# 1. Week 11: Cuspidal representations

Recall that, in Week 2, we investigated cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$. We first defined principal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ by considering the induction from Borel subgroups, and then defined the cuspidality. The aim of this week is to first generalize the notion of the cuspidality to any finite group of Lie type and investigate it from the viewpoint of Deligne–Lusztig theory.

## 1.1. Parabolic subgroups.

Let $G$ be a connected reductive group over $k = \mathbb{F}_q$.

**Proposition/Definition 1.1.**
(1) Let $P$ be a $k$-rational closed subgroup of $G$. We say that $P$ is a *$k$-rational parabolic subgroup* of $G$ if $P_{\overline{k}}$ contains a Borel subgroup of $G_{\overline{k}}$.
(2) For any $k$-rational parabolic subgroup $P$ of $G$, there exists a $k$-rational connected reductive subgroup $L$ of $P$ such that $P$ is the semi-direct product $P = L \ltimes U_P$, where $U_P$ is the unipotent radical of $P$. We call such an $L$ a *$k$-rational Levi subgroup* of $P$. We call the decomposition $P = L \ltimes U_P$ a *Levi decomposition*.

**Remark 1.2.**
(1) By definition, $G$ and any $k$-rational Borel subgroup of $G$ are obviously parabolic subgroups; these are maximal/minimal parabolic subgroups.
(2) Note that a Levi subgroup of a given parabolic subgroup is not unique in general.

**Example 1.3.** Let $G = \mathrm{GL}_3$.

(1) We put
$$P := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset G.$$
Then this is a $k$-rational parabolic subgroup of $G$. The unipotent radical of $P$ is given by
$$U_P = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \subset P.$$
Hence, for example, a Levi subgroup of $P$ can be taken to be
$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & * \end{pmatrix} \subset P.$$

(2) We put
$$P' := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset G.$$
Then this is a $k$-rational parabolic subgroup of $G$. The unipotent radical of $P'$ is given by
$$U_{P'} = \begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \subset P'.$$

Hence, for example, a Levi subgroup of $P'$ can be taken to be

$$\begin{pmatrix} * & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset P'.$$

Recall that there always exists a $k$-rational Borel subgroup of $G$ since $k = \mathbb{F}_q$; let us fix such a $B_0$. We call a $k$-rational parabolic subgroup $P$ *standard parabolic* if $P$ contains $B_0$.

**Fact 1.4.** *Any $k$-rational parabolic subgroup of $G$ is $G(k)$-conjugate to a $k$-rational standard parabolic subgroup of $G$.*

The above definition of a parabolic subgroup is too abstract. So let us also introduce a concrete description of (standard) parabolic subgroups. In the following (in this subsection), we assume that $G$ is split for simplicity. But the theory does not change even when $G$ is non-split.

Recall that reductive groups are classified by root data "$(X, R, X^\vee, R^\vee)$". Let us first review how $(X, R, X^\vee, R^\vee)$ is associated to $G$ (Week 4). We let $T_0$ be a split $k$-rational maximal torus of $G$ contained in $B_0$. Then $X$ and $X^\vee$ are defined to be $X^*(T_0)$ and $X_*(T_0)$. The sets $R$ and $R^\vee$ are finite subsets of $X$ and $X^\vee$; these are called the sets of roots and coroots. An element $\alpha \in X$ belongs to $R$ if and only if there exists a closed subgroup $U_\alpha$ of $G$ such that

- $U_\alpha$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$ (fix $\iota\colon \mathbb{G}_{\mathrm{a}} \cong U_\alpha$), and
- $U_\alpha$ is normalized by $T_0$-conjugation and satisfies
$$t \cdot \iota(x) \cdot t^{-1} = \iota(\alpha(t) \cdot x)$$
for any $t \in T_0$ and $x \in \mathbb{G}_{\mathrm{a}}$.

Let us call a root $\alpha \in R$ a *positive root* if its associated root subgroup $U_\alpha$ is contained in the unipotent radical $U_0$ of the fixed Borel subgroup $B_0$. We write $R_+$ for the subset of $R$ of positive roots. We put $R_- := -R_+$ and call an element of $R_-$ a *negative root*. Note that $R_-$ is also a subset of $R$ since we have $-R = R$.

**Fact 1.5.** *(1) We have $R = R_+ \sqcup R_-$.*
*(2) There exists a unique subset $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of $R_+$ such that any positive root is uniquely written as a $\mathbb{Z}_{\geq 0}$-linear combination of $\alpha_1, \ldots, \alpha_l$; $\alpha = \sum_{i=1}^{l} n_i\alpha_i$ $(n_i \in \mathbb{Z}_{\geq 0})$.*

We call $\Delta$ the set of *simple roots*. Note that, by this fact and the definition of $R_-$, any negative root is uniquely written as a $\mathbb{Z}_{\leq 0}$-linear combination of simple roots.

**Remark 1.6.** Recall that, the construction of root datum $(X, R, X^\vee, R^\vee)$ depends on the choice of $T_0$, but does not on $B_0$. On the other hand, the notions of a positive root and a simple root depends on $B_0$.

Now let $I$ be any subset of $\Delta$. We consider the following subset $R_I$ of $R$:

$$R_I := \Bigl\{\alpha = \sum_{i=1}^{l} n_i\alpha_i \in R \Bigm| n_i \geq 0 \text{ if } i \notin I\Bigr\}.$$

We define a $k$-rational closed subgroup $P_I$ of $G$ by

$$P_I := \langle T_0, U_\alpha \mid \alpha \in R_I \rangle.$$

For example:

- When $I = \Delta$, we have $R_\Delta = R$ and $P_\Delta = \langle T_0, U_\alpha \mid \alpha \in R \rangle = G$.
- When $I = \emptyset$, we have $R_\emptyset = R_+$ and $P_\emptyset = \langle T_0, U_\alpha \mid \alpha \in R_+ \rangle = B_0$.

In particular, in general, $P_I$ is a $k$-rational closed subgroup of $G$ containing $B_0$, hence a standard parabolic subgroup.

**Fact 1.7.** *The above construction gives an order-preserving bijection*

$$\{I \subset \Delta\} \xrightarrow{1:1} \{k\text{-rational standard parabolic subgroups of } G\} \colon I \mapsto P_I.$$

*Moreover, each $P_I$ is equipped with a natural Levi subgroup (we call the "standard Levi subgroup") $L_I$, which is given by*

$$L_I = \langle T_0, U_\alpha \mid \alpha \in R_I^0 \rangle,$$

*where*

$$R_I^0 := \Big\{ \alpha = \sum_{i=1}^{l} n_i \alpha_i \in R \,\Big|\, n_i = 0 \text{ if } i \notin I \Big\}.$$

**Example 1.8.** Let $G = \mathrm{GL}_3$. Let $T_0$ be its diagonal maximal torus. As usual, we choose $B_0$ to be the upper-triangular one.

$$B_0 := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix},$$

then both $P$ and $P'$ contains $B_0$, hence are standard. The set $R$ of roots is given by

$$R = \{\pm(e_1 - e_2), \pm(e_2 - e_3), \pm(e_1 - e_3)\}.$$

The corresponding root subgroups are as follows:

$$U_{e_1-e_2} = \begin{pmatrix} 1 & * & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_2-e_1} = \begin{pmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$U_{e_2-e_3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_3-e_2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & * & 1 \end{pmatrix},$$

$$U_{e_1-e_3} = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{e_3-e_1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ * & 0 & 1 \end{pmatrix}.$$

Thus the positive roots are $e_1 - e_2$, $e_2 - e_3$, $e_1 - e_3$. The negative roots are $e_2 - e_1$, $e_3 - e_2$, $e_3 - e_1$. The set of simple roots $\Delta$ in this case is given by $\{e_1 - e_2, e_2 - e_3\}$ (indeed, we have $e_1 - e_3 = (e_1 - e_2) + (e_2 - e_3)$). We can check that the standard parabolic subgroups corresponding to subsets of $\Delta$ are as follows:

$$P_\Delta = \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix},$$

$$P_{\{e_1-e_2\}} = \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix}, \quad P_{\{e_2-e_3\}} = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix},$$

$$P_\emptyset = \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix}.$$

1.2. **Parabolic induction.** Let $P$ be a $k$-rational parabolic subgroup of $G$. Let $L$ be a $k$-rational Levi subgroup of $P$ and $U_P$ be the unipotent radical of $P$; hence we have $P = L \ltimes U_P$. Note that, as $U_P$ is connected, we can check that $(P/U_P)^F \cong P^F/U_P^F$ by the usual argument via Lang's theorem. In particular, we have a canonical surjection (quotient)

$$P^F \twoheadrightarrow P^F/U_P^F \cong (P/U_P)^F \cong L^F.$$

**Definition 1.9.** Suppose that $\sigma$ is a representation of $L^F$. By pulling back $\sigma$ to $P^F$ via $P^F \twoheadrightarrow L^F$, we regard $\sigma$ as a representation of $P^F$. We call its induction $\mathrm{Ind}_{P^F}^{G^F} \sigma$ to $G^F$ the *parabolic induction* of $\sigma$.

**Example 1.10.** Recall that a Borel subgroup is a minimal parabolic subgroup. Thus let us take $P$ to be $B_0$. In this case, a Levi subgroup of $B_0$ can be taken to be $T_0$. The parabolic induction of a 1-dimensional representation ($\mathrm{Ind}_{B_0^F}^{G^F} \chi$ for a character $\chi\colon B_0^F \to \mathbb{C}^\times$) is nothing but the principal series representation, which was introduced before.

**Definition 1.11.** Let $\rho$ be a representation of $G^F$. We say that $\rho$ is *cuspidal* if there does not exist a pair $(P, \sigma)$ of a proper $k$-rational parabolic subgroup $P \subsetneq G$ with a Levi $L$ and a representation $\sigma$ of $L^F$ such that $\langle \rho, \mathrm{Ind}_{P^F}^{G^F} \sigma\rangle \neq 0$.

We explain why the cuspidal representations are so important. Suppose that $\rho$ is a **non**-cuspidal irreducible representation of $G^F$. Then, by definition, there exists a pair $(P_1 \subsetneq G, \sigma_1)$ such that $\rho$ is contained in $\mathrm{Ind}_{P_1^F}^{G^F} \sigma_1$. We may assume that such a $\sigma_1$ is irreducible. Let us consider what will happen if $\sigma_1$ is not a cuspidal representation of $L_1^F$. Then, again by definition, $\sigma_1$ is contained in $\mathrm{Ind}_{P_2^F}^{L_1^F} \sigma_2$ for some proper parabolic subgroup $P_2 \subsetneq L_1$ with a Levi $L_2$ and an irreducible representation $\sigma_2$ of $L_2^F$. We can continue this procedure, but not forever because there cannot exist an infinite chain of proper parabolic subgroups. In other words, eventually we arrive at a pair $(P, \sigma)$, where $\sigma$ is a cuspidal irreducible representation of $L^F$.

**Exercise 1.12.** Prove the associativity of the parabolic induction.

**Proposition 1.13.** *Let $\rho$ be a representation of $G^F$. The following are equivalent:*

*(1) $\rho$ is cuspidal;*

*(2) for any $k$-rational parabolic subgroup $P$ with a $k$-rational Levi $L$, we have $\langle \rho, \mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1}\rangle = 0$.*

*Proof.* Note that

$$\mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1} \cong \mathrm{Ind}_{P^F}^{G^F}(\mathrm{Ind}_{U_P^F}^{P^F} \mathbb{1}) \cong \bigoplus_{\sigma\in\mathrm{Irr}(L^F)} \mathrm{Ind}_{P^F}^{G^F} \sigma,$$

where the sum is over all irreducible representations of $L^F$ (we used that $P^F/U_P^F \cong L^F$). Therefore, we have $\langle \rho, \mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1}\rangle = 0$ if and only if $\langle \rho, \mathrm{Ind}_{P^F}^{G^F} \sigma\rangle = 0$ for all irreducible representations $\sigma$ of $L^F$. $\square$

1.3. **DL's cuspidality criterion.** Suppose that $T$ is a $k$-rational maximal torus of $G$ contained in a $k$-rational parabolic subgroup $P$ of $G$. Let $L$ be a Levi subgroup of $P$ and $U_P$ the unipotent radical of $P$. Then, under the map $P \twoheadrightarrow P/U_P \cong L$, $T$ is mapped to a $k$-rational maximal torus of $L$ isomorphically (the kernel of the map is $T \cap U_P$, which is semisimple and unipotent, hence trivial). Let us again write $T$ for the $k$-rational maximal torus of $L$ determined in this way.

**Proposition 1.14.** *For any character $\theta\colon T^F \to \mathbb{C}^\times$, we have $\mathrm{Ind}_{P^F}^{G^F}(R_T^L(\theta)) \cong R_T^G(\theta)$.*

*Proof.* Let us fix a Borel subgroup $B$ of $G$ such that $T \subset B \subset P$. Note that we can always find such a Borel subgroup. (Indeed, by definition, $P$ contains a Borel subgroup of $G$, say $B'$. Let $T'$ be any maximal torus contained in $B'$. Then, as any two maximal tori of a connected linear algebraic group are conjugate, $T$ and $T'$ are $P$-conjugate, say $T = pT'p^{-1}$. By putting $B := pB'p^{-1}$, we get a desired Borel.) Let $U$ be the unipotent radical of $B$.

We introduce a set $\mathcal{P}$ as follows:

$$\mathcal{P} := \{P' \subset G \mid P' \text{ is a parabolic subgroup of } G \text{ which is } G^F\text{-conjugate to } P\}.$$

Note that we have a bijection $G^F/P^F \xrightarrow{1:1} \mathcal{P}$ given by $y \mapsto yPy^{-1}$ (here, we use a fact that, for any parabolic subgroup $P$, its normalizer group $N_G(P)$ is $P$ itself). Recall that the Deligne–Lusztig variety $\mathcal{X}_{T\subset B}^G$ is defined by

$$\mathcal{X}_{T\subset B}^G := \{x \in G \mid x^{-1}F(x) \in F(U)\}.$$

For each $P' \in \mathcal{P}$, we define a subvariety $\mathcal{X}_{T\subset B}^G(P')$ of $\mathcal{X}_{T\subset B}^G$ by

$$\mathcal{X}_{T\subset B}^G(P') := \{x \in G \mid x^{-1}F(x) \in F(U),\, xPx^{-1} = P'\}.$$

**Claim.** We have $\mathcal{X}_{T\subset B}^G = \bigsqcup_{P' \in \mathcal{P}} \mathcal{X}_{T\subset B}^G(P')$.

*Proof of Claim.* The union on the right-hand side is obviously contained in the left-hand side and also disjoint. Thus it is enough to check the converse inclusion. Let $x \in \mathcal{X}$. Then our task is to show that there exists $P' \in \mathcal{P}$ satisfying $xPx^{-1} = P'$. In other words, it suffices to show that there exists $y \in G^F$ satisfying $xPx^{-1} = yPy^{-1}$. Since we have $x^{-1}F(x) \in F(U) \subset F(B) \subset F(P) = P$, we have an element $z \in P$ such that $x^{-1}F(x) = z$. By applying Lang's lemma to $z \in P$, we can find an element $p \in P$ satisfying $x^{-1}F(x) = p^{-1}F(p)$, or equivalently, $xp^{-1} \in G^F$. Then we have $xPx^{-1} = (xp^{-1})P(xp^{-1})^{-1}$. So $y$ can be taken to be $xp^{-1}$. $\square$

Here, we appeal to a general fact that $B_L := L \cap B$ is a Borel subgroup of $L$ with unipotent radical $L \cap U$. Thus it makes sense to talk about the Deligne–Lusztig variety $\mathcal{X}_{T\subset B_L}^L$ associated to $T \subset B_L \subset L$.

Now let us suppose that $x \in \mathcal{X}_{T\subset B}^G(P')$, where $P' = yPy^{-1}$ with $y \in G^F$. Then, since $xPx^{-1} = P' = yPy^{-1}$, we have $y^{-1}x \in N_G(P) = P$. If we again write $y^{-1}x$ for the image of $yx^{-1}$ in $L$ under the map $P \to P/U_P \cong L$, then we have $(y^{-1}x)^{-1}F(yx^{-1}) = x^{-1}F(x) \in F(U)$, hence $(y^{-1}x)^{-1}F(yx^{-1}) \in L \cap F(U) = F(L \cap U)$. In other words, $yx^{-1}$ belongs to $\mathcal{X}_{T\subset B_L}^L$. Thus we obtain a morphism

$$\mathcal{X}_{T\subset B}^G(P') \to \mathcal{X}_{T\subset B_L}^L : x \mapsto y^{-1}x,$$

which is an isomorphism whose inverse is simply given by $yx \leftarrow\!\shortmid x$.

Therefore, in summary, we get a decomposition

$$\mathcal{X}_{T\subset B}^G = \bigsqcup_{P' \in \mathcal{P}} \mathcal{X}_{T\subset B}^G(P') = \bigsqcup_{y \in G^F/P^F} y\mathcal{X}_{T\subset B_L}^L.$$

It is not difficult to check that this decomposition implies that the representation of $G^F$ realized on $H_c^i(\mathcal{X}_{T\subset B}^G, \overline{\mathbb{Q}}_\ell)$ is nothing but the induced representation of the representation of $P^F$ realized on $H^i_c(\mathcal{X}^L_{T\subset B_L}, \overline{\mathbb{Q}}_\ell)$ (through the map $P^F \to L^F$). By also noting that the above decomposition is equivariant with respect to the right $T^F$-translation action, we conclude that $R^G_T(\theta) \cong \mathrm{Ind}^{G^F}_{P^F}(R^L_T(\theta))$. □

**Definition 1.15.** We say that a $k$-rational maximal torus $T$ of $G$ is *elliptic* if $T$ is not contained in any proper $k$-rational parabolic subgroup of $G$.

**Corollary 1.16.** *If the Deligne–Lusztig representation $R^G_T(\theta)$ contains a cuspidal irreducible constituent, then $T$ must be elliptic.*

*Proof.* Let us suppose that $T$ is not elliptic, hence there exists a proper $k$-rational parabolic subgroup $P$ with a $k$-rational Levi subgroup $L$. Let $\rho$ be any irreducible representation contained in $R^G_T(\theta)$. Then, by the previous proposition, we have $R^G_T(\theta) \cong \mathrm{Ind}^{G^F}_{P^F} R^L_T(\theta)$. This means that there exists an irreducible representation $\rho_L$ of $L^F$ contained in $R^L_T(\theta)$ such that $\rho$ is contained in $\mathrm{Ind}^{G^F}_{P^F} \rho_L$. Hence $\rho$ is not cuspidal. □

Then, how about the converse statement? In fact, when the Deligne–Lusztig representation is irreducible (recall that we call such representation "regular"), the situation is understandable:

**Proposition 1.17.** *Suppose that $S$ is an elliptic $k$-rational maximal torus of $G$. If $\eta\colon S^F \to \mathbb{C}^\times$ is a regular character, then $(-1)^{r_G - r_S} R^G_S(\eta)$ is an irreducible cuspidal representation of $G^F$.*

*Proof.* Recall that, in the proof of the exhaustion theorem, we established a formula

$$\frac{1}{\mathrm{St}_G(s)} \sum_{s\in T\in\mathcal{T}_G} \sum_{\theta\in(T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot R^G_T(\theta) = |(G^F)_s| \cdot \mathbb{1}_{[s]}$$

for any $s \in G^F_{\mathrm{ss}}$. In particular, when $s = 1$, we get

$$\frac{1}{\mathrm{St}_G(1)} \sum_{T\in\mathcal{T}_G} \sum_{\theta\in(T^F)^\vee} (-1)^{r_G - r_T} \cdot R^G_T(\theta) = |G^F| \cdot \mathbb{1}_{\{1\}}.$$

Note that we have $|G^F| \cdot \mathbb{1}_{\{1\}} = \mathrm{Ind}^{G^F}_{\{1\}} \mathbb{1}$. We utilize this formula for any $k$-rational Levi subgroup $L$ of a $k$-rational parabolic subgroup $P$:

$$\frac{1}{\mathrm{St}_L(1)} \sum_{T\in\mathcal{T}_L} \sum_{\theta\in(T^F)^\vee} (-1)^{r_L - r_T} \cdot R^L_T(\theta) = \mathrm{Ind}^{L^F}_{\{1\}} \mathbb{1}.$$

We apply the parabolic induction from $P^F$ to $G^F$ to the both sides. Since we have $\mathrm{Ind}^{G^F}_{P^F}(\mathrm{Ind}^{L^F}_{\{1\}} \mathbb{1}) \cong \mathrm{Ind}^{G^F}_{U^F_P} \mathbb{1}$ (Exercise), the previous proposition implies that

$$\frac{1}{\mathrm{St}_L(1)} \sum_{T\in\mathcal{T}_L} \sum_{\theta\in(T^F)^\vee} (-1)^{r_L - r_T} \cdot R^G_T(\theta) = \mathrm{Ind}^{G^F}_{U^F_P} \mathbb{1}.$$

Since

- there are $|U^F_P|$-many lifts of a $k$-rational maximal torus $T$ of $L$ to a $k$-rational maximal torus of $P$,
- $\mathrm{St}_G(1) = \mathrm{St}_L(1) \cdot |U^F_P|$,
- $r_G = r_L$ (this follows from that $L$ is a $k$-rational Levi),

we get

$$\frac{1}{\mathrm{St}_G(1)} \sum_{T \in \mathcal{T}_P} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot R_T^G(\theta) = \mathrm{Ind}_{U_P^F}^{G^F} \mathbb{1}.$$

Now we prove the cuspidality of the irreducible representation $(-1)^{r_G - r_S} R_S^G(\eta)$. (Recall that the irreducibility follows from the regularity of $\eta$ and the dimension formula.) Our task is to show that, for any proper $k$-rational parabolic subgroup $P$ of $G$, we have

$$\langle (-1)^{r_G - r_S} R_S^G(\eta), \mathrm{Ind}_{U_P^F}^{G_F} \mathbb{1} \rangle = 0.$$

By using the previous decomposition, we have

$$\langle (-1)^{r_G - r_S} R_S^G(\eta), \mathrm{Ind}_{U_P^F}^{G_F} \mathbb{1} \rangle = \frac{1}{\mathrm{St}_G(1)} \sum_{T \in \mathcal{T}_P} \sum_{\theta \in (T^F)^\vee} (-1)^{r_T - r_S} \cdot \langle R_S^G(\eta), R_T^G(\theta) \rangle.$$

By the inner product formula, each summand is given by

$$|\{ w \in W_{G^F}(S, T) \mid {}^w\eta = \theta \}|.$$

However, by the assumption that $S$ is elliptic, $S$ cannot conjugate to any $k$-rational maximal torus $T$ of $P$; in particular, this summand is zero. □

8:49am, November 26, 2024