1. Week 1: Course overview

1.1. **Introduction.** Suppose that a group G is given and that we want to understand the group G. But then what exactly does it mean to "understand" G? There is a rich framework which enables us to "define" a reasonable answer to this problem; it is *representation theory.* Recall that a *representation* of a group G is a vector space V, say $\mathbb{C}$-coefficient here, equipped with an action of G.

> Let us say that "we understand the group G" when we understand all the representations of G.

The aim of this course is to give an introduction to "Deligne–Lusztig theory" (established in [DL76]), which provides a realization of all representations of finite groups of Lie type.

1.2. **Quick review of representation theory of finite groups.** The basic reference of this subsection is Serre's book [Ser77].

In the following, we let G be a finite group.

Definition 1.1 (representation)**.** We say that (ρ, V) is a *representation* of G if V is a finite-dimensional $\mathbb{C}$-vector space equipped with an action ρ of G, i.e., ρ is a homomorphism $G \to \mathrm{GL}_{\mathbb{C}}(V) := \mathrm{Aut}_{\mathbb{C}}(V)$. We often only write ρ or V for a representation (ρ, V).

Definition 1.2 (homomorphism)**.** Let (ρ_1, V_1) and (ρ_2, V_2) be representations of G. We say that a $\mathbb{C}$-linear map $f\colon V_1 \to V_2$ is a *homomorphism* from (ρ_1, V_1) to (ρ_2, V_2) if it is equivariant with respect to the actions ρ_1 and ρ_2 of G, i.e., we have $f(\rho_1(g)(v)) = \rho_2(g)(f(v))$ for any $g \in G$ and $v \in V_1$.

$$\begin{tikzcd} V_1 \arrow[r, "f"] \arrow[d, "\rho_1(g)"'] \arrow[dr, phantom, "\circlearrowleft"] & V_2 \arrow[d, "\rho_2(g)"] \\ V_1 \arrow[r, "f"] & V_2 \end{tikzcd}$$

We write $\mathrm{Hom}_G(\rho_1, \rho_2)$ for the set of homomorphisms from ρ_1 to ρ_2 (this has a natural $\mathbb{C}$-vector space structure). We say that (ρ_1, V_1) and (ρ_2, V_2) are *isomorphic* if there exists an isomorphism $f\colon V_1 \to V_2$ (i.e., homomorphism which is isomorphic as a $\mathbb{C}$-linear map).

Definition 1.3 (subrepresentation)**.** Let (ρ, V) be a representation of G. We say that a subspace W of V is a *subrepresentation* of V if it is stable under the action ρ of G.

Definition 1.4 (irreducible representation)**.** Let V be a representation of G. We say that V is *irreducible* if $V \neq \{0\}$ and there is no subrepresentation of V except for V itself and $\{0\}$.

Note that basic operations on vector spaces can be considered also for representations. For example, when (ρ_1, V_1) and (ρ_2, V_2) are representations of G, we define their *direct sum* $(\rho_1 \oplus \rho_2, V_1 \oplus V_2)$, which is a representation of G, by

$$(\rho_1 \oplus \rho_2)(g)(v_1 + v_2) := \rho_1(g)(v_1) + \rho_2(g)(v_2)$$

for any $g \in G$ and $v_1 \in V_1$, $v_2 \in V_2$. Similarly, we define the tensor product $\rho_1 \otimes \rho_2$, which is a representation of G, by

$$(\rho_1 \otimes \rho_2)(g)(v_1 \otimes v_2) := \rho_1(g)(v_1) \otimes \rho_2(g)(v_2).$$

We also often use the "box-tensor product" $\rho_1 \boxtimes \rho_2$, which is a representation of $G \times G$ defined by

$$(\rho_1 \boxtimes \rho_2)(g_1, g_2)(v_1 \otimes v_2) := \rho_1(g_1)(v_1) \otimes \rho_2(g_2)(v_2).$$

(Note that this definition works for, more generally, representations ρ_1 of G_1 and ρ_2 of G_2; in this case, $\rho_1 \boxtimes \rho_2$ is a representation of $G_1 \times G_2$.)

The following theorem is very fundamental and important in representation theory of finite groups.

Theorem 1.5 (semisimplicity of representations)**.** *Let V be a representation of G. Then there is a unique (up to permutation) way to write*

$$V \cong \bigoplus_{i=1}^{r} W_i^{\oplus n_i},$$

where W_i's are pairwise inequivalent irreducible representations of G and n_i's are positive integers determined only by V.

By this theorem, the problem of understanding representations of G can be divided into the following two steps:

(1) Classify all irreducible representations of G.
(2) Find a systematic way of determining each n_i from a given V.

Let us list some fundamental facts on the first part (1):

Theorem 1.6. *(1) The number of conjugacy classes of G equals the number of isomorphism classes of irreducible representations of G.*

(2) We have

$$|G| = \sum_{\rho} \dim(\rho)^2,$$

where ρ runs over isomorphism classes of irreducible representations of G.

The key to the part (2) is the following:

Theorem 1.7 (Schur's lemma)**.** *Let (ρ_1, V_1) and (ρ_2, V_2) be irreducible representations of G. Then we have*

$$\mathrm{Hom}_G(\rho_1, \rho_2) \cong \begin{cases} \mathbb{C} & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \ncong \rho_2. \end{cases}$$

By Schur's lemma, each multiplicity n_i of an irreducible representation V_i in the irreducible decomposition of a representation V of G is given by $\dim_{\mathbb{C}} \mathrm{Hom}_G(V, V_i)$ (or $\dim_{\mathbb{C}} \mathrm{Hom}_G(V_i, V)$). Then, how can we determine this number for each V_i? Theory of *characters* provides a satisfactory answer to this question.

Definition 1.8 (character)**.** Let (ρ, V) be a representation of G. The *character* of (ρ, V), for which we write Θ_ρ (or Θ_V), is the function $G \to \mathbb{C}$ defined by $\Theta_\rho(g) := \mathrm{Tr}\,\rho(g)$. Namely, $\Theta_\rho(g)$ is the trace of the representation matrix of $\rho(g)$ (with respect to any $\mathbb{C}$-basis of V).

Note that Θ_ρ is constant on each conjugacy class of G. Such a function is called a *class function* on G. Let $C(G)$ denote the set of $\mathbb{C}$-valued class functions on G. Then $C(G)$ has a natural $\mathbb{C}$-vector space structure equipped with an inner product $\langle -, - \rangle$ given by

$$\langle f_1, f_2 \rangle := \frac{1}{|G|} \sum_{g \in G} f_1(g) \cdot \overline{f_2(g)}.$$

Theorem 1.9. *The set of characters of irreducible representations of G forms an orthonormal basis of $C(G)$ with respect the inner product $\langle -, - \rangle$. In particular, for irreducible representations (ρ_1, V_1) and (ρ_2, V_2) of G, we have*

$$\langle \Theta_{\rho_1}, \Theta_{\rho_2} \rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \not\cong \rho_2. \end{cases}$$

Note that, by this theorem, it is enough to compute $\langle \Theta_V, \Theta_{W_i} \rangle$ to get the multiplicity n_i of W_i in V.

From these discussion, we could say that our ultimate goal in representation theory of G is to get a list of the character values of all irreducible representations on all conjugacy classes of G. Such a list is called the *character table* of G.

1.3. **Warmup example: $\mathfrak{S}_3$.** When G is a finite abelian group, all irreducible representations of G are 1-dimensional, i.e., characters. Thus there exists $|G|$ irreducible representations of G; all of them can be described explicitly by, e.g., the structure theorem of finite abelian groups.

So let us look at the non-abelian group of the smallest order, i.e., the permutation group of three letters:

$$\mathfrak{S}_3 = \{1, (12), (23), (31), (123), (132)\}.$$

Since this group has three conjugacy classes

$$\{1\}, \quad \{(12), (23), (31)\}, \quad \{(123), (132)\},$$

there should be three irreducible representations. Firstly, we have the trivial representation of $\mathfrak{S}_3$, which is 1-dimensional. Secondly, the signature character $\mathrm{sgn} \colon \mathfrak{S}_3 \to \{\pm 1\}$ gives another 1-dimensional representation. [1]

So, what is the remaining representation? We let r be its dimension. Then we should have

$$1^2 + 1^2 + r^2 = |\mathfrak{S}_3| = 6,$$

i.e., r must be 2. Let us find the remaining 2-dimensional irreducible representation. Almost by definition, $\mathfrak{S}_3$ acts on the set of three letters $X := \{1, 2, 3\}$. Thus, if we let $V := \mathbb{C}[X]$ be the space of $\mathbb{C}$-valued functions on X, then $\mathfrak{S}_3$ also acts on V (via pull-back of functions). This representation is 3-dimensional and contains the trivial representation as its subrepresentation. Indeed, the subspace of constant functions on X is stable under the action $\mathfrak{S}_3$; let us write W for it. We claim that V/W, which is 2-dimensional, is an irreducible representation of $\mathfrak{S}_3$. To check this, it is enough to show that $\langle \Theta_{V/W}, \Theta_{V/W} \rangle = 1$.

Let us first compute the character Θ_V of V. Since Θ_V is a class function, it is enough to compute the traces of the actions of 1, (12), and (123). Let $\mathbb{1}_i$ denote the characteristic function of $\{i\} \subset X$ for $i = 1, 2, 3$. Then $\{\mathbb{1}_i \mid i = 1, 2, 3\}$ is a $\mathbb{C}$-basis of V and the representation matrices of the actions of 1, (12), and (123) with respect to this basis is given by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

[1] Recall that, in general, the signature character of $\mathfrak{S}_3$ associates $+1$ (resp. -1) to a permutation expressed by the product of even (resp. odd) number of transpositions.

Hence we have

$$\Theta_V(1) = 3, \quad \Theta_V((12)) = 1, \quad \Theta_V((123)) = 0.$$

As we have $\Theta_W(1) = 1, \Theta_W((12)) = 1, \Theta_W((123)) = 1$, we get

$$\Theta_{V/W}(1) = 2, \quad \Theta_{V/W}((12)) = 0, \quad \Theta_{V/W}((123)) = -1.$$

Therefore, we have

$$\begin{aligned}\langle \Theta_{V/W}, \Theta_{V/W} \rangle &= \frac{1}{6} \sum_{g \in \mathfrak{S}_3} \Theta_{V/W}(g) \cdot \overline{\Theta_{V/W}(g)} \\ &= \frac{1}{6}\big(2^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2\big) = 1.\end{aligned}$$

1.4. **What is Deligne–Lusztig theory?** When a group G is finite, we win if we can find all irreducible representations "by hand" in any way. However, we immediately notice that it's not easy in general. (We will look at the example of $\mathrm{GL}_2(\mathbb{F}_q)$ in the next week. We construct its all irreducible representations by hand, by we can see that it's already not obvious at all.)

In fact, we can find an idea in the above example of $\mathfrak{S}_3$. This example suggests that, more generally, we can try to construct representations of a given group G according to the following steps:

(1) First, introduce a "space" X equipped with an action of G.
(2) Second, find a "functorial linearization" $X \mapsto V_X$, i.e., an operation which associates a vector space to each space X which is functorial in X. Then, the action of G on X induces an action of G on V_X.

Deligne–Lusztig theory exactly realizes this idea for so-called finite groups of Lie type. What is a finite group of Lie type? To explain this, let us first recall the definition of a *general linear group*:

$$\mathrm{GL}_n(\mathbb{C}) := \{g \in M_n(\mathbb{C}) \mid g \text{ is invertible}\}.$$

So $\mathrm{GL}_n(\mathbb{C})$ is the set of all invertible n-by-n matrices whose entries are complex numbers; this has a group structure with respect to the usual multiplication of matrices. The point here is that the definition of a general linear group completely makes sense even if we replace the field $\mathbb{C}$ with any field (or even any ring!). Thus, in some sense, we may think of GL_n as a "machine" which associates a group to any ring;

$$R \mapsto \mathrm{GL}_n(R) := \{g \in M_n(R) \mid g \text{ is invertible}\}.$$

In particular, by taking R to be a finite field $\mathbb{F}_q$, we obtain a finite group $\mathrm{GL}_n(\mathbb{F}_q)$.

In general, this kind of machine is called an *algebraic group.* Among algebraic groups, there is a particular class called *reductive groups.* The general linear group is one of the most typical examples of a reductive group. A finite group of Lie type is a finite group obtained by letting R be a finite field $\mathbb{F}_q$ for a reductive group G which can be "defined over $\mathbb{F}_q$". (In the case of GL_n, its definition makes sense over $\mathbb{Z}$, hence also over $\mathbb{F}_q$.)

Let us introduce more examples. Recall that the symplectic (resp. orthogonal) group is the group consisting of symplectic (resp. orthogonal) matrices:

$$\mathrm{Sp}_{2n}(\mathbb{C}) := \{g \in \mathrm{GL}_{2n}(\mathbb{C}) \mid {}^t g J_{2n} g = J_{2n}\},$$

$$\mathrm{O}_n(\mathbb{C}) := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t g g = I_n\}.$$

Here, J_{2n} (resp. I_n) denotes the anti-diagonal matrix whose $(i, 2n+1-i)$-entry is given by $(-1)^{i-1}$ (resp. the identity matrix). The defining equations of these groups only uses 1 and -1, hence it makes sense to replace $\mathbb{C}$ with $\mathbb{F}_q$; then we get $\mathrm{Sp}_{2n}(\mathbb{F}_q)$ and $\mathrm{O}_n(\mathbb{F}_q)$.

Let us also introduce a bit more tricky example. The unitary group is the group consisting of unitary matrices:

$$\mathrm{U}_n := \{g \in \mathrm{GL}_n(\mathbb{C}) \mid {}^t\overline{g}g = I_n\}.$$

Here, $\overline{g}$ denotes the entry-wise complex conjugate of g. Note that the complex conjugation is nothing but the nontrivial element of the Galois group of the quadratic extension $\mathbb{C}/\mathbb{R}$. This viewpoint suggests that we can define a unitary group in the same way as long as a quadratic extension of fields is given. In particular, by taking a finite field $\mathbb{F}_q$ and its quadratic extension $\mathbb{F}_{q^2}$, we can define

$$\mathrm{U}_n(\mathbb{F}_q) := \{g \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid {}^tF(g)g = I_n\}.$$

Here, F denotes the nontrivial element of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$; this is so-called the Frobenius, which is given by taking (entry-wise) q-th power.

Now let us also mention the "space X" and the "functorial linearization $X \mapsto V_X$". The space X in the context of Deligne–Lusztig theory is called the *Deligne–Lusztig variety*. The definition of the Deligne–Lusztig variety depends on a finite group of Lie type $G(\mathbb{F}_q)$ (with its additional structure). It originates from a very concrete curve with $\overline{\mathbb{F}}_q$-coefficient called the Drinfeld curve, whose defining equation is given by $xy^q - x^qy = 1$. However, the general Deligne–Lusztig variety is defined based on a very sophisticated language of the theory of reductive groups. We have to make full use of the structure theory of reductive groups to analyze its geometric structure.

On the other hand, the role of "functorial linearization $X \mapsto V_X$" is played by the theory of étale cohomology. More precisely, by choosing a prime number ℓ different to the characteristic p of $\mathbb{F}_q$, we obtain the (compactly supported) ℓ-adic cohomology $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ of X. This cohomology $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is a finite dimensional $\overline{\mathbb{Q}}_\ell$-vector space, where $\overline{\mathbb{Q}}_\ell$ is an algebraic closure of the ℓ-adic number field $\mathbb{Q}_\ell$. The point here is that $\overline{\mathbb{Q}}_\ell$ is abstractly isomorphic to $\mathbb{C}$, hence we can regard $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ as a finite dimensional $\mathbb{C}$-vector space. In particular, we obtain a representation of $G(\mathbb{F}_q)$. In order to analyze the structure of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ as a representation of $G(\mathbb{F}_q)$, we also need to appeal to various fundamental properties of the étale cohomology.

1.5. **Why Deligne–Lusztig theory?** Then, why is Deligne–Lusztig theory so important? The first reason is that Deligne–Lusztig theory is only a framework (at present) which enables us to construct all irreducible representations of finite groups of Lie type in a uniform way. A lot of important examples of finite groups are contained in the class "finite groups of Lie type". However, irreducible representations had been classified only in the case of $\mathrm{GL}_n(\mathbb{F}_q)$ (due to Green in 1955) before the work of Deligne–Lusztig. Moreover, even in that case, Green's method is based on heavy combinatorial arguments, hence it is quite nontrivial whether it can be generalized to other finite groups of Lie type. Let us cite a comment of Shoji from his book [Sho04] (written in Japanese):

> *A preprint by Deligne–Lusztig was released when I was a student. I was shocked about it; it was like that an iron-made steamship suddenly appeared in a peaceful small village which was only based*

> *on the handicraft industry before. For people peacefully living with* $\mathrm{GL}_n(\mathbb{F}_q)$ *at that time, Deligne–Lusztig theory was so surprising, almost like the devil's work.*

The second reason is that Deligne–Lusztig theory is expected to have an application to the local Langlands correspondence. The local Langlands correspondence is also called the non-abelian class field theory; roughly speaking, it predicts a natural connection between representations of p-adic reductive groups (such as $\mathrm{GL}_n(\mathbb{Q}_p)$, $\mathrm{Sp}_{2n}(\mathbb{Q}_p)$, etc...) and Galois representations. The expectation is that a certain case of the local Langlands correspondence can be made from Deligne–Lusztig theory (e.g., [DR09]). [2]

References

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DR09] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901.

[Ser77] J.-P. Serre, *Linear representations of finite groups*, Springer-Verlag, New York-Heidelberg, 1977, Translated from the second French edition by Leonard L. Scott, Graduate Texts in Mathematics, Vol. 42.

[Sho04] 庄司 俊明, 「ドリーニュ-ルスティック指標を訪ねて」, 群論の進化, 朝倉書店, 2004.

9:00pm, September 2, 2024

[2] But nothing about this will be explained in this course! Maybe next semester???