THEORY OF ALGEBRAIC GROUPS
(2025 FALL @ NTU)

MASAO OI

Contents

2025-12-09 08:53:59+08:00

1. Week 1: Course overview

1.1. **Why algebraic groups?** If you have ever studied the theory of manifolds, you might have encountered the notion of a Lie group. A Lie group is a mathematical object equipped with two different kinds of mathematical structures in a consistent way; the one is a manifold structure, and the other is a group struture. An "algebraic group" is an algebraic version of the notion of a Lie group, where a "manifold structure" is replaced with an "algebraic variety structure".

The theory of algebraic groups is interesting in its own right, but it also plays a very important role in applications. For example, much of modern representation theory is founded on the theory of algebraic groups. Nowadays, theory of algebraic groups has became an indispensable "language" for developing representation theory.

The aim of this course is to learn basics of the theory of algebraic groups, mainly following the textbooks [Bor91, Spr09, Mil17].

1.2. **Algebraic varieties.** Before introducing the definition of an algebraic group, we briefly review the notion of schemes. See any textbook on algebraic geometry for more details, for example, [Har77], [Liu02], etc...

Definition 1.1. For a ring[1] R, we put $\operatorname{Spec} R$ to be the set of all prime ideals of R. We call $\operatorname{Spec} R$ the *spectrum* of R.

Let R be a ring. For any ideal $I \subset R$, we define a subset $V(I)$ of $\operatorname{Spec} R$ by

$$V(I) := \{\mathfrak{p} \in \operatorname{Spec} R \mid I \subset \mathfrak{p}\}.$$

When I is a principal ideal (f) generated by an element $f \in R$, we simply write $V(f)$ instead of $V((f))$. Also, we put $D(f) := \operatorname{Spec} R \smallsetminus V(f)$.

Lemma 1.2.
(1) *For any ideals $I, J \subset R$, we have $V(I) \cup V(J) = V(I \cap J)$.*
(2) *For any family of ideals $\{I_\lambda\}_{\lambda\in\Lambda}$ of R, we have $\bigcap_{\lambda\in\Lambda} V(I_\lambda) = V(\sum_{\lambda\in\Lambda} I_\lambda)$.*
(3) *We have $V(R) = \varnothing$ and $V(0) = \operatorname{Spec} R$.*

Exercise 1.3. Prove this lemma.

The above lemma shows that the family $\{V(I) \mid I \subset R\colon \text{ ideal}\}$ defines a topology on $\operatorname{Spec} R$ such that the closed subsets are the sets of the form $V(I)$. We call the topology on $\operatorname{Spec} R$ defined in this way the *Zariski topology*.

Note that, from the above definition, the closed points of $\operatorname{Spec} R$ are nothing but the maximal ideals of R.

Example 1.4. Let k be an algebraically closed field. We put $\mathbb{A}^n_k := \operatorname{Spec} k[x_1, \ldots, x_n]$ (where $k[x_1, \ldots, x_n]$ is the polynomial ring with n variables over k). Then $\mathbb{A}^n_k$ is called the *n-dimensional affine space* over k.

(1) Let us first consider the subset of closed points of $\mathbb{A}^k_n$. By the Hilbert's Nullstellensatz, any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$ (note that, for this, it is needed that k is algebraically closed).
(2) Let us next consider a closed subset $V(I) \subset \mathbb{A}^n_k$ for an ideal $I = (f_1, \ldots, f_r)$ of R generated by $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$. Let $x \in \mathbb{A}^n_k$ be a closed point corresponding to a maximal ideal $\mathfrak{m} = (x_1 - a_1, \ldots, x_n - a_n)$. Then $x \in V(I)$ if and only if $\mathfrak{m} \supset I$, which is furthermore equivalent to $f_1(a_1, \ldots, a_n) = \cdots = f_r(a_1, \ldots, a_n) = 0$. In other words, the subset of closed points of $V(I)$ is identified with the set of simultaneous solutions to polynomial equations $f_1 = \cdots = f_r = 0$ in k^n.

Definition 1.5. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups (resp. rings) on X is a contravariant functor from the category of open sets of X to the category of abelian groups (resp. rings). More precisely, $\mathcal{F}$ associates an abelian group (resp. a ring) $\mathcal{F}(U)$ to each open set $U \subset X$ such that

(1) $\mathcal{F}(\varnothing) = 0$,

[1] In this lecture, the word "ring" always means a commutative ring with unit.

(2) for any open subsets $V \subset U \subset X$, we have a group homomoprhism (resp. ring homomorphism) $\rho_{U,V} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ (called the *restriction* homomorphism) satisfying
   - $\rho_{U,U} = \mathrm{id}_U$ for any open subset $U \subset X$,
   - $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for any open subsets $W \subset V \subset U \subset X$.

For each open set $U \subset X$, we call an element $s \in \mathcal{F}(U)$ a *section* of $\mathcal{F}$ over U. We write $s|_V$ in short for $\rho_{V,U}(s)$.

Definition 1.6. We say that a presheaf $\mathcal{F}$ on X is a *sheaf* if it satisfies the following conditions:

(1) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i \in I}$, if a section $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.
(2) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i \in I}$, if a family of sections $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ satisfies $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ satisfying $s|_{U_i} = s_i$.

Now we let $X = \operatorname{Spec} R$ for a ring R. Then we can construct a (unique) sheaf of rings $\mathcal{O}_X$ on X such that

- $\mathcal{O}_X(D(f)) = R_f$ for any $f \in R$ (R_f denotes the localization of R with respect to f), and
- for any $f, g \in R$ such that $D(g) \subset D(f)$, the restriction $\rho_{D(f),D(g)} \colon \mathcal{O}_X(D(f)) \to \mathcal{O}_X(D(g))$ is given by the natural homomorphism $R_f \to R_g$ (note that f is invertible in R_g when $D(g) \subset D(f)$).

We call the sheaf $\mathcal{O}_X$ the *structure sheaf* of X.

Definition 1.7. We call the pair $(X = \operatorname{Spec} R, \mathcal{O}_X)$ the *affine scheme* associated to the ring R. We refer to R as the *coordinate ring* of X.

In general, a topological space equipped with a sheaf of rings is called a "ringed space". For ringed spaces, we can define the notion of a morphism. When a ringed space $(X, \mathcal{O}_X)$ is locally isomorphic to affine schemes (more precisely, there exists an open covering $\{U_i\}_{i \in I}$ of X such that each $(X, \mathcal{O}_X)|_{U_i}$ is isomorphic to an affine scheme), we call $(X, \mathcal{O}_X)$ a *scheme*. (We often omit the symbol $\mathcal{O}_X$ of the structure sheaf and simply write "X" for a scheme $(X, \mathcal{O}_X)$.)

Note that, when we have a ring homomorphism $\varphi \colon R \to S$, we can naturally define a continuous map $\varphi^\sharp \colon \operatorname{Spec} S \to \operatorname{Spec} R$ by $\varphi^\sharp(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \operatorname{Spec} S$. This map furthemore naturally induces a morphism between ringed spaces $(X := \operatorname{Spec} S, \mathcal{O}_X) \to (Y := \operatorname{Spec} R, \mathcal{O}_Y)$.

Fact 1.8. *The association $R \mapsto (X = \operatorname{Spec} R, \mathcal{O}_X)$ gives a contravariant equivalence between*

- *the category of rings and*
- *the category of affine schemes.*

The inverse is given by $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$.

When a ring R is a k-algebra, we say that the affine scheme $\operatorname{Spec} R$ is "over k". When X is an affine scheme over k, its coordinate ring (i.e., the ring R when $X = \operatorname{Spec} R$) is often denoted by $k[X]$.

When a scheme is made from affine schemes over k (such that any restriction morphism is a k-algebra homomorphism), we say that the scheme is over k. Any scheme X over k is equipped with a moprhism $X \to \operatorname{Spec} k$; locally, this is a morphism of affine schemes corresponding to the structure morphism $k \to R$ of a k-algebra R. We call $X \to \operatorname{Spec} k$ the "structure morphism" of X.

Definition 1.9. Let k be an algebraically closed field.[2].

(1) When R is a reduced finitely generated k-algebra, we call $\operatorname{Spec} R$ an *affine algebraic variety* over k.
(2) When a scheme X over k has a finite open covering $\{U_i\}_{i\in I}$ such that each U_i is an affine algebraic variety, we call X an *algebraic variety* over k.

As long as k is fixed and there is no confusion, we often omit the word "over k".

1.3. **Definition and examples of algebraic groups.** For any schemes X and Y over k, there uniquely (up to a unique isomorphism) exists their "fibered product" $X \times_k Y$, which is a scheme over k equipped with morphisms $p_1 \colon X \times_k Y \to X$ and $p_2 \colon X \times_k Y \to Y$ over k satisfying the following "universal property":

> for any scheme Z over k equipped with morphisms $q_1 \colon Z \to X$ and $q_2 \colon Z \to Y$ over k, there uniquely exists a morphism $f \colon Z \to X \times_k Y$ over k such that $q_1 = p_1 \circ f$ and $q_2 = p_2 \circ f$.

Note that, when $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ for k-algebras R and S, their fibered product is simply given by $\operatorname{Spec}(R \otimes_k S)$ (the morphisms p_1 and p_2 are given by the natural k-algebra homomorphisms $R \to R \otimes_k S$ and $S \to R \otimes_k S$).

Definition 1.10 (algebraic group)**.** Let G be an algebraic variety over k. We say that G is an *algebraic group over* k if G is equipped with a group structure, i.e., morphisms of schemes over k

- $m \colon G \times_k G \to G$ ("multiplication morphism"),
- $i \colon G \to G$ ("inversion morphism"), and
- $e \colon \operatorname{Spec} k \to G$ ("unit element")

satisfying the axioms of groups. More precisely, the following diagrams are commutative:

$$\begin{tikzcd}
G \times_k G \times_k G \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times m"'] \arrow[dr, phantom, "\circlearrowleft"] & G \times_k G \arrow[d, "m"] \\
G \times_k G \arrow[r, "m"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "\mathrm{id} \times e"] \arrow[d, "e \times \mathrm{id}"'] \arrow[dr, "\mathrm{id}"] & G \times_k G \arrow[d, "m"] \\
G \times_k G \arrow[r, "m"] & G
\end{tikzcd}$$

$$\begin{tikzcd}
G \times_k G \arrow[d, "\mathrm{id} \times i"'] \arrow[dr, phantom, "\circlearrowleft"] & G \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\epsilon"] \arrow[dr, phantom, "\circlearrowleft"] & G \times_k G \arrow[d, "i \times \mathrm{id}"] \\
G \times_k G \arrow[r, "m"] & G & G \times_k G \arrow[l, "m"']
\end{tikzcd}$$

Here, ϵ denotes the composition of the structure morphism $G \to \operatorname{Spec} k$ and $e \colon \operatorname{Spec} k \to G$.

Definition 1.11. Let G and H be algebraic group over k. We say that a morphism $f \colon G \to H$ over k is a *homomorphism* of algebraic groups if the following diagram

[2] In this lecture, for the definition of an algebraic variety, we always assume that k is an algebraically closed field. Also, please be careful that the definition of the word "algebraic variety" heavily depends on textbooks. The definition given here may not be very universal.

is commutative:

$$\begin{tikzcd} G \times_k G \arrow[r, "f \times f"] \arrow[d, "m"'] \arrow[dr, phantom, "\circlearrowleft"] & H \times_k H \arrow[d, "f"] \\ G \arrow[r, "m"] & H \end{tikzcd}$$

Here, the left vertical arrow denotes the multiplication morphism for G and the right one denotes that for H.

Remark 1.12. Suppose that G is an affine algebraic variety with coordinate ring $k[G]$ (i.e., $G = \operatorname{Spec} k[G]$). Recall that the category of affine schemes is equivalent to the category of rings. Thus giving G an algebraic group structure is equivalent to defining k-algebra homomorphisms

- $m\colon k[G] \to k[G] \otimes_k k[G]$,
- $i\colon k[G] \to k[G]$,
- $e\colon k[G] \to k$.

In general, a commutative ring equipped with such an additional structure is called a *Hopf algebra*.

Example 1.13. (1) We put $\mathbb{G}_{\mathrm{a}} := \operatorname{Spec} k[x]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x] \to k[x] \otimes_k k[x]; \quad x \mapsto x \otimes 1 + 1 \otimes x$,
- $i\colon k[x] \to k[x]; \quad x \mapsto -x$,
- $e\colon k[x] \to k; \quad x \mapsto 0$.

Then $\mathbb{G}_{\mathrm{a}}$ is an algebraic group over k with respect to the corresponding morphisms. We call $\mathbb{G}_{\mathrm{a}}$ the *additive group* over k.

(2) We put $\mathbb{G}_{\mathrm{m}} := \operatorname{Spec} k[x, x^{-1}]$ and define m, i, and e at the level of rings as follows:
- $m\colon k[x, x^{-1}] \to k[x, x^{-1}] \otimes_k k[x, x^{-1}]; \quad x \mapsto x \otimes x$,
- $i\colon k[x, x^{-1}] \to k[x, x^{-1}]; \quad x \mapsto x^{-1}$,
- $e\colon k[x, x^{-1}] \to k; \quad x \mapsto 1$.

Then $\mathbb{G}_{\mathrm{m}}$ is an algebraic group over k with respect to the corresponding morphisms. We call $\mathbb{G}_{\mathrm{m}}$ the *multiplicative group* over k.

(3) We put $\mathrm{GL}_n := \operatorname{Spec} k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n]$, where $D := \det(x_{ij})_{1 \leq i,j \leq n}$. We define m, i, and e at the level of rings as follows:
- $m(x_{ij}) := \sum_{k=1}^{n} x_{ik} \otimes x_{kj}$,
- $i(x_{ij}) :=$ the (i, j)-entry of the inverse of the matrix $(x_{ij})_{1 \leq i,j \leq n}$,
- $e(x_{ij}) := \delta_{ij}$ (Kronecker's delta).

Then GL_n is an algebraic group over k with respect to the corresponding morphisms. We call GL_n the *general linear group (of rank n)* over k. (Note that $\mathrm{GL}_1 \cong \mathbb{G}_{\mathrm{m}}$.)

Now we explain a "functorial" viewpoint of algebraic groups, which is more practical.

Let $X = \operatorname{Spec} k[X]$ be an affine scheme over k. We consider a functor $X(-)$ from the category of k-algebras to the category of sets given by

$$X(R) := \mathrm{Mor}_k(\operatorname{Spec} R, X)$$

for any k-algebra R, where $\mathrm{Mor}_k(-, -)$ denotes the set of morphisms of affine schemes over k. Since the category of affine schemes is equivalent to the category of rings, we have

$$\mathrm{Mor}_k(\mathrm{Spec}\, R, X) \cong \mathrm{Hom}_k(k[X], R),$$

where the latter $\mathrm{Hom}_k(-,-)$ denotes the set of k-algebra homomorphisms. In fact, the affine scheme X is determined by the functor $X(-)$. Therefore, we may regard the affine scheme X as a "machine" which associate to each k-algebra R a set $X(R)$ in a functorial way. (More precisely, the association $X \mapsto X(-)$ gives a fully faithful functor from the category of affine schemes over k to the category of functors from the category of affine schemes over k to the category of sets; this is so-called "Yoneda's lemma".)

We call an element of $X(R)$ an *R-valued point* or an *R-rational point* of X.

Example 1.14. Let $X = \mathrm{Spec}\, k[x,y]/(y^2 - x^3)$. Then, for any k-algebra R, we have

$$X(R) \cong \mathrm{Hom}_k(k[x,y]/(y^2 - x^3), R).$$

Note that, any k-algebra homomorphism f from $k[x,y]/(y^2 - x^3)$ to R is uniquely detemined by the images $f(x), f(y) \in R$ of x, y. Since x and y satisfies the equation $y^2 - x^3 = 0$ in the coordinate ring $k[x,y]/(y^2 - x^3)$, their images must satisfy $f(y)^2 - f(x)^3 = 0$. Conversely, for any elements $(a,b) \in R^2$ satisfying the equation $b^2 - a^3 = 0$, we can define a k-algebra homomorphism $f\colon k[x,y]/(y^2 - x^3) \to R$ by $f(x) = a$ and $f(y) = b$. Therefore, we get

$$X(R) \cong \mathrm{Hom}_k(k[x,y]/(y^2 - x^3), R) \cong \{(a,b) \in R^2 \mid b^2 - a^3 = 0\}.$$

In other words, we can think of X as a machine which associates to each R the set of solutions to the equation $y^2 - x^3 = 0$ in R^2.

Now let G be an algebraic group over k. Then the multiplication morphism $m\colon G \times_k G \to G$ induces a map $m_R\colon G(R) \times G(R) \to G(R)$ for each k-algebra R. Indeed, let $g_1, g_2 \in G(R) = \mathrm{Mor}_k(\mathrm{Spec}\, R, G)$. Then we can define an element $m_R(g_1, g_2) \in G(R)$ to by

$$m_R(g_1, g_2)\colon\ \mathrm{Spec}\, R \xrightarrow{(g_1,g_2)} G \times_k G \xrightarrow{m} G.$$

(Here, (g_1, g_2) denotes the morphism induced from g_1 and g_2 by the universal property of the fibered product $G \times_k G$.) Similarly, we also have a map $i_R\colon G(R) \to G(R)$ induced by i. Furthermore, the unit morphism $e\colon \mathrm{Spec}\, k \to G$ induces an element $e_R \in G(R)$ given by $e_R\colon \mathrm{Spec}\, R \to \mathrm{Spec}\, k \xrightarrow{e} G$, where the first arrow is the structure morphism for $\mathrm{Spec}\, R$. Then, it can be easily checked that the axiom of an algebraic group implies that $G(R)$ is a group in the usual sense with respect to the map m_R with inversion map i_R and unit element e_R. As a result, $G(-)$ gives a functor from the category of k-algebras to the category of groups.

Example 1.15. (1) For a k-algebra R, we have $\mathbb{G}_\mathrm{a}(R) \cong R$, where the group structure on R is given by the additive structure of R. Indeed, we have

$$\mathbb{G}_\mathrm{a}(R) = \mathrm{Mor}_k(\mathrm{Spec}\, R, \mathbb{G}_\mathrm{a}) \cong \mathrm{Hom}_k(k[x], R) \cong R,$$

where the last map is given by $f \mapsto f(x)$. The multiplication map m_R induced on $\mathbb{G}_\mathrm{a}(R)$ corresponds to the addition on R. Indeed, let us take any elements $g_1, g_2 \in \mathbb{G}_\mathrm{a}(R)$, hence $m_R(-,-)$ is given by the composition

$$m_R(g_1, g_2)\colon\ \mathrm{Spec}\, R \xrightarrow{(g_1,g_2)} G \times_k G \xrightarrow{m} G.$$

At the level of rings, this amounts to the composition

$$k[x] \xrightarrow{m} k[x] \otimes_k k[x] \xrightarrow{g_1 \otimes g_2} R.$$

Since $m(x) = x \otimes 1 + 1 + \otimes x$ by definition, we get

$$(g_1 \otimes g_2) \circ m(x) = (g_1 \otimes g_2)(x \otimes 1 + 1 + \otimes x) = g_1(x) + g_2(x).$$

This is why $\mathbb{G}_{\mathrm{a}}$ is called the "additive group".

(2) For a k-algebra R, we have $\mathbb{G}_{\mathrm{m}}(R) \cong R^\times$, where $R^\times$ denotes the unit group of R with respect to the multiplicative structure of R. Indeed, we have

$$\mathbb{G}_{\mathrm{m}}(R) = \mathrm{Mor}_k(\operatorname{Spec} R, \mathbb{G}_{\mathrm{m}}) \cong \mathrm{Hom}_k(k[x, x^{-1}], R) \cong R^\times,$$

where the last map is given by $f \mapsto f(x)$. In a similar manner to above, we can check that the multiplication map m_R on $\mathbb{G}_{\mathrm{m}}(R)$ corresponds to the multiplication on $R^\times$. This is why $\mathbb{G}_{\mathrm{m}}$ is called the "multiplicative group".

(3) For a k-algebra R, we have

$$\mathrm{GL}_n(R) \cong \{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}.$$

Indeed, by definition, we have

$$\mathrm{GL}_n(R) = \mathrm{Mor}_k(\operatorname{Spec} R, \mathrm{GL}_n) \cong \mathrm{Hom}_k(k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n], R).$$

The right-hand side is isomorphic to (at least as sets) $\{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}$ by the map $f \mapsto (f(x_{ij}))_{i,j}$. It is a routine work to check that this bijection is indeed a group isomorphism.

(4) The *symplectic group* Sp_{2n} is an affine algebraic group such that the group of its R-valued points is given as follows:

$$\mathrm{Sp}_{2n}(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_{2n}(R) \mid {}^t g J_{2n} g = J_{2n}\},$$

where J_{2n} denotes the antidiagonal matrix whose antidiagonal entries are given by 1 and -1 alternatively:

$$J_{2n} := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}.$$

Here, we don't explain how to define the coordinate ring of Sp_{2n} and also how to introduce the group structure at the level of the coordinate ring. Only the important viewpoint here is what kind of groups are associated as the groups of R-valued points! So, in this course, let us just believe that the functor Sp_{2n} is indeed *representable*, i.e., realized as the functor of points of some affine algebraic groups. This remark is always applied to any affine algebraic group which we will encounter in the future.

2. Week 2: Very basic properties of general algebraic groups

Recall that, in general, a *scheme* X is a topological space equipped with a sheaf of rings $\mathcal{O}_X$ ("structure sheaf") which is locally isomorphic to affine schemes ("$\operatorname{Spec} A$" for a commutative ring A).

In the following, we let k be an algebraically closed field. Also, when we say "an algebraic variety", it always means "an algebraic variety over k". Here, recall that we say that a scheme X is an algebraic variety over k if it is locally isomoprhic to $\operatorname{Spec} A$ for a finitely generated reduced k-algebra (hence, in particular, A is of the form $k[x_1, \ldots, x_n]/I$ for an ideal I of $k[x_1, \ldots, x_n]$).

For any algebraic variety X over k, the subset of closed points of X can be identified with the set $X(k)$ of k-rational points of X; for any k-rational point $\operatorname{Spec} k \to X$, the image of the unique point of $\operatorname{Spec} k$ is a closed point of X, and vice versa. From now on, we freely identify the set of closed points of X with $X(k)$. Moreover, the subset of closed points of X is dense in X because k is algebraically closed. (Both these facts are consequences of Hilbert's "nullstellensatz", which asserts that any maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $(x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$; this fact assumes that k is algebraically closed.)

2.1. Identity component subgroup.

Let G be an algebraic group over k. Recall that, in particular, G is equipped with a unit element $e \in G(k)$. Let G° denote the connected component of G containing the closed point e.

Proposition 2.1. *The subset G° is a subgroup of G. Moreover, G° is normal of finite index in G.*

Proof. We have to show that G° is closed under the multiplication morphism $m\colon G \times G \to G$ and the inversion morphism $i\colon G \to G$. More precisely, our task is to check that $m(G^\circ, G^\circ) \subset G^\circ$ and $i(G^\circ) \subset G^\circ$. But both statements follow by combining a general fact that the image of a connected set under a continuous map is again connected with that $m(e,e) = e$ and $i(e) = e$.

To show the second assertion, let us take $g \in G(k)$. (By definition, being normal means that $gG^\circ g^{-1} \subset G^\circ$ for any $g \in G(k)$.) Then it can be easily checked that $gG^\circ g^{-1}$ is a subgroup of G which is connected and contains the unit element. Hence we get $gG^\circ g^{-1} \subset G^\circ$. The finite-index property follows from that the set of connected components of an algebraic variety is finite. □

Definition 2.2. We call the algebraic subgroup G° of G the *identity component* of G.

2.2. Smoothness of algebraic groups.

Let us first look at the following example: we consider an affine algebraic variety $X := \operatorname{Spec} k[x,y]/(y^2 - x^3)$, i.e., X is the spectrum of the quotient ring of $k[x,y]$ by the ideal generated by $(y^2 - x^3)$. Recall that, X represents the space of solutions to the equation $y^2 - x^3 = 0$. More precisely, for any k-algebra R, the set $X(R)$ of R-rational points of X is equal to the set of solutions to $y^2 - x^3 = 0$ in R. If we try to draw a picture of the set $X(\mathbb{R}) \subset \mathbb{R}^2$, then we can immediately notice that the resulting curve is "smooth" except for the origin $(0,0)$; at the origin, the curve has a "singular point"[3].

[3]Because we assume k is algebraically closed in this lecture, it's not actually allowed to take R to be $\mathbb{R}$. If you want to be rigorous please take the coefficient k to be any smaller field, for example, $\mathbb{Q}$.

In fact, the difference between the point $(0, 0)$ and the other points in this example can be explained in terms of ring-theoretic properties of the coordinate ring $k[x, y]/(y^2 - x^3)$. Let us explain how to introduce the notion of a "smooth point" and also a "singular point" for general schemes in the following.

Let X be a scheme. For any point $x \in X$, we define a ring $\mathcal{O}_{X,x}$ by

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U),$$

where the inductive limit is over open sets U of X containing $x \in X$ (the structure morphisms are given by the restriction maps $\mathcal{O}_X(V) \to \mathcal{O}_X(U)$ for any $x \in V \subset U$). This ring is a local ring and called the *stalk* of X at $x \in X$. If $x \in X$ is contained in an affine open subscheme $U \subset X$ isomorphic to $\operatorname{Spec} A$, where x is identified with a prime ideal $\mathfrak{p}$ of A, then the stalk $\mathcal{O}_{X,x}$ is nothing but the localization $A_{\mathfrak{p}}$ of A with respect to $\mathfrak{p}$.

For any $x \in X$, we write $\mathfrak{m}_x$ for the unique maximal ideal of the stalk $\mathcal{O}_{X,x}$. We put $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ and call $\kappa(x)$ the *residue field* of X at $x \in X$.

Definition 2.3. Let X be an algebraic variety over k.

(1) We say that a point $x \in X$ is *smooth* if the local ring $\mathcal{O}_{X,x}$ of X at x is a regular local ring, i.e., we have

$$\dim(\mathcal{O}_{X,x}) = \dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

Here, the left-hand side denotes the Krull dimension of the ring $\mathcal{O}_{X,x}$ and the right-hand side denotes the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$-vector space.

(2) We say that X is *smooth* if every point of X is smooth.

Fact 2.4. *Let X be an algebraic variety over k. Then the subset of smooth points of X is open dense in X.*

The subset of smooth point of X is often referred to as the *smooth locus* of X.

Proposition 2.5. *Let G be an algebraic group over k. Then G is smooth.*

Proof. Let U be the smooth locus of G, which is open dense in G by the above fact. Let us show that any closed point g of G is contained in U. If we can show this, then the assertion follows. Indeed, the complement $G \setminus U$ is a closed subset of G; if this is not empty, then it contains at least one closed point of G, hence a contradiction.

Firstly, U contains at least one closed point g_0 of G because, otherwise, $G \setminus U$ is a closed subset of G containing all closed points, hence equal to G by the density of closed points. Next, for any closed point g of G, we consider the (gg_0^{-1})-multiplication morphism

$$G \to G : x \mapsto gg_0^{-1}x.$$

(Precisely speaking, for any $h \in G(k)$, the h-multiplication morphism is defined to be the composition $G \cong \operatorname{Spec} k \times_k G \to G \times_k G \to G$, where the second arrow is the fibered product of $h : \operatorname{Spec} k \to G$ and id_G and the last arrow is the multiplication morphism of G. At the level of k-rational points, this realizes the intuitive map $x \mapsto hx$.) Then, because this is an isomoprhism of algebraic varieties, any smooth point is mapped to a smooth point. In particular, g, which is the image of the smooth point g_0, is also smooth. Thus U contains g. $\square$

Remark 2.6. The word "smooth" usually means a property of a morphism of schemes $f\colon X \to Y$; the definition introduced above is usually referred to as the regularity (non-singularity) of X (at x), which is an "absolute" notion depending only on X. When $Y = \operatorname{Spec} k$ (where k is an algebraically closed field), the smoothness for the morphism f is equivalent to the regularity (non-singularity) of X. In general, we must be careful about the difference between the regularity and the smoothness; see, e.g., [Mil17, §1.b].

2.3. **Homomorphism between algebraic groups.** Let us investigate a homomorphism between algebraic groups over k.

Proposition 2.7. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then the image $\alpha(G)$ is a closed subgroup of G'.*

To show this proposition, let us first review some general notions for topological spaces.

Definition 2.8. Let X be a topological spaces.

(1) We say that a subset Z of X is *locally closed* if Z is an intersection of an open subset of X and a closed subset of X.
(2) We say that a subset Z of X is *constructible* if Z is a finite union of locally closed subsets of X.
(3) We say that X is *noetherian* if any open subset of X is quasi-compact.

Remark 2.9. In the above definition, the word "quasi-compact" just means "compact", i.e., any open covering has a finite subcovering. This is because, sometimes (depending on areas), the word "compact" is used to mean "Hausdorff and compact". In the context of algebraic geometry, we often use the word "quasi-compact" to emphasize that the Hausdorff property is not assumed.

The following fact is a general nonsense on topological spaces:

Lemma 2.10. *Let X be a noetherian topological space. Let Y be a constructible subset of X. Then Y contains an open dense subset of its closure $\overline{Y}$ in X.*

Exercise 2.11. Prove the above lemma.

Note that, any algebraic variety over k is a noetherian topological space, hence the above lemma can be applied.

On the other hand, the following fact is much deeper:

Fact 2.12. *Let $f\colon X \to Y$ be a morphism between algebraic varieties over k. Then the image of any constructible subset under f is a constructible subset of Y.*

Let us utilize these facts to deduce some useful facts on algebraic groups.

Lemma 2.13. *Let G be an algebraic group over k. Then, for any open dense sets U and V of G, we have $U \cdot V = G$, where we put $U \cdot V := \{u \cdot v \in G \mid u \in U, v \in V\}$.*

Proof. It is enough to show that the open subset $U \cdot V$ contains every closed point g of G. Let $g \in G$ be a closed point. Then both U and $g \cdot V^{-1}$ are dense open subsets of G, hence have a nonempty open intersection. By the density of closed points, there exists a closed point in $U \cap (g \cdot V^{-1})$. In other words, there exists closed points $u \in U$ and $v \in V$ satisfying $u = hv^{-1}$, hence $h = uv \in U \cdot V$. $\square$

Proposition 2.14. *Let G be an algebraic group over k. Then any constructible subgroup H of G is closed.*

Proof. By Lemma 2.10, H contains an open dense subset U of its closure $\overline{H}$ in G. Since H is a subgroup of G, we obtain

$$U \cdot U \subset H \cdot H \subset H.$$

By the above lemma, we have $U \cdot U = \overline{H}$, hence $H = \overline{H}$. □

Corollary 2.15. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then the image $\alpha(G)$ is a closed subgroup of G'.*

Proof. By Fact 2.12, $\alpha(G)$ is a constructible subset of G'. Since $\alpha(G)$ is a subgroup of G', the above proposition implies that $\alpha(G)$ is closed. □

Remark 2.16. The notion of a "kernel" in the context of algebraic groups is quite subtle. Scheme-theoretically, the kernel of α is defined to be the fibered product of $\alpha\colon G \to G'$ and $e'\colon \operatorname{Spec} k \to G'$, where e' denotes the unit element of G'. However, the problem is that this fibered product is not necessarily reduced, hence not necessarily an algebraic variety in our sense. For example, consider the morphism $\mathbb{G}_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m}}\colon x \mapsto x^p$ for the multiplicative group defined over an algebraically closed field k of characteristic $p > 0$. Then, as "points", its kernel is equal to $\mu_p(k) := \{x \in k \mid x^p = 1\} = \{1\}$. However, the fibered product is isomorphic to $\operatorname{Spec} k[x]/(x-1)^p$, which is not reduced. This observation suggests that, for a better treatment of algebraic groups, we should work with more general notion of group schemes.

2.4. **Dimension of algebraic groups.**

Definition 2.17. Let X be an algebraic variety. We say that a closed subset Y of X is *irreducible* if Y is non-empty and cannot be written as $Y = Z_1 \cup Z_2$ for non-empty proper closed subsets $Z_1, Z_2 \subsetneq Y$. We call a maximal irreducible subset of X an *irreducible component* of X.

Definition 2.18. For an algebraic variety X, we define the *dimension* $\dim X$ of X to be the maximum of the length d of a chain

$$Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d$$

of irreducible subsets of Y_d.

In fact, the dimension of an algebraic variety is related to the Krull dimension of its stalks in the following sense: let Y be an irreducible component of X. Then, for any $x \in X$, we have $\dim \mathcal{O}_{X,x} = \dim Y$.

Fact 2.19. *Let $\alpha\colon G \to G'$ be a homomorphism between algebraic groups over k. Then we have*

$$\dim G = \dim \operatorname{Ker}(\alpha) + \dim \alpha(G).$$

Here, as noted above, $\alpha(G)$ is a closed subgroup of G while $\operatorname{Ker}(\alpha)$ is not in general because it might not be reduced. So the (ad hoc) meaning of "$\operatorname{Ker}(\alpha)$" is that it is the set-theoretic preimage of the unit element $e' \in G'$ under α. Since α is continuous and e' is closed, the preimage is closed in G, hence it makes sense to talk about its dimension.

For the proof of this fact, see [Mil17, Proposition 1.63].

2.5. **Algebraic group action on algebraic varieties.**

Definition 2.20. Let G be an algebraic group over k and X an algebraic variety over k. We say that G *acts on* X if there exists a morphism of algebraic varieties $\alpha\colon G \times X \to X$ satisfying the usual axioms of group actions, i.e., the following diagrams are commutative:

$$\begin{tikzcd} G \times_k G \times_k X \arrow[r, "m \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times \alpha"'] \arrow[dr, phantom, "\circlearrowleft"] & G \times_k X \arrow[d, "\alpha"] \\ G \times_k X \arrow[r, "\alpha"] & G \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "\mathrm{id}"'] & G \times_k X \arrow[d, "m"] \arrow[dl, phantom, "\circlearrowleft"] \\ & X \end{tikzcd}$$

We can also consider the usual notion on the group action such as normalizer, stabilizer, and so on, in the context of algebraic groups.

Proposition-Definition 2.21. *Suppose that an algebraic group G acts on an algebraic variety X.*

(1) *For any closed subvarieties Y and Z of X, there exists a closed subvariety $N_G(Y,Z)$ satisfying*
$$N_G(Y,Z)(R) = N_{G(R)}(Y(R), Z(R)) := \{n \in G(R) \mid nY(R) \subset Z(R)\}$$
for any k-algebra R. We call $N_G(Y,Z)(R)$ the transporter from Y to Z in G.

(2) *When $Y = Z$, we call the transporter $N_G(Y,Y)$ the normalizer of Y in Z and write $N_G(Y) := N_G(Y,Y)$. Note that the normalizer is a subgroup of G.*

(3) *When Y consists of a single closed point $x \in X$, we call the normalizer group $N_G(\{x\})$ the stabilizer group of x in G and write $G_x := N_G(\{x\})$. More generally, for any closed subvariety $Y \subset X$, we put $G_Y := \cap_{x\in Y} G_x$.*[4]

The subtle point of the above definition is that, so that the resulting "subfunctor" $N_G(Y,Z)$ is indeed given by a "subvariety" (more naively speaking, the subset $\{n \in G \mid nY \subset Z\}$ has a natural subscheme structure), we need to assume that the subsets Y and Z are *closed* subvarieties of G. See [Mil17, 1.79] for the details.

Proposition 2.22 ("Closed orbit lemma")**.** *Let G be an algberaic group acting on an algebraic variety X. For any closed point $x \in X$, let Gx denote its orbit.*

(1) *Each Gx is a smooth variety which is open in its closure $\overline{Gx}$ in X.*

(2) *The boundary $\overline{Gx} \smallsetminus Gx$ is a union of orbits of strictly smaller dimension.*

Proof. Note that $G\cdot x$ is (by definition) the image of the morphism $G \to X\colon g \mapsto gx$. Using the fact that the image of any constructible set is again constructible (Fact 2.12), we see that Gx contains a dense open subset U of its closure $\overline{Gx}$. Here note that both Gx and $\overline{Gx}$ are stable under the G-action. In particular, we have
$$Gx = \bigcup_{g\in G(k)} gU.$$
(Precisely speaking, we first see that the closed points contained in $\bigcup_{g\in G(k)} gU$ are the same as those of Gx. Then, by the density of closed points, we get the equality

[4] When $X = G$ and the action of G on X is the conjugation, we call the stabilizer G_X the *centralizer* of X in G.

as subvarieties.) Each gU is open in $\overline{Gx}$, hence this equality implies that Gx is open in $\overline{Gx}$. The smoothness follows from the same argument as in the proof of the smoothness of algebraic groups, i.e., use the open-density of the smooth locus and that G acts on Gx transitively.

It can be easily checked that any dense open subset of a noetherian space intersects every irreducible component. In particular, the boundary $\overline{Gx} \setminus Gx$ does not contain any irreducible component $\overline{Gx}$. In other words, $\overline{Gx} \setminus Gx$ is a closed subset of $\overline{Gx}$ of strictly smaller dimension. Since $\overline{Gx} \setminus Gx$ is G-stable, it can be written as the union of its G-orbits. $\square$

Corollary 2.23. *Let G be an algberaic group acting on an algebraic variety X. Then any G-orbit of minimal dimension is closed. In particular, X always has a closed G-orbit.*

Proof. If the dimension of a G-orbit Gx is minimal, then the boundary $\overline{Gx} \setminus Gx$ must be empty by the above proposition. Hence Gx is closed. $\square$

Example 2.24. A typical example of the application of the above proposition is the following. Let $G = \mathrm{GL}_n$. We consider $\mathcal{N} := \{N \in M_n \mid (N - I_n)^r = 0 \text{ for some } r \in \mathbb{Z}_{\leq 0}\}$. In other words, $\mathcal{N}$ is an algebraic subvariety of $M_n \cong \mathbb{A}_k^{n^2}$ (the affine space of n-by-n matrices) consisting of nilpotent matrices. Then G acts on $\mathcal{N}$ via conjugation. By the theory of Jordan normal form, each nilpotent G-orbit corresponds to a partition of n. It is known that the "closure relation" on $\mathcal{N}$ (i.e., when a G-orbit Gx is contained in the closure of another G-orbit $\overline{Gy}$) can be described in terms of the combinatorics on the partition of n.

3. Week 3: Jordan decomposition

As before, we let k be an algebraically closed field. When we say "an algebraic variety", it always means "an algebraic variety over k".

3.1. Linear algebraic groups.

The first goal of this week is to show the following.

Proposition 3.1. *Let G be an algebraic group. Then G is affine if and only if there exists a closed immersion $G \hookrightarrow \mathrm{GL}_n$ for some $n \in \mathbb{Z}_{>0}$.*

Here let us give some comments on the notion of a "closed immersion" of schemes. When we say that a map between topological spaces $f\colon X \to Y$ is a "closed immersion", we usually mean that f is a homeomorphism onto a closed subset of Y. In the context of schemes, a "morphism $f\colon X \to Y$ between schemes" is defined to be a pair of a continuous map $f\colon X \to Y$ between the underlying topological spaces and a morphism of sheaves $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y. Here, $f_*\mathcal{O}_X$ is the sheaf of rings on Y defined by $f_*\mathcal{O}_X(V) := \mathcal{O}_X(f^{-1}(V))$ for any open subset V of Y (called the direct image of $\mathcal{O}_X$ by f). Such a morphism f is said to be a closed immersion if it is a closed immersion as a map between topological spaces and also the map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves is surjective (i.e., surjective on each open set $V \subset Y$).

Recall that, when schemes X and Y are affine (say $X = \operatorname{Spec} S$ and $Y = \operatorname{Spec} R$), any morphism $f\colon X \to Y$ corresponds to a ring homomorphism $f^*\colon R \to S$ (although I have not explained how to associate f to f^* in this lecture). In fact, in this case, f is a closed immersion of schemes if and only if the ring homomorphism is surjective. See, for example, [Liu02, §2.3.2] for more details.

Note that the general linear group GL_n is affine by definition. In general, any closed subscheme of an affine scheme is again affine (see [Liu02, Proposition 3.20]). So what is really nontrivial in Proposition 3.1 is that its converse (any affine algebraic group G can be embedded into GL_n for some $n \in \mathbb{Z}_{>0}$) also holds once the group structure is given. For this reason, we call any affine algebraic group *a linear algberaic group*.

3.2. Translation on coodinate rings.

Before we start proving the above proposition, we investigate the notion of the "translation" action induced by an action of an algberaic group. Let G be an affine algebraic group with coordinate ring $k[G]$. Suppose that G acts on an affine algebraic variety X with coordinate ring $k[X]$. Let $\alpha\colon G \times_k X \to X$ be the morphism defining the action and

$$\alpha^*\colon k[X] \to k[G] \otimes_k k[X]$$

the corresponding k-algebra homomorphism.

Let $g \in G(k)$; recall that, scheme-theoretically, this is a morphism $g\colon \operatorname{Spec} k \to G$, which corresponds to a k-algebra homomorphism $g^*\colon k[G] \to k$. We define a k-algebra endomorphism λ_g on $k[X]$ to be $((g^{-1})^* \otimes \mathrm{id}) \circ \alpha^*$:

$$\lambda_g\colon k[X] \xrightarrow{\alpha^*} k[G] \otimes_k k[X] \xrightarrow{(g^{-1})^*\otimes\mathrm{id}} k[X]$$

Then this defines an action $\lambda_{(-)}$ of the group $G(k)$ on the k-vector space $k[X]$. We call this action the *left translation action* of $G(k)$ on $k[X]$.

Remark 3.2. (1) Note that the "action" of $G(k)$ on $k[X]$ is not in the sense of algebraic group actions defined in the last week since $k[X]$ is an inifinite-dimensional k-vector space. (But, of course, if you look at a finite-dimensional

$G(k)$-stable subspace, then you can ask if the action is algebraic or not when the finite-dimensional subspace is regarded as an affine space.)

(2) The morphism $X \to X$ induced by the k-algebra endomorphism λ_g is nothing but the left-translation by g^{-1}, i.e., at the level of R-rational points for any k-algebra,

$$X(R) \to X(R); \quad x \mapsto g^{-1} \cdot x,$$

where the dot denotes the induced action of $G(k) \subset G(R)$ on $X(R)$.

(3) One important perspective is to look at the coordinate ring $k[X]$ of an affine algebraic variety X as a set of functions on X. To be more precise, for any $f \in k[X]$ and a closed point $x \in X(k)$ which corresponds to $x^* \colon k[X] \to k$, we put $f(x) := x^*(f)$. Then f is regarded as a k-valued function on $X(k)$. With this viewpoint, the left-translation action of $G(k)$ on $k[X]$ can be literally thought of as the "left-translation", i.e., for any $f \in k[X]$, we have

$$(\lambda_g(f))(x) = f(g^{-1} \cdot x)$$

for any $x \in X(k)$.

Lemma 3.3. *Let V be a finite-dimensional subspace of $k[X]$. Then,*

(1) *V is $G(k)$-stable if and only if $\alpha^*(V) \subset k[G] \otimes_k V$;*

(2) *there exists a V' be a $G(k)$-stable finite-dimensional subspace of $k[X]$ such that $V \subset V'$.*

Proof. Let us first show (1). We fix a k-basis $\{f_1 \dots, f_r\}$ of V and extend it to a k-basis $\{f_1 \dots, f_r, f_{r+1}, \dots\}$ of $k[X]$. For any $f \in k[X]$, we write $\alpha^*(f) \in k[G] \otimes_k k[X]$ as

$$\alpha^*(f) = \sum_i h_i \otimes f_i,$$

where $h_i \in k[G]$. Then we have

$$\lambda_g(f) = ((g^{-1})^* \otimes \mathrm{id}) \circ \alpha^*(f) = \sum_i (g^{-1})^*(h_i) \cdot f_i = \sum_i h_i(g^{-1}) \cdot f_i$$

Now suppose that V is $G(k)$-stable. Then, for any $f \in V$ and any $g \in G(k)$, we have $\sum_i h_i(g^{-1}) \cdot f_i \in V$. In particular, we have $h_i(g^{-1}) = 0$ for any $i > r$. Since this holds for any $g \in G(k)$, we get $h_i = 0$ for any $i > r$, which means that $\alpha^*(f) \in k[G] \otimes_k V$. Conversely, let us suppose that $\alpha^*(V) \subset k[G] \otimes_k V$. Then, for any $f \in V$, we have $\alpha^*(f) = \sum_{i=1}^r h_i \otimes f_i$. Hence, again by the computation as above, we get that $\lambda_g(f) \in V$.

We next show (2). It is enough to consider only the case where V is one-dimensional, which is generated by $f \in k[X]$. Let us write

$$\alpha^*(f) = \sum_{i=1}^s h_i \otimes f_i,$$

where $h_i \in k[G]$ and $f_i \in k[X]$, hence

$$\lambda_g(f) = \sum_{i=1}^s h_i(g^{-1}) \cdot f_i$$

for any $g \in G(k)$. This implies that the subspace

$$\mathrm{Span}_k\{\lambda_g(f) \in k[X] \mid g \in G(k)\}$$

of $k[X]$ is at least contained in $\mathrm{Span}_k\{f_1, \dots, f_s\}$, hence must be finite-dimensional. Since $\mathrm{Span}_k\{\lambda_g(f) \in k[X] \mid g \in G(k)\}$ is obviously $G(k)$, we can choose V' to be this subspace.

□

Now we consider the case where $X = G$. In this case, we can consider two kinds on actions of G on $X = G$; $g \cdot x = gx$ and $g \cdot x = xg^{-1}$. As above, we call the action of $G(k)$ on $k[G]$ induced by the former action ($g \cdot x = gx$) the *left-translation* and write λ_g. We call the action $G(k)$ on $k[G]$ induced by the latter one ($g \cdot x = xg^{-1}$) the *right-translation* and write ρ_g. If we regard elements of $k[G]$ as functions on $G(k)$, these actions can be described as

$$\lambda_g(f)(x) = f(g^{-1}x), \quad \rho_g(f)(x) = f(xg).$$

Proposition 3.4. *Any finite-dimensional subspace of $k[G]$ is contained in a finite-dimensional subspace of $k[G]$ stable under both left and right translations.*

Proof. This immediately follows by applying Lemma 3.3 to the action of $G \times G$ on G given by $(g, h) \cdot x = gxh^{-1}$. □

Remark 3.5. In general, we say that a representation $G(k)$ on a k-vector space is *locally finite* if any element $v \in V$ is contained in a $G(k)$-stable finite-dimensional subspace of V. By the discussion so far, for any affine algebraic group G, the representation (either left-translation or right-translation) of $G(k)$ on the coordinate ring $k[G]$ is locally finite.

3.3. **Proof of the linearity of affine algebraic groups.** Now let us prove Proposition 3.1.

Proof of Proposition 3.1. Let G be an affine algebraic group with coordinate ring $k[G]$. We let α be the action of G on G given by $g \cdot x := xg^{-1}$.

Since $k[G]$ is finitely generated, we can write $k[G] = k[f_1, \dots, f_n]$ with $f_i \in k[G]$. Here, by Lemma 3.3, we may assume that $V := \mathrm{Span}_k\{f_1, \dots, f_n\}$ is $G(k)$-stable. (Indeed, otherwise, by applying Lemma 3.3 to the subspace V, we can find a $G(k)$-stable finite-dimensional subspace V' which contains V, hence generates $k[G]$ as a k-algebra. Hence, it is enough to replace $f_1, \dots, f_n$ with any k-basis of V'.)

As V is $G(k)$-stable, $\alpha^*(V) \subset k[G] \otimes_k V$ (Lemma 3.3). Note that, this is equivalent to that $m^*(V) \subset V \otimes_k k[G]$, where m denotes the multiplication morphism $G \times_k G \to G$. Indeed, this can be checked by noting that the following diagram is commutative, where i is the inversion morphism and sw denotes the morphism swapping the first and second entries:

$$\begin{tikzcd} G \times G \arrow[r, "\alpha"] \arrow[d, "i \times \mathrm{id}"'] & G \\ G \times G \arrow[r, "\mathrm{sw}"'] & G \times G \arrow[u, "m"'] \end{tikzcd} \qquad \begin{tikzcd} (g, x) \arrow[r, mapsto] \arrow[d, mapsto] & xg^{-1} \\ (g^{-1}, x) \arrow[r, mapsto] & (x, g^{-1}) \arrow[u, mapsto] \end{tikzcd}$$

Hence, for each $1 \le i \le n$, we can write

$$m^*(f_i) = \sum_{j=1}^{n} f_j \otimes h_{ji} \in V \otimes_k k[G]$$

with $h_{ji} \in k[G]$ $(j = 1, \dots, n)$. Note that then, for each $g \in G(k)$, we have

$$\begin{aligned}\rho_g(f_i) &= ((g^{-1})^* \otimes \mathrm{id}) \circ \alpha^*(f_i) \\ &= (g^* \otimes \mathrm{id}) \circ (i^* \otimes \mathrm{id}) \circ \alpha^*(f_i) \\ &= (g^* \otimes \mathrm{id}) \circ \mathrm{sw}^* \circ m^*(f_i) = \sum_{j=1}^{n} h_{ji}(g) \cdot f_j.\end{aligned}$$

In other words, for any $g \in G(k)$, the matrix $(h_{ji}(g))_{ji}$ represents the k-linear automorphism ρ_g of V with respect to the basis $\{f_1, \dots, f_n\}$. In particular, this shows that the resulting map

$$G(k) \to \mathrm{GL}_n(k) \colon g \mapsto (h_{ji}(g))_{ji}$$

is a group homomorphism.

Recall that the coordinate ring of GL_n is $k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n]$, where $D := \det(x_{ij})_{1 \leq i,j \leq n}$. Using the coefficients h_{ij}'s, we define a k-algebra homomorphism

$$\iota^* \colon k[\mathrm{GL}_n] \to k[G]$$

by $\iota^*(x_{ji}) := h_{ji}$. Then the corresponding morphism of algebraic varieties $G \to \mathrm{GL}_n$ obviously realizes the above map $G(k) \to \mathrm{GL}_n(k) \colon g \mapsto (h_{ji}(g))_{ji}$ at the level of k-rational points. Since ι is thus a group homomorpshim on k-rational points, the density of k-rational points implies that ι is a homomorphism between algebraic groups (see Exercise below).

Hence, to complete the proof, it suffices to check that ι^* is surjective. For this, let us evaluate the function $\rho_g(f_i) = \sum_{j=1}^{n} h_{ji}(g) \cdot f_j \in k[G]$ at the unit element $e \in G(k)$. From the left-hand side, we get

$$\rho_g(f_i)(e) = f_i(e \cdot g) = f_i(g).$$

From the right-hand sider, we get

$$\sum_{j=1}^{n} h_{ji}(g) \cdot f_j(e).$$

In other words, as a function on $G(k)$, we can express f_i as k-linear combination of $h_j i$'s. Since f_j's generates $k[G]$ as a k-algebra, this means that ι^* is surjective. $\square$

Exercise 3.6. Let G and H be affine algebraic groups. Let $f \colon G \to H$ be a morphism as algberaic varieties. Prove that, if the induced map $G(k) \to H(k)$ on k-rational points is a group homomorphism, then f is a homomorphism of algebraic groups. (Hint: Use that the diagonal subset $\Delta X := \{(x, x) \in X \times_k X\}$ of $X \times_k X$ is closed for any affine scheme X over k (any affine scheme is "separated").)

Remark 3.7. From the above proof, we can see that the infinite-dimensional representation $\rho_{(-)}$ of $G(k)$ on $k[G]$ is faithful, i.e., $\rho_{(g)}$ is not trivial for any $g \neq e$. Indeed, if ρ_g is trivial, then its restriction to $V \subset k[G]$ is also trivial, where V is as in the above proof. However, this means that the image of g under the closed immersion $\mathrm{GL}_n \hookrightarrow \mathrm{GL}_n$ constructed above is trivial, hence $g = e$.

3.4. **Jordan decomposition for** GL_n**.**

Definition 3.8. Let Let g be an element of $\mathrm{GL}_n(k)$. We say that g is

(1) *semisimple* if g is diagonalizable in $\mathrm{GL}_n(\overline{k})$;
(2) *nilpotent* if all the eigenvalues of g are 0 (equivalently, g^n is zero),
(3) *unipotent* if all the eigenvalues of g are 1 (equivalently, $g - I_n$ is nilpotent).

Proposition 3.9. *For any $g \in \mathrm{GL}_n(k)$, there exists a unique decomposition $g = g_s + g_n$ such that*

- $g_s g_n = g_n g_s$,
- $g_s \in \mathrm{GL}_n(k)$ *is semisimple, and*
- $g_n \in \mathrm{GL}_n(k)$ *is nilpotent.*

Proof. We regard $g \in \mathrm{GL}_n(k)$ as an endomorphism of $V := k^{\oplus n}$. We let $\{\alpha_1, \ldots, \alpha_r\}$ be the set of eigenvalues of g. Recall that the generalized eigenspace of g with respect to its eigenvalue α_i is defined by

$$V_i := \mathrm{Ker}(g - \alpha_i \cdot I_n)^{n_i},$$

Then, by the theorem of Cayley–Hamilton, we have $V = \bigoplus_{i=1}^r V_i$.

We put $g_i := g|_{V_i} \in \mathrm{End}_k(V_i)$. If we put $g_{i,s} := \alpha_i \cdot I_{\dim V_i}$ and $g_{i,n} := g_i - g_{i,s}$, then we have

- $g_{i,s}$ is semisimple,
- $g_{i,n}$ is nilpotent, and
- $g_{i,s} g_{i,n} = g_{i,n} g_{i,s}$.

Thus, by putting $g_s := \bigoplus_{i=1}^r g_{i,s}$ and $g_n := \bigoplus_{i=1}^r g_{i,n}$, we get a decomposition $g = g_s + g_n$ satisfying the desired conditions.

To check the uniqueness of such a decomposition, suppose that we have another such decomposition $g = g'_s + g'_n$. Then, since g'_s commutes with g, g'_s preserves each V_i. By noting that $g_i - (g'_s)|_{V_i} = (g'_n)|_{V_i}$, which is nilpotent, we see that g and g'_s have the same eigenvalues on V_i. As g'_s is semisimple, this implies that g'_s must be equal to $\alpha_i \cdot I_{\dim V_i}$. Hence we also get $g_n = g'_n$. □

We call the decomposition $g = g_s + g_n$ the *additive Jordan decomposition* of g.

Corollary 3.10. *For any $g \in \mathrm{GL}_n(k)$, there exists a unique decomposition $g = g_s g_u$ such that*

- $g_s g_u = g_u g_s$,
- g_s *is semisimple, and*
- g_u *is unipotent.*

Proof. Let $g = g_s + g_n$ be the additive Jordan decomposition of g. Then we have $g = g_s(I_n + g_s^{-1} g_n)$. Since $g_s^{-1} g_n$ is nilpotent (use that g_s and g_n commute), $I_n + g_s^{-1} g_n$ is unipotent. Let us put $g_u := I_n + g_s^{-1} g_n$. As g_s commutes with g_u, $g = g_s g_u$ is a desired decomposition.

To check the uniqueness, let us assume that $g = g'_s g'_u$ is another such decomposition. Then, by putting $g'_n := g'_s(g'_u - I_n)$, we get the additive Jordan decomposition $g = g'_s + g'_n$. By the uniqueness of the additive Jordan decomposition, we have $g'_s = g_s$ and $g'_u = g_u$. □

We call the decomposition $g = g_s g_u$ the *Jordan decomposition* of g.

3.5. **Jordan decomposition for linear algebraic groups.** We next investigate the Jordan decomposition for linear algebraic groups. The key is to consider the right-translation action of $G(k)$ on $k[G]$.

Let G be a linear algebraic group. Recall that the right-translation action $\rho_{(-)}$ of $G(k)$ on $k[G]$ is locally finite. Hence, we can write $V = \sum_i V_i$ with $G(k)$-stable finite-dimensional subspaces V_i of $k[G]$. For any $g \in G(k)$, since the restriction of ρ_g to V_i is an element of $\mathrm{GL}_k(V_i)$, we can consider its Jordan decomposition $\rho_g|_{V_i} = (\rho_g|_{V_i})_s(\rho_g|_{V_i})_u$. Since the Jordan decomposition is unique, these decompositions are consistent on any intersection $V_i \cap V_j$, hence defines a decomposition $\rho_g = (\rho_g)_s(\rho_g)_u$ of ρ_g (i.e., $(\rho_g)_s$ and $(\rho_g)_u$ are unique automorphisms of $k[G]$ such that $(\rho_g)_s|_{V_i} = (\rho_g|_{V_i})_s$ and $(\rho_g)_u|_{V_i} = (\rho_g|_{V_i})_u$ for each i). Moreover, again by the uniqueness of the Jordan decomposition, we also see that this definition of $(\rho_g)_s$ and $(\rho_g)_u$ does not depend on the choice of the decomposition $k[G] = \sum_i V_i$. We also call this decomposition the *Jordan decomposition* of ρ_g.

Before we proceed, let us note the following:

Lemma 3.11. *The right-translation representation is faithful, i.e., the homomorphism $\rho_{(-)}\colon G(k) \to \mathrm{Aut}_k(k[G])$ is injective.*

Proof. Let us show the kernel of ρ is trivial. For this, we suppose that $g \in G(k)$ satisfies $\rho_g = \mathrm{id}$. Then, for any $f \in k[G]$, we have $f = \rho_g(f)$. In other words, we have $f(x) = f(xg)$ for any $x \in G(k)$. Since this holds for any $f \in k[G]$, we necessarily have $x = xg$. Hence $g = e$. □

Also, note that the Jordan decomposition for GL_n is consistent with the Jordan decomposition of the corresponding right-translation action in the following sense:

Proposition 3.12. *Let $G = \mathrm{GL}_n$. Let g be an element of $G(k)$ with Jordan decomposition $g = g_s g_u$. Let $\rho_g = (\rho_g)_s(\rho_g)_u$ be the Jordan decomposition of $\rho_g \in \mathrm{Aut}_k(k[G])$. Then $(\rho_g)_s = \rho_{g_s}$ and $(\rho_g)_u = \rho_{g_u}$. Moreover, g_s and g_u are characterized as the unique elements of $G(k)$ satisfying this condition.*

Proof. Since $\rho_{(-)}$ is a group homomorphism, we have $\rho_g = \rho_{g_s}\rho_{g_u} = \rho_{g_u}\rho_{g_s}$. Thus, by the uniqueness of the Jordan decomposition of ρ_g, the task is to show that ρ_{g_s} is semisimple and ρ_{g_u} is unipotent. This is actually not quite obvious, but please allow me to omit in this course; see, e.g., [Bor91, §4.3]. □

Now let us discuss the Jordan decomposition for general linear algebraic groups.

Theorem 3.13. *Let G be a linear algebraic group. For any $g \in G(k)$, let $\rho_g = (\rho_g)_s(\rho_g)_u$ be the Jordan decomposition of $\rho_g \in \mathrm{Aut}_k(k[G])$. There exist unique elements $g_s, g_u \in G(k)$ such that $g = g_s g_u = g_u g_s$, $(\rho_g)_s = \rho_{g_s}$, and $(\rho_g)_u = \rho_{g_u}$.*

Proof. We first note that, if such elements g_s and g_u really exist, then they must be unique by the faithfulness of $\rho_{(-)}$ and the conditions $(\rho_g)_s = \rho_{g_s}$ and $(\rho_g)_u = \rho_{g_u}$.

Let us take a closed immersion $\iota\colon G \hookrightarrow \mathrm{GL}_n$ for some $n \in \mathbb{Z}_{>0}$. We let $\iota^*\colon k[\mathrm{GL}_n] \twoheadrightarrow k[G]$ be the corresponding surjective k-algebra homomorphism. Let $I \subset k[\mathrm{GL}_n]$ be the kernel of ι^*. Then, via ι, $G(k)$ identified with the subgroup $\{g \in \mathrm{GL}_n(k) \mid f(g) = 0 \text{ for any } f \in I\}$ of $\mathrm{GL}_n(k)$. Conversely, $I = \{f \in k[\mathrm{GL}_n] \mid f(G(k)) = 0\}$. This observation implies that $G(k)$ can be also thought of as the subgroup $\{g \in \mathrm{GL}_n(k) \mid \rho_g(I) = I\}$. Moreover, we see that the right-translation actions for $G(k)$ and $\mathrm{GL}_n(k)$ are consistent under the maps ι and ι^* in the sense that the following holds for any $g \in G(k)$:

$$\rho_{\iota(g)} = \rho_g.$$

Here, the right-hand side is an element of $\mathrm{Aut}_k(k[G])$ while the left-hand side is a priori an element of $\mathrm{Aut}_k(k[\mathrm{GL}_n])$ but regarded as an element of $\mathrm{Aut}_k(k[G])$ by noting that $\rho_g(I) = I$.

$$\begin{tikzcd}
G(k) \arrow[r, hook, "\iota"] \arrow[d, hook, "\rho"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathrm{GL}_n(k) \arrow[d, hook, "\rho"] \\
\mathrm{Aut}_k(k[G]) & \mathrm{Aut}_k(k[\mathrm{GL}_n]) \arrow[l, squiggly, "\iota^*"]
\end{tikzcd}$$

We take the Jordan decomposition $\iota(g) = \iota(g)_s \iota(g)_u$ of $\iota(g) \in \mathrm{GL}_n(k)$. Then, $\iota(g)_s$ and $\iota(g)_u$ belong to $G(k)$. Indeed, by the discussion in the previous paragraph, $\rho_{\iota(g)}$ preserves I. This implies that $\rho_{\iota(g)}|_I$ also has a Jordan decomposition and its nothing but the restictions of $\rho_{\iota(g)_s}$ and $\rho_{\iota(g)_u}$ to I (recall the construction of the Jordan decomposition for locally finite automorphisms). In particular, $\rho_{\iota(g)_s}$ and $\rho_{\iota(g)_u}$ also preserve I, hence $\iota(g)_s$ and $\iota(g)_u$ belong to $G(k)$.

Let us put $g_s := \iota^{-1}(\iota(g)_s)$ and $g_u := \iota^{-1}(\iota(g)_u)$. The remaining thing is to check that $(\rho_g)_s = \rho_{g_s} \in \mathrm{Aut}_k(k[G])$, and $(\rho_g)_u = \rho_{g_u} \in \mathrm{Aut}_k(k[G])$. By this definition, $\rho_{g_s} \in \mathrm{Aut}_k(k[G])$ is nothing but the automorphism induced by $\rho_{\iota(g_s)} = \rho_{\iota(g)_s} \in \mathrm{Aut}_k(k[\mathrm{GL}_n])$ (recall the discussion above the diagram). On the other hand, $(\rho_g)_s$ is the automorphism induced by $(\rho_{\iota(g)})_s \in \mathrm{Aut}_k(k[\mathrm{GL}_n])$. Since we have $\rho_{\iota(g)_s} = (\rho_{\iota(g)})_s$ by the previous proposition, this completes the proof. □

We call the decomposition $g = g_s g_u$ as in the above theorem the *Jordan decomposition* of g.

Proposition 3.14. *The Jordan decomposition is preserved under any homomorphism of linear algebraic groups. To be more precise, let G and H be linear algebraic groups and $f \colon G \to H$ a homomorphism of algebraic groups. Then, for any $g \in G(k)$ with Jordan decomposition $g = g_s g_u$, the Jordan decomposition of $f(g) \in H(k)$ is given by $f(g) = f(g_s) f(g_u)$.*

Proof. Recall that the image of any homomorphism of algebraic groups is closed (Corollary 2.15). Hence, we may decompose any homomorphism into a surjective homomorphism and a closed immersion. Thus it suffices to show the claim only for the homomorphisms of these two types. In the case where f is a closed immersion, we can check the claim by a routine argument. In the case where f is surjective, we only need one additional fact that the corresponding k-algebra homomorphism $f^* \colon k[H] \to k[G]$ is injective (this is a general property of "dominant" morphisms between reduced affine schemes), but the remaining part is the same. □

Exercise 3.15. We consider the following matrix:

$$g := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

(1) Compute the Jordan decomposition of g when g is regarded as an element of $\mathrm{GL}_2(\mathbb{C})$.
(2) Compute the Jordan decomposition of g when g is regarded as an element of $\mathrm{GL}_2(\overline{\mathbb{F}}_p)$.

4. Week 4: Unipotent groups, Solvable groups, Nilpotent groups

4.1. **Simultaneous Jordan decomposition for commutative groups.** As usual, let k be an algebraically closed field. We consider a linear algberaic group G (over k) with coordinate ring $k[G]$. Let $\rho_{(-)}$ denote the right-translation action of $G(k)$ on $k[G]$. Then, as ρ_g is a locally finite automorphism of $k[G]$, we can take its Jordan decomposition $\rho_g = (\rho_g)_s(\rho_g)_u$.

In the last week, we proved that any element $g \in G(k)$ has a unique decomposition $g = g_s g_u$ called the Jordan decomposition of g. Here, $g_u \in G(k)$ and $g_s \in G(k)$ are characterized by the following properties:

- $g = g_u g_s = g_s g_u$,
- $\rho_{g_s} = (\rho_g)_s$,
- $\rho_{g_u} = (\rho_g)_u$.

Definition 4.1. For $g \in G(k)$, we say that g is

- *semisimple* if $g = g_s$, and
- *unipotent* if $g = g_u$.

Lemma 4.2. *For $g \in G(k)$, g is semisimple (resp. unipotent) if and only if $\iota(g)$ is semisimple (resp. unipotent) for some embedding $G \hookrightarrow \mathrm{GL}_n$. This is furthermore equivalent to that $\iota(g)$ is semisimple (resp. unipotent) for any embedding $G \hookrightarrow \mathrm{GL}_n$.*

Proof. Recall that the Jordan decomposition is preserved under any group homomorphism. In particular, for any embedding ι, g is semisimple (resp. unipotent) if and only if so is $\iota(g)$. $\square$

Let us consdier the subsets

$$G(k)_s := \{g \in G(k) \mid g = g_s\} \quad \text{and} \quad G(k)_u := \{g \in G(k) \mid g = g_u\}.$$

We call $G(k)_s$ and $G(k)_u$ the *semisimple locus* (resp. *unipotent locus*) of $G(k)$.

Note that if we fix an embedding $\iota\colon G \hookrightarrow \mathrm{GL}_n$, then, for $g \in G(k)$, g belongs to $G(k)_u$ if and only if $(\iota(g) - I_n)^n = 0$. The equation $(\iota(g) - I_n)^n = 0$ can be expressed via polynomial in the coordinates of GL_n (i.e., entries of n-by-n matrices). In particular, we see that $G(k)_n$ is the set of k-rational points of a closed subset G_u of G. We also call G_u the unipotent locus of G. On the other hand, note that $G_s(k)$ cannot be thought of as k-rational points of a closed subset in general. Also, note that $G(k)_s$ and $G(k)_u$ are not subgroups of $G(k)$ in general.

However, when G is commutative, the following particular fact holds:

Theorem 4.3. *Let G be a commutative linear algebraic group. Then G_u is a closed subgroup of G and $G_s(k)$ is also regarded as the group of k-rational points of a closed subgroup G_s of G. Furthermore, the multiplication map $m\colon G_s \times G_u \to G$ is an isomorphism of algebraic groups.*

Proof. The commutativity implies that $G(k)_u$ is a subgroup of $G(k)$, hence G_u is a subgroup of G as an algebraic group. (In general, it can be easily checked that a closed subvariety of an algebraic group is closed under the multiplication and inversion at the level of k-rational points, then it is closed under the multiplication and inversion; this follows from the density of closed points.)

On the other hand, it is a basic fact in linear algebra that any commutative family of semisimple (diagonalizable) elements of $\mathrm{GL}_n(k)$ can be simultaneously diagonalized. Thus, we can find an embedding $\iota\colon G \hookrightarrow \mathrm{GL}_n$ such that the image of $G(k)_s$ is contained in the diagonal subgroup $T_n(k) \subset \mathrm{GL}_n(k)$ by replacing ι by $\mathrm{GL}_n(k)$-conjugation if necessary. Since any element of $T_n(k)$ is semisimple, we then have $G(k)_s = \iota^{-1}(T_n(k)) = \iota^{-1}(T_n)(k)$. In other words, if we define $G_s := \iota^{-1}(T_n)$, then G_s is a closed subgroup of G whose k-rational points realize $G(k)_s$.

Now let us check that $G_s \times G_u \to G$ is an isomorphism of algebraic groups. (At least we already know that this is bijective at the level of k-rational points; but we have to show that this homomorphism has an inverse homomorphism as algebraic groups.) For this, we again appeal to a more genral fact in linear algebra, that is, any commutative family of elements of $\mathrm{GL}_n(k)$ can be simultaneously triangulated. In other words, if we let B_n be the closed subgroup of GL_n consisting of upper-triangular matrices, then we may suppose that the image of ι is contained in B_n. We define a closed subgroup U_n of B_n to be $(B_n)_u$, i.e., upper-triangular unipotent matrices. Then G_s and G_u are nothing but the preimages of T_n and U_n under ι. For the morphism $T_n \times U_n \to B_n$ we can check that it is an isomorphism of algebraic varieties, hence we have its inverse. Under ι, this inverse induces the inverse of $G_s \times G_u \to G$. □

4.2. **Simultaneous triangulation for unipotent groups.**

Definition 4.4. We say that a linear algebraic group G is *unipotent* if $G = G_u$.

Remark 4.5 (CAUTION)**.** Later, we will introduce the notion of a "semisimple group", but it does not mean $G = G_s$. In fact, the condition $G = G_s$ is quite strong so that it implies G is commutative.

Theorem 4.6. *Let G be a closed unipotent subgroup of* GL_n. *Then G is conjugate to a subgroup of U_n.*

Proof. It is enough to show the assertion at the level of k-rational points. Let $V = k^{\oplus n}$ and identify $\mathrm{GL}_n(k) \cong \mathrm{Aut}_k(V)$. We appeal to the induction on dimension of V. The assertion is clear when $\dim_k V = 1$, thus let us consider the case where $\dim_k V > 1$.

It is enough to find an element $v \in V \setminus \{0\}$ which is a simultaneous eigenvector for all elements of G (i.e., $g \cdot v = v$ for any $g \in G(k)$). Indeed, if we can do this, then we can apply the induction hypothesis to V/kv.

If V is not irreducible as a representation of $G(k)$ (i.e., there exists a nonzero proper subspace $0 \subsetneq W \subsetneq V$), then we can apply the induction hypothesis of W. Especially, we can find a simultaneous eigenvector in W. Thus, the essential case is when the action of $G(k)$ on V is irreducible.

The key fact is the following (so-called "Burnside's theorem'):

> Let V be a finite dimensional k-vector space over an algebraically closed field k. Let $\rho\colon G \to \mathrm{Aut}_k(V)$ be an irreducible representation of a group G (in the abstract sense). Then we have $\mathrm{End}_k(V) = \mathrm{Span}_k\{\rho(g) \mid g \in G\}$.

Now we claim that $G(k) = \{I_n\}$. For the sake of contradiction, we assume that $g \in G(k)$ such that $g \neq I_n$ and write $g = I_n + h$ (hence h is a nilpotent matrix). For any $x \in G(k)$, we have

$$\mathrm{Tr}(hx) = \mathrm{Tr}\big((g - I_n)x\big) = \mathrm{Tr}(gx) - \mathrm{Tr}(x) = 0$$

(in the last equality, note that gx and x are elements of $G(k)$, hence unipotent; in particular, their traces are $\dim_k(V)$). Since $\mathrm{End}_k(V) = \mathrm{Span}_k\{\rho(g) \mid g \in G\}$ by the Burnside's theorem, this implies that $\mathrm{Tr}(hx) = 0$ for any $x \in \mathrm{End}_k(V)$. But this forces that $x = 0$.

As $G(k) = \{I_n\}$, the dimension of its irreducible representation V must be 1. This completes the proof. □

Corollary 4.7. *Let G be a unipotent algebraic group. Then G is isomorphic to a closed subgroup of U_n for some n.*

Proof. We choose any embedding ι of G into GL_n. Then apply the previous theorem to $\iota(G) \subset \mathrm{GL}_n$ and replace ι with the suitably conjugated one. □

4.3. **Solvable and Nilpotent groups.** We next introduce the notions of "solvable" and "nilpotent" algebraic groups. Let us first recall their definitions in the context of abstract group theory.

Definition 4.8. Let G be an abstract group. For any subgroups H_1 and H_2 of G, we let $[H_1, H_2]$ denote the subgroup of G generated by (i.e., the smallest subgroup containing) $\{h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2\}$.

(1) We put $D^0 G = G$ and $D^{n+1} G := [D^n G, D^n G]$. If $D^n G = \{e\}$ for $n \gg 0$, we say that G is *solvable*.
(2) We put $C^0 G = G$ and $C^{n+1} G := [G, C^n G]$. If $C^n G = \{e\}$ for $n \gg 0$, we say that G is *nilpotent*.

(Note that $D^n G \subset C^n G$, in particular, any nilpotent group is solvable.)

Now let G be a linear algebraic group. We define the solvability and the nilpotency for G in the exactly same way as above by replacing "the smallest subgroup containing" with "the smallest algebraic subgroup containing" and also "$\{h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2\}$" with "the image of $H_1 \times H_2$ under the commutator morphism $G \times G \to G\colon (x, y) \mapsto xyx^{-1}y^{-1}$".

Note that $[H_1(k), H_2(k)]$ is dense in $[H_1, H_2]$. This implies that $D^n(G(k))$ and $C^n(G(k))$ are dense in $D^n(G)$ and $C^n(G)$, respectively. In particular, in order to check if a given algebraic group is solvable or nilpotent, it suffices to look at only k-rational points.

Example 4.9. (1) Any commutative linear algebraic group G satisfies $[G, G] = \{e\}$, hence must be nilpotent. In particular, $\mathbb{G}_{\mathrm{a}}$ and $\mathbb{G}_{\mathrm{m}}$ are nilpotent.
(2) The closed subgroup B_n of GL_n consisting of upper-triangular matrices is solvable. This is well-known at the level of k-rational points. Note that B_n is not nilpotent.
(3) The closed subgroup U_n of B_n consisting of upper-triangular unipotent matrices is nilpotent. This is well-known at the level of k-rational points.

Proposition 4.10. *Any unipotent linear algebraic group is nilpotent.*

Proof. Recall that any unipotent linear algebraic group can be embedded into $U_n \subset \mathrm{GL}_n$. Then we have $C^m G \subset C^m U_n$. As U_n is nilpotent, we have $C^m U_n = \{e\}$ for $e \gg 0$, hence $C^m G = \{e\}$ for $e \gg 0$. □

The following are generalizations of the results for unipotent groups which we introduced this week. (We will explain the proofs in the future.)

Theorem 4.11 (Lie–Kolchin's theorem)**.** *Let G be a connected closed subgroup of GL_n. If G is solvable, then G is conjugate to a subgroup of B_n.*

5. Week 5: Diagonalizable groups

5.1. **Diagonalizable groups.** As usual, let k be an algebraic closed field. We start by fixing some notations:

- For algebraic varieties X and Y over k, we write $\mathrm{Mor}_k(X,Y)$ for the set of morphisms between X and Y over k.
- For algebraic groups G and H over k, we write $\mathrm{Hom}_k(G,H)$ for the set of homomorphisms between G and H over k.
- For k-algebras R and S, we write $\mathrm{Hom}_k(R,S)$ for the set of k-algebra homomorphisms.
- For Hopf k-algebras R and S, we write $\mathrm{Hom}^*_k(R,S)$ for the set of Hopf k-algebra homomorphisms.

Recall that, when X and Y are affine varieties, we have

$$\mathrm{Mor}_k(X,Y) \cong \mathrm{Hom}_k(k[Y],k[X])\colon \phi \leftrightarrow \phi^*,$$

where $k[X]$ and $k[Y]$ are coordinate rings of X and Y, respectively. When G and H are linear algebraic groups, we have

$$\mathrm{Hom}_k(G,H) \cong \mathrm{Hom}^*_k(k[H],k[G])\colon \phi \leftrightarrow \phi^*.$$

Remark 5.1. Let us give another viewpoint of "regular functions". Let X be an affine algebraic variety with coordinate ring $k[X]$. Recall that any element $f \in k[X]$ can be viewed as a function on $X(k)$ by

$$f\colon X(k) \to k; \quad x \mapsto f(x) := x^*(f),$$

where $x^*\colon k[X] \to k$ denotes the k-algebra homomorphism corresponding to $x \in X(k) = \mathrm{Mor}_k(\operatorname{Spec} k, X)$. We call the functions $X(k) \to k$ obtained in this manner *regular functions* on $X(k)$. On the other hand, suppose that $f \in \mathrm{Mor}_k(X,\mathbb{A}^1_k)$; then, f induces a map $X(k) \to \mathbb{A}^1_k(k) \cong k$ at the level of k-rational points. In fact, these association gives an identification between the set of regular functions and $\mathrm{Mor}_k(X,\mathbb{A}^1_k)$:

$$k[X] \cong \{\text{regular functions } X(k) \to k\} \cong \mathrm{Mor}_k(X,\mathbb{A}^1_k)$$

(This can be checked by going back to all the definitions.)

For any linear algebraic group G, we write $X^*(G)$ for the set of homomorphisms from G to $\mathbb{G}_\mathrm{m}$ and call it the set of *characters*:

$$X^*(G) := \mathrm{Hom}_k(G,\mathbb{G}_\mathrm{m}).$$

Note that the multiplicative group $\mathbb{G}_\mathrm{m}$ is naturally regarded as an open subvariety of $\mathbb{A}^1_k$ (we may write $\mathbb{G}_\mathrm{m} = \mathbb{A}^1_k \smallsetminus \{0\}$):

$$\mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1_k \quad \leftrightsquigarrow \quad k[t^{\pm 1}] \hookleftarrow k[t].$$

In particular, this implies that any character of G can be viewed as a regular function on $X(k)$ which is not zero anywhere:

$$X^*(G) := \mathrm{Hom}_k(G,\mathbb{G}_\mathrm{m}) \subset \mathrm{Mor}_k(G,\mathbb{G}_\mathrm{m}) \hookrightarrow \mathrm{Mor}_k(G,\mathbb{A}^1_k) \cong k[G].$$

Also note that $X^*(G)$ has a natural structure of an abelian group. To be more precise, for characters $\phi_1,\phi_2 \in X^*(G)$, we define a morphism "$\phi_1+\phi_2$" from G to $\mathbb{G}_\mathrm{m}$ to be

$$G \xrightarrow{(\phi_1,\phi_2)} \mathbb{G}_\mathrm{m} \times_k \mathbb{G}_\mathrm{m} \xrightarrow{m} \mathbb{G}_\mathrm{m},$$

where m denotes the multiplication morphism of $\mathbb{G}_{\mathrm{m}}$. Then we can easily check that this is again a character, i.e., a homomorphism of algebraic groups. If ϕ_1 and ϕ_2 are $[g \mapsto \phi_1(g)]$ and $[g \mapsto \phi_2(g)]$ at the level of k-rational points, then $\phi_1 + \phi_2$ is just $[g \mapsto \phi_1(g) \cdot \phi_2(g)]$.

Definition 5.2. We say that a linear algebraic group G is *diagonalizable* if $k[G]$ is spanned by $X^*(G)$ as k-vector space.

Example 5.3. Let

$$T := \mathbb{G}_{\mathrm{m}}^r := \underbrace{\mathbb{G}_{\mathrm{m}} \times_k \mathbb{G}_{\mathrm{m}} \times_k \cdots \times_k \mathbb{G}_{\mathrm{m}}}_{r}.$$

Let us check that T is diagonalizable. We have

$$k[T] = k[t^{\pm 1}] \otimes_k \cdots \otimes_k k[t^{\pm 1}] \cong k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$$

On the other hand, we have

$$\begin{aligned} X^*(T) &= \mathrm{Hom}_k(T, \mathbb{G}_{\mathrm{m}}) \\ &\cong \mathrm{Hom}_k^*(k[t^{\pm 1}], k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]) \\ &\subset \mathrm{Hom}_k(k[t^{\pm 1}], k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]). \end{aligned}$$

Any k-algebra homomorphism from $k[t^{\pm 1}]$ to $k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$ is determined uniquely by the image of t, which must be invertible, hence of the form $ax_1^{n_1} \cdots x_r^{n_r}$, where $a \in k^\times$ and $n_1, \ldots, n_r \in \mathbb{Z}$. If we let $f^* \in \mathrm{Hom}_k(k[t^{\pm 1}], k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}])$ be the k-algebra homomorphism such that $f^*(t) = ax_1^{n_1} \cdots x_r^{n_r}$, then the corresponding morphism $f \colon T \to \mathbb{G}_{\mathrm{m}}$ is given by $(a_1, \ldots, a_r) \mapsto aa_1^{n_1} \cdots a_r^{n_r}$ at the levet of k-rational points (i.e., as a regular function $(k^\times)^r \to k^\times$). So that this is a group homomorphism, we must have $a = 1$; conversely, whenever $a = 1$, it defines a homomorphism as algebraic groups. In summary, if we write $e_i \colon T \to \mathbb{G}_{\mathrm{m}}$ for the character whose regular function is $(a_1, \ldots, a_r) \mapsto a_i$ (in other words, $e_i^*(t) = x_i$ at the level of coordinate rings), then we have

$$X^*(T) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r.$$

Each $n_1e_1 + \cdots + n_re_r \in X^*(T)$ corresponds to the element $x_1^{n_1} \cdots x_r^{n_r} \in k[T]$ when we regard $X^*(T) \subset k[T]$. In particular, $k[T]$ is spanned by $X^*(T)$ (even strongly, $k[T]$ is isomorphic to the group algebra $k[X^*(T)]$). Hence T is diagonalizable.

Exercise 5.4. Show that $\mathbb{G}_{\mathrm{a}}$ is not diagonalizable by tracing the argument in the above example.

5.2. **Basic properties of diagonalizable groups.**

Lemma 5.5. *If G is a diagonalizable group, then $k[G]$ is the group algebra $k[X^*(G)]$. In particular, $X^*(G)$ is a finitely generated abelian group.*

Proof. By definition, $k[G]$ is spanned by $X^*(G)$. Moreover, the addition and multiplication on the k-algebra $k[G]$ is consistent with those of the group algebra $k[X^*(G)]$; this can be seen by looking at the descriptions of these functions as regular functions on $G(k)$. Hence the only task is to show that all the elements of $X^*(G)$ are linearly independent (as k-valued functions on H).

This follows from so-called "Dedekind's lemma"

> Let H be an abstract group. The set of group homomorphisms $H \to k^\times$ is linearly independent.

Let us explain the proof of this lemma, which is very famous.

Suppose that the set is linearly dependent for the sake of contradiction. Let $n \in \mathbb{Z}_{>1}$ be the minimal number such that there exist linearly dependent group homomorphisms $\chi_1, \dots, \chi_n \colon H \to k^\times$. Suppose that

$$f := a_1\chi_1 + \cdots + a_n\chi_n = 0$$

as k-valued functions on H for some $a_1, \dots, a_r \in k^\times$. We may suppose that $a_n = 1$ by dividing the both sides of the above equation by a_n. Since $\chi_1 \neq \chi_n$, we can choose $h_0 \in H$ such that $\chi_1(h_0) \neq \chi_n(h_0)$. Then, for any $h \in H$, we have

$$\begin{aligned} f(h_0h) - \chi_n(h_0)f(h) &= \sum_{i=1}^{n} a_i\chi_i(h_0)\chi_i(h) - \chi_n(h_0)\sum_{i=1}^{n} a_i\chi_i(h) \\ &= \sum_{i=1}^{n-1} a_i\big(\chi_i(h_0) - \chi_n(h_0)\big)\chi_i(h). \end{aligned}$$

Note that this is 0 since $f = 0$. Moreover, since $\chi_1(h_0) \neq \chi_n(h_0)$, at least the coefficient of $\chi_1(h)$ is not zero. In other words, we get $\sum_{i=1}^{n-1} a_i(\chi_i(h_0) - \chi_n(h_0))\chi_i = 0$, which is a nontrivial linearly dependent equation. This contradicts to the minimality of n. $\square$

Lemma 5.6. *Let G be a diagonalizable group and H its closed subgroup. Then H is also diagonalizable. Moreover, any character on H can be extended to a character on G.*

Proof. By the previous lemma, we have $k[G] = k[X^*(G)]$. Moreover, by Dedekind's lemma, we also have that $k[X^*(H)] \subset k[H]$. The k-algebra homomorphism $k[G] \to k[H]$ corresponding to $H \hookrightarrow G$ is given by restricting regular functions on $G(k)$ to $H(k)$. Thus, the fact that $k[G] \to k[H]$ is surjective implies that the restriciton homomorphism $X^*(G) \to X^*(H)$ is surjective and also that $k[H] = k[X^*(H)]$, which means that H is diagonalizable. $\square$

Proposition 5.7. *The association $G \mapsto X^*(G)$ gives a fully faithful contravariant functor from the category of diagonalizable groups to the category of finitely generated abelian groups. In other words, for any diagonalizable groups G and H, the natural map $\mathrm{Hom}_k(G, H) \to \mathrm{Hom}(X^*(H), X^*(G))$ is bijective.*

Proof. Let G and H be diagonalizable groups. We have the following commutative diagram:

$$\begin{tikzcd} & \mathrm{Hom}_k(G,H) \arrow[dl, leftrightarrow, "\alpha"'] \arrow[dr, "\gamma"] & \\ \mathrm{Hom}^*_k(k[H], k[G]) & & \mathrm{Hom}(X^*(H), X^*(G)) \arrow[ll, "\beta"'] \end{tikzcd}$$

Here,

- α is the isomorphism coming from the equivalence between the categories of affine algebraic groups and Hopf algebras,
- β is the natural homomorphism induced by the fact that $k[G] = k[X^*(G)]$ and $k[H] = k[X^*(H)]$, and
- γ is the natural homomorphism of our interest.

It is not difficult to see that, through $k[G] = k[X^*(G)]$, the Hopf algebra structure on $k[G]$ is given by the diagonal map $X^*(G) \to X^*(G) \times X^*(G)$ and the inversion map $X^*(G) \to X^*(G)$. Noting this, we can see that β is bijective. Hence γ is bijective. $\square$

Later, we will see that $G \mapsto X^*(G)$ in fact gives an equivalence of categories.

5.3. **Characterization of diagonalizable groups.**

Proposition 5.8. *For linear algebraic group G, the following are equivalent:*

(1) *G is diagonalizable;*
(2) *G is a closed subgroup of a torus;*
(3) *for any group homomorphism $\pi\colon G \to \mathrm{GL}_n$, the image $\pi(G)$ is conjugate to a subgroup of the diagonal torus T_n.*

Proof. Let us first show (1) $\implies$ (2). For this, we go back to the proof of the fact that "affine algebraic groups are linear" (Week 3). Recall that, for any affine algebraic group G, we can find a finite-dimensional $G(k)$-stable (with respect to the right translation action) subspace V of $k[G]$. Let $\{f_1, \ldots, f_n\}$ be a k-basis of V and write

$$m^*(f_i) = \sum_{j=1}^{n} f_j \otimes h_{ji} \in V \otimes_k k[G]$$

with $h_{ji} \in k[G]$ ($j = 1, \ldots, n$), where $m^*\colon k[G] \to k[G] \otimes k[G]$ denotes the k-algebra homomorphism corresponding to the multiplication homomorphism $G \times_k G \to G$. Then, for any $g \in G(k)$, the right translation action ρ_g on V is given by

$$\rho_g(f_i) = \sum_{j=1}^{n} h_{ji}(g) \cdot f_j.$$

In other words, for any $g \in G(k)$, the matrix $(h_{ji}(g))_{ji}$ represents the k-linear automorphism ρ_g of V with respect to the basis $\{f_1, \ldots, f_n\}$. The association

$$G(k) \to \mathrm{GL}_n(k)\colon g \mapsto (h_{ji}(g))_{ji}$$

gives the closed immersion of G into GL_n at the level of k-rational points.

Now suppose that G is diagonalizable. We choose $f_1, \ldots, f_n$ to be a generator of the finitely generated abelian group $X^*(G)$. Then the subspace of $k[G] = k[X^*(G)]$ spanned by $f_1, \ldots, f_n$ is $G(k)$-stable. Indeed, for any $g \in G(k)$, the translated regualr function $\rho_g(f_i)$ is given by, for $x \in G(k)$,

$$\big(\rho_g(f_i)\big)(x) = f_i(xg) = f_i(g)f_i(x)$$

(we used that f_i is a character in the last equality). Moreover, this also shows that the closed immersion $G \hookrightarrow \mathrm{GL}_n$ associated to $\{f_1 \ldots, f_n\}$ is given by $g \mapsto \mathrm{diag}(f_1(g), \ldots, f_n(g))$. In other words, the image lies in the diaognal torus T_n of GL_n.

The implication (2) $\implies$ (1) follows from Example 5.3 and Lemma 5.6.

The implication (3) $\implies$ (2) follows by choosing π to be any closed immersion of G into GL_n.

We finally show (2) $\implies$ (3). Suppose that G is a closed subgroup of a torus and $\pi\colon G \to \mathrm{GL}_n$ is a homomorphism. We first note that any k-rational point of a torus is semisimple. Moreover, any torus is commutative. Hence, G is necessarily commutative and all k-rational points of G are semisimple. Thus, $\pi(G)$ is a closed commutative subgroup of GL_n whose k-rational points are all semisimple. By applying the simultaneous diagonalization to $\pi(G)(k)$, we see that $\pi(G)(k)$ is conjugate to a subgroup of $T_n(k)$. Hence $\pi(G)$ is also conjugate to a subgroup of T_n by the density of closed points. $\square$

5.4. **Characterization of tori.**

Proposition 5.9. *For linear algebraic group G, the following are equivalent:*

(1) *G is a torus $\mathbb{G}_{\mathrm{m}}^n$,*
(2) *G is a connected diagonalizable group of dimension n,*
(3) *G is a diagonalizable group with $X^*(G) \cong \mathbb{Z}^{\oplus n}$.*

Proof. The implication (1) $\implies$ (2) follows from Example 5.3 (and the basic facts from algebraic geometry that the product of connected varieties is connected and that the dimension of the product is the sum of the dimensions).

Let us check (2) $\implies$ (3). Let $\chi \colon G \to \mathbb{G}_{\mathrm{m}}$ be a character. Then, since G is connected, the image $\pi(G)$ is a closed connected subgroup of $\mathbb{G}_{\mathrm{m}}$, which implies that $\pi(G) = \{e\}$ or $\pi(G) = \mathbb{G}_{\mathrm{m}}$ (this is due to that $\mathbb{G}_{\mathrm{m}}$ is dimension one; the point here is that the connectedness and the irreducibility coincide for algebraic groups). In patricular, this means that $X^*(G)$ is torsion-free, hence free of finite rank. Recall that $k[G] = k[X^*(G)]$; this especially means that the rank of $X^*(G)$ is equal to the transcendental degree of $k[G]$ over k, which is equal to the dimension of X.

Finally let us show (3) $\implies$ (1). Let $e_1, \ldots, e_n$ be a basis of $X^*(G)$, hence we have $k[G] = k[e_1^{\pm 1}, \ldots, e_n^{\pm 1}]$ (where each character e_i is regarded as a regular function on $G(k)$). Then the map

$$k[G] = k[e_1^{\pm 1}, \ldots, e_n^{\pm 1}] \leftrightarrow k[x_1^{\pm 1}, \ldots, x_n^{\pm 1}] = k[\mathbb{G}_{\mathrm{m}}^n] \colon e_i \leftrightarrow x_i$$

gives an isomorphism of Hopf algebras, hence $G \cong \mathbb{G}_{\mathrm{m}}^n$. $\square$

Proposition 5.10. *Let G be a diagonalizable group. Then the identity component group G° is a torus. Moreover, we have a product decomposition $G \cong G^\circ \times F$, where F is a finite group.*

Proof. The first assertion follows from Lemma 5.6 and Proposition 5.9. To be more precise, any closed subgroup of a diagonalizable subgroup is again diagonalizable and any connected diagonalizable group is a torus.

To show the second assertion, let us first take a closed immersion of G into an n-dimensional torus T. Recall that the restriction homomorphism $X^*(T) \to X^*(G^\circ)$ is surjective (Lemma 5.6). Since $X^*(T)$ and $X^*(G^\circ)$ are free of finite rank (say, the rank of $X^*(G^\circ)$ is r), the surjection $X^*(T) \twoheadrightarrow X^*(G^\circ)$ splits. Let us choose a basis $e_1, \ldots, e_n$ of $X^*(T)$ so that $e_1, \ldots, e_r$ forms a basis of $X^*(G^\circ)$ ("the surjection $X^*(T) \twoheadrightarrow X^*(G^\circ)$ splits" exactly means that such a choice is possible). At the level of diagonalizable groups, this means that we have a decomposition $T \cong G^\circ \times \mathbb{G}_{\mathrm{m}}^{n-r}$.

By restricting this decomposition to G, we get $G \cong G^\circ \times (\mathbb{G}_{\mathrm{m}}^{n-r} \cap G)$. If we put $F := (\mathbb{G}_{\mathrm{m}}^{n-r} \cap G)$, then this gives a decomposition as desired. Indeed, we have $F \cong G/G^\circ$, hence F is finite. $\square$

5.5. **Equivalence between categories.**

Theorem 5.11. *Let p be the characteristic of k. The functor $G \mapsto X^*(G)$ gives a contravariant equivalence between the category of diagonalizable groups and*

- *the category of finitely generated abelian groups if $p = 0$,*

- *the category of finitely generated p-torsion-free abelian groups if $p > 0$.*

Proof. Recall that we already showed that the functor is fully faithful. Hence, to show that this functor gives an equivalence, it only suffices to check the essential-surjectivity, i.e., any finitely generated (p-torsion-free) abelian group is realized as $X^*(G)$ for some diagonalizable group G.

Let M be such a group and put R to be the group algebra $k[M]$, which has a natural Hopf algebra structure. We let $G := \operatorname{Spec} R$ be the corresponding "linear algebraic group". By examining all the discussions and constructions so far, it is almost obvious that $X^*(G) \cong M$. Only the subtle point is whether R is a reduced ring. (Recall that affine algebraic variety corresponds to a finitely generated and reduced k-algebra.)

To check this, we have to show that R does not contain any nonzero nilpotent element. When $k = 0$, it is easy to see that the group algebra does not contain any nonzero nilpotent element (just be careful that the unit element of M is not zero in R). When $k = p$, the argument is more complicated, but still elementary (if M has an element m of order p, then the p-th power of $1 - m \in R$ is zero. Conversely, if M has no element of order p, R is semisimple algebra, which is in particular reduced; this is the content of so-called Maschke's theorem). $\square$

6. Week 6: Lie algebras of algebraic groups

The aim of this week is to investigate the notion of the "Lie algbera" of an algebraic group.

6.1. **Review: Lie algebras of Lie groups.** We first introduce the following purely algebraic notion:

Definition 6.1. Let k be any field and R be a k-algebra. Let M be an R-module. A *derivation* from R to M is a k-linear homomorphism $D\colon R \to M$ satisfying the "Leibniz rule"

$$D(fg) = f \cdot D(g) + g \cdot D(f)$$

for any $f, g \in R$. We write $D_k(R, M)$ for the k-vector space consisting of derivations from R to M.

Recall that, for any real manifold X and its point $p \in X$, the *tangent space* T_pX of X at p is defined by

$$T_pX := D_{\mathbb{R}}(C^\infty(X), \mathbb{R}_p),$$

where $\mathbb{R}_p$ denotes $\mathbb{R}$ regarded as a $C^\infty(X)$-module through the map $\mathrm{ev}_p\colon C^\infty(X) \twoheadrightarrow \mathbb{R}\colon f \mapsto f(p)$ (i.e., $f \cdot r := f(p)r$). Elements of T_pX are called tangent vectors of X at p.

Remark 6.2. Note that the tangent space has a nice description in terms of local coordinates. If we let $x_1, \dots, x_n$ be the local coordinates of a real manifold X around a point $p \in X$, then T_pX is the n-dimensional $\mathbb{R}$-vector space spanned by

$$\left(\frac{\partial}{\partial x_1}\right)(p), \dots, \left(\frac{\partial}{\partial x_n}\right)(p),$$

where $(\frac{\partial}{\partial x_i})(p) : f(x) \mapsto (\frac{\partial f}{\partial x_i})(p)$. For this reason, T_pX is often defined as $\oplus_{i=1}^n \mathbb{R}(\frac{\partial}{\partial x_i})(p)$. But here we prefer the above purely algebraic definition to make its similarity to the tangent space of algebraic varieties clearer.

Also recall that, a *vector field* on M is an association $X \ni p \mapsto v_p \in T_pX$ such that v_p "varies smoothly in $p \in X$" (i.e., $v_{(-)}$ is a global section of the tangent bundle $TX \to X$). Let $\mathfrak{X}(X)$ be the set of vector fields on X. For any $v \in \mathfrak{X}(X)$, we define an $\mathbb{R}$-linear homomorphism $v\colon C^\infty(X) \to C^\infty(X)$ by $v(f)(p) := v_p(f)$. Then we can check that $v(-)$ satisifes the Leibniz rule, i.e., $v \in D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$. In fact, this procedure gives an identification between $\mathfrak{X}(X)$ and $D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$:

$$\begin{tikzcd}
\mathfrak{X}(X) \arrow[r, "\cong"] \arrow[d, "v \mapsto v_p"'] & D_{\mathbb{R}}(C^\infty(X), C^\infty(X)) \arrow[d, "\text{push via } \mathrm{ev}_p"] \\
T_pM \arrow[r, equal] & D_{\mathbb{R}}(C^\infty(X), \mathbb{R}_p)
\end{tikzcd}$$

For any $u, v \in \mathfrak{X}(X)$, we define their *bracket product* $[u, v] \in \mathfrak{X}(X)$ by $[u, v] := u \circ v - v \circ u$, where u and v are regarded as elements of $D_{\mathbb{R}}(C^\infty(X), C^\infty(X))$. It is not difficult to check that $(\mathfrak{X}(X), [-,-])$ forms a *Lie algbera* in the abstract sense. Here, recall:

Definition 6.3. Let k be a field and $\mathfrak{g}$ a k-vector space equipped with a k-bilinear map $[-,-]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$. We say that $(\mathfrak{g}, [-,-])$ is a *Lie algebra* if the following are satisfied:

(1) $[x,x]=0$ for any $x\in\mathfrak{g}$,
(2) $[x,[y,z]]+[y,[x,z]]+[z,[x,y]]=0$ for any $x,y,z\in\mathfrak{g}$.

Now let us suppose that G is a Lie group (a manifold equipped with a group structure which is compatible with the manifold structure). In this case, G acts on G itself via left and right translations, hence $\mathfrak{X}(G)$ also has the left and right translations by G. Let $\mathfrak{X}_l(G)$ (resp. $\mathfrak{X}_r(G)$) denote the subset of left-invariant (resp. right-invariant) vector fields on G. Then $\mathfrak{X}_l(G)$ and $\mathfrak{X}_r(G)$ are identified with T_eG via $v\mapsto v_p$;

$$\mathfrak{X}_l(G)\xrightarrow{\cong}T_eG\quad\text{and}\quad\mathfrak{X}_r(G)\xrightarrow{\cong}T_eG.$$

Note that, $\mathfrak{X}_l(G)$ and $\mathfrak{X}_r(G)$ are obviously closed under the bracket product, hence are Lie subalgebras of $\mathfrak{X}(G)$. In particular, the tangent space at the unit element T_eG also gets a Lie algebra structure through this identification. We write $\operatorname{Lie}G:=T_eG$ and call it the *Lie algbera* of the Lie group G.

6.2. **Zariski tangent spaces and Lie algebras.** Now let us move on to algebraic varieties and algebraic groups. In fact, we can also define the notions of tangent spaces and Lie algebras just by imitating those for Lie groups.

As usual, let k be an algebraically closed field. Let X be an affine algebraic variety over k. In this context, we may think of the coordinate ring $k[X]$ of X (ring of "regular functions on X") as an analogue of $C^\infty(X)$ for a real manifold X. Thus it is natural to introduce the following notion:

Definition 6.4. For any closed point $x\in X(k)$, we define the *Zariski tangent space* T_xX of X at x by

$$T_xX:=D_k(k[X],k_x),$$

where k_x denotes k regarded as a $k[X]$-module through $k[X]\to k\colon f\mapsto f(x)$[5].

In this way, we can perform the same discussion as before to define the Lie algebra of an algebraic group. Namely, we define

$$\mathfrak{X}(X):=D_k(k[X],k[X]).$$

Then, with the bracket produce $[u,v]:=u\circ v-v\circ u$, we get a Lie algebra $(\mathfrak{X}(X),[-,-])$ over k (in the abstract sense).

When $X=G$ is an algebraic group, we can consider the right and left translation action of $G(k)$ on the coordinate ring $k[G]$. More precisely, for any $g\in G(k)$, we have a k-algebra automorphisms λ_g and ρ_g of $k[G]$. Using this, we define automorphisms $\lambda(g)$ and $\rho(g)$ on $\mathfrak{X}(G)$ by

$$\lambda(g)(D):=\lambda_g\circ D\circ\lambda_g^{-1}\quad\text{and}\quad\rho(g)(D):=\rho_g\circ D\circ\rho_g^{-1}.$$

We define $\mathfrak{X}_l(G)$ and $\mathfrak{X}_r(G)$ to be the subspaces of $\mathfrak{X}(G)$ consisting of left-invariant and right-invariant derivations (i.e., invariant under $\lambda(g)$ and $\rho(g)$ for any $g\in G(k)$), respectively. Then $\mathfrak{X}_l(G)$ and $\mathfrak{X}_r(G)$ are Lie subalgebras of $\mathfrak{X}(G)$.

Fact 6.5. *The natural map*

$$\mathfrak{X}_l(G)\to T_eG\colon D\mapsto\operatorname{ev}_e\circ D$$

is an isomorphism as k-vector spaces. The same is true for $\mathfrak{X}_r(G)$.

[5]Recall that, here each element $f\in k[X]$ is regarded as a function on $X(k)$ by $f(x):=x^*(f)$, where $x^*\in\operatorname{Hom}_k(k[X],k)$ corresponds to $x\in X(k)=\operatorname{Mor}_k(\operatorname{Spec}k,X)$. So, the evaluation map "ev_x" in this context is nothing but x^*.

Especially, T_eG gets a Lie algebra structure over k. We write $\operatorname{Lie} G := T_eG$ and call it the *Lie algbebra* of G.

6.3. **Zariski cotangent space.** We next investigate another expression of Zariski tangent spaces. Let X be an affine algebraic varity with coordinate ring $k[X]$. Let $x \in X(k)$ its closed point, hence x is a morphism $\operatorname{Spec} k \to X$. Let $x^* \colon k[X] \twoheadrightarrow k$ be the corresponding k-algebra homomorphism. If we put $\mathfrak{m}_x := \operatorname{Ker}(x^*)$, then $\mathfrak{m}_x$ is a maximal ideal of $k[X]$ because the quotient $k[X]/\mathfrak{m}_x \cong k$ is a field.

Now let us take a tangent vector $D \in T_xX = D_k(k[X], k_x)$. Then we have $D(\mathfrak{m}_x^2) = 0$. Indeed, for any $f, g \in \mathfrak{m}_x$, the Leibniz rule implies that $D(f \cdot g) = f \cdot D(g) + g \cdot D(f)$. But $\mathfrak{m}_x$ acts on k_x via zero, we must have $f \cdot D(g) + g \cdot D(f) = 0$. Hence, the restriction $D|_{\mathfrak{m}_x}$ of D to $\mathfrak{m}_x$ induces a k-linear homomorphism from $\mathfrak{m}_x/\mathfrak{m}_x^2 \to k_x$. In other words, $D|_{\mathfrak{m}_x}$ is regarded as an element of the k-linear dual $(\mathfrak{m}_x/\mathfrak{m}_x^2)^* := \operatorname{Hom}_k(\mathfrak{m}_x/\mathfrak{m}_x^2, k)$.

Lemma 6.6. *The map*

$$T_xX = D_k(k[X], k_x) \to (\mathfrak{m}_x/\mathfrak{m}_x^2)^* \colon D \mapsto D|_{\mathfrak{m}_x}$$

is a k-linear isomorphism.

By this lemma, we can think of the k-vector space $\mathfrak{m}_x/\mathfrak{m}_x^2$ as the dual $(T_xX)^*$ of the tangent space T_xX. For this reason, we call $\mathfrak{m}_x/\mathfrak{m}_x^2$ the *cotangent space* of X at x.

Remark 6.7. Recall that a point $x \in X(k)$ is called smooth if $\dim(\mathcal{O}_{X,x}) = \dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2)$. By the above explanation, $\dim_k(\mathfrak{m}_x/\mathfrak{m}_x^2)$ is nothing but the dimension of the tangent space T_xX.

Example 6.8.
(1) Put $X := \mathbb{A}_k^n = \operatorname{Spec} k[x_1, \dots, x_n]$. Let us consider a point $x = (a_1, \dots, a_n) \in X(k)$, which corresponds to the maximal ideal $\mathfrak{m}_x = (x_1 - a_1, \dots, x_n - a_n)$ of $k[x_1, \dots, x_n]$. Then we have $\mathfrak{m}_x/\mathfrak{m}_x^2 = \oplus_{i=1}^n k(\overline{x_i - a_i})$, where $\overline{x_i - a_i}$ is the image of $x_i - a_i$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$. In particular, we see that $T_xX = (\mathfrak{m}_x/\mathfrak{m}_x^2)^*$ is n-dimensional.
(2) Put $X := \operatorname{Spec} k[x_1, x_2]/(x_1x_2)$. Let us determine the dimension of T_xX for $x = (0,0), (0,1) \in X(k)$.
   - When $x = (0,0)$, which corresponds to the maximal ideal $\mathfrak{m}_x = (x_1, x_2)$ of $k[x_1, x_2]/(x_1x_2)$, we have $\mathfrak{m}_x = (x_1^2, x_1x_2, x_2^2)$. Hence we have $\mathfrak{m}_x/\mathfrak{m}_x^2 = k(\overline{x_1}) \oplus k(\overline{x_2})$, where $\overline{x_1}$ and $\overline{x_2}$ are the images of x_1 and x_2 in $\mathfrak{m}_x/\mathfrak{m}_x^2$, respectively. In particular, T_xX is 2-dimensional.
   - When $x = (0,1)$, which corresponds to the maximal ideal $\mathfrak{m}_x = (x_1, x_2 - 1)$ of $k[x_1, x_2]/(x_1x_2)$, we have $\mathfrak{m}_x^2 = (x_1^2, x_1(x_2 - 1), (x_2 - 1)^2)$. By noting that $x_1x_2 = 0$, we have $\mathfrak{m}_x^2 = (x_1^2, -x_1, (x_2 - 1)^2) = (x_1, (x_2 - 1)^2)$. Hence we have $\mathfrak{m}_x/\mathfrak{m}_x^2 = k(\overline{x_2 - 1})$, where $\overline{x_2 - 1}$ is the image of $x_2 - 1$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$. In particular, T_xX is 1-dimensional.

Exercise 6.9. Put $X := \operatorname{Spec} k[x, y]/(y^2 - x^3)$. Compute T_xX for $x = (0,0)$ and $x = (1,1)$.

6.4. **Differential modules.** Recall that, in the manifold setting (let temorarily X denote a manifold here), we call an association $X \ni p \mapsto \omega_p \in (T_pX)^*$ such that ω_p "varies smoothly in $p \in X$" (i.e., $v_{(-)}$ is a global section of the cotangent bundle $T^*X \to X$) a *first differential form (1-form)* on X. Let us write $\Omega(X)$ for the set of

1-forms. Note that this has a $C^\infty(X)$-module structure by $(f\cdot\omega)_p := f(p)\omega_p$. On the other hand, the space $\mathfrak{X}(X)$ also has a $C^\infty(X)$-module structure by $(f\cdot v)_p := f(p)v_p$. It is a basic fact that then $\Omega(X)$ is naturally identified the the $C^\infty(X)$-dual of $\mathfrak{X}(X)$, i.e., $\mathrm{Hom}_{C^\infty(X)}(\mathfrak{X}(X), C^\infty(X))$:

$$\Omega(X) \xrightarrow{\cong} \mathrm{Hom}_{C^\infty(X)}(\mathfrak{X}(X), C^\infty(X))\colon \omega \mapsto [v \mapsto [p \mapsto \langle v_p, \omega_p\rangle]].$$

Another viewpoint of this isomorphism is as follows: the dual of this isomorphism as $C^\infty(X)$-modules is given by

$$\mathrm{Hom}_{C^\infty(X)}(\Omega(X), C^\infty(X)) \cong \mathfrak{X}(X) = D_{\mathbb{R}}(C^\infty(X), C^\infty(X))\colon \phi \mapsto \phi\circ d,$$

where $d\colon C^\infty(X) \to \Omega(X)$ is the exterior differential.

Now let us go back to the setting of algebraic varieties (so again X denotes an affine algebraic variety). In this context, we can also define an object completely analogous to $\Omega^1(X)$ as follows. Let us write $R := k[G]$ in short. We consider a surjective k-algebra homomorphism

$$q\colon R\otimes_k R \twoheadrightarrow R\colon x\otimes y \mapsto xy.$$

Then $\mathrm{Ker}(q)$ is an ideal of $R\otimes_k R$ such that $(R\otimes_k R)/\mathrm{Ker}(q) \cong R$. We define

$$\Omega_X := \mathrm{Ker}(q)/\mathrm{Ker}(q)^2.$$

Note that Ω_X has an action of $(R\otimes R)/\mathrm{Ker}(q)$, hence regarded as an R-module. We call Ω_X the *(Kähler) differential module* of R.

We can also define the "exterior differential map" $d\colon R\to\Omega_X$ as follows. For any $x\in R$, we have $x\otimes 1 - 1\otimes x \in \mathrm{Ker}(q)$; we let dx be its image in $\Omega_X = \mathrm{Ker}(q)/\mathrm{Ker}(q)^2$.

Proposition 6.10. *For any R-module M, we have*

$$\mathrm{Hom}_R(\Omega_X, M) \xrightarrow{\cong} D_k(R, M)\colon \phi\mapsto\phi\circ d.$$

In particular, by taking M to be R, we have

$$\mathrm{Hom}_R(\Omega_X, R) \xrightarrow{\cong} D_k(R, R) = \mathfrak{X}(X)\colon \phi\mapsto\phi\circ d.$$

Hence we see that Ω_X is completely analogous to the space of 1-forms $\Omega(X)$ in the theory of manifolds.

6.5. **Morphisms between Lie algebras.** Suppose that $f\colon X\to Y$ is a morphism of algebraic varieties with corresponding k-algebra homomorphism $f^*\colon k[Y]\to k[X]$. For any $x\in X(k)$, we let df_x denote the k-linear homomorphism naturally induced on the tangent spaces:

$$df_x\colon T_xX \to T_{f(x)}Y\colon D\mapsto D\circ f^*.$$

When $f\colon G\to H$ is a homomorphism of algebraic groups, we write $df := df_e$.

The aim of this subsection is to show the following.

Proposition 6.11. *Let $f\colon G\to H$ be a homomorphism of algebraic groups. Then the induced map $df\colon \mathrm{Lie}\,G\to\mathrm{Lie}\,H$ is a homomorphism of Lie algebras.*

The following is a special (but a refined) version of the above proposition:

Lemma 6.12. *Let $f\colon G\to H$ be a closed immersion of algebraic groups. Then the induced map $df\colon \mathrm{Lie}\,G\to\mathrm{Lie}\,H$ is an injective homomorphism of Lie algebras. Moreover, if we write I for the kernel of $f^*\colon k[H]\twoheadrightarrow k[G]$ (i.e., $k[G]\cong k[H]/I$), then we have $df(\mathrm{Lie}\,G) = \{v\in T_eH \mid v(I)=0\}$.*

Here let us give some supplementary comments on the fibered product.

Definition 6.13. Let X and Y be schemes over k. We say that a scheme Z over k is the *fibered product* of X and Y over k if it is equipped with morphisms $p_1\colon Z \to X$ and $p_2\colon Z \to Y$ over k satisfying the following universality property:

> For any scheme Z' over k equipped with morphisms $p_1'\colon Z' \to X$ and $p_2'\colon Z' \to Y$ over k, there exists a unique morphism $q\colon Z' \to Z$ over k such that $p_1' = p_1 \circ q$ and $p_2' = p_2 \circ q$.

$$\begin{tikzcd}
Z' \arrow[dr, dashed, "q"] \arrow[drr, "p_2'"] \arrow[ddr, "p_1'"'] & & \\
& Z \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d] \\
& X \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

By this property, (Z, p_1, p_2) is determined uniquely up to unique isomorphisms if exists. We write $X \times_k Y$ for the fibered product of X and Y over k. We often write (p_1', p_2') for q.

In fact, the fibered product always exists. In the case where X and Y are affine, it is simply given by the affine scheme associated to the tensor product $k[X] \otimes_k k[Y]$ of coordinate rings of X and Y. (Indeed, the tensor product has the universality property for the diagram opposite to the above one.)

Remark 6.14. The rational points of the fibered product can be identified with the product of rational points, i.e., $(X \times_k Y)(R) \cong X(R) \times Y(R)$. In terms of the above diagram, this identification is given by $q \mapsto (p_1 \circ q, p_2 \circ q)$.

Proof of Proposition 6.11. Let $f\colon G \to H$ be a homomorphism of affine algebraic groups. We consider the following diagram:

$$\begin{tikzcd}
G \arrow[dr, dashed, "(\mathrm{id}, f)"] \arrow[drr, "f"] \arrow[ddr, "\mathrm{id}"'] & & \\
& G \times_k H \arrow[r, "p_2"] \arrow[d, "p_1"] & H \arrow[d] \\
& G \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

In particular, f is decomposed as $p_2 \circ (\mathrm{id}, f)$. Since both p_2 and (id, f) are homomorphisms of algebraic groups, it is enough to check the assertion for these homomorphism.

Concerning p_2, in general, we can show that $T_{(e,e)}(G \times_k H)$ is isomorphic to $T_eG \oplus T_eH$ as Lie algebras. In particular, the map $dp_2\colon \mathrm{Lie}(G \times_k H) \to \mathrm{Lie}(H)$ is nothing but the projection of $\mathrm{Lie}(G \times_k H) \cong \mathrm{Lie}(G) \oplus \mathrm{Lie}(H)$ onto the second part, hence the Lie algbera homomorphism.

On the other hand, the morphism (id, f) is a closed immersion. Indeed, at the level of rings, if we write $f^*\colon k[H] \to k[G]$ for the k-algebra homomorphism corresponding to f, then (id, f) corresponds to

$$(\mathrm{id}, f)^*\colon k[G \times_k H] = k[G] \otimes_k k[H] \to k[G]\colon x \otimes y \mapsto x f^*(y),$$

which is obviously surjective. Thus, this case is already treated in the above lemma. □

6.6. **Examples.**

6.6.1. *Additive group.* Let us consider the case of $G = \mathbb{G}_{\mathrm{a}}$. Recall that $k[\mathbb{G}_{\mathrm{a}}] = k[x]$. Let us first compute

$$\mathfrak{X}(\mathbb{G}_{\mathrm{a}}) = D_k(k[x], k[x]).$$

Any element $D \in \mathfrak{X}(\mathbb{G}_{\mathrm{a}})$ is determined by $D(x) \in k[x]$. Indeed, by Leibniz rule, we must have

$$D(x^2) = 2xD(x), \quad D(x^3) = 3x^2 D(x), \quad \ldots, \quad D(x^n) = nx^{n-1}D(x), \quad \ldots$$

Conversely, for any polynomial $f \in k[x]$, we can define an element $D \in \mathfrak{X}(\mathbb{G}_{\mathrm{a}})$ by the formula $D(x^n) := nx^{n-1}f$.

The right translation action of $a \in G(k) = k$ on $k[G] = k[x]$ is given by $x \mapsto x+a$. Hence, with the above notation, an element $D \in \mathfrak{X}(\mathbb{G}_{\mathrm{a}})$ is translation-invarinat if and only if $f(x+a) = f(x)$ for any $a \in k$. It is easy to check that this condition is equivalent to that $f(x) = c$ for some $c \in k$.

Recall that we have an isomorphism

$$\mathfrak{X}_r(\mathbb{G}_{\mathrm{a}}) \xrightarrow{\cong} T_0\mathbb{G}_{\mathrm{a}} = D_k(k[x], k_0) \colon D \mapsto \mathrm{ev}_0 \circ D.$$

The evaluation map $\mathrm{ev}_0 \colon k[x] \to k_0$ is just given by $x \mapsto 0$. Thus, when $f(x) = c$, we have $\mathrm{ev}_0 \circ D(x) = c$. This enables us to identify $T_0\mathbb{G}_{\mathrm{a}}$ with k. Furthermore, by this description, we immediately see that the bracket structure on $\mathfrak{X}_r(\mathbb{G}_{\mathrm{a}}) \xrightarrow{\cong} T_0\mathbb{G}_{\mathrm{a}}$ is trivial (zero).

In summary, $\operatorname{Lie}\mathbb{G}_{\mathrm{a}}$ is the 1-dimensional vector space k with trivial Lie bracket.

6.6.2. *Multiplicative group.*

Exercise 6.15. Determine the Lie algebra of $\mathbb{G}_{\mathrm{m}}$ by imitating the computation as in the case of $\mathbb{G}_{\mathrm{a}}$.

6.6.3. *General linear group.* Let us consider the case of $G = \mathrm{GL}_n$. Recall that $k[\mathrm{GL}_n] = k[\{x_{ij} \mid 1 \le i,j \le n\}, D^{-1}]$.

We consider the Lie algebra $\mathfrak{gl}_n(k)$, which is the k-vector space $M_n(k)$ of n-by-n matrices with entries in k equipped with the bracket product $[A,B] := AB - BA$. For any $A = (a_{ij})_{ij} \in \mathfrak{gl}_n(k)$, we consider an element D_A of $\mathrm{Hom}_k(k[\mathrm{GL}_n], k[\mathrm{GL}_n])$ defined By

$$D_A(x_{ij}) := -\sum_{l=1}^{n} x_{il}a_{lj}$$

In fact, this element satisfies the Leibniz rule, hence an element of $\mathfrak{X}(\mathrm{GL}_n) = D_k(k[\mathrm{GL}_n], k[\mathrm{GL}_n])$. Furthermore, it can be checked that D_A is invariant under translation, hence an element of $\operatorname{Lie}\mathrm{GL}_n$.

The k-linear map $\mathfrak{gl}_n(k) \to \operatorname{Lie}\mathrm{GL}_n \colon A \mapsto D_A$ is obviously injective by construction. Since the dimensions of both spaces $\mathfrak{gl}_n(k) \to \operatorname{Lie}\mathrm{GL}_n$ are n^2, this is bijective. It is also a routine work to check that the map $A \mapsto D_A$ preserves the bracket products, i.e., $D_{AB-BA} = D_A \circ D_B - D_B \circ D_A$.

7. Week 7: Reductive groups and root data

As usual, let k be an algebraically closed field.

7.1. **Definition of reductive and semisimple groups.**

Proposition 7.1. *Let G be a connected linear algebraic group over k. There uniquely exists a maximal closed connected normal solvable subgroup.*

Proof. Suppose that H_1 and H_2 are closed connected normal solvable subgroups of G. We define a subvariety H_1H_2 of G to be the closure of the image of the multiplication morphism $m\colon H_1 \times_k H_2 \to G$. We claim that H_1H_2 is again a closed connected normal solvable subgroup of G; if we can show this, the unique existence of a maximal such subgroup follows immediately.

First, H_1H_2 is by definition closed. Since H_1 and H_2 are connected, $m(H_1 \times_k H_2)$ is also connected, hence so is H_1H_2. Note that $H_1(k) \times H_2(k) \cong (H_1 \times_k H_2)(k)$ is dense in $H_1 \times_k H_2$, hence its image under m (i.e., $H_1(k) \cdot H_2(k) := \{h_1h_2 \in G(k) \mid h_1 \in H_1(k), h_2 \in H_2(k)\}$) is also dense in H_1H_2. Hence, to check that H_1H_2 is a normal solvable subgroup, it is enough to check that $H_1(k) \cdot H_2(k)$ is a normal solvable subgroup of $G(k)$.

Thus now the claim is reduced to a purely group-theoretic problem. Since $H_1(k)$ and $H_2(k)$ are normal subgroup in $G(k)$, so is $H_1(k) \cdot H_2(k)$. Indeed, for any $h_1, h_1' \in H_1(k)$ and $h_2, h_2' \in H_2(k)$, we have $(h_1h_2) \cdot (h_1'h_2') = (h_1h_1') \cdot (h_1'^{-1}h_2h_1') \cdot (h_2') \in H_1(k) \cdot H_2(k)$, hence $H_1(k) \cdot H_2(k)$ is closed under the multiplication. Also, for any $h_1 \in H_1(k)$ and $h_2 \in H_2(k)$, the inverse of h_1h_2 is given by $h_2^{-1}h_1^{-1} = (h_2^{-1}h_1^{-1}h_2) \cdot h_2^{-1} \in H_1(k) \cdot H_2(k)$, hence lies in $H_1(k) \cdot H_2(k)$. Furthermore, for any $g \in G(k)$ and $h_1 \in H_1(k)$, $h_2 \in H_2(k)$, we have $gh_1h_2g^{-1} = (gh_1g^{-1})(gh_2g^{-1}) \in H_1(k) \cdot H_2(k)$, which means that $H_1(k) \cdot H_2(k)$ is normal.

Let us finally check that $H_1(k) \cdot H_2(k)$ is solvable. Recall that solvable groups are closed under extensions, i.e., if we have a short exact sequence $1 \to N \to H \to Q \to 1$ of groups such that N and Q are solvable, then so is H. We apply this fact to the following short exact sequence:

$$1 \to H_1(k) \to H_1(k) \cdot H_2(k) \to H_1(k) \cdot H_2(k)/H_1(k) \to 1.$$

The first term $H_1(k)$ is solvable by assumption. On the other hand, we have an isomorphism $H_1(k) \cdot H_2(k)/H_1(k) \cong H_2(k)/(H_1(k) \cap H_2(k))$; the latter is a quotient of a solvable group $H_2(k)$, hence also solvable. Thus we conclude that $H_1(k) \cdot H_2(k)$ is solvable. □

We call the subgroup as in this proposition the *radical* of G and write $R(G)$.

We next define the notion of the "unipotent radical". For this, we utilize the following (Week 4; here, we again postpone the proof):

Theorem 7.2 (Lie–Kolchin's theorem)**.** *Let G be a connected closed subgroup of GL_n. If G is solvable, then G is conjugate to a subgroup of the upper-triangular matrices B_n.*

Proposition 7.3. *Let G be a connected linear algebraic group over k. There uniquely exists a maximal closed connected normal unipotent subgroup, which is given by the unipotent locus $R(G)_u$ of $R(G)$.*

Proof. Recall that a linear algebraic group is called "unipotent" if all its k-rational points are unipotent. Also recall that $R(G)_u$ is defined to be the closed subvariety of $R(G)$ satisfying $R(G)_u(k) = R(G)(k)_u$ (the latter is the subset of unipotent elements of $R(G)(k)$).

Note that, in general the unipotent locus of a linear algebraic group might not be a subgroup. However, we claim that $R(G)_u$ is a subgroup. Indeed, by Lie–Kolchin's theorem, we may regard $R(G)$ as a subgroup of $B_n \subset \mathrm{GL}_n$ for some n. The unipotent locus of B_n is the subgroup of upper-triangular unipotent matrices U_n. Hence $R(G)_u = R(G) \cap U_n$ is a subgroup of $R(G)$. Note that, this can be also viewed as the image of $R(G)$ under the projection $B_n \cong T_n \times U_n \twoheadrightarrow U_n$; in particular, $R(G)_u$ is the image of a connected group, hence connected. Moreover, since $R(G)$ is normal in G and "being unipotent" is preserved by the conjugation action, $R(G)_u$ is a normal subgroup of G. In summary, we have checked that $R(G)_u$ is a closed connected normal unipotent subgroup of G.

Let us show the maximality of $R(G)_u$. Suppose that H is another closed connected normal unipotent subgroup of G. Since any unipotent subgroup is solvable (or even nilpotent; see Proposition 4.10 in Week 4), H is contained by $R(G)$ by the maximality of $R(G)$. Again by noting that H is unipotent, H is necessarily contained in $R(G)_u$. □

We call the subgroup as in this proposition the *unipotent radical*.

Definition 7.4 (semisimple/reductive groups)**.** Let G be a connected linear algebraic group over k.

(1) We say that G is *semisimple* if $R(G)$ is trivial.
(2) We say that G is *reductive* if $R(G)_u$ is trivial.

Note that any semisimple group is necessarily reductive.

7.2. **Examples.**

7.2.1. *Commutative groups.* Let G be any connected commutative linear algebraic group. Then G itself is solvable, hence we have $R(G) = G$. This means that G can never be semisimple unless $G = \{1\}$.

If G is a torus, i.e., $G \cong \mathbb{G}_\mathrm{m}^r$, all elements of $G(k)$ are semisimple. Hence $R_u(G) = G_u = \{1\}$. Thus G is not semisimple but reductive.

If G is the additive group $\mathbb{G}_\mathrm{a}$, all elements of $G(k)$ are unipotent. For example, this can be seen by choosing the following embedding of $\mathbb{G}_\mathrm{a}$ into GL_2:

$$\mathbb{G}_\mathrm{a} \hookrightarrow \mathrm{GL}_2 \colon x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}.$$

Hence $R_u(\mathbb{G}_\mathrm{a}) = (\mathbb{G}_\mathrm{a})_u = \mathbb{G}_\mathrm{a}$. Thus $\mathbb{G}_\mathrm{a}$ is not reductive.

More generally, recall that any commutative linear algebraic group G has a decomposition $G \cong G_s \times G_u$. Hence G is reductive if and only if $G = G_s$, which is equivalent to that G is a torus (because G_s is connected).

7.2.2. *General linear group.* Let $G := \mathrm{GL}_n$. Since $R(G)$ is solvable, $R(G)$ is conjugate to a subgroup of B_n by Lie–Kolchin's theorem. However, as $R(G)$ is normal in G, this implies that $R(G)$ already lies in B_n. Let us consider the following element:

$$w := \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix} \in G(k) = \mathrm{GL}_n(k).$$

Since $R(G)$ is normal in G, we must have $R(G) = wR(G)w^{-1} \subset wB_nw^{-1} = \overline{B}_n$, where $\overline{B}_n$ is the subgroup of lower-triangular matrices. Therefore, we have $R(G) \subset B_n \cap \overline{B}_n = T_n$. At least T_n does not have any (nontrivial) unipotent element, hence we can conclude that $R_u(G)$ is trivial, which means that GL_n is reductive.

Let us also determine $R(G)$. We claim that $R(G) = Z$, where Z denotes the subgroup consisting of scalar matrices (the center of G). As Z is a closed connected normal subgroup, we must have $Z \subset R(G)$. For the sake of contradiction, let us assume $Z \subsetneq R(G) \subset T_n$. Then we can find an element $\mathrm{diag}(t_1, \ldots, t_n) \in R(G)(k)$ such that at least some two entries are different, say $t_i \neq t_j$. If we put, for example, $g := I_n + E_{ij}$, where I_n is the identity matrix and E_{ij} is the matrix whose entries are zero expect for (i, j) and 1 for (i, j), then we can check that gtg^{-1} is not a diagonal matrix (with our choice of g, the (i, j)-entry survives). In particular, this means that $gtg^{-1} \notin R(G)$, which contradicts the normality of $R(G)$.

Exercise 7.5. Let $G := \mathrm{SL}_n$. Recall that this is a closed subgroup GL_n such that

$$\mathrm{SL}_n(R) = \{g \in \mathrm{GL}_n(R) \mid \det(g) = 1\}$$

for any k-algebra R. Show that SL_n is semisimple.

In fact, the following holds:

Fact 7.6. *A connected reductive group is semisimple if and only if its center is finite.*

7.2.3. *Classical groups.* Here I just emphasize that *classical groups* are also very important examples of reductive groups. But we will investigate them later, after learning more about the notion of "root systems".

7.3. **Classification theorem.** Now let us state the classification theorem of connected reductive groups, which is the main goal of this course:

Theorem 7.7 (Classification of reductive groups)**.** *There exists a bijection between the set of*

- *isomorphism classes of connected reductive groups and*
- *isomoprhism classes of reduced root data.*

At this point, the statement of Theorem 7.7 is not quite clear. Firstly, we have not defined the notion of a root datum. Secondly, it is not explained whether the bijection can be given explicitly.

In the following, we explain what a root datum is. Also, we explain how a root datum can be associated to a connected reductive group in the case of GL_n. However, we cannot explain the procedure for general connected reductive groups; for it, we need more about generalities on the structure theory of connected reductive groups, especially, "Borel subgroups" and "maximal tori". So the aim of this week is to provide enough motivation to tackle them.

7.4. **Root systems and root data.**

Definition 7.8. A *root system* is a pair (V, R) of a finite-dimensional $\mathbb{R}$-vector space V and its finite subset $R \subset V$ satisfying the following:

1. $0 \notin R$ and $V = \mathrm{Span}_{\mathbb{R}}(R)$;
2. for each $\alpha \in R$, there exists an $\alpha^\vee \in V^\vee$ such that
   (a) $\langle \alpha, \alpha^\vee \rangle = 2$,

(b) $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$,
(c) $s_\alpha(R) = R$ for any $\alpha \in R$, where $s_\alpha \colon V \to V$ denotes the "reflection" with respect to α:

$$s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha.$$

Each $\alpha^\vee$ is called the *coroot* of α.

We say that a root datum (V, R) is *reduced* if for any $\alpha \in R$, we have $R \cap \mathbb{Q}\alpha = \{\pm\alpha\}$.

Remark 7.9. Depending on the context, a root system is sometimes defined in terms of an inner product $(-,-)$ on V instead of a canonical pairing $\langle -, - \rangle$.

A root datum is an enhancement of a root system as follows.

Definition 7.10 (root datum)**.** A *root datum* is a quadruple $(X, R, X^\vee, R^\vee)$, where

- X and $X^\vee$ are free abelian groups of finite rank equipped with a perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$ and
- R and $R^\vee$ are finite subsets of X and $X^\vee$ (called the sets of *roots* and *coroots*) equipped with a bijection $R \leftrightarrow R^\vee \colon \alpha \mapsto \alpha^\vee$

satisfying

(1) for any $\alpha \in R$, we have $\langle \alpha, \alpha^\vee \rangle = 2$,
(2) for any $\alpha \in R$, we have $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

Here, s_α and $s_\alpha^\vee$ denote the automorphisms of X and $X^\vee$ given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^\vee(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle \alpha^\vee.$$

We say that a root datum $(X, R, X^\vee, R^\vee)$ is *reduced* if for any $\alpha \in R$, we have $R \cap \mathbb{Q}\alpha = \{\pm\alpha\}$.

For a root datum $(X, R, X^\vee, R^\vee)$, we call $Q := \mathrm{Span}_{\mathbb{Z}}(R) \subset X$ the *root lattice.* By looking at the definitions, we can easily see that $(Q \otimes_{\mathbb{Z}} \mathbb{R}, R)$ forms a root system. The point here is that the associted root system $(Q \otimes_{\mathbb{Z}} \mathbb{R}, R)$ does not remember the integral structure X of the original root datum. In particular, it could happen that two non-isomorphic root data have the same (isomorphic) root system.

In fact, from the viewpoint of the classification theorem, taking the root systems associated to root data amounts to taking the Lie algebras of connected reductive groups. There is a puley algebraic notion of a "reductive Lie algebra"; it can be proved that the Lie algebra of any connected reductive group is a reductive Lie algebra. Moreover, there is a classification of reductive Lie algebras in terms of reduced root systems. Theorem 7.7 is consistent with this classification of reductive Lie algebras. (I hope to explain all these stories in more detail eventually.)

Here let us introduce the notion of the rank and simple roots of a root system, which plays an important role in classifying root systems.

Fact 7.11. *For any root system (V, R), there exists a finite set of roots $\{\alpha_1, \ldots, \alpha_l\}$ satisfying the following property:*

Any root $\alpha \in R$ can be uniquely written as follows:

$$\alpha = n_1\alpha_1 + \cdots + n_l\alpha_l,$$

where $n_1, \ldots, n_l \in \mathbb{Z}$ are either all-positive or all-negative.

Definition 7.12. We call a subset of roots as in the above fact a set of *simple roots* of (V, R). When the coefficient of a root $\alpha \in R$ in the linear combination expression via simple roots are all-positive (resp. all-negative), we say α is a *positive* (resp. *negative*) root.

Note that a set of simple roots is NOT unique. Therefore, the notion of positive/negative roots is NOT canonical; depends on the choice of a set of simple roots. However, the number of simple roots is independent of the choice of a set of simple roots. We call the number of simple roots the *rank* of the root system.

7.5. **Root datum of** GL_n**.** Now let us explain how to produce a root datum (with a set of simple roots) from G in the case where $G = \mathrm{GL}_n$. The key of the construction is the subgroups $T_n \subset B_n \subset \mathrm{GL}_n$.

The diagonal torus T_n acts on GL_n by conjugation. Thus each element of $T_n(k)$ gives an algebraic group automorphism of GL_n, which induces a Lie algebra automorphism of $\mathfrak{g}$, where $\mathfrak{g} := \mathrm{Lie}\, \mathrm{GL}_n \cong \mathfrak{gl}_n(k)$. In other words, we get an action of $T_n(k)$ on $\mathfrak{g}$. This action is called the *adjoint* action. The point here is that we can take a simlutaneous eigenspace decomposition of $\mathfrak{g}$ with respect to this action of $T(k)$.

More concretely, the decomposition is described as follows. We choose a k-basis of $\mathfrak{g}$ to be

$$\{E_{ij} \mid 1 \leq i, j \leq n\},$$

where E_{ij} denotes the n-by-n matrix whose (i, j)-entry is 1 and all the other entries are 0. Then the decomposition

$$\mathfrak{g} = \mathfrak{gl}_n(k) = \bigoplus_{1 \leq i,j \leq n} kE_{ij}$$

gives the simlutaneous eigenspace decomposition of the action of $T_n(k)$. To see this, let us first note that the action of $T_n(k)$ on $\mathfrak{g} = \mathfrak{gl}_n(k)$ is given by the usual conjugation of matrices, i.e., for any $t \in T_n(k)$ and $A \in \mathfrak{g}(k)$, we have $t \cdot A = tAt^{-1}$. Thus, if we write $t = \mathrm{diag}(t_1, \ldots, t_n)$ with $t_i \in k^\times$, then we have

$$t \cdot E_{ij} \cdot t^{-1} = t_i \cdot t_j^{-1} \cdot E_{ij}.$$

In other words, the action of t on each subspace kE_{ij} is given though the map

$$T_n(k) \to k^\times \colon t = \mathrm{diag}(t_1, \ldots, t_n) \mapsto t_i/t_j.$$

Here, let us recall that the space $X := \mathrm{Hom}_k(T, \mathbb{G}_\mathrm{m})$ of characters of T_n is given by

$$X \cong \mathrm{Hom}_k(\mathbb{G}_\mathrm{m}^n, \mathbb{G}_\mathrm{m}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}e_i,$$

where e_i denotes the character $T_n \to \mathbb{G}_\mathrm{m}$ given by $e(\mathrm{diag}(t_1, \ldots, t_n)) \mapsto t_i$ at the level of k-rational points (Week 5). In particular, the above action of $T_n(k)$ on each simultaneous eigenspace kE_{ij} is actually algebraic and given by the chracter $e_i - e_j$. Also note that, when $i = j$, the action of $T_n(k)$ on kE_{ij} is trivial. The subspace $\bigoplus_{i=1}^n kE_{ii}$ is naturally identifies with the Lie subalgebra $\mathfrak{t} := \mathrm{Lie}\, T_n$ of $\mathfrak{g} = \mathrm{Lie}\, \mathrm{GL}_n$.

Therefore, we have obtained the simultaneous eigenspaces decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{1 \leq i \neq j \leq n} \mathfrak{g}_{e_i - e_j},$$

where $\mathfrak{g}_{e_i-e_j}$ denotes the 1-dimensional subspace where $T_n(k)$ acts through the character $e_i - e_j$. We define

$$R := \{e_i - e_j \mid 1 \leq i \neq j \leq n\},$$

which is a finite subset of X.

We define

$$X^\vee := \mathrm{Hom}_k(\mathbb{G}_\mathrm{m}, T_n) \cong \bigoplus_{i=1}^{n} \mathbb{Z}e_i^\vee,$$

where $e_i^\vee$ denotes the cocharacter $\mathbb{G}_\mathrm{m} \to T_n$ given by $x \mapsto \mathrm{diag}(1, \ldots, 1, x, 1, \ldots, 1)$ at the level of k-rational points and also

$$R^\vee := \{e_i^\vee - e_j^\vee \mid 1 \leq i \neq j \leq n\}.$$

Then, in fact, $(X, R, X^\vee, R^\vee)$ forms a root datum.

Furthermore, we consider the Lie subalgebra $\mathfrak{b} := \mathrm{Lie}\, B_n$ of $\mathfrak{g}$ associated to the upper-triangular subgroup B_n. Note that it also has a simultaneous decomposition with respect to the action of $T_n(k)$ and that the "roots" contained in $\mathfrak{b}$ are only $e_i - e_j$'s such that $i < j$:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{1 \leq i < j \leq n} \mathfrak{g}_{e_i - e_j}.$$

In fact, these roots $\{e_i - e_j \mid 1 < i \neq j \leq n\}$ forms the subset of positive roots of R with respect to the following set ot simple roots:

$$\{e_1 - e_2, e_2 - e_3, \ldots, e_{n-1} - e_n\}.$$

Let us summarize what have happened so far.

(1) We consider the conjugate action of T_n on GL_n, which induces an algebraic action on the Lie algbera $\mathfrak{gl}_n$.
(2) We decompose the adjoint action into the eigenspaces labeled by characters of T_n.
(3) We define $X := \mathrm{Hom}_k(T_n, \mathbb{G}_\mathrm{m})$ and let $R \subset X$ be the subset of characters appearing in the decomposition.
(4) We define $X^\vee := \mathrm{Hom}_k(\mathbb{G}_\mathrm{m}, T_n)$ and also $R^\vee$ to be the "dual" of $R^\vee$ to get a root datum $(X, R, X^\vee, R^\vee)$.
(5) We also consider the Lie algbera of B_n to get a set of simple roots.

Note that thus our construction crucially relies on the subgroups $T_n \subset B_n$ of GL_n. In fact, all the procedures here can be generalized to any connected reductive groups once we can generalize "T_n" and "B_n" in a conceptual way to any conenected reductive groups.

The torus T_n of GL_n is generalized to a "maximal torus" of a connected reductive group. Its definition is simple; it is just a torus contained in GL_n which is maximal among all such tori. On the other hand, the notion of a "Borel subgroup", which generalizes B_n of GL_n, is more difficult to state (this will be given next week, hopefully). Furthermore, even if we can arrive at the definition of a Borel subgroup and imitate all the above constructions for general connected reductive groups, it is not clear at all whether the resulting root datum is determined canonically in any sense. For example, the choices of a maximal torus and a Borel subgroup of a given connected reductive group are not unique at all. In fact, the root datum obained by this construction is, up to isomorphism of root data, independent of such choices. However, to prove it, we need to appeal some deep group-theoretic properties of maximal tori or Borel subgroups.

In the next few weeks, we will learn the definitions and properties of these subgroups. Please keep in mind that one of the motivations is to establish a connection between connected reductive groups and root data.

8. Week 8: Quotients of algebraic groups

In the classical group theory, taking the quotient of a group by its subgroup is a very basic operation. The aim of this week is to discuss the notion of the "quotient" in the context of algebraic groups. To be more precise, we want to consider the quotient of a linear algebraic group G by its closed subgroup H.

8.1. **Quotient of topological spaces.** Let us start with reviewing the quotients for topological spaces. Let X be a topological space. We consider an equivalence relation $\sim$ on X and consider the quotient space $\tilde{X} := X/{\sim}$. To be more precise, $\tilde{X}$ is the set of equivalence classes of X with respect to $\sim$. Hence we have a natural surjective map

$$\pi\colon X \twoheadrightarrow \tilde{X} = X/{\sim}.$$

The topology on $\tilde{X}$ ("quotient topology") is given so that a subset $U \subset \tilde{X}$ is open if and only if $\pi^{-1}(U) \subset X$ is open. In other words, it is the finest topology on $\tilde{X}$ such that π is continuous.

In particular, when a group G acts on a topological space X, we may consider the quotient of X with respect to the equivalence relation given by the action of G (i.e., $x \sim y$ if and only if $y = g \cdot x$ for some $g \in G$). In this case, we write X/G for the quotient topological space.

8.2. **Quotient of algebraic varieties.** Next we consider the "quotient" for algebraic varieties, especially in the case of a group action. Suppose that X is an algebraic variety (defined over an algebraically closed field k, as usual). We also assume that an algebraic group G acts on X algebraically, i.e., we have a morphism

$$\alpha\colon G \times_k X \to X$$

of algebraic varieties satisfying the axiom of a group action (Week 2). Let us consider whether the quotient of X by G exists in this situation. However, here X has more structures, i.e., X is a topological space equpped with a sheaf. So the meaning of the "quotient" is not clear a priori. To investigate the existence of the quotient, we first have to clarify what the "quotient" does mean in this context.

Recall that, for each k-algebra R, we get a group action (in the classical sense) of $G(R)$ on $X(R)$. Keeping this in mind, one could try to define the "quotient" as follows:

> When we have a morphism $\pi\colon X \to Y$ of algebraic varieties such that the induced map $\pi_R\colon X(R) \to Y(R)$ is equal to the quotient map $X(R) \twoheadrightarrow X(R)/G(R)$ for each k-algebra R, we call Y the "quotient" of X by G.

However, this definition of a quotient is in fact too strong. Especially, in constrast to the case of topological spaces, it can happen that the quotient does not exists!

To see this, let us consider the following example.

Example 8.1. Let $G := \mathbb{G}_{\mathrm{m}}$ and $X := \mathbb{A}_k^2$. We define an action of G on X by

$$\alpha\colon G \times_k X \to X; \quad (z, (x, y)) \mapsto (zx, zy)$$

(at the level of R-points for each R). Let us consider what will happen if there exists the "quotient variety" $\pi\colon X \twoheadrightarrow X/G$ in the above sense. At least we must have $(X/G)(k) \cong X(k)/G(k)$ and π_k is the quotient map $X(k) \to X(k)/G(k)$. Recall that, for algebraic varieties, the set of k-rational points are nothing but the set of closed points. Therefore, the fiber $\pi^{-1}(p)$ of any k-rational point $p \in (X/G)(k)$ must be closed. However, each fiber $\pi^{-1}(p)(k)$ is

- a 1-dimensional line $k^\times \cdot (x,y)$ minus $(0,0)$ for some $(x,y) \neq (0,0)$, or
- the origin $(0,0)$.

In particular, $(0,0)$ is contained in the closure of any fiber $\pi^{-1}(p)(k)$ of the form $k^\times \cdot (x,y) \smallsetminus \{(0,0)\}$. This means that $\pi^{-1}(p)(k)$ is not closed unless it is $(0,0)$, hence we get a contradiction. Thus the "quotient X/G" in the above sense cannot exist.

Thus it seems better to try to seek another idea of defining the "quotients" for an algebraic variety. For this, let us again examine the quotients for topological spaces. We first note that the quotient $\tilde{X} = X/\sim$ of a topological space X by an equivalence relation $\sim$ is characterized as a unique (up to homeomorphism) topological space satisfying the following universal property:

> For any continuous map $f\colon X \to Y$ from X to a topological space Y such that $f(x) = f(x')$ whenever $x \sim x'$, there uniquely exists a continuous map $g\colon \tilde{X} \to Y$ satisfying $f = g' \circ \pi$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ \tilde{X} \arrow[r, dotted, "\exists! g"'] & Y \end{tikzcd}$$

When the equivalence relation $\sim$ is given by an action of a group G on X (write $\alpha\colon G \times X \to X$ for the action map), we can also rewrite the condition on "$f(x) = f(x')$ whenever $x \sim x'$" as follows:

> A map $f\colon X \to Y$ satisfies $f(x) = f(x')$ for any x and x in the same G-orbit if and only if the maps $f \circ \alpha$ and $f \circ \mathrm{pr}_2$ from $G \times X \to Y$ coincide.

$$\begin{tikzcd} G \times X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Based on these observations, let us try another definition of the quotient as follows. Again let G be an algebraic group acting on an algebraic variety X.

Definition 8.2. We say that a morphism $f\colon X \to Y$ between algebraic varieties is *G-invariant* if the morphisms $f \circ \alpha$ and $f \circ \mathrm{pr}_2$ from $G \times X \to Y$ coincide.

$$\begin{tikzcd} G \times X \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\mathrm{pr}_2"'] & X \arrow[r, "f"] & Y \end{tikzcd}$$

Definition 8.3. When an algebraic variety Z equipped with a G-invariant morphism $\pi\colon X \to Z$ satisfies the following universal property, we call Z the *categorical quotient* of X by G and write $X/\!/G$ for it:

> For any G-invariant morphism $f\colon X \to Y$, there uniquely exists a morphism $g\colon Z \to Y$ satisfying $f = g \circ \pi$.

$$\begin{tikzcd} X \arrow[d, "\pi"'] \arrow[dr, "f"] & \\ Z \arrow[r, dotted, "\exists! g"'] & Y \end{tikzcd}$$

Remark 8.4. The meaning of the word "categorical" can be made precise in terms of category theory. The above universal property is nothing but the universal property of a "coequalizar", which is a very special case of a "colimit". More previsely, the categorical quotient $X \to X/\!/G$ is the coequalizer of the morphisms α and pr_2 in the category of algebraic varieties over k.

8.3. **GIT quotient.** Even we define the notion of the quotient in the above way, the existence of the quotient is not guaranteed in general. However, we can make the following observation in the case where everything is affine. Let us suppose that

- G is affine,
- X is affine,
- the categorical quotient $X/\!/G$ exists and it is an affine variety.

Recall that the action $\alpha\colon G \times_k X \to X$ induces an action of $G(k)$ on the coordinate ring $k[X]$. Then, by going back to the definitions, we can see that a morphism $f\colon X \to Y$ to an affine algebraic variety Y is G-invariant if and only if the corresponding morphism on the coordinatering $f^*\colon k[Y] \to k[X]$ has the image in $k[X]^{G(k)}$. Hence the universal property says, for any such f^*, there uniquely exists $g^*\colon k[Y] \to k[X/\!/G]$ satisfying $\pi^* \circ g^* = f^*$.

$$
\begin{tikzcd}
k[X]^{G(k)} \arrow[r, hookrightarrow] & k[X] \\
k[X/\!/G] \arrow[u, "\pi^*"] & k[Y] \arrow[l, dotted, "\exists! g^*"] \arrow[ul, "f^*"']
\end{tikzcd}
$$

From this observation, we obviously see that a canonical candidate for the categorical quotient $X/\!/G$ is the spectrum of the invariant ring $k[X]^{G(k)}$ itself. Only the issue here is that whether $k[X]^{G(k)}$ is a reduced k-algebra of finite type so that it is the coordinate ring of an affine algebraic variety in our sense. In other words, as long as $k[X]^{G(k)}$ is a reduced k-algebra of finite type, the affine algebraic variety $\mathrm{Spec}(k[X]^{G(k)})$ has the desired universal property at least for G-invariant morphisms $f\colon X \to Y$ to any affine algebraic variety Y.

In fact, the following is known:

Theorem 8.5 (Hilbert's finiteness theorem)**.** *When G is a connected reductive group acting on an affine algebraic variety X, the invariant ring $k[X]^{G(k)}$ is reduced of finite type. Moreover, the morphism $X = \mathrm{Spec}\, k[X] \to \mathrm{Spec}(k[X]^{G(k)})$ induced by the natural inclusion $k[X]^{G(k)} \hookrightarrow k[X]$ gives the categorical quotient $X \to X/\!/G$.*

In the situation of this theorem, we also call $X/\!/G$ the *GIT quotient* of X by G.

Remark 8.6. This observation shows that if we work in the category of affine schemes over k, the coequalizer of α, pr_2 always exists; it is given by $\mathrm{Spec}\, k[X]^{G(k)}$.

Although the above theorem is certainly great, actually we cannot be so satisfies at this point. Firstly, the above quotient is sometimes still far from the one we initially expected as follows:

Example 8.7. Let us consider the situation of Example 8.1, where $G = \mathbb{G}_{\mathrm{m}}$ acting on $X = \mathbb{A}^2_k$ by $z \cdot (x, y) = (zx, zy)$. Note that $\mathbb{G}_{\mathrm{m}}$ is reductive, hence the categorical quotient $X/\!/G$ is given by $\mathrm{Spec}(k[X]^{G(k)})$. However, it is not difficult to see that $k[X]^{G(k)} = k$. Hence $X/\!/G$ is just the point $\mathrm{Spec}\, k$; $(X/\!/G)(k)$ very different from $X(k)/G(k)$.

Secondly, G must be reductive in the above theorem. Recall that what we want is theory of the quotient for any affine algebraic group G by its closed subgroup H (i.e., we want to take 'G' in the theorem to be H and 'X' in the theorem to be G). On the other hand, in the theorem, X can be general affine variety. This suggests that probably we have been working with too general situation. We should furthermore investigate what we can show in the specific situation of our interest.

(Indeed, what I wanted to do by our discussion so far is just to emphasize how subtle the notion of the quotient is...)

8.4. **Homogeneous spaces.** Here let us make an easy observation on topological spaces. Let G be a topological group acting on a topological space X continuously. Suppose that X is a G-*homogeneous space* in the sense that the action is transitive, i.e., $Gx = X$ for any $x \in X$. Then, by fixing any point $x \in X$, we get a continuous map

$$\pi_x \colon G \to X; \quad g \mapsto g \cdot x.$$

We consider the quotient space G/G_x of G by the translation action of the stabilizer subgroup G_x. Then the universal property of the quotient space induces a continuous map $G/G_x \to X$. This map is obviously bijective. Therefore, if π_x is an open map (i.e., the image of any open subset is open), the bijective continuous map $G/G_x \to X$ is a homeomorphism.

Let us go back to the setting of algebraic groups. Let G be an algebraic group acting on an algebraic variety X. We also define the notion of a homogeneous space as follows:

Definition 8.8. We say that X is a G-*homogeneous space* if the G-action is transitive in the sense that $X = Gx$ for any $x \in X(k) = \mathrm{Mor}_k(\operatorname{Spec} k, X)$.

In the following, we construct the quotient "G/H" for any linear algebraic group G and its closed subgroup H. The idea of the construction is to find a G-homogeneous space X equipped with a point $x \in X(k)$ such that $G_x \cong H$.

8.5. **Construction of $G/\!/G_x$ as a G-orbit.** Let G be an algebraic group acting on an algebraic variety X. For $x \in X(k)$, we put $\pi_x := \alpha \circ (\mathrm{id}, x)$:

$$\pi_x \colon G \xrightarrow{(\mathrm{id},x)} G \times_k X \xrightarrow{\alpha} X.$$

We call the image of π_x the G-orbit of $x \in X(k)$ and write Gx for it. Recall that each G-orbit is a locally closed subset of X (Week 2). In particular, we can let Gx be equipped with a natural structure of an algebraic variety.

Theorem 8.9. *The natural morphism $\pi_x \colon G \to Gx$ is equal to the categorical quotient $G \to G/\!/G_x$ of G by the stabilizer group G_x of x in G. Furthermore, we have $G(k)/G_x(k) \cong Gx(k)$.*

Proof. By definition, the morphism $\pi_x \colon G \to Gx$ is surjective. This implies that the map on the k-rational points $G(k) \to Gx(k)$ is also surjective. Moreover, fibers are nothing but the $G_x(k)$-orbits. Hence our task is to show the first statement.

For this, we recall our discussion on the GIT quotient. Note that π_x is a G_x-invariant dominant (surjective) morphism. Hence, for any open subset $V \subset Gx$, its preimage $U := \pi_x^{-1}(V) \subset G$ is nonempty and $G_x(k)$-stable. Thus we again get a G_x-invariant dominant morphism

$$\pi_x \colon U \to V.$$

This implies that the k-algebra homomorphism between the sections of structure sheaves

$$\mathcal{O}_{Gx}(V) \to \mathcal{O}_G(U)$$

has the image in $\mathcal{O}_G(U)^{G_x(k)}$. By the dominance of $\pi_x \colon U \to V$, this homomorphism is injective. It is enough to show that the homomprphism

$$\mathcal{O}_{Gx}(V) \to \mathcal{O}_G(U)^{G_x(k)}$$

is in fact an isomorphism. (When both U and V are affine, the homomorphism $\mathcal{O}_{Gx}(V) \to \mathcal{O}_G(U)^{G_x(k)}$ is nothing but $k[V] \to k[U]^{G_x(k)}$.)

Let us take any $f \in \mathcal{O}_G(U)^{G_x(k)}$. Recall that, for any affine algebraic variety $\operatorname{Spec} R$, its coordinate ring R (the ring of regular functions) is identified with $\operatorname{Mor}_k(\operatorname{Spec} R, \mathbb{A}^1_k)$. This identification can be generalized to any algebraic variety; in the current situation, $\mathcal{O}_G(U)$ and $\mathcal{O}_{Gx}(V)$ are naturally identified with $\operatorname{Mor}_k(U, \mathbb{A}^1_k)$ and $\operatorname{Mor}_k(V, \mathbb{A}^1_k)$, respectively. Hence, it suffices to show that, when we regard f as a morphism $f \colon U \to \mathbb{A}^1_k$, there exists a morphism $f' \colon V \to \mathbb{A}^1_k$ satisfying $f = f' \circ \pi_x$.

Let Γ_f be the graph of f, i.e., the image of the morphism $(\mathrm{id}, f) \colon U \to U \times_k \mathbb{A}^1_k$ induced by the universality of the fibered product.

$$\begin{tikzcd}
U \arrow[dr, dashed, "(\mathrm{id}, f)"] \arrow[drr, "f"] \arrow[ddr, "\mathrm{id}"'] & & \\
& U \times_k \mathbb{A}^1_k \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathbb{A}^1_k \arrow[d] \\
& U \arrow[r] & \operatorname{Spec} k
\end{tikzcd}$$

Note that since $(\mathrm{id}, f) \colon U \to U \times_k \mathbb{A}^1_k$ is a closed immersion (this follows from that its composition with p_1 is the identity), Γ_f is a closed subvariety of $U \times_k \mathbb{A}^1_k$. In fact, it can be proved that the morphism $\pi_x \times \mathrm{id} \colon U \times_k \mathbb{A}^1_k \to V \times_k \mathbb{A}^1_k$ is open. (This fact is a kind of “open-image theorems”, which are actually deep, but we skip its proof. See, for example, [Spr09, Theorem 5.3.2].) By combining this fact with that f is G_x-invariant, we see that $\Gamma'_f := (\pi_x \times \mathrm{id})(\Gamma_f) \subset V \times_k \mathbb{A}^1_k$ is a closed subvariety of $V \times_k \mathbb{A}^1_k$. Moreover, by choosing V to be irreducible from the beginning, we may assume that Γ'_f is also irreducible.

We consider the morphism

$$\Gamma'_f \hookrightarrow V \times_k \mathbb{A}^1_k \xrightarrow{\mathrm{pr}_1} V.$$

By looking at the construction, we can easily check that this morphism is bijective. We note that V is smooth; this can be proved by using that G acts transitively on V (the same proof as that for the smoothness of any algebraic group works). With these conditions, we can utilize so-called “Zariski's main theorem”, which implies that the bijective morphism $\Gamma'_f \hookrightarrow V \times_k \mathbb{A}^1_k \xrightarrow{\mathrm{pr}_1} V$ is in fact an isomorphism. By letting φ be its inverse, we define a morphism $f' \colon V \to \mathbb{A}^1_k$ to be $\mathrm{pr}_2 \circ \varphi$. Then we see that f' is a morphism as desired. $\square$

8.6. **Construction of G/H.** Before we further proceed, let us give a comment on the *projective space*. We define a set $\mathbb{P}^n_k(k)$ to be the quotient of $\mathbb{A}^{n+1}_k(k) \setminus \{0\}$ by the equivalence relation $\sim$ such that $(x_0, \ldots, x_n) \sim (y_0, \ldots, y_n)$ if and only if

$(y_0, \ldots, y_n) = (zx_0, \ldots, zx_n)$ for some $z \in k^\times$. We often write $[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k(k)$ for the equivalence class of $(x_0, \ldots, x_n) \in \mathbb{A}^{n+1}_k(k)$.

In fact, the set $\mathbb{P}^n_k(k)$ can be defined as the set of k-rational points of an algberaic variety. To be more precise, there exists an algebraic variety $\mathbb{P}^n_k$ whose set of k-rational points is exactly given by $\mathbb{P}^n_k(k)$. It can be constructed by gluing affine spaces. For example, the subset $U_0(k) \subset \mathbb{P}^n_k(k)$ given by

$$U_0(k) := \{[x_0 : x_1 : \cdots : x_n] \in \mathbb{P}^n_k(k) \mid x_0 \neq 0\}$$

is bijective to $\mathbb{A}^n_k(k)$ by the map

$$[x_0 : x_1 : \cdots : x_n] \mapsto \left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

If we similarly define $U_1(k), \ldots, U_n(k)$, then we see that $\mathbb{P}^n_k(k)$ is covered by $U_i(k)$'s. This "gluing" construction using the affine spaces can be done at the level of algebraic varieties. The resulting algebraic varieties is written by $\mathbb{P}^n_k$ and called the *projective space*.

Note that, each point $[x_0 : x_1 : \cdots : x_n]$ of $\mathbb{P}^n_k(k)$ defines a 1-dimensional line in $\mathbb{A}^{n+1}(k) = k^{n+1}$, i.e., the line passing through the origin and $(x_0, \ldots, x_n)$. Conversely, any a 1-dimensional line gives rise to a point of $\mathbb{P}^n_k(k)$. This association obviously defines a bijection. Hence $\mathbb{P}^n_k(k)$ can be thought of as the space of 1-dimensional lines in k^{n+1}. Noting this, we can also define a projective space $\mathbb{P}(V)$ for any finite-dimensional k-vector space V; if we fix a basis of V and identify V with k^{n+1} $(n = \dim V - 1)$, then $\mathbb{P}(V)$ is just identified with $\mathbb{P}^{n+1}_k$.

Definition 8.10. Let X be an algebraic variety.

1. (1) We say that X is *projective* if it has a closed immersion into a projective space.
2. (2) We say that X is *quasi-projective* if it has an open immersion into a projective variety.

Theorem 8.11. *Let G be a linear algebraic group and H its closed subgroup. Then the categorical quotient $G//H$ always exists. Moreover, $G//H$ is a quasi-projective variety satisfying $G(k)/H(k) \cong (G//H)(k)$.*

Proof. If we can find a quasi-projective G-homogeneous space X equipped with $x \in X(k)$ such that $H = G_x$, then the previous theorem immediately implies this theorem. To construct such a space X, we consider the right-translation action $\rho_{(-)}$ of $G(k)$ on the coordinate ring $k[G]$. Recall that we can always find a finite-dimensional k-subspace $V \subset k[G]$ such that V is $G(k)$-stable and generates $k[G]$ as a k-algebra. If we define its subspace W to be $V \cap I$, where I is the kernel of $k[G] \twoheadrightarrow k[H]$, then we can check that

$$H(k) = \{g \in G(k) \mid \rho_g(W) = W\}$$

(cf. the discussion in Week 3).

Let d be the dimension of W and consider $L := \wedge^d W \subset \wedge^d V$ (note that L is a 1-dimensional line). The right-translation action $\rho_{(-)}$ of $G(k)$ on V induces an action $\rho'_{()} := \wedge^d \rho_{(-)}$. We can check the following (the exercise below):

$$\rho_g(W) = W \iff \rho'_g(L) = L$$

for any $g \in G(k)$.

We consider the projective space $\mathbb{P}(\wedge^d V)$ associated to $\wedge^d V$:

$$\mathbb{P}(\wedge^d V) = (\wedge^d V \smallsetminus \{0\})/k^\times \cong \{L' \subset \wedge^d V \mid \dim(L') = 1\}.$$

If we let n be the dimension of $\wedge^d V$, then we may identify $\mathbb{P}(\wedge^d V)$ with $\mathbb{P}^n_k(k)$ (by choosing a basis of $\wedge^d V$). Hence we may regard L as a k-rational point x of $\mathbb{P}^n_k$. The action of $G(k)$ on $\mathbb{P}(\wedge^d V)$ can be realized as an action of an algebraic group G on $\mathbb{P}^n_k$. Since any G-orbit is locally closed (recall: Week 2), the image of the morphism

$$\pi_x \colon G \to \mathbb{P}^n_k \colon g \mapsto gx$$

is a G-homogeneous space which is quasi-projective. Furthermore, the stabilizer group G_x is H. $\square$

We simply write G/H for the algebraic variety $G/\!/H$. Because of the second assertion of the above theorem, the notation is not strange.

Exercise 8.12. Let V be a finite-dimensional k-vector space. Let W be a d-dimensional k-subspace of V. Put $L := \wedge^d W$. Let $x \in \mathrm{Aut}_k(V)$. Show that $x(W) = W$ if and only if $\wedge^d(x)(L) = L$.

Fact 8.13. *For any linear algberaic group G and its closed subgroup H, we have*

$$\dim(G/H) = \dim G - \dim H.$$

Proof. See [Spr09, Corollary 5.5.6]. $\square$

8.7. **The case when H is normal.** When G is a linear algebraic group and H is a closed normal subgroup, the quotient variety G/H has a natural structure of an algebraic group. In fact, we furthermore have that G/H is affine, hence a *linear* algebraic group. Here let us just state it without proof.

Theorem 8.14. *When G is a linear algebraic group and H is its closed normal subgroup, the quotient variety G/H is a linear algberaic group with respect to its natural group structure.*

Proof. See [Spr09, Proposition 5.5.10]. $\square$

Also note that, in this case, the natural morphism of the categorical quotient $G \to G/H$ is a surjective group homomorphism.

9. Week 9: Borel subgroups and maximal tori

Recall that, in our construction of a root datum (with simple roots) associated to GL_n, we utilized structural properties of the diagonal torus T_n and the upper-triangular subgroup B_n of GL_n. These subgroups can be generalized to any algebraic group G as follows.

Definition 9.1. For a linear algebraic group G, we call a torus $T \subset G$ which is maximal with respect to the inclusion a *maximal torus*.

Definition 9.2. For a linear algebraic group G, we call a closed connected and solvable subgroup $B \subset G$ which is maximal with respect to the inclusion a *Borel subgroup*.

For a given linear algebraic group G, there always exist a maximal torus and a Borel subgroup by definition. However, note that they are not unique in general. Indeed, if T is a maximal torus of G, then its conjugate xTx^{-1} is also a maximal torus of G for any $x \in G$. (Of course, the same is true for a Borel subgroup.) In fact, the converse is true; the aim of this week is to understand the following theorem:

Theorem 9.3. *Let G be a linear algberaic group.*

(1) *All Borel subgroup of G are G-conjugate.*
(2) *All maximal tori of G are G-conjugate.*

9.1. Parabolic subgroups.

Before we define a parabolic subgroup, we introduce one notion from algebraic geometry.

Definition 9.4. We say that an algebraic variety is *complete* if the projection morphism $\mathrm{pr}_2 \colon X \times_k Y \to Y$ is closed (i.e., the image of any closed subset is closed) for any algebraic variety Y.[6]

We just briefly summarize several facts on complete varieties, which will be used later.

Fact 9.5.

(1) *An algebraic variety is projective if and only if it is quasi-projective and complete.*
(2) *Any closed subvariety of a complete variety is again complete.*
(3) *If $f \colon X \to Y$ is a moprhism of algebraic varieties such that X is complete, then the image is closed in Y and complete.*

Recall that, in the last week, we investigated the quotient of a linear algebraic group G by its any closed subgroup H; the quotient variety G/H always exists and is quasi-projective.

Definition 9.6. Let G be a linear algebraic group. We call a closed subgroup $P \subset G$ a *parabolic subgroup* if the quotient variety G/P is complete (or equivalently, projective by Fact 9.5 (1)). We say that a parabolic subgroup P is a *proper* parabolic subgroup if $P \neq G$.[7]

Remark 9.7. It is known that any connected algebraic group which is complete is automatically commutative. Such an algebraic group is called an "abelian variety",

[6] In the modern language of scheme theory, the completeness is referred to as the *properness*.
[7] Be careful that this "proper" does not mean the properness in the scheme-theoretic sense!

which generalizes elliptic curves. Since an affine variety is not complete (as long as its dimension is not zero), in theory of 'linear' algebraic groups, we only treat non-complete algebraic groups. However, it is convenient to consider also non-affine algebraic varieties as the definition of a parabolic subgroup shows.

Lemma 9.8. *Let G be a linear algebraic group. Any closed subgroup of G containing a parabolic subgroup is parabolic.*

Proof. Let $Q \subset G$ be a closed subgroup containing a parabolic subgroup P of G. The universality of the quotient G/P implies that we have a natural morphism $G/P \twoheadrightarrow G/Q$. Since G/P is complete, Fact 9.5 (3) implies that G/Q is also complete. Hence Q is a parabolic subgroup. $\square$

Being a parabolic subgroup satisfies the following chain rule:

Lemma 9.9. *Let G be a linear algebraic group. Let $P \subset Q \subset G$ be closed subgroups such that P is parabolic in Q and Q is parabolic in G. Then P is parabolic in G.*

Proof. We appeal to the open-imageness of the quotient morphism $G \times_k Y \twoheadrightarrow G/H \times_k Y$ (for any closed subgroup H and an algebraic variety Y). We consider the diagram

$$\begin{tikzcd}
Q \times_k G \times_k Y \arrow[r] \arrow[dd, "\text{open}"'] & G \times_k Y \arrow[r, two heads, "\text{open}"] & G/P \times_k Y \arrow[dr, "\mathrm{pr}_2"] & \\
& & & Y \\
Q/P \times_k G \times_k Y \arrow[r, "(\mathrm{pr}_2, \mathrm{pr}_3)"] & G \times_k Y \arrow[r, two heads, "\text{open}"] & G/Q \times_k Y \arrow[ur, "\mathrm{pr}2"] &
\end{tikzcd}$$

where the top-left horizontal arrow is given by $(q, g, y) \mapsto (gq, y)$. Using the commutativity of this diagram, it is a routine work to check that the projection morphism $G/P \times_k Y \to Y$ is closed for any algebraic variety Y. $\square$

9.2. **Borel's fixed point theorem.**

Proposition 9.10. *Let G be a connected linear algebraic group. Then G has a proper parabolic subgroup if and only if G is non-solvable.*

Proof. $\Longleftarrow$: We first show that G is solvable when G does not have a proper parabolic subgroup. We take a closed immersion $G \hookrightarrow \mathrm{GL}_n \cong \mathrm{Aut}_k(V)$, where V is an n-dimensional k-vector space. Since $\mathrm{GL}_n \cong \mathrm{Aut}_k(V)$ naturally acts on the projective space $\mathbb{P}_k^{n-1} \cong \mathbb{P}(V)$, we get an action of G on $\mathbb{P}(V)$. By the closed orbit lemma (Week 2), there exists a closed orbit $Gx \subset \mathbb{P}(V)$. By the discussion in the last week, the quotient variety G/G_x is isomoprhic to Gx, which is projective (any closed subvariety of a projective variety is again projective). In other words, G_x is a parabolic subgroup of G. The assumption implies that G_x is necessarily equal to G.

If we write L for the 1-dimensional line in V corresponding to $x \in \mathbb{P}(V)$. Then, L is stabilized by G since $G_x = G$. Hence, if we put $V' := V/L$, then G acts on V'. By applying the same discussion to V', we can furthermore find a line $L' \subset V'$ stabilized by G. Repeating this procedure, we finally obtain a sequence of subspaces

$$\{0\} \subsetneq L = L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n = V,$$

such that each L_i is i-dimensional and stabilized (as a subspace) by the action of G. (For example, we let L_2 be the preimage of $L' \subset V/L$ under $V \twoheadrightarrow V/L$.) This means that, if we choose a k-basis $\{v_1, \ldots, v_n\}$ of V such that each v_i is contained in V_i, then the image of G in $\mathrm{Aut}_k(V) \cong \mathrm{GL}_n$ is contained in the upper-triangular subgroup B_n, hence solvable.

$\Longrightarrow$: We next show that any solvable group G does not have a proper parabolic subgroup by induction on $\dim G$. For the sake of contradiction, let us suppose that G has a proper parabolic subgroup $P \subsetneq G$. We furthermore assume that $\dim P$ is maximal among all such subgroups. Note that, P° is a parabolic subgroup of P since P/P° is a finite group (0-dimensional points), hence projective. Thus, by Lemma 9.9, P° is also a parabolic subgroup of G. By replacing P with P°, we may assume that P is a connected proper parabolic subgroup of G.

Let $G' := [G, G]$. Since G is solvable, G' is a proper closed normal solvable subgroup of G. We consider a subgroup $Q := PG'$ of G, which is closed and connected (note that G is conenected, hence so is G'). Thus Lemma 9.8 implies that Q is a parabolic subgroup of G. However, the maximality of P implies that $Q = P$ or $Q = G$.

If $Q = P$, then we have $G' = [G, G] \subset P$. This implies that P is a normal subgroup of G, hence the quotient G/P has a structure of a linear algebraic group; in particular, G/P is affine. This contradicts the projectivity of G/P.

If $Q = G$, then the natural morphism

$$G'/(G' \cap P) \to G/P$$

induced by the universal property of the quotient $G'/(G' \cap P)$ is bijective. In fact, in this situation, the completeness of G/P implies that of $G'/(G' \cap P)$ ([Spr09, Lemma 6.2.1]). Hence $G'/(G' \cap P)$ is projective, which means that $G' \cap P$ is a parabolic subgroup of G'. By the induction hypothesis (applied to G'), G' cannot have a proper parabolic subgroup. Thus we must have $G' \cap P = G'$, i.e., $G' \subset P$. But this implies that $G = Q = PG' = P$, hence a contradiction. □

Theorem 9.11. *Suppose that G is a connected solvable group acting on a proper variety X. Then X has a fixed closed point, i.e., a closed point $x \in X(k)$ such that $gx = x$ for any $g \in G(k)$.*

Proof. By the closed orbit lemma (Week 2), X has a closed G-orbit, say $Gx \subset X$. Since X is projective, so is Gx. By the discussion in the last week, the G-orbit Gx is isomorphic to the quotient G/G_x. In particular, G/G_x is a projective variety. In other words, G_x is a parabolic subgroup of G. By the previous Proposition, G does not have any proper parabolic subgroup. Hence we must have $G_x = G$, which means that x is a fixed point. □

Our arguments so far actually contains the proof of Lie–Kolchin's theorem:

Corollary 9.12 (Lie–Kolchin's theorem)**.** *Let G be a connected closed solvable subgroup of GL_n. Then G is conjugate to the upper-triangular subgroup B_n.*

Proof. By applying Borel's fixed point theorem to the action of $G \subset \mathrm{GL}_n \cong \mathrm{Aut}_k(V)$ on $\mathbb{P}(V)$, where $V := k^n$, we can find a fixed point $x \in \mathbb{P}(V(k))$. Letting $L \subset V$ be the line corresponding to x, which is stable under the G-action, we apply the same argument to the action of G on V/L to find a G-stable line of V/L.

Repeating this procedure, we obtain a sequence of subspaces

$$\{0\} \subsetneq L = L_1 \subsetneq L_2 \subsetneq \cdots \subsetneq L_n = V,$$

such that each L_i is i-dimensional and stabilized (as a subspace) by the action of G. This means that, if we choose a k-basis $\{v_1, \ldots, v_n\}$ of V such that each v_i is contained in V_i, then the image of G in $\mathrm{Aut}_k(V) \cong \mathrm{GL}_n$ is contained in B_n. $\square$

We also obtain the following characterization of a parabolic subgroup.

Proposition 9.13. *Let G be a linear algebraic group. A closed subgroup $P \subset G$ is parabolic if and only if it contains a Borel subgroup of G. In particular, a Borel subgroup is a minimal parabolic subgroup.*

Proof. We may assume that G is connected.

We first consider the "only if" direction. Let us take a Borel subgroup B and a parabolic subgroup P. Then the quotient variety G/P is projective, hence has a fixed point with respect to the translation action by B by Borel's fixed point theorem. We let $x \in G(k)$ be a point representing the fixed point $\bar{x}$ of $(G/P)(k) \cong G(k)/P(k)$. Then, since $B\bar{x} = \bar{x}$, we have $BxP = xP$. In particular, this implies that $x^{-1}Bx \subset P$. Since $x^{-1}Bx$ is a Borel subgroup, this completes the only if part.

Next let us prove the "if" direction by induction on $\dim G$. Since any closed subgroup containing a parabolic subgroup is again parabolic (Lemma 9.8), it suffices to show that a Borel subgroup is parabolic. Let $B \subset G$ be a Borel subgroup. If G is solvable, then B is equal to G itself by deifnition, hence obviously a parabolic subgroup. If G is not solvable, then G has a proper parabolic subgroup P by Proposition 9.10. By the discussion in the previous paragraph, we can find a conjugate $x^{-1}Bx$ which is contained in P. Thus, by replacins B with $x^{-1}Bx$, we may assume that B is contained in P from the beginning. Then B is also a Borel subgroup of P. Since $P \subsetneq G$, we have $\dim P < \dim G$. Hence the induction hypothesis implies that B is a parabolic subgroup of P. By Lemma 9.9, we see that B is parabolic in G. $\square$

9.3. **Structure theorem for solvable groups.** We investigate connected solvable groups in detail. (A connected solvable group considered here is supposed to be taken to be a Borel subgroup of a linear algebraic group later.)

The following is a generalization of a result proved for commutative linear algebraic groups in Week 4:

Proposition 9.14. *Let G be a connected nilpotent group. Then the following hold:*

1. *the semisimple locus G_s is a closed subtorus contained in the center of G,*
2. *the unipotent locus G_u is a closed connected subgroup of G,*
3. *the product map $G_s \times_k G_u \to G$ is an isomoprhism of algebraic groups.*

Proof. We recall that the semisimple locus of any solvable group can be defined algebraically, i.e., as a closed subvariety. Let us first show that G_s is contained in the center (note that this implies that G_s is a closed subgroup of G). Letting $s \in G_s(k)$ be any semisimple element, we consider the commutator morphism

$$\phi_s \colon G \to G; \quad g \mapsto sgs^{-1}g^{-1}.$$

Since G is nilpotent, $\phi_s^n(G) = \{e\}$ for sufficiently large $n \gg 0$. Thus the differential $d\phi \colon \mathfrak{g} \to \mathfrak{g}$ induced by ϕ on $\mathfrak{g} := \mathrm{Lie}\, G$ satisfies $(d\phi)^n = 0$, which means that $d\phi$ is a nilpotent endomorphism of $\mathfrak{g}$. On the other hand, if we let $\mathrm{Ad}(s) \colon \mathfrak{g} \to \mathfrak{g}$

denote the differential of the automorphism $G \to G\colon g \mapsto sgs^{-1}$, then we also have $d\phi = \mathrm{Ad}(s) - \mathrm{id}$. In general, the endomorphism $\mathrm{Ad}(s)$ associated to a semisimple element s is semisimple (the "adjoint representation" $\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g})$ is a homomorphism of algebraic groups, hence preserves the Jordan decomposition). Hence $d\phi = \mathrm{Ad}(s) - \mathrm{id}$ is nilpotent and semisimple, which necessarily means that $d\phi = 0$, or equivalently, $\mathrm{Ad}(s) = \mathrm{id}$. It can be checked that, in general, $\mathrm{Lie}\, Z_G(s)$ is equal to the centralizer of $\mathrm{Ad}(s)$ in $\mathfrak{g}$ (whenever s is semisimple). Therefore, we get $\mathrm{Lie}\, Z_G(s) = \mathfrak{g}$. As G is connected, this implies that $Z_G(s) = G$ (compare the dimensions of both groups), hence s is central in G.

We take a closed immersion $G \hookrightarrow \mathrm{GL}_n \cong \mathrm{Aut}_k(V)$, where $V := k^n$. As G_s is central in G, hence is a diagonalizable group, we may choose a simultaneous diagonalization of $G_s(k)$. If we let $V = \bigoplus_{i=1}^r V_i$ be the decomposition of V into the eigenspaces, then each V_i is preserved by the action of G since G_s is central in G. In other words, by replacing the embedding $G \subset \mathrm{GL}_n$ (using the eigenvectors as k-basis), we may assume that the image of G is contained in $\prod_{i=1}^r \mathrm{GL}_{n_i}$, where $n_i := \dim V_i$. If we let $G_i := G \cap \mathrm{GL}_{n_i}$, then we have $G \cong \prod_{i=1}^r G_i$.

By Lie–Kolchin's theorem, we may also assume that each G_i is a subgroup of $B_{n_i} \subset \mathrm{GL}_{n_i}$. Since the product morphism $T_{n_i} \times_k U_{n_i} \to \mathbf{B}_{n_i}$ is an isomoprhism of algebraic varieties, its restriction induces an isomorphism

$$(G_i \cap T_{n_i}) \times_k (G_i \cap U_{n_i}) \to G_i.$$

Here, $G_i \cap U_{n_i}$ is nothing but the unipotent locus $(G_i)_u$ of G_i. Also, as the action of $G_s(k)$ on V_i is given by scalar multiplications, we have $(G_i)_s = G_i \cap T_{n_i}$. Hence the left-hand sider of the above isomorphism is $(G_i)_s \times_k (G_i)_u$. Furthremore, this is an isomoprhism as algebraic group since G_s is central in G.

We finally note that G_s and G_u can be thought of as the images of the projections of $G \cong G_s \times G_u$ onto both factors. In particular, as G is connected, both are connected. □

Corollary 9.15. *Let G be a connected solvable group. Then the following hold:*

(1) *$[G, G]$ is a connected nilpotent group,*
(2) *the unipotent locus G_u is a closed connected nilpotent normal subgroup,*
(3) *the quotient G/G_u is a torus.*

Proof. By Lie–Kolchin's theorem, we may assume that G is a closed subgroup of B_n. Then the statement (1) immediately follows from that $[B_n, B_n]$ equals U_n, which is nilpotent.

We show (2) and (3). Note that U_n is a closed nilpotent normal subgroup of B_n. Since $G_u = G \cap U_n$, G_u is also a closed nilpotent normal subgroup of G. The universal property of the quotient G/G_u induces a natural homomorphism $G/G_u \to B_n/U_n$. By looking at the closed points, we see that this homomorphism is injective. Since B_n/U_n is isomorphic to T_n, which is a torus, this implies that G/G_u consists only of semisimple elements. Moreover, G/G_u is connected as G is connected. Therefore, G/G_u is a torus.

Our remaining task is to check that G_u is connected. Since G_u is normal in G, so is G_u°. If we put $\bar{G} := G/G_u^\circ$, then $(\bar{G})_u = G_u/G_u^\circ$ (use that the natural surjection $G \twoheadrightarrow \bar{G}$ preserves the Jordan decomposition). As G_u/G_u° is a finite group, by replacing G with $\bar{G}$, we may assume that G_u is finite from the beginning. (So, what we have to show now is G_u is trivial.)

Let us show that G is nilpotent. (If we can show this, then Proposition 9.14 implies that G_u is connected, hence G_u is trivial.) Since G/G_u is a torus, $[G/G_u, G/G_u] = \{e\}$, or equivalently, $[G,G] \subset G_u$. As G_u is normal, we have $[G, G_u] \subset G_u$. We use the following (see [Spr09, Corollary 2.2.8]):

Fact 9.16. *For closed subgroups H_1 and H_2 of an algebraic group H, their commutator product subgroup $[H_1, H_2]$ is connected if one of H_1, H_2 is connected.*

Since we are assuming that G_u is finite, this fact implies that $[G, G_u] \subset G_u^\circ = \{e\}$. Thus, in summary, we obtained

$$[G, [G, G]] \subset [G, G_u] = \{e\},$$

which means that G is a (2-step) nilpotent group. $\square$

Lemma 9.17. *Let G be a connected solvable group. If G is not a torus, then there exists a closed normal subgroup $N \subset G$ suth that N is contained in the center of G_u and $N \cong \mathbb{G}_\mathrm{a}$.*

Proof. By Lie–Kolchin's theorem, we may assume that G is a subgroup of $B_n \subset \mathrm{GL}_n$. Then G_u is identified with $G \cap U_n$. By Corollary 9.15, G_u is a connected nilpotent normal subgroup of G.

We take (see Exercise below) a sequence of closed subgroups

$$U_n = U_n^{(0)} \supsetneq U_n^{(1)} \supsetneq \cdots \supsetneq U_n^{(r)} = \{e\}$$

such that, for each i,

(i) U_n^i is a normal subgroup of B_n,
(ii) $[U_n, U_n^{(i)}] \subset U_n^{(i+1)}$,
(iii) $U_n^{(i)}/U_n^{(i)} \cong \mathbb{G}_\mathrm{a}$.

If we put $G_u^{(i)} := (G \cap U_n^{(i)})^\circ$, then we get a sequence

$$G_u = G_u^{(0)} \supset G_u^{(1)} \supset \cdots \supset G_u^{(r)} = \{e\}$$

of closed normal subgroups by the condition (i). By the condition (ii), the dimension of each graded quotient $G_u^{(i)}/G_u^{(i+1)}$ is at most 1. Hence, in particular, if we choose N to be the smallest $G_u^{(i)}$ which is not trivial, then N is 1-dimensional unipotent group. In fact, it is known that such an algebraic group is necessarily equal to $\mathbb{G}_\mathrm{a}$. Furthermore, by the condition (ii), we have $[G_u, N] = \{e\}$. Since $G \cong G_s \times G_u$ and G_s is central by Corollary 9.15, this implies that $[G, N] = \{e\}$, which means that N is central. $\square$

Exercise 9.18. Prove that we can indeed take a sequece $U_n = U_n^{(0)} \supsetneq U_n^{(1)} \supsetneq \cdots \supsetneq U_n^{(r)} = \{e\}$ as in the proof. (Hint: Compute the derived series $D^1U_n = [U_n, U_n]$, $D^2U_n = [D^1U_n, D^1U_n]$,... Then divide each $D^iU_n/D^{i+1}U_n$ into further smaller pieces.)

10. Week 10: Borel subgroups and maximal tori: continued

10.1. **Proof of the conjugacy theorem.** Now let us try to prove Theorem 9.3. Recall that the statement is that for any linear algberaic group G, we have the following:

(1) All Borel subgroup of G are G-conjugate.
(2) All maximal tori of G are G-conjugate.

We first show (1). Let B and B' are Borel subgroups. Since B is parabolic, by the proof of Proposition 9.13, we can find $x \in G(k)$ such that $x^{-1}B'x \subset B$. As $x^{-1}B'x$ is also a Borel subgroup, the maximality of a Borel subgroup implies that $x^{-1}B'x = B$.

We next show (2). Let T and T' be maximal tori of G. As a torus is connected and solvable, we may find Borel subgroups B and B' of G such that $T \subset B$ and $T \subset B'$ (by the maximality of a Borel subgroup). By (1), there exists $x \in G(k)$ such that $B' = xBx^{-1}$. Hence xTx^{-1} is a maximal torus contained in B'. Therefore, it is enough to establish the assertion for B', which is a connected solvable group.

For convenience, we introduce the following temporary terminology:

Definition 10.1. Let G be a connected solvable group. By Corollary 9.15, the quotient G/G_u is a torus. When a subtorus T of G has the same dimension as G/G_u, we say that T is a *max-dimensional torus* of G.

Note that, in contrast to a maximal torus, a priori it is not clear whether there exists a max-dimensional torus. (However, later it will turn out that being a max-dimensional torus is equivalent to being a maximal torus.)

Remark 10.2. We can easily check that a max-dimensional torus must be a maximal torus. Indeed, if we let T be a torus of G, then the quotient morphism $G \twoheadrightarrow G/G_u$ should restrict to an injective morphism $T \hookrightarrow G/G_u$. In particular, we must have $\dim T \leq \dim(G/G_u)$. Hence, if a max-dimensional torus T is contained in another torus T', then we have $\dim(G/G_u) = \dim T \leq \dim T' \leq \dim(G/G_u)$. Thus we get $\dim T = \dim T'$, which implies that $T = T'$.

Theorem 10.3. *Let G be a connected solvable group.*

(1) *Any semisimple element $s \in G$ is contained in a max-dimensional torus. In particular, there exists a max-dimensional torus.*
(2) *All max-dimensional tori of G are G-conjugate.*

Proof. We note that all the statements are clear if $\dim G_u = 0$, or equivalently, G is a torus.

The case where $\dim G_u = 1$ can be treated by using a fact that any 1-dimensional linear algebraic group is isomorphic to $\mathbb{G}_{\mathrm{a}}$ or $\mathbb{G}_{\mathrm{m}}$. The idea is to note the conjugate action of G/G_u on G_u; G/G_u is a torus by Corollary 9.15 and G_u is isomorphic to $\mathbb{G}_{\mathrm{a}}$. We omit the details; see [Spr09, Theorem 6.3.5].

We next consider the general case by induction on $\dim G$. When $\dim G = 1$, we necessarily have $\dim G_u \leq 1$, hence we may assume that $\dim G > 1$. Also, we may assume that G is not a torus. Then we may apply Lemma 9.17 to choose a subgroup N contained in the center of G_u and isomorphic to $\mathbb{G}_{\mathrm{a}}$. We consider the natural surjection $\pi\colon G \twoheadrightarrow G/N =: \bar{G}$. We consider the quotient $\pi\colon G \twoheadrightarrow G/N =: \bar{G}$. Then we have $\dim \bar{G} = \dim G - \dim N = \dim G - 1 < \dim G$, hence we may apply the induction hypothesis to $\bar{G}$.

To show (1), let us take any semisimple element $s \in G$ with image $\bar{s}$ in $\bar{G}$. By the induction hypothesis, we can take a max-dimensional torus $\bar{T} \subset \bar{G}$ containing $\bar{s}$. We put $T' := \pi^{-1}(\bar{T})$. Since T' is a closed subgroup of G whose quotient by N is $\bar{T}$, T' is connected. In particular, T' is a connected solvable group such that $T'_u = 1$, which is the case already treated.[8] Thus we can find a max-dimensional torus $T \subset T'$ containing s. Let us show that T is a max-dimensional torus also of G. Note that

$$\dim(T'/T'_u) = \dim T' - 1 = \dim \bar{T} = \dim \bar{G}/\bar{G}_u.$$

Since $\pi^{-1}((\bar{G})_u)$ is nothing but G_u, the universal property of π induces a morphism $G/G_u \to \bar{G}/(\bar{G})_u$ which is bijective on closed points.[9] Thus we necessarily have that $\dim G/G_u = \dim \bar{G}/(\bar{G})_u$. Therefore, T is also a max-dimensional torus of G.

We next show (2). We take two max-dimensional tori T_1 and T_2 of G. If we let $\bar{T}_1$ and $\bar{T}_2$ be their images in $\bar{G}$, then we can find an element $\bar{g} \in \bar{G}(k)$ satisfying $\bar{T}_2 = \bar{g}\bar{T}_1\bar{g}^{-1}$ by the induction hypothesis. We choose any lift $g \in G(k)$ of $\bar{g}$. If we let T'_1 and T'_2 be the preimages of $\bar{T}_1$ and $\bar{T}_2$ via π, respectively, then we have $T'_2 = gT'_1g^{-1}$. In particular, both gT_1g^{-1} and T_2 are dimension-maximal tori of T'_2. By applying the induction hypothesis to the connected solvable group T'_2, we can find an element $h \in T'_2(k)$ such that $T_2 = hgT_1g^{-1}h^{-1}$.

□

The following theorem is also important, but we omit its proof in this course.

Theorem 10.4 ([Spr09, Theorem 6.3.5])**.** *Let G be a connected solvable group. Then, for any semisimple element s, its centralizer $Z_G(s)$ is connected.*

Exercise 10.5. In general, the centralizer $Z_G(s)$ of a semisimple element $s \in G(k)$ could be disconneted. Find such an example.

Proposition 10.6. *Let G be a connected solvable group. Let H be a subgroup of G such that $H = H_s$. Then H is contained in a max-dimensional torus of G.*

Proof. We prove the assertion by induction on $\dim G$. (The base case is $\dim G = 0$, where the statement is obvious.) Let Z denote the center of G.

We first consider the case where $H \subset Z$. For any $s \in H$, by Theorem 10.3 (1), there exists a max-dimensional torus T of G. Furthermore, for any other max-dimensional torus T', Theorem 10.3 (2) implies that there exists $g \in G$ satisfying $T' = gTg^{-1}$. As $gsg^{-1} = s$, we have that $s \in T'$. In other words, s is contained in any max-dimensional torus. Hence H is contained in any max-dimensional torus.

We next consider the case where $H \not\subset Z$. Since $H \cap G_u = \{1\}$, the quotient morphism $\pi\colon G \to G/G_u$ induces an injective morphism $H \to G/G_u$. Since G/G_u is a torus, hence commutative, H is also commutative. In particular, for any $s \in H$, we have $H \subset Z_G(s)$. Let us take s such that $s \in H \setminus Z$. Then we have $Z_G(s) \subsetneq G$. Moreover, by Theorem 10.4, $Z_G(s)$ is connected. Hence we can apply the induction hypothesis to $Z_G(s)$; we can find a max-dimensional torus T of $Z_G(s)$ containing H. On the other hand, we let T' be a max-dimensional torus of G containing s (which exists by Theorem 10.3 (1)). Note that $s \in T'$ implies that $T' \subset Z_G(s)$. Thus $Z_G(s)$ contains two tori; T satisfying $\dim T = \dim(Z_G(s)/Z_G(s)_u)$ and T'

[8] Actually, this is a bit tricky point of this proof. If we try to proceed only with the induction on $\dim G$, then we get stuck here because it is possible that $T' = G$.

[9] Be careful that, in general, this may not imply that two algebraic groups are isomorphic.

satisfying $\dim T' = \dim(G/G_u)$. As we have an injection $Z_G(s)/Z_G(s)_u \hookrightarrow G/G_u$, we get $\dim T \leq \dim T'$. Hence the dimensional maximality of T in $Z_G(s)$ implies that $\dim T = \dim T'$. This means that T is also max-dimensional in G. $\square$

Now we complete the proof of the conjugacy theorem.

Theorem 10.7. *Let G be a connected solvable group. Then the notions of a maximal torus and a max-dimensional torus coincide. In particular, all maximal tori of G are G-conjugate (by Theorem 10.3).*

Proof. Recall that any max-dimensional torus is a maximal torus. So let us show the converse; let T be a maximal torus of G. Then, by Proposition 10.6, we can find a max-dimensional torus T' of G containing T. By the maximality of T, we must have $T = T'$. $\square$

We can also deduce the conjugacy theorem for maximal nilpotent subgroups.

Corollary 10.8. *Let G be a connected linear algebraic group. All maximal connected unipotent closed subgroups of G are G-conjugate.*

Proof. Let U and U' be such subgroups. Then, by the maximality of Borel subgroups, we can find Borel subgroups B and B' of G containing U and U', respectively. Note that, since $U \subset B_u$ and $U' \subset B'_u$, the maximality of U and U' imply that $U = B_u$ and $U' = B'_u$. The conjugacy theorem for Borel subgroups implies that B and B' are G-conjugate, hence so are $B_u = U$ and $B'_u = U'$. $\square$

Exercise 10.9. Discuss whether this corollary still holds even if "unipotent" is replaced with "nilpotent". (That is, please prove it if it is still true; please provide a counterexample if it is not true.)

10.2. **Some facts on Borel subgroups.** Here we just introduce two facts on Borel subgroups. Both are technically very important, but we give up explaining their proofs in this course. (Please see the cited reference for details.)

Theorem 10.10 ([Spr09, Theorem 6.4.5])**.** *Let G be a connected linear algebraic group. Then we have the following.*

(1) *$G = \bigcup_B B$, where B runs all Borel subgroups;*
(2) *$G_s = \bigcup_T T$, where T runs all maximal tori;*
(3) *$G_u = \bigcup_U U$, where U runs all maximal connected closed nilpotent subgroups.*

Theorem 10.11 ([Spr09, Theorem 6.4.7])**.** *Let G be a connected linear algebraic group. Let S be any subtorus of G.*

(1) *The centralizer group $Z_G(S)$ is connected.*
(2) *For any Borel subgroup $B \subset G$, its restriction $Z_G(S) \cap B$ is a Borel subgroup of $Z_G(S)$.*
(3) *Conversely, any Borel subsgroup of $Z_G(S)$ is of the form $B \cap Z_G(S)$ for a Borel subgroup B of G.*

10.3. **Normalizer theorem.**

Theorem 10.12. *Let G be a connected linear algebraic group. For any Borel subgroup B of G, we have $N_G(B) = B$.*

Proof. We prove the statement by induction on $\dim G$.

Let B be a Borel subgroup of G. We fix a maximal torus T of B. Let $x \in N_G(B)$ (so our goal is to show that $x \in B$). Since $xTx^{-1} \subset xBx^{-1} = B$, both T and xTx^{-1} are maximal tori of B. Hence we can find an element $b \in B$ satisfying $bxTx^{-1}b^{-1} = T$. By replacing x with bx, we may assume that $xTx^{-1} = T$, i.e., $x \in N_G(T)$.

We consider an algberaic group homomorphism

$$\phi\colon T \to T; \quad t \mapsto xtx^{-1}t^{-1}.$$

Let us first investigate what will happen if ϕ is not surjective. We let $S \subset T$ be a subgroup defined by $S := (\operatorname{Ker}\phi)^\circ$. Then S is a nontrivial subtorus. Indeed, ϕ induces a homomorphism $\phi^*\colon X^*(T) \to X^*(T)$ on the character groups. Then ϕ is surjective if and only if ϕ^* is injective, which is furthermore equivalent to that the cokernel of ϕ^* is finite (torsion). As S is nothing but the torus corresponding to the maximal free quotient of $\operatorname{Cok}(\phi^*)$ (recall that we have a categorical equivalence between diagonalizable groups and finitely generated abelian groups), we conclude that S is trivial if and only if ϕ is surjectivce.

By Theorem 10.11, $Z_G(S) \cap B$ is a Borel subgroup of a connected linear algberaic group $Z_G(S)$. By definition, obviously we have $x \in Z_G(S)$, hence $x \in N_{Z_G(S)}(Z_G(S) \cap B)$. If $Z_G(S) \subsetneq G$, then the induction hypothesis implies that $x \in N_{Z_G(S)}(Z_G(S) \cap B) = Z_G(S) \cap B \subset B$. If $Z_G(S) = G$, then S is central in G, hence contained in any Borel subgroup; in particular, we get $S \subset B$. As we have $\dim(G/S) = \dim G - \dim S < \dim G$, we can apply the induction hypothesis to the Borel subgroup B/S of G/S to get $N_{G/S}(B/S) = B/S$. Again noting that S is central, this is equivalent to that $N_G(B) = B$.

We next consider the case where ϕ is surjective. We take an algebraic group homomorphism

$$\rho\colon G \to \mathrm{GL}_n \cong \mathrm{Aut}_k(V)$$

equipped with $v \in V \smallsetminus \{0\}$ such that $N_G(B)(k) = \{g \in G(k) \mid \rho(g)v \in k \cdot v\}$. (The argument is the same as the one given in the discussion on the quotient; we first choose V to be a finite-dimensional $G(k)$-stable subspace which generates $k[G]$ as k-algebra, and consider a subspace W defined to be the intersection of V and the ideal of definition of the closed subgroup $N_G(B)$. Then we replace V and W with their d-th exterior product, where $d = \dim W$.)

Let us consider the morphism

$$\rho_v\colon G \to V\ (\cong \mathbb{A}^n_k); \quad g \mapsto \rho(g)v.$$

Then ρ_v is B-invariant. Indeed, for any $g \in B_u(k)$, all the eigenvalues of $\rho(g)$ are 1. Thus, if $\rho(g)v \in k \cdot v$, we must have $\rho(g)v = v$. Also, for any $g \in T(k)$, the surjectivity assumption on ϕ implies that there exists $t \in T(k)$ such that $g = xtx^{-1}t^{-1}$. Recalling that $x \in N_G(B)$, both x and t acts on v by scalar multiplications, hence the action of $xtx^{-1}t^{-1}$ on v is trivial. As we have $B = TB_u$, we get that ρ_v is B-invariant.

Therefore, the universal property of the quotient $G \twoheadrightarrow G/B$ induces a morphism $\bar{\rho}_v\colon G/B \to V$ such that ρ_v is the composition of the quotient morphism and $\bar{\rho}_v$. Since B is a Borel subgroup, hence parabolic, the quotient variety G/B is complete. Hence its image under $\bar{\rho}_v$ is also complete and closed in V. However, as V is affine, only the possibility is that the image is a point (note that G is irreducible, hence so is $\bar{\rho}_v(G/B)$). Hence we get $\rho_v(G) = \{v\}$, which means that $N_G(B) = G$. In other words, B is normal in G. This gurantees that G/B is affine. Again by noting that G/B is complete, we get $G = B$. $\square$

Remark 10.13. The projective variety G/B is referred to as the *flag variety* of G. By the conjugacy theorem, all Borel subgroups of G are G-conjugate to each other. Hence, the map

$$G(k) \to \{\text{Borel subgroups of } G\}\colon g \mapsto gBg^{-1}$$

is surjective. Moreover, by Theorem 10.12, this map factors through a bijection

$$G(k)/B(k) \xrightarrow{1:1} \{\text{Borel subgroups of } G\}.$$

Therefore, the flag variety G/B can be thought of as the moduli space of Borel subgroups of G.

Exercise 10.14. Let G be a connected linear algebraic group. Prove that, for any parabolic subgroup P of G, we have $N_G(P) = P = P^\circ$. (Hint: take a Borel subgroup of P to apply Theorem 10.12.)

10.4. **Descent property of Jordan decomposition.**

Theorem 10.15. *Let G be a connected linear algberaic group. Let $g \in G$ be any element with Jordan decomposition $g = g_s g_u$. Then $g, g_s, g_u \in Z_G(g_s)^\circ$.*

Proof. By Theorem 10.10 (1), we can choose a Borel subgroup B containing g. Then we have $g_s, g_u \in B$, hence $g, g_s, g_u \in Z_B(g_s)$. Since $Z_B(g_s)$ is connected by Theorem 10.4, the obvious inclusion $Z_B(g_s) \subset Z_G(g_s)$ implies that $Z_B(g_s) \subset Z_G(g_s)^\circ$. $\square$

11. Week 11: Construction of root data for reductive groups

Recall that, in Week 7, we introduced the notions of root systems and root data. Then we also explained that root data give a complete classification of connected reductive groups.

Theorem 11.1 (Thoerem 7.7)**.** *There exists a bijection between the set of*

- *isomorphism classes of connected reductive groups and*
- *isomoprhism classes of reduced root data.*

Today's aim is to explain how the bijective map is given and also look at several examples.

11.1. Weyl group and Bruhat decomposition.

Definition 11.2. Let G be a linear reductive group. We fix a maximal torus T of G. The *Weyl group* $W_G(T)$ of G with respect to T is defined to be the quotient group

$$W_G(T) := N_G(T)/Z_G(T).$$

Recall that, when G is connected, all maximal tori are conjugate to each other (Theorem 9.3). Hence the Weyl group does not depend on the choice of a maximal torus up to isomorphism.

Fact 11.3. *The Weyl group $W_G(T)$ is a finite group.*

Fact 11.4 (Bruhat decomposition)**.** *Let G be a connected reductive group. We fix a Borel subgroup B of G and a maximal torus T of B. Then we have the following decomposition of algebraic varieties:*

$$G = \bigsqcup_{w\in W_G(T)} B\dot{w}B,$$

where, for each $w \in W_G(T)$, we take any representative $\dot{w} \in N_G(T)(k)$ of w.

11.2. **Examples of reductive groups.** Recall that, for a linear algebraic group G, we define its radical $R(G)$ and unipotent radical $R_u(G)$ to be the unique maximal closed connected solvable normal subgroup and the unique maximal closed connected unipotent normal subgroup, respectively. Then we say that G is reductive (resp. semisimple) if $R_u(G)$ (resp. $R(G)$) is trivial.

So far, we only explained that tori and GL_n are reductive (and also stated that SL_n is semisimple as an exercise). Let us see more examples.

11.2.1. *General linear group.* We first again consider the general linear group GL_n. We already saw that GL_n is reductive in Week 9. Recall that we put B_{2n} and T_{2n} to be the subgroups of upper-triangular matrices and diagonal matrices, respectively.

Lemma 11.5. *We have the following:*

1. *B_n is a Borel subgroup of GL_n;*
2. *T_n is a maximal torus of GL_n.*

Proof. To show (1), take any Borel subgroup B of GL_n, i.e., a maximal connected solvable closed subgroup. Then, by Lie–Kolchin's theorem, there exists $x \in \mathrm{GL}_n(k)$ such that $xBx^{-1} \subset B_n$. Since B_n is a connected solvable closed subgroup of GL_n, the maximality of B implies that $xBx^{-1} = B_n$. This implies that B_n is also a Borel subgroup.

To show (2), we recall that a subtorus of a connected solvable group is maximal if and only if it is maximal-dimensional (Theorem 10.7). By definition, the dimension of any maximal-dimensional torus of B_n is the dimension of B_n/U_n, where U_n denotes the subgroup of upper-triangular unipotent matrices. Since $B_n/U_n \cong T_n$, we obviously have $\dim T_n = \dim(B_n/U_n)$. Thus T_n is a maximal torus of B_n. In general, any maximal torus of a Borel subgroup is also a maximal torus of the whole group (this follows from that any maximal torus of the whole group is contained in a Borel subgroup and that all maximal tori are conjugate). Hence T_n is also a maximal torus of GL_n. □

Remark 11.6. We can also show (2) more directly as follows. Suppose that $T \subset \mathrm{GL}_n$ is a torus containing T_n. Then, by taking the centralizers of $T_n \subset T$, we get $Z_{\mathrm{GL}_n}(T_n) \supset Z_{\mathrm{GL}_n}(T)$. By concrete calculation, we can check that $Z_{\mathrm{GL}_n}(T_n) = T_n$. On the other hand, as T is commutative, we have $T \subset Z_{\mathrm{GL}_n}(T)$. Thus we get $T \subset Z_{\mathrm{GL}_n}(T) \subset Z_{\mathrm{GL}_n}(T_n) = T_n \subset T$, which implies that $T = T_n$.

Lemma 11.7. *The Weyl group $W_{\mathrm{GL}_n}(T_n)$ is isomorphic to the symmetric group $\mathfrak{S}_n$.*

Proof. The conjugate action of $N_{\mathrm{GL}_n}(T_{2n})$ on T_{2n} induces a faithful algebraic action of a finite group $W_{\mathrm{GL}_{2n}}(T_{2n})$ on T_{2n}. Let $w \in W_{\mathrm{GL}_{2n}}(T_{2n})(k)$; then its action on T_{2n} is given by

$$\mathrm{diag}(t_1, \ldots, t_n) \mapsto \mathrm{diag}(s_1, \ldots, s_n),$$

where each s_i is of the form $t_1^{r_1^{(i)}} \cdots t_n^{r_n^{(i)}}$ with $r_j^{(i)} \in \mathbb{Z}$. Let consider a diagonal matrix $t = \mathrm{diag}(x_1, \ldots, x_n) \in T_{2n}(k)$ such that $x_i \neq x_j$ for any $i \neq j$ (this is possible since k is an infinite set).

Then, in particular, we have $wtw^{-1} \in T_{2n}(k)$. Since both t and wtw^{-1} are diagonal matrices which are conjugate, the set of diagonal entries of wtw^{-1} must be the same as that of t. This means that $(s_1, \ldots, s_n)$ is a permutation of $(t_1, \ldots, t_n)$.

Conversely, for any permutation $\sigma \in \mathfrak{S}_n$, we can find an element $\dot{w} \in N_{\mathrm{GL}_n}(T_n)(k)$ whose action on T_n is given by σ; we may choose $\dot{w}$ to be the permutation matrix corresponding to σ. □

11.2.2. *Symplectic group.* The *symplectic group* Sp_{2n} is a linear algebraic group whose R-valued points are given as follows:

$$\mathrm{Sp}_{2n}(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_{2n}(R) \mid {}^t g J_{2n} g = J_{2n}\},$$

where J_{2n} denotes the antidiagonal matrix whose antidiagonal entries are given by 1 and -1 alternatively:

$$J_{2n} := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \iddots & & & \end{pmatrix}.$$

The symplectic group Sp_{2n} is reductive. To show this, we note that Sp_{2n} is a subgroup of GL_{2n} by definition. Recall that B_{2n} and T_{2n} are a Borel subgroup and a maximal torus of GL_{2n}, respectively.

Lemma 11.8. *We have the following:*

(1) $B_{2n} \cap \mathrm{Sp}_{2n}$ *is a Borel subgroup of* Sp_{2n}*;*

(2) $T_{2n} \cap \mathrm{Sp}_{2n}$ *is a maximal torus of* Sp_{2n}.

Proof. We put $B := B_{2n} \cap \mathrm{Sp}_{2n}$. Let us show that B is a Borel subgroup of Sp_{2n}. As B is a subgroup of B_{2n}, B is solvable. Moreover, B is connected; this can be checked directly by computing the defining equations of B. Hence we can take a Borel subgroup B' of Sp_{2n} containing B. By Lie–Kolchin's theorem, there exists an element $x \in \mathrm{GL}_{2n}(k)$ such that $xB'x^{-1} \subset B_{2n}$.

we put

$$g_i := \begin{cases} I_{2n} + E_{i,i+1} + E_{2n-i,2n+1-i} & 1 \leq i \leq n-1, \\ I_{2n} + E_{n,n+1} & i = n, \end{cases}$$

and $g := g_1 \cdots g_n$. (The explicit description of g as a matrix is not important; only note that each $(i, i+1)$-entry $(1 \leq i \leq 2n-1)$ of g is nonzero.) At least we can check easily that g_i belongs to B, hence so does their product g. In particular, we have $g \in B_{2n}$ and $xgx^{-1} \in B_{2n}$.

Now we use the Bruhat decomposition of GL_{2n} with respect to B_{2n}:

$$\mathrm{GL}_{2n} = \bigsqcup_{w \in W_{\mathrm{GL}_{2n}}(T_{2n})} B_{2n} \dot{w} B_{2n}.$$

We can find elements $b, b' \in B_{2n}$ and a permutation matrix w such that $x = b'wb$. Then the condition $xgx^{-1} \in B_{2n}$ is equivalent to $w(bgb^{-1})w^{-1} \in B_{2n}$. We note that each $(i, i+1)$-entry $(1 \leq i \leq 2n-1)$ of bgb^{-1} is nonzero (since $b \in B_{2n}$ and g satisfies this condition). Thus, for $w(bgb^{-1})w^{-1}$ to be upper-triangular, w must be the identity element. Hence we have $x \in B_{2n}$. This implies that B' is contained in B_{2n}. Hence $B' \subset B_{2n} \cap \mathrm{Sp}_{2n} = B$, which implies $B = B'$.

The statement (2) can be shown in a similar manner as in the case of GL_n. $\square$

Proposition 11.9. *The symplectic group* Sp_{2n} *is reductive.*

Proof. The group $B := B_{2n} \cap \mathrm{Sp}_{2n}$ is a Borel subgroup of Sp_{2n} by the above lemma. Similarly, its opposite $\overline{B} := \overline{B}_{2n} \cap \mathrm{Sp}_{2n}$ is also a Borel subgroup of Sp_{2n},

Since the unipotent radical $R_u(\mathrm{Sp}_{2n})$ is connected and unipotent, hence solvable, the maximality of Borel subgroups implies that $R_u(\mathrm{Sp}_{2n})$ is contained in a Borel subgroup of Sp_{2n}. Moreover, as $R_u(\mathrm{Sp}_{2n})$ is normal in Sp_{2n}, the fact that all Borel subgroups are conjugate to each other implies that $R_u(\mathrm{Sp}_{2n})$ is contained in any Borel subgroup of Sp_{2n}. In particular, we get $R_u(\mathrm{Sp}_{2n}) \subset B \cap \overline{B} = T$. Since this furthemore implies that $R_u(\mathrm{Sp}_{2n}) \subset T_u = \{1\}$, Sp_{2n} is reductive. $\square$

Remark 11.10. For the definition of the symplectic group, we often use the matrix

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$$

instead of J_{2n} (especially, in the context of Siegel modular forms). Both definitions give isomorphic algebraic groups, but description of Borel subgroups are different. In particular, be careful that $B_{2n} \cap \mathrm{Sp}_{2n}$ is not a Borel subgroup of Sp_{2n} if we define Sp_{2n} using this matrix.

11.2.3. *Orthogonal group.* Let us assume that the characteristic of k is not 2. The *orthogonal group* O_n is a linear algebraic group whose R-valued points are given as follows:

$$\mathrm{O}_n(R) \cong \{g = (g_{ij})_{i,j} \in \mathrm{GL}_n(R) \mid {}^t g J'_n g = J'_n\},$$

where J'_n denotes the antidiagonal matrix whose antidiagonal entries are 1. We define the *special orthogonal group* SO_n to be O_n°. By the same argument as in the symplectic case, we can show the following:

Proposition 11.11. *We have the following:*

(1) *$B_n \cap \mathrm{SO}_n$ is a Borel subgroup of SO_n;*
(2) *$T_n \cap \mathrm{SO}_n$ is a maximal torus of SO_n;*
(3) *The special orthogonal group SO_n is reductive.*

11.3. **Construction of root data.** We first recall the definitions of a root datum and a root system from Week 7 (Definition 7.10). We call a pair (V, R) a root system if V is a finite-dimensional $\mathbb{R}$-vector space and R is its finite subset satisfying the following:

(1) $0 \notin R$ and $V = \mathrm{Span}_{\mathbb{R}}(R)$;
(2) for each $\alpha \in R$, there exists an $\alpha^\vee \in V^\vee$ such that
   (a) $\langle \alpha, \alpha^\vee \rangle = 2$,
   (b) $\langle R, \alpha^\vee \rangle \subset \mathbb{Z}$,
   (c) $s_\alpha(R) = R$ for any $\alpha \in R$, where $s_\alpha \colon V \to V$ denotes the "reflection" with respect to α:

$$s_\alpha(x) := x - \langle x, \alpha^\vee \rangle \alpha.$$

   Each $\alpha^\vee$ is called the *coroot* of α.

We call a quadruple $(X, R, X^\vee, R^\vee)$ a *root datum* if

- X and $X^\vee$ are free abelian groups of finite rank equipped with a perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$ and
- R and $R^\vee$ are finite subsets of X and $X^\vee$ (called the sets of *roots* and *coroots*) equipped with a bijection $R \leftrightarrow R^\vee \colon \alpha \mapsto \alpha^\vee$

satisfying

(1) for any $\alpha \in R$, we have $\langle \alpha, \alpha^\vee \rangle = 2$,
(2) for any $\alpha \in R$, we have $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

Here, s_α and $s_\alpha^\vee$ denote the automorphisms of X and $X^\vee$ given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^\vee(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle \alpha^\vee.$$

In the following, we explain how to associate a root datum to a connected reductive group. Here we only give the outline; see [Spr09, Chapter 7] for details.

Let G be a connected reductive group. We first take a maximal torus T of G. We put $X := X^*(T)$ and $X^\vee := X_*(T)$. Note that then X and $X^\vee$ have a natural perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$. To be more precise, we put

$$\langle \chi, \chi^\vee \rangle := \chi \circ \chi^\vee \in \mathrm{Hom}_k(\mathbb{G}_{\mathrm{m}}, \mathbb{G}_{\mathrm{m}}) \cong \mathbb{Z}.$$

We consider the conjugate action of T on G, i.e., $T \times_k G \to G \colon (t, g) \mapsto tgt^{-1}$. Then this induces an algebraic group action of T on $\mathfrak{g} := \mathrm{Lie}\, G$. Since T is a torus, especially, a diagonalizable group, any algebraic representation is diagonalizable (to be more precise, the image of any homomorphism $T \to \mathrm{GL}_n$ is conjugate to a subgroup of T_n; see Proposition 5.9). By applying this property to the action of T on $\mathfrak{g}$, we obtain a simultaneous eigenspace decomposition

$$\mathfrak{g} = \bigoplus_{\chi \in X^*} \mathfrak{g}_\chi,$$

where $\mathfrak{g}_\chi$ is an affine subspace of $\mathfrak{g}$ where T acts via character χ. We let $R := \{\alpha \in X \mid \mathfrak{g}_\alpha \neq 0\} \smallsetminus \{0\}$ and $V := \mathrm{Span}_{\mathbb{R}}(R)$.

Fact 11.12. *We have $\mathfrak{g}_0 = \mathfrak{t} := \mathrm{Lie}\, T$ and each $\mathfrak{g}_\alpha$ is isomorphic to $\mathbb{G}_{\mathrm{a}}$. In particular, we have*

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha.$$

Furthermore, (V, R) forms a reduced root system.

If we furthermore choose a Borel subgroup B containing T, then the conjugate action of T on G preserves B. Therefore, we may consider a subset R_+ of R consisting of roots α such that $\mathfrak{g}_\alpha \subset \mathrm{Lie}\, B$. In fact, this determines a set of simple roots. More previsely, R_+ is the set of positive roots with respect to a set of simple roots of (V, R).

For each root $\alpha \in R$, we can find a unique subgroup U_α of G satisfying the following property. First, U_α is isomorphic to $\mathbb{G}_{\mathrm{a}}$ and normalized by T. Second, by fixing an isomorphism $\iota\colon \mathbb{G}_{\mathrm{a}} \xrightarrow{\cong} U_\alpha$, the conjugate action of T on U_α is identified with α, i.e.,

$$t \cdot \iota(x) \cdot t^{-1} = \iota(\alpha(t) \cdot x)$$

for any $x \in \mathbb{G}_{\mathrm{a}}(k)$ and $t \in T(k)$. We call U_α the *root subgroup* of α.

It can be proved that $-\alpha$ is also a root when α is a root. Moreover, the subgroup $\langle U_\alpha, U_{-\alpha} \rangle$ generated by U_α and $U_{-\alpha}$ is isomorphic to SL_2 or $\mathrm{PGL}_2 := \mathrm{GL}_2 / \mathbb{G}_{\mathrm{m}}$. Furthermore, in any case, there exists a homomorphism $\phi\colon \mathrm{SL}_2 \to \langle U_\alpha, U_{-\alpha} \rangle$ satisfying

$$\phi\left(\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right) = U_\alpha \quad \text{and} \quad \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\right) = U_{-\alpha}.$$

This homomorphism ϕ maps any diagonal element of SL_2 into T. Thus, we can define a cocharacter $\alpha^\vee \in X^\vee$ by

$$\alpha^\vee(y) := \phi\left(\left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)\right).$$

We call $\alpha^\vee$ the *coroot associated to* α. We put $R^\vee$ to be the set of all coroots obtained in this way.

Fact 11.13. *The quadruple $(X, R, X^\vee, R^\vee)$ forms a reduced root datum.*

We note that, in the definitions of a root system and a root datum, we consider the *reflection* s_α for each root $\alpha \in R$; recall that s_α is an automorphism of X. We let $W(X, R)$ be the group of automorphisms of X generated by $\{s_\alpha \mid \alpha \in R\}$. On the other hand, also recall that we have an action of the Weyl group $W_G(T)$ on $X = X^*(T)$. In fact, $W(X, R)$ and $W_G(T)$ are equal (as subgroups of the automorphism group of X).

We also note that, this construction of a root datum $(X, R, X^\vee, R^\vee)$ depends on the choice of T. However, as all maximal tori of G are conjugate to each other, the resulting root data is well-defined up to isomorphism.

11.4. **Examples of root data.**

11.4.1. GL_n *case again.* In Week 7, we examined root datum of GL_n; let us re-examine it here. The simultaneous eigenspace decomposition of $\mathfrak{g} := \operatorname{Lie}\mathrm{GL}_n \cong \mathfrak{gl}_n$ is given by

$$\mathfrak{gl}_n = \bigoplus_{1\le i,j\le n} \mathfrak{g}_{ij},$$

where $\mathfrak{g}_{ij}$ denotes the 1-dimensional subspace of $\mathfrak{g}$ whose only (i,j)-entry is nonzero. The action of T_n on $\mathfrak{g}_{ij}$ is given by $e_i - e_j \in X := X^*(T_n)$, where e_i denotes the projection to the i-th diagonal entry. Thus the set of roots is given by

$$R := \{e_i - e_j \mid 1 \le i \ne j \le n\}.$$

In Week 7, we 'defined' the set of coroots by

$$R^\vee := \{e_i^\vee - e_j^\vee \mid 1 \le i \ne j \le n\}$$

and claimed that $(X, R, X^\vee, R^\vee)$ forms a root datum. However, now we can see that this explanation is not really following the above construction of a root datum. What we have to do is to, for each root $\alpha \in R$, determine the subgroup $\langle U_\alpha, U_{-\alpha}\rangle$. Moreover, we choose a homomorphism $\phi\colon \mathrm{SL}_2 \to \langle U_\alpha, U_{-\alpha}\rangle$ satisfying

$$\phi\left(\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right) = U_\alpha \quad \text{and} \quad \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\right) = U_{-\alpha}.$$

Then this homomorphism ϕ defines the coroot $\alpha^\vee \in X^\vee$ by the formula

$$\alpha^\vee(y) := \phi\left(\left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)\right).$$

In the case of GL_n, we can check that the subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ is isomoprhic to SL_2. If $\alpha = e_i - e_j$, then a homomorphism (isomorphism) $\phi\colon \mathrm{SL}_2 \to \langle U_\alpha, U_{-\alpha}\rangle$ can be taken so that

$$\phi\left(\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)\right) = I_n + xE_{ij} \quad \text{and} \quad \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ x & 1 \end{smallmatrix}\right)\right) = I_n + xE_{ji},$$

where E_{ij} denotes the matrix whose (i,j)-entry is 1 and all the others are 0. Then it is easy to see that $\alpha^\vee(y)$ is a diagonal matrix whose i-the entry is y, j-th entry is y^{-1}, and all the others are 1. In other words, $\alpha^\vee = e_i^\vee - e_j^\vee$, where $e_i^\vee$ denotes the injection of $\mathbb{G}_{\mathrm{m}}$ into the i-th entry of T_n.

11.4.2. Sp_{2n}. We next consider the case of $G = \mathrm{Sp}_{2n}$. Our first task is to compute the Lie algebra of Sp_{2n}.

Let $i\colon \mathrm{Sp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$ be the closed immersion of Sp_{2n} into GL_{2n}. Then i induces an injective Lie algebra homomorphism $di\colon \operatorname{Lie}\mathrm{Sp}_{2n} \hookrightarrow \operatorname{Lie}\mathrm{GL}_{2n}$. Moreover, its image is identified with

$$\{D \in \operatorname{Lie}\mathrm{GL}_{2n} \mid D(I) = 0\},$$

where I denotes the kernel of the k-algebra homomorphism $k[\mathrm{GL}_{2n}] \twoheadrightarrow k[\mathrm{Sp}_{2n}]$ on the coordinate rings (see Lemma 6.12 in Week 6). If we write $k[\mathrm{GL}_{2n}] = k[\{x_{ij}\}_{ij}, \delta]$, where[10] $\delta := \det(x_{ij})_{ij}$, then I is its ideal generated by all entries of the matrix ${}^tXJ_{2n}X - J_{2n}$ $(X := (x_{ij})_{ij})$.

By definition and Lemma 6.6 in Week 6, we have

$$\operatorname{Lie}\mathrm{GL}_{2n} = T_{I_{2n}}\mathrm{GL}_{2n} = D_k(k[\mathrm{GL}_{2n}], k_{I_{2n}}) \cong (\mathfrak{m}_{I_{2n}}/\mathfrak{m}_{I_{2n}}^2)^*,$$

where $T_{I_{2n}}\mathrm{GL}_{2n}$ is the Zariski tangent space of GL_{2n} at the origin I_{2n} and $\mathfrak{m}_{I_{2n}}$ is the maximal ideal of $k[\mathrm{GL}_{2n}]$ corresponding to $I_{2n} \in \mathrm{GL}_{2n}(k)$ $(k_{I_{2n}} = k[\mathrm{GL}_{2n}]/\mathfrak{m}_{I_{2n}})$.

[10] Usually I have used D for the symbol of determinant, but here I avoid using because D is also used for elements of the Lie algebra.

As quickly explained in Section 6.6.3, $\operatorname{Lie}\mathrm{GL}_n$ can be identified with the Lie algebra $\mathfrak{gl}_n$ of n-by-n matrices by

$$\mathfrak{gl}_n(k) \to \operatorname{Lie}\mathrm{GL}_n\colon A \mapsto D_A,$$

where D_A is expressed by $D_A(x_{ij}) = -\sum_{l=1}^{n} x_{il}a_{lj}$ for $A = (a_{ij})_{ij} \in \mathfrak{gl}_n(k)$ as an element of $D_k(k[\mathrm{GL}_n], k[\mathrm{GL}_n])$. Note that, in other words, $D_A(x_{ij}) = (-XA)_{ij}$.

Under this identification, an element $A \in \mathfrak{gl}_n(k)$ belongs to $\operatorname{Lie}\mathrm{Sp}_{2n}$ if and only if $D_A(I) = 0$, which is equivalent to that

$$D_A(({}^tXJ_{2n}X - J_{2n})_{ij}) = 0$$

for all (i,j). Using Leibniz rule, we can compute the left-hand side as follows:

$$\begin{aligned} D_A(({}^tXJ_{2n}X - J_{2n})_{ij}) &= D_A(({}^tXJ_{2n}X)_{ij}) \\ &= (D_A({}^tX)J_{2n}X + {}^tXD_A(J_{2n}X))_{ij} \\ &= ({}^tD_A(X)J_{2n}X + {}^tXJ_{2n}D_A(X))_{ij}. \end{aligned}$$

Here, we simply write $D_A((f_{ij})_{ij}) := (D_A(f_{ij}))_{ij}$. Recalling that $D_A(X) = (-XA)$, this equals

$${}^t(-XA)J_{2n}X + {}^tXJ_{2n}(-XA) = 0.$$

The left-hand side is written as an element of $D_k(k[\mathrm{GL}_n], k[\mathrm{GL}_n])$. If we regard it as an element of $D_k(k[\mathrm{GL}_n], k_{I_{2n}})$, i.e., substitute $X = I_{2n}$ for it, then we get

$${}^tAJ_{2n} + J_{2n}A = 0.$$

In summary, $\operatorname{Lie}\mathrm{Sp}_{2n}$ can be thought of as the following subspace of $\mathfrak{gl}_{2n}$:

$$\mathfrak{sp}_{2n} := \{A \in \mathfrak{gl}_{2n} \mid {}^tAJ_{2n} + J_{2n}A = 0\}.$$

Once we obtain this description, it is a routine work to determine the root datum. As explained before, $T := T_{2n} \cap \mathrm{Sp}_{2n}$ is a maximal torus of Sp_{2n}. Explicitly, this is given by

$$T = \{\operatorname{diag}(t_1, \ldots, t_n, t_n^{-1}, \ldots, t_1^{-1}) \mid t_i \in \mathbb{G}_\mathrm{m}\}.$$

Using these explicit coordinates, we can check that the set of roots is given by

$$\{e_i - e_j \mid 1 \le i \ne j \le n\} \sqcup \{\pm 2e_i \mid 1 \le i \le n\}.$$

Moreover, the set of coroots is given by

$$\{e_i - e_j \mid 1 \le i \ne j \le n\} \sqcup \{\pm e_i \mid 1 \le i \le n\}.$$

Exercise 11.14. Complete the above "routine work", i.e., justify that the sets of roots and coroots are indeed given as above.

12. Week 12: Classification of reduced root systems

Recall that we have constructed a map

$$\{\text{connected reductive groups}\}/\sim\ \to \{\text{reduced root data}\}/\sim.$$

To get the classification theorem (Theorem 7.7), we need to show the injectivity and the surjectivity of this map. The key of the proof of the surjectivity part is to classify reduced root data. Recall that a root datum is a quadruple $(X, R, X^\vee, R^\vee)$. For any root datum, the pair $(X \otimes_{\mathbb{Z}} \mathbb{R}, R)$ forms a root system; so a root system is weaker than a root datum (in the sense that it has less information). What we will do this week is to first classify reduced root systems. What is surprising is that this problem is eventually reduced to a simple elimination game on weighted graphs.

12.1. **Another definition of a root system.** Recall that a root system is a pair (V, R) of a finite-dimensional $\mathbb{R}$-vector space V and its finite subset R satisfying several axioms, which especially requires the existence of a coroot $\alpha^\vee \in V^\vee$ for each root $\alpha \in R$. Also recall that we define the Weyl group $W := W(V, R)$ of (V, R) to be the group of automorphism of V generated by all reflections s_α associated to $\alpha \in R$.

Fact 12.1. *Let (V, R) be a root system. There exists a unique inner product (positive-definite symmetric bilinear form) $(-,-)\colon V \times V \to \mathbb{R}$ such that*

1. *$(-,-)$ is W-invariant, i.e. $(wv, wv') = (v, v')$ for any $w \in W$, $v, v' \in V$,*
2. *the isomorphism $V \cong V^\vee$ determined by $(-,-)$ identifies $\alpha^\vee$ with $2\alpha/(\alpha,\alpha)$.*

In the following discussion, we always regard coroots $\alpha^\vee$ as elements of V by this fact. (Depending on the textbooks, a coroot is defined to be an element of V from the beginning; in that case, by reversing this discussion, we can obtain a root system in our sense.)

12.2. **From reduced root system to Cartan matrix.** Let us first start from introducing the notion of an abstract Cartan matrix.

Definition 12.2. We call an l-by-l matrix $A = (a_{ij})_{ij} \in M_l(\mathbb{Z})$ a *Cartan matrix* if it satisfies the following:

1. $a_{ii} = 2$ for any $1 \le i \le l$,
2. $a_{ij} \le 0$ for any $1 \le i \ne j \le l$,
3. $a_{ij} = 0$ if and only if $a_{ji} = 0$,
4. there exists a diagonal matrix D with positive diagonal entries such that DAD^{-1} is a symmetric and positive definite.

We say that two Cartan matrices A and A' are *equivalent* if there exists a permutation matrix P such that $A' = PAP^{-1}$.

Remark 12.3. We say that a Cartan matrix is indecomposable if it is not equivalent to a Cartan matrix of a nontrivial block-diagonal form. As long as Cartan matrix is indecomposable, a diagonal matrix D as in (4) must be unique up to scalar ($\mathbb{R}_{>0}$).

To any reduced root system (V, R), we associate a Cartan matrix in the following way. We fix a set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ of (V, R). We define entries a_{ij} by

$$a_{ij} := \frac{2(\alpha_i, \alpha_j)}{|\alpha_i|^2}.$$

We put $A := (a_{ij})_{ij}$.

Example 12.4. The reduced root system constructed from GL_4 is $V = \mathrm{Span}_{\mathbb{R}}(R)$, where

$$R := \{e_i - e_j \mid 1 \leq i \neq j \leq 4\}.$$

Its set of simple roots can be taken to be $\{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$. Also, its inner product is the restriction of the standard Euclidean inner product on $\mathbb{R}e_1 \oplus \cdots \mathbb{R}e_4$. Hence we can compute its Cartan matrix as follows;

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

Lemma 12.5. *The matrix A determined by (V, R) as above is a Cartan matrix. Moreover, the equivalence class of A does not depend on the choice of Δ (and also on the labeling $\{\alpha_1, \dots, \alpha_l\}$).*

Proof. For a fixed set of simple roots Δ, if we change its labeling, obviously the associated matrix A is changed only via permutation of rows and columns, hence the equivalence class does not change.

To check that the choice of a set of simple roots does not matter, we use the following fact:

> For any sets of simple roots Δ and Δ', there uniquely exists an element $w \in W$ such that $\Delta' = w(\Delta)$.

For $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and $\Delta' = \{\alpha'_1, \dots, \alpha'_l\}$, let $w \in W$ be an element such that $\alpha'_i = s(\alpha_i)$. Then we have

$$\frac{(\alpha'_i, \alpha'_j)}{|\alpha'_i|^2} = \frac{(s(\alpha_i), s(\alpha_j))}{|s(\alpha_i)|^2} = \frac{(\alpha_i, \alpha_j)}{|\alpha_i|^2}$$

by using that the inner product $(-,-)$ is W-invariant. $\square$

Remark 12.6. (1) If we have root systems (V, R) and (V', R'), then $(V \oplus V', R \sqcup R')$ is again a root system. If a root system cannot be written as a 'sum' of nonzero root systems, we say that it is *irreducible*. We can easily see that a reduced root system is irreducible if and only if the associated Cartan matrix A is indecomposable.

(2) If A is a Cartan matrix constructed from a reduced root system in the above manner, a diagonal matrix D as in (4) of the definition of a Cartan matrix can be chosen to be $\mathrm{diag}(|\alpha_1|, \dots, |\alpha_l|)$. In particular, the matrix D tells us the ratio of absolute values of simple roots $|\alpha_1| : \cdots : |\alpha_l|$.

This fact means that we have defined a map from the set of isomorphism classes of reduced root systems to the set of equivalence classes of Cartan matrices.

Fact 12.7. *This map is injective.*

Proof. We just give a brief sketch. Suppose that we have two reduced root systems (V, R) and (V', R') with sets of simple roots Δ and Δ', respectively, such that the associated Cartan matrices are the same. Write $\Delta = \{\alpha_1, \dots, \alpha_l\}$ and $\Delta' = \{\alpha'_1, \dots, \alpha'_l\}$. We may assume that both (V, R) and (V', R') are irreducible.

As remarked above, the Cartan matrix enables us to recover the ratios $|\alpha_1| : \cdots : |\alpha_l|$ and $|\alpha'_1| : \cdots : |\alpha'_l|$. Hence we firstly see that these ratios are the same.

By scaling all roots in R' (this operation does not change the isomorphism class of (V', R')), we may assume that $|\alpha_i| = |\alpha'_i|$ for all $1 \le i \le l$. Then, by looking at the (i,j)-entry of the Cartan matrix, we also get $(\alpha_i, \alpha_j) = (\alpha'_i, \alpha'_j)$ for all i and j. We define an R-vector space isomorphism $\varphi \colon V \to V'$ by $\varphi(\alpha_i) = \alpha'_i$; note that this preserves the inner products on V and V'. Hence it is enough to check that $\varphi(R) = R'$. This follows from the following general facts on reduced root systems:

(1) $R = \bigcup_{i=1}^{l} W(\alpha_i)$,
(2) $W = \langle s_{\alpha_i} \mid i = 1, \dots, l \rangle$.

$\square$

12.3. **From Cartan matrix to Dynkin diagram.** Let $A = (a_{ij})_{ij}$ be a Cartan matrix. Let l be the size of the matrix A. Let $D = \operatorname{diag}(d_1, \dots, d_l)$ be a diagonal matrix with positive diagonal entries such that DAD^{-1} is symmetric. Recall that such D exists uniquely up to scalar multiplication. So, we normalize D so that the smallest positive diagonal entry is 1.

We associate a weighted graph to A in the following way (the meaning of a "weighted graph" is exactly the one explained below):

- We put l vertices; let us call them $v_1, \dots, v_l$.
- We put $a_{ij} \cdot a_{ji}$ edges between v_i and v_j (for $i \ne j$).
- We let the weight of v_i be d_i^2.

We call a weighted graph obtained from a reduced root system a *Dynkin diagram*. Note that equivalent Cartan matrices define the same Dynkin diagram.

Lemma 12.8. *If two Cartan matrices define the same Dynkin diagram, then they are equivalent.*

Proof. Let $A = (a_{ij})_{ij}$ be a Cartan matrix. It is enough to check that $A = (a_{ij})_{ij}$ can be recovered from its Dynkin diagram up to equivalence. Firstly, all a_{ii} must be 2 by the definition of a Cartan matrix. Since DAD^{-1} is symmetric, we must have

$$\frac{d_i}{d_j} \cdot a_{ij} = \frac{d_j}{d_i} \cdot a_{ji}$$

for any $1 \le i, j \le l$. On the other hand, for any $1 \le i, j \le l$, $a_{ij} \cdot a_{ji}$ can be read off from the number of edges between the vertices α_i and α_j. These are enough to determine a_{ij} and a_{ji} uniquely. $\square$

From our discussion so far, we have obtained

$$\{\text{reduced root systems}\}/\!\sim\ \hookrightarrow \{\text{Cartan matrices}\}/\!\sim\ \xrightarrow{1:1} \{\text{Dynkin diagrams}\}.$$

(Note that a Cartan matrix is indecomposable if and only if the associated Dynkin diagram is connected.)

Our strategy to classify the isomorphism classes of reduced root systems consists of the following steps:

(1) We first narrow down the possibilities of Dynkin diagrams, i.e., investigate necessary conditions for a weighted graph being a Dynkin diagram.
(2) We show that all the weighted graphes eventually left in the step (1) are indeed coming from Cartan matrices.
(3) For each Cartan matrix, we construct a reduced root system with that Cartan matrix "by hand".

12.4. **Classification of Dynkin diagrams.**

Lemma 12.9. *Let Γ be a Dynkin diagram. If two vertices are joined by a single edge, then they have the same weight.*

Proof. Let $A = (a_{ij})_{ij}$ be a Cartan matrix which gives rise to Γ. Let v_i and v_j be such vertices. As observed in the proof of Lemma 12.8, we have $\frac{d_i}{d_j} \cdot a_{ij} = \frac{d_j}{d_i} \cdot a_{ji}$, or equivalently, $d_i^2 \cdot a_{ij} = d_j^2 \cdot a_{ji}$. On the other hand, the single edge condition implies that $a_{ij} \cdot a_{ji} = 1$. Since a_{ij} and a_{ji} are non-positive integers, we must have $a_{ij} = a_{ji} = -1$. Hence we get $d_i^2 = d_j^2$. □

We consider the folowing two operations on weighted graphs:

Operation (I): Remove a vertex and all edges containing the vertex.
Operation (II): Contract two vertices joined by a single edge (and put the same weight as the original one).

In fact, when Γ is a Dynkin diagram and Γ' is the weighted graph obtained by applying (I) or (II) to Γ, we can show that Γ' is also a Dynkin diagram. More precisely, the following hold.

Proposition 12.10. *Let $A = (a_{ij})_{ij}$ be a Cartan matrix of size l with Dynkin diagram Γ.*

(1) *Let A' be the $(l-1)$-by-$(l-1)$ matrix obtained by removing i-th row and i-th column of A. Then A is a Cartan matrix. Moreover, its Dynkin diagram is the one obtained by applying (I) to the i-th vertex of Γ.*
(2) *Let i and j be indices such that $a_{ij} \cdot a_{ji} \neq 0$. Let A'' be the $(l-1)$-by-$(l-1)$ matrix obtained by*
   - *contracting the 2-by-2 minor matrices consisting of (i,i), (i,j), (j,i), (j,j)-entries to the 1-by-1 matrix (2), and*
   - *summing i-th row and j-th row, and also i-th column and j-th column.*

   Then A'' is a Cartan matrix. Moreover, its Dynkin diagram is the one obtained by applying (II) to the i-th and j-th vertices of Γ.

Example 12.11. Consider the following Cartan matrix (this is called the Cartan matrix of type D_5):

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{pmatrix}.$$

If we apply the operation as in Proposition 12.10 (I) to the 5th row and column, we get the Cartan matrix of type A_4:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}.$$

If we apply the operation as in Proposition 12.10 (II) to the second and third rows and columns, we get the Cartan matrix of type D_4:

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}.$$

Exercise 12.12. Prove Proposition 12.10. (Hint: (1) is straightforward. For (2), note that, if we choose i and j to be $l-1$ and l, the matrix A'' is given by $E \cdot A \cdot {}^tE$, where $E := \left(\begin{smallmatrix} I_{l-2} & 0 & 0 \\ 0 & 1 & 1 \end{smallmatrix}\right)$.)

Lemma 12.13. *Let A be a Cartan matrix. For any $1 \le i \ne j \le l$, we have*

(1) $a_{ij} \cdot a_{ji} < 4$.
(2) $a_{ij} = 0, -1, -2, -3$.

Proof. By applying (I) to all the rows and columns except for i-th and j-th, we can reduce to the case of size 2. In this case, A is just $\left(\begin{smallmatrix} 2 & a_{12} \\ a_{21} & 2 \end{smallmatrix}\right)$. By the definition of a Cartan matrix, this matrix is conjugate to a symmetric positive definite matrix. In particular, the determinant is positive. Hence we have $4 - a_{12}a_{21} > 0$. This inequality (combined with axioms of a Cartan matrix) implies both assertions. □

For the later discussion, let us introduce some additional objects. (As explained at the beginning, our final goal is to show that any Cartan matrix is coming from a reduced root system. So, we eventually want to construct a vector space "V" and a set of roots "R". What we do in the following is to construct only "V" and a set of simple roots "Δ".)

Let $A = (a_{ij})_{ij} \in M_l(\mathbb{Z})$ be a Cartan matrix. Let $D = \operatorname{diag}(d_1, \dots, d_l)$ be a diagonal matrix as in the definition of a Cartan matrix. If we put $Q := \frac{1}{2}DAD^{-1}$, then Q is a symmetric positive definite matrix whose all diagonal entries are 1. We write $Q = (q_{ij})_{ij}$. By linear algebra, we can find a symmetric positive-definite matrix $\sqrt{Q} \in M_l(\mathbb{R})$ satisfying $(\sqrt{Q})^2 = Q$. We consider an l-dimensional $\mathbb{R}$-vector space $V := \mathbb{R}^l$ with standard basis $\{e_1 \dots, e_l\}$. We put $\varphi_i := \sqrt{Q} \cdot e_i \in V$. Then

$$(\varphi_i, \varphi_j) = (\sqrt{Q}e_i, \sqrt{Q}e_j) = {}^t(\sqrt{Q}e_i) \cdot (\sqrt{Q}e_j) = {}^te_i \cdot Q \cdot e_j = q_{ij}.$$

In particular, $\{\varphi_1, \dots, \varphi_l\}$ is an $\mathbb{R}$-basis of V such that $|\varphi_i| = 1$ for all $1 \le i \le l$. Finally, we put $\alpha_i := d_i \cdot \varphi_i$. Then we have

$$a_{ij} = 2 \cdot (D^{-1}QD)_{ij} = 2 \cdot \frac{d_j}{d_i} \cdot q_{ij} = 2 \cdot \frac{d_j}{d_i} \cdot \langle \varphi_i, \varphi_j \rangle = \frac{2(\alpha_i, \alpha_j)}{|\alpha_i|^2},$$

where, in the last equality, we note that $d_i = |\alpha_i|$.

Proposition 12.14. *Let Γ be a Dynkin diagram with l vertices. Then the following hold:*

(1) *There are at most $(l-1)$ pairs of vertices joined by at least one edge.*
(2) *Γ has no loop.*
(3) *Each vertice has at most 3 edges.*

Proof. We first show (1). Let $A = (a_{ij})_{ij}$ be a Cartan matrix which realizes Γ. Put

$$\alpha := \sum_{i=1}^{l} \frac{\alpha_i}{|\alpha_i|}.$$

Since $\alpha \neq 0$, we get

$$0 < |\alpha|^2 = \sum_{1\le i,j\le l} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle = \sum_{1\le i\le l} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_i}{|\alpha_i|} \right\rangle + 2 \sum_{1\le i<j\le l} \left\langle \frac{\alpha_i}{|\alpha_i|}, \frac{\alpha_j}{|\alpha_j|} \right\rangle$$
$$= l + \sum_{1\le i<j\le l} \frac{2\langle \alpha_i, \alpha_j\rangle}{|\alpha_i| \cdot |\alpha_j|}$$
$$= l - \sum_{1\le i<j\le l} \sqrt{a_{ij} \cdot a_{ji}}.$$

Here, in the last equality, note that $\langle \alpha_i, \alpha_j\rangle \le 0$ for any $i \neq j$.

By Lemma 12.13, we have $a_{ij} \cdot a_{ji} = 0, 1, 2, 3$. Recall that vertices v_i and v_j are joined if and only if this number is not zero. Therefore, we get

$$0 < l - \sum_{1\le i<j\le l} \sqrt{a_{ij} \cdot a_{ji}} \le l - \sum_{\substack{1\le i<j\le l \\ v_i \text{ and } v_j \text{ are joined}}} 1.$$

In other words, the number of pairs of vertices joined by at least one edge is smaller than l.

We next show (2). For the sake of contradiction, let us suppose that Γ contains a loop. Then, by applying the operation (I) to Γ again and again, we can obtain a Dynkin diagram with three vertices containing a loop. However, such a graph obviously has 3 pairs of vertices joined by at least one edge. This contradicts (1).

We finally show (3). Let us note one vertex v_i of Γ. Let $v_{i_1}, \dots, v_{i_r}$ be the vertices joined with v_i. Let ℓ_k be the number of edges joining v_i and v_{i_k} (so our task is to show that the sum of ℓ_k over $1 \le k \le r$ is at most 3).

We consider the subspace $W := \mathrm{Span}_{\mathbb{R}}\{\alpha_i, \alpha_{i_1}, \dots, \alpha_{i_r}\}$, which is $(r+1)$-dimensional. Since γ has no loop by (2), we must have $\langle \alpha_{i_k}, \alpha_{i_m}\rangle = 0$ for any $k \neq m$. In other words, $\{\alpha_{i_1}/|\alpha_{i_1}|, \dots, \alpha_{i_k}/|\alpha_{i_k}|\}$ forms an orthonormal subset of W. We choose an orthonormal basis of W extending it, say, $\{\alpha_{i_1}/|\alpha_{i_1}|, \dots, \alpha_{i_k}/|\alpha_{i_k}|, \beta\}$. Since α_i is not contained in $\mathrm{Span}_{\mathbb{R}}\{\alpha_{i_1}, \dots, \alpha_{i_r}\}$, we then must have $\langle \alpha_i, \beta\rangle \neq 0$. Hence,

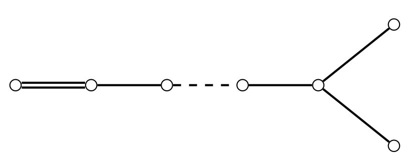
$$|\alpha_i|^2 = \sum_{k=1}^{r} \left\langle \alpha_i, \frac{\alpha_{i_k}}{|\alpha_{i_k}|} \right\rangle^2 + \langle \alpha_i, \beta\rangle^2 > \sum_{k=1}^{r} \left\langle \alpha_i, \frac{\alpha_{i_k}}{|\alpha_{i_k}|} \right\rangle^2.$$

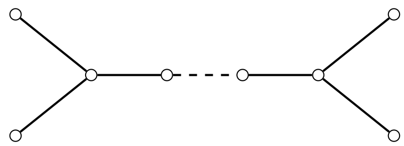
Equivalently,

$$1 > \sum_{k=1}^{r} \frac{\langle \alpha_i, \alpha_{i_k}\rangle}{|\alpha_i|^2 \cdot |\alpha_{i_k}|^2} = \sum_{k=1}^{r} \frac{1}{4} a_{i i_k} \cdot a_{i_k i} = \sum_{k=1}^{r} \frac{1}{4} \ell_k.$$

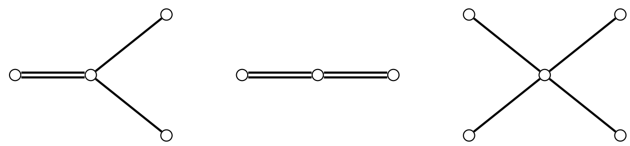
This completes the proof. $\square$

Lemma 12.15. *The following graphs cannot contained as a subgraph in a Dynkin diagram.*





Proof. By applying the operation (II) repeatedly to the middle vertices, the graphs as above are transformed into the following graphs:



However, all of these have a vertex whose having four edges. This contradicts Proposition 12.14 (3). ☐

Let Γ be a connected Dynkin diagram (recall that this condition amounts to that the corresponding Cartan matrix $A = (a_{ij})_{ij}$ is not decomposable). By Proposition 12.14 (3) and Lemma 12.15, we only have the following possibilities:

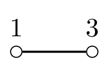
(a) Triple edge (G_2): If Γ has a triple edge, by Proposition 12.14 (3), each vertex cannot contain any more edge. Hence Γ must be of the following shape[11]:

In this case, we say Γ if of "type G_2".

Recall that, in general, the number of edges between i-th and j-th vartices is given by $a_{ij} \cdot a_{ji}$. Hence, in this case, we see that $a_{12} = -1$ and $a_{21} = -3$ (up to permutation). In other words, the Cartan matrix A is represented by

$$\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}.$$

The diagonal matrix $D = \mathrm{diag}(d_1, d_2)$ satisfies that DAD^{-1} is symmetric, which implies that $\frac{d_1}{d_2} a_{12} = \frac{d_2}{d_1} a_{21}$. Thus we get $3d_1^2 = d_2^2$. Therefore, if we normalize D so that $d_1 = 1$, we have $d_2^2 = 3$. Hence the weights of vertices are as follows:



(b) Double edge (type BCF): If Γ has a double edge, by Proposition 12.14 (3), each vertex containing the edge can have at most one more edge, which must be single in that case. Hence, by using Lemma 12.15, Γ is necessarily of the following shape:

In this case, we say Γ if of "type BCF".

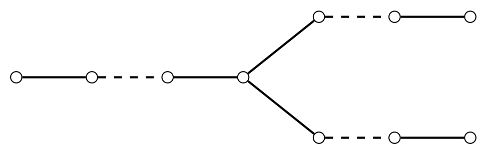
Recall that if two vertices are joined by a single edge, they have the same weight (Lemma 12.9). By the same argument as in the previous case, we

[11]Because I don't know how to write a triple line, I'm just writing a single edge!

can check that the weights of the vertices having double edges are given by 1 and 2. Thus the weights of vertices are:

(c) Single edge ("simply-laced", type ADE): In the case where Γ only has single edges, we consider whether Γ has a vetex joined with three other vertices (let us call it a "triple point") or not.

- If Γ has a triple point, Lemma 12.15 implies that Γ must be of the following shape:



In this case, we say Γ if of "type DE".

- If Γ does not have a triple point, Γ must be of the following shape:

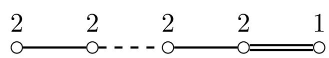
In this case, we say Γ if of "type A".

Note that Lemma 12.9 implies that all the weights are 1 in any case.

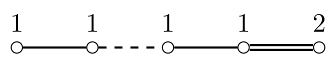
Let us first investigate the type BCF case.

Proposition 12.16. *Let Γ be a Dynkin diagram associated to a Cartan matrix $A = (a_{ij})_{ij}$. If it is of type BCF, then it must be one of the following (the subscript "n" denotes the number of vertices):*

Type B_n:

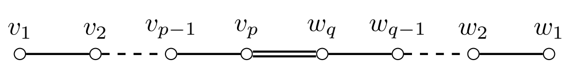


Type C_n:



Type F_4:

Proof. Let us give names to the vertices as follows:



Then it is enough to show that $(p-1)(q-1) < 2$. Recall that the Cartan matrix A gives rise to an $\mathbb{R}$-vector space V with a basis $\{\alpha_1, \dots, \alpha_l\}$ and an inner product $(-,-)$. To be more precise, α_i's sastisfies $a_{ij} = 2(\alpha_i, \alpha_j)/|\alpha_i|^2$ and $|\alpha_i| = d_i$. For convenience, we write $\{\alpha_1, \dots, \alpha_l\} = \{\alpha_1, \dots, \alpha_p\} \sqcup \{\beta_1, \dots, \beta_q\}$ so that α_i and β_j correspnd to v_i and w_j, respectively (hence $l = p + q$). We put

$$\alpha := \sum_{i=1}^{p} i \cdot \alpha_i \quad \text{and} \quad \beta := \sum_{j=1}^{q} j \cdot \beta_j.$$

The idea is to compare $(\alpha, \beta)^2$ with $|\alpha|^2$ and $|\beta|^2$.

Let us first compute the inner products of α_i's or β_j's corresponding to adjacent vertices.

- Since $v_1, \dots, v_p$ have the same weights (1), we have $|\alpha_1|^2 = \cdots = |\alpha_p|^2 = 1$. Moreover, the single edge condition implies that $a_{i,i+1} = a_{i+1.i} = -1$ for any $1 \le i \le p-1$. This implies that $(\alpha_i, \alpha_{i+1}) = -\frac{1}{2}$ for any $1 \le i \le p-1$.
- By exactly the same discussion (using $|\alpha_1|^2 = \cdots = |\alpha_p|^2 = 2$), we also get $(\beta_j, \beta_{j+1}) = -1$ for any $1 \le j \le q-1$.
- Since we have $2 = a_{p,p+1}a_{p+1,p} = 4(\alpha_p, \beta_q)^2/(|\alpha_p|^2 \cdot |\beta_q|^2)$, we get $(\alpha_p, \beta_q) = -1$.

Therefore, we get

$$(\alpha, \beta) = \sum_{i=1}^{p} \sum_{j=1}^{q} ij \cdot (\alpha_i, \beta_j) = -pq.$$

On the other hand, we have

$$\begin{aligned} |\alpha|^2 = \sum_{i=1}^{p} \sum_{i'=1}^{p} ii' \cdot (\alpha_i, \alpha_{i'}) &= \sum_{i=1}^{p} i^2 |\alpha_i|^2 + 2\sum_{i=1}^{p-1} i(i+1) \cdot (\alpha_i, \alpha_{i+1}) \\ &= \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) \\ &= p^2 - \sum_{i=1}^{p-1} i = \frac{p(p+1)}{2}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |\beta|^2 = \sum_{j=1}^{q} \sum_{j'=1}^{q} jj' \cdot (\beta_j, \beta_{j'}) &= \sum_{j=1}^{q} j^2 |\beta_j|^2 + 2\sum_{j=1}^{q-1} j(j+1) \cdot (\beta_j, \beta_{j+1}) \\ &= 2\sum_{j=1}^{q} i^2 - 2\sum_{i=1}^{q-1} j(j+1) \\ &= 2q^2 - 2\sum_{j=1}^{q-1} j = q(q+1). \end{aligned}$$

By the Cauchy–Schwarz inequality, we have

$$(\alpha, \beta)^2 \le |\alpha|^2 \cdot |\beta|^2.$$

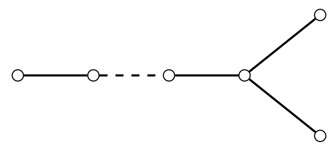
Note that the equality does not hold since α and β are note linear independent. Therefore, the above computation implies that

$$(-pq)^2 < \frac{p(p+1)}{2} \cdot q(q+1).$$

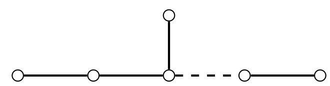
Noting that p and q are postive integers, this is equivalent to $(p-1)(q-1) < 2$. $\square$

Proposition 12.17. *Let Γ be a Dynkin diagram associated to a Cartan matrix $A = (a_{ij})_{ij}$. If it is of type DE, then it must be one of the following (the subscript "n" denotes the number of vertices):*

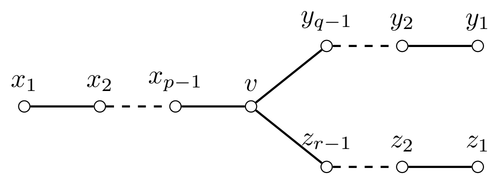
Type D_n:



Type E_n, where $6 \leq n \leq 8$:



Proof. The argument is similar to the type BCF case. We give names to vertices as follows:



To get the assertion, it is enough to establish the inequality (see Exercise below).

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1.$$

We write $\{\alpha_1, \dots, \alpha_l\} = \{\alpha_1, \dots, \alpha_{p-1}\} \sqcup \{\beta_1, \dots, \beta_{q-1}\} \sqcup \{\gamma_1, \dots, \gamma_{r-1}\} \sqcup \{\delta\}$ so that

- $\alpha_1, \dots, \alpha_{p-1}$ correspond to $x_1 \dots, x_{p-1}$,
- $\beta_1, \dots, \beta_{q-1}$ correspond to $y_1 \dots, y_{q-1}$,
- $\gamma_1, \dots, \gamma_{r-1}$ correspond to $z_1 \dots, z_{r-1}$, and
- δ corresponds to v.

We put

$$\alpha := \sum_{i=1}^{p-1} i \cdot \alpha_i, \quad \beta := \sum_{j=1}^{q-1} j \cdot \beta_j, \quad \beta := \sum_{k=1}^{r-1} k \cdot \gamma_k.$$

Let W be the 4-dimensional subspace of V generated by α, β, γ, δ. Note that α, β, and γ are orthogonal to each other. Hence $\{\alpha/|\alpha|, \beta/|\beta|, \gamma/|\gamma|\}$ forms an orthonormal subset of W. We extend it to an orthonormal basis $\{\alpha/|\alpha|, \beta/|\beta|, \gamma/|\gamma|, \varepsilon\}$ of V. Then we have

$$|\delta|^2 = (\delta, \alpha/|\alpha|)^2 + (\delta, \beta/|\beta|)^2 + (\delta, \gamma/|\gamma|)^2 + (\delta, \varepsilon)^2.$$

Note that δ is not contained in the subspace spanned by $\{\alpha/|\alpha|, \beta/|\beta|, \gamma/|\gamma|\}$, hence $(\delta, \varepsilon) \neq 0$. As $|\delta|^2 = 1$, we get

$$(\delta, \alpha/|\alpha|)^2 + (\delta, \beta/|\beta|)^2 + (\delta, \gamma/|\gamma|)^2 < 1.$$

Let us compute the left-hand side. By a similar, but simpler, computation as in the type BCF case, we can check that $(\alpha_i, \alpha_{i+1}) = -\frac{1}{2}$ for any $1 \leq i < p-1$, which implies that $|\alpha|^2 = \frac{1}{2}p(p-1)$. Also note that the product of $2(\alpha_{p-1}, \delta)/|\alpha_{p-1}|^2$ and $2(\alpha_{p-1}, \delta)/|\delta|^2$ gives the number of edges joining x_{p-1} and v, that is, 1. As $|\alpha_{p-1}|^2 = |\delta|^2 = 1$, this implies that $(\alpha_{p-1}, \delta) = -\frac{1}{2}$. Hence we get

$$(\alpha, \delta) = ((p-1)\alpha_{p-1}, \delta) = -\frac{p-1}{2}.$$

Similarly, we also get

$$(\beta, \delta) = -\frac{q-1}{2} \quad \text{and} \quad (\gamma, \delta) = -\frac{r-1}{2}.$$

Therefore, the above inequality can be rewritten as

$$\frac{p-1}{2p} + \frac{q-1}{2q} + \frac{r-1}{2r} < 1,$$

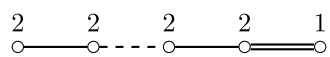
which is equivalent to the inequality as desired. □

Exercise 12.18. Complete the above proof, i.e., show that only the possibilities of (p, q, r) satisfying the inequality $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1$ are the ones of diagrams of types D_n, E_6, E_7, E_8.

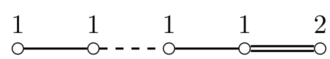
Let us summarize our discussion so far. If Γ is a Dynkin diagram, it must be one of the following (note that, at this point, we still do not know if all of them are indeed realized as a Dynkin diagram):

Type A_n:

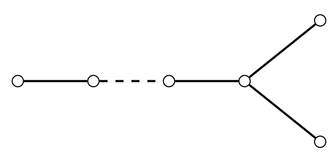
Type B_n:



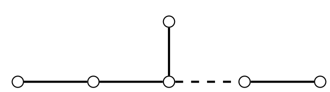
Type C_n:



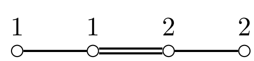
Type D_n:



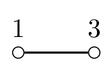
Type E_n, where $6 \leq n \leq 8$:



Type F_4:



Type G_2:



Remark 12.19. The infinite series A_n, B_n, C_n, and D_n are often referred to as "classical types", while the finitely many exceptional cases E_6, E_7, E_8, F_4, and G_2 are referred to as "exceptional types". Sometimes Dynkin diagrams are written using arrows instead of denoting weights on vertices. In that case, the direction of each arrow is from a heavier vertex to a lighter vertex. (Note that, as performed in the above discussion, the weights can be read off by looking at the number of edges as long as we know which vertices are lighter.)

12.5. **From Dynkin diagram to reduced root system.**

Theorem 12.20. *The weighted graphs A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 are Dynkin diagrams, i.e., associated to Cartan matrices.*

Proof. We can check that the following are Cartan matrices realizing all the Dynkin diagrams:

$$A_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix}, \quad B_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -2 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{pmatrix},$$

$$C_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -2 & 2 \end{pmatrix},$$

$$D_n = \begin{pmatrix} 2 & -1 & 0 & \cdots & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & & \vdots \\ 0 & -1 & 2 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & -1 & 0 & 0 \\ \vdots & & \ddots & -1 & 2 & -1 & -1 \\ \vdots & & & 0 & -1 & 2 & 0 \\ 0 & \cdots & \cdots & 0 & -1 & 0 & 2 \end{pmatrix},$$

$$E_6 := (\text{left-upper } 6\times 6 \text{ minor of } E_8), \quad E_7 := (\text{left-upper } 7\times 7 \text{ minor of } E_8),$$

$$E_8 = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix},$$

$$F_4 = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \quad G_2 = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}.$$

□

Theorem 12.21. *The Dynkin diagrams A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 are associated to reduced root systems.*

Proof. The proof can be done by case-by-case computation. Here, we only present an example of F_4; see [Hum78, §12.1] for other cases.

We let $V := \mathbb{R}^4$ with standard basis $\{e_1, \ldots, e_4\}$. We set

$$R := \{\pm e_i \mid 1 \le i \le 4\} \sqcup \{\pm e_i \pm e_j \mid 1 \le i < j \le 4\} \sqcup \left\{\frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\right\}$$

with all possible sign combinations. Then we can check that this pair (V, R) is a root system by complicated, but elementary, computation. As a set of simple roots, for example, we can choose

$$\Delta := \left\{e_2 - e_3,\, e_3 - e_4,\, e_4,\, \frac{1}{2}(e_1 - e_2 - e_3 - e_4)\right\}.$$

It is easy to check that the associated Cartan matrix is exactly F_4. □

In summary, we have obtained the following:

Theorem 12.22. *The above constructions give bijective maps*

$$\{\textit{reduced root systems}\}/\sim \xrightarrow{1:1} \{\textit{Cartan matrices}\}/\sim \xrightarrow{1:1} \{\textit{Dynkin diagrams}\}.$$

When a reduced root system (V, R) maps to a Cartan matrix A and a Dynkin diagram Γ, the following are equivalent:

(1) *(V, R) is irreducible,*
(2) *A is indecomposable,*
(3) *Γ is connected.*

13. Week 13: Classification of connected reductive groups

References

[Bor91] A. Borel, *Linear algebraic groups*, second ed., Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Har77] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, vol. No. 52, Springer-Verlag, New York-Heidelberg, 1977.

[Hum78] J. E. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York-Berlin, 1978, Second printing, revised.

[Liu02] Q. Liu, *Algebraic geometry and arithmetic curves*, Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, 2002, Translated from the French by Reinie Erné, Oxford Science Publications.

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, The theory of group schemes of finite type over a field.

[Spr09] T. A. Springer, *Linear algebraic groups*, second ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009.

Department of Mathematics, National Taiwan University, Astronomy Mathematics Building 5F, No. 1, Sec. 4, Roosevelt Rd., Taipei 10617, Taiwan.

Email address: `masaooi@ntu.edu.tw`