1. Week 7: Deligne–Lusztig character formula

Let G be a connected reductive group over $k = \mathbb{F}_q$ and F its associated Frobenius endomorphism. We fix a k-rational maximal torus T of G and a Borel subgroup B of G containing T. We also fix a character $\theta\colon T^F \to \mathbb{C}^\times$. Then we have the Deligne–Lusztig virtual representation $R^G_{T\subset B}(\theta)$ of G^F. By abuse of notation, we also write $R^G_{T\subset B}(\theta)$ for the Deligne–Lusztig virtual character, which is a class function $G^F \to \mathbb{C}$ defined to be the trace of the Deligne–Lusztig virtual representation. Today's aim is to establish a character formula for $R^G_{T\subset B}(\theta)$.

1.1. **Deligne–Lusztig character formula.**

We write G^F_{ss} and G^F_{unip} for the set of semisimple (equivalently, prime-to-p order) and unipotent elements of G^F (equivalently, p-power order), respectively. In the following, for any $g \in G$ and $h \in G$, we write ${}^g h = ghg^{-1}$. Similarly, for any $g \in G$ and a subgroup $H \subset G$, we write ${}^g H = gHg^{-1}$.

Definition 1.1. We define a function $Q^G_T\colon G^F_{\mathrm{unip}} \to \mathbb{C}$ by $Q^G_T := R^G_{T\subset B}(\mathbb{1})|_{G^F_{\mathrm{unip}}}$. We call Q^G_T the *Green function (of G associated to T).*

We note that, for notational convenience, we simply write "Q^G_T" although a priori Q^G_T depends on the choice of a Borel subgroup B containing T. (But, in fact, later it will turn out that Q^G_T does not depend on B!)

To state the Deligne–Lusztig character formula, let us recall that any element $g \in G^F$ has the Jordan decomposition $g = su$, where $s \in G^F$ is a semisimple element and $u \in G^F$ is a unipotent element such that $su = us$.

Theorem 1.2 (Deligne–Lusztig character formula)**.** *Let $g \in G^F$ with Jordan decomposition $g = su$. We shortly write G_s for the centralizer of s in G, i.e., $G_s = Z_G(s) = \{x \in G \mid xs = sx\}$. Then we have*

$$R^G_T(\theta)(g) = \frac{1}{|(G^\circ_s)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G^\circ_s}_{{}^x T}(u).$$

Let us explain why the right-hand side of this formula makes sense. We first note the following result (see [Car85, 1.14]).

Lemma 1.3. *(1) For any $s \in G^F_{\mathrm{ss}}$, the identity component G°_s of its centralizer G_s is a connected reductive group defined over k.*
(2) Any unipotent element of G_s lies in G°_s. In particular, when $g \in G^F$ has the Jordan decomposition $g = su$, its unipotent part u belongs to $(G^\circ_s)^F$.

Let us look at the index set of the sum in the Deligne–Lusztig character formula. When $x^{-1}sx \in T$, we necessarily have the opposite inclusion $Z_G(x^{-1}sx) \supset Z_G(T)$. Here, it is easy to check that $Z_G(x^{-1}sx) = x^{-1}Z_G(s)x$. On the other hand, it is known that the centralizer of a maximal torus in a connected reductive group is the maximal torus itself, i.e., $Z_G(T) = T$ (see [Spr09, 7.6.4]). Hence, we have $x^{-1}Z_G(s)x \supset T$, or equivalently, ${}^x T = xTx^{-1} \subset Z_G(s) = G_s$. Since T is connected, this furthermore implies that ${}^x T \subset G^\circ_s$. Furthermore, it is known that $(B \cap G^\circ_s)^\circ$ is a Borel subgroup of G°_s and $U \cap G^\circ_s$ is its unipotent radical.[1]

[1] Here, note that $U \cap G^\circ_s$ is already connected!

In summary, when $x^{-1}sx \in T$, we obtain a k-rational maximal torus ${}^xT$ of a connected reductive group G_s°. Thus it makes sense to consider the Green function $Q_{{}^xT}^{G_s^\circ}$ of G_s° associated to ${}^xT$ and $(B \cap G_s^\circ)^\circ$. Since u belongs to $(G_s^\circ)^F_{\mathrm{unip}}$, it also makes sense to look at the value of $Q_{{}^xT}^{G_s^\circ}$ at u.

Thus the Deligne–Lusztig character formula reflects an inductive nature of the theory of reductive groups. The contribution of the semisimple part s is given just by θ, which is very simple. On the other hand, the contribution of the unipotent part u is given by the Green function, which is independent of θ and taken in G_s°. Hence, ultimately, the Deligne–Lusztig characters of G are governed by the Green functions for G and all its "smaller" reductive subgroups.

1.2. **Outline of the proof of DL character formula.** The key of the proof of the Deligne–Lusztig character formula is the following general result, which is called *Deligne–Lusztig's fixed point formula*:

Theorem 1.4 (Deligne–Lusztig fixed point formula)**.** *Let X be an algebraic variety over k and g is an automorphism of X of finite order. Let s and u be automorphisms of X such that s is of prime-to-p order, u is of p-power order, and $g = su = us$. Then we have $\mathcal{L}(g, X) = \mathcal{L}(u, X^s)$, where $X^s := \{x \in X \mid s(x) = x\}$.*

Now suppose that $g \in G^F$ has the Jordan decomposition $g = su = us$. As disxussed in the last week, we have

$$R^G_{T\subset B}(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g,t), \mathcal{X}^G_{T\subset B}).$$

Let us compute each $\mathcal{L}((g,t), \mathcal{X}^G_{T\subset B})$ using the Deligne–Lusztig fixed point formula.

Recall that the action of (g,t) on $\mathcal{X}^G_{T\subset B} = \{x \in G \mid x^{-1}F(x) \in F(U)\}$ is given by $x \mapsto gxt$. We note that the order of T^F is prime-to-p. (Indeed, if we suppose that T splits over $\mathbb{F}_{q^n}$, i.e., $T_{\mathbb{F}_{q^n}} = \mathbb{G}_{\mathrm{m}}^r$ for some r, we have $T^F = T(\mathbb{F}_q) \subset T_{\mathbb{F}_{q^n}}(\mathbb{F}_{q^n}) \cong (\mathbb{F}_{q^n}^\times)^r$.) Hence the order of t is also prime-to-p. Thus, the decomposition $(g,t) = (s,t) \circ (u,1)$ satisfies the assumption of the Deligne–Lusztig fixed point formula.

We determine $(\mathcal{X}^G_{T\subset B})^{(s,t)}$. In the following, we simply write $\mathcal{X} := \mathcal{X}^G_{T\subset B}$.

Proposition 1.5. *We have*

$$\mathcal{X}^{(s,t)} = \bigsqcup_{\substack{x \in G^F/(G_t^\circ)^F \\ {}^xt = s^{-1}}} \mathcal{X}^{(s,t)}(x),$$

where we put $\mathcal{X}^{(s,t)}(x) := \mathcal{X}^{(s,t)} \cap xG_t^\circ$.

Proof. Suppose that $y \in \mathcal{X}^{(s,t)}$, i.e., $y \in G$ is an element satisfying $syt = y$ and $y^{-1}F(y) \in F(U)$ (say $y^{-1}F(y) = v$). By applying F to $syt = y$, we get $sF(y)t = F(y)$, thus $syvt = yv$. Combining $syvt = yv$ with $syt = y$, we get $yt^{-1}vt = yv$, hence $t^{-1}vt = v$. This means that u belongs to $G_t = Z_G(t)$. As u is unipotent, u furthermore lies in G_t°. Let us apply Lang's theorem to G_t°, which asserts that the map

$$G_t^\circ \to G_t^\circ \colon z \mapsto z^{-1}F(z)$$

is surjective; we can find an element $z \in G_t^\circ$ satisfying $z^{-1}F(z) = v$.

We put $x := yz^{-1}$. Then $F(x) = F(y)F(z)^{-1} = yvv^{-1}z^{-1} = yz^{-1} = x$, i.e., $x \in G^F$. Note that we have $y \in xG_t^\circ$. Furthermore, we have

$$xz = y = syt = s(xz)t = sxtz$$

(use that $z \in G_t$ in the last equality), hence ${}^x t = s^{-1}$.

From the disxussion so far, we have obtained

$$\mathcal{X}^{(s,t)} = \bigcup_{\substack{x \in G^F/(G_t^\circ)^F \\ {}^x t = s^{-1}}} \mathcal{X}^{(s,t)}(x).$$

It is obvious that the union is disjoint. $\square$

Let us investigate each summand $\mathcal{X}^{(s,t)}(x)$. Note that, since $t \in T^F$, we have $T \subset G_t^\circ$. Moreover, $B_t^\circ := (B \cap G_t^\circ)^\circ$ is a Borel subgroup of G_t° with unipotent radical $U \cap G_t^\circ$ (see the paragraph after Lemma 1.3). Therefore, it makes sense to consider the Deligne–Lusztig variety for G_t° associated to $T \subset B_t^\circ$:

$$\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} = \{y' \in G_t^\circ \mid y'^{-1}F(y') \in U \cap G_t^\circ\}.$$

This is a variety equipped with an action of $(G_t^\circ)^F \times T^F$. On the other hand, $\mathcal{X}^{(s,t)}(x)$ is stable under the action of the subgroup $(G_s^\circ)^F \times T^F$ of $G^F \times T^F$ on $\mathcal{X}$.

Proposition 1.6. *Let $x \in G^F$ be an element satisfying ${}^x t = s^{-1}$. Then have an isomorphism of varieties*

$$\varphi_x \colon \mathcal{X}^{(s,t)}(x) \xrightarrow{\cong} \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} : y \mapsto x^{-1}y,$$

which is equivariant with respect to the actions of $(G_s^\circ)^F \times T^F$ on $\mathcal{X}^{(s,t)}(x)$ and $(G_t^\circ)^F \times T^F$ on $\mathcal{X}_{T \subset (B \cap G_t^\circ)^\circ}^{G_t^\circ}$. Here, $(G_s^\circ)^F \times T^F$ and $(G_t^\circ)^F \times T^F$ are identified by $(z, t') \mapsto (x^{-1}zx, t')$.

Proof. Suppose that $y \in \mathcal{X}^{(s,t)}(x)$, i.e., $y \in xG_t^\circ$ is an element satisfying $syt = y$ and $y^{-1}F(y) \in F(U)$. Then we have $x^{-1}y \in G_t^\circ$ and thus

$$(x^{-1}y)^{-1}F(x^{-1}y) = y^{-1}F(y) \in F(U) \cap G_t^\circ = F(U \cap G_t^\circ).$$

In other words, $\varphi_x(y) = x^{-1}y$ belongs to $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$. Conversely, for any element $y' \in \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$, we can check that $\varphi_x^{-1}(y') = xy' \in \mathcal{X}^{(s,t)}(x)$.

Let us check the assertion on the equivariance. What we have to prove is that, for any $(z, t') \in (G_s^\circ)^F \times T^F$ and $y \in \mathcal{X}^{(s,t)}(x)$, we have

$$\varphi_x((z, t') \cdot y) = (x^{-1}zx, t') \cdot \varphi_x(y).$$

The left-hand side is given by $\varphi_x((z, t') \cdot y) = \varphi_x(zyt') = x^{-1}zyt'$. The right-hand side is given by $(x^{-1}zx, t') \cdot \varphi_x(y) = (x^{-1}zx, t') \cdot (x^{-1}y) = x^{-1}zx(x^{-1}y)t' = x^{-1}zyt'$. So these indeed coincide. $\square$

Now let us start the proof of the Deligne–Lusztig character formula:

Proof of Theorem 1.2. We have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

By applying the Deligne–Lusztig fixed point theorem to $(g,t)=(s,t)\circ(u,1)$, we get

$$\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B})=\mathcal{L}((u,1),(\mathcal{X}^G_{T\subset B})^{(s,t)}).$$

By combining the above propositions, we get

$$\mathcal{L}(u,(\mathcal{X}^G_{T\subset B})^{(s,t)})=\sum_{\substack{x\in G^F/(G_t^\circ)^F\\ {}^x t=s^{-1}}}\mathcal{L}(x^{-1}ux,\mathcal{X}^{G_t^\circ}_{T\subset B_t^\circ}).$$

Hence we get

$$\begin{aligned}R^G_{T\subset B}(\theta)(g)&=\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\sum_{\substack{x\in G^F/(G_t^\circ)^F\\ {}^x t=s^{-1}}}\mathcal{L}(x^{-1}ux,\mathcal{X}^{G_t^\circ}_{T\subset B_t^\circ})\\ &=\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\cdot\frac{1}{|(G_t^\circ)^F|}\sum_{\substack{x\in G^F\\ {}^x t=s^{-1}}}\mathcal{L}(x^{-1}ux,\mathcal{X}^{G_t^\circ}_{T\subset B_t^\circ}).\end{aligned}$$

Note that the internal sum is nonzero only when there exists an element $x\in G^F$ satisfying $t=x^{-1}s^{-1}x$. In this case, $|(G_t^\circ)^F|=|(G_s^\circ)^F|$, hence the above equals

$$\frac{1}{|T^F|\cdot|(G_s^\circ)^F|}\sum_{t\in T^F}\sum_{\substack{x\in G^F\\ {}^x t=s^{-1}}}\theta(t)^{-1}\cdot\mathcal{L}(x^{-1}ux,\mathcal{X}^{G_t^\circ}_{T\subset B_t^\circ}).$$

We note that the set $\{(t,x)\in T^F\times G^F\mid {}^x t=s^{-1}\}$ is bijective to $\{x\in G^F\mid x^{-1}sx\in T^F\}$ by $(t,x)\mapsto x$. By also noting that $\mathcal{L}(x^{-1}ux,\mathcal{X}^{G_t^\circ}_{T\subset B_t^\circ})=\mathcal{L}(u,\mathcal{X}^{G_s^\circ}_{{}^x T\subset B_s^\circ})$, we rewrite the above double sum:

$$\begin{aligned}&\frac{1}{|T^F|\cdot|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}s^{-1}x)^{-1}\cdot\mathcal{L}(u,\mathcal{X}^{G_s^\circ}_{{}^x T\subset B_s^\circ})\\ &=\frac{1}{|T^F|\cdot|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\cdot\mathcal{L}(u,\mathcal{X}^{G_s^\circ}_{{}^x T\subset B_s^\circ}).\end{aligned}$$

Here, in general, we have

$$Q^G_T(u)=\frac{1}{|T^F|}\cdot\mathcal{L}(u,\mathcal{X}^G_{T\subset B}).$$

Indeed, by definition, we have

$$Q^G_T(u)=\frac{1}{|T^F|}\sum_{t\in T^F}\mathcal{L}((u,t),\mathcal{X}^G_{T\subset B}).$$

By using the Deligne–Lusztig fixed point formula to $(u,t)=(u,1)\cdot(1,t)$, we have $\mathcal{L}((u,t),\mathcal{X}^G_{T\subset B})=\mathcal{L}((u,1),(\mathcal{X}^G_{T\subset B})^{(1,t)})$. However, $(\mathcal{X}^G_{T\subset B})^{(1,t)}$ is nonempty only when $t=1$ (indeed, $x\in\mathcal{X}^G_{T\subset B}$ is fixed by $(1,t)$ if and only if $xt=x$). Thus we get

$$Q^G_T(u)=\frac{1}{|T^F|}\mathcal{L}(u,\mathcal{X}^G_{T\subset B}).$$

Therefore, we finally obtain

$$R^G_{T\subset B}(g) = \frac{1}{|T^F|\cdot|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\cdot Q^{G_s^\circ}_{{}^xT}(u).$$

□

Corollary 1.7. *We have $R^G_{T\subset B}(\theta)|_{G^F_{\mathrm{unip}}} = Q^G_T$ for any character $\theta\colon T^F\to\mathbb{C}^\times$.*

Proof. Let $g\in G^F_{\mathrm{unip}}$ (hence its semisimple part s is 1 and unipotent part u is g). Then, by applying the Deligne–Lusztig character formula to g, we get

$$\begin{aligned} R^G_{T\subset B}(\theta)(g) &= \frac{1}{|(G_s^\circ)^F|}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}}\theta(x^{-1}sx)\cdot Q^{G_s^\circ}_{{}^xT}(u)\\ &= \frac{1}{|G^F|}\sum_{x\in G^F} Q^G_{{}^xT}(u). \end{aligned}$$

It is not difficult to check that, in general, we have $R^G_{T\subset B}(\theta)(g) = R^G_{{}^xT\subset{}^xB}({}^x\theta)({}^xg)$, where ${}^x\theta$ denotes the character of ${}^xT^F$ defined by ${}^x\theta({}^xt)=\theta(t)$. In particular, when $\theta=\mathbb{1}$, hence get $Q^G_T(u) = Q^G_{{}^xT}({}^xu)$. By also noting that the Green function is invariant under conjugation (since it is the restriction of a Deligne–Lusztig character, which is a class function), we get $Q^G_T(u) = Q^G_{{}^xT}({}^xu) = Q^G_{{}^xT}(u)$. Hence the most right-hand side of the above equalities is $Q^G_T(u)$. □

Definition 1.8. We say that a semisimple element $s\in G$ is *regular* if G_s° is a maximal torus of G.

Example 1.9. Let $G=\mathrm{GL}_2$. Let T be the diagonal maximal torus of G. We consider an element $s=\left(\begin{smallmatrix} a & 0\\ 0 & b\end{smallmatrix}\right)\in T$. Then, since s is already diagonalized, s is semisimple. Let us compute the centralizer $G_s = Z_G(s)$ of s in G.

- When $a=b$, s commutes with any element of G. Thus $G_s=G$, hence $G_s^\circ = G^\circ = G$. Hence s is not regular in this case.
- Suppose that $a\neq b$. If $g=\left(\begin{smallmatrix} x & y\\ z & w\end{smallmatrix}\right)\in Z_G(s)$, we have $sgs^{-1}=g$. Since

$$sgs^{-1} = \begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}\begin{pmatrix} x & y\\ z & w\end{pmatrix}\begin{pmatrix} a & 0\\ 0 & b\end{pmatrix}^{-1} = \begin{pmatrix} x & ayb^{-1}\\ a^{-1}zb & w\end{pmatrix},$$

  we necessarily have $y=z=0$, i.e., $g\in T$. Conversely, we obviously have $T\subset Z_G(s)$. Hence we get $G_s=T$, so $G_s^\circ = T$, which means that s is regular.

Exercise 1.10. Let $G=\mathrm{GL}_n$ and $g\in G$. Prove that g is regular semisimple if and only if the characteristic polynomial of g has n distinct roots. (Hint: compute the centralizer of g in G by looking at the Jordan normal form of g.)

Exercise 1.11. Let $G=\mathrm{GL}_n$ and T be the diagonal maximal torus of G. Count the number of regular semisimple elements in $T^F = T(\mathbb{F}_q)$.

Corollary 1.12. *Suppose that $s\in G^F$ is a regular semisimple element. If s is not conjugate to any element of T^F, then we have $R^G_{T\subset B}(\theta)(s)=0$. If s is conjugate to any element of T^F (suppose that s itself belongs to T^F), then we have*

$$R^G_{T\subset B}(\theta)(s) = \sum_{x\in W_{G^F}(T)}\theta(x^{-1}sx),$$

where $W_{G^F}(T) := N_{G^F}(T)/T^F$.

Proof. By the Deligne–Lusztig character formula, we have

$$R^G_{T\subset B}(\theta)(s) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G_s^\circ}_{{}^xT}(1).$$

Since the index set is empty if s is not conjugate to any element of T^F, we get the first assertion.

To show the second assertion, let us suppose that $s \in T^F$. Then, we must have $G_s^\circ = Z_G(s)^\circ \supset Z_G(T) = T$. Since G_s° is a maximal torus of G, this implies that $G_s^\circ = T$. By the same argument, the condition $x^{-1}sx \in T^F$ of the index set implies that $T = x^{-1}Tx$. In other words, $x \in N_{G^F}(T)$. Conversely, any element $x \in N_{G^F}(T)$ satisfies $x^{-1}sx \in T^F$. Thus, by noting that $Q^T_T(1) = 1$ (this can be checked by going back to the definition), we get

$$R^G_{T\subset B}(\theta)(s) = \frac{1}{|T^F|} \sum_{x\in N_{G^F}(T)} \theta(x^{-1}sx) = \sum_{x\in W_{G^F}(T)} \theta(x^{-1}sx).$$

□

References

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