CMSI 485 – Classwork 3
Solution

Instructions:
This worksheet will not only provide you with practice problems for your upcoming exam, but will add to your deep understanding of the mechanics of many probabilistic reasoning systems.

- Provide answers to each of the following questions and write your responses in the blanks. If you are expected to show your work in arriving at a particular solution, space will be provided for you.
- Place the names of your group members below:

Group Members:

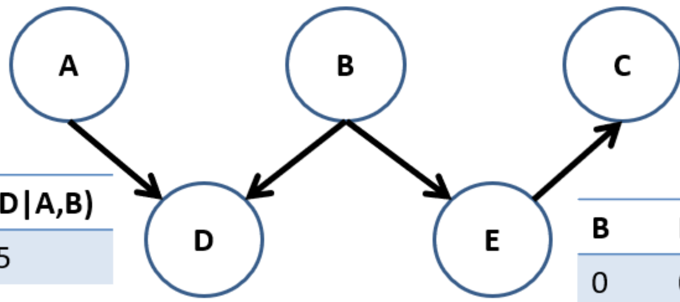
1. ________________________________
2. ________________________________
3. ________________________________
4. ________________________________

Problem 1 – Bayesian Network Exact Inference

Consider the following Bayesian Network and use it to answer the questions that follow.

B	P(B)
0	0.6

A	P(A)
0	0.8



E	C	P(C|E)
0	0	0.4
1	0	0.2

A	B	D	P(D|A,B)
0	0	0	0.5
0	1	0	0.7
1	0	0	0.8
1	1	0	0.1

B	E	P(E|B)
0	0	0.9
1	0	0.4

While examining Exact Inference in Bayesian Networks, we saw some methods for *simplifying* queries and the resulting computations that can yield large performance improvements when implemented. E.g., variables whose CPTs never affect the query outcome can be ignored.

1.1. For each of the following queries, determine *which variables' CPTs* will at all affect the answer to the query. Justify your responses in the boxes that follow. Hint: in your justification, express the query in terms of: $P(A|B) = \alpha \sum P(\dots)$ with $\alpha = \frac{1}{P(e)}$

1.1.1. $P(A|B = b, D = d)$

CPTs Used: ✓ A ✓ B C ✓ D E

Justification:

$$P(A|b,d) = \alpha \sum_e \sum_c P(A,b,c,d,e) = \alpha \sum_e \sum_c P(A)P(d|A,b)P(b)P(e|b)P(c|e)$$
$$= \alpha\, P(A)P(d|A,b)P(b) \sum_e P(e|b) \sum_c P(c|e) = \alpha P(A)P(d|A,b)P(b)$$

1.1.2. $P(E|D = d)$

CPTs Used: ✓ A ✓ B C ✓ D ✓ E

Justification:

$$P(E|d) = \alpha \sum_a \sum_b \sum_c P(a,b,c,d,E) = \alpha \sum_a \sum_b \sum_c P(a)P(d|a,b)P(b)P(E|b)P(c|E)$$
$$= \alpha \sum_b P(b)P(E|b) \sum_a P(a)P(d|a,b) \sum_c P(c|E) = \alpha \sum_b P(b)P(E|b) \sum_a P(a)P(d|a,b)$$

1.2. Using the Bayesian Network on the previous page, find the solutions to the following.

1.2.1. $P(A=0|B=1, D=1)$ *(Box your answer once finished)*

Step 1 – Label Vars: Q = query, e = evidence, Y = hidden

$$Q=\{A\}, e=\{B=1, D=1\}, Y=\{C, D\}$$

Step 2 – Find $P(Q,e)=\sum_y P(Q,e,y)$ (taken from 1.1.1 above):

$$P(A|B=1, D=1)=\alpha\sum_e\sum_c P(A, B=1, D=1, C=c, E=e)=\alpha P(A)P(d|A,b)P(b)$$

$$A=0 \Rightarrow P(A,b,d)=P(A=0)P(D=1|A=0,B=1)P(B=1)=0.8*0.3*0.4=0.096$$

$$A=1 \Rightarrow P(A,b,d)=P(A=1)P(D=1|A=1,B=1)P(B=1)=0.2*0.9*0.4=0.072$$

Step 3 – Find $P(e)=\sum_q P(Q=q,e)$*:*

$$\begin{aligned}P(e)&=P(B=1,D=1)\\&=\sum_a P(a,b,d)=P(A=0,B=1,D=1)+P(A=1,B=1,D=1)\\&=0.096+0.072=0.168\end{aligned}$$

Step 4 – Solve query: $P(Q|e)=\frac{P(Q,e)}{P(e)}=\frac{\{Step\ 2\}}{\{Step\ 3\}}$

$$\therefore P(A=0|B=1, D=1)=\frac{P(A=0, B=1, D=1)}{P(B=1, D=1)}=\frac{0.096}{0.168}\approx 0.571$$

1.2.2 $P(B=1|A=0,D=1,E=0,C=1)$ *(Box your answer once finished)*

Step 1 – Label Vars:

$$Q=\{B\}, e=\{A=0,C=1,D=1,E=0\}, Y=\emptyset$$

Step 2 – Find $P(Q,e)=\sum_y P(Q,e,y)$:

$$P(B|A=0,D=1,E=0,C=1)$$
$$=\alpha\, P(A=0)P(B)P(D=1|A=0,B)P(E=0|B)P(C=1|E=0)$$

$B=0\Rightarrow P(B,a,c,d,e)$
$=P(A=0)P(B=0)P(D=1|A=0,B=0)P(E=0|B=0)P(C=1|E=0)$
$=0.8*0.6*0.5*0.9*0.6=0.1296$

$B=1\Rightarrow P(B,a,c,d,e)$
$=P(A=0)P(B=1)P(D=1|A=0,B=1)P(E=0|B=1)P(C=1|E=0)$
$=0.8*0.4*0.3*0.4*0.6=0.02304$

Step 3 – Find $P(e)=\sum_q P(Q=q,e)$:

$$P(e)=P(A=0,D=1,E=0,C=1)=\sum_b P(B,a,c,d,e)$$
$$=P(B=0,A=0,D=1,E=0,C=1)+P(B=1,A=0,D=1,E=0,C=1)$$
$$=0.1296+0.02304=0.15264$$

Step 4 – Solve query: $P(Q|e)=\frac{P(Q,e)}{P(e)}=\frac{\{Step\ 2\}}{\{Step\ 3\}}$:

$$\therefore P(B=1|A=0,D=1,E=0,C=1)=\frac{P(B=1,A=0,D=1,E=0,C=1)}{P(A=0,D=1,E=0,C=1)}$$
$$=\frac{0.02304}{0.15264}\approx 0.151$$

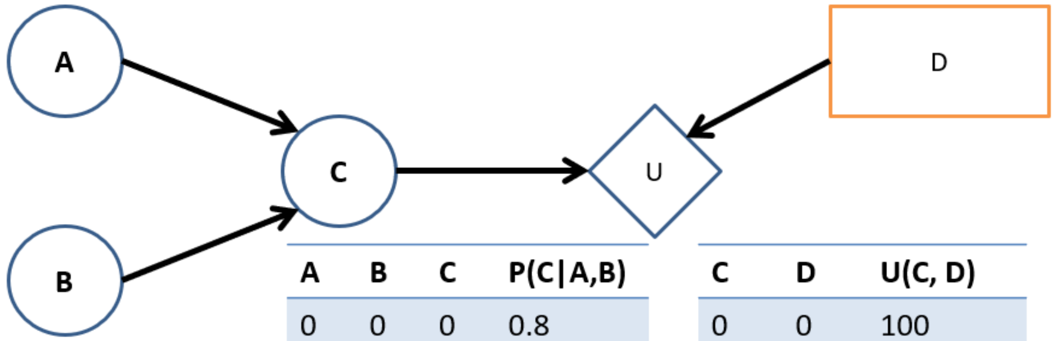
(sorry for the messy probability values there, oh well… life isn't always nice and neat!)

Problem 2 – Decision Networks & MEU

Use the following Decision Network with chance nodes A, B, C, decision node $D \in \{0,1\}$, and utility node U to answer the questions that follow.

A	P(A)
0	0.2

B	P(B)
0	0.5



A	B	C	P(C|A,B)
0	0	0	0.8
0	1	0	0.6
1	0	0	0.1
1	1	0	0.5

C	D	U(C, D)
0	0	100
0	1	20
1	0	40
1	1	80

2.1. Find the $MEU(B=0)$ (i.e., the Maximum Expected Utility with evidence $B=0$).

Reminder: (for evidence e, actions a, and chance nodes that are parents of utility s):

$$MEU(e) = \max_a EU(a|e) = \sum_s P(s|a,e)U(s,a)$$

$$MEU(B=0) = \max_d EU(d|B=0) = \max_d \sum_c P(c|d,B=0)U(c,d)$$

Since $P(c|d,B=0)$ is not in terms of our network parameters (the CPTs) we'll transform it:

$$\sum_c P(c|d,B=0)U(c,d) = \sum_c P(c|B=0)U(c,d) = \sum_c \sum_a P(c|a,B=0)P(a)U(c,d)$$

Now, computed for each available action:

$$EU(D=0|B=0) = \sum_c \sum_a P(c|a,B=0)P(a)U(c,D=0)$$

$$\begin{aligned}
&= P(C=0|A=0,B=0)P(A=0)U(C=0,D=0)\\
&+ P(C=0|A=1,B=0)P(A=1)U(C=0,D=0)\\
&+ P(C=1|A=0,B=0)P(A=0)U(C=1,D=0)\\
&+ P(C=1|A=1,B=0)P(A=1)U(C=1,D=0)\\
&= 0.8*0.2*100 + 0.1*0.8*100 + 0.2*0.2*40 + 0.9*0.8*40 = 54.4
\end{aligned}$$

$$EU(D=1|B=0) = \sum_c \sum_a P(c|a,B=0)P(a)U(c,D=1)$$

$$\begin{aligned}
&= P(C=0|A=0,B=0)P(A=0)U(C=0,D=1)\\
&+ P(C=0|A=1,B=0)P(A=1)U(C=0,D=1)\\
&+ P(C=1|A=0,B=0)P(A=0)U(C=1,D=1)\\
&+ P(C=1|A=1,B=0)P(A=1)U(C=1,D=1)\\
&= 0.8*0.2*20 + 0.1*0.8*20 + 0.2*0.2*80 + 0.9*0.8*80 = 65.6
\end{aligned}$$

$$\therefore MEU(B=0) = 65.6$$

[!] Note: The above derivation solved for the computable query $P(C|B = 0)$ using the insight that summing over A would yield expressions in terms of the CPTs – it did NOT use enumeration inference, which you can always do, like in the following derivation:

Step 1 – Label Vars:

$$Q = \{C\}, e = \{B = 0\}, Y = \{A\}$$

Step 2 – Find $P(Q, e) = \sum_y P(Q, e, y)$:

$$P(C, B = 0) = \sum_a P(C, B = 0, A = a) = P(B = 0) \sum_a P(A = a) P(C|B = 0, A = a)$$

$$C = 0 => \ P(C = 0, B = 0) = 0.12$$
$$C = 1 => \ P(C = 1, B = 0) = 0.38$$

Step 3 – Find $P(e) = \sum_q P(Q = q, e)$:

$$P(e) = P(B = 0) = \sum_c P(C = c, B = 0)$$
$$= 0.12 + 0.38 = 0.5$$

(which you can double check from the CPT, since the $P(e)$ happened to be immediately answerable in this particular problem).

Step 4 – Solve query: $P(Q|e) = \frac{P(Q,e)}{P(e)} = \frac{\{Step\ 2\}}{\{Step\ 3\}}$:

$$P(C = 0|B = 0) = \frac{P(C = 0, B = 0)}{P(B = 0)} = \frac{0.12}{0.5} = 0.24$$
$$P(C = 1|B = 0) = \frac{P(C = 1, B = 0)}{P(B = 0)} = \frac{0.38}{0.5} = 0.76$$

Back to the MEU Computation…

$$MEU(B = 0) = \max_d EU(d|B = 0) = \max_d \sum_c P(c|B = 0) U(c, d)$$

$$EU(D = 0|B = 0) = P(C = 0|B = 0)U(C = 0, D = 0) + P(C = 1|B = 0)U(C = 1, D = 0)$$
$$= 0.24 * 100 + 0.76 * 40 = 54.4$$
$$EU(D = 1|B = 0) = P(C = 0|B = 0)U(C = 0, D = 1) + P(C = 1|B = 0)U(C = 1, D = 1)$$
$$= 0.24 * 20 + 0.76 * 80 = 65.6$$
$$\therefore MEU(B = 0) = 65.6$$

2.2. Given your computations above, what decision should your agent make by MEU?

Since $d^* = argmax_d EU(d|B = 0) = 1$, our agent should choose action $D = 1$.

Problem 3 – Value of Perfect Information

Using the network and your answer from the previous problem, we're going to compute the Value of Perfect Information (VPI) of knowing the state of variable A when $B = 0$ is given. Let's do so step-by-step:

3.1. Find $MEU(A = 0, B = 0)$.

$$MEU(A = 0, B = 0) = \max_d EU(d|A = 0, B = 0) = \max_d \sum_c P(c|A = 0, B = 0)U(c, d)$$

$$EU(D = 0|A = 0, B = 0) = \sum_c P(c|A = 0, B = 0)U(c, D = 0)$$
$$= P(C = 0|A = 0, B = 0)U(C = 0, D = 0) + P(C = 1|A = 0, B = 0)U(C = 1, D = 0)$$
$$= 0.8 * 100 + 0.2 * 40 = 88$$

$$EU(D = 1|A = 0, B = 0) = \sum_c P(c|A = 0, B = 0)U(c, D = 1)$$
$$= P(C = 0|A = 0, B = 0)U(C = 0, D = 1) + P(C = 1|A = 0, B = 0)U(C = 1, D = 1)$$
$$= 0.8 * 20 + 0.2 * 80 = 32$$

$$\therefore MEU(A = 0, B = 0) = 88$$

3.2. Using the your answer to 4.1 and knowledge that $MEU(A = 1, B = 0) = 74$ (freebee!) find $MEU(A, B = 0)$.

We have to weight the $MEU(A = a, B = 0)$ by the likelihood of $A = a|B = 0$:

$$MEU(A, B = 0) = \sum_a P(a|B = 0)MEU(a, B = 0) = \sum_a P(a)MEU(a, B = 0)$$
$$= P(A = 0)MEU(A = 0, B = 0) + P(A = 1)MEU(A = 1, B = 0)$$
$$= 0.2 * 88 + 0.8 * 74 = 76.8$$

3.3. Compute the $VPI(A|B = 0)$.

$$VPI(A|B = 0) = MEU(A, B = 0) - MEU(B = 0) = 76.8 - 65.6 = 11.2$$

3.4. If the utility scores represent dollar amounts, what would be a fair price for A when $B = 0$?
The fair price is given by the VPI! That's its definition. So, the fair cost for info on A is: \$11.20. If we bought it for less, it would be a good deal, and for more, a ripoff.