Directed Seach Models (a.k.a. Competitive Search Models)

704a Macroeconomics
Lecture 6

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Directed Search Model

- In the last lecture, we saw
  1. when different jobs search in the pooled market, composition externality arises
  2. when different jobs exogenously search in the segmented market, it vanishes

- Today: different jobs **endogenously** segment into different markets

- Called **directed search** (**competitive search**) model
  - Pioneered by Moen (1997), and popularized by Menzio and Shi (2011)

Environment

Environment

- Start from no heterogeneity
- Populated by a unit measure of workers and firms
  - Both have discount factor β and linear preferences: $\sum_{t=0}^{\infty} \beta^t c_t$
- Worker is endowed with a unit of labor
  - earn b when unemployed
- Firm operates linear technology in labor with productivity z
  - job exogenously separates with prob s

Submarkets

- There is a continuum of **submarkets** indexed by w
  - Matching function within each submarket, $M(u, v)$
  - Let $\theta(w) \equiv v(w)/u(w)$ denote the market tightness for submarket w
  - When matched, firms offer the wage w
- Firms can post vacancy in each submarket at cost c
  - Find a worker with prob. $q(\theta(w))$ where $q(\theta) \equiv M(1/\theta,1) = M(u, v)/v$
- Workers can choose which submarket to search (can choose only one)
  - Find a job with prob. $f(\theta(w))$ where $f(\theta) \equiv M(1,\theta)$
- Timing:

  firms post vacancy → workers apply → match → produce → separate

Interpretation

- Firms post and **commit** to wage offer w
  - Post a job ad saying "we will pay $15/hr"
- Workers see all available job postings and decide which job to apply for
  - again, can apply for only one job
- Note the contrast to DMP: communication and commitment
  - In DMP, workers had no ex-ante info about wage offers
  - In DMP, firms had no commitment to future wages

Equilibrium

Worker's Problem

- Throughout, we focus on the steady state
- Value of unemployed workers searching in a submarket w:

$$U(w) = b + \beta[f(\theta(w))E(w) + (1 - f(\theta(w)))U]$$

where

$$E(w) = w + \beta[(1 - s)E(w) + sU]$$

- Workers arbitrage between markets implies:

$$U(w) = U \qquad \Rightarrow \qquad f(\theta(w))[E(w) - U] = [(1 - \beta)U - b]/\beta \equiv \Lambda$$

  - $E(w)$ is increasing in $w \Rightarrow f(\theta(w))$ is decreasing in w
  - Better jobs are harder to find in equilibrium

Firm's Problem

- Firms decide which submarket to post a vacancy (what wage to post)

$$\max_{w} q(\theta(w))J(w) \tag{4}$$

$$\text{s.t.} \quad f(\theta(w))[E(w) - U] = \Lambda \tag{IC}$$

where

$$J(w) = z - w + \beta[(1 - s)J(w) + sV]$$

$$V = -c + \beta[q(w^*)J(w^*) + (1 - q(w^*))V] = 0$$

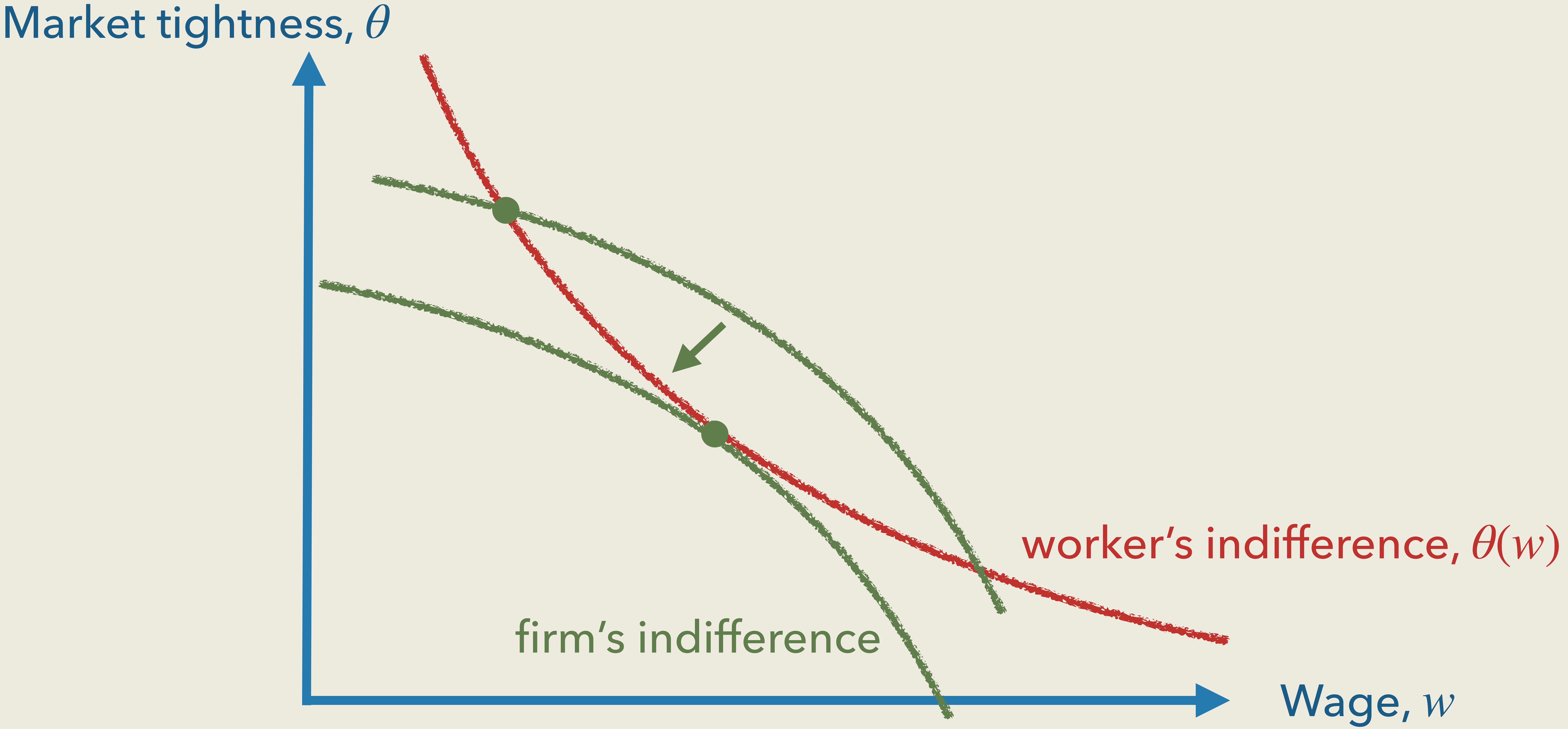
- The (IC) constraint captures subgame perfection
  - Firms rationally anticipate how many workers will apply when posting w
- Tradeoff: higher wage (i) attracts more workers (ii) but is costly.

Equilibrium Definition

A competitive search equilibrium is a tuple $\{U, E(w), V, J(w), \theta(w), w^*\}$ such that

1. U solves
$$U = b + \beta[f(\theta(w))E(w) + (1 - f(\theta(w)))U] \quad \text{for all } w$$
2. $E(w)$ solves
$$E(w) = w + \beta[(1 - s)E(w) + sU]$$
3. w^* is the solution to
$$\max_w q(\theta(w))J(w) \qquad \text{s.t.} \quad f(\theta(w))[E(w) - U] = \Lambda$$
4. $J(w)$ solves
$$J(w) = z - w + \beta[(1 - s)J(w)]$$
5. $\theta(w)$ satisfies
$$c = \beta q(\theta(w))J(w) \quad \text{for } w = w^*$$
$$c > \beta q(\theta(w))J(w) \quad \text{for } w \neq w^*$$

Indifference Curve Figure



Wage Determination

- The first order condition w.r.t. w gives

$$q'(\theta(w))\theta'(w)J(w) + J'(w)q(\theta(w)) = 0$$

- Totally differentiating (IC),

$$f'(\theta(w))\theta'(w)[E(w) - U] + f(\theta(w))E'(w) = 0$$

- Combining the above two conditions and manipulating,

$$\alpha J(w^*) = (1 - \alpha)[E(w^*) - U]$$

where $\alpha = d \ln f / d \ln \theta \,|_{\theta=\theta(w^*)} = d \ln M / d \ln u$

- Defining $S = J(w) + E(w) - U$ (note S independent of wage) as the match surplus,

$$J(w^*) = \alpha S, \qquad E(w^*) - U = (1 - \alpha)S$$

- Looks familiar?

Equilibrium is Efficient

$$J(w^*) = \alpha S, \qquad E(w^*) - U = (1-\alpha)S$$

- Hosios condition is endogenously achieved in equilibrium!
- As a result, entry is at the efficient level

$$c = \beta q(\theta(w)) \underbrace{J(w)}_{(1-\alpha)S} \quad \text{for } w = w^*$$

  and no vacancy is created for $w \neq w^*$

- Result: Competitive search equilibrium is efficient
- Collorary:
  Competitive search equilibrium results in the same allocation as DMP with Hosios

Reason for Efficiency

- In DMP, eqm was not efficient
  1. Planner cares about how much additional vacancy congests the market
  2. Firms care about how much an additional vacancy generates profit
- With ex-post bargaining, no reason 1 and 2 coincide
- Here, firms set wages facing the same trade-off just as the planner does
  1. To hire more workers, firms have to be in a less congested market
  2. But a less congested market generates lower profits

Job Heterogeneity

Job Heterogeneity

- Now introduce heterogeneous firm types, $\{z_1, z_2, \ldots, z_J\}$
- Each firm type decides which submarket w to post with vacancy cost $c(v_i)$
- Workers decide which submarket to search for a job (same as before)
- The firm's optimal choice of w solves

$$\max_{w_i} q(\theta(w_i))J_i(w) \qquad \text{s.t.} \quad f(\theta(w_i))[E(w_i) - U] = \Lambda$$

$$J_i(w_i) = z_i - w_i + \beta[(1-s)J_i(w_i)]$$

Solution: $E(w_i) - U = \alpha_i S_i, \quad J(w_i) = (1-\alpha_i)S_i$, where $\alpha_i = \partial \ln M(u_i, v_i)/\partial \ln u_i$

- Optimal vacancy creation:

$$c'(v_i) = \beta q(\theta(w_i))J(w_i)$$

Efficiency with Job Heterogeneity

- Combining, in equilibrium,

$$S_i^{DE} = z_i - b + \beta(1 - s - \alpha_i f_i^{DE})S_i^{DE}$$

$$c'(v_i^{DE}) = (1 - \alpha)\beta q_i^{DE} S_i^{DE}$$

$$\Lambda^{DE} = \alpha_i f_i S_i^{DE}$$

- The planner's problem is the same as the segmented market case:

$$S_i^{SP} = z_i - b + \beta(1 - s - \alpha_i f_i^{SP})S_i^{SP}$$

$$c'(v_i^{SP}) = (1 - \alpha)\beta q_i^{SP} S_i^{SP}$$

$$\Lambda^{SP} = \alpha_i f_i S_i^{SP}$$

- Equilibrium is efficient even with heterogeneity (firms endogenously segment)
- Productive firms endogenously sort into less-tight but high-wage submarket

Block Recursivity

Block Recursivity

- Go back to the homogenous case
- Introduce the notion of block recursivity
- A useful tool to solve search models numerically even with heterogeneity

Equivalence

- It is often more tractable to solve equilibrium in the following way.

Proposition

1. Suppose $\{U, w^*, \theta(w^*)\}$ is an equilibrium. Then $\{w^*, \theta(w^*)\}$ solve

$$U = \max_{\theta,w} b + \beta[f(\theta)E(w) + (1 - f(\theta))U]$$

$$\text{s.t.} \quad \beta q(\theta)J(w) \geq c$$

2. Conversely, suppose $\{U^*, w^*, \theta^*\}$ solve the above problem. Then there exists and equilibrium $\{U, w, \theta(w)\}$ with $\theta(w^*) = \theta^*$, $U = U^*$, and $w = w^*$.

Proof Sketch for Part 1

- Assume to the contrary: there exists w_t', θ_t' such that

$$b + \beta[f(\theta_t')E(w_t') + (1 - f(\theta_t'))U_{t+1}] > U_t$$
$$\beta q(\theta_t')J(w_t') \geq c$$

- The utility the worker obtains in equilibrium by searching for wage w_t' is

$$b + \beta[f(\theta_t(w_t'))E(w_t') + (1 - f(\theta_t(w_t')))U_{t+1}]$$

- From free entry, $\beta q(\theta_t(w_t'))J(w_t') \leq c \leq \beta q(\theta_t')J(w_t')$. This implies $\theta_t(w_t') \geq \theta_t'$.

- Since $f(\theta)$ is strictly increasing in θ,

$$b + \beta[f(\theta_t(w_t'))E(w_t') + (1 - f(\theta_t(w_t')))U_{t+1}] > b + \beta[f(\theta_t')E(w_t') + (1 - f(\theta_t'))U_{t+1}]$$
$$> U_t$$

- This contradicts worker's optimality where searching for w_t^* was optimal

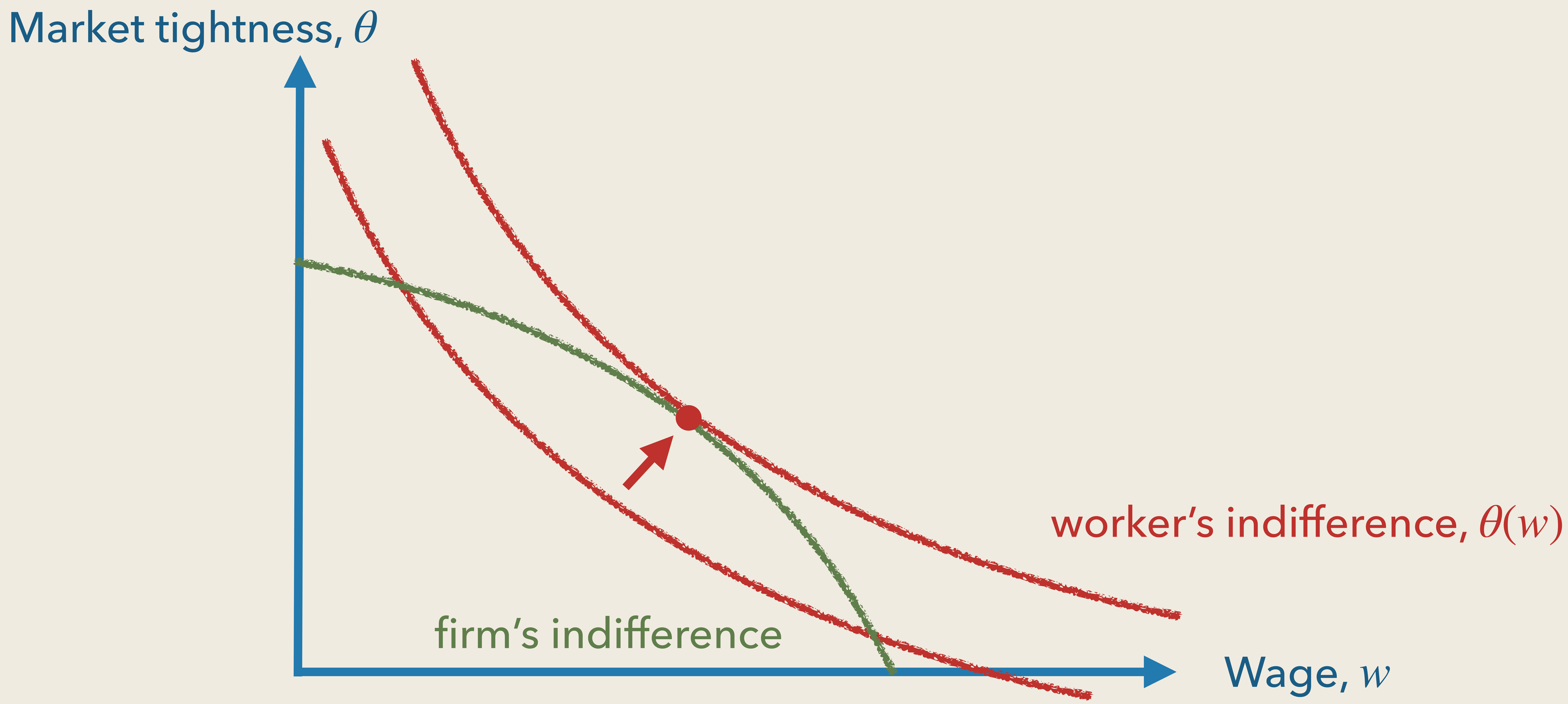
Proof Sketch for Part 2

- Construct $\theta_t(w_t)$ be such that

$$b + \beta[f(\theta_t(w_t))E(w_t) + (1 - f(\theta_t(w_t)))U_{t+1}] = U_t^*$$

- By construction, workers are indifferent across all submarkets.
- Now we confirm the firm's optimality: $\beta q(\theta(w_t^*))J(w_t^*) = c$
- Suppose to the contrary there exists w' such that $\beta q(\theta_t(w_t'))J(w_t') > \beta q(\theta(w_t^*))J(w_t^*) = c$.
- Then there exists $\theta_t' > \theta_t(w_t')$ such that $\beta q(\theta_t')J(w_t') = c$.
- Then $b + \beta[f(\theta_t')E(w_t') + (1 - f(\theta_t'))U_{t+1}] > U_t^*$ but this contradicts θ^*, w^* were optimal.

Indifference Curve Figure



Block Recursivity

- Why is this useful? Not much if workers are homogenous.
- Suppose workers are heterogeneous $i \in \{1,2,\ldots,N\}$ with different b_i.
- Then the same logic implies that $\{w_i, \theta_i, U_i\}$ is an equilibrium if and only if

$$U_{it} = \max_{\theta_{it}, w_{it}} b_i + \beta[f(\theta_{it})E(w_{it}) + (1 - f(\theta_{it}))U_{it+1}]$$

$$\text{s.t.} \quad \beta q(\theta_{it})J(w_{it}) \geq c$$

- Realize that the above problem is completely separable across i
- This property is called Block Recursivity
  – the equilibrium value & policy functions are independent of distribution
- After solving the above problem, one can compute $u_{it+1} = su_{it} - f(\theta_{it})(1 - u_{it})$

Taking Stock

- Competitive search makes search economy efficient
- Key ingredients:
  - Firms commit to wage offers
  - Workers perfectly observe all possible offers
  - Different wage offers operate in a different labor market
- As a result, firms/workers internalize congestion & endogenously segment
- With free-entry, it features block recursivity – a useful computational trick
- But raises several questions:
  - If everyone knows the offer and whom to meet, why is there search friction?
  - Search frictions usually a short-hand for incomplete information, screening, etc.
- Reality is somewhere between random and directed search