, $\{u_t\}$, market tightness, $\{\theta_t\}$, and reservation match quality $\{\underline{z}_t\}$ such that (6), (7), (16), (17), (18), (19), and (20) hold. The steady-state equilibrium is the one where all variables are constant over time.*

3.2 Social planner's problem

The social planner directly controls the offer-acceptance decisions of all workers and thus controls all worker flows (subject to matching frictions), as well as the creation of vacancies. Let $\mathbb{I}_t^{UE}(z)$ be an indicator function that takes a value of 1 if an unemployed worker meeting a job with match quality z accepts an offer. Likewise, let $\mathbb{I}_t^{EE}(z,z')$ be an indicator function that takes a value of 1 if an employed worker with match quality z meeting a job with match quality z' accepts an offer. Finally, ς_t denotes the mass of workers for whom the planner resolves the match at time t.

The social planner's problem is to choose $\{\theta_t, n_t(z), \mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z,z'), \varsigma_t\}$ to maximize

$$\int_0^\infty e^{-rt}\left[\int_0^\infty A_t z n_t(z)dz + h\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\theta_t\left(1-\int_0^\infty n_t(z)dz + \int_0^\infty \zeta n_t(z)dz\right)\right]dt$$

subject to

$$\dot{n}_t(z) = \left(1-\int n_t(z')dz'\right)p(\theta_t)g(z)\mathbb{I}_t^{UE}(z) + \int_0^\infty p(\theta_t)g(z)\mathbb{I}_t^{EE}(z',z)\zeta n_t(z')dz' - \int_0^\infty p(\theta_t)g(z')\mathbb{I}_t^{EE}(z,z')\zeta n_t(z)dz' - \sigma n_t(z) - \varsigma_t. \tag{21}$$

The first term in the square brackets in the objective function is the production by active matches, the second term is the home production, and the third term is the vacancy-posting cost κv_t. For the constraint, the change in the number of matches with the match quality z is due to (i) matches created with workers moving from U to E plus (ii) E to E movements from another z' to z minus (iii) E to E movements out of z (to another z') and separation.

The current-value Hamiltonian for this problem is

$$\begin{aligned} H = &\int_0^\infty A_t z n_t(z)dz + h\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\theta_t\left(1-\int_0^\infty (1-\zeta)n_t(z)dz\right) \\ &+ \int_0^\infty \mu_t(z)\Bigg[\left(1-\int_0^\infty n_t(z')dz'\right)p(\theta_t)g(z)\mathbb{I}_t^{UE}(z) + \zeta\int_0^\infty p(\theta_t)g(z)\mathbb{I}_t^{EE}(z',z)n_t(z')dz' \\ &\qquad - \zeta\int_0^\infty p(\theta_t)g(z')\mathbb{I}_t^{EE}(z,z')dz' - \sigma n_t(z) - \varsigma_t\Bigg]dz, \end{aligned}$$

where $\mu_t(z)$ is the costate variable that represents the shadow value of the constraint (21). Thus, $\mu_t(z)$ is the shadow value of creating one unit of match with match quality z. For this reason, we refer to $\mu_t(z)$ as the match surplus from the planner's perspective.

The optimality conditions for $\{\mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z,z')\}$ are

$$\mathbb{I}_t^{UE}(z) = \begin{cases} 1 & \mu_t(z) > 0 \\ 0 & \mu_t(z) \le 0 \end{cases}, \qquad \mathbb{I}_t^{EE}(z,z') = \begin{cases} 1 & \mu_t(z') > \mu_t(z) \\ 0 & \mu_t(z') \le \mu_t(z). \end{cases}$$

The optimality condition for endogenous separation ς_t implies $\mu_t(z) \ge 0$ for all z with $n_t(z) > 0$.

The first-order optimality condition on $n_t(z)$ is

$$(r+\sigma)\mu_t(z) = A_t z - h - \int_{\underline{z}_t}^\infty p(\theta_t)g(z')\mu_t(z')dz' + \zeta p(\theta_t)\int_z^\infty g(z')(\mu_t(z')-\mu_t(z))dz' + \kappa\theta_t(1-\zeta) + \dot{\mu}_t(z), \tag{22}$$

where we have already imposed the fact that $\mu_t(z)$ is increasing in z. The reservation match quality $\underline{z}_t$, above which the planner forms a match, satisfies

$$\mu_t(\underline{z}_t) = 0.$$

We can rewrite the first-order optimality condition for θ_t as

$$\kappa = (1-\eta(\theta_t))q(\theta_t)\left[f_t^u \int_{\underline{z}_t}^{\infty} g(z')\mu(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{z}^{\infty} f_t(z)g(z')\left(\mu_t(z') - \mu_t(z)\right)dz'dz\right], \tag{23}$$

where f_t^u and $f_t(z)$ are defined as (6) and $\eta(\theta) \equiv -\theta q'(\theta)/q(\theta)$. The left-hand side is the cost of posting one additional vacancy, and the right-hand side is the marginal increase in the matching probability $p'(\theta_t) = (1-\eta(\theta_t))q(\theta_t)$ times the value of the match the vacancy creates. With frequency f^u, the vacancy meets an unemployed worker and generates value $\mu_t(z')$ with probability (density) $g(z')$ (when $\mu_t(z') > 0$). With frequency $f_t(z)$, it meets with an employed worker with quality z and draws the new match quality z' with density $g(z')$. When $\mu_t(z') > \mu_t(z)$, the worker moves to the new job, and the social value $\mu_t(z') - \mu_t(z)$ is generated.

3.3 Efficiency

Following the same steps as in Section 2, we organize the source of the discrepancy between the planner's problem and the market equilibrium into two margins: the investment margin and the valuation margin.

3.3.1 Investment margin

We first consider the investment margin. In equilibrium, condition (17) represents the firm's incentive to invest in vacancy. In the planner's problem, the equation for this margin is (23). Let us compare these two equations. Suppose the Hosios condition (1) holds. If f_t^u and $f_t(z)$ are common and $\mu_t(z) = S_t(z)$, the equilibrium θ_t and the optimal θ_t coincide when the wage-setting protocol follows the sequential auction, that is, $\omega = 1$. If the wage-setting rules are different, the efficiency of the equilibrium θ_t is not guaranteed even in this situation. For example, with Nash bargaining, that is, $\omega = 0$, the value of equilibrium θ_t is too high compared to the social optimum. Intuitively, this result is because of the *"worker-stealing" externality*: the poaching firm does not internalize the loss of the poached firm.[7] With the sequential auction, the poaching firm pays for the loss (not to the poached firm, but to the worker) in the form of higher wages. Note that from (18) and (20)

[7] This externality is analogous to the business-stealing externality in the economic growth literature. Gautier et al. (2010) emphasize a similar externality that arises in their model. However, Gautier et al. (2010) do not formally show the relationship between this externality and the sequential auction, as we do here. Note that Gautier et al. (2010) do not analyze the valuation margin, which is an essential element of our analysis.

and the definitions of f^u and $f_t(z)$ (i.e., (6)), the steady-state values of f^u and $f(z)$ are the same when θ is the same. Thus, an important question for efficiency under the Hosios condition (and sequential auction in particular) is whether $\mu_t(z) = S_t(z)$ holds.

3.3.2 Valuation margin

To see how $\mu_t(z)$ and $S_t(z)$ are determined, let us look at the valuation margin. On the valuation margin, we rewrite the equation for the social planner (22) as:

$$\begin{aligned}(r+\sigma)\mu_t(z) - \dot{\mu}_t(z) = A_t z - h - p(\theta_t)\int_{\underline{z}_t} g(z')\mu_t(z')dz + \zeta p(\theta_t)\int_{z}^{\infty} g(z')(\mu_t(z') - \mu_t(z))dz' \\ + \Delta^{SP}(\theta_t),\end{aligned} \tag{24}$$

where

$$\begin{aligned}\Delta^{SP}(\theta) \equiv & \underbrace{(1-\eta(\theta))p(\theta)\left[f_t^u \int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{\tilde{z}}^{\infty} f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right]}_{\text{positive externality of not searching off the job} \equiv \Delta^{SP,UE}(\theta)} \\ & \underbrace{-\zeta(1-\eta(\theta_t))p(\theta)\left[f_t^u \int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{\tilde{z}}^{\infty} f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right]}_{\text{negative externality of searching on the job} \equiv \Delta^{SP,EE}(\theta)}\end{aligned} \tag{25}$$

is the term reflecting the net externality from forming an employment relationship. The first term $A_t z - h$ on the right-hand side of (24) is the static gain from moving one unemployed worker to employment with the match quality z. The second term is the opportunity cost of not being able to conduct an off-the-job search. This cost is offset by the possibility of an on-the-job search, which is the last term in the first line.

The term $\Delta^{SP}(\theta)$ summarizes net externalities from forming an employment relationship. The first term inside $\Delta^{SP}(\theta)$ is the positive externality of not doing an off-the-job search, which raises the probability of a match for all the other workers. The second term inside $\Delta^{SP}(\theta)$ is the negative externality of an on-the-job search, which lowers the probability of a match for all the other workers as long as $\zeta > 0$. Note that when $\zeta = 1$ (employed workers have the same search efficiency as unemployed workers), the two externalities cancel out so that $\Delta^{SP}(\theta) = 0$. When $\zeta = 1$, workers impose the same congestion externality on the economy regardless of whether they are employed or unemployed. We will revisit this case later.

For the equilibrium, we rewrite (15) to facilitate with the comparison with (24):

$$\begin{aligned}(r+\sigma)S_t(z) - \dot{S}_t(z) = A_t z - h - p(\theta_t)\int_{\underline{z}_t} g(z')S_t(z')dz' + \zeta p(\theta_t)\int_{z}^{\infty} g(z')\left(S_t(z') - S_t(z)\right)dz' \\ + \Delta^{DE}(z,\theta_t),\end{aligned} \tag{26}$$

where

$$\Delta^{DE}(z,\theta) \equiv \begin{array}{c} \underbrace{(1-\gamma)p(\theta)\int_{\underline{z}_t}^{\infty} g(z')S_t(z')dz'}_{\text{cost of not searching off the job that is unaccounted for } \equiv \Delta^{DE,UE}(z,\theta)} \\ \underbrace{-\zeta(1-\gamma)p(\theta)\int_{z}^{\infty} g(z')\left(S_t(z')-\omega S_t(z)\right)dz'}_{\text{benefit of searching on-the-job that is unaccounted for } \equiv \Delta^{DE,EE}(z,\theta)} \end{array} . \tag{27}$$

The first line in (26) is analogous to that in the planner's solution (24). The term $\Delta^{DE}(z,\theta)$ summarizes the net opportunity cost that is unaccounted for by the match. The first term inside $\Delta^{DE}(z,\theta)$ is the opportunity cost of an off-the-job search that is unaccounted for. By being employed, a worker loses the opportunity for an off-the-job search, but that opportunity cost shows up only as a γ fraction because $(1-\gamma)$ fraction of the surplus accrues to the firm due to Nash bargaining. Similarly, the second term inside $\Delta^{DE}(z,\theta)$ is the benefit of an on-the-job search that is unaccounted for. Among the total value of the new match $S_t(z')$, the amount $(1-\gamma)(S_t(z')-\omega S_t(z))$ accrues to *the new employer*, corresponding to (14). (The rest goes to the worker.) This amount is not accounted for when the "opportunity gain" of the on-the-job search is counted in the surplus.

The comparison between $\Delta^{SP}(\theta)$ and $\Delta^{DE}(z,\theta)$ reveals these two are not necessarily equal even under the Hosios condition. In fact, this point is the fundamental reason that the efficiency in the valuation margin, $\mu_t(z)=S_t(z)$, is difficult to achieve. First and foremost, there is a fundamental difference between $\Delta^{SP}(\theta)$ and $\Delta^{DE}(z,\theta)$: $\Delta^{SP}(\theta)$ does not depend on z, while $\Delta^{DE}(z,\theta)$ does. A worker's on-the-job search imposes externality to *all other* employed workers. The (unaccounted-for) opportunity cost in equilibrium is fundamentally about *the current match*. Consider a match with a high z. The private gain from the on-the-job search on such a job is small, and thus, the unaccounted benefit is small. As a consequence, such a match has a large $\Delta^{DE}(z,\theta)$ compared to a low-z match. In contrast, the externality that one employed worker imposes on others is the same regardless of the value of z. The heterogeneity of z across workers is a ubiquitous feature of a job-ladder model because there are no social gains from job-to-job transitions without such heterogeneity. This heterogeneity necessarily creates a disagreement between the externality on other matches and the opportunity cost within the own match.

To further facilitate the intuition and direction of inefficiency, we highlight the two special cases that provide a sharp characterization of the difference between $\Delta^{SP}(\theta)$ and $\Delta^{DE}(z,\theta)$.

Case 1: $\zeta=1$. Consider the case with $\zeta=1$; that is, the efficiency of an on-the-job search is the same as that of an off-the-job search. Under $\zeta=1$, as we observed above,

$$\Delta^{SP}(\theta)=0.$$

From the planner's perspective, forming a match has a net externality of zero because both employed and unemployed workers equally congest the matching market. Meanwhile, when $\zeta = 1$,

$$\Delta^{DE}(z,\theta) > 0$$

for any z and θ. This inequality reflects that the imperfect appropriation of surplus from an on-the-job search is always smaller than that from an off-the-job search because the surplus gain from an on-the-job search is lower than that from an off-the-job search.

First, consider the case with $\omega = 0$. Although both situations divide the surplus from new employment $S_t(z')$, on-the-job searchers have a better outside option and thus only accept the jobs with $z' > z$, whereas unemployed workers accept offers with $z' > \underline{z}_t$. Because $(1-\gamma)$ fraction of the surplus from the new match belongs to the new employer in both cases, the "undervaluation of the opportunity cost" from off-the-job search is always larger than the "undervaluation of the opportunity gain" from on-the-job search, resulting in

$$\Delta^{DE}(z,\theta) > \Delta^{SP}(\theta) = 0.$$

Positive offer-matching, $\omega > 0$, further reduces the surplus that belongs to the new match, and exacerbates the gap between $\Delta^{DE}(z,\theta)$ and $\Delta^{SP}(\theta)$. Note that this conclusion holds for any value of γ and $\eta(\theta)$.

We can make two further observations, which will be useful later. First, as we explained earlier, $\Delta^{DE}(z,\theta)$ is increasing in z, and thus, the gap between $\Delta^{DE}(z,\theta)$ and $\Delta^{SP}(\theta)$ is larger when z is larger. This relationship implies that the over-valuation of the match is particularly pronounced at the top of the job ladder. Second, the gap between $\Delta^{DE}(z,\theta)$ and Δ^{SP} is increasing in market tightness θ and is larger when the match surplus $S_t(z)$ is larger. This relationship explains why the discrepancy between the planner's solution and the equilibrium is exacerbated during the labor market booms driven by an increase in the aggregate productivity, which we show in Section 5.4.

Case 2: Degenerate $g(z)$. Another case that sheds insight is when $g(z)$ is degenerate, that is, when z is the same across all workers at $z = \bar{z}$ (or consider the limit where the variance of z is vanishingly small). In such a case,

$$\Delta^{SP}(\theta) = (1-\zeta)(1-\eta(\theta))p(\theta)f_t^u\mu_t(\bar{z})$$

and

$$\Delta^{DE}(\bar{z},\theta) = (1-\gamma)p(\theta)S_t(\bar{z}) - \zeta(1-\gamma)p(\theta)(1-\omega)S_t(\bar{z}),$$

for any value of θ. It is straightforward to see that, whenever the Hosios condition ($\eta(\theta) = \gamma$) holds and $\zeta \in (0,1)$,

$$\Delta^{DE}(\bar{z},\theta) > \Delta^{SP}(\theta).$$

When all jobs are homogenous, there is no social gain from on-the-job search. Therefore, the externality is imposed only on unemployed workers, reflected in $f_t^u < 1$ term in $\Delta^{SP}(\theta)$. In contrast, when $\omega = 0$, the difference in the surplus from off-the-job and on-the-job search that is unaccounted for is $(1-\zeta)(1-\gamma)p(\theta)S_t(\bar{z})$. Due to the absence of $f_t^u < 1$, the net opportunity cost of an on-the-job search that is not accounted for by a private agent is always larger than the net externality from an on-the-job search. This results in the over-valuation of the job. Offer-matching $\omega > 0$ further reduces the surplus that accrues to the new matches and thus only makes the difference between $\Delta^{DE}(\bar{z},\theta)$ and $\Delta^{SP}(\theta)$ larger. Once again, this gap tends to be exacerbated when the match surplus is larger.

3.4 The role of market segmentation

In the previous section, we highlighted the importance of two factors that impede efficiency along the valuation margin. First, moving a worker from unemployment to employment imposes externalities both to the unemployed workers and other employed workers. Second, because employed workers are heterogeneous, offsetting the externalities to other workers by proportionally reducing the opportunity cost of employment for one worker is difficult.

To clarify these two factors further, in Appendix B, we analyze a model where (i) the matching functions for the unemployed and the employed are segmented, and (ii) for the employed, the matching functions for different values of z are segmented. We find that the equilibrium outcome is efficient with the Hosios condition and sequential auction ($\omega = 1$).[8] Both are needed for efficiency—just segmenting the unemployed market and the employed market is not sufficient. The results are analogous to the efficiency result in Menzio and Shi (2011), who show the efficiency of a job-ladder model with directed search. Our model features a random search, and for efficiency, in addition to segmentation, the Hosios condition and sequential auction mechanism are required.

4 Analytical characterizations

The intuition highlighted in the previous section extends to a broader range of models with a similar structure. In this section, we exploit the simplicity of the current model to derive further analytical results. All proofs are contained in Appendix C.

The main result is Proposition 1 below. It shows that, under the Hosios condition and a wide range of wage-setting mechanisms, the equilibrium vacancies are excessive compared with the social efficiency. This result underscores that the negative externalities an employed worker (especially the one with a high z) imposes on other workers (both unemployed workers and the other

[8] We thank Xincheng Qiu for suggesting this exercise.

employed workers) are difficult to offset with other inefficiencies in equilibrium.

Proposition 1 is important, particularly in the context of policy evaluations. It implies assuming the Hosios condition is not "neutral" in a job-ladder model, as it automatically favors a policy that discourages vacancy posting. This result highlights the fundamental difference from the model without an on-the-job search, where the Hosios condition implies no intervention is optimal. Beyond isolating the role of on-the-job search as a source of inefficiency, Proposition 1 is directly relevant for existing applied works, as it has been common to impose Hosios condition in the calibration of quantitative DMP literature.[9] In the case of a job-ladder model, this calibration implies policies that encourage or discourage vacancy posting give rise to inefficiency even without other distortions.

4.1 Market equilibrium

The decentralized equilibrium can be characterized by a system of two linear partial differential equations. Taking the derivative of (15) with respect to z, the match surplus solves

$$\begin{aligned}[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))]\,\partial_z S_t(z)&\\ = A_t+(1-\gamma)(1-\omega)\zeta p(\theta_t)g(z)S_t(z)+\partial_z \dot{S}_t(z),&\end{aligned}$$

with a boundary condition $S_t(\underline{z}_t)=0$. The reservation match quality $\underline{z}_t$ satisfies

$$0=A\underline{z}_t-h-\gamma(1-\zeta)p(\theta_t)\int_{\underline{z}_t}^{\infty} g(z')S_t(z')dz'.$$

The second partial differential equation is the evolution of employment distribution (18).

Evaluating these partial differential equations at the steady state, the steady-state equilibrium can be characterized by two equations with two unknowns, which are useful later.

Lemma 1 *The steady-state equilibrium market tightness and reservation match quality, $\{\theta,\underline{z}\}$, jointly solve*

$$0=A\underline{z}-h-A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z} \tag{28}$$

and

$$\begin{aligned}\kappa=A(1-\gamma)q(\theta)\Bigg[&\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz\\ &+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg],\end{aligned} \tag{29}$$

[9] In the models without on-the-job search, this calibration is frequently used since Shimer (2005). Recent examples that employ this calibration strategy in a model with on-the-job search are Engbom (2019) and Faberman et al. (2022).

where $\Gamma(\tilde{z};\theta) \equiv r + \sigma + (1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))$. *Moreover,* (28) *defines a weakly increasing relationship between* $\underline{z}$ *and* θ*, which we write as* $\underline{z}^R(\theta)$*, and* (29) *defines a strictly decreasing relationship between* $\underline{z}$ *and* θ*, which we write as* $\underline{z}^{FE}(\theta)$*.*

Given $\{\underline{z}, \theta\}$, the rest of the equilibrium can be obtained as follows. The steady-state match surplus is given by

$$S(z) = A\int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}} d\tilde{z} \quad \text{for} \quad z \geq \underline{z},$$

the steady-state employment distribution is

$$N(z) = \frac{(G(z)-G(\underline{z}))\, p(\theta) u}{\sigma + (1-G(z))\zeta p(\theta)}, \tag{30}$$

and the steady state unemployment rate is

$$u = \frac{\sigma}{\sigma + (1-G(\underline{z}))p(\theta)}. \tag{31}$$

4.2 Social planner's problem

As in the decentralized equilibrium, the social value of a job satisfies the following partial differential equation, which we obtain by taking the derivative of (22) with respect to z:

$$[r + \sigma + \zeta p(\theta_t)]\, \partial_z \mu_t(z) = A_t + \partial_z \dot{\mu}_t(z),$$

with a boundary codnition $\mu_t(\underline{z}_t) = 0$. The reservation match quality $\underline{z}$ solves

$$0 = A_t \underline{z}_t - h - (1-\zeta)p(\theta_t)\int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \kappa\theta_t(1-\zeta).$$

Evaluating these equations at the steady state, the efficient steady-state allocation can be tractably characterized as follows.

Lemma 2 *The efficient steady-state market tightness and reservation match quality* $\{\theta^{SP}, \underline{z}^{SP}\}$ *jointly solve*

$$\begin{aligned} 0 = {} & A\underline{z}^{SP} - h \\ & - A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma\eta(\theta^{SP}) + (1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma + (1-G(\tilde{z}))\zeta p(\theta^{SP})} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z} \end{aligned} \tag{32}$$

and

$$\kappa = A(1-\eta(\theta))q(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty} \frac{\sigma}{\sigma + \zeta p(\theta^{SP})(1-G(\tilde{z}))} \frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))} d\tilde{z}. \tag{33}$$

Moreover, there exists $k < 0$ *such that if* $\eta'(\theta) > k$*,* (32) *defines a weakly increasing relationship between* $\underline{z}^{SP}$ *and* θ*, which we write as* $\underline{z}^{SP,R}(\theta)$*, and* (33) *defines a strictly decreasing relationship between* $\underline{z}^{SP}$ *and* θ*, which we write as* $\underline{z}^{FE,R}(\theta)$*.*

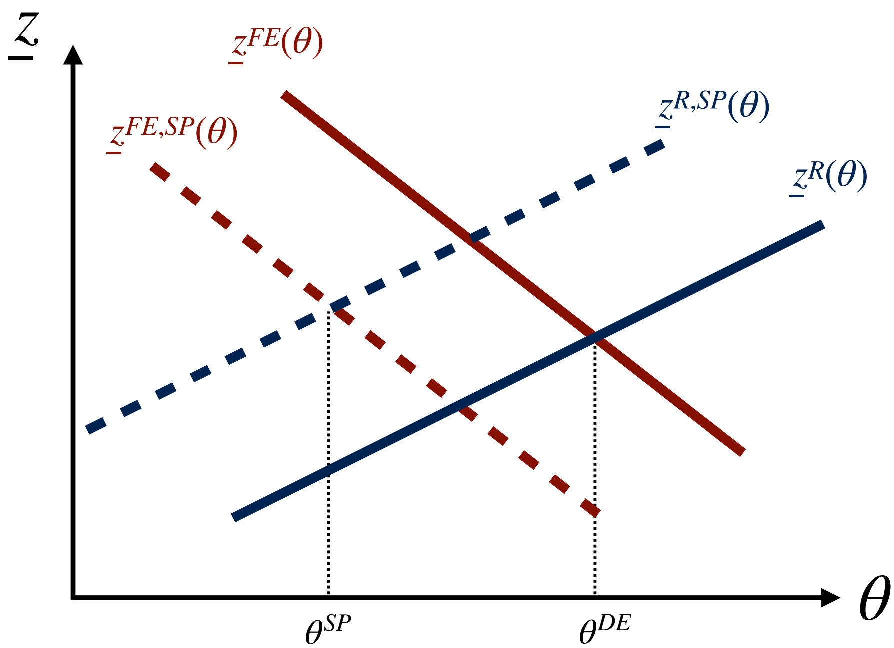

Figure 1: Inefficiency with on-the-job search

Given $\{\underline{z}^{SP}, \theta^{SP}\}$, the social value of a job can be obtained as

$$\mu(z) = A \int_{\underline{z}^{SP}}^{z} \frac{1}{r + \sigma + \zeta p(\theta^{SP})(1 - G(\tilde{z}))} d\tilde{z},$$

and the employment distribution is given by (30) and (31).

4.3 Results

In Section 3.3, through several special cases, we have demonstrated that the presence of on-the-job search is a force toward the over-creation of jobs both from the valuation and investment margins. The following proposition shows that the conclusion carries over much more generally.

Proposition 1 *Suppose $\gamma = \eta(\theta^{SP})$ for all θ. Then, the steady-state equilibrium market tightness θ in the market equilibrium is higher than the socially efficient level when $\zeta > 0$.*

In Appendix C.3, we prove a more general version of the result that there exists $\hat{\gamma} > \eta(\theta^{SP})$ such that if $\gamma < \hat{\gamma}$, the steady-state equilibrium market tightness θ in the market equilibrium is higher than the socially efficient level.

The logic underlying the proof can be graphically explained in Figure 1. The downward-sloping red solid line plots the relationship $\underline{z}^{FE}(\theta)$ in the decentralized equilibrium, derived from (29). The downward-sloping red dotted line plots the corresponding relationship for the planner's solution, denoted $\underline{z}^{FE,SP}(\theta)$, derived from (33). These lines are downward sloping because a larger $\underline{z}$ implies a less frequent match formation, and thus, a lower return from posting vacancy. It can be shown that the solid line ($\underline{z}^{FE}(\theta)$) lies to the right of the solid line ($\underline{z}^{FE,SP}(\theta)$) as long as $\zeta > 0$. Both equations (29) and (33) represent the optimal vacancy-posting condition. For a given reservation match quality ($\underline{z}$), the value of a vacancy is higher in the decentralized economy than in the social optimum.

The upward-sloping blue solid line represents the equilibrium relationship $\underline{z}^{R}(\theta)$, derived from (28). The upward-sloping blue dotted line is the corresponding relationship for the social planner, $\underline{z}^{R,SP}(\theta)$, derived from (32). These lines are upward-sloping because a higher frequency of match allows the match formation to be more selective. Once again, the equilibrium relationship lies to the right of the social planner's relationship. The decentralized equilibrium values a job more than the planner, and therefore, they are more likely to form a match given θ. From the graph, we can see the outcome is a higher market tightness in the decentralized equilibrium than the efficient level, that is, $\theta^{DE} > \theta^{SP}$.

In the previous section, we have shown that the overvaluation of the match surplus occurs in two special cases: (i) $\eta = 1$ and (ii) $g(z)$ is degenerate. Proposition 1 generalizes these results. In these special cases, we emphasized three reasons for overvaluation. First, the imperfect appropriation problem on the valuation margin is more severe for off-the-job search than on-the-job search because (i) job-to-job transitions occur less frequently than job finding from unemployment, and (ii) even when they occur, a smaller share of the surplus goes to the workers. This asymmetry leads to the undervaluation of the opportunity cost of moving a worker from unemployment to employment. Second, the positive externality that is imposed on other workers is totally (when $\eta = 1$) or partially (when $\eta < 1$) offset by the negative externality due to on-the-job search. Third, the externality imposed on employed workers does not have as large an impact on the social value as the externality on unemployed workers (which is a small part of the population) because the gain is incremental. All three intuitions apply to more general situations, which contributes to Proposition 1.

The natural next question is whether combinations of parameter values exist such that the decentralized equilibrium is efficient. Such situations can relatively easily be described in a special case with $\zeta = 1$. Under $\zeta = 1$ (the efficiency of on-the-job search is the same as unemployed), the reservation match quality in the decentralized equilibrium always coincides with the efficient level with $\underline{z} = h/A$. In such a situation, we can guarantee (the proof is omitted here) that a worker bargaining parameter γ exists such that efficiency is achieved. It can also be shown that, in such a case, γ is higher than what is prescribed by the Hosios condition. When $\zeta < 1$, two parameters (γ, ω) have to ensure the two equilibrium variables $(\underline{z}, \theta)$ are at the efficient level.

sectionDecentralization

In this section, we consider the implementation of the efficient allocation through taxes and subsidies. There are two goals. First, this exercise is of direct policy relevance because knowing how much (and what kind of) taxes should be imposed and how they depend on the economic situation is of practical use. Second, expressing the magnitude of inefficiencies through "wedges," or implicit taxes, as in Chari et al. (2007), clarifies the economic forces behind the inefficiencies.

Having these two goals in mind, we consider two types of implementations. The first is a simple implementation that utilizes two fiscal instruments: an entry tax and an unemployment insurance benefit. The second is a more general implementation that directly corrects the wedges characterized in the previous section.

4.4 Implementation with simple instruments

We first consider a simple implementation of efficient allocation using two instruments: an entry tax (a vacancy tax) and an unemployment insurance benefit. The government imposes a vacancy tax χ_t^e on firms so that the total cost of posting a vacancy is $\kappa + \chi_t^e$. The government also imposes an unemployment insurance benefit, b_t, on unemployed workers so that the flow value of unemployment is $h + b_t$. The government finances these fiscal instruments through a lump-sum tax or subsidy on workers.

With these taxes and benefits, the joint match surplus (15) is now given by

$$
\begin{aligned}
(r+\sigma)S_t(z) = {} & A_t z - h - b_t \\
& + \zeta p(\theta) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right) dz' \\
& - p(\theta_t) \int_{\underline{z}_t}^\infty g(z')\gamma S_t(z') dz' + \dot{S}_t(z).
\end{aligned}
$$

The free entry condition (17) is now modified as

$$
\kappa + \chi_t^e = (1-\gamma)q(\theta_t)\left[f_t^u \int_{\underline{z}} g(z)S_t(z)dz + \int_0^\infty \int_z^\infty f_t(z)g(z')\left[S_t(z') - \omega S_t(z)\right] dz'dz\right], \quad (34)
$$

and the following result holds.

Proposition 2 *There exists entry tax χ^e and unemployment insurance benefit b that implement the efficient allocation in the steady state. When the Hosios condition $\gamma = \eta(\theta)$ holds and in the presence of on-the-job search ($\zeta > 0$), the entry tax and the unemployment insurance benefit are both positive, that is, $\chi^e > 0$ and $b \geq 0$ (with strict inequality if $\zeta < 1$).*

Because there are only two endogenous variables (the reservation match quality $\underline{z}$ and the market tightness θ), these two instruments are sufficient to implement the efficient allocation, as the first part of Proposition 2 demonstrates. The second part of the proposition shows that, under the Hosios condition, both the entry tax and the unemployment insurance benefit are positive whenever there is an on-the-job search. As we show in Proposition 1, the equilibrium market tightness is higher than the efficient level. The positive entry tax corrects the over-creation of jobs. Moreover, as the decentralized equilibrium tends to overvalue a job, the positive unemployment insurance benefit raises the reservation match quality, thereby preventing the formation of too unproductive jobs.

4.5 Implementation that directly acts on wedges

Although the simple tax instruments above implement efficient allocation, they have shortcomings in that they do not directly map to the wedges that we characterize as the valuation margin and the investment margin. Moreover, the simple tax instruments are not robust in the sense that they fail to deliver efficient allocation in an environment with endogenous effort, as we show in Appendix E. Here, we consider an implementation that directly corrects the wedges induced by the search.

The government imposes taxes (or subsidies) on workers and firms. The government imposes a tax of $\tau^{s,u}$ on unemployed workers and $\zeta\tau^{s,e}(z)$ on employed workers. The government also imposes an entry tax χ^e so that the vacancy-posting cost that the firms face is given by $\chi^e_t + \kappa$. The government finances these fiscal instruments through a lump-sum tax (or subsidy) on workers. With these taxes, the joint match surplus (15) is now given by

$$\begin{aligned}(r+\sigma)S_t(z) = {} & A_t z - h + \tau^{s,u} - \zeta\tau^{s,e}(z) \\ & + \zeta p(\theta)\int_z^\infty g(z')\left(\omega S_t(z) + \gamma\left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right)dz' \\ & - p(\theta_t)\int_{\underline{z}_t}^\infty g(z')\gamma S_t(z')dz' + \dot{S}_t(z).\end{aligned} \tag{35}$$

The free entry condition is given by (34). In order for the decentralized equilibrium to achieve $S_t(z) = \mu_t(z)$ as well as $\theta^{DE}_t = \theta^{SP}_t$, the taxes on unemployed workers has to satisfy

$$\begin{aligned}\tau^{s,u}_t = {} & (1-\eta(\theta^{SP}_t))p(\theta^{SP}_t)\left[f^u_t\int_{\underline{z}_t}^\infty g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^\infty\int_{\tilde{z}}^\infty f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right] \\ & - (1-\gamma)p(\theta^{SP}_t)\int_{\underline{z}_t}^\infty g(z')\mu_t(z')dz'\end{aligned}, \tag{36}$$

the tax on employed workers has to satisfy

$$\begin{aligned}\tau^{s,e}_t(z) = {} & (1-\eta(\theta^{SP}_t))p(\theta^{SP}_t)\left[f^u_t\int_{\underline{z}_t}^\infty g(z')\mu_t(z')dz' + \int_{\underline{z}_t}^\infty\int_{\tilde{z}}^\infty f_t(\tilde{z})g(z')\left(\mu_t(z') - \mu_t(\tilde{z})\right)dz'd\tilde{z}\right] \\ & - (1-\gamma)p(\theta^{SP}_t)\int_z^\infty g(z')\left(\mu_t(z') - \omega\mu_t(z)\right)dz'\end{aligned}, \tag{37}$$

and the vacancy tax has to satisfy

$$\chi_t^e = (\eta(\theta_t^{SP}) - \gamma)\left[f_t^u \int_{\underline{z}_t} g(z')\mu(z')dz' + \int_{\underline{z}_t}^{\infty}\int_{z}^{\infty} f_t(z)g(z')\left(\mu_t(z') - \mu_t(z)\right)dz'dz\right] \\ + (1-\omega)(1-\eta(\theta_t^{SP}))\int_{\underline{z}_t}^{\infty}\int_{z}^{\infty} f_t(z)g(z')\left(\mu_t(z)\right)dz'dz. \tag{38}$$

The following result summarizes the implementation.

Proposition 3 *Suppose that the taxes $\{\tau_t^{s,u}, \tau_t^{s,e}(z), \chi_t^e\}$ are set so that (36), (37), and (38) hold. Then, the decentralized equilibrium implements the efficient allocation. In such an equilibrium, $S_t(z) = \mu_t(z)$ for all z.*

The proposition demonstrates that the decentralized equilibrium implements an efficient allocation and that the match surplus is valued in the same manner as the social planner's solution.

The following proposition characterizes the properties of the wedges under the Hosios condition.

Proposition 4 *Suppose that the Hosios condition $\gamma = \eta(\theta)$ holds and there is an on-the-job search, that is, $\zeta > 0$. Then, the following holds:*

1. *The tax on unemployed workers' search is strictly negative $\tau^{s,u} < 0$.*
2. *The tax on employed workers' search is strictly increasing in z. Furthermore, there exists z^* such that $\tau_t^{s,e}(z) < 0$ for $z < z^*$ and $\tau_t^{s,e}(z) > 0$ for $z > z^*$.*
3. *The entry tax is (weakly) positive, that is, $\chi^e \geq 0$. It is zero when the offer-matching is perfect, that is, $\omega = 1$.*

Taxes on unemployed workers and employed workers jointly correct the valuation margin. Unemployed workers receive positive transfer. Intuitively, under the Hosios condition, the optimal tax on unemployed workers is zero without on-the-job searches. With on-the-job searches, the effect on the opportunity cost is stronger than the change in externality because a portion of the positive externality is allocated to on-the-job search, which generates less surplus. Because the positive externality is now weaker, it is beneficial to increase the value of unemployment to lower the market value of surplus. Note that a positive unemployment insurance benefit would have the same effect.

The employed workers at the bottom of the job ladder are also subsidized. At the top of the job ladder, employed workers are taxed. Moreover, the tax is strictly increasing as a worker moves up the job ladder. For a low-z match, the missing opportunity gain is large, and it dominates the negative externality of on-the-job search. Thus, it is helpful to increase the value of employment to raise the market surplus of a low-z match. As z increases, the missing opportunity gain decreases,

Parameter	Description	Value	Target/Source
PANEL A. ASSIGNED PARAMETERS			
r	Discount rate	0.004	Annual interest rate 5%
η	Elasticity of matching function	0.5	Standard
γ	Worker bargaining power	0.5	Standard
ω	Offer matching	1.0	Dey and Flinn (2005),Cahuc et al. (2006)
h	Home production value	0.6	Normalization
A	Aggregate productivity	1.0	Normalization
PANEL B. CALIBRATED PARAMETERS			
σ	Separation rate	0.024	EU rate 2.4%
α	Pareto distribution shape parameter	5.2	UI elasticity of job-finding rate 0.37
z_{min}	Pareto distribution location parameter	2.5	Job-acceptance rate 49.4%
κ	Cost of vacancy creation	1.5	Market tightness 1
m	Matching efficiency parameter	0.83	Unemployment rate 5.5%
ζ	On-the-job search efficiency	0.25	EE rate 2.5%

Table 1: Parameter values

and at some point, the negative externality begins to dominate. At that point, the on-the-job search has to be taxed.

The entry tax corrects the investment margin. In particular, it corrects the worker-stealing externality when the offer matching is imperfect. The entry tax is zero when the offer-matching is perfect, $\omega = 1$. The entry tax is (weakly) positive because the worker-stealing externality induces an over-creation of vacancies in equilibrium.

5 Quantitative exploration

In this section, we quantitatively explore the difference between the efficient allocation and the equilibrium outcome. We calibrate the economy in a standard manner; that is, we stay as close as possible to the existing macroeconomic literature. Knowing whether the difference between the efficient allocation and equilibrium outcome is small or large is of interest because the policy implications would be different. The magnitude of desired policy interventions, for example, depends on the magnitude of distortions in the economy.

5.1 Calibration

Table 1 summarizes our parameterization. We interpret one period as a month and set the discount rate to 5% annually, $r = 0.05/12$. The exogenous separation rate is set to 2.4% at a monthly frequency, that is, $\sigma = 0.024$. We assume the Cobb-Douglas matching function, $M(u, v) = mu^{\eta}v^{1-\eta}$

with $\eta = 0.5$, and the bargaining power is such that $\gamma = \eta$ (i.e., the Hosios condition). This assumption ensures that without an on-the-job search, the equilibrium is efficient, thereby isolating the role of an on-the-job search as a source of inefficiency. We measure units of vacancy so that $\theta \equiv v/u = 1$ in the steady state. We assume the productivity distribution is given by the Pareto distribution: $G(z) = 1 - (z/z_{min})^{-\alpha}$, which is consistent with a notion of a balanced growth path in Martellini and Menzio (2020). The flow value of unemployment is normalized to $h = 0.6$, and the aggregate productivity is normalized to $A = 1$.[10] We set the offer-matching parameter to $\omega = 1$, which corresponds to the sequential-auction protocol of Dey and Flinn (2005) and Cahuc et al. (2006) and is widely used as a benchmark in the subsequent literature.

We then choose five parameters $\{m, \alpha, z_{min}, \kappa, \zeta\}$ to jointly match the following five moments: the steady-state unemployment rate of 5.5%; the monthly employment-to-employment transition rate of 2.5%; the job-acceptance rate of unemployed of 0.494, as reported in Faberman et al. (2022); and the partial-equilibrium elasticity of the job-finding rate with respect to the unemployment benefit, h, of 0.37, which corresponds to median estimates in the literature as surveyed by Schmieder and Von Wachter (2016). We compute the partial-equilibrium elasticity of the job-finding rate with respect to the unemployment benefit by simulating a model with a small change in h and calculating the percentage change in the job-finding rate while holding the market tightness θ constant.

Although all parameters are jointly calibrated, we provide a heuristic argument for how each moment identifies each parameter. First, the matching-efficiency parameter m is identified from the steady-state unemployment rate. The efficiency of on-the-job search, ζ, is inferred from the employment-to-employment transition rate. The lower bound of the productivity distribution, z_{min}, is identified from the job-acceptance rate of the unemployed. The cost of vacancy creation, κ, is chosen to ensure our normalization of $\theta = 1$. The elasticity of the job-finding rate with respect to the unemployment benefit identifies the Pareto-distribution shape parameter, α, because it controls the mass of workers near the reservation match quality.[11]

5.2 Results: Decentralized equilibrium vs. efficient allocation

Now, we compare the outcomes of the decentralized equilibrium and the efficient allocation. Table 2 compares the aggregate variables. The equilibrium allocation has too many vacancies; θ is too high. As a result, the unemployment rate is too low and the aggregate productivity (the average z among the employed) is too high. The resulting consumption is lower in the equilibrium allocation, although the difference (about 0.07%) is small. The unemployment rate is 0.9 percentage

[10] With a Pareto match-quality distribution and perfect offer matching ($\omega = 1$), z_{min}, κ, and h are not separately identified, because any combination of (z_{min}, h, κ) with the same z_{min}/h and κ/h result in the same labor market allocation. Therefore, the choice of h merely amounts to normalization.

[11] Our calibration of α falls in the middle of the values explored in Martellini and Menzio (2020).

	Equilibrium	Efficient
Market tightness	1.00	0.78
Reservation match quality	2.865	2.889
Consumption	3.7725	3.7751
UE rate	0.412	0.349
EE rate	0.025	0.023
Unemployment rate	5.5%	6.4%

Table 2: Comparison between equilibrium vs. efficient allocation

Note: The table shows the aggregate variable in the decentralized equilibrium and the efficient allocation.

points higher in the efficient allocation. The difference in the unemployment rate, therefore, is economically significant.

The discrepancy in the aggregate variables masks the underlying discrepancy in the distribution across job ladders. The top-left panel of Figure 2 compares the density function of z across the job ladder in the decentralized equilibrium with that in the planner's solution. The figure shows two features. First, because θ is larger in equilibrium, the distribution at the top is thicker for the market equilibrium. Second, at the bottom, the social planner is pickier in forming a match; thus, the bottom part of the distribution is more extended in the market equilibrium. All in all, the job ladder in the market equilibrium features more dispersion in z. In the market equilibrium, too much inequality in earnings exists.

The result of too many productive jobs in the decentralized equilibrium contrasts with models featuring ex-ante job heterogeneity, such as Acemoglu (2001), where too few "good jobs" tend to exist in equilibrium compared to the efficient allocation.

In Section 3.3, we established that the valuation margin causes the discrepancy between the equilibrium allocation and the optimal allocation (in the case of the Hosios condition and sequential auction). To visualize the discrepancy, the top-right panel of Figure 2 compares the surplus in the decentralized equilibrium ($S(z)$; solid line in the left panel) with the planner's valuation ($\mu(z)$; dashed line in the left panel). Consistent with our earlier discussion, the planner systematically places a lower match value for each z.

To further investigate the difference between $S(z)$ and $\mu(z)$, in the bottom two panels in Figure 2, we plot the two terms inside $\Delta^{SP}(\theta)$ and $\Delta^{DE}(z,\theta)$, that is, term $\Delta^{SP,UE}$ in (25) versus term $\Delta^{DE,UE}$ in (27) and term $\Delta^{SP,UE}$ in (25) versus term $\Delta^{DE,EE}$ in (27). For $\Delta^{SP,UE}$ and $\Delta^{DE,UE}$ (the bottom-left panel), the equilibrium value is significantly larger, reflecting the fact that the externality from off-the-job search is smaller in the presence of on-the-job search, while the imperfect appropriation of surplus from off-the-job search remains unchanged. For $\Delta^{SP,EE}$ versus $\Delta^{DE,EE}$ (the bottom-right

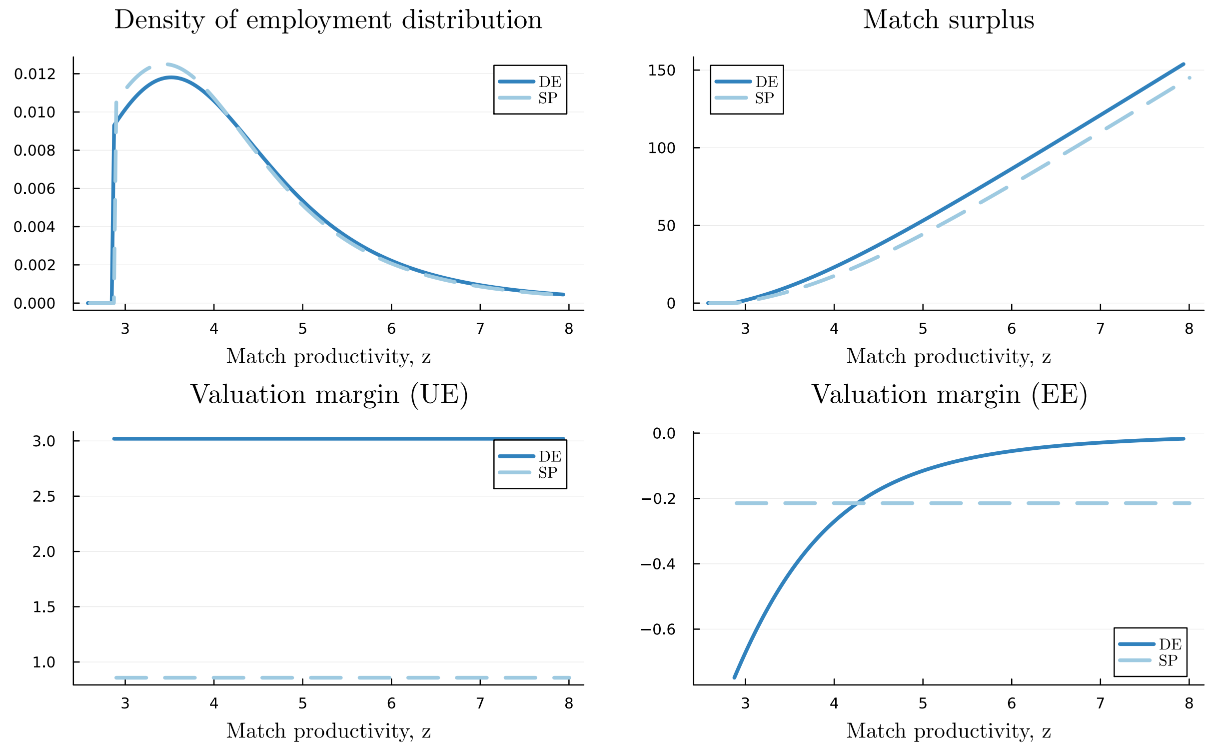

Figure 2: Employment distribution and match surplus: Equilibrium vs. planner

Note: The top-left panel compares the density of employment across the job ladder in the decentralized equilibrium and the efficient allocation, $n(z)$. The top-right panel compares the match surplus in the decentralized equilibrium, $S(z)$, and the planner's valuation of a match, $\mu(z)$. The bottom-left panel compares terms $\Delta^{SP,UE}$ in (25) and $\Delta^{DE,UE}$ in (27). The bottom-left panel compares terms $\Delta^{SP,EE}$ in (25) and $\Delta^{DE,EE}$ in (27).

panel), note first that $\Delta^{DE,EE}$ is a function of z and $\Delta^{SP,EE}$ is independent of z. In equilibrium, a worker with high z has small job-to-job gains, and thus, the "missing private gain" due to the initial Nash bargaining is not large (in absolute value). By contrast, because of the random-search assumption, the externality one person imposes due to on-the-job search is the same regardless of the value of z. For workers with large z, therefore, the effect of the negative externality they impose on others is significant.

In sum, the value of employment in equilibrium is excessively high for two reasons: (i) The negative externality an unemployed worker imposes on others is relatively small compared with the effect that in equilibrium, the value of unemployed search is undervalued, and (ii) for a high-z employed, the negative externality that they impose on others by their on-the-job search is larger (in absolute value) relative to the effect that in equilibrium, on-the-job search for these workers is privately undervalued.

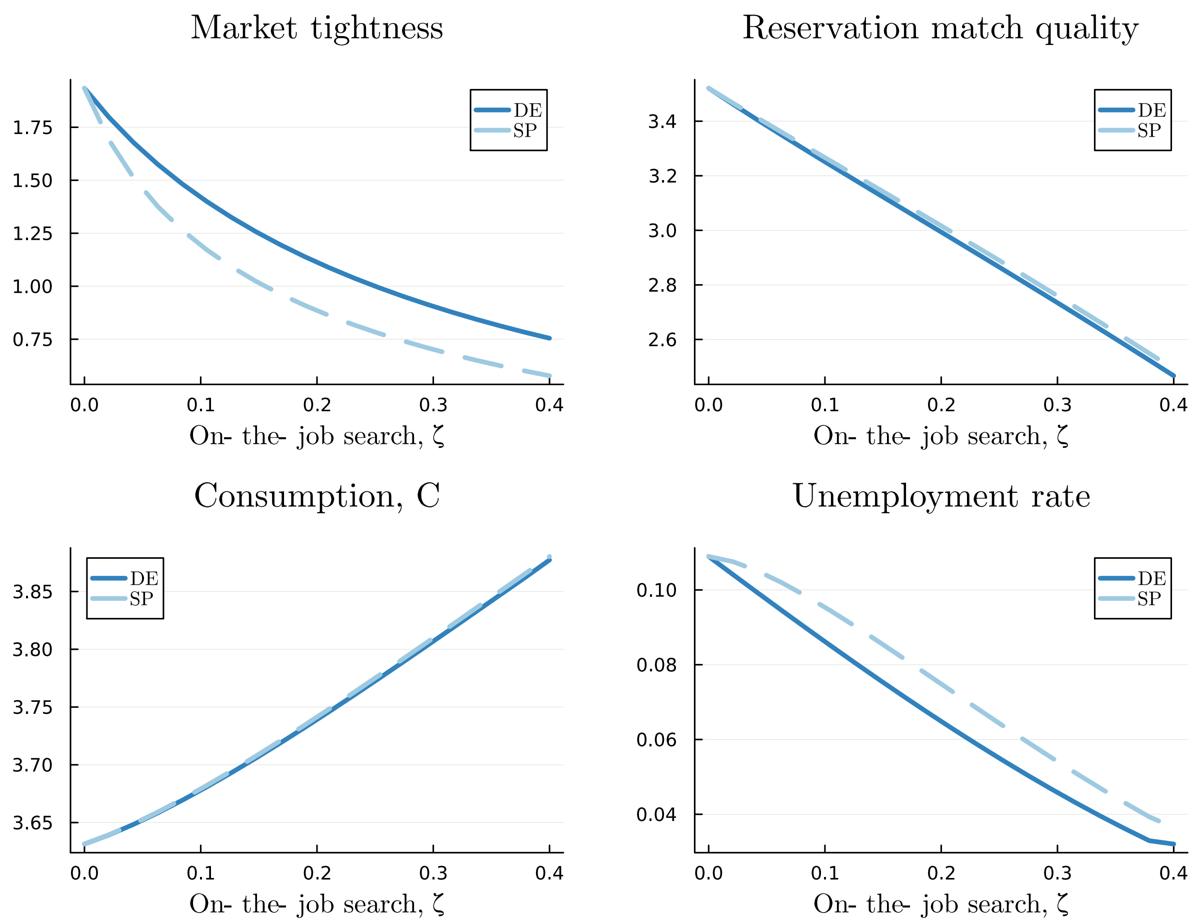

Figure 3: Varying the on-the-job search parameter ζ: equilibrium vs. planner

5.3 Steady-state comparative statics

This section conducts several comparative statics in terms of key parameters. We ask two questions. The first question is: how does the existence of on-the-job searches matter? We vary ζ from 0 (no on-the-job search) to 1 (on-the-job search is as efficient as off-the-job search) to examine this question. Second, how does the wage matching by the poaching firm matter? We have seen in Section 3.3 that when $\omega = 0$, the investment margin suffers from the worker-stealing externality; however, the perfect offer matching (sequential auction) with $\omega = 1$ alleviates this issue.

For the first question, Figure 3 varies the degree of on-the-job search ζ. The other parameters are kept constant at their baseline values. Because we impose the Hosios condition, the market equilibrium is efficient when $\zeta = 0$ (no on-the-job search). What is striking here is that the labor market tightness in the market equilibrium significantly departs from the optimal outcome even with a small amount of on-the-job search; for example, $\zeta = 0.1$.

For the second question, Figure 4 varies the degree of offer matching ω. We have seen our

	Calibration	Efficient
Worker bargaining power, γ	0.50	0.57
Offer matching, ω	1.00	0.98

Table 3: Efficient Bargaining Parameters

Note: The table shows the bargaining parameters used in our calibration and the ones that achieve the efficient allocation.

	Calibration	Efficient
Unemployment Insurance (% of output per worker)	0.0%	5.7%
Entry tax (% of vacancy cost κ)	0.0%	20.8%

Table 4: Efficient Unemployment Insurance and Entry Tax

Note: The table shows the unemployment insurance and the entry tax that achieve the efficient allocation.

benchmark, $\omega = 1$, achieves optimal investment in vacancy when valuation is correct. Therefore, it is not surprising that the consumption tends to be farther away from the efficient outcome when ω is lower. Although the labor market tightness increases with ω, the reservation match quality $\underline{z}$ also increases. The latter effect dominates in shaping the unemployment rate. The difference in the equilibrium unemployment rate between $\omega = 0$ and $\omega = 1$ is approximately two percentage points, indicating that the worker-stealing externality is economically significant.

Reservation match quality increases with ω because workers become choosier with a higher θ. But why does θ increase? This result seems counterintuitive given that a higher ω implies a poaching firm has to pay a higher wage, discouraging the vacancy posting—θ should fall with ω for a *given* $S(z)$. However, $S(z)$ is not given. The valuation-margin equation (15), in fact, reveals $S(z)$ increases with ω. This result holds because workers are forward-looking: the workers' surplus (thus, the current match surplus) includes the future possibility of receiving a higher wage from the poaching firm. This increase in $S(z)$ (for given θ and $\underline{z}$) provides a higher motivation for vacancy posting. In our calibration, the latter effect dominates.

Finally, we ask whether a pair of bargaining parameters, (γ, ω), exists such that the steady-state allocation is efficient. The answer turns out to be yes. Table 3 shows the values of worker bargaining power, γ, and the offer-matching parameter, ω, that induce the same allocation (same θ and $\underline{z}$) in the steady state and contrast them with the calibration. We find that the worker bargaining power γ that is higher than the Hosios condition and offer-matching parameter ω that is lower than the sequential auction achieve the efficient allocation.

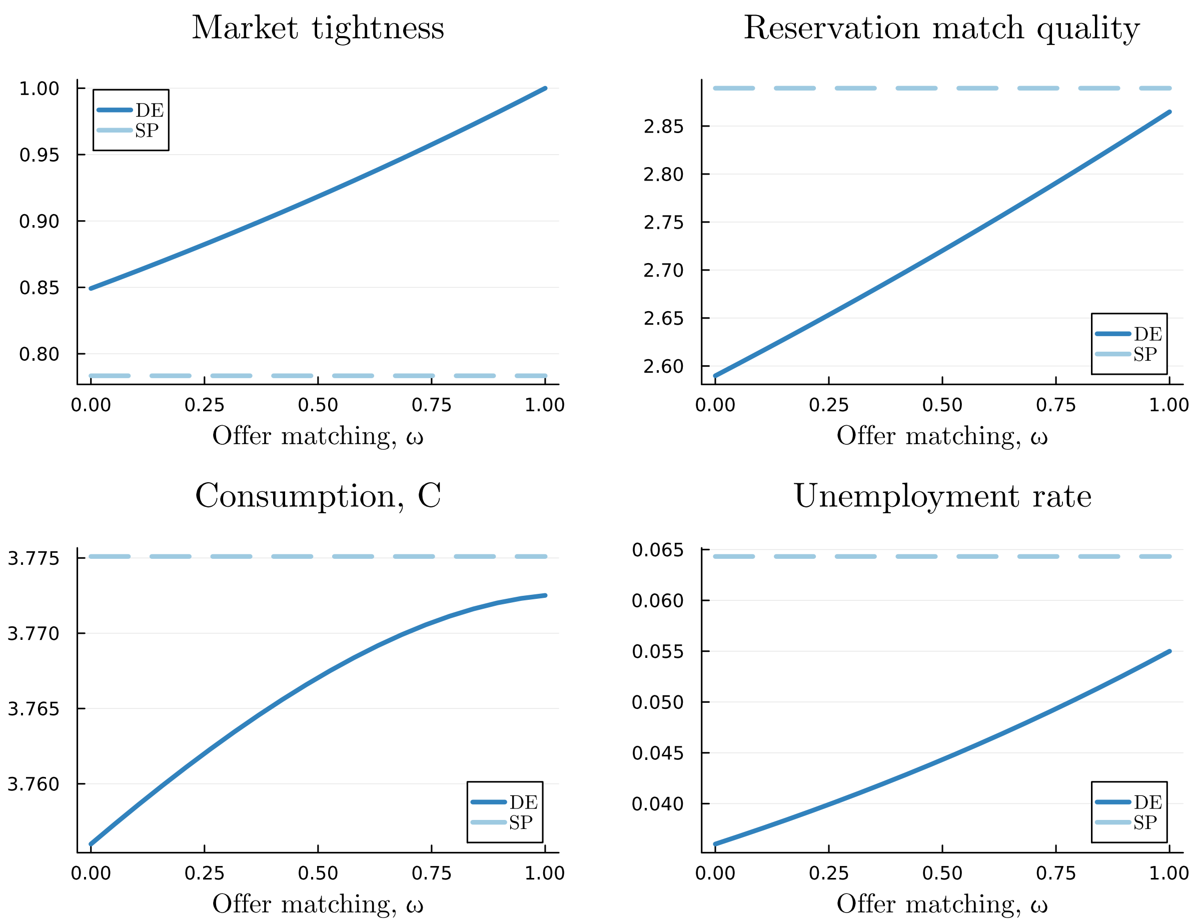

Figure 4: Varying the offer matching parameter ω: equilibrium vs. planner

5.4 Transition dynamics

So far, our focus has been on the steady state. We now consider the economy's response to the productivity shock. This experiment is important in evaluating how the efficiency properties interact with various phases of the business cycle.

We consider a one-time unanticipated shock ("MIT shock") to the aggregate productivity, A_t, that increases by 1% at $t = 0$, which decays with rate ρ_A:

$$d \ln A_t = -\rho_A \ln(A_t/A)dt.$$

We set $\rho_A = 0.04$, which corresponds to the autocorrelation of 0.96 at the monthly frequency. We compare the response of the social planner's solution with that of the decentralized equilibrium, starting from the steady state described earlier. We solve the first-order approximation of transition dynamics around the steady state in sequence space, following the approach of sequence space Jacobian algorithm by Auclert et al. (2021). We describe the details of algorithms in Ap-

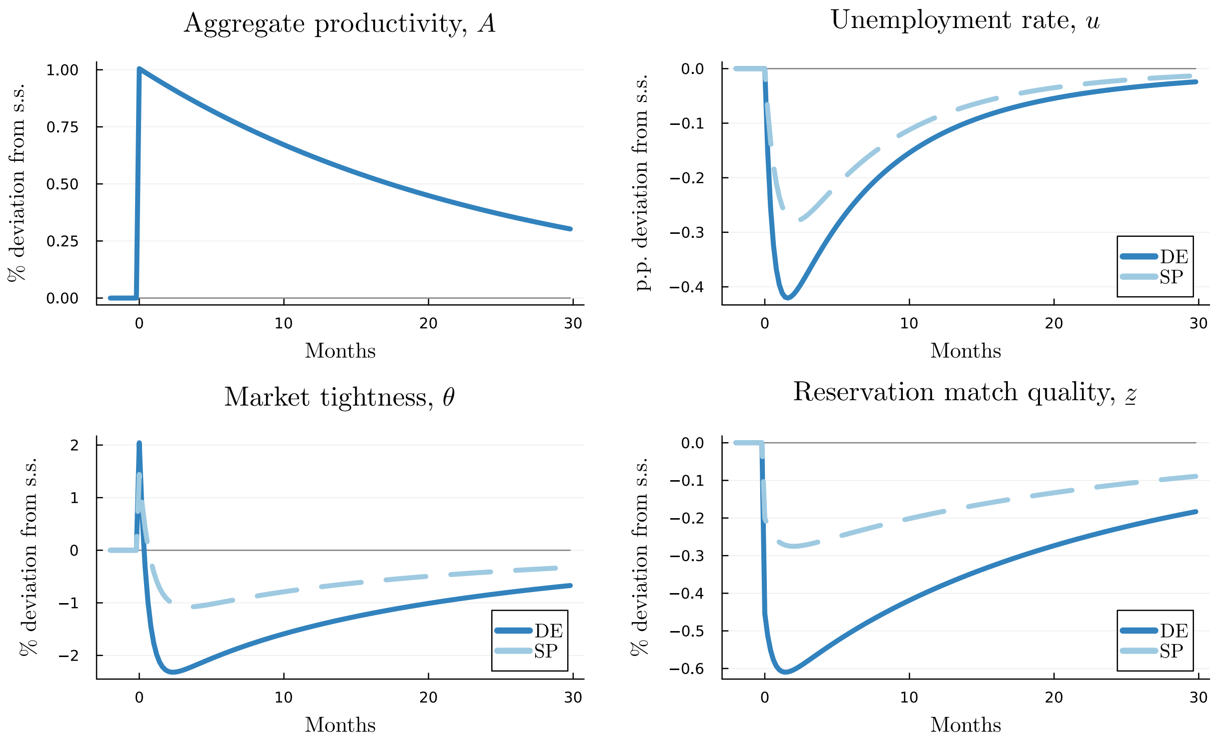

Figure 5: Response to productivity shock: Equilibrium and planner solution

Note: The figure plots the response of the decentralized equilibrium and the social planner's solution to a 1% initial increase in aggregate productivity at $t = 0$. The solid line represents the decentralized equilibrium, and the dashed line represents the social planner's solution.

pendix D.[12]

The top-left panel of Figure 5 shows the path of aggregate productivity. The top-right panel shows that the unemployment rate decreases more in the decentralized equilibrium than in the planner's solution. Given the steady-state unemployment rate was already lower in the equilibrium than in the planner's solution, the positive productivity shock exacerbates the gap in the unemployment rate. The bottom two panels show this difference is driven by a larger initial rise in market tightness and a persistent larger fall in reservation match quality. Both of these effects are driven by the fact that a positive productivity shock induces the further over-valuation of the match in the equilibrium allocation relative to the planner's allocation. As we explained in Section 3.3.2, the inefficiency along the valuation margin is exacerbated when the match surplus is larger and the market tightness is higher, which is the case when the aggregate productivity is higher.

[12]To the best of our knowledge, we are the first to apply the sequence space Jacobian algorithm to a model with on-the-job search and wage bargaining with non-zero worker bargaining power. Payne et al. (2025) globally solves this class of model by approximating the equilibrium dynamics with finite-dimensional parameters using deep learning. Our approach, similar to Auclert et al. (2021), relies on the first-order approximation but does not require any additional approximations.

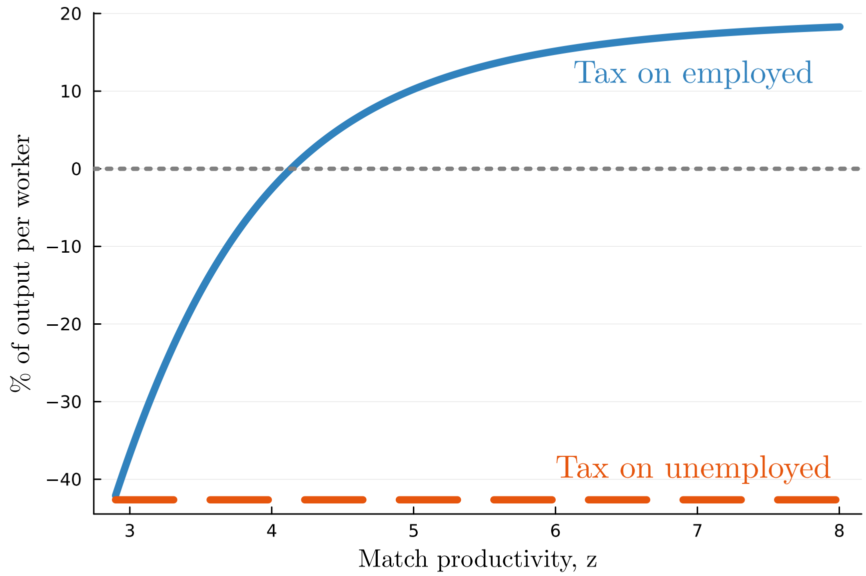

Figure 6: Optimal steady-state tax on unemployed workers ($\tau^{s,u}$) and on employed workers ($\tau^{s,e}(z)$)

Note: The figure plots the optimal steady-state tax on unemployed workers, $\tau^{s,u}$ and on employed workers, $\tau^{s,e}(z)$, against match productivity, z, which we characterize in (36) and (37). The solid blue line represents the tax on employed workers, and the dashed red line represents the tax on unemployed workers.

5.5 Decentralization of the efficient allocation

We now quantitatively illustrate the decentralization of the efficient allocation that we characterized in Section 4.3. We first consider implementation through unemployment insurance and entry tax, as in Section 4.4, and then consider an implementation that acts on wedges, as in Section 4.5.

Table 4 shows the unemployment insurance and the entry tax that achieve the efficient allocation. Both the optimal unemployment insurance and the entry tax are quantitatively substantial: the optimal unemployment insurance is 5.7%of the average output per person, and the entry tax is 20.8%of the cost of vacancy creation. Note also that they are both positive in line with Proposition 2, as we assume Hosios condition holds here.

Figure 6 shows an alternative implementation of the efficient allocation through the tax on workers and firms that we characterize in (36)-(38). Here, they are expressed relative to the home production value h. In line with Proposition 4, the tax on unemployed workers is negative; the tax on employed workers starts at a negative value with low z, increases as one moves up the job ladder, and eventually becomes a positive tax. Quantitatively, these taxes are substantial, ranging from -40% to 20% of the output per worker. Note that the optimal entry tax is zero, as we assume Hosios condition and sequential auction protocol (see Proposition 4).

6 Conclusion

This paper examines the efficiency of market equilibrium in a DMP-style job ladder model. We identify two margins where inefficiencies can arise: the investment margin and the valuation margin. In the presence of an on-the-job search, the Hosios condition does not ensure the efficiency of market equilibrium. Even when we impose a wage-setting protocol that ensures the investment margin remains undistorted, the valuation margin remains distorted. We show that, for a wide range of wage-setting protocols, too many vacancies are posted in market equilibrium under the Hosios condition. These results have important implications for policy evaluations that utilize job-ladder models.

The quantitative exercise focuses on a situation where the economy is calibrated as in the standard macroeconomic literature, and the Hosios condition is satisfied. It reveals that even with a small amount of on-the-job search, the optimal unemployment rate and the equilibrium unemployment rate can differ significantly. Thus, the inefficiencies highlighted in the earlier sections are quantitatively important. We also examine how efficient allocations can be decentralized and provide a sharp characterization of the optimal taxes.

The intuitions on inefficiencies would carry over to more complex models with various other factors. When considering more complex models, a crucial factor is how labor markets for different workers are segmented. The segmentation can be in different dimensions—here, we focused on the employment status (unemployed vs. employed) and the match quality. The actual labor market can be segmented across various other dimensions, such as gender, race, education, experience, and geography. Our paper shows understanding the degree of segmentation in the economy is an essential input in considering desirable policies under a frictional labor market.

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Appendix

A Pissarides (1985) model

In this Appendix, we outline the entire Pissarides (1985) model and derive the Hosios condition by comparing the social planner's solution with the market outcome.

A.1 Social planner's problem

The social planner's problem is

$$\max_{n_t,\theta_t} \int_0^\infty e^{-rt}[zn_t + h(1-n_t) - \kappa\theta_t(1-n_t)]dt$$

subject to

$$\dot{n}_t = p(\theta_t)\,(1-n_t) - \sigma n_t.$$

The current-value Hamiltonian for this problem can be written as

$$H = zn_t + h\,(1-n_t) - \kappa\theta_t\,(1-n_t) + \mu_t\,(p(\theta_t)\,(1-n_t) - \sigma n_t)\,.$$

Thus, first-order conditions are

$$\kappa(1-n_t) = p'(\theta_t)\mu_t(1-n_t) \tag{39}$$

and

$$z - h + \kappa\theta_t - p(\theta_t)\mu_t - (r+\sigma)\mu_t + \dot{\mu}_t = 0. \tag{40}$$

From (39), noting $p'(\theta) = (1-\eta(\theta))q(\theta)$,

$$\kappa = (1-\eta(\theta))q(\theta)\mu_t.$$

This equation corresponds to (2) in the main text. From (40) in the steady state where $\dot{\mu}_t = 0$,

$$(r+\sigma)\mu_t = z - h - (p(\theta_t)\mu_t - \kappa\theta_t)$$

holds. This equation corresponds to (4) in the main text.

A.2 Market equilibrium

In the market equilibrium, the Hamilton-Jacobi-Bellman (HJB) equation for an unemployed worker is

$$rU_t = h + p(\theta_t)(W_t - U_t) + \dot{U}_t, \tag{41}$$

where U_t is the value of an unemployed worker and W_t is the value of an employed worker. The HJB equation for an employed worker is

$$rW_t = w_t - \sigma(W_t - U_t) + \dot{W}_t, \tag{42}$$

where w_t is the wage at time t. On the firm side, the HJB equation for a matched job is

$$rJ_t = z - w_t - \sigma(J_t - V_t) + \dot{J}_t, \tag{43}$$

where J_t is the value of a matched job and V_t is the value of a vacancy. The value of vacancy satisfies

$$rV_t = -\kappa + q_t(J_t - V_t) + \dot{V}_t. \tag{44}$$

We assume anyone can post a vacancy (free entry). Therefore, the value of vacancy is driven down to zero in an equilibrium where the vacancy posting is strictly positive (we focus on such an equilibrium):

$$V_t = 0. \tag{45}$$

The wages are determined by Nash bargaining, which implies

$$W_t = \gamma S_t + U_t \tag{46}$$

and

$$J_t = (1 - \gamma)S_t + V_t, \tag{47}$$

where $S_t \equiv W_t + J_t - U_t - V_t$ is the joint surplus from a match, and γ is workers' baragining power. Thus, an employed worker and a matched firm share the surplus with the fractions γ and $(1 - \gamma)$, in addition to their outside options. Using (45) and (47) in (44), we obtain

$$\kappa = (1 - \gamma)q(\theta_t)S_t.$$

This equation corresponds to (3) in the main text. Adding up (41), (42), and (43) and using (45) and (46) (also imposing the steady-state condition $\dot{S}_t = 0$), we obtain

$$(r + \sigma)S_t = z - h - p(\theta_t)\gamma S_t,$$

which corresponds to (5) in the main text.

B Segmented market

We modify our baseline model by assuming the labor market is segmented by workers' employment status as well as the match quality of current jobs if workers are employed. We denote the market tightness that unemployed workers face as θ^u and the market tightness that employed workers with match quality z face as $\theta^e(z)$. Free entry of firms is assumed in each of the segmented markets.

The value of being unemployed is given by

$$rU_t = h + p(\theta_t^u) \int g(z) \max\{W_t(z, U_t) - U_t, 0\} dz + \dot{U}_t.$$

The value function of workers employed with match quality z and an outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned}
rW_t(z, \bar{O}) = {} & w_t(z, \bar{O}) \\
& + \zeta p(\theta_t^e(z)) \int_z^\infty g(z') \left(W_t(z', U_t + \omega S_t(z)) - W_t(z, \bar{O}) \right) dz' \\
& + \zeta p(\theta_t^e(z)) \int_0^z g(z') \left(W(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - W(z, \bar{O}) \right) dz' \\
& + \sigma(U_t - W_t(z, \bar{O})) + \dot{W}_t(z, \bar{O}).
\end{aligned}$$

The value of filled job with productivity z and outside option $\bar{O} \in [U, U + \omega S(z)]$ is given by

$$\begin{aligned}
rJ_t(z, \bar{O}) = {} & A_t z - w_t(z, \bar{O}) \\
& + \zeta p(\theta_t^e(z)) \int_z^\infty g(z') \left(V_t - J_t(z', \bar{O}) \right) dz' \\
& + \zeta p(\theta_t^e(z)) \int_0^z g(z') \left(J_t(z, \max\{U_t + \omega S_t(z'), \bar{O}\}) - J_t(z, \bar{O}) \right) dz' \\
& + \sigma \left(V_t - J_t(z, \bar{O}) \right) + \dot{J}_t(z, \bar{O}).
\end{aligned}$$

The value of vacancy in the market for unemployed workers is given by

$$V^u = -\kappa + q(\theta_t^u) \int_0^\infty g(z) \max\{J_t(z, U) - V_t, 0\} dz + \dot{V}^u,$$

and the value of vacancy in the market for employed workers with match quality z is given by

$$V^e(z) = -\kappa + q(\theta_t^e(z)) \int_z^\infty f(z') \max\{J_t(z', U + \omega S_t(z)) - V_t, 0\} dz' + \dot{V}^e(z).$$

The free-entry condition is

$$V_t = \max\{V^u, \max_z V^e(z)\} = 0.$$

With Nash bargaining, the surplus from a match is

$$\begin{aligned}
(r + \sigma) S_t(z) = {} & A_t z - h \\
& + \zeta p(\theta^e(z)) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z) \right] - S_t(z) \right) dz' \\
& - p(\theta_t^u) \int_{\underline{z}_t}^\infty g(z') \gamma S_t(z') dz' + \dot{S}_t(z)
\end{aligned} \tag{48}$$

for $z \geq \underline{z}_t$, where $\underline{z}_t$ satisfies

$$S(\underline{z}_t) = 0.$$

The matches with $z < \underline{z}_t$ are not formed.

The free-entry conditions are (assuming all markets have strictly positive vacancy postings)

$$\kappa = (1-\gamma)q(\theta_t^u)\int_{\underline{z}} g(z)S_t(z)dz \tag{49}$$

and

$$\kappa = (1-\gamma)q(\theta_t^e(z))\int_z^\infty g(z')\left[S_t(z') - \omega S_t(z)\right]dz'. \tag{50}$$

Now, we turn to the social planner's problem. The social planner's problem is to choose $\{\theta_t^u, \theta_t^e(z), n_t(z), \mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z,z'), \varsigma_t\}$ to maximize

$$\int_0^\infty e^{-rt}\left[\int_0^\infty A_t z n_t(z)dz + h\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\theta_t^u\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\int_0^\infty \theta_t^e(z)\zeta n_t(z)dz\right]dt$$

subject to

$$\begin{aligned}\dot{n}_t(z) = &\left(1-\int n_t(z')dz'\right)p(\theta_t^u)g(z)\mathbb{I}_t^{UE}(z) + \int_0^\infty p(\theta_t^e(z'))g(z)\mathbb{I}_t^{EE}(z',z)\zeta n_t(z')dz' \\ &- \int_0^\infty p(\theta_t^e(z))g(z')\mathbb{I}_t^{EE}(z,z')\zeta n_t(z)dz' - \sigma n_t(z) - \varsigma_t.\end{aligned}$$

The current-value Hamiltonian for this problem is

$$\begin{aligned}H = &\int_0^\infty A_t z n_t(z)dz + h\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\theta_t^u\left(1-\int_0^\infty n_t(z)dz\right) - \kappa\int_0^\infty \theta_t^e(z)\zeta n_t(z)dz \\ &+ \int_0^\infty \mu_t(z)\left[\left(1-\int_0^\infty n_t(z')dz'\right)p(\theta_t^u)g(z)\mathbb{I}_t^{UE}(z) + \zeta\int_0^\infty p(\theta_t^e(z'))g(z)\mathbb{I}_t^{EE}(z',z)n_t(z')dz'\right. \\ &\left. - \zeta\int_0^\infty p(\theta_t(z))g(z')n_t(z)\mathbb{I}_t^{EE}(z,z')dz' - \sigma n_t(z) - \varsigma_t\right]dz,\end{aligned}$$

where $\mu_t(z)$ is the costate variable that represents the shadow value of the constraint (21). Thus, $\mu_t(z)$ is the shadow value of creating one unit of match with match quality z.

The optimality condition for $\{\mathbb{I}_t^{UE}(z), \mathbb{I}_t^{EE}(z,z')\}$ is

$$\mathbb{I}_t^{UE}(z) = \begin{cases}1 & \mu_t(z) > 0 \\ 0 & \mu_t(z) \le 0\end{cases},$$

and

$$\mathbb{I}_t^{EE}(z,z') = \begin{cases}1 & \mu_t(z') > \mu_t(z) \\ 0 & \mu_t(z') \le \mu_t(z)\end{cases}.$$

The optimality condition for endogenous separation ς_t implies $\mu_t(z) \ge 0$ for all z with $n_t(z) > 0$.

The first-order optimality condition on $n_t(z)$ is

$$\begin{aligned}(r+\sigma)\mu_t(z) = &A_t z - h - \int_{\underline{z}_t}^\infty p(\theta_t^u)g(z')\mu_t(z')dz' + \zeta p(\theta_t^e(z))\int_z^\infty g(z')(\mu_t(z') - \mu_t(z))dz' \\ &+ \kappa\theta_t^u - \kappa\zeta\theta_t^e(z) + \dot{\mu}_t(z),\end{aligned} \tag{51}$$

where we have already imposed the fact that $\mu_t(z)$ is increasing in z. The reservation match quality $\underline{z}_t$ satisfies

$$\mu_t(\underline{z}_t) = 0.$$

The first-order optimality condition for θ^u_t is

$$\kappa = (1-\eta(\theta^u_t))q(\theta^u_t) \int_{\underline{z}}^{\infty} \mu_t(z')g(z')dz' \tag{52}$$

The first-order condition with respect to $\theta^e_t(z)$ is

$$\begin{aligned}\kappa = {} & (1-\eta(\theta^e_t(z)))q(\theta^e_t(z)) \int_0^{\infty} \mu_t(z')g(z')\mathbb{I}^{EE}_t(z,z')dz' \\ & - (1-\eta(\theta^e_t(z)))q(\theta^e_t(z)) \int_0^{\infty} g(z')\mathbb{I}^{EE}_t(z,z')dz'\mu_t(z),\end{aligned} \tag{53}$$

which we can rewrite as

$$\kappa = (1-\eta(\theta^e_t(z)))q(\theta^e_t(z)) \int_z^{\infty} g(z') \left[\mu_t(z') - \mu_t(z)\right] dz'.$$

Using free-entry conditions, we can rewrite (51) as

$$(r+\sigma)\mu_t(z) = A_t z - h - \eta(\theta^u_t)p(\theta^u_t) \int_{\underline{z}_t}^{\infty} g(z')\mu_t(z')dz' + \zeta\eta(\theta^e_t(z))p(\theta^e_t(z)) \int_z^{\infty} g(z')(\mu_t(z') - \mu_t(z))dz'. \tag{54}$$

Comparing (48), (49), and (50) with (54), (52), and (53), we can see that the social planner's problem is equivalent to the decentralized equilibrium as long as Hosios condition holds, $\eta(\theta) = \gamma$ for all θ, and offer-matching is perfect, $\omega = 1$. We summarize the results as follows.

Proposition 5 *Consider the environment with segmented labor markets described above. The decentralized equilibrium is efficient if the Hosios condition holds, that is, $\eta(\theta) = \gamma$ for all θ, and the offer-matching is perfect, $\omega = 1$.*

C Proofs

C.1 Proof of Lemma 1

In the steady state, the match surplus $S(z)$ satisfies

$$\begin{aligned}(r+\sigma)S(z) = {} & Az - h \\ & + \zeta p(\theta) \int_z^{\infty} g(z') \left((\omega-1)S(z) + \gamma\left[S(z') - \omega S(z)\right]\right) dz' \\ & - p(\theta) \int_{\underline{z}}^{\infty} g(z')\gamma S(z')dz'.\end{aligned} \tag{55}$$

Taking a derivative with respect to z, we have

$$[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))]\,S'(z)+\zeta p(\theta)g(z)(\omega-1)(1-\gamma)S(z)=A.$$

With a boundary condition $S(\underline{z})=0$, solving this ODE, we obtain

$$S(z;\underline{z},\theta)=\begin{cases} A\int_{\underline{z}}^{z}\frac{[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))]^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}. & \text{for} \quad z\geq\underline{z} \\ 0 & \text{for} \quad z<\underline{z}. \end{cases}$$

Evaluating (55) at $z=\underline{z}$ yields

$$0=A\underline{z}-h-\gamma(1-\xi)p(\theta)\int_{\underline{z}}^{\infty}g(z')S(z';\underline{z},\theta)dz'.$$

Applying integration by parts, we have

$$\int_{\underline{z}}^{\infty}g(z')S(z';\underline{z},\theta)dz'=A\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z},$$

where

$$\Gamma(z;\theta)\equiv r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z)).$$

Plugging it back, we have

$$\begin{aligned}0&=A\underline{z}-h-A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}\\&=H^R(\underline{z},\theta).\end{aligned}$$

The solution to the above equation can be used to define the mapping $\underline{z}^R(\theta)$. Its derivative is

$$\frac{d\underline{z}^R(\theta)}{d\theta}=-\frac{\frac{\partial H^R(\underline{z},\theta)}{\partial\theta}}{\frac{\partial H^R(\underline{z},\theta)}{\partial\underline{z}}}.$$

The denominator is positive, that is, $\partial H^R(\underline{z},\theta)/\partial\underline{z}>0$. To sign the numerator,

$$\begin{aligned}\frac{\partial H^R(\underline{z},\theta)}{\partial\theta}&=A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}\\&-A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))}{\Gamma(\tilde{z};\theta)^2}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}\left(\frac{\gamma}{(1-\omega(1-\gamma))}\right)-G(\tilde{z})\right)d\tilde{z}\\&\geq A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}\\&-A\gamma(1-\zeta)p'(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}\\&=0.\end{aligned}$$

Therefore, $z^R(\theta)$ is weakly increasing.

Let $P(z)$ denote the mass of workers employed with match quality below z or unemployed, where $P(\underline{z})$ corresponds to the unemployment rate. Its law of motion in the steady state is

$$0 = -(1 - G(z))p(\theta)P(\underline{z}) - (P(z) - P(\underline{z}))(1 - G(z))\zeta p(\theta) + \sigma(1 - P(z)).$$

Solving for $P(z)$, we have

$$P(z) = \begin{cases} \frac{\sigma-(1-\zeta)(1-G(z))p(\theta)P(\underline{z})}{\sigma+(1-G(z))\zeta p(\theta)} & \text{for} \quad z \geq \underline{z} \\ \frac{\sigma}{p(\theta)(1-G(\underline{z}))+\sigma} & \text{for} \quad z < \underline{z}. \end{cases}$$

Define the probability that a vacancy meets with a worker already employed with match quality below z or unemployed is

$$\begin{aligned} F^\ell(z) &\equiv \frac{P(\underline{z}) + \zeta(P(z) - P(\underline{z}))}{P(\underline{z}) + \zeta(1 - P(\underline{z}))} \\ &= \begin{cases} \frac{\sigma}{\sigma+(1-G(z))\zeta p(\theta)} & \text{for} \quad z \geq \underline{z} \\ \frac{\sigma}{\sigma+(1-G(\underline{z}))\zeta p(\theta)} & \text{for} \quad z < \underline{z} \end{cases} \equiv F^\ell(z; \theta, \underline{z}). \end{aligned}$$

The associated density function is

$$f(z) = \frac{\sigma g(z)\zeta p(\theta)}{(\sigma + (1 - G(z))\zeta p(\theta))^2}.$$

Using $F^\ell(z; \theta, \underline{z})$, the free-entry condition in the steady state can be written as

$$\kappa = (1 - \gamma)q(\theta) \int_0^\infty \int_{z'}^\infty g(z) \left[S(z; \underline{z}, \theta) - \omega S(z'; \underline{z}, \theta)\right] dz dF^\ell(z'; \theta, \underline{z}). \tag{56}$$

This defines a mapping $\underline{z}^{FE}(\theta)$.

Taking derivative,

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta} = -\frac{\frac{\partial}{\partial\theta}\left[q(\theta) \int_0^\infty \int_{z'}^\infty g(z) \left[S(z; \underline{z}, \theta) - \omega S(z'; \underline{z}, \theta)\right] dz dF^\ell(z'; \theta, \underline{z})\right]}{q(\theta) \int_0^\infty \int_{z'}^\infty g(z) \left[\frac{\partial S(z;\underline{z},\theta)}{\partial \underline{z}} - \omega \frac{\partial S(z';\underline{z},\theta)}{\partial \underline{z}}\right] dz dF^\ell(z'; \theta, \underline{z})}. \tag{57}$$

We would like to sign the above object. Note the denominator of (57) is negative because

$$\begin{aligned} \frac{\partial^2 S(z; \underline{z}, \theta)}{\partial z \partial \underline{z}} = -A(1 - \omega)(1 - \gamma)\zeta p(\theta)g(z) \frac{[r + \sigma + (1 - \omega(1 - \gamma))\zeta p(\theta)(1 - G(z))]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))} - 1}}{[r + \sigma + (1 - \omega(1 - \gamma))\zeta p(\theta)(1 - G(\underline{z}))]^{\frac{\gamma}{(1-\omega(1-\gamma))}}} \\ \leq 0, \end{aligned}$$

and as a result,

$$\begin{aligned} &q(\theta) \int_0^\infty \int_{z'}^\infty g(z) \left[\frac{\partial S(z; \underline{z}, \theta)}{\partial \underline{z}} - \omega \frac{\partial S(z'; \underline{z}, \theta)}{\partial \underline{z}}\right] dz dF^\ell(z'; \theta, \underline{z}) \\ &\qquad \leq q(\theta) F^\ell(\underline{z}; \theta, \underline{z}) \int_0^\infty g(z) \frac{\partial S(z; \underline{z}, \theta)}{\partial \underline{z}} dz < 0. \end{aligned}$$

The numerator of (57) can be decomposed into (i) the effect through $q(\theta)$, (ii) the effect through $F^{\ell}(z';\theta,\underline{z})$, and (iii) the effect through $S(z;\underline{z},\theta)$. Note the first two are negative because $q'(\theta)<0$ and $F^{\ell}(z';\theta',\underline{z})$ first-order stochastically dominates $F^{\ell}(z';\theta,\underline{z})$ for $\theta'>\theta$. To sign the third effect, note

$$\frac{\partial S(z;\underline{z},\theta)}{\partial\theta}=AB_1(z)+AB_2(z),$$

where

$$B_1(z)=-\eta(\theta)\frac{(1-\omega)(1-\gamma)\zeta p(\theta)(1-G(z))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))}S(z)$$

and

$$B_2(z)=$$

$$-\eta(\theta)\int_{\underline{z}}^{z}\frac{\gamma\zeta p(\theta)(1-G(\tilde{z}))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))}\frac{[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z))]^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{[r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(\tilde{z}))]^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}.$$

Then,

$$\begin{aligned}\int_{z'}^{\infty}B_1(z)g(z)dz-\omega\int_{z'}^{\infty}B_1(z')g(z)dz&<-\eta(\theta)\frac{(1-\omega)(1-\gamma)\zeta p(\theta)(1-G(z'))}{r+\sigma+(1-\omega(1-\gamma))\zeta p(\theta)(1-G(z'))}\\&\qquad\times\left[\int_{z'}^{\infty}S(z)g(z)dz-\omega\int_{z'}^{\infty}S(z')g(z)dz\right]\\&<0\end{aligned}$$

and

$$\int_{z'}^{\infty}B_2(z)g(z)dz-\omega\int_{z'}^{\infty}B_2(z')g(z)dz<0$$

because $B_2(z)$ is strictly decreasing. Therefore, the numerator is strictly negative. Putting it all together, we have

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta}<0.$$

Now, we seek to rewrite (56). We can rewrite the integral as follows:

$$\int_{\underline{z}}^{\infty}\int_{z'}^{\infty}g(z)\left[S(z)-\omega S(z')\right]dzdF^{\ell}(z')=\int_{\underline{z}}^{\infty}F^{\ell}(z)g(z)S(z)dz-\omega\int_{\underline{z}}^{\infty}(1-G(z))S(z)f(z)dz.$$

Using integration by parts,

$$\begin{aligned}\int_{\underline{z}}^{\infty}F^{\ell}(z)g(z)S(z)dz&=-\left[F^{\ell}(z)(1-G(z))S(z)\right]_{\underline{z}}^{\infty}+A\int_{\underline{z}}^{\infty}f(z)(1-G(z))S(z)dz\\&\qquad+A\int_{\underline{z}}^{\infty}F^{\ell}(z)(1-G(z))S'(z)dz\\&=A\int_{\underline{z}}^{\infty}f(z)(1-G(z))\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz+A\int_{\underline{z}}^{\infty}F^{\ell}(z)(1-G(z))\frac{1}{\Gamma(z,\theta)}dz.\end{aligned}$$

Substituting the above expressions back and using the expressions for $F^{\ell}(z)$ and $f(z)$, we have

$$\int_0^\infty \int_{z'}^\infty g(z)\left[S(z)-\omega S(z')\right]dz dF^{\ell}(z') = A\int_{\underline{z}}^\infty \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz$$
$$+A(1-\omega)\int_{\underline{z}}^\infty \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^z \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz.$$

Now, (56) can be rewritten as follows:

$$\kappa=(1-\gamma)q(\theta)A\left[\int_{\underline{z}}^\infty \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz\right.$$
$$\left.+(1-\omega)\int_{\underline{z}}^\infty \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^z \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\right].$$

C.2 Proof of Lemma 2

Note $F^{\ell}(z;\theta)$ can be characterized in the same way as Lemma 1. We omit superscript SP for notational simplicity.

We start from equation (24) evaluated at the steady state:

$$\begin{aligned}(r+\sigma)\mu(z) &= Az-h\\ &\quad -p(\theta)\int_{\underline{z}}^\infty g(z')\mu(z')dz+\zeta p(\theta)\int_z^\infty g(z')(\mu(z')-\mu(z))dz'\\ &\quad +(1-\zeta)(1-\eta(\theta))p(\theta)\left[\int_0^\infty\int_{z'}^\infty g(\tilde{z})\left(\mu(\tilde{z})-\mu(z')\right)d\tilde{z}dF^{\ell}(z';\theta,\underline{z})\right].\end{aligned} \tag{58}$$

Taking the derivative with respect to z,

$$\left[r+\sigma+\zeta p(\theta)(1-G(z))\right]\mu'(z)=A.$$

Solving for $\mu(z)$ with a boundary condition $\mu(\underline{z})=0$ gives

$$\mu(z;\underline{z},\theta)=\begin{cases} A\int_{\underline{z}}^z \frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}. & \text{for} \quad z\geq\underline{z}\\ 0 & \text{for} \quad z<\underline{z}\end{cases}$$

Evaluating (58) at $z=\underline{z}$ gives

$$\begin{aligned}0&=A\underline{z}-h\\ &\quad-(1-\zeta)p(\theta)\int_{\underline{z}}^\infty g(z')\mu(z';\underline{z},\theta)dz\\ &\quad+(1-\zeta)(1-\eta(\theta))p(\theta)\left[\int_0^\infty\int_{z'}^\infty g(\tilde{z})\left(\mu(\tilde{z};\underline{z},\theta)-\mu(z';\underline{z},\theta)\right)d\tilde{z}dF^{\ell}(z';\theta,\underline{z})\right].\end{aligned} \tag{59}$$

Now, we seek to simplify the above expression. We rewrite the second line using integration by parts:

$$\int_{\underline{z}}^{\infty} g(z')\mu(z';\underline{z},\theta)dz = A\int_{\underline{z}}^{\infty}(1-G(\tilde{z}))\frac{1}{r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}, \tag{60}$$

We rewrite the last term as

$$\begin{aligned}
&\int_{0}^{\infty}\int_{z'}^{\infty} g(\tilde{z})\left(\mu(\tilde{z})-\mu(z';\underline{z},\theta)\right)d\tilde{z}dF^{\ell}(z') \\
&= \int_{\underline{z}}^{\infty} F^{\ell}(z')g(z')\mu(\tilde{z};\underline{z},\theta)d\tilde{z} - \int_{\underline{z}}^{\infty}(1-G(z'))\mu(z';\underline{z},\theta)f(z')dz' \\
&= A\int_{\underline{z}}^{\infty} F^{\ell}(\tilde{z})(1-G(\tilde{z}))\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\
&= A\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z},
\end{aligned} \tag{61}$$

where the second line changes the order of integration, and the third line uses integration by parts. Plugging (60) and (61) into (59), we obtain

$$\begin{aligned}
0 &= A\underline{z} - h - A(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)}{\sigma+(1-G(\tilde{z}))\zeta p(\theta)}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\
&\equiv H^R(\underline{z},\theta).
\end{aligned}$$

The solution to the above equation gives a mapping $\underline{z}^R(\theta)$, and its derivative is given by

$$\frac{d\underline{z}^R(\theta)}{d\theta} = -\frac{\frac{\partial H^R(\underline{z},\theta)}{\partial\theta}}{\frac{\partial H^R(\underline{z},\theta)}{\partial\underline{z}}}.$$

Clearly, the denominator is positive, that is, $\frac{\partial H^R(\underline{z},\theta)}{\partial\underline{z}} > 0$. To sign the numerator,

$$\begin{aligned}
\frac{\partial H^R(\underline{z},\theta)}{\partial\theta} &= A(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{1-G(\tilde{z})}{(\sigma+(1-G(\tilde{z}))\zeta p(\theta))^2(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))^2} \\
&\times\Big[\sigma\eta(\theta)\left(\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)\right)\left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right) \\
&\quad+\sigma\eta'(\theta)\left(\sigma+(1-G(\tilde{z}))\zeta p(\theta)\right)\left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right) \\
&\quad+(1-G(\tilde{z}))\zeta\eta(\theta)p(\theta)\left(\sigma+(1-G(\tilde{z}))\zeta p(\theta)\right)\left(r+\sigma(1-\eta(\theta))\right)\Big]d\tilde{z}.
\end{aligned}$$

This is weakly positive if $\eta'(\theta) \geq k^R$, where

$$\begin{aligned}
k^R &\equiv \max_{\tilde{z}} -\frac{\eta(\theta)\left(\sigma\eta(\theta)+(1-G(\tilde{z}))\zeta p(\theta)\right)}{\left(\sigma+(1-G(\tilde{z}))\zeta p(\theta)\right)} - \frac{(1-G(\tilde{z}))\zeta\eta(\theta)p(\theta)(r+\sigma(1-\eta(\theta)))}{\sigma\left(r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))\right)} \\
&< 0.
\end{aligned}$$

Therefore, $\underline{z}^R(\theta)$ is weakly increasing if $\eta'(\theta) \geq k^R$.

We can rewrite (23) to obtain:

$$\kappa = (1-\eta(\theta))q(\theta)\left[\int_0^\infty \int_{z'}^\infty g(\tilde{z})\left(\mu(\tilde{z};\underline{z},\theta) - \mu(z';\underline{z},\theta)\right) d\tilde{z} dF^\ell(z';\theta,\underline{z})\right].$$

Substituting (61), we have

$$\begin{aligned}\kappa &= A(1-\eta(\theta))q(\theta)\int_{\underline{z}}^\infty \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\ &\equiv H^{FE}(\underline{z},\theta).\end{aligned}$$

This defines a mapping $\underline{z}^{FE}(\theta)$. To sign this,

$$\frac{d\underline{z}^{FE}(\theta)}{d\theta} = -\frac{\frac{\partial H^{FE}(\underline{z},\theta)}{\partial \theta}}{\frac{\partial H^{FE}(\underline{z},\theta)}{\partial \underline{z}}}.$$

Clearly, $\frac{\partial H^{FE}(\underline{z},\theta)}{\partial \underline{z}} < 0$, and the denominator is negative. To sign the numerator,

$$\begin{aligned}&\frac{\partial H^{FE}(\underline{z},\theta)}{\partial \theta} \\ &= -A\eta'(\theta)q(\theta)\int_{\underline{z}}^\infty \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\ &\quad + (1-\eta(\theta))q'(\theta)\int_{\underline{z}}^\infty \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z} \\ &\quad - (1-\eta(\theta))q(\theta)\int_{\underline{z}}^\infty \frac{\sigma(1-G(\tilde{z}))\zeta p'(\theta)(1-G(\tilde{z}))}{(\sigma+\zeta p(\theta)(1-G(\tilde{z})))}\frac{1}{(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))} \\ &\qquad \times\left(\frac{1}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}+\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right)d\tilde{z}.\end{aligned}$$

This is strictly negative if $\eta'(\theta) > k^{FE}$, where

$$\begin{aligned}k^{FE} &\equiv -(1-\eta(\theta))\eta(\theta) \\ &\qquad - \frac{1}{\int_{\underline{z}}^\infty \frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}}\times \\ &\qquad (1-\eta(\theta))\int_{\underline{z}}^\infty \frac{\sigma(1-G(\tilde{z}))\zeta p'(\theta)(1-G(\tilde{z}))}{(\sigma+\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))} \\ &\qquad \times\left(\frac{1}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}+\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right)d\tilde{z} \\ &< 0.\end{aligned}$$

Therefore, if $\eta'(\theta) > k^{FE}$, $z^{FE}(\theta)$ is strictly decreasing. Setting $k \equiv \max\{k^{FE}, k^R\} < 0$ completes the proof.

C.3 Proof of Proposition 1

We state and prove a more general version of Proposition 1.

Proposition 6 *There exists $\hat{\gamma} > \eta(\theta^{SP})$ such that for $\gamma < \hat{\gamma}$, the steady-state equilibrium market tightness θ in the market equilibrium is higher than the socially efficient level when $\zeta > 0$.*

Proof. We first show $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$ for all θ. Let

$$H^R(\underline{z},\theta) \equiv A\underline{z} - h - A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty} \frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z})\right)d\tilde{z}$$

and

$$H^{R,SP}(\underline{z},\theta) \equiv A\underline{z} - h - A(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty} \frac{\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)}{\sigma + (1-G(\tilde{z}))\zeta p(\theta)}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}.$$

Because $\underline{z}^R(\theta)$ and $\underline{z}^{R,SP}(\theta)$ are the solutions to $H^R(\underline{z},\theta) = 0$, and $H^{R,SP}(\underline{z},\theta) = 0$ and both H^R and $H^{R,SP}$ are increasing in $\underline{z}$, $H^R(\underline{z},\theta) > H^{R,SP}(\underline{z},\theta)$ for all $(\underline{z},\theta)$ is sufficient for $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$. Note that $H^R(\underline{z},\theta) > H^{R,SP}(\underline{z},\theta)$ holds if

$$\frac{\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)}{\sigma + (1-G(\tilde{z}))\zeta p(\theta)}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z})\right) \geq 0$$

for all $\tilde{z}$.

To show the above inequality,

$$\begin{aligned}
&\frac{\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)}{\sigma + (1-G(\tilde{z}))\zeta p(\theta)}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}} - G(\tilde{z})\right)\\
&\geq (1-G(\tilde{z}))\left(\frac{\sigma\eta(\theta) + (1-G(\tilde{z}))\zeta p(\theta)}{\sigma + (1-G(\tilde{z}))\zeta p(\theta)}\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))} - \frac{\gamma}{r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z}))}\right)\\
&= (1-G(\tilde{z}))\frac{(\eta(\theta)-\gamma)\sigma\left(r+\sigma+(1-G(\tilde{z}))\zeta p(\theta)\right) + (1-G(\tilde{z}))\zeta p(\theta)(r+\sigma)(1-\gamma)}{(r+\sigma+\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z})))(r+\sigma+\gamma\zeta p(\theta)(1-G(\tilde{z})))},
\end{aligned}$$

which is strictly positive whenever $\eta(\theta) \geq \gamma$. Therefore $\underline{z}^R(\theta) \leq \underline{z}^{R,SP}(\theta)$ for all θ.

Next, we show $\underline{z}^{FE}(\theta) \geq \underline{z}^{FE,SP}(\theta)$ for all θ. They are given by the solutions to $H^{FE}(\underline{z},\theta) = 0$ and $H^{FE,SP}(\underline{z},\theta) = 0$, respectively, where

$$\begin{aligned}
H^{FE}(\underline{z},\theta) = A(1-\gamma)q(\theta)\Bigg[&\int_{\underline{z}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz\\
&+ (1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z} \frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg] - \kappa
\end{aligned}$$

and

$$H^{FE,SP}(\underline{z},\theta) = A(1-\eta(\theta))q(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}-\kappa.$$

Because $H^{FE}(\underline{z},\theta)$ and $H^{FE,SP}(\underline{z},\theta)$ are both increasing in $\underline{z}$, showing $H^{FE}(\underline{z},\theta) > H^{FE,SP}(\underline{z},\theta)$ for all $(\underline{z},\theta)$ is sufficient for $\underline{z}^{FE}(\theta) \geq \underline{z}^{FE,SP}(\theta)$. To show this,

$$\begin{aligned}
&H^{FE}(\underline{z},\theta) - H^{FE,SP}(\underline{z},\theta)\\
&= A(1-\gamma)q(\theta)\left[\int_{\underline{z}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\left(\frac{1}{\Gamma(z,\theta)}-\frac{1}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\right)dz\right.\\
&\qquad\left.+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\right]\\
&\qquad+A(\eta(\theta)-\gamma)q(\theta)\int_{\underline{z}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta)(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta)(1-G(\tilde{z}))}d\tilde{z}.
\end{aligned}$$

The right-hand side is strictly positive whenever $\eta(\theta) \geq \gamma$.

By continuity with respect to γ, there exists $\hat{\gamma} > \eta(\theta)$ such that for $\gamma < \hat{\gamma}$, $\underline{z}^{R}(\theta) < z^{R,SP}(\theta)$ and $\underline{z}^{FE}(\theta) > z^{FE,SP}(\theta)$. Putting all together, we have shown θ is strictly higher in the decentralized equilibrium than the efficient level, as in Figure 1. ■

C.4 Proof of Proposition 2

We consider an alternative implementation of efficient allocation through unemployment insurance and entry tax. Unemployment insurance provides b units of consumption goods for unemployed workers so that the flow value of being unemployed is now given by $h+b$. The entry tax is given by τ^e, and the cost of entry inclusive of tax is $\chi^e+\kappa$. These two policy instruments are sufficient to implement efficient allocation.

Proposition 7 *There exists a pair of unemployment insurance b and entry tax χ^e that implement the efficient allocation in the steady state. Moreover, there exists $\bar{\gamma} \geq \eta(\theta)$ such that for $\gamma < \bar{\gamma}$, the unemployment insurance and the entry tax are both positive, $b > 0$ and $\chi^e > 0$.*

Proof. In the steady state of the decentralized equilibrium with the above policy instruments, $(\underline{z},\theta)$ solve

$$0 = A\underline{z} - h + b - A\gamma(1-\zeta)p(\theta)\int_{\underline{z}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta)}\left(\frac{\Gamma(\tilde{z};\theta)^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}$$

and

$$\kappa + \chi^e = A(1-\gamma)q(\theta)\Bigg[\int_{\underline{z}}^{\infty} \frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta)}\frac{1}{\Gamma(z,\theta)}dz$$
$$+(1-\omega)\int_{\underline{z}}^{\infty} \frac{\sigma g(z)\zeta p(\theta)(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta)^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta)^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg].$$

We compare the above two conditions with those for the efficient allocation:

$$0 = A\underline{z}^{SP} - h - A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma\eta(\theta^{SP})+(1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma+(1-G(\tilde{z}))\zeta p(\theta^{SP})}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}$$

and

$$\kappa = A(1-\eta(\theta))q(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}.$$

Therefore, we can obtain the equivalence by setting

$$b = A\gamma(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{1}{\Gamma(\tilde{z};\theta^{SP})}\left(\frac{\Gamma(\tilde{z};\theta^{SP})^{\frac{(1-\omega)(1-\gamma)}{(1-\omega(1-\gamma))}}}{(r+\sigma)^{\frac{(1-\omega)(1-\gamma)}{1-\omega(1-\gamma)}}}-G(\tilde{z})\right)d\tilde{z}$$
$$-A(1-\zeta)p(\theta^{SP})\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma\eta(\theta^{SP})+(1-G(\tilde{z}))\zeta p(\theta^{SP})}{\sigma+(1-G(\tilde{z}))\zeta p(\theta^{SP})}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}$$

and

$$\chi^e = (1-\gamma)\Bigg[\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma(1-G(z))}{\sigma+(1-G(z))\zeta p(\theta^{SP})}\frac{1}{\Gamma(z,\theta^{SP})}dz$$
$$+(1-\omega)\int_{\underline{z}}^{\infty}\frac{\sigma g(z)\zeta p(\theta^{SP})(1-G(z))}{(\sigma+(1-G(z))\zeta p(\theta^{SP}))^2}\int_{\underline{z}}^{z}\frac{\Gamma(z,\theta^{SP})^{\frac{(\omega-1)(1-\gamma)}{(1-\omega(1-\gamma))}}}{\Gamma(\tilde{z},\theta^{SP})^{\frac{\gamma}{(1-\omega(1-\gamma))}}}d\tilde{z}dz\Bigg]$$
$$-(1-\eta(\theta^{SP}))\int_{\underline{z}^{SP}}^{\infty}\frac{\sigma}{\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}\frac{1-G(\tilde{z})}{r+\sigma+\zeta p(\theta^{SP})(1-G(\tilde{z}))}d\tilde{z}$$

The proof of Proposition 1 already shows that when $\gamma \leq \eta(\theta^{SP})$,

$$b \geq 0, \quad \chi^e > 0.$$

By continuity in γ, there exists $\bar{\gamma} > \eta(\theta^{SP})$ such that for all $\gamma \leq \bar{\gamma}$, $b \geq 0$, and $\chi^e > 0$ hold. Note that when $\zeta = 1$, $b = 0$. ■

C.5 Proof of Proposition 3

Substituting the expression for taxes (36)-(38) into (34) and (35), we recover the same conditions as the planner's problem, (23) and (24), and the result follows.

C.6 Proof of Proposition 4

This proposition immediately follows from expressions (36)-(38).

D Computational algorithms for transition dynamics

We describe the computational algorithms for the transition dynamics. Throughout, we focus on one-time, unanticipated shocks that start from the steady state. We describe the transition dynamics for the decentralized equilibrium, allowing for the presence of taxes. The transition dynamics for the planner's problem can be obtained in the same manner. The transition dynamics for the decentralized equilibrium with taxes are characterized by $\{S_t(z), \underline{z}_t, \theta_t, N_t(z), u_t\}$, which jointly solve (16), (18), (20), (35), and (34).

We solve the first-order approximation of the transition dynamics around the steady state in sequence space, following the approach of Auclert et al. (2021). We discretize time with the time interval Δ and the truncated horizon of the transition dynamics with finite period T.

We first solve the joint match surplus backward in time using (35), starting from the terminal condition $dS_T(z) = 0$, in response to small changes in aggregate productivity and market tightness at the terminal period, $\{dA_T, d\theta_T\}$. This procedure gives the following derivatives:

$$\frac{dS_{T-s}(z)}{dA_T}, \quad \frac{dS_{T-s}(z)}{d\theta_T}.$$

We then represent the first-order response of the match surplus as a function of an arbitrary sequence of $\{A_s, \theta_s\}_s$ as follows:

$$d\boldsymbol{S}(z) = \mathcal{J}^{S,\theta}(z)d\boldsymbol{\theta} + \mathcal{J}^{S,A}(z)d\boldsymbol{A}, \tag{62}$$

where $d\boldsymbol{S}(z) \equiv [dS_t(z)]_t, d\boldsymbol{\theta} \equiv [d\theta_t]_t$, and $d\boldsymbol{A} \equiv [dA_t]_t$ are all $T/\Delta \times 1$ vectors, and $\mathcal{J}^{S,X}(z)$ is a $T/\Delta \times T/\Delta$ Jacobian matrix of match surplus with productivity z, $\boldsymbol{S}(z)$ with respect to $\boldsymbol{X}$. Each element of Jacobian can be obtained from a single backward iteration mentioned above as

$$\mathcal{J}^{S,X}_{t,t+s}(z) = \begin{cases} \frac{dS_{T-s}(z)}{dX_T} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},$$

for $X = \theta$ and A, which is the core insight of Auclert et al. (2021).

Second, we represent the response of reservation match quality as

$$\begin{aligned} d\underline{\boldsymbol{z}} &= \frac{1}{S'(\underline{z})} d\boldsymbol{S}(\underline{z}) \\ &= \frac{1}{S'(\underline{z})} \left[\mathcal{J}^{S,\theta}(\underline{z})d\boldsymbol{\theta} + \mathcal{J}^{S,\theta}(\underline{z})d\boldsymbol{A} \right] \end{aligned} \tag{63}$$

which follows from total differentiation of $S_t(\underline{z}_t) = 0$.

The third step is to compute the derivatives of the employment distribution with respect to the aggregate productivity, market tightness, and reservation match quality at time 0 to obtain the following objects:

$$\frac{\partial u_t}{\partial A_0}, \quad \frac{\partial u_t}{\partial \theta_0}, \quad \frac{\partial u_t}{\partial \underline{z}_0}, \quad \frac{\partial N_t(z)}{\partial A_0}, \quad \frac{\partial N_t(z)}{\partial \theta_0}, \quad \frac{\partial N_t(z)}{\partial \underline{z}_0}.$$

The first-order response of the distribution as a function of an arbitrary sequence of $\{A_s, \theta_s\}_s$ can be expressed as follows.

$$d\boldsymbol{u} = \mathcal{J}^{u,\theta} d\boldsymbol{\theta} + \mathcal{J}^{u,A} d\boldsymbol{A} + \mathcal{J}^{u,\underline{z}} d\underline{\boldsymbol{z}}, \tag{64}$$

$$d\boldsymbol{N}(z) = \mathcal{J}^{N,\theta}(z) d\boldsymbol{\theta} + \mathcal{J}^{N,A}(z) d\boldsymbol{A} + \mathcal{J}^{N,\underline{z}}(z) d\underline{\boldsymbol{z}}, \tag{65}$$

where the Jacobian matrix can be obtained as

$$\mathcal{J}^{u,X}_{t+s,t} = \begin{cases} \frac{du_s}{dX_0} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},$$

$$\mathcal{J}^{N,X}_{t+s,t}(z) = \begin{cases} \frac{dN_s(z)}{dX_0} & \text{for } s \geq 0 \\ 0 & \text{for } s < 0 \end{cases},$$

for $X = \theta$, A, and $\underline{z}$.

Because the free-entry condition, (34), is a function of $\{u_t, S_t(z), N_t(z), \theta_t, \underline{z}_t\}$, we can write it as $\tilde{\mathcal{H}}^{FE}(u_t, S_t(z), N_t(z), \theta_t, \underline{z}_t) = \kappa$. The linearized free-entry condition is

$$\frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t} du_t + \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial \theta_t} d\theta_t + \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial \underline{z}_t} d\underline{z}_t + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')} dS_t(z') dz' + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial N_t(z')} dN_t(z') dz' = 0.$$

Substituting (62), (64), and (65) into the above equation, we can write the linearized free entry condition as,

$$\mathcal{H}^{FE,\theta} d\boldsymbol{\theta} + \mathcal{H}^{FE,A} d\boldsymbol{A} + \mathcal{H}^{FE,\underline{z}} d\underline{\boldsymbol{z}} = 0, \tag{66}$$

where $\mathcal{H}^{FE,\theta}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t} \mathcal{J}^{u,\theta}_{t,s}$, $\mathcal{H}^{FE,A}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t} \mathcal{J}^{u,A}_{t,s} + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')} \mathcal{J}^{S,A}_{t,s}(z') dz'$, and $\mathcal{H}^{FE,\underline{z}}_{t,s} = \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial u_t} \mathcal{J}^{u,\underline{z}}_{t,s} + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial S_t(z')} \mathcal{J}^{S,\underline{z}}_{t,s}(z') dz' + \int \frac{\partial \tilde{\mathcal{H}}^{FE}}{\partial N_t(z')} \mathcal{J}^{N,\underline{z}}_{t,s}(z') dz'$.

Stacking (63) and (66), the first-order response of $d\underline{\boldsymbol{z}}, d\boldsymbol{\theta}$ solves

$$\begin{bmatrix} \mathcal{H}^{FE,\theta} & \mathcal{H}^{FE,\underline{z}} \\ \frac{1}{S'(\underline{z})} \mathcal{J}^{S,\theta}(\underline{z}) & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} d\boldsymbol{\theta} \\ d\underline{\boldsymbol{z}} \end{bmatrix} = - \begin{bmatrix} \mathcal{H}^{FE,A} \\ \frac{1}{S'(\underline{z})} \mathcal{J}^{S,A}(\underline{z}) \end{bmatrix} d\boldsymbol{A},$$

where $\boldsymbol{I}$ is $T/\Delta \times T/\Delta$ identity matrix. We can solve for $d\boldsymbol{\theta}, d\underline{\boldsymbol{z}}$ by inverting the matrix on the left-hand side. Given $d\boldsymbol{\theta}, d\underline{\boldsymbol{z}}$, the rest of the objects can be obtained using (62), (64), and (65).

E Extension with endogenous effort in maintaining match

Consider an extension of the baseline model with taxes to an environment with endogenous effort choice that determines the separation rate. In particular, we assume workers can choose the level of effort, e, which determines the separation rate of a match, $\sigma(e)$. The separation rate is decreasing and concave in effort. The unit cost of effort is ι. The equilibrium joint match surplus in this environment is given by

$$
\begin{aligned}
rS_t(z) = \max_e\, & A_t z(1-\tau_t(z)) - h - \iota e - \sigma(e)S_t(z) \\
& + \zeta p(\theta) \int_z^\infty g(z') \left(\omega S_t(z) + \gamma \left[S_t(z') - \omega S_t(z)\right] - S_t(z)\right) dz' \\
& - p(\theta_t) \int_{\underline{z}_t}^\infty g(z')\gamma S_t(z')dz' + \dot{S}_t(z).
\end{aligned}
$$

The optimal choice of effort of the job with match quality z satisfies

$$\iota = \sigma'(e)S_t(z).$$

We denote the solution to the above expression as $e_t(z)$. The rest of the equilibrium conditions are the same as the baseline model except that the separation rate σ is replaced with $\sigma(e(z))$ for each match quality z.

Likewise, the valuation of the job in the planner's solution is given by

$$
\begin{aligned}
r\mu_t(z) = \max_e\, & A_t z - h - \iota e - \sigma(e)\mu_t(z) \\
& - \int_{\underline{z}_t}^\infty p(\theta_t) g(z')\mu_t(z')dz' + \zeta p(\theta_t) \int_z^\infty g(z')(\mu_t(z') - \mu_t(z))dz' \\
& + \kappa\theta_t(1-\zeta) + \dot{\mu}_t(z).
\end{aligned}
$$

The optimal choice of effort of the job with match quality z satisfies

$$\iota = \sigma'(e)\mu_t(z).$$

We denote the solution to the above expression as $e_t^{SP}(z)$.

The output tax $\tau(z)$ that ensures $\mu_t(z) = S_t(z)$ will also ensure the effort choices are efficient $e_t(z) = e_t^{SP}$. By contrast, because unemployment insurance or entry tax alone does not ensure $\mu_t(z) = S_t(z)$, the effort choices will not be efficient.