1. Week 8: Inner product formula for Deligne–Lusztig representations

1.1. **Inner product formula for Deligne–Lusztig representations.** Let G be a connected reductive group over $k = \mathbb{F}_q$. Recall that the $\mathbb{C}$-vector space $C(G^F)$ of class functions on G^F has an inner product $\langle -, - \rangle$ given by

$$\langle f_1, f_2 \rangle := \frac{1}{|G^F|} \sum_{g \in G^F} f_1(g) \cdot \overline{f_2(g)}.$$

Our next aim is to compute the inner product of two Deligne–Lusztig representations. To state the theorem, we introduce some notations. For k-rational maximal tori T and T' of G, we put

$$N_{G^F}(T, T') := \{ n \in G^F \mid {}^nT = T' \},$$
$$W_{G^F}(T, T') := N_{G^F}(T, T')/T^F \cong T'^F \backslash N_{G^F}(T, T').$$

(Recall that, in our notation, ${}^nT$ denotes nTn^{-1}.) Note that, for any $w \in W_{G^F}(T, T')$ and a character $\theta \colon T^F \to \mathbb{C}^\times$, we can define a character ${}^w\theta$ of T'^F by

$${}^w\theta(t') := \theta(w^{-1}t'w).$$

(This definition is independent of the choice of a representative of w.)

Theorem 1.1 (Inner product formula)**.** *Let T and T' be k-rational maximal tori of G. Let $B = TU$ and $B' = T'U'$ be Borel subgroups of G containing T and T', respectively. For any characters $\theta \colon T^F \to \mathbb{C}^\times$ and $\theta' \colon T'^F \to \mathbb{C}^\times$, we have*

$$\langle R^G_{T \subset B}(\theta), R^G_{T' \subset B'}(\theta') \rangle = |\{ w \in W_{G^F}(T, T') \mid {}^w\theta = \theta' \}|.$$

Before we prove this theorem, we explain several important consequences.

Corollary 1.2. *The Deligne–Lusztig representation $R^G_{T \subset B}(\theta)$ is independent of the choice of a Bore subgroup $B \subset T$. The Green function Q^G_T is also independent of $B \subset T$.*

Proof. Recall that $Q^G_T := R^G_{T \subset B}(\mathbb{1})|_{G^F_{\mathrm{unip}}}$. Thus it is enough to show the first assertion.

Let us take any Borel subgroup B and B' containing T. Our task is to show that $R^G_{T \subset B}(\theta) = R^G_{T \subset B'}(\theta)$ (here, both are regarded as class functions on G^F). Equivalently, it suffices to show that

$$\langle R^G_{T \subset B}(\theta) - R^G_{T \subset B'}(\theta), R^G_{T \subset B}(\theta) - R^G_{T \subset B'}(\theta) \rangle = 0.$$

The left-hand side equals

$$\langle R^G_{T \subset B}(\theta), R^G_{T \subset B}(\theta) \rangle - 2\langle R^G_{T \subset B}(\theta), R^G_{T \subset B'}(\theta) \rangle + \langle R^G_{T \subset B'}(\theta), R^G_{T \subset B'}(\theta) \rangle.$$

This equals 0 since we have

$$\langle R^G_{T \subset B}(\theta), R^G_{T \subset B}(\theta) \rangle = \langle R^G_{T \subset B}(\theta), R^G_{T \subset B'}(\theta) \rangle = \langle R^G_{T \subset B'}(\theta), R^G_{T \subset B'}(\theta) \rangle$$

by the inner product formula. □

From now on, let us simply write $R^G_T(\theta)$ instead of $R^G_{T \subset B}(\theta)$. (But, in the proof of the inner product formula, we will again write $R^G_{T \subset B}(\theta)$.)

Corollary 1.3. *Suppose that T and T' are k-rational maximal tori of G which are not G^F-conjugate. Then, for any characters θ of T and θ' of T', we have*

$$\langle R^G_T(\theta), R^G_{T'}(\theta') \rangle = 0.$$

Proof. This is clear from the inner product formula; if T and T' are not G^F-conjugate, then $N_{G^F}(T,T')$ is empty. $\square$

Remark 1.4. Note that even if $\langle R_T^G(\theta), R_{T'}^G(\theta')\rangle = 0$, it might happen that $R_T^G(\theta)$ and $R_{T'}^G(\theta')$ have a common irreducible constituent. For example, the inner product of virtual representations $\pi_1+\pi_2$ and $\pi_1-\pi_2$ is zero, when π_1 and π_2 are irreducible.

Corollary 1.5. *If we write*

$$R_T^G(\theta) = \sum_\rho n_\rho \rho,$$

where ρ runs all isomorphism classes of irreducible representations of G^F, we have

$$\sum_\rho n_\rho^2 = |\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\}|.$$

In particular, $R_T^G(\theta)$ is irreducible up to sign if and only if we have $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$.

Proof. This follows from the inner product formula (choose (T', θ') to be (T,θ)) and the general fact that, for irreducible representations ρ_1 and ρ_2 of G^F, we have

$$\langle \Theta_{\rho_1}, \Theta_{\rho_2}\rangle = \begin{cases} 1 & \text{if } \rho_1 \cong \rho_2, \\ 0 & \text{if } \rho_1 \ncong \rho_2. \end{cases}$$

$\square$

Definition 1.6. We say that a character $\theta\colon T^F \to \mathbb{C}^\times$ is *regular (in general position)* if $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$. (Note that, by the above corollary, this is equivalent to that $R_T^G(\theta)$ is irreducible up to sign.)

1.2. **Weyl groups of k-rational maximal tori.** The inner product formula suggests that it is practically very important to determine the set $W_{G^F}(T,T')$ and its "action" on T[1]. Suppose that $N_{G^F}(T,T')$ is not empty. If we fix any element $n_0 \in N_{G^F}(T,T')$, then we get a bijection

$$N_{G^F}(T) \xrightarrow{\cong} N_{G^F}(T,T')\colon n \mapsto n_0 n.$$

Similarly, if we fix any element $w_0 \in W_{G^F}(T,T')$ (as long as this set is not empty), then we get a bijection

$$W_{G^F}(T) \xrightarrow{\cong} W_{G^F}(T,T')\colon w \mapsto w_0 w.$$

Therefore, it is essentially enough to investigate the action of $W_{G^F}(T)$ on T.

Recall that $W_{G^F}(T) := N_{G^F}(T)/T^F$. We also introduce $W_G(T)^F := (N_G(T)/T)^F$. Note the following lemma:

Lemma 1.7. *We have $W_{G^F}(T) \cong W_G(T)^F$.*

Proof. Let $N_{G^F}(T) \hookrightarrow N_G(T)$ be the natural inclusion, which induces an inclusion $N_{G^F}(T)/T^F \hookrightarrow N_G(T)/T$. The image of this inclusion is obviously fixed by F, thus we get a natural inclusion

$$W_{G^F}(T) = N_{G^F}(T)/T^F \hookrightarrow (N_G(T)/T)^F = W_G(T)^F.$$

To show the surjectivity, let us take an element $w \in W_G(T)^F$ and its representative $n \in N_G(T)$. Since w is fixed by F, there exists an element $t \in T$

[1] Since $W_{G^F}(T,T')$ is not a group, it is better to say "how $W_{G^F}(T,T')$ transports T to T'"

satisfying $F(n) = nt$. We apply Lang's theorem to $t \in T$; then we can find an element $s \in T$ satisfying $s^{-1}F(s) = t$. We let $n' := ns^{-1}$. As we have $F(n') = F(n)F(s)^{-1} = F(n)t^{-1}s^{-1} = ns^{-1} = n'$, we have $n' \in N_{G^F}(T)$. Moreover, obviously n' and n maps to w. This completes the proof. $\square$

Based on this lemma, let us consider $W_G(T)^F$ instead of $W_{G^F}(T)$. We review how the (G^F-conjugacy classes of) k-rational maximal tori of G are classified. Let B_0 be a k-rational Borel subgroup G and T_0 be a k-rational maximal torus of G contained in B_0. We write W_0 for the Weyl group $W_0 := W_G(T_0) := N_G(T_0)/T_0$.[2] Note that this is a finite group on which F (the Frobenius endomorphism of G) acts. In Week 5, we (Cheng-Chiang) discussed that there exists a bijection

$$\{k\text{-rational maximal tori of } G\}/G^F\text{-conj.} \to W_0/F\text{-conj.}$$

Let $w \in W_0$. Let us recall how to produce a k-rational maximal torus T_w corresponding to w. We take a representative $n \in N_G(T_0)$ of w and apply the Lang's theorem to n; we can find $g \in G$ satisfying $g^{-1}F(g) = n$. If we put $T_w := {}^gT_0 = gT_0g^{-1}$, then T gives a k-rational maximal torus of G corresponding to (the F-conjugacy class of) w under the above bijection. The action of F on T_w is described as follows:

$$\begin{tikzcd} T_w \arrow[d, "F"'] & T_0 \arrow[l, "\mathrm{Int}(g)"'] \\ T_w \arrow[r, "\mathrm{Int}(g)^{-1}"'] & T_0 \end{tikzcd} \qquad \begin{tikzcd} gtg^{-1} \arrow[d, mapsto] & & t \arrow[ll, mapsto] \\ F(g)F(t)F(g)^{-1} \arrow[r, mapsto] & g^{-1}F(g)F(t)F(g)^{-1}g = \mathrm{Int}(w) \circ F(t) \end{tikzcd}$$

Hence, in particular, we have an isomorphism

$$\mathrm{Int}(g)\colon T_0^{\mathrm{Int}(w)\circ F} \xrightarrow{\cong} T_w^F; \quad t \mapsto gtg^{-1}.$$

Note that $\mathrm{Int}(g)$ also gives an identification $W_0 = W_G(T_0) \xrightarrow{\cong} W_G(T_w)\colon w \mapsto gwg^{-1}$, which induces

$$\mathrm{Int}(g)\colon W_0^{\mathrm{Int}(w)\circ F} \xrightarrow{\cong} W_G(T_w)^F; \quad w \mapsto gwg^{-1}.$$

Example 1.8. Let $G = \mathrm{GL}_n$ and T_0 be the diagonal maximal torus of G. Then W_0 is naturally identified with $\mathfrak{S}_n$, which is realized as the subgroup of permutation matrices in $\mathrm{GL}_n(\mathbb{F}_q)$. In this case, the Frobenius action F on W_0 is trivial.

(1) When $w = 1$, we have

$$T_0^{\mathrm{Int}(w)\circ F} = T_0^F = \{\mathrm{diag}(t_1, \ldots, t_n) \mid t_i \in \mathbb{F}_q^\times\}.$$

The action of $W_0^{\mathrm{Int}(w)\circ F} = W_0 = \mathfrak{S}_n$ on this group is given by the natural permutation action.

(2) When w is the cyclic permutation $(1\,2\,\ldots\,n)$, we have

$$T_0^{\mathrm{Int}(w)\circ F} = \{\mathrm{diag}(t_1, t_1^q \ldots, t_1^{q^{n-1}}) \mid t_1 \in \mathbb{F}_{q^n}^\times\}$$

(see Week 5 notes for details). Note that $W_0^{\mathrm{Int}(w)\circ F} = W_0^{\mathrm{Int}(w)}$ is nothing but the centralizer of $w = (1\,2\,\ldots\,n)$ in $\mathfrak{S}_n$. We can check that it is the subgroup $\langle w\rangle$ generated by w. Since $w(t_1, t_1^q \ldots, t_1^{q^{n-1}}) = (t_1^q \ldots, t_1^{q^{n-1}}, t_1) =$

[2] Caution: this is the "absolute" Weyl group taken in G, while we consider the "relative" Weyl group taken in G^F in the inner product formula.

$(t_1^q \dots, t_1^{q^{n-1}}, t_1^{q^n})$, the action of $\langle w \rangle$ on $T_0^{\mathrm{Int}(w)\circ F}$ is identified with the action of $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ on $\mathbb{F}_{q^n}^\times$.

1.3. **Example: the case of** GL_2**.** Let $G = \mathrm{GL}_2$. Recall that we exactly have two non-isomorphic k-rational maximal tori of G (up to G^F-conjugacy): the split one T and the non-split one S.

(1) For the split one T, we have $T^F = T(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^2$ and $W_{G^F}(T) \cong \mathfrak{S}_2$; $\mathfrak{S}_2$ acts on $(\mathbb{F}_q^\times)^2$ by swapping two entries. Therefore, for any character $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ of $(\mathbb{F}_q^\times)^2$, we have that
   - $R_T^G(\boldsymbol{\chi})$ is irreducible (up to sign) if $\chi_1 \neq \chi_2$ ($\boldsymbol{\chi}$ is regular), and
   - $R_T^G(\boldsymbol{\chi})$ consists of two irreducible representations (up to sign) if $\chi_1 = \chi_2$.

(2) For the non-split one S, we have $S^F = S(\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times$ and $W_{G^F}(S) = \mathbb{Z}/2\mathbb{Z}$; $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{F}_{q^2}^\times$ via $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. Therefore, for any character θ of $\mathbb{F}_{q^2}^\times$, we have that
   - $R_S^G(\theta)$ is irreducible (up to sign) if $\theta^q \neq \theta$ (θ is regular), and
   - $R_S^G(\theta)$ consists of two irreducible representations (up to sign) if $\theta^q = \theta$.

Recall that, in Week 6, we proved that $R_{T\subset B}^G(\boldsymbol{\chi}) \cong \mathrm{Ind}_B^G(\boldsymbol{\chi})$. Also recall that, in Week 2, we proved that $\mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \boldsymbol{\chi}$ is irreducible when $\chi_1 \neq \chi_2$ and consists of two irreducible representations when $\chi_1 = \chi_2$. Therefore, the computation in the bove example is perfectly consistent with those!

Exercise 1.9. For any θ of S^F satisfying $\theta^{q-1} \neq \mathbb{1}$, we have $R_S^G(\theta) \cong -\pi_\theta$.

Hint: Recall that the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are classified as follows (see Week 2 notes):

(1) Characters of $\mathrm{GL}_2(\mathbb{F}_q)$; $\chi \circ \det$ for a character $\chi \colon \mathbb{F}_q^\times$.
(2) Character twists of the Steinberg representation; $\mathrm{St}_G \otimes (\chi \circ \det)$ for a character $\chi \colon \mathbb{F}_q^\times$.
(3) Irreducible principal series representations; $\mathrm{Ind}_B^G \boldsymbol{\chi}$ for $\boldsymbol{\chi} = \chi_1 \boxtimes \chi_2$ where $\chi_1 \neq \chi_2$.
(4) Irreducible cuspidal representations; $\pi_{\theta'}$ for a character θ' of $\mathbb{F}_{q^2}^\times$ satisfying $\theta'^q \neq \theta'$.

Exclude the first three possibilities by using the inner product formula for $R_T^G(\boldsymbol{\chi})$ and $R_T^G(\theta)$, which implies that necessarily have $R_S^G(\theta) \cong \pm\pi_{\theta'}$ for some θ'. Then compute the characters of $R_S^G(\theta)$ at regular semisimple elements using the Deligne–Lusztig character formula. Compare it with the character computation on $\pi_{\theta'}$ demonstrated in Week 2.

1.4. **Proof of inner product formula for DL representations.** We first prove the inner product formula for Deligne–Lusztig representations by admitting the following:

Theorem 1.10 (Orthogonality relation for Green functions)**.** *Let T and T' be k-rational maximal tori of G. Let B and B' be Borel subgroup of G containing T and Q_T^G and $Q_{T'}^G$ associated Green functions. Then we have*

$$\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q_T^G(u) \cdot Q_{T'}^G(u) = \frac{|N_{G^F}(T, T')|}{|T^F| \cdot |T'^F|}.$$

Proof of Theorem 1.1. Recall that the Jordan decomposition implies that we have the following bijection:

$$\bigsqcup_{s\in G^F_{\mathrm{ss}}} (G^\circ_s)^F_{\mathrm{unip}} \xrightarrow{1:1} G^F : u \mapsto su.$$

By using the Deligne–Lusztig character formula, we have

$$\begin{aligned}
&\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle \\
&= \frac{1}{|G^F|}\sum_{g\in G^F} R^G_{T\subset B}(\theta)(g)\cdot \overline{R^G_{T'\subset B'}(\theta')(g)} \\
&= \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\sum_{u\in (G^\circ_s)^F_{\mathrm{unip}}} \frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)Q^{G^\circ_s}_{{}^xT}(u)\sum_{\substack{y\in G^F\\ y^{-1}sy\in T'^F}} \overline{\theta'(y^{-1}sy)Q^{G^\circ_s}_{{}^yT'}(u)} \\
&= \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}} \frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}} \theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\sum_{u\in (G^\circ_s)^F_{\mathrm{unip}}} Q^{G^\circ_s}_{{}^xT}(u)\overline{Q^{G^\circ_s}_{{}^yT'}(u)}.
\end{aligned}$$

Here, note that the values of Green functions are integer (exercise). By applying the orthogonality relation for Green functions (for G°_s), this equals

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}} \frac{1}{|(G^\circ_s)^F|}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}} \theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\frac{|N_{(G^\circ_s)^F}({}^xT, {}^yT')|}{|{}^xT^F|\cdot|{}^yT'^F|}$$

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}} \frac{1}{|(G^\circ_s)^F|\cdot|T^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}} \theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\cdot|N_{(G^\circ_s)^F}({}^xT, {}^yT')|.$$

Here, we note that the following two sets are bijective by the map $(x,y,n)\mapsto (x, y^{-1}nx, n)$ and its inverse $(x, nxn'^{-1}, n) \leftarrow\!\shortmid (x,n',n)$:

$$\{(x,y,n)\in G^F\times G^F\times G^F \mid x^{-1}sx\in T^F, y^{-1}sy\in T'^F, n\in N_{(G^\circ_s)^F}({}^xT, {}^yT')\},$$

$$\{(x,n',n)\in G^F\times N_{G^F}(T,T')\times (G^\circ_s)^F \mid x^{-1}sx\in T^F\}.$$

Hence, the above sum equals

$$\frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}} \frac{1}{|(G^\circ_s)^F|\cdot|T^F|^2}\sum_{\substack{x\in G^F\\ n'\in N_{G^F}(T,T')\\ n\in (G^\circ_s)^F\\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\overline{\theta'((nxn'^{-1})^{-1}s(nxn'^{-1}))}.$$

As n commutes with s, we have

$$\theta'((nxn'^{-1})^{-1}s(nxn'^{-1})) = \theta'(n'x^{-1}sxn'^{-1}) = {}^{n'^{-1}}\theta'(x^{-1}sx).$$

In particular, this is independent of $n\in(G^\circ_s)^F$. Thus we get

$$\frac{1}{|G^F|\cdot|T^F|^2}\sum_{s\in G^F_{\mathrm{ss}}}\sum_{\substack{x\in G^F\\ n'\in N_{G^F}(T,T')\\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\overline{{}^{n'^{-1}}\theta'(x^{-1}sx)}.$$

We finally note that the following map is surjective

$$\{(s,x) \in G_{\mathrm{ss}}^F \times G^F \mid x^{-1}sx \in T^F\} \twoheadrightarrow T^F \colon (s,x) \mapsto x^{-1}sx.$$

Moreover, each fiber is of order $|G^F|$. Therefore, we get

$$\begin{aligned}
&\frac{1}{|G^F| \cdot |T^F|^2} \sum_{s \in G_{\mathrm{ss}}^F} \sum_{\substack{x \in G^F \\ n' \in N_{G^F}(T,T') \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \overline{{}^{n'^{-1}}\theta'(x^{-1}sx)} \\
&= \frac{1}{|T^F|^2} \sum_{t \in T^F} \sum_{n' \in N_{G^F}(T,T')} \theta(t) \overline{{}^{n'^{-1}}\theta'(t)} \\
&= \sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t) \overline{{}^{w^{-1}}\theta'(t)} \\
&= \sum_{w \in W_{G^F}(T,T')} \begin{cases} 1 & \text{if } \theta = {}^{w^{-1}}\theta', \\ 0 & \text{if } \theta \neq {}^{w^{-1}}\theta', \end{cases} \\
&= |\{w \in W_{G^F}(T,T') \mid {}^{w}\theta = \theta'\}|.
\end{aligned}$$

□

Exercise 1.11. For any connected reductive group G over k and its k-rational maximal torus T, prove that the Green function $Q_T^G(-)$ is $\mathbb{Z}$-valued.

Hint: Describe the Green function using a Lefschetz number by going back to the definition. Then utilize the fact that the Lefschetz number is an integer.

9:00am, October 29, 2024