Deligne–Lusztig theory and the local Langlands correspondence

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What is the local Langlands correspondence?

- **Langlands correspondence** (conj.) predicts a natural connection between automorphic representations and Galois representations.
- **Local Langlands correspondence** (conj.) is the local version of the Langlands correspondece.

- F: a local field of characteristic 0 (i.e., a finite extension of $\mathbb{Q}_p$ or $\mathbb{R}$).
- W_F: the Weil group of F.
- G: a reductive group over F.
- $\hat{G}$: the Langlands dual group of G over $\mathbb{C}$ ($\curvearrowleft W_F$).

G	GL_n	U_n	SO_{2n+1}	Sp_{2n}	SO_{2n}
$\hat{G}$	GL_n	GL_n	Sp_{2n}	SO_{2n+1}	SO_{2n}

- $\Pi(G)$: the set of irreducible admissible representations of $G(F)$.
- $\Phi(G)$: the set of L-parameters of G ($W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G} \rtimes W_F$).

Local Langlands correspondence (rough form)

Local Langlands correspondence (conjectural/rough form)

There exists a "natural" map from $\mathrm{LLC}_G\colon \Pi(G) \to \Phi(G)$.

- The meaning of "natural" is not clear in general.
  ($\rightsquigarrow$ It is not known even how to characterize LLC_G!)
- Known for $F = \mathbb{R}, \mathbb{C}$ due to Langlands ('89).
  ($\rightsquigarrow$ We focus on a p-adic F in the following.)
- $\exists$ various results for specific G:
  - $G = \mathrm{GL}_n$: Harris–Taylor ('01), Hennairt ('00).
  - $G = \mathrm{Sp}_{2n}, \mathrm{SO}_n$: Arthur ('13).
  - $G = \mathrm{U}_n$: Mok ('15).
  - ...
- But $\exists$ still many things to study even in these cases.

Want: an explicit description (construction) of LLC_G.

Regular supercuspidal representations

Another approach to LLC: Instead of specifying a group G, restrict ourselves to some specific class of representations.

- $\Pi_{\mathrm{sc}}(G)$: the set of supercuspidal representations of $G(F)$.
- Kaletha ('19) introduced a subclass $\Pi_{\mathrm{reg}}(G)$ of $\Pi_{\mathrm{tame}}(G)$ (called "regular supercuspidal representations") and parametrized $\Pi_{\mathrm{reg}}(G)$ via "regular pairs" (S, θ).
  - S: a tamely ramified elliptic maximal torus of G over F.
  - $\theta\colon S(F) \to \mathbb{C}^\times$: a character satisfying a certain "regularity" condition.

$$\Pi(G) \supset \Pi_{\mathrm{sc}}(G) \supset \Pi_{\mathrm{reg}}(G) \overset{1:1}{\longleftrightarrow} \{(S, \theta) \mid \text{regular pairs}\}$$
$$\pi_{(S,\theta)} \longleftrightarrow (S, \theta)$$

LLC for regular supercuspidal representations

- Then Kaletha constructed "LLC_G" on $\Pi_{\mathrm{reg}}(G)$:
  - Apply the LLC for S ($\fallingdotseq$ local class field theory) to θ to get the L-parameter $\phi_\theta \colon W_F \to \hat{S} \rtimes W_F$.
  - Apply the "Langlands–Shlestad construction" to get an embedding $\hat{S} \rtimes W_F \hookrightarrow \hat{G} \rtimes W_F$.
  - By composing them, get an L-parameter of G (say ϕ):
  $$\phi \colon W_F \xrightarrow{\phi_\theta} \hat{S} \rtimes W_F \hookrightarrow \hat{G} \rtimes W_F$$
  - Define $\phi := \mathrm{LLC}_G(\pi_{(S,\theta)})$.

Theorem (O.–Tokimoto, 2021)

Kaletha's LLC = Harris–Taylor and Henniart's LLC for $\Pi_{\mathrm{reg}}(\mathrm{GL}_n)$.

Theorem (O., 2023)

Kaletha's LLC = Arthur's LLC for "toral" regular supercuspidal representatations of Sp_{2n} or SO_n when $p \gg 0$.

$\rightsquigarrow$ provides one answer to the problem of explicit LLC!

Deligne–Lusztig theory

Questions

1. The parametrization $(S,\theta) \leftrightarrow \pi_{(S,\theta)}$ is constructive (based on Yu's theory ('01)). Is there a more conceptual way to understand it?
2. Is it possible to establish such a parametrization outside $\Pi_{\mathrm{reg}}(G)$?

We may seek a hint in **Deligne–Lusztig theory**.

- Deligne–Lusztig ('76) constructed all irreducible representations of $\mathbb{G}(\mathbb{F}_q)$ for any reductive group $\mathbb{G}$ over $\mathbb{F}_q$.
  - Take any $\mathbb{F}_q$-rational maximal torus $\mathbb{S}$ of $\mathbb{G}$.
  - Introduce a variety $X_{\mathbb{S}}^{\mathbb{G}} \curvearrowleft \mathbb{G}(\mathbb{F}_q) \times \mathbb{S}(\mathbb{F}_q)$ ("Deligne–Lusztig variety").
  - Take its ℓ-adic cohomology $H_c^i(X_{\mathbb{S}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)$ (repn of $\mathbb{G}(\mathbb{F}_q) \times \mathbb{S}(\mathbb{F}_q)$).
  - Take any character $\theta\colon \mathbb{S}(\mathbb{F}_q) \to \mathbb{C}^\times (\cong \overline{\mathbb{Q}}_\ell^\times)$.
  - Cut & take the alternating sum: $R_{\mathbb{S}}^{\mathbb{G}}(\theta) := \sum_i (-1)^i H_c^i(X_{\mathbb{S}}^{\mathbb{G}}, \overline{\mathbb{Q}}_\ell)[\theta]$.
- Input/output look similar to $(S,\theta) \leftrightarrow \pi_{(S,\theta)}$!

p-adic version of Deligne–Lusztig theory

- Chan–Ivanov ('21) established a p-adic version of the story of Deligne–Lusztig theory.
- To any pair (S, θ) of
  - any unramified F-rational maximal torus S of G,
  - any character $\theta\colon S(F) \to \mathbb{C}^\times$,

  they associated a virtual representation $\pi^{\mathrm{CI}}_{(S,\theta)}$ of $G(F)$.

Both Kaletha–Yu & Chan–Ivanov associate a repn of $G(F)$ to (S, θ). But,

- Kaletha–Yu: S must be tame and elliptic; θ must be regular.
- Chan–Ivanov: S must be unramified, but can be non-elliptic; θ can be any.

Q. Relation between those on the "intersection" of their domains?

Comparison result

Theorem (Chan–O., '21, '24)

Suppose that S is an unramified elliptic maximal torus of G and $\theta\colon S(F) \to \mathbb{C}^{\times}$ is a regular character. Then $\pi_{(S,\theta)} \cong \pi^{\mathrm{CI}}_{(S,\theta)}$ under some assumptions on q (the order of the residue field of F).

Idea of proof.

- $\exists$ explicit character formulas:
  - $\pi_{(S,\theta)}$: Adler–DeBacker ('09), DeBacker–Spice ('18), Spice ('18, '21).
  - $\pi^{\mathrm{CI}}_{(S,\theta)}$: Chan–Ivanov ('21).
- Both are too complicated to be compared in general, but are drastically simplified on "very regular" semisimple elements.
- If $q \gg 0$, we have enough many very regular semisimple elements so that the coincidence of the characters only on such elements implies the coincidence of representations.

$\square$

Future direction

Q. Is there a unified parametrization $(S, \theta) \leftrightarrow \pi_{(S,\theta)}$ which works on the "union" of the domains of Kaletha–Yu and Chan–Ivanov?

$$
\begin{tikzcd}[column sep=small]
 & \{\text{tame } S, \text{ any } \theta\} & \\
\{\text{ur } S, \text{ any } \theta\} \arrow[ur, phantom, sloped, "\subset"] & & \{\text{tame ell } S, \text{ reg } \theta\} \arrow[ul, phantom, sloped, "\supset"] \\
 & \{\text{ur ell } S, \text{ reg } \theta\} \arrow[ul, phantom, sloped, "\subset"] \arrow[ur, phantom, sloped, "\subset"] &
\end{tikzcd}
$$