
Labor Reallocation and Misallocation

741 Macroeconomics
Topic 7

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Firm Employment is Log-Linear in TFP

- In Hopenhayn-Rogerson, firm-level employment is given by

$$\begin{aligned} n &= \underbrace{(z^{1-\alpha} \alpha/w)^{\frac{1}{1-\alpha}}}_{\equiv Z} \\ &\Leftrightarrow \log n = \frac{1}{1-\alpha} \log Z + const \end{aligned}$$

- Taking the first difference,

$$\Delta \log n = \frac{1}{1-\alpha} \Delta \log Z$$

⇒ Firms react symmetrically to positive and negative TFP shocks

- Is this true in the data?

Ilut, Kehrig & Schneider (2018)

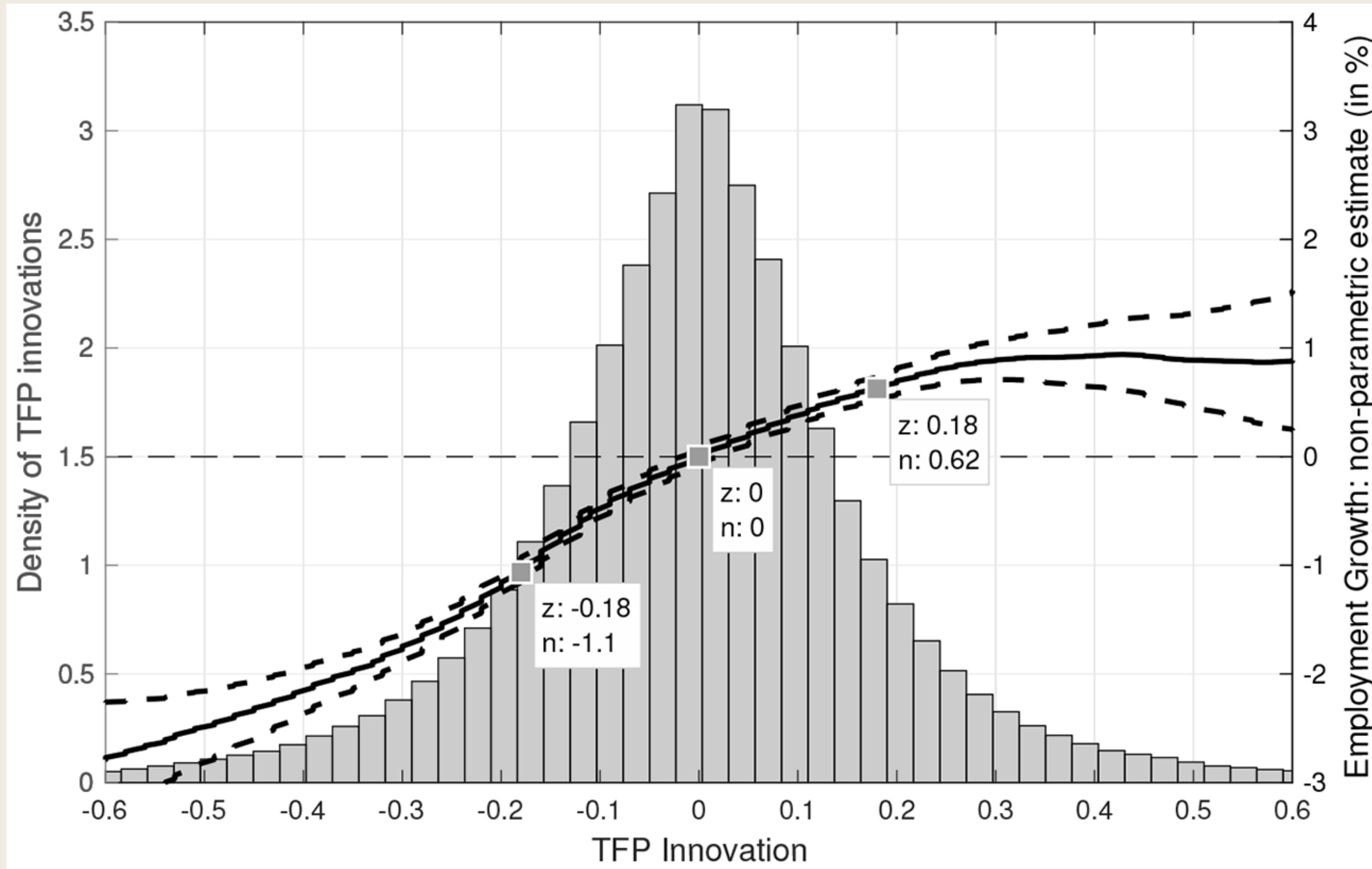
- Focus on US manufacturing establishments (Census data)
- Construct firm-level TFP using Solow residual:

$$\log sr_{it} = \log y_{it} - (\beta_n \log n_{it} + \beta_k \log k_{it} + \beta_m \log m_{it})$$

- Construct firm-level TFP shocks, Z_{it} , assuming
- Q: How does firm-level employment respond to TFP shocks?

$$\Delta \log n_{it} = h(\Delta \log Z_{it}) + \gamma' X_{it} + \epsilon_{it}$$

Concave Hiring Rule



Firm Dynamics with Labor Adjustment Costs – Hopenhayn & Rogerson (1993)

Slow to Hire, Quick to Fire

- The simplest explanation:
 - it is costly to hire workers
 - less so to fire workers

- We incorporate employment adjustment costs into Hopenhayn-Rogerson

Labor Adjustment Cost

- Suppose that employment stock is costly to adjust
- Every period, $\delta \in [0,1]$ fraction of workers exogenously separate
- Firms can hire $h \times n$ workers with hiring cost $\Phi(h, n)$
 - $h < 0$ corresponds to firing
- The stock-flow equation of employment:

$$n_t = n_{t-1}(1 - \delta + h_t)$$

Bellman Equation

■ Bellman equation:

$$v(n_{-1}, z) = \max \{ v^*(n_{-1}, z), -\Phi(-(1-\delta), n_{-1}) \}$$

where v^* is the continuation value

$$\begin{aligned} v^*(n_{-1}, z) &= \max_{h, n} f(n, z) - wn - c_f - \Phi(h, n_{-1}) + \beta \mathbb{E}v(n, z') \\ \text{s.t. } n &= n_{-1}(1 - \delta + h) \end{aligned}$$

■ Policy functions:

- $\chi(n_{-1}, z) \in \{0,1\}$: whether to exit or not
- $h(n_{-1}, z)$: hiring rate
- $n(n_{-1}, z)$: employment

Rest of the Equilibrium Conditions

- Assume that the initial firm size is given by n_0

- The free-entry condition is

$$\int v(n_0, z) \psi_0(z) dz = c_e$$

- Let $g(n_{-1}, z)$ denote the steady-state distribution, which satisfies

$$g(n, z') = \iint \Pi(z' | z)(1 - \chi(n_{-1}, z)) \mathbb{I}[n(n_{-1}, z) = n] g(n_{-1}, z) dz dn_{-1} + m\psi_0(z) \mathbb{I}[n = n_0]$$

- The labor market clearing condition is

$$\iint n(n_{-1}, z) g(n_{-1}, z) dn_{-1} dz = L$$

Equilibrium Definition

- Recursive equilibrium: $\{v(n, z), \chi(n, z), n'(n, z), w\}$ and $\{g(n_{-1}, z), m\}$ such that:
 1. Given w , $\{v(n, z), \chi(n, z), n'(n, z)\}$ solve the Bellman equation
 2. Free entry holds, $\int v(n_0, z) \psi_0(z) dz = c_e$
 3. $\{g(n_{-1}, z), m\}$ satisfies the steady state law of motion
 4. Labor market clears
- The equilibrium retains the same structure as before:
 1. Block recursive property: value and policy functions independent of distribution
 2. $g(n_{-1}, z)$ homogenous in m :
 - can solve for $\hat{g} \equiv g/m$ first \Rightarrow solve for m using labor market clearing

Equilibrium Definition

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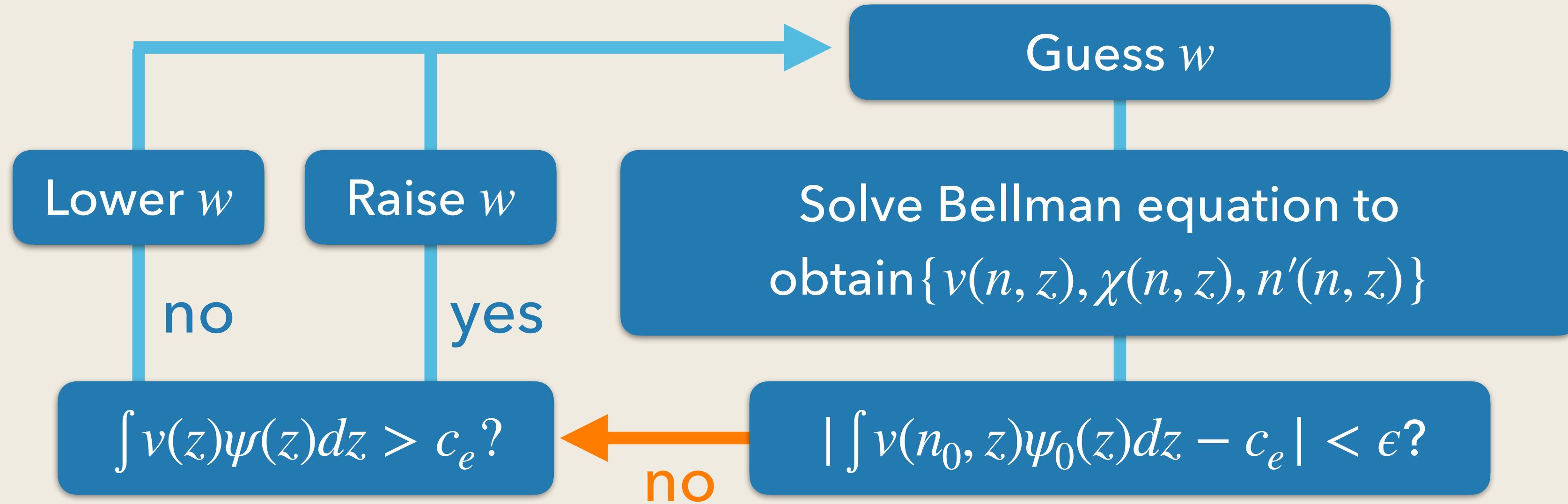
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1 & 2 alone
 $\Rightarrow \{v(n, z), \chi(n, z), n'(n, z), w\}$

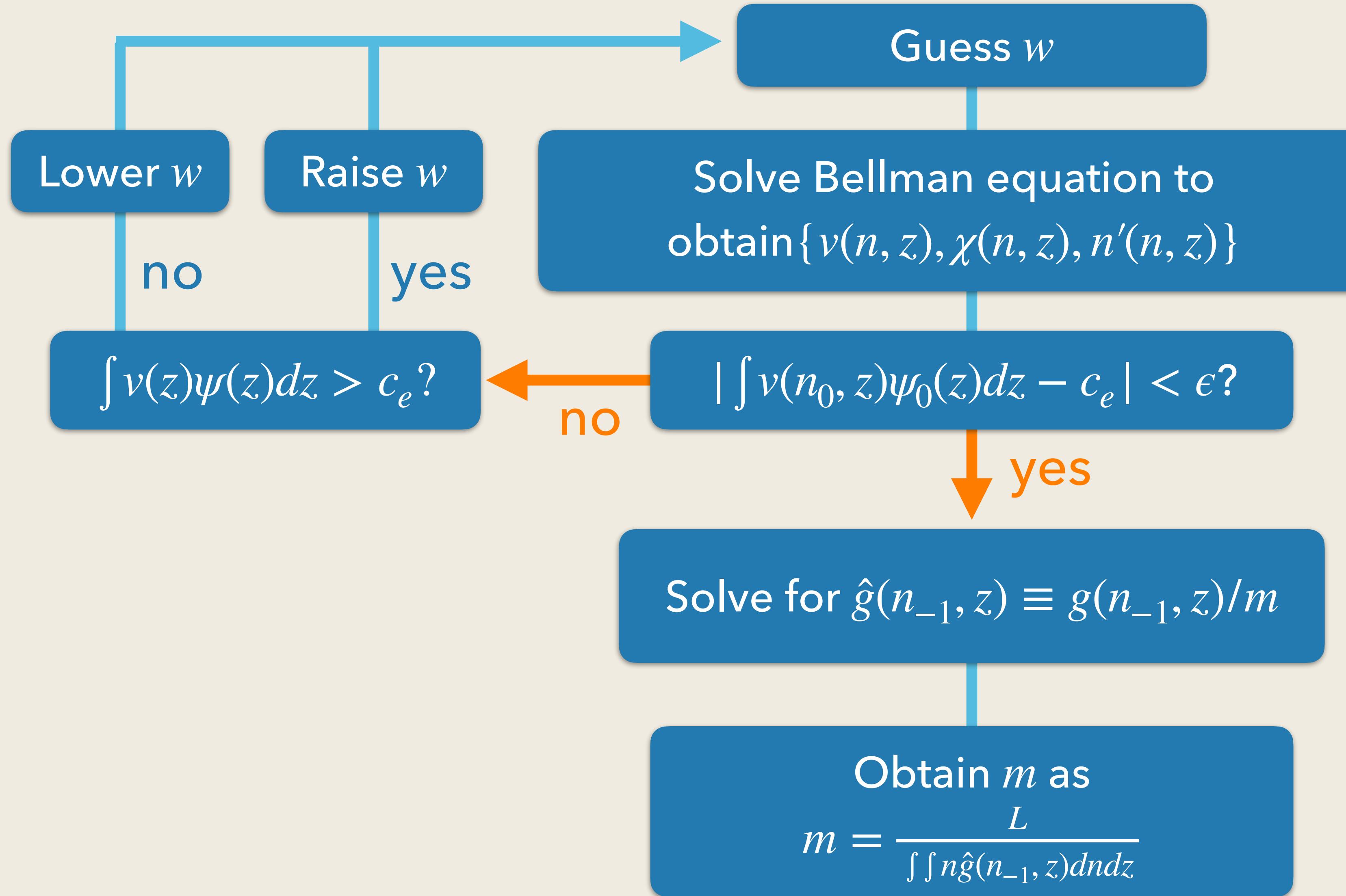
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Computational Algorithm



Computational Algorithm



Calibration

- Same parameter values for those that appear in the previous lecture note
- Set

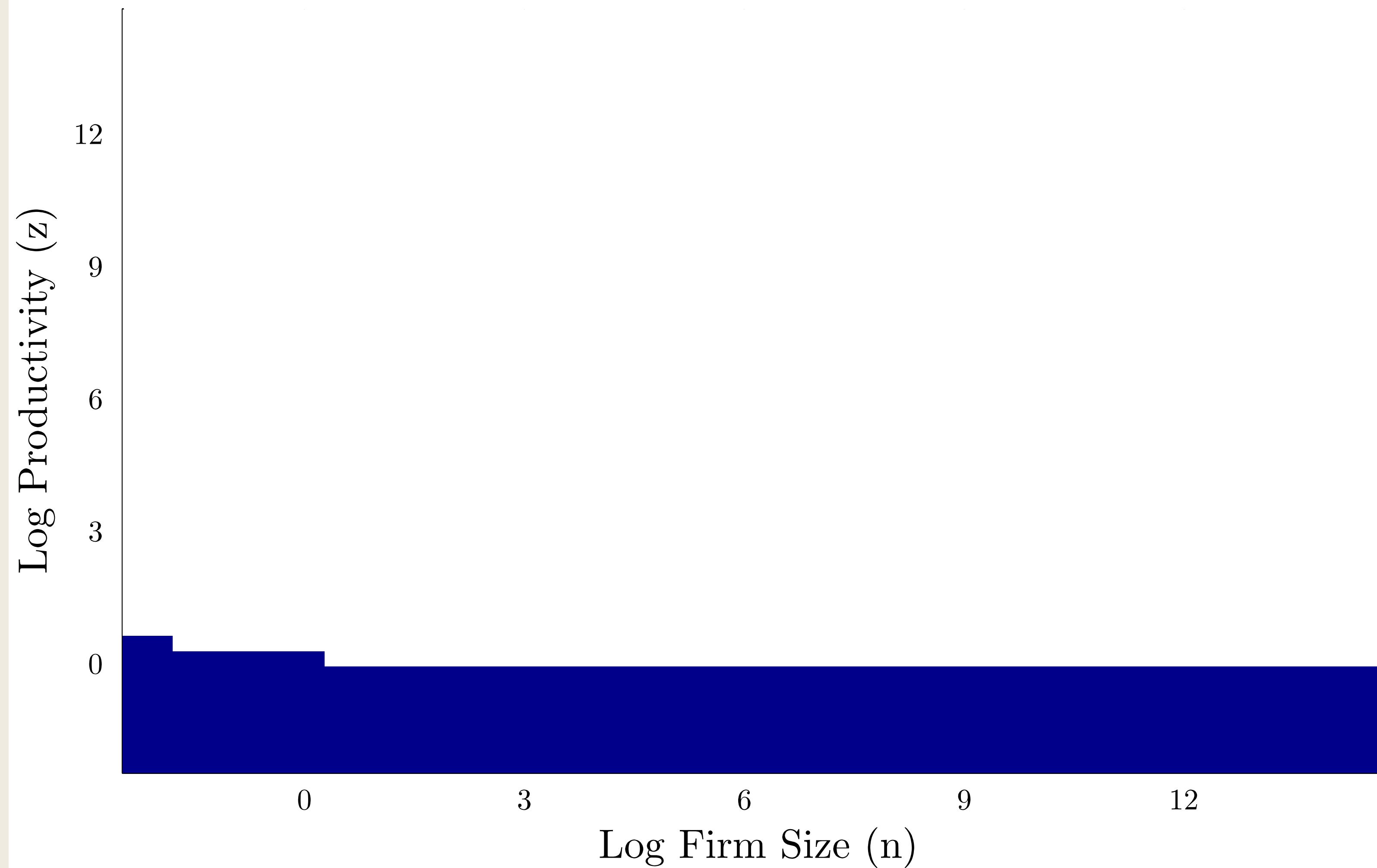
$$\Phi(h, n_{-1}) = \begin{cases} \frac{\phi_+}{2} h^2 n_{-1} & \text{if } h > 0 \\ \frac{\phi_-}{2} h^2 n_{-1} & \text{if } h < 0 \end{cases}$$

and assume $\phi_- = 0$ and $\phi_+ = 5$

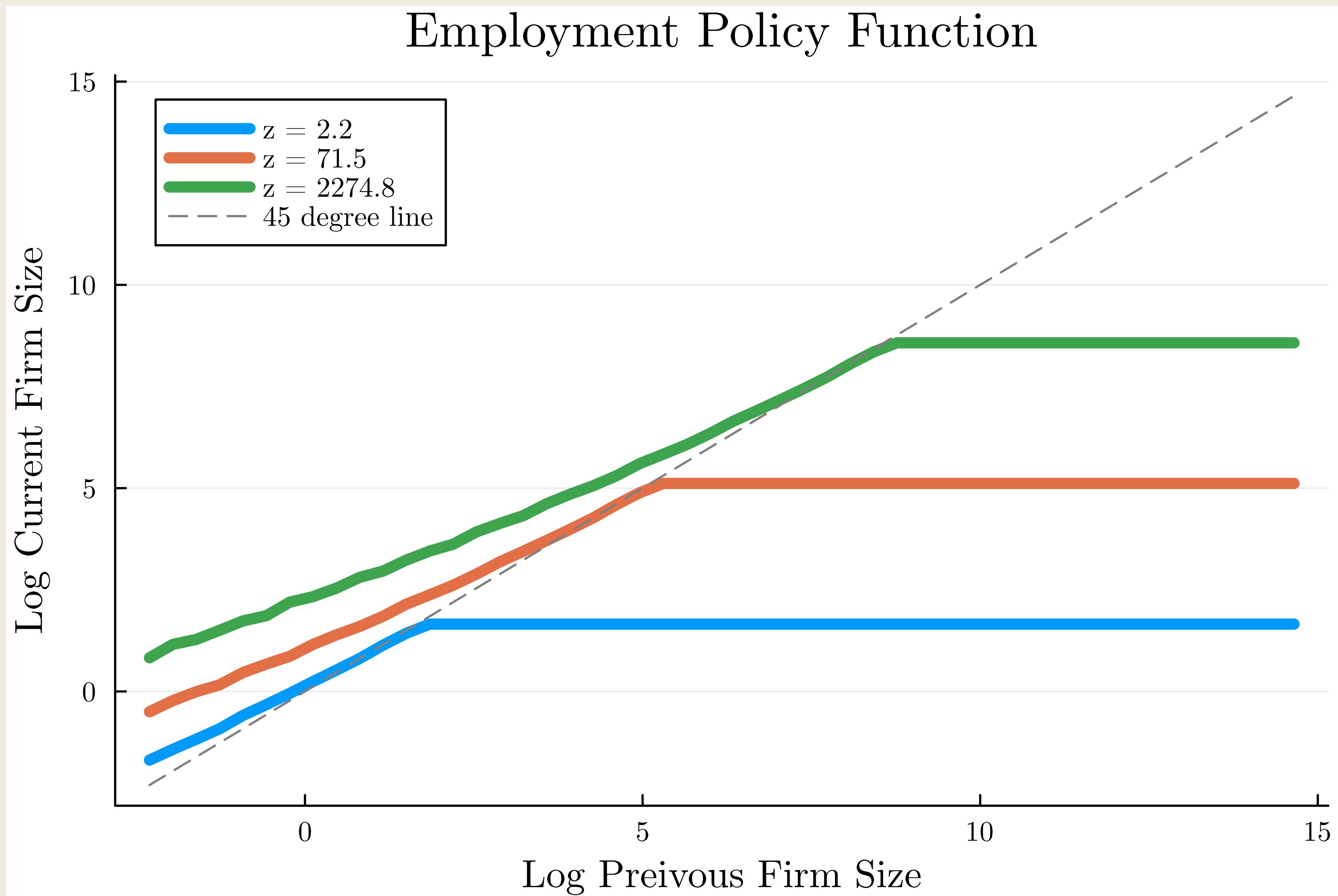
- Set $\delta = 0.1$
- Assume $n_0 = 5$ to roughly match the initial firm size

Exit Policy

Exit Policy (Blue = Exit, White = Continue)

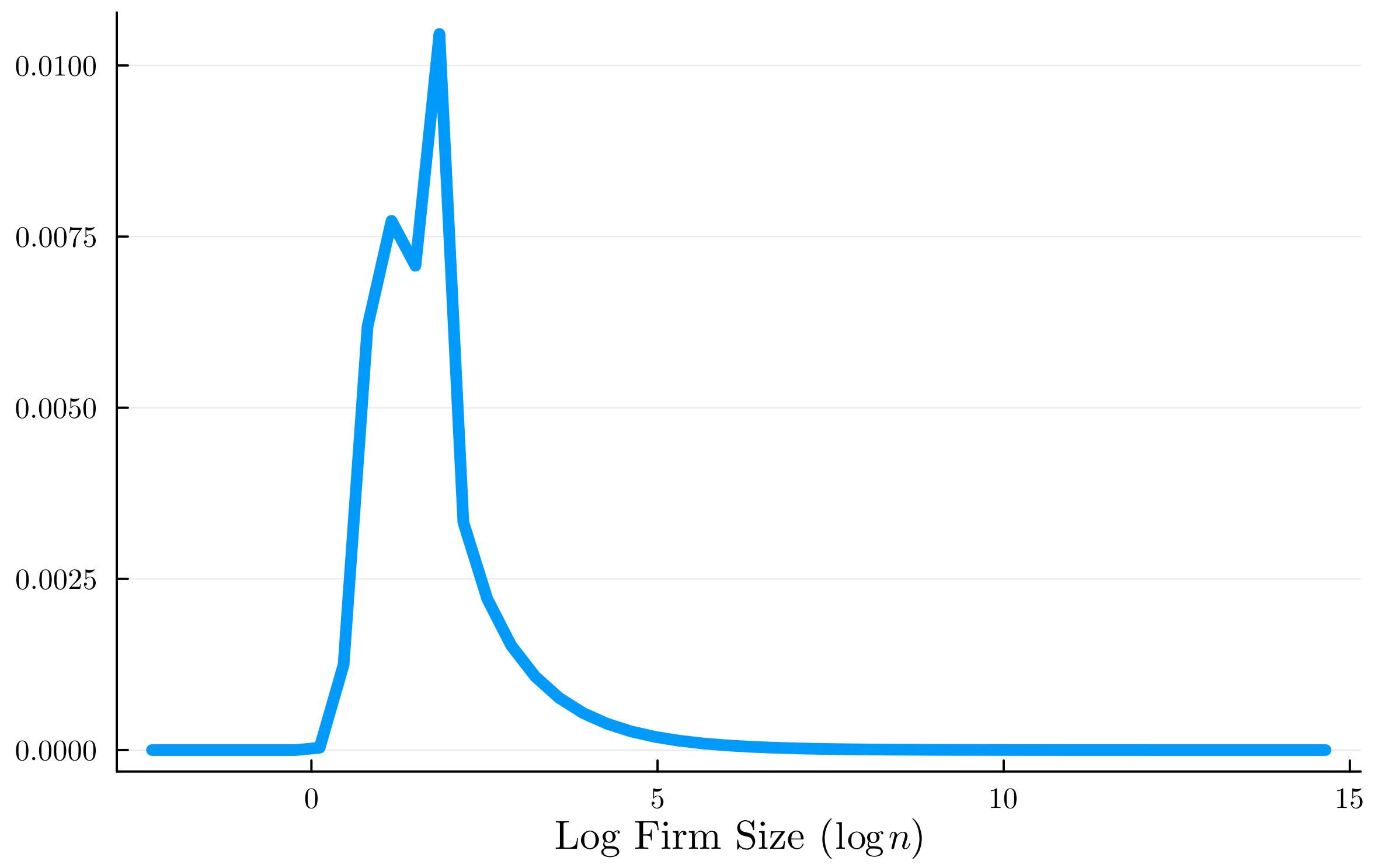


Employment Policy

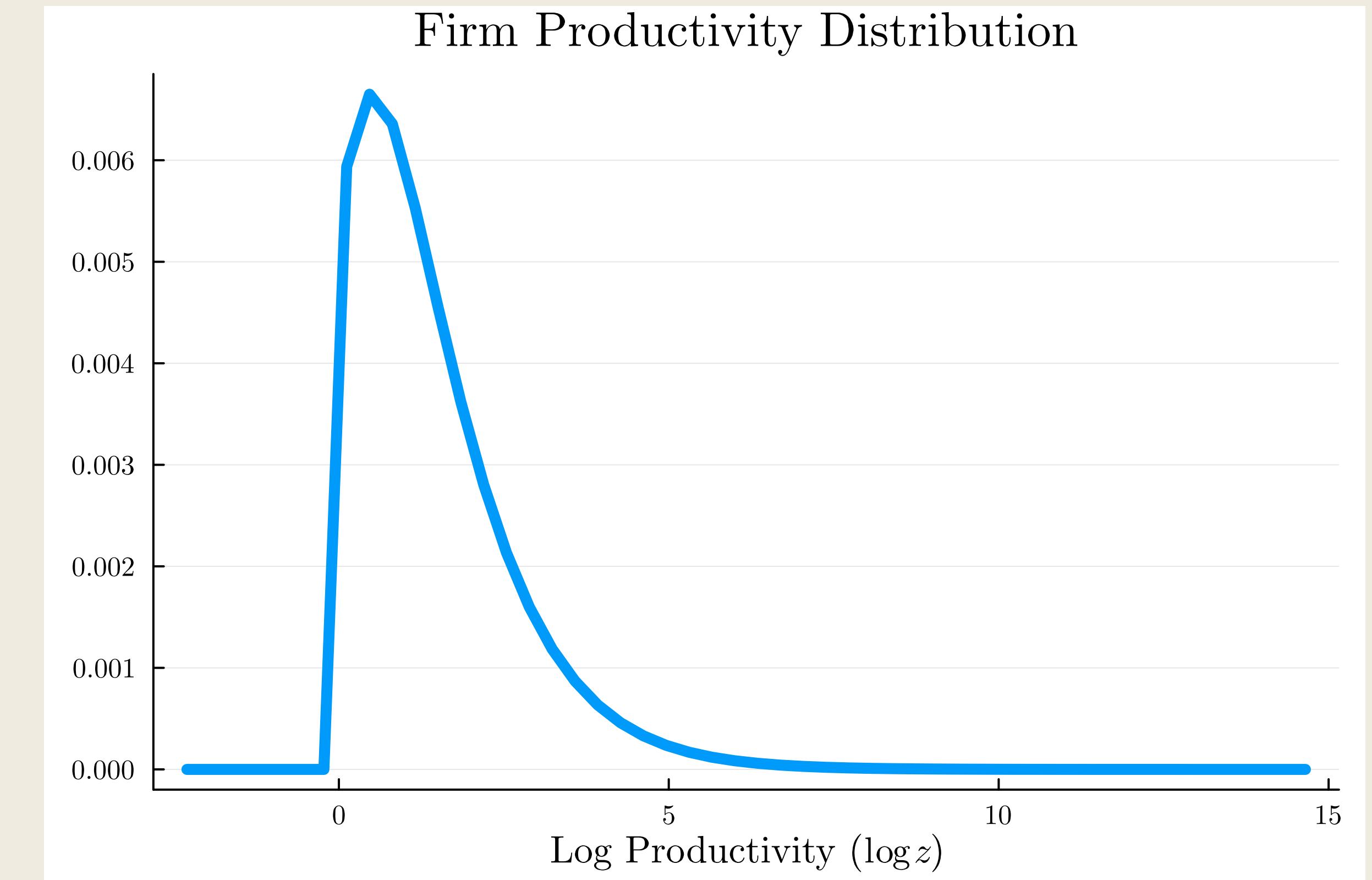


Distribution

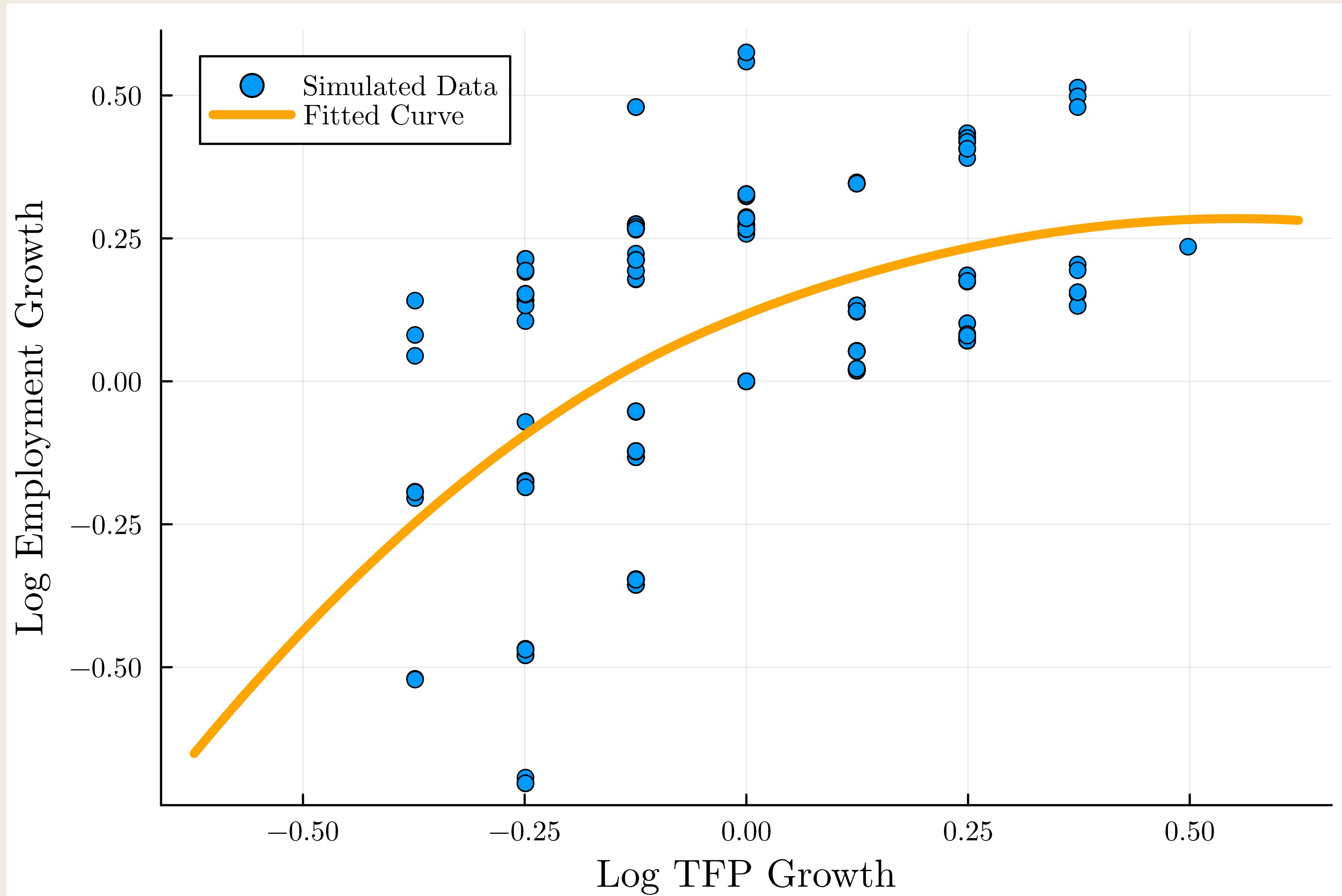
Firm Size Distribution



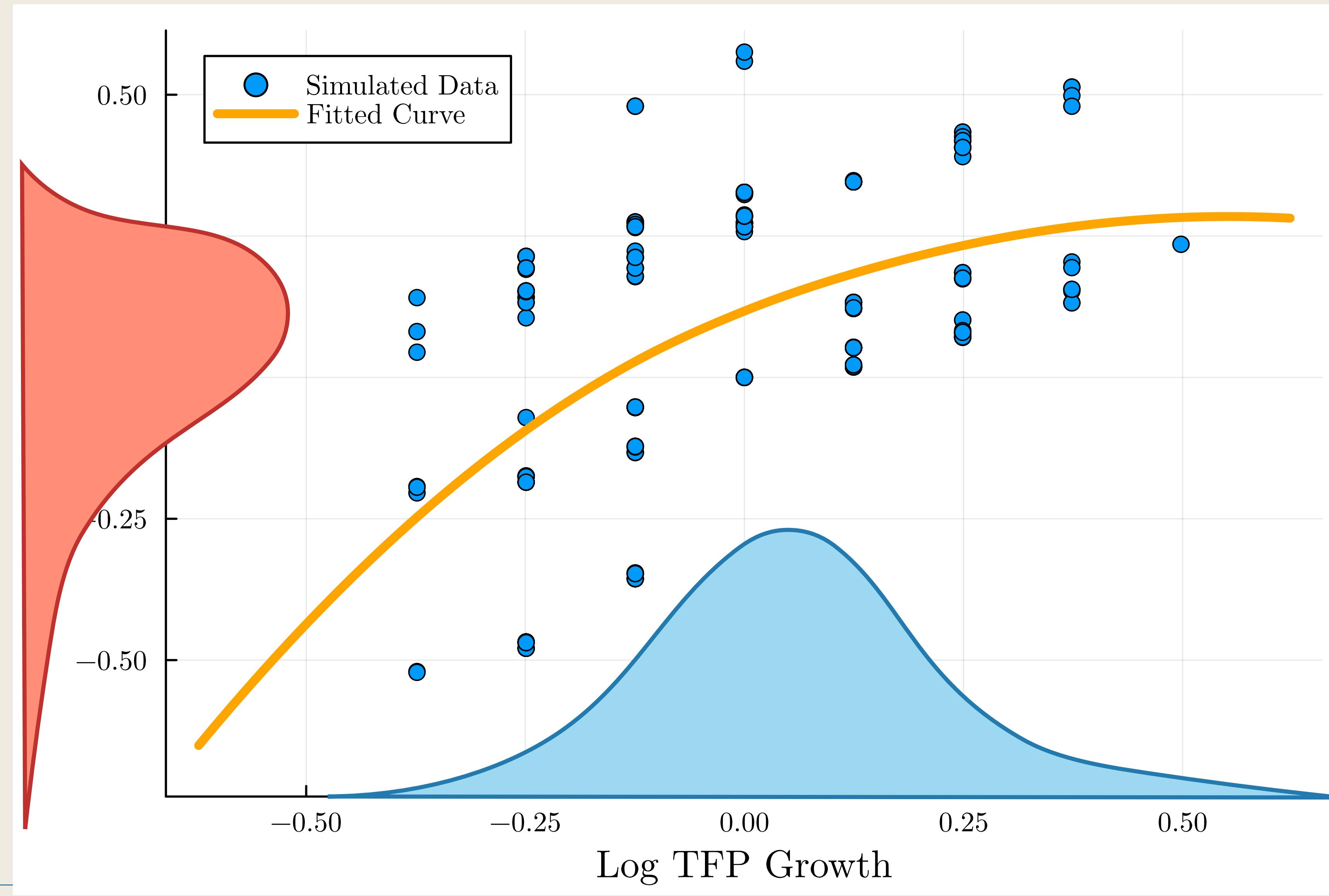
Firm Productivity Distribution



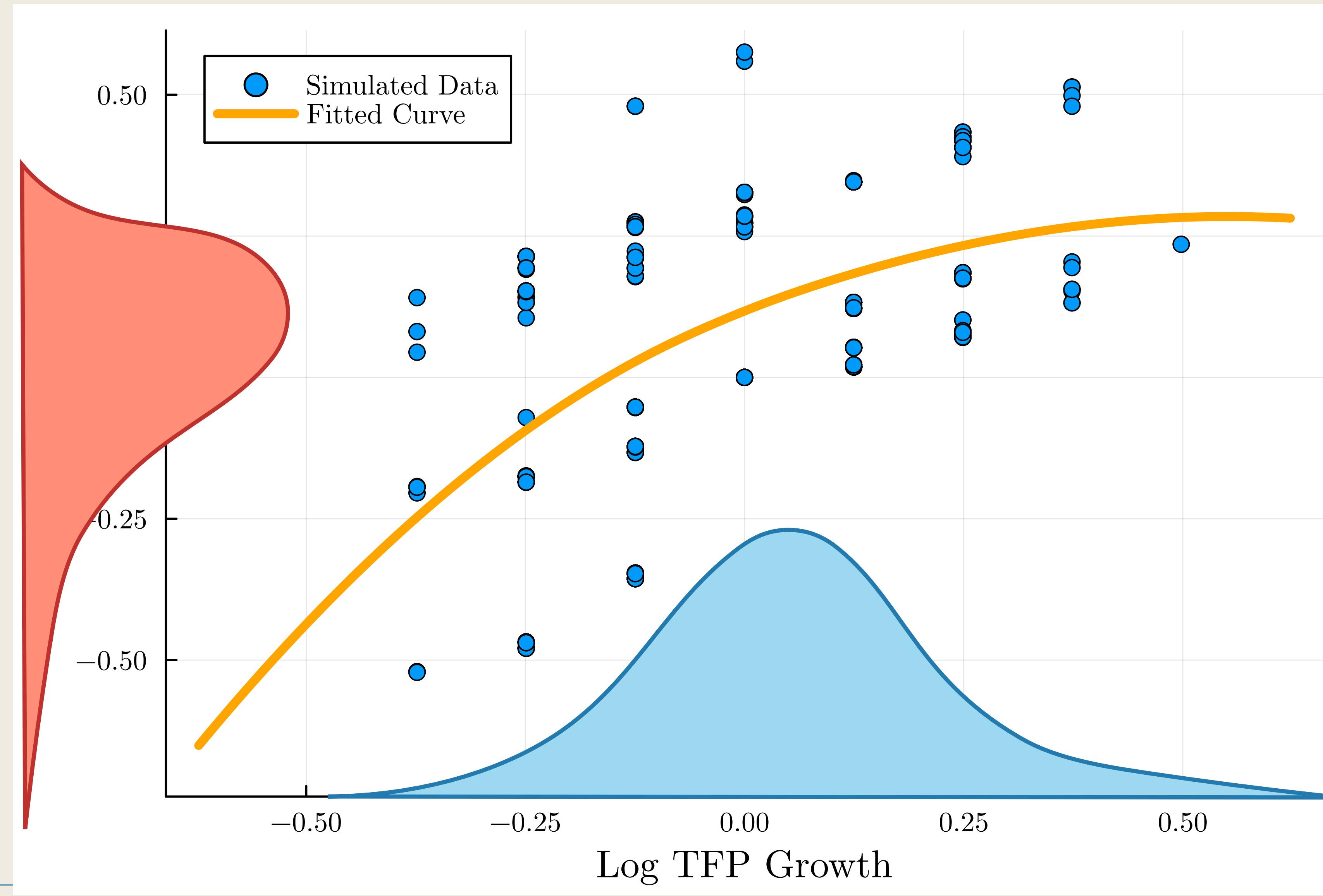
Concave Hiring Rule



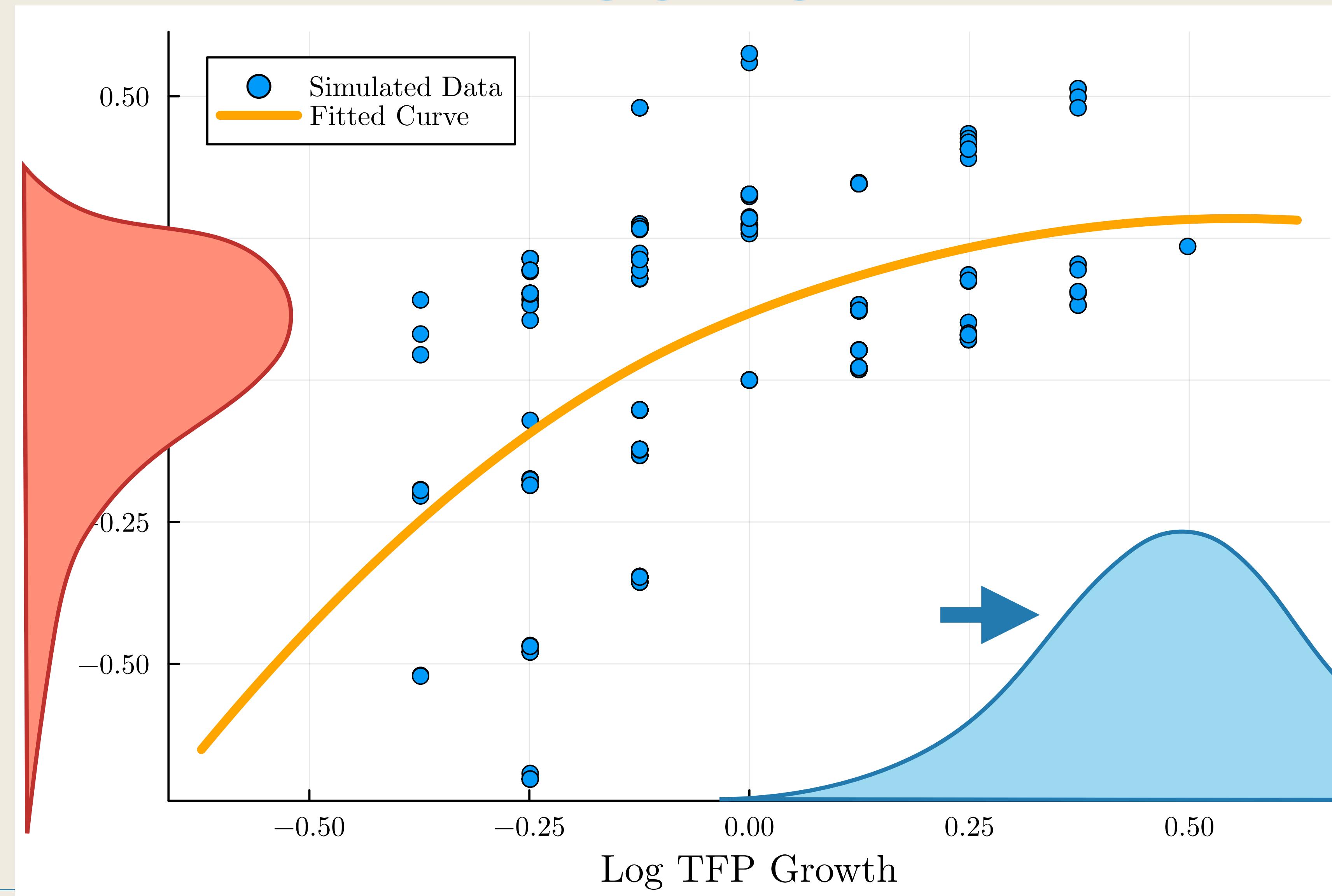
Distribution



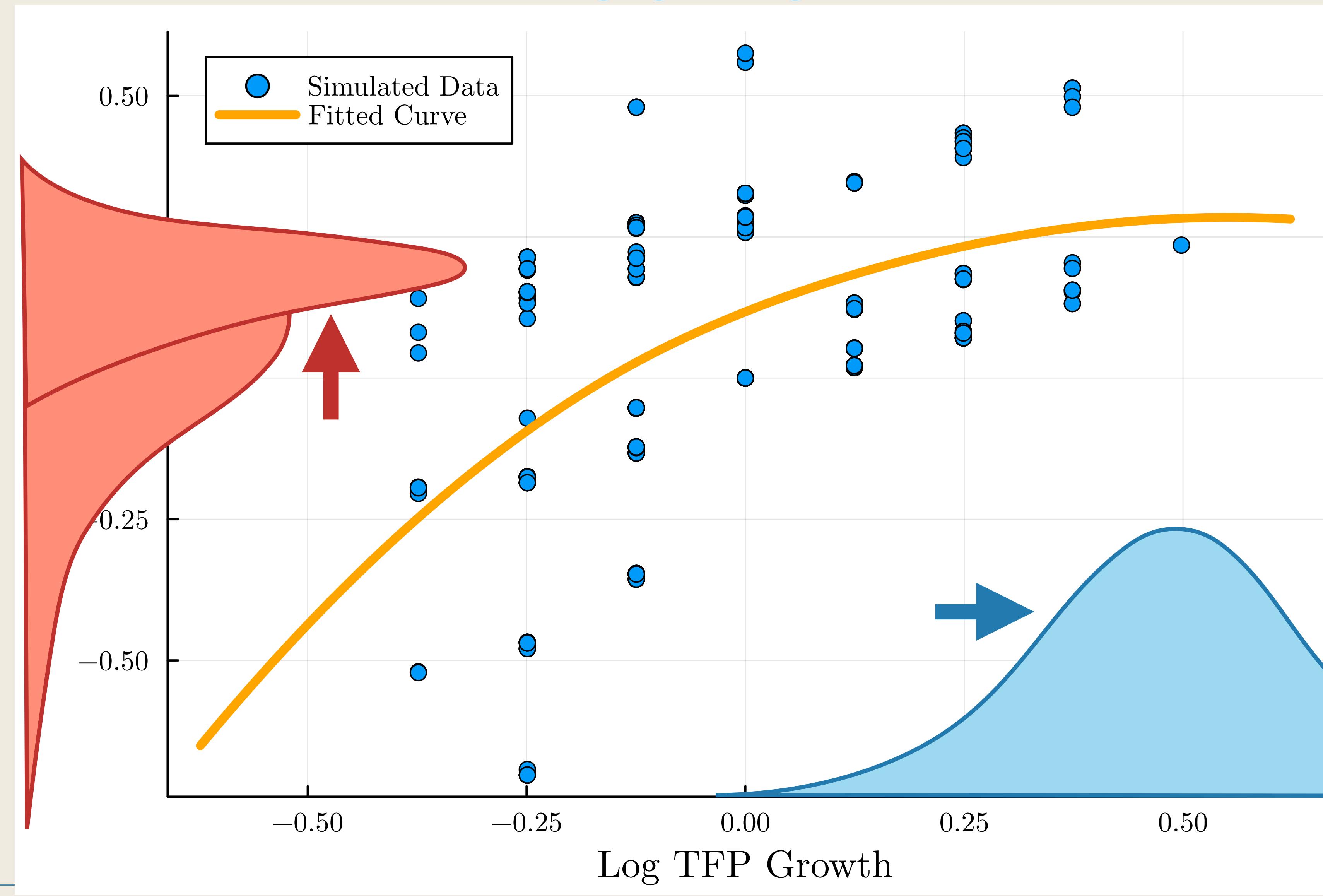
Positive Aggregate Shock



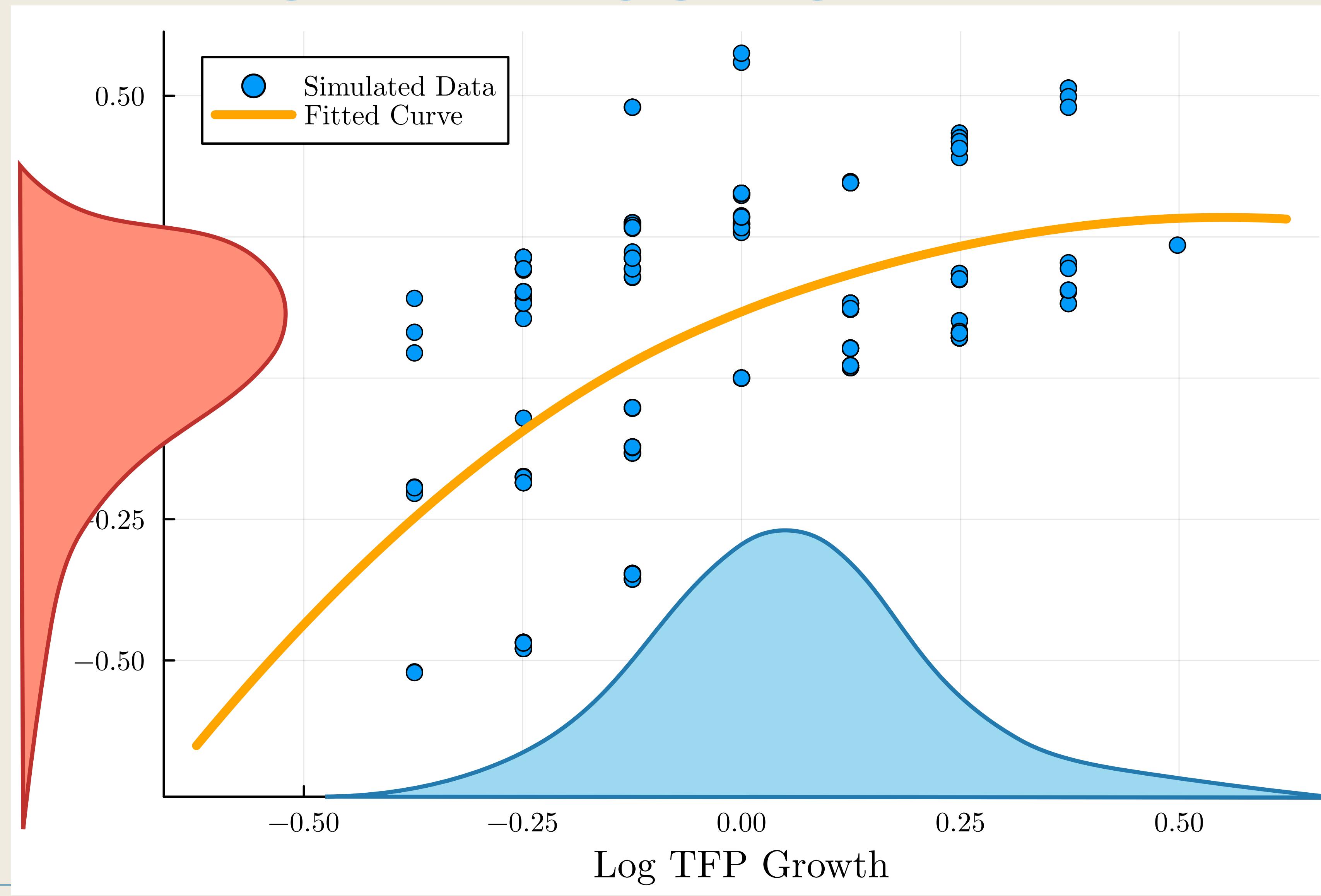
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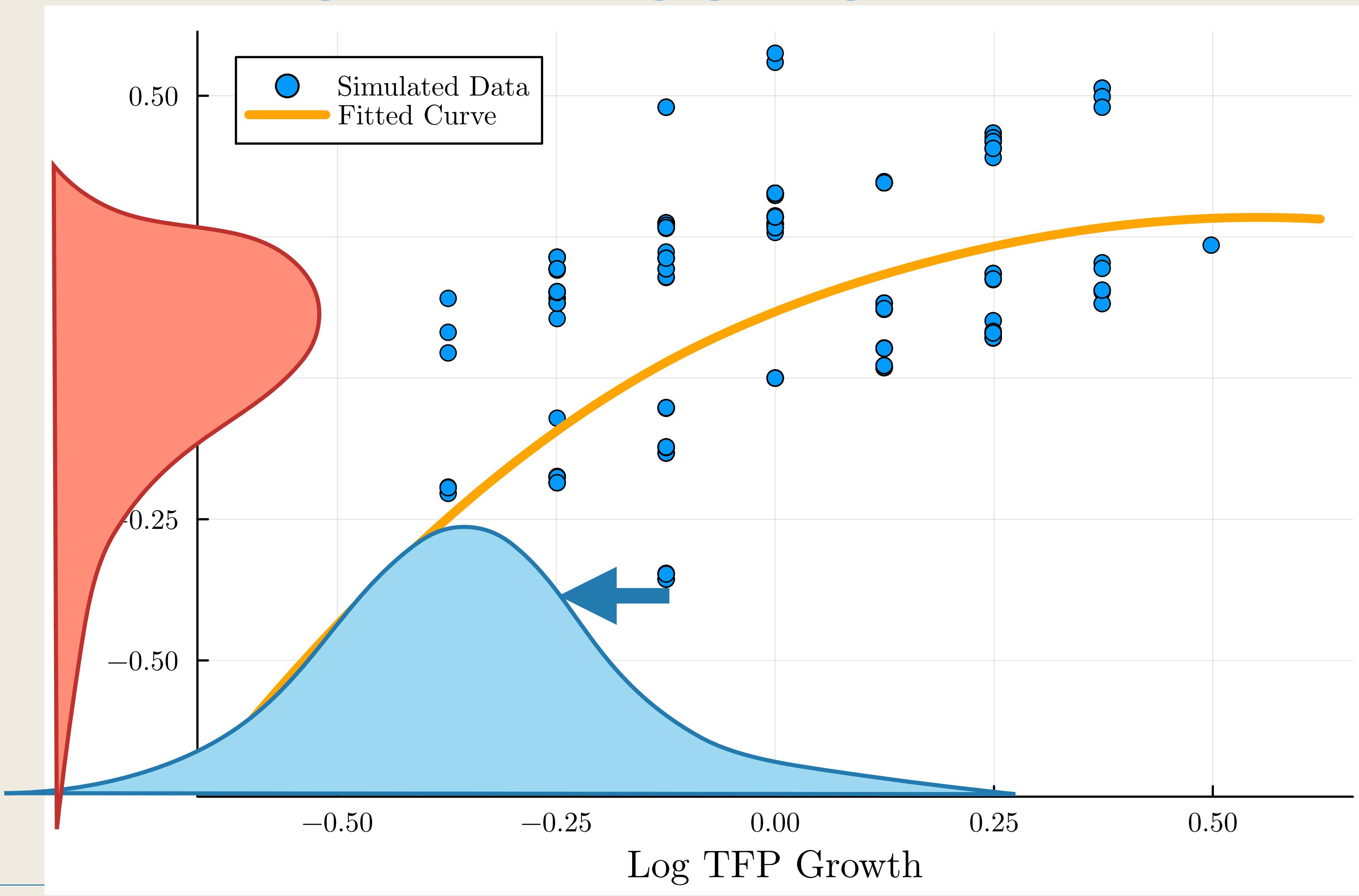
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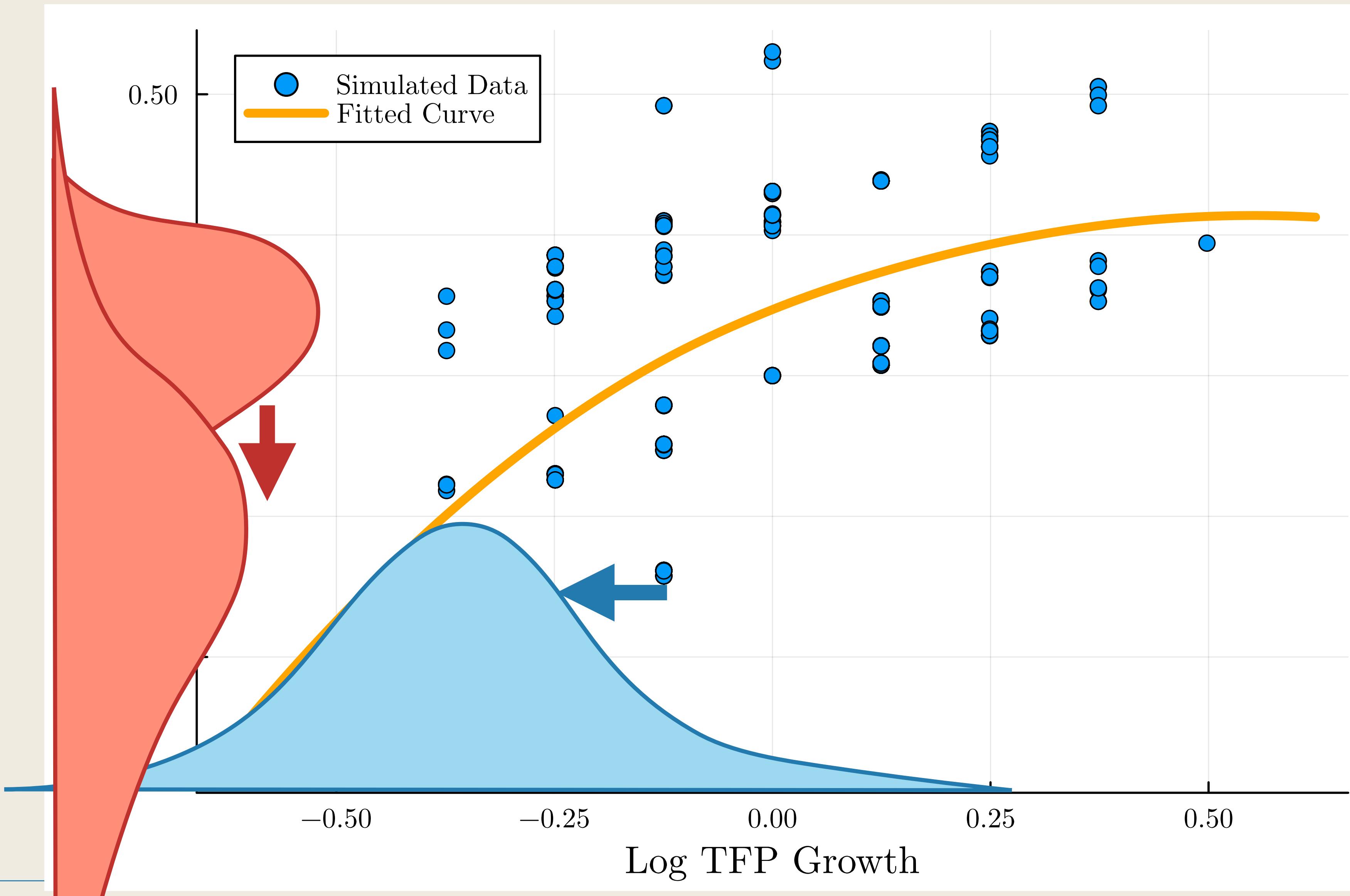
Negative Aggregate Shock



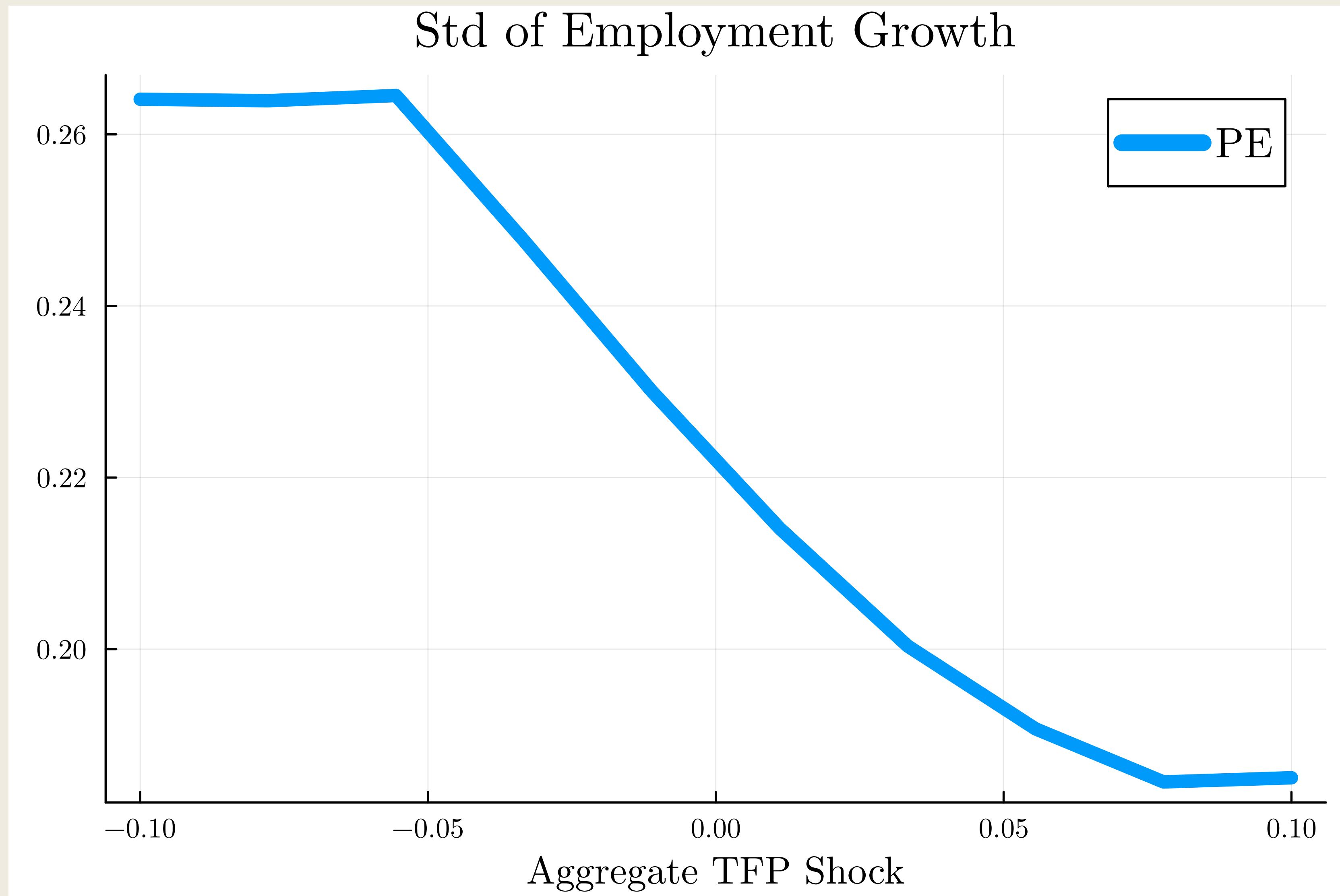
Negative Aggregate Shock



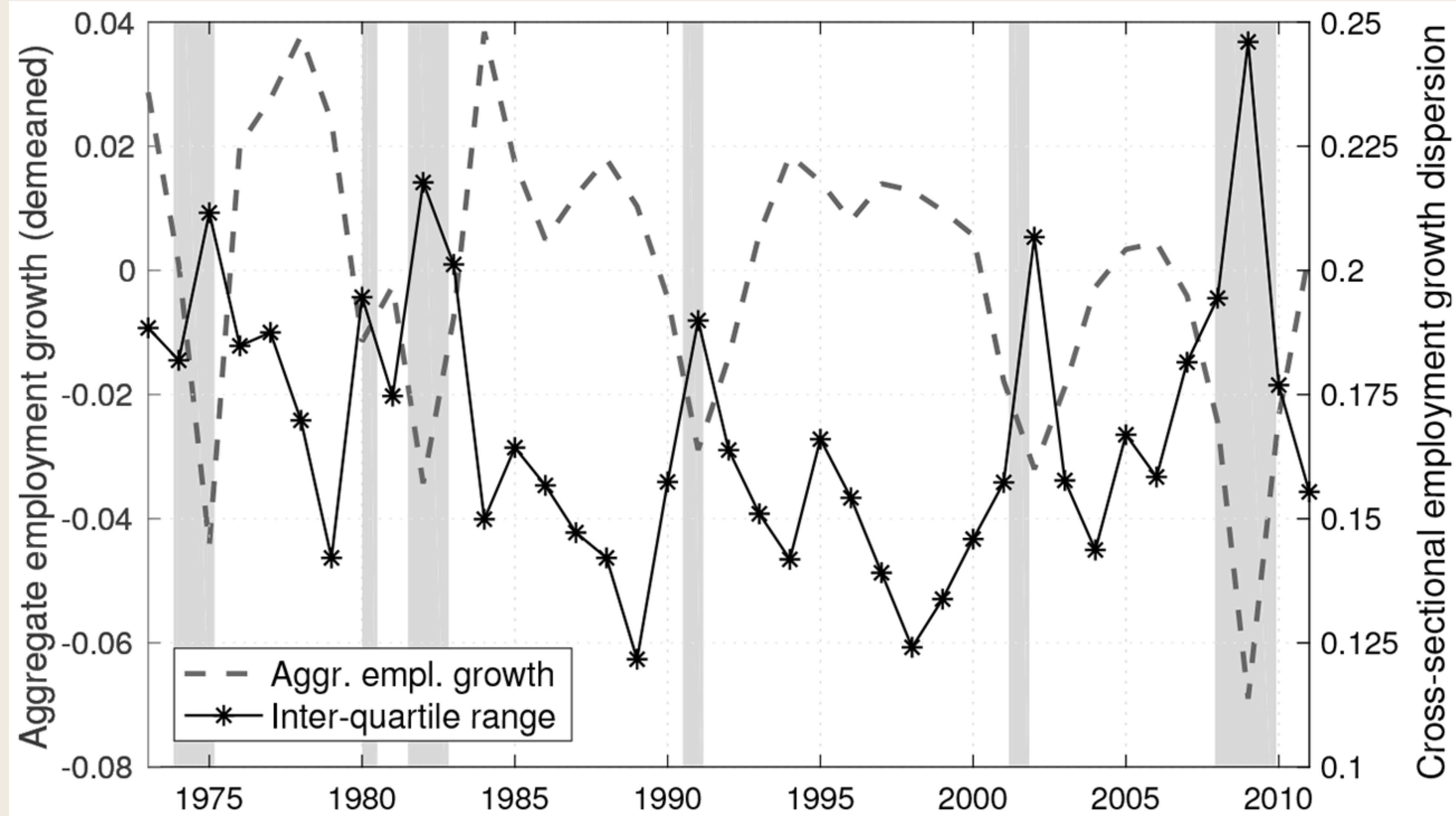
Negative Aggregate Shock



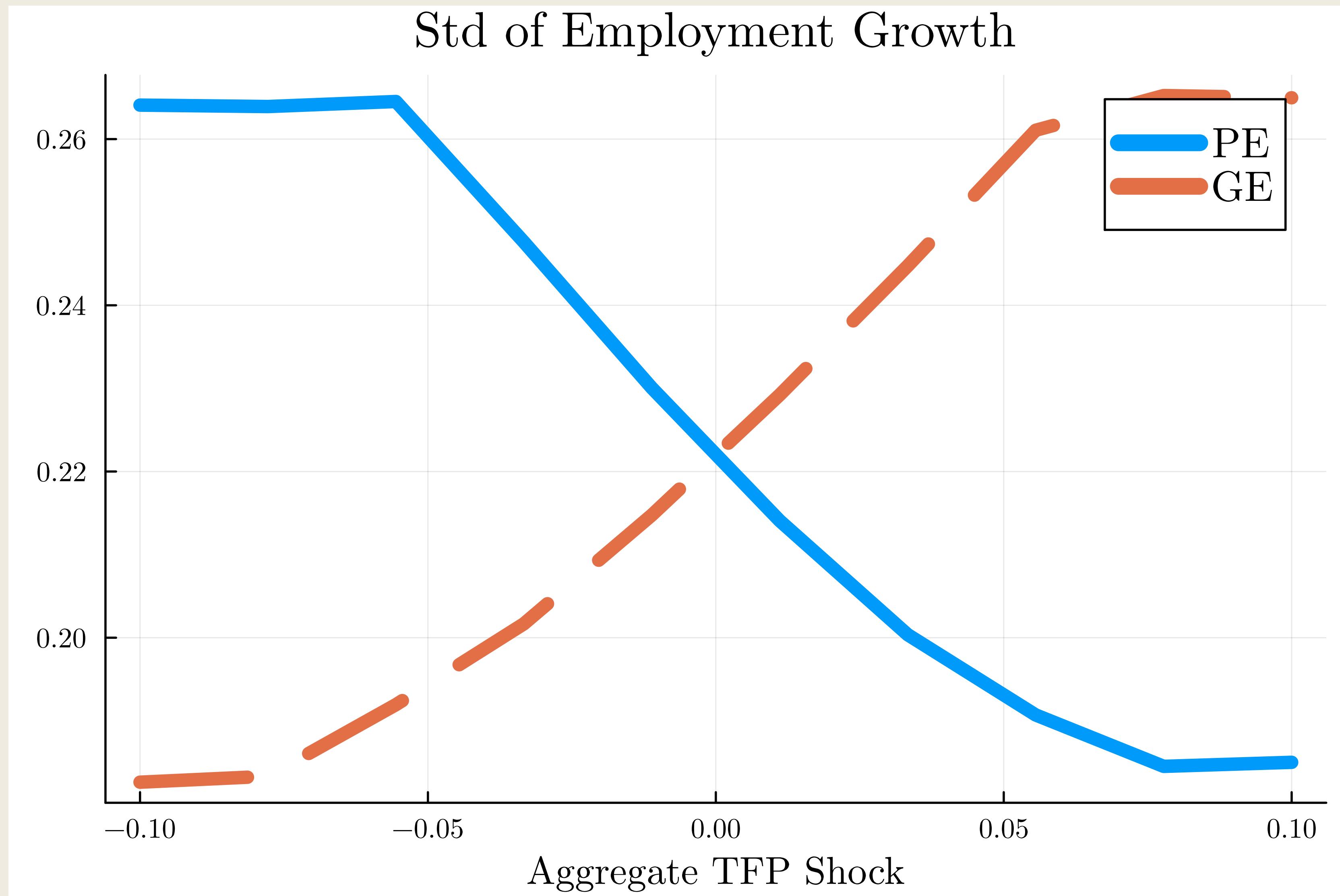
Partial Equilibrium



Countercyclical Volatility



Results Flip in GE...



...because Hiring Rule Changes



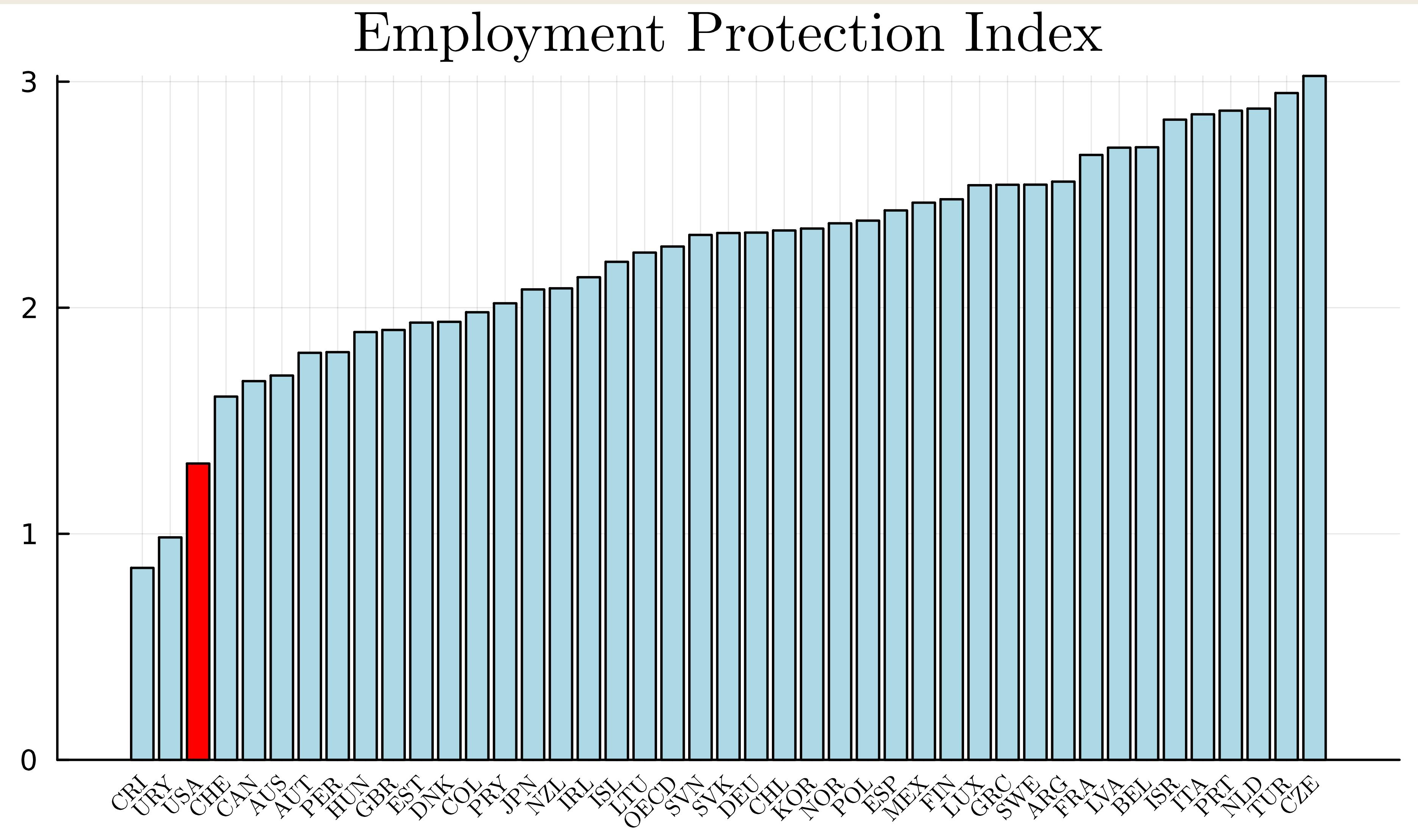
- GE adjustment in wages:
 1. The flat part of the hiring rule is steeper in booms
 2. The steep part of the hiring rule is flatter in recessions

Firing Cost and Misallocation

– Hopenhayn & Rogerson (1993)

Employment Protection Index

Employment Protection Index



Question

- What is the cost of strict firing regulations?
- Suppose that in order to fire a worker, firms have to pay $\tau \times$ annual wage salary
 - US: $\tau = 0$
 - Europe: high τ
- Firing costs take the form of taxes
 - Distinct from adjustment cost Φ , which is a part of the technology
- The collected tax revenue is rebated back to households as lump-sum transfers

Bellman Equation

- Bellman equation:

$$v(n_{-1}, z) = \max \left\{ v^*(n_{-1}, z), -\Phi(- (1 - \delta), n_{-1}) - \tau w (1 - \delta) n \right\}$$

where v^* is the continuation value

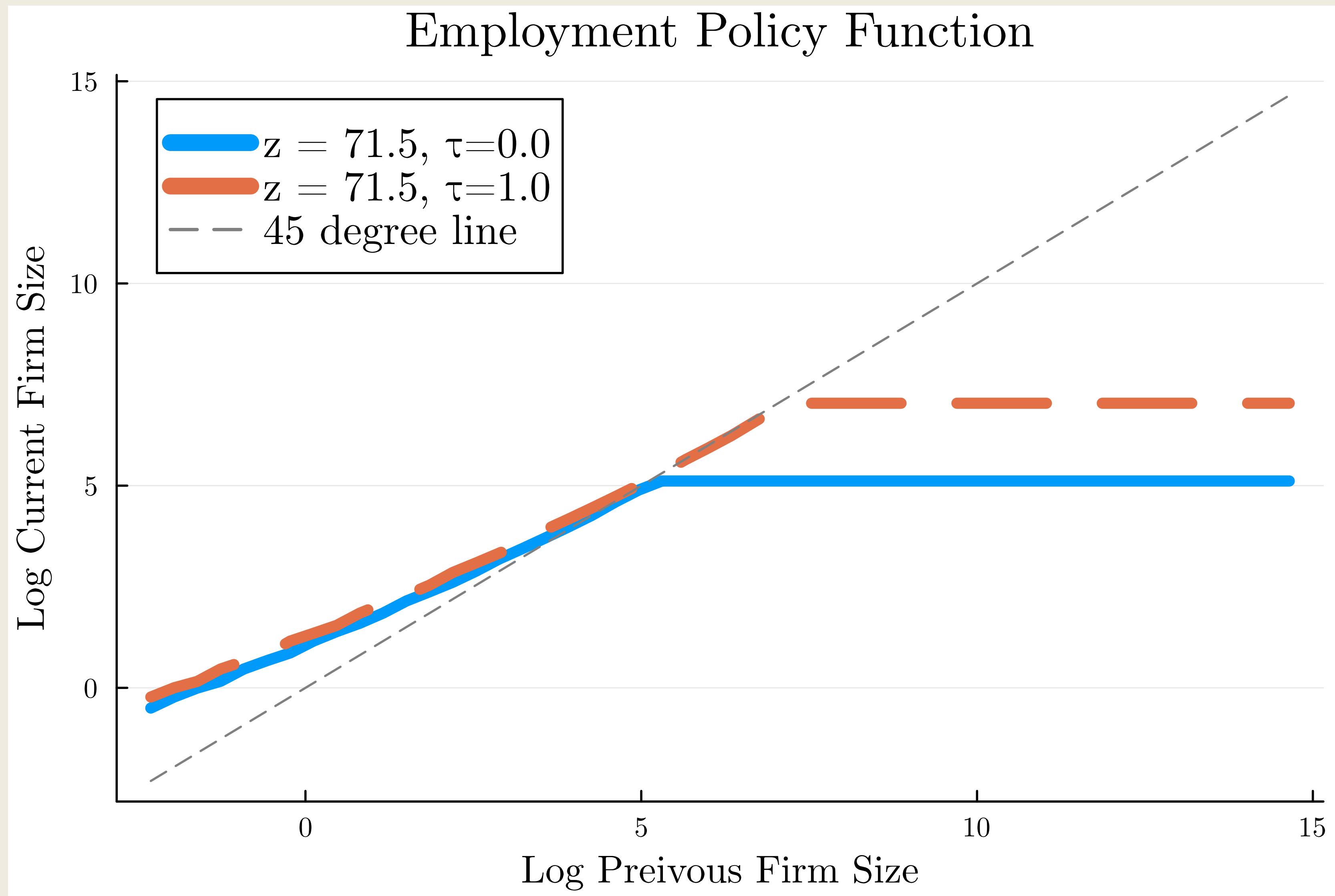
$$v^*(n_{-1}, z) = \max_{h, n} f(n, z) - wn - c_f - \Phi(h, n_{-1}) + \tau wnh \mathbb{I}[h < 0] + \beta \mathbb{E}v(n, z')$$

$$\text{s.t. } n = n_{-1} (1 - \delta + h)$$

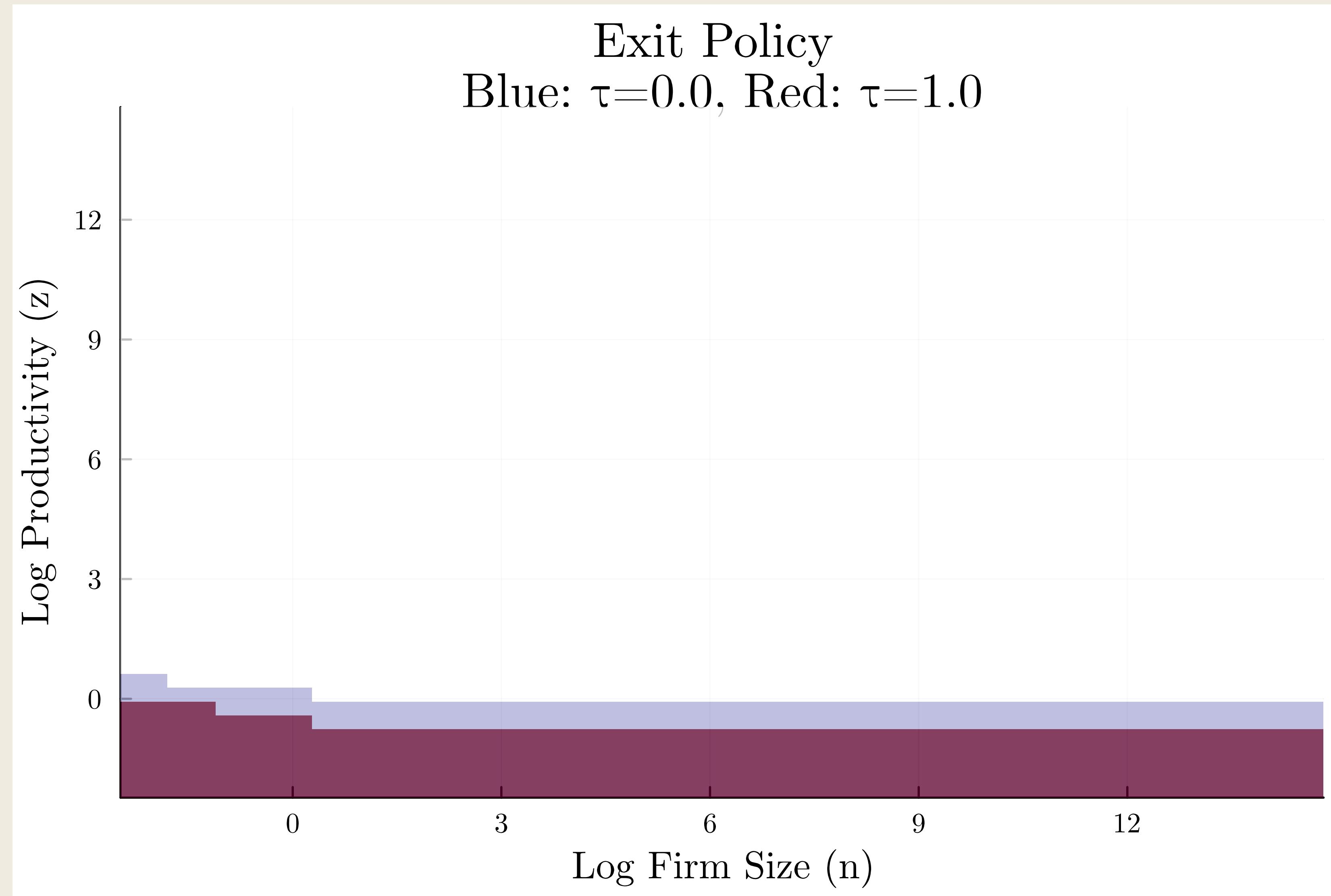
- The rest of the model is unchanged

Firms Fire Less

Employment Policy Function

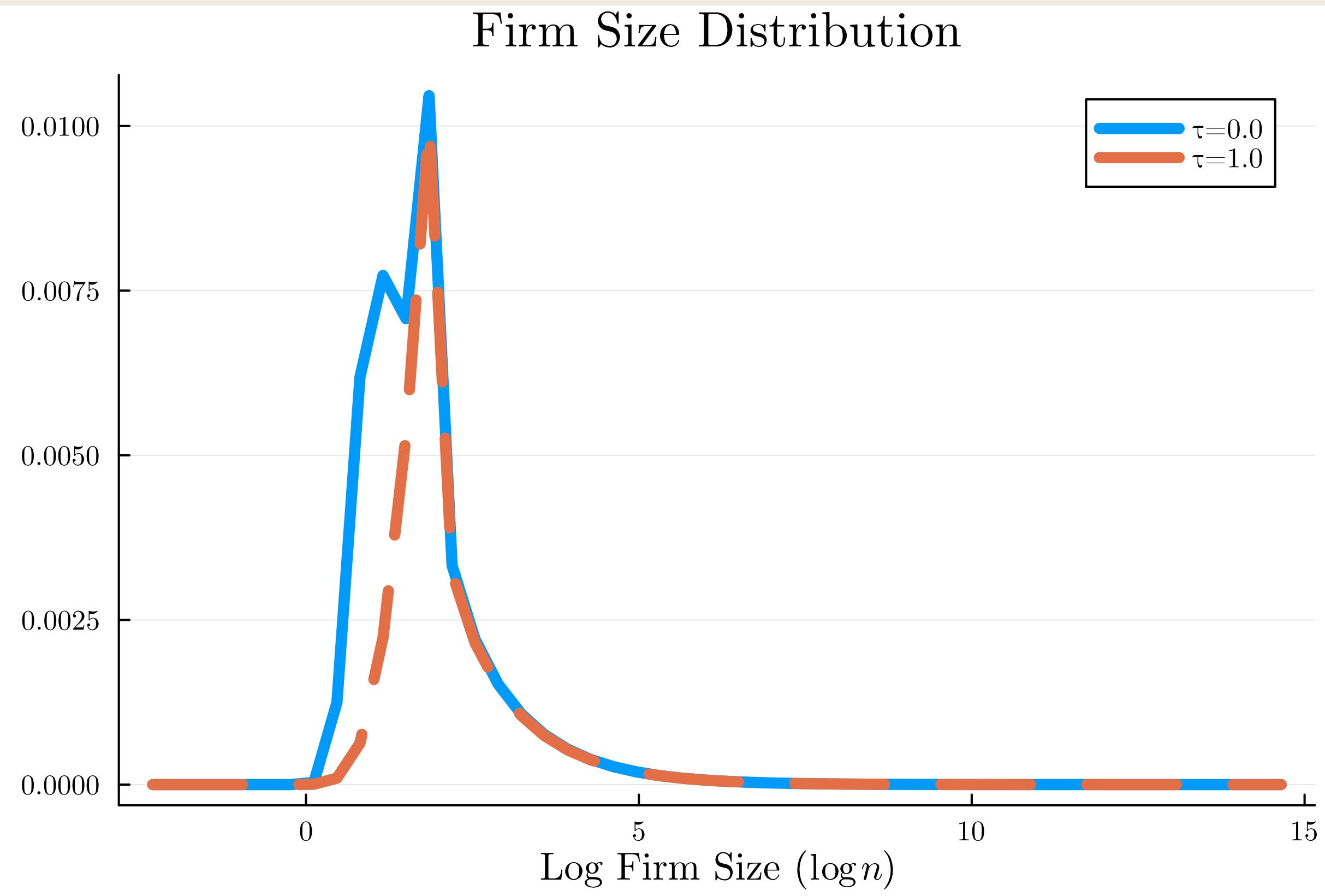


Firms Exit Less

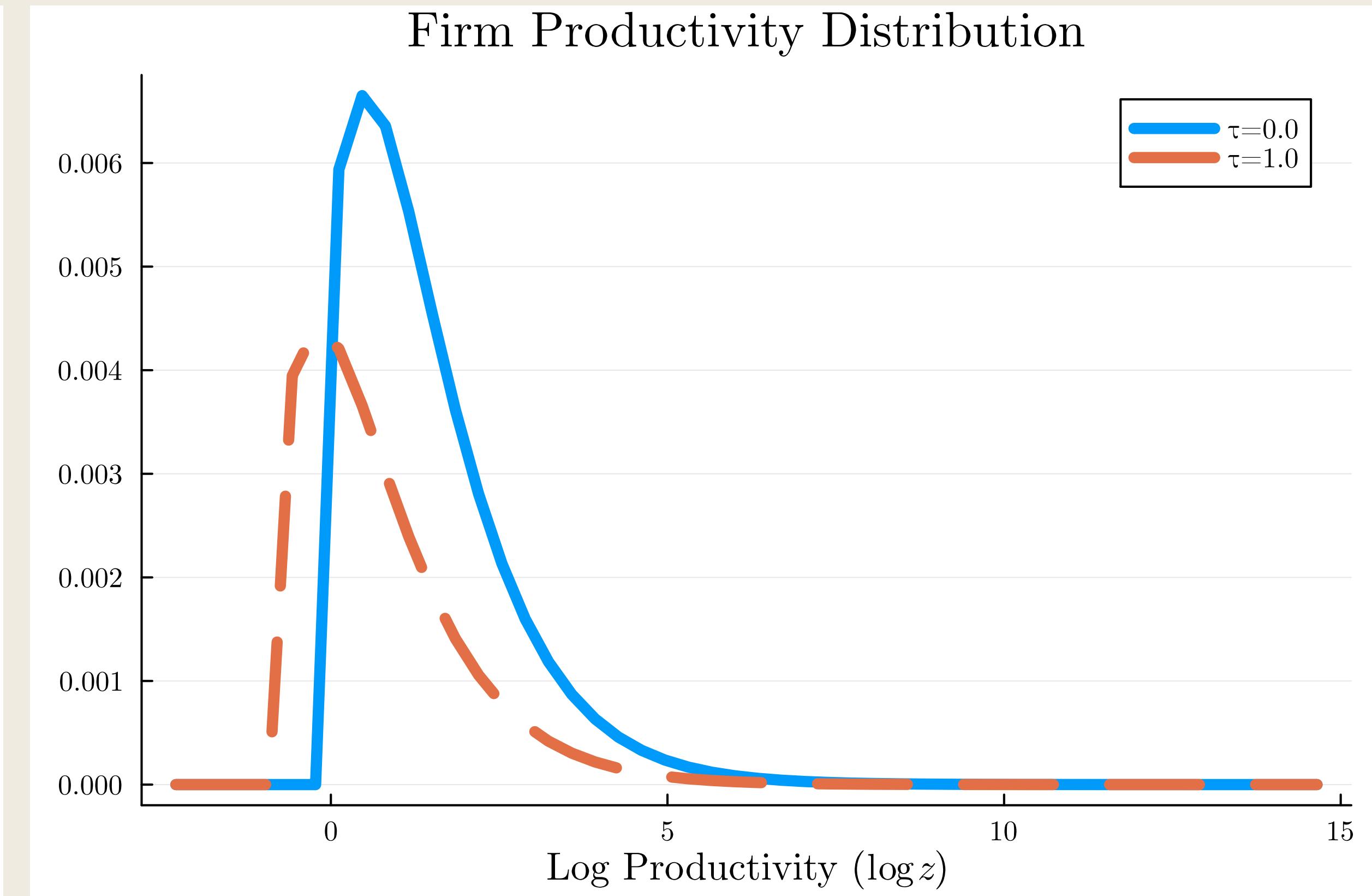


Firms Are Larger and Less Productive

Firm Size Distribution

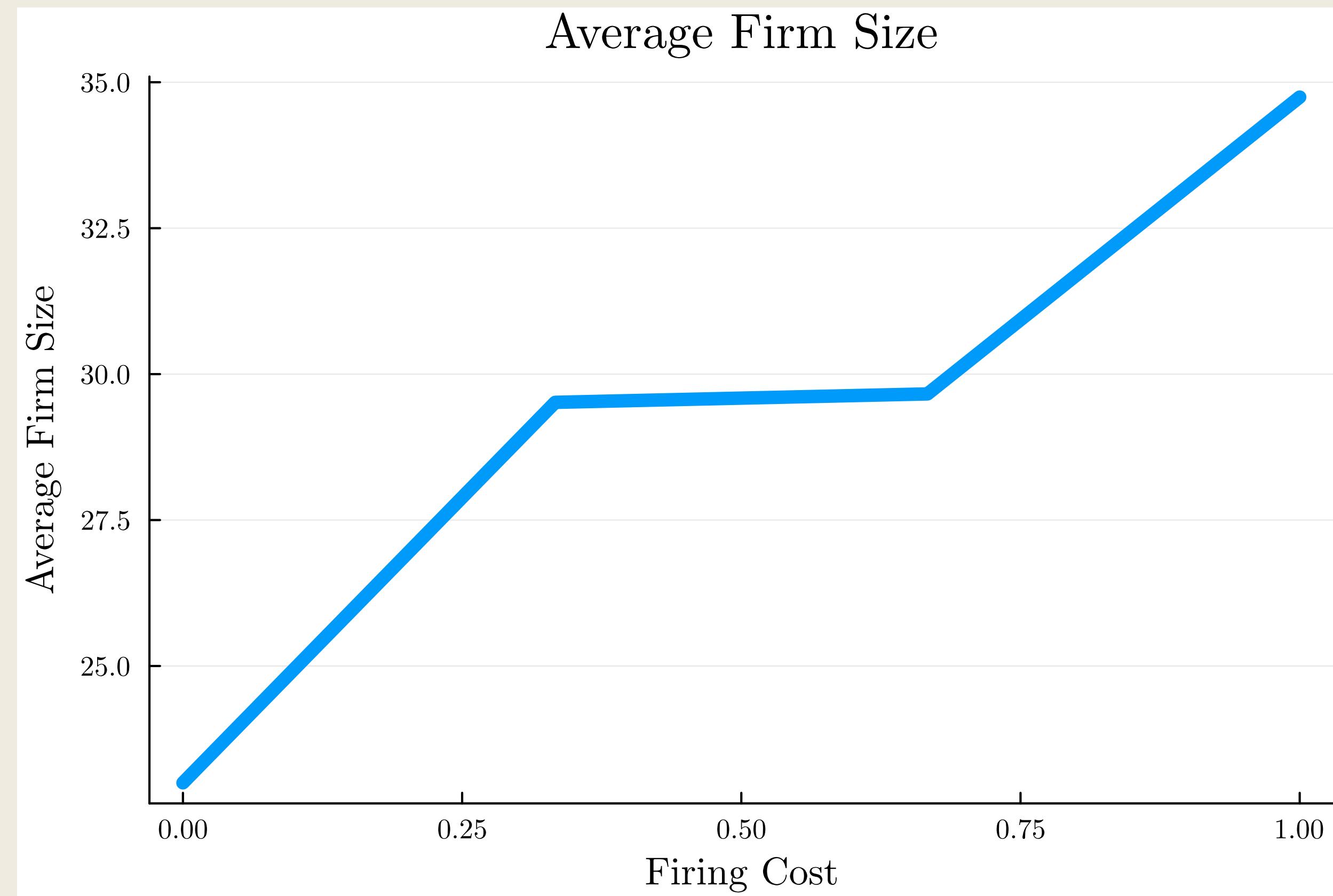


Firm Productivity Distribution

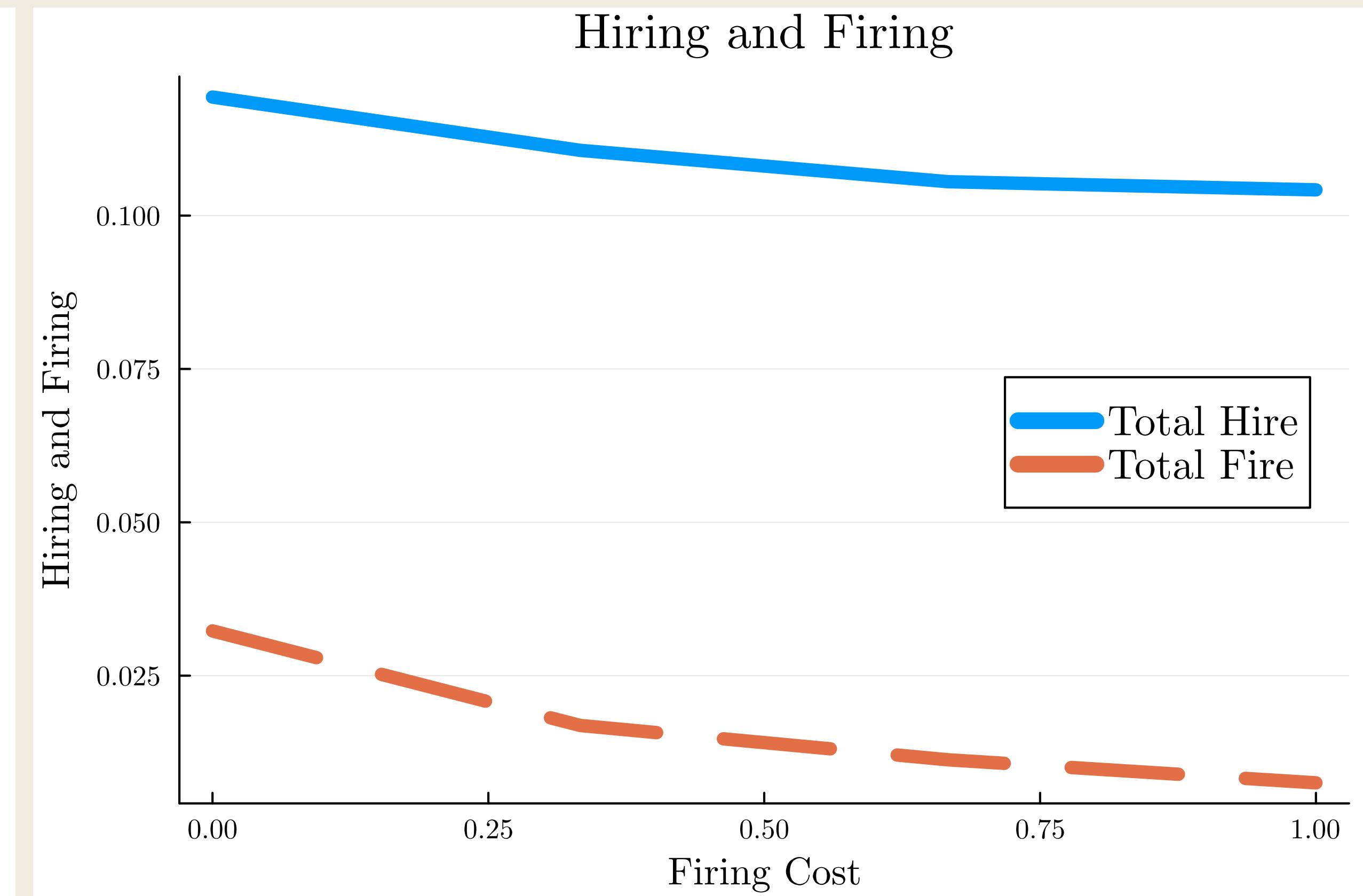


Firing Cost $\uparrow \Rightarrow$ Labor Reallocation \downarrow

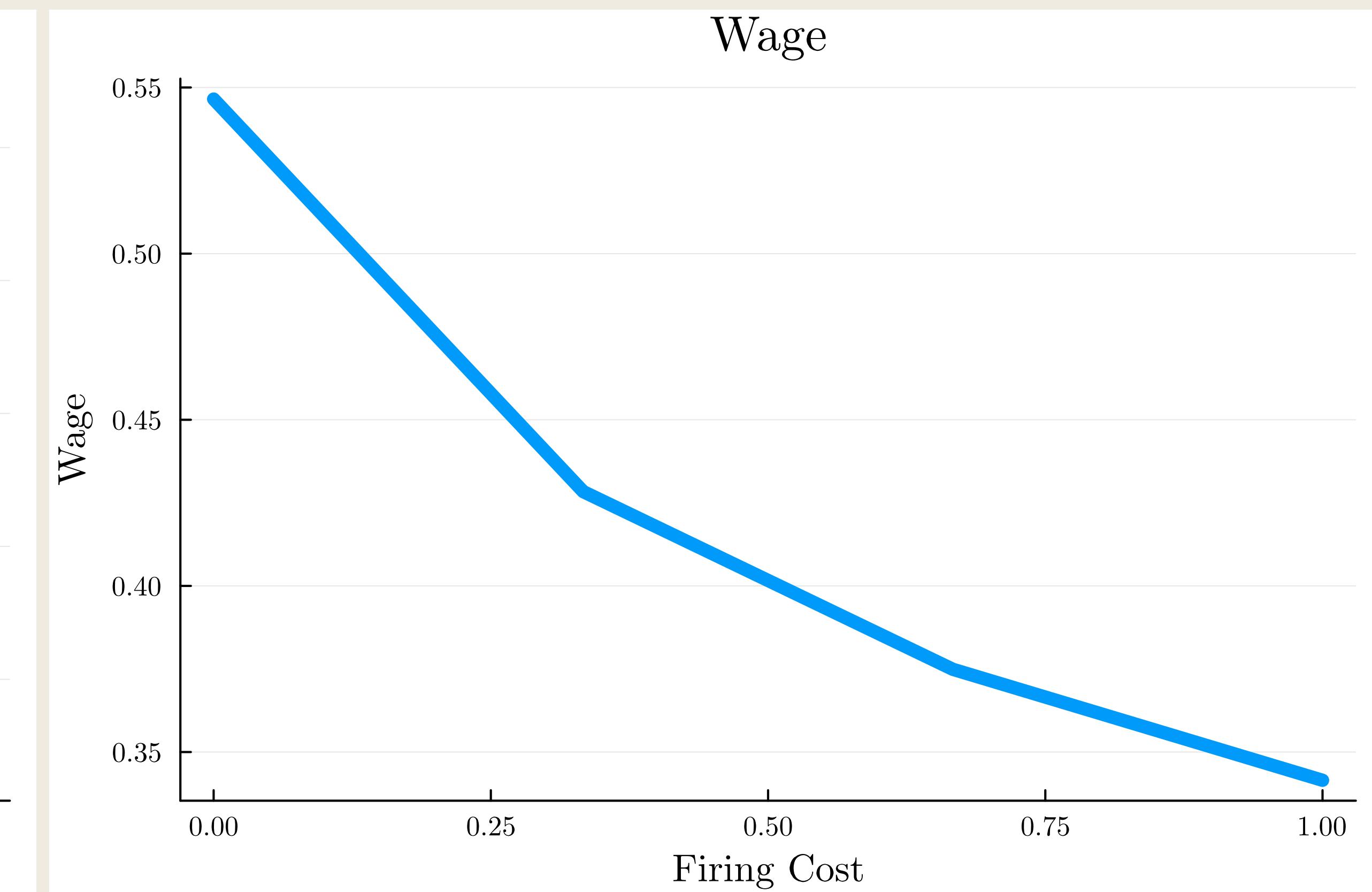
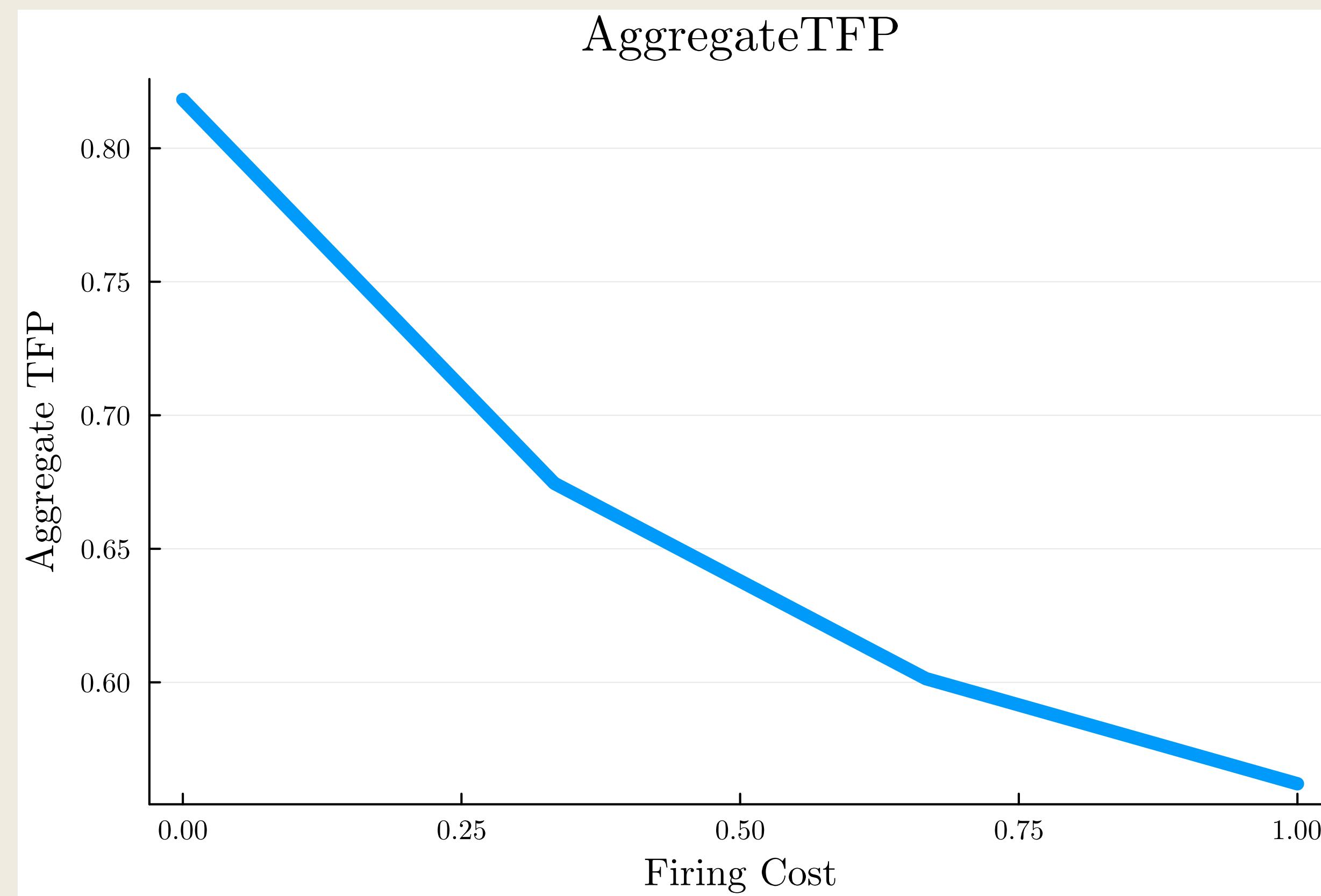
Average Firm Size



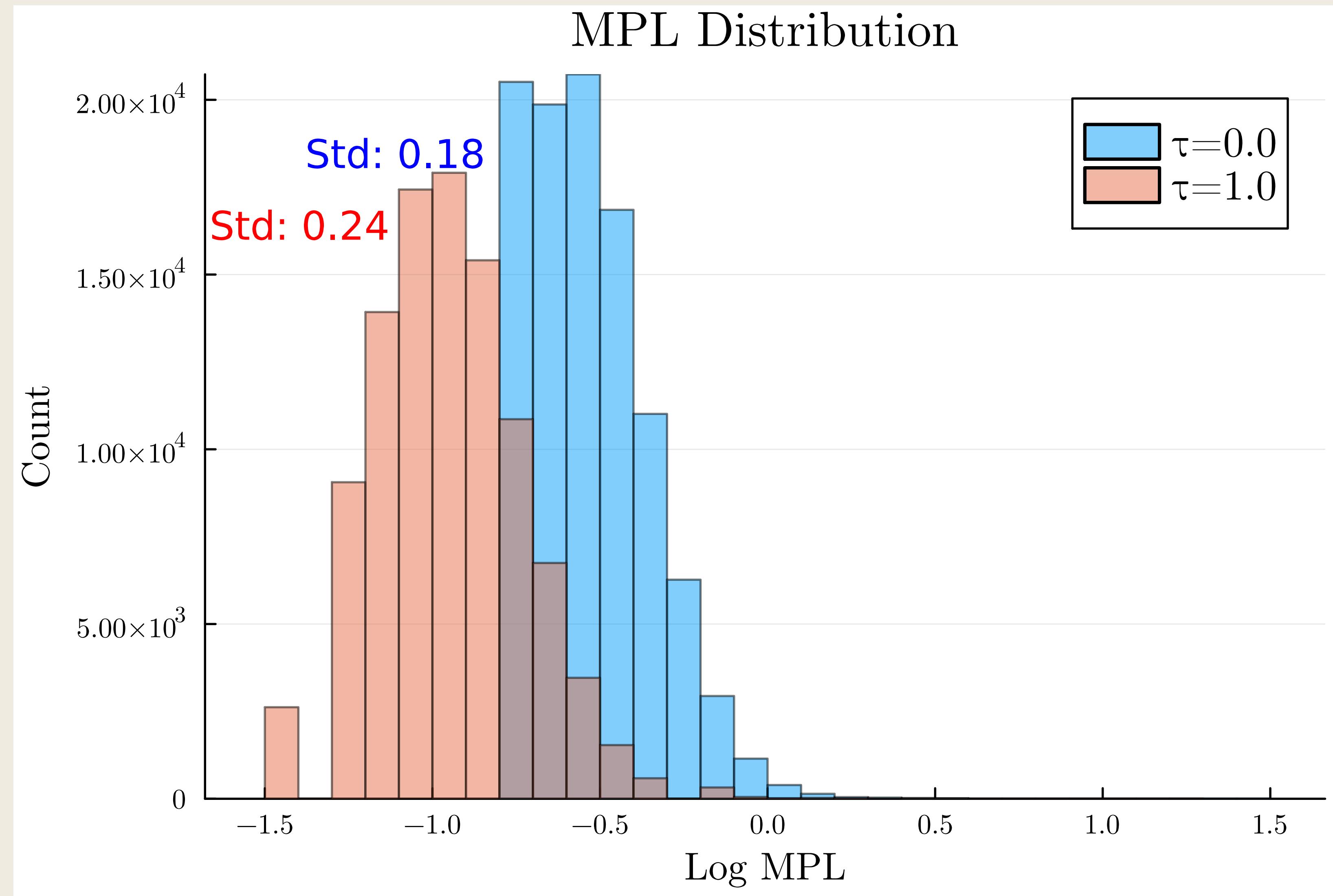
Hiring and Firing



Firing Cost $\uparrow \Rightarrow$ TFP \downarrow & Wage \downarrow



More Misallocation



Question

- Why, then, do so many countries regulate firing?
- An interesting idea is that firing cost could be a commitment device for firms
(Karabay & McLaren, 2011; Créchet, 2024; Souchier, 2023)
- Normative aspects of labor market institutions seem underexplored

Broader Questions on Hopenhayn-Rogerson

- What is z ?
 - Many argue z relates to the customer base
(e.g., Einav, Klenow, Levin & Murciano-Goroff, 2022; Foster, Haltiwanger, Syverson, 2015; Argente, Fitzgerald, Moreira & Priolo, 2021)
- Is the entry really free?
 - Cagetti and De Nardi (2006): a model of entrepreneurship with financial friction
- Does the model get the age distribution right?
 - The average age of Walmart/Amazon size class in the model is 100 years
 - Walmart is 60 years old, and Amazon is 30 years old
- In the model, large firms are large just by luck (ex-ante homogenous). Are they?
 - Hurst & Pugsley (2011) and Pugsley, Sedláček & Sterk (2020) argue not

Non-Parametric Identification of Misallocation

– Carrillo, Donaldson, Pomeranz, & Singhal (2023)

What is the Cost of Misallocation?

- How large is the cost of misallocation in the data?
- Let us step back and consider a **static** model with a **fixed mass** of firms
- Each firm i produces using

$$y_i = f_i(n_i) \tag{7}$$

- The efficient allocation solves

$$\begin{aligned} Y^* &\equiv \max_{\{n_i\}} \int f_i(n_i) di \\ \text{s.t. } & \int n_i di = L \end{aligned}$$

- The solution features equalization of MPL:

$$f'_i(n_i) = w \quad \text{for all } i$$

Variance of MPL as a Sufficient Statistics

- Take arbitrary allocation $\{n_i\}$. Up to a second order around the efficient allocation

$$\log Y - \log Y^* \approx -\frac{1}{2} \text{Var}_{\lambda/\epsilon} [\log(MPL_i)]$$

where $\text{Var}_{\lambda/\epsilon}[X_i] = \sum_{i=1}^N \frac{\lambda_i^*}{\epsilon_i^*} (X_i - \mathbb{E}_{\lambda/\epsilon}[X_i])^2$, $\mathbb{E}_{\lambda/\epsilon}[X_i] = \sum_{i=1}^N \frac{\lambda_i^*}{\epsilon_i^*} X_i$, $MPL_i = f'_i(n_i)$,
 $\lambda_i = w_i^* n_i^* / Y^*$, and $\epsilon_i \equiv -\frac{d \log MPL_i}{d \log n_i}$.

- (Weighted) variance of MPL is the key moment for the cost of misallocation
- Testing the presence of misallocation \Leftrightarrow testing $\text{Var}(MPL_i) = 0$
- How do we get the distribution of MPL?
 1. Assume $f_i(n_i) = Z_i n_i^\alpha$, and then $MPL_i = \alpha \frac{y_i}{n_i}$ (Hsieh & Klenow, 2009)
 2. Nonparametrically identify the distribution of MPL (Carrillo et al. 2023)

Nonparametric Identification

- Taking the first-order approximation of equation (7),

$$\Delta y_i = \beta_i \Delta n_i + \varepsilon_i$$

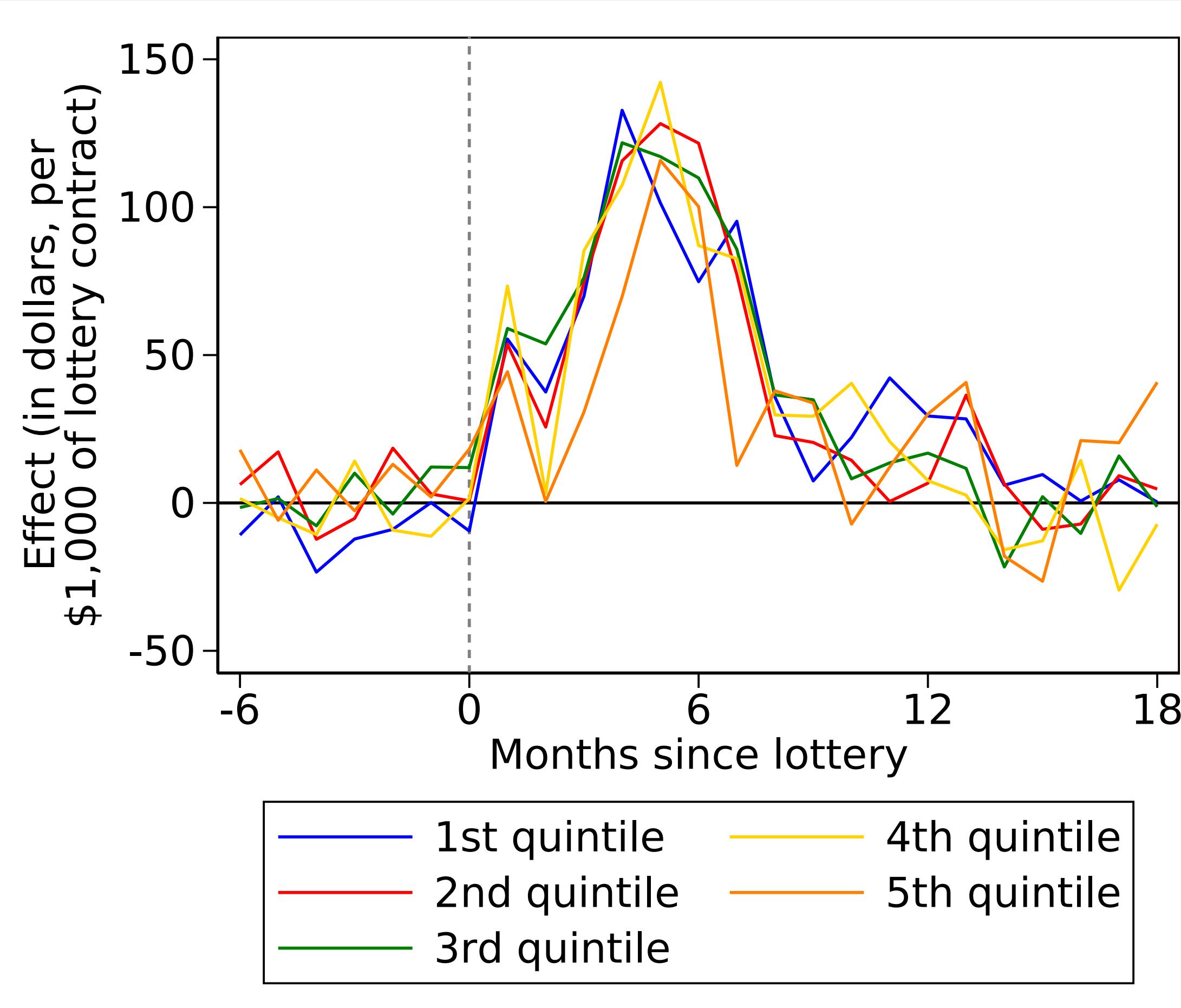
- ε_i : technology shocks (i.e., changes in $f_i(\cdot)$)
 - $\beta_i = f'_i(n_i) = MPL_i$: treatment effect of exogenously increasing n_i on y_i
- With suitable instruments Z_i that exogenously shift n_i , $\mathbb{E}[\beta_i^k]$ ($k = 1, 2, \dots$) are identified (Masten & Torgovitsky, 2016)

Empirical Implementation

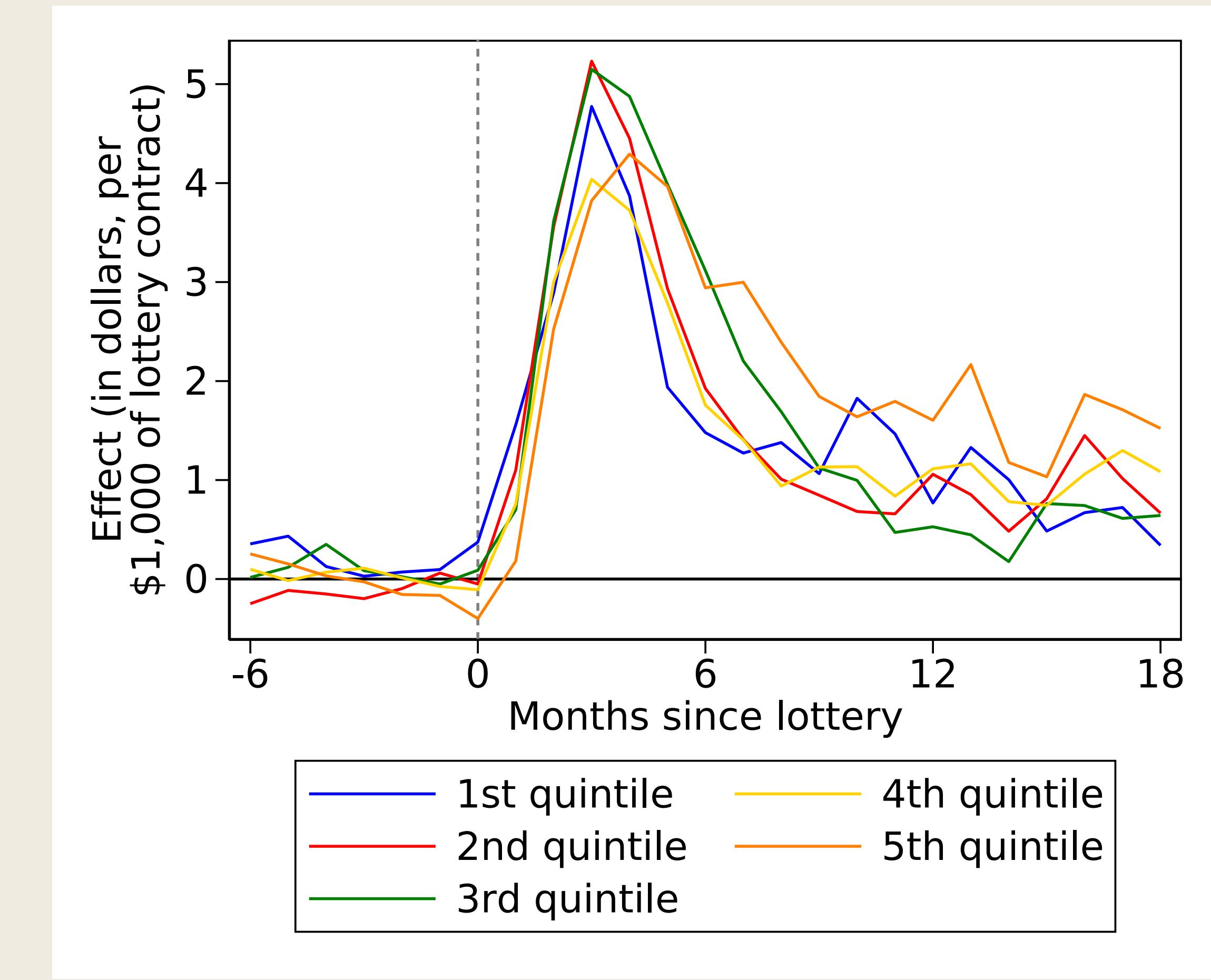
- Construction sector in Ecuador, 2009-2014
- Public construction projects were allocated through a randomized lottery
- Lottery serves as an ideal instrument
 - **exogeneity**: orthogonal to technology shocks ε_i or MPL_i
 - **relevance**: winning a lottery does shift n_i

Heterogenous Treatment Effects by Firm Size?

Sales



Labor Inputs



Small Cost of Misallocation

Table 4: Estimated Cost of Misallocation

| | $\mathbb{E}_{\bar{\lambda}}[\bar{\mu}]$ (1) | $\text{Var}_{\bar{\lambda}}[\bar{\mu}]$ (2) | $\frac{\Delta W}{W}$ (3) |
|--|--|--|-----------------------------|
| Panel (a): IVCRC estimates | | | |
| Baseline | 1.126 [1.093, 1.161] | 0.014 [0, 0.341] | 0.016 [0, 0.261] |
| Panel (b): Alternative procedure assuming common scale elasticities | | | |
| Constant returns-to-scale ($\gamma = 1$) | 1.240 [1.223, 1.257] | 0.611 [0.544, 0.730] | 0.479 [0.427, 0.572] |

- Assume $\epsilon_i = 3$ for all i
- The welfare cost of misallocation is 1.6%
- Hsieh-Klenow type calculation implies 48% of welfare loss in the same dataset

Question

Does this approach capture the full story of misallocation in Hopenhayn-Rogerson?

– No, for two reasons:

1. Entry & exit
2. Dynamics

Two Open Questions

1. Entry & exit

- In HR, distortion \Rightarrow fewer firms enter
 - This reduces the mean of MPL (not just about variance!)
- In HR, exit threshold changes \Rightarrow the selection of the active firms change
 - This again shifts the mean of MPL

2. Dynamics

- Laissez-faire of Hopenhayn-Rogerson with labor adjustment costs is efficient
- But, MPL is not equalized in a static sense
- Firms hire workers until
(present discounted value of hiring a worker) = (hiring cost today)
- Hiring a worker is an investment

How do we incorporate these two issues without imposing strong assumptions?

Hopenhayn-Rogerson with Search Frictions

- Based on McCrary (2022)

Search Friction

- Only unemployed workers search for a job
- Firms need to post vacancies to hire workers
- Assume firing is costless
 - Can easily extend to the case with costly firing
- Each vacancy meets an unemployed worker with probability λ^F
- Let U be the unemployed's value function

Value and Policy Functions

■ Policy functions:

$\chi(n, z) \in \{0, 1\}$: exit, $w(n, z)$: wage, $s(n, z)$: firing, $v(n, z)$: vacancy

■ The value function of the firm:

$$J(n, z) = (1 - \chi(n, z))J^*(n, z)$$

$$J^*(n, z) = f(n, z) - w(n, z)n - c_f - \Phi(v(n, z), n) + \beta \mathbb{E} J(n', z')$$

$$\text{s.t. } n' = (1 - \delta - s(n, z))n + \lambda^F v(n, z)n$$

■ The value function of the workers:

$$W(n, z) = (1 - \chi(n, z)) \left[w(n, z) + \beta(1 - \delta - s(n, z)) \mathbb{E} W(n', z') + \beta(\delta + s(n, z))U \right] + \chi(n, z)U$$

Wage Bargaining

- In each period, a coalition of workers and a firm bargains to determine w, v, s, χ
- We assume Nash bargaining with worker bargaining power γ :

$$\max_{\chi(n,z), v(n,z), w(n,z), s(n,z)} (W(n, z)n - Un)^\gamma J(n, z)^{1-\gamma} \quad (1)$$

- Noting $\frac{\partial W(n, z)}{\partial w} = -\frac{\partial J(n, z)}{\partial w}$, FOC w.r.t. w gives

$$(W(n, z)n - Un) = \gamma S(n, z), \quad J(n, z) = (1 - \gamma)S(n, z) \quad (2)$$

where $S(n, z) \equiv J(n, z) + (W(n, z) - U)n$ is the joint match surplus

- Substituting (2) back into (1), we have

$$\max_{\chi(n,z), v(n,z), s(n,z)} \gamma^\gamma (1 - \gamma)^{1-\gamma} S(n, z)$$

- **Result:** vacancy, firing, and exit policies maximize joint match surplus

Bellman Equation for Surplus

- The joint match surplus solves

$$S(n, z) = \max\{S^*(n, z), 0\} \quad (3)$$

where

$$S^*(n, z) = \max_{v, s, n'} f(n, z) - c_f - \Phi(v, n) - (1 - \beta)U_n + \beta \left(1 - \frac{\gamma \lambda^F v}{(1 - \delta - s) + \lambda^F v} \right) \mathbb{E} S(n', z')$$

$$\text{s.t. } n' = (1 - \delta - s)n + \lambda^F v n$$

- Why is there extra discounting $1 - \frac{\gamma \lambda^F v}{(1 - \delta - s) + \lambda^F v}$?

- New hires will “steal” a portion of next-period surplus $S(n', z')$

Matching

- The total number of matches is dictated by a CRS matching function:

$$\mathcal{M}(u, v)$$

- The meeting probabilities are

$$\lambda^U = \frac{\mathcal{M}(u, v)}{u} \equiv \mathcal{M}(1, \theta), \quad \lambda^F = \frac{\mathcal{M}(u, v)}{v} \equiv \mathcal{M}(1/\theta, 1) \quad (4)$$

where $\theta \equiv v/u$ is the market tightness and

$$u = 1 - \iint n g(n, z) dndz$$

$$v = \iint v(n, z) g(n, z) dndz$$

Rest of the Model

- The value of unemployment is

$$U = b + \beta \lambda^U \iint \frac{v(n, z)}{v} \left[\gamma S(n'(n, z), z') \frac{1}{n'(n, z)} + U \right] g(n, z) dndz \quad (5)$$

where b is the UI benefit, and $g(n, z)$ is the mass of firms with (n, z)

- The free-entry condition is

$$\int (1 - \gamma) S(n_0, z) \psi_0(z) dz = c_e \quad (6)$$

- The steady-state distribution satisfies

$$g(n', z') = \iint (1 - \chi(n, z)) \Pi(z' | z) \mathbb{I}[n'(n, z) = n'] g(n, z) dndz + m \mathbb{I}[n' = n_0] \psi_0(z') \quad (7)$$

Equilibrium Definition

A steady-state recursive equilibrium consists of

- value and policy functions: $\{S(n, z), n'(n, z), v(n, z), s(n, z), \chi(n, z)\}$
- the market tightness and meeting probabilities: $\{\theta, \lambda^U, \lambda^F\}$
- the value of unemployment: U
- the steady state distribution: $g(n, z)$

such that

1. value and policy functions solve the Bellman equation (3)
2. the meeting probabilities satisfy (4)
3. the value of unemployment satisfies (5)
4. Free-entry holds (6)
5. the steady state distribution satisfies (7)

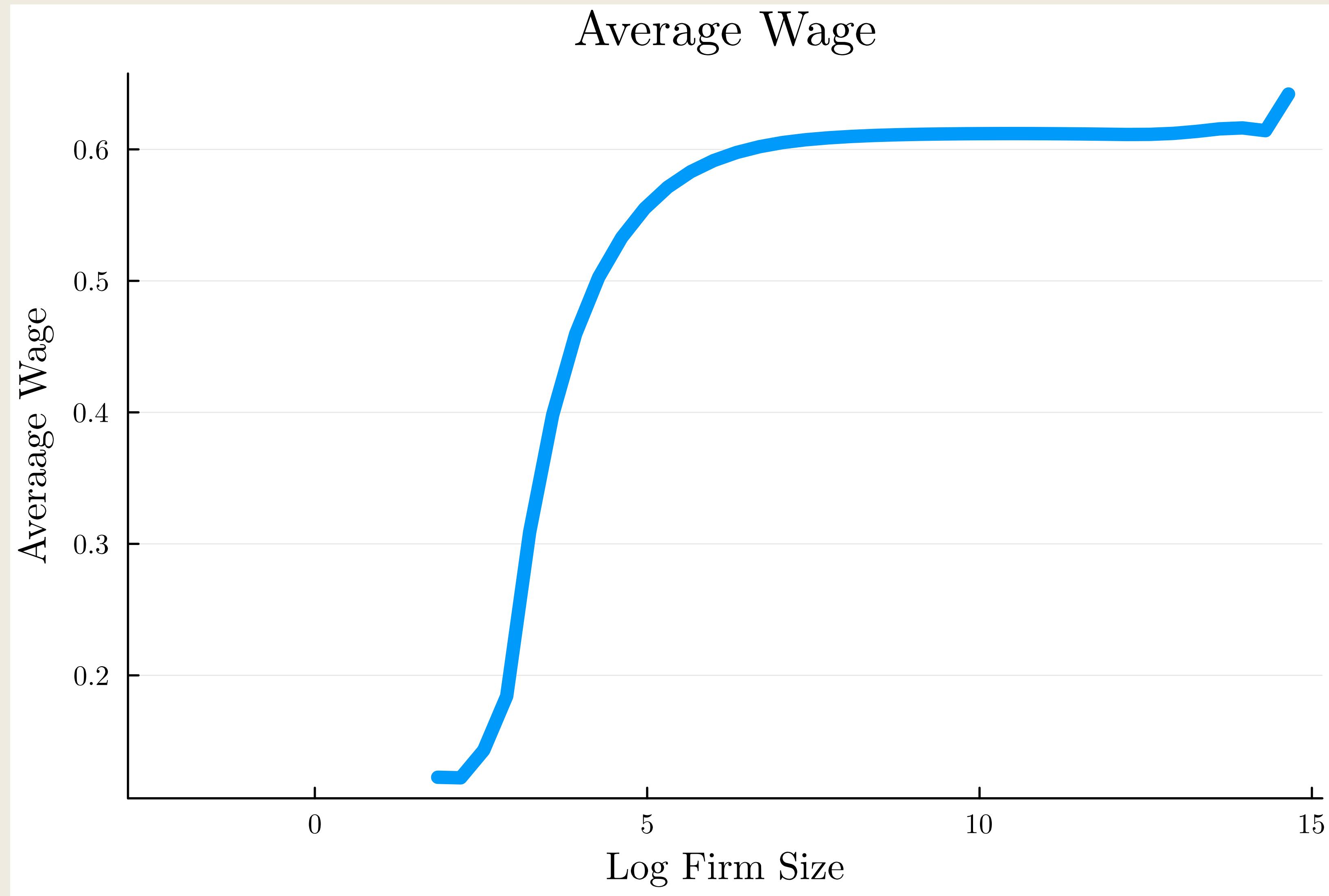
Implications for Firm Wage

- Search frictions allow us to talk about the firm wage
 - This wasn't possible in Hopenhayn-Rogerson
- The wage of firm (n, z) is given by

$$w(n, z) = \gamma \frac{1}{n} \left\{ f(n, z) - c_f - \Phi(v, n) \right\} + \beta(1 - \gamma) \frac{\gamma \lambda^F v}{n'} \mathbb{E} S(n', z') + (1 - \gamma)(1 - \beta) U$$

- Recall, in Burdett-Mortensen, wage and firm size were related monotonically
- Is it possible to break it?

Low Adjustment Cost



High Adjustment Cost

