Using Sequence Space Jacobians to Solve Heterogeneous Agent Models

Auclert, Bardóczy, Rognlie & Straub (2021, ABRS)

741 Macroeconomics
Topic 9

Masao Fukui

Fall 2025

Question

- Modern macro puts emphasis on heterogeneity
  - households' income/wealth/consumption/friction
  - firms' productivity/size/friction
  - space/sectors/occupations
- Many interesting macro questions are about dynamics
  - IRF to monetary/fiscal policy shocks or business cycle shocks
  - Transition dynamics following the technological/policy change
- How do we solve the dynamics of macro models with heterogeneity?

Hugget Model (from 704)

- Households solve

$$V_t(a_t, e_t) = \max_{c_t,\, a_{t+1} \geq -\phi} u(c_t) + \beta \mathbb{E}_t V_{t+1}(a_{t+1}, e_{t+1})$$

$$\text{s.t.} \quad c_t + a_{t+1} = (1 + r_{t-1})a_t + e_t Y_t$$

  - Policy functions: $c_t(a, e), a_{t+1}(a, e)$
  - Normalize $\mathbb{E}e = 1$ unconditionally

- Distribution evolves

$$\mu_{t+1}(a, e_j) = \sum_i \mu_t(a'^{-1}_{t+1}(a, e_i), e_i)\text{Prob}(e_j \mid e_i)$$

- Goods market clears (bonds market clears by Warlas' law):

$$\int c_t(a, e) d\mu_t = Y_t$$

Equilibrium Counterfactuals

Neoclassical:

- Given $\{Y_t\}$, recursive eqm consists of $\{c_t, a_{t+1}, V_t\}$, $\{\mu_t\}$, and $\{r_t\}$ such that
  - $\{c_t, a_{t+1}, V_t\}$ solve Bellman & $\{\mu_t\}$ follows the law of motion
  - Goods market clears

HANK:

- Given $\{r_t\}$, recursive eqm consists of $\{c_t, a_{t+1}, V_t\}$, $\{\mu_t\}$, and $\{Y_t\}$ such that
  - $\{c_t, a_{t+1}, V_t\}$ solve Bellman & $\{\mu_t\}$ follows the law of motion
  - Goods market clears

Q: How do we compute the IRFs in response to $\{Y_t\}$ shocks or $\{r_t\}$ shocks?

Two Approaches

1. **State-space approach** (Krusell & Smith, 1998; Reiter, 2009; Bilal, 2023)
    - Find the relevant state variables and drop time subscript t
    - Dominant approach when I was a grad student
2. **Sequence-space approach** (Boppart, Krusell & Mitman, 2018; ABRS, 2021)
    - Carry t and solve for a sequence of $\{r_t\}$ or $\{Y_t\}$
    - State-of-the-art method

State-Space Approach in One Slide

Aggregate state variable: distribution $\mu_t(a, e)$!

- No household directly cares about distribution – they care about $\{r_t\}$ and $\{Y_t\}$
- Why is distribution $\mu_t(a, e)$ a state variable?
  – because they need distribution to forecast the future $\{r_t\}$ or $\{Y_t\}$
- μ is an infinite-dimensional object
  $\Rightarrow$ infeasible to carry μ_t as a state variable on the computer
- Needs a dimensional reduction:
  1. Krusell-Smith (1998): Replace μ_t with the first K moments ($K = 1$ in practice)
  2. Reiter (2009): Replace μ_t with the histogram & linearize w.r.t. agg. shocks
- Both are computationally expensive

Sequence-Space Approach

Sequence Space Approach

- Define the aggregate consumption function as

$$\mathscr{C}_t(\boldsymbol{r}, \boldsymbol{Y}) = \int c_t(a, e) d\mu_t$$

where $\boldsymbol{r} \equiv [r_t]_t$ and $\boldsymbol{Y} \equiv [Y_t]_t$

- The key observation: $\mathscr{C}$ depends only on a small number of aggregate vectors
  - Cruicially, it does not (directly) depend on distribution μ

- The goals is to find $\boldsymbol{r}$ or $\boldsymbol{Y}$ that solve

$$\mathscr{C}_t(\boldsymbol{r}, \boldsymbol{Y}) = Y_t$$

which we can stack across t to write

$$\boldsymbol{\mathscr{C}}(\boldsymbol{r}, \boldsymbol{Y}) = \boldsymbol{Y}$$

Linearized Solution

- We look for the linearized solutions around the steady state w.r.t. agg. shocks:

$$\boldsymbol{M}^r\, d\boldsymbol{r} + \boldsymbol{M}^Y\, d\boldsymbol{Y} = d\boldsymbol{Y}$$

where $\boldsymbol{M}^r \equiv \left[\frac{\partial \mathscr{C}_t}{\partial r_s}\right]_{t,s}$ and $\boldsymbol{M}^Y \equiv \left[\frac{\partial \mathscr{C}_t}{\partial Y_s}\right]_{t,s}$ are **seqeuence-space Jacobians (SSJ)**

- Both $\boldsymbol{M}^r$ and $\boldsymbol{M}^Y$ have clear economic interpretations:
  - (t, s) element of $\boldsymbol{M}^r$: PE response of agg. C at time t to an increase in r at time s
  - (t, s) element of $\boldsymbol{M}^Y$: PE response of agg. C at time t to an increase in Y at time s

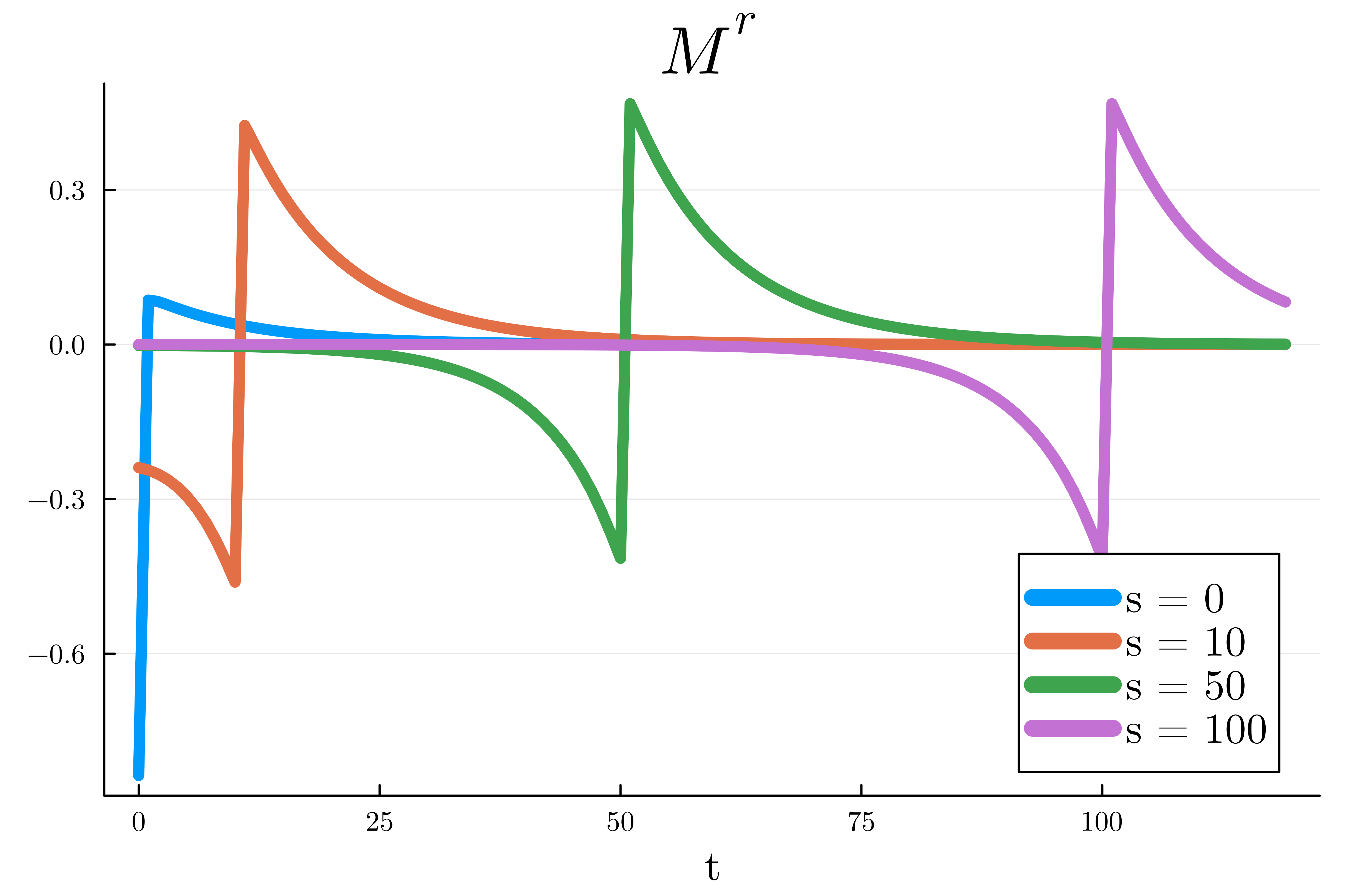
- If we know the SSJ, we can solve the linear systems to find the solutions:

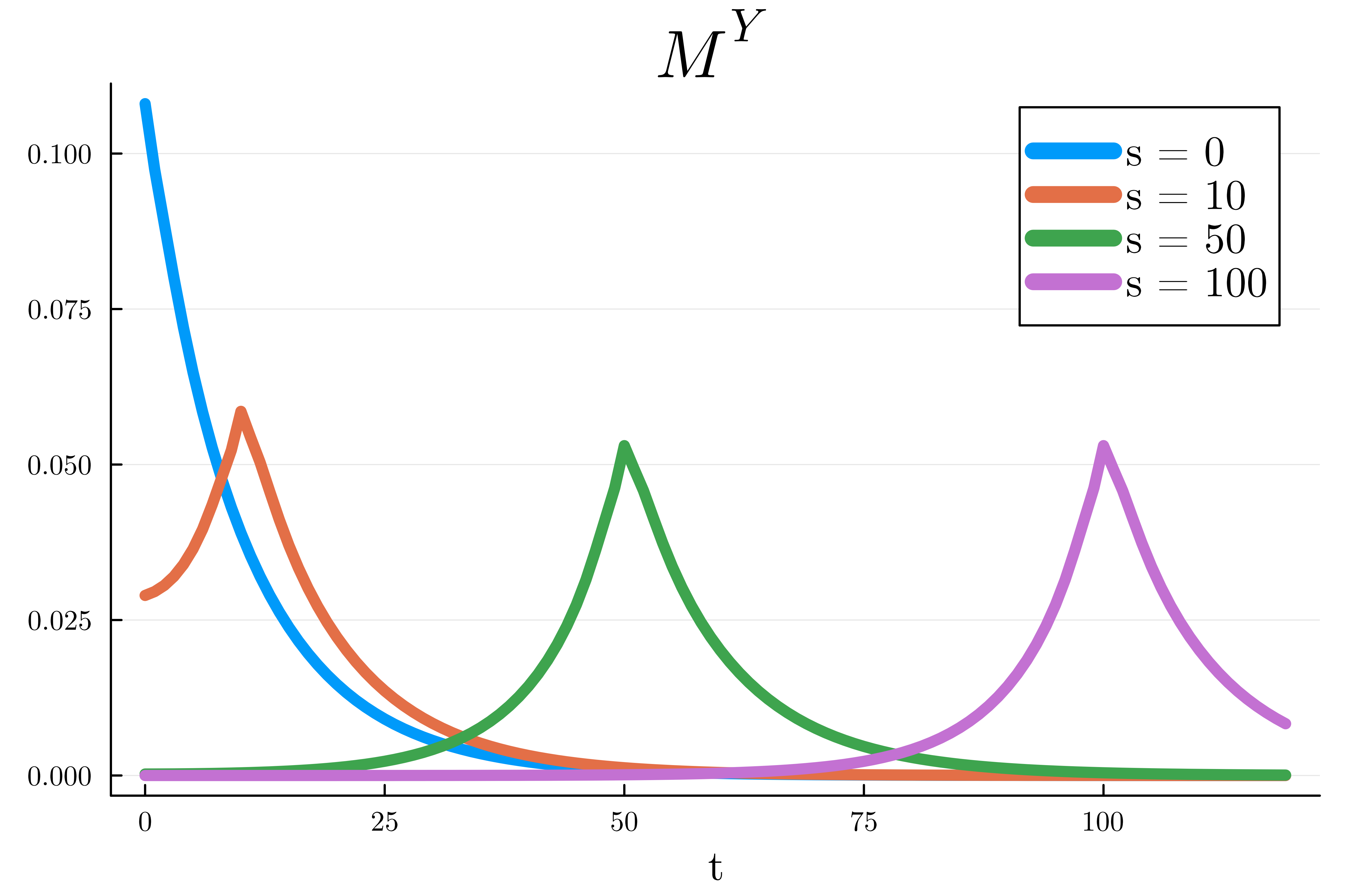
$$d\boldsymbol{r} = [M^r]^{-1}\left[d\boldsymbol{Y} - \boldsymbol{M}^Y\, d\boldsymbol{Y}\right] \qquad \text{for neolcassilcal}$$

or

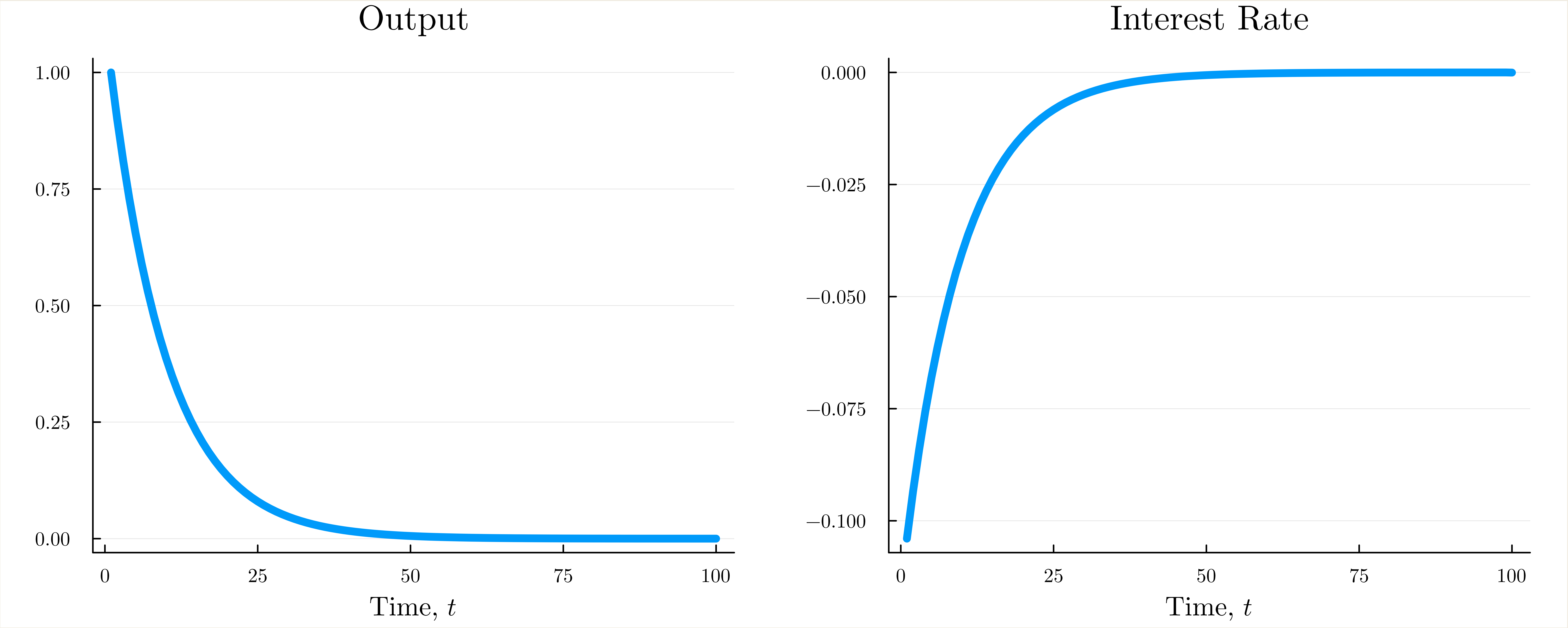
$$d\boldsymbol{Y} = [I - M^Y]^{-1}\left[\boldsymbol{M}^r\, d\boldsymbol{r}\right] \qquad \text{for HANK}$$

Visualizing M^r and M^Y

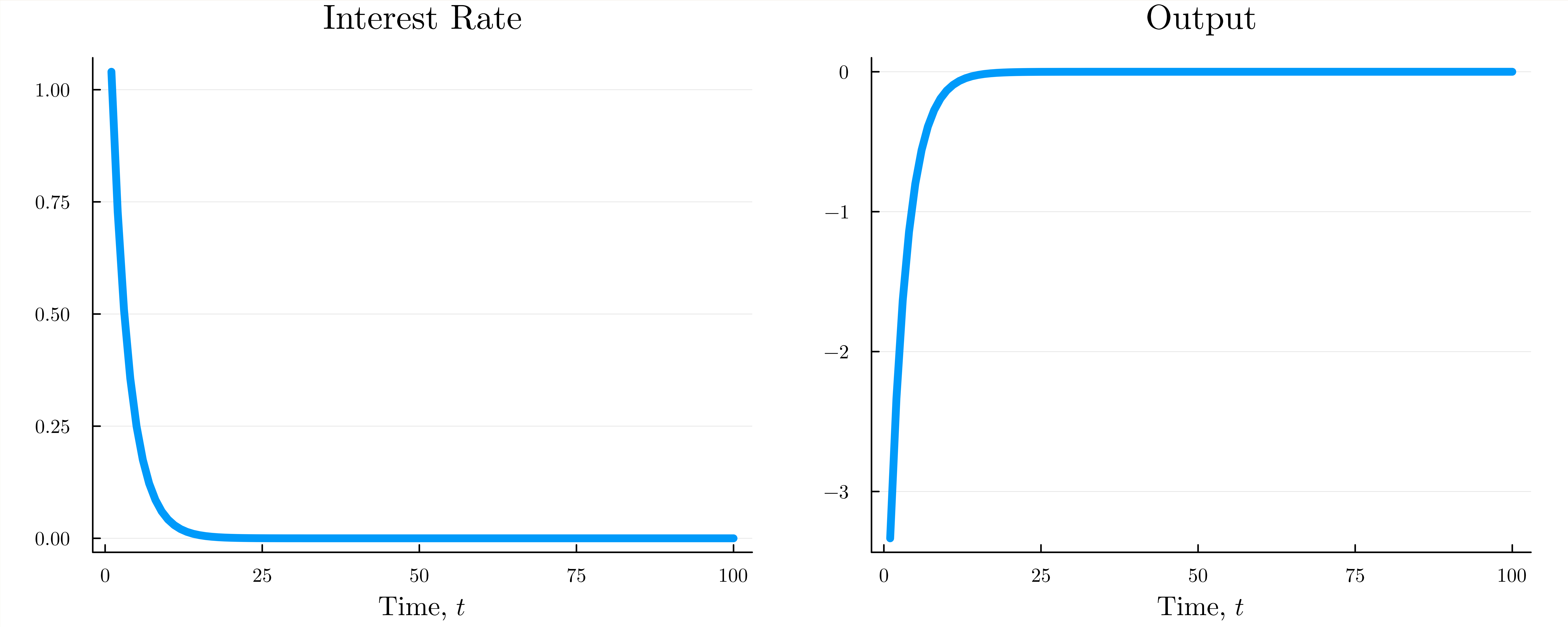




IRF of Neoclassical Economy



IRF of HANK



Interest Rate
1.00
0.75
0.50
0.25
0.00
0
25
50
75
100
Time, t
Output
0
-1
-2
-3
0
25
50
75
100
Time, t

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r + dr_0 \\ r \\ \vdots \\ r \end{bmatrix}}_{T\times 1 \text{ vector}}$$

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r + dr_0 \\ r \\ \vdots \\ r \end{bmatrix}}_{T\times 1 \text{ vector}}$$

→

Set $c_{T+1} = c_{ss}$

for $t = T : -1 : 1$

backward iteration of the Euler:

$u'(c_t) \geq \beta(1 + r_t)\mathbb{E}u'(c_{t+1})$

$c_{t+1} \Rightarrow c_t$

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r + dr_0 \\ r \\ \vdots \\ r \end{bmatrix}}_{T\times 1 \text{ vector}}$$

→

Set $c_{T+1} = c_{ss}$

for $t = T : -1 : 1$

backward iteration of the Euler:

$u'(c_t) \geq \beta(1 + r_t)\mathbb{E}u'(c_{t+1})$

$c_{t+1} \Rightarrow c_t$

→

Given $\mu_0 \equiv \mu_{ss}$ & $\{c_t, a_{t+1}\}_t$,

for $t = 1 : T$

forward iteration of the law of motion:

$\mu_{t-1} \Rightarrow \mu_t$

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r + dr_0 \\ r \\ \vdots \\ r \end{bmatrix}}_{T\times 1 \text{ vector}}$$

→

Set $c_{T+1} = c_{ss}$

for $t = T : -1 : 1$

backward iteration of the Euler:

$u'(c_t) \geq \beta(1 + r_t)\mathbb{E}u'(c_{t+1})$

$c_{t+1} \Rightarrow c_t$

→

Given $\mu_0 \equiv \mu_{ss}$ &

$\{c_t, a_{t+1}\}_t$,

for $t = 1 : T$

forward iteration of the law of motion:

$\mu_{t-1} \Rightarrow \mu_t$

→

Given $\{c_t, \mu_t\}$

$\Rightarrow C_t = \int c_t(a, e)d\mu_t$

$\Rightarrow dC_t = C_t - C_{ss}$

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r \\ r + dr_1 \\ \vdots \\ r \end{bmatrix}}_{T\times 1 \text{ vector}}$$

→

Set $c_{T+1} = c_{ss}$

for $t = T : -1 : 1$

backward iteration of the Euler:

$u'(c_t) \geq \beta(1 + r_t)\mathbb{E}u'(c_{t+1})$

$c_{t+1} \Rightarrow c_t$

→

Given $\mu_0 \equiv \mu_{ss}$ & $\{c_t, a_{t+1}\}_t$,

for $t = 1 : T$

forward iteration of the law of motion:

$\mu_{t-1} \Rightarrow \mu_t$

→

Given $\{c_t, \mu_t\}$

$\Rightarrow C_t = \int c_t(a, e) d\mu_t$

$\Rightarrow dC_t = C_t - C_{ss}$

How Do We Know SSJ?

- First, SSJs are infinite-dimensional objects, so truncate to $T \times T$ matrices
  - This is the only approximation involved in SSJ method
- How do we obtain $\boldsymbol{M}_r$ and $\boldsymbol{M}_Y$?
- The brute force approach:

$$\boldsymbol{r} := \underbrace{\begin{bmatrix} r \\ r \\ \vdots \\ r + dr_T \end{bmatrix}}_{T\times 1 \text{ vector}}$$

→

Set $c_{T+1} = c_{ss}$

for $t = T : -1 : 1$

backward iteration of the Euler:

$u'(c_t) \geq \beta(1+r_t)\mathbb{E}u'(c_{t+1})$

$c_{t+1} \Rightarrow c_t$

→

Given $\mu_0 \equiv \mu_{ss}$ & $\{c_t, a_{t+1}\}_t$,

for $t = 1 : T$

forward iteration of the law of motion:

$\mu_{t-1} \Rightarrow \mu_t$

→

Given $\{c_t, \mu_t\}$

$\Rightarrow C_t = \int c_t(a,e) d\mu_t$

$\Rightarrow dC_t = C_t - C_{ss}$

Speeding Up the SSJ Computation

- The brute force approach is prohibitively expensive
  - We need T backward iterations & T forward iterations
- Can we do better? Yes, a lot better!

Speeding Up the Backward Iteration

- The first key insight:

$$\frac{dc_t(a,e)}{dr_s} = \begin{cases} 0 & s < t \\ \dfrac{dc_{T-(s-t)}(a,e)}{dr_T} & s \geq t \end{cases}$$

- Bellman equation is entirely forward-looking! This encompasses:

1. Past shocks do not change the policy **functions**
    - It does change the consumption path, but that is through changes in asset a
2. Future shocks change the policies, but only the distance to the shock matters!
    - Calendar time is irrelevant

- This implies that a **single** backward iteration is enough to compute $\frac{dc_t(a,e)}{dr_s}$ for all t, s

Matrix Notation

- With discretized μ ($N_a N_y \times 1$ vector), the law of motion is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Lambda}_t' \boldsymbol{\mu}_t \quad \text{where } \Lambda_t \text{ is the Markov transition matrix}$$

- To a first-order

$$\frac{d\boldsymbol{\mu}_{t+1}}{dr_s} = \left[\frac{d\boldsymbol{\Lambda}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}_{ss} \frac{d\boldsymbol{\mu}_t}{dr_s}$$

- Discretize the policy ($\boldsymbol{c}$ being $N_a N_y \times 1$ vector) to write the agg. consumption as

$$\mathscr{C}_t = \boldsymbol{c}_t' \boldsymbol{\mu}_t$$

- To a first-order,

$$M_{t,s}^r \equiv \frac{d\mathscr{C}_t}{dr_s} = \left[\frac{d\boldsymbol{c}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} \quad + \quad \boldsymbol{c}_{ss}' \left[\frac{d\boldsymbol{\mu}_t}{dr_s}\right]$$

Matrix Notation

- With discretized μ ($N_a N_y \times 1$ vector), the law of motion is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Lambda}_t' \boldsymbol{\mu}_t \quad \text{where } \Lambda_t \text{ is the Markov transition matrix}$$

- To a first-order

$$\frac{d\boldsymbol{\mu}_{t+1}}{dr_s} = \left[\frac{d\boldsymbol{\Lambda}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}_{ss} \frac{d\boldsymbol{\mu}_t}{dr_s}$$

- Discretize the policy ($\boldsymbol{c}$ being $N_a N_y \times 1$ vector) to write the agg. consumption as

Impact through changes in policy holding dist. fixed

$$\boldsymbol{c}_t' \boldsymbol{\mu}_t$$

- To a first-order,

$$M_{t,s}^r \equiv \frac{d\mathscr{C}_t}{dr_s} = \left[\frac{d\boldsymbol{c}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}_{ss}' \left[\frac{d\boldsymbol{\mu}_t}{dr_s}\right]$$

Matrix Notation

- With discretized μ ($N_a N_y \times 1$ vector), the law of motion is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Lambda}_t' \boldsymbol{\mu}_t \quad \text{where } \Lambda_t \text{ is the Markov transition matrix}$$

- To a first-order

$$\frac{d\boldsymbol{\mu}_{t+1}}{dr_s} = \left[\frac{d\boldsymbol{\Lambda}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}_{ss} \frac{d\boldsymbol{\mu}_t}{dr_s}$$

- Discretize the policy ($\boldsymbol{c}$ being $N_a N_y \times 1$ vector) to write the agg. consumption as

$$\mathscr{C}_t = \boldsymbol{c}_t' \boldsymbol{\mu}_t$$

- To a first-order,

$$M_{t,s}^r \equiv \frac{d\mathscr{C}_t}{dr_s} = \left[\frac{d\boldsymbol{c}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} \quad + \quad \boldsymbol{c}_{ss}' \left[\frac{d\boldsymbol{\mu}_t}{dr_s}\right]$$

Matrix Notation

- With discretized μ ($N_a N_y \times 1$ vector), the law of motion is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Lambda}_t' \boldsymbol{\mu}_t \quad \text{where } \Lambda_t \text{ is the Markov transition matrix}$$

- To a first-order

$$\frac{d\boldsymbol{\mu}_{t+1}}{dr_s} = \left[\frac{d\boldsymbol{\Lambda}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}_{ss} \frac{d\boldsymbol{\mu}_t}{dr_s}$$

- Discretize the policy ($\boldsymbol{c}$ being $N_a N_y \times 1$ vector) to write the agg. consumption as

$$\mathscr{C}_t = \boldsymbol{c}_t' \boldsymbol{\mu}_t$$

Impact through changes in dist.
holding policy fixed

- To a first-order,

$$M^r_{t,s} \equiv \frac{d\mathscr{C}_t}{dr_s} = \left[\frac{d\boldsymbol{c}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}_{ss}' \left[\frac{d\boldsymbol{\mu}_t}{dr_s}\right]$$

Matrix Notation

- With discretized μ ($N_a N_y \times 1$ vector), the law of motion is

$$\boldsymbol{\mu}_{t+1} = \boldsymbol{\Lambda}_t' \boldsymbol{\mu}_t \quad \text{where } \Lambda_t \text{ is the Markov transition matrix}$$

- To a first-order

$$\frac{d\boldsymbol{\mu}_{t+1}}{dr_s} = \left[\frac{d\boldsymbol{\Lambda}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}_{ss} \frac{d\boldsymbol{\mu}_t}{dr_s}$$

- Discretize the policy ($\boldsymbol{c}$ being $N_a N_y \times 1$ vector) to write the agg. consumption as

$$\mathscr{C}_t = \boldsymbol{c}_t' \boldsymbol{\mu}_t$$

- To a first-order,

$$M^r_{t,s} \equiv \frac{d\mathscr{C}_t}{dr_s} = \left[\frac{d\boldsymbol{c}_t}{dr_s}\right]' \boldsymbol{\mu}_{ss} \quad + \quad \boldsymbol{c}_{ss}' \left[\frac{d\boldsymbol{\mu}_t}{dr_s}\right]$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$
\begin{aligned}
\mathcal{M}^r_{t,s} - \mathcal{M}^r_{t-1,s-1} &= \underbrace{\left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]'}_{=\,0} \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right] \\
&= \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right] \\
&= \boldsymbol{c}_{ss}{}'\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right] \\
&= \boldsymbol{c}_{ss}{}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right] \\
&= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}
\end{aligned}
$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= 0$$

$$= \boldsymbol{c}_{ss}\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}_{ss}{}'\boldsymbol{\Lambda}_{ss}\left[\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t\frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

The difference btwn :

1. Response at t to shock at s
2. Response at $t-1$ to shock at $s-1$

Recursive Structure of SSJ

- The SSJs have a particular recursive structure:

$$M^r_{t,s} - M^r_{t-1,s-1} = \left[\frac{d\boldsymbol{c}_t}{dr_s} - \frac{d\boldsymbol{c}_{t-1}}{dr_{s-1}}\right]' \boldsymbol{\mu}_{ss} + \boldsymbol{c}'_{ss}\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$\left[\frac{d\boldsymbol{\mu}_t}{dr_s} - \frac{d\boldsymbol{\mu}_{t-1}}{dr_{s-1}}\right]$$

$$\left[\frac{d\boldsymbol{\Lambda}_{t-1}}{dr_s}\boldsymbol{\mu}_{ss} + \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-1}}{dr_s} - \frac{d\boldsymbol{\Lambda}_{t-2}}{dr_{s-1}}\boldsymbol{\mu}^{ss} - \boldsymbol{\Lambda}^{ss}\frac{d\boldsymbol{\mu}_{t-2}}{dr_{s-1}}\right]$$

$$\left[\frac{d\boldsymbol{\mu}_{t-1}}{\ldots} \quad \frac{d\boldsymbol{\mu}_{t-2}}{\ldots}\right]$$

$$= \boldsymbol{c}'_{ss}(\boldsymbol{\Lambda}_{ss})^t \frac{d\boldsymbol{\mu}_1}{dr_s} \equiv \mathscr{F}_{t,s}$$

The difference btwn :

1. Response at t to shock at s
2. Response at $t-1$ to shock at $s-1$

Households had one more period in advance to prepare for the shock

Initial Condition for SSJ

- The initial conditions are

$$M^r_{0,s} = \left[\frac{d\boldsymbol{c}_0}{dr_s}\right]' \boldsymbol{\mu}_{ss}$$

- Now we know

1. $M^r_{t,s} - M^r_{t-1,s-1} \equiv \mathscr{F}_{t,s}$
2. $M^r_{0,s}$

$\Rightarrow$ We can compute any element of M^r!

Filling the SSJ

$$\boldsymbol{M}^r \equiv \begin{bmatrix} M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\ & & & & & & \vdots \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix}$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix} M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\ & M^r_{1,1} = M^r_{0,0} + \mathscr{F}_{1,1} & & & & & \vdots \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix}
$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix}
M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\
 & \color{red}{M^r_{1,1}} & & & & & \vdots \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & &
\end{bmatrix}
$$

Filling the SSJ

$$\boldsymbol{M}^r \equiv \begin{bmatrix} M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\ & M^r_{1,1} & M^r_{1,2} = M^r_{0,1} + \mathscr{F}_{1,2} & & & & \vdots \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix}$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix}
M^r_{0,0} & M^r_{0,1} & & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\
& M^r_{1,1} & M^r_{1,2} & & & & & \vdots \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & &
\end{bmatrix}
$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix}
M^r_{0,0} & M^r_{0,1} & & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\
M^r_{1,0} = \mathscr{F}^r_{1,0} M^r_{1,1} & & M^r_{1,2} & & & & & \vdots \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & & \\
\vdots & & & & & & &
\end{bmatrix}
$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix} M^r_{0,0} & M^r_{0,1} & & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\ M^r_{1,0} & M^r_{1,1} & M^r_{1,2} & & & & & \vdots \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \\ \vdots & & & & & & & \end{bmatrix}
$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix}
M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\
M^r_{1,0} & M^r_{1,1} & \cdots & \cdots & \cdots & \cdots & M^r_{1,T} \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & &
\end{bmatrix}
$$

Filling the SSJ

$$\boldsymbol{M}^r \equiv \begin{bmatrix} M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\ M^r_{1,0} & M^r_{1,1} & \cdots & \cdots & \cdots & \cdots & M^r_{1,T} \\ \color{red}{M^r_{2,0}} & \color{red}{M^r_{2,1}} & \cdots & \cdots & \cdots & \cdots & \color{red}{M^r_{2,T}} \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \\ \vdots & & & & & & \end{bmatrix}$$

Filling the SSJ

$$
\boldsymbol{M}^r \equiv \begin{bmatrix}
M^r_{0,0} & M^r_{0,1} & \cdots & \cdots & \cdots & \cdots & M^r_{0,T} \\
M^r_{1,0} & M^r_{1,1} & \cdots & \cdots & \cdots & \cdots & M^r_{1,T} \\
M^r_{2,0} & M^r_{2,1} & \cdots & \cdots & \cdots & \cdots & M^r_{2,T} \\
\vdots & & & & & & \\
\vdots & & & & & & \\
\vdots & & & & & & \\
M^r_{T,0} & M^r_{T,1} & \cdots & \cdots & \cdots & \cdots & M^r_{T,T}
\end{bmatrix}
$$

Sequence Space Jacobian Algorithm

1. Solve Euler backward in response to a shock at the terminal period, dr_T
   - This gives $\left\{\frac{d\boldsymbol{c}_t}{dr_s}, \frac{d\boldsymbol{a}_{t+1}}{dr_s}\right\}$ for any t, s because $\frac{dx_t}{dr_s} = \frac{dx_T}{dr_{T-(t-s)}}$

2. For each $s = 0,1,\ldots,T$
   - Compute $M^r_{0,s} \equiv \left[\frac{d\boldsymbol{c}_0}{dr_s}\right]' \boldsymbol{\mu}_{ss}, \quad \frac{d\boldsymbol{\mu}_1}{dr_s} = \frac{d\boldsymbol{\Lambda}_0}{dr_s} \times \boldsymbol{\mu}_{ss}, \quad \mathcal{F}_{t,s} = \boldsymbol{c}_{ss}'(\boldsymbol{\Lambda}_{ss})^t \frac{d\boldsymbol{\mu}_1}{dr_s}$
   - For each $t = 0,1,\ldots,T$, compute $M^r_{t,s}$ recursively using

$$M^r_{t,s} = M^r_{t-1,s-1} + \mathcal{F}_{t,s}$$

⚠ For $t < 0$ or $s < 0$, set $M^r_{t,s} = 0$

The Sources of Speed Gains

Brute-force algorithm

1. Backward iteration from $t = T$ to $t = 0$ for eash $s = 1,\ldots,T$
2. Forward iteration from $t = 0$ to $t = T$ for each $s = 1,\ldots,T$

ARBS algorithm ("fake-news algorithm")

1. A **single** backward iteration from $t = T$ to $t = 0$
2. (Non-SS) forward iteration **only at** $t = 0$ for each $s = 1,\ldots,T$

This cuts the computation time by a factor of T

- It takes ≈ 0.03 seconds to compute SSJ on my laptop

Tips

- You cannot do a grid search for Bellman iteration
  - The optimal grid never moves in response to a small shock
  - Use EGM, FOC, or continuous optimization
- Use automatic differentiation whenever you can!
  - This is easy in Julia
- I encounter numerical issues whenever T is big
  - The small error in the market clearing in the SS accumulates as $T \to \infty$
  - Need to separately take care of the error coming from the SS tolerance
- ARBS' Python package: https://github.com/shade-econ/sequence-jacobian
  - I'd discourage using it. Don't constrain what you do with what the package can do!

Scope and Frontiers

Scope of the Methodology

- So far, MIT shock. However, to a first order, certainty equivalence implies

  IRF in a stochastic economy = IRF to an unanticipated shock

  under the incomplete market w.r.t. aggregate shocks

- The methodology naturally extends to a much broader classes of models
  - Other shocks (e.g., ϕ shock & β shock)
  - Many other classes of heterogeneous agent models (e.g., heterogenous firms)

- A key restriction is that individuals do not directly care about the distribution
  - This fails in many classes of job-ladder models
  - Still, the SSJ methodology extends as long as the job ladder is rank preserving (Fukui, 2021; Fukui & Mukoyama, 2025)

SSJs are Useful Not Only for Computation

- SSJs are, in principle, directly measurable from the data
  - Sufficient statistics that strongly discipline the model
    (e.g., Auclert-Rognlie-Straub, 2024, “Intertemporal Keynesian Cross”)
- SSJs serve as the basis for the policy rule counterfactuals
  (McKay and Wolf, 2023; Barnichon and Mesters, 2023)
- SSJs guide the optimal macro policies
  (e.g, McKay and Wolf, 2025; Auclert, Cai, Rognlie, and Straub, 2024)

Frontier

1. Allow agents to hedge against the aggregate shocks
   (Aculert, Rognlie, Straub, Ťapák, 2024; Donald, Fukui, Miyauchi, 2025)

2. 2nd- and higher-order approximation
   (“Beyond Certainty Equivalence: Second-Order Solutions in the Sequence Space” by ARS+)

3. Other approaches:

   - Machine learning to obtain global solutions:
     Fernández-Villaverde, Hurtado, Nuno (2023), Payne, Rebei, Yang (2025)

   - State-space: Bhandari-Bourany-Golosov-Sargent (2023)

   - State-space + master equation: Bilal (2024)