1. Week 13: Example session

1.1. Algebraic characterization of regular Deligne–Lusztig representations. In this course, we have studied Deligne–Lusztig's construction of a virtual representation $R_T^G(\theta)$, which is critically based on very deep geometric discussions. The motivating problem we want to discuss here is the following:

Q1. Is there a purely-algebraic characterization of $R_T^G(\theta)$?

Let us recall the Deligne–Lusztig character formula:

Theorem 1.1 (Deligne–Lusztig character formula)**.** *Let $g \in G^F$ with Jordan decomposition $g = su$. Then we have*

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{{}^xT}^{G_s^\circ}(u).$$

Since any virtual representation is uniquely determined by its character, we can think of this formula as the characterization of the Deligne–Lusztig virtual representation $R_T^G(\theta)$. However, the right-hand side contains the Green functions $Q_{{}^xT}^{G_s^\circ}$. Remember that it is the restriction of the character of $R_{{}^xT}^{G_s^\circ}(\mathbb{1})$ to the set of unipotent elements; so its definition unavoidably depends on geometry.

But then, how about looking at the character values only on regular semisimple elements? Recall the following (an easy consequence of the Deligne–Lusztig character formula):

Corollary 1.2. *Suppose that $s \in G_{\mathrm{rs}}^F$ (the set of regular semisimple elements of G^F).*

1. *If s is not conjugate to any element of T^F, then we have $R_T^G(\theta)(s) = 0$.*
2. *If s is conjugate to an element of T^F (suppose that $s \in T^F$), then we have*

$$R_T^G(\theta)(s) = \sum_{w \in W_{G^F}(T)} {}^w\theta(s),$$

where $W_{G^F}(T) := N_{G^F}(T)/T^F$.

The right-hand side of this formula only consists of purely algebraic quantities! So we next come up with the following question:

Q2. Is the above character formula on G_{rs}^F enough to characterize $R_T^G(\theta)$?

In general, to determine a given representation from its character, we have to look at all its character values. However, sometimes (depending on a group and a representation), it is possible to determine a given representation by only looking at its character values on some special elements. For example, recall that $\mathfrak{S}_3$ has two 1-dimensional irreducible representations and only 2-dimensional representation. This means that, to distinguish the 2-dimensional irreducible representation from the others, it is only enough to look at their character values at 1! This example is maybe too stupid, but in any case we can hope that we could give an affirmative answer to Q2 in some cases.

Indeed, we can find the following "reasonable" answer[1]:

[1] This result is due to Charlotte Chan and I (joint work), which is based on a preceding work of Guy Henniart.

Theorem 1.3. *Let $\theta\colon T^F \to \mathbb{C}^\times$ be a regular character, i.e., $\{w \in W_{G^F}(T) \mid {}^w\theta = \theta\} = \{1\}$. Suppose that the following inequality holds:*

$$\frac{|T^F|}{|T^F \smallsetminus T^F_{\mathrm{rs}}|} > 2 \cdot |W_{G^F}(T)|.$$

Then $R^G_T(\theta)$ is the unique irreducible representation (up to sign) such that, for any $s \in G^F_{\mathrm{rs}}$,

$$(*) \qquad R^G_T(\theta) = \begin{cases} 0 & \text{if } s \text{ is not conjugate to elements of } T^F, \\ \sum_{w \in W_{G^F}(T)} {}^w\theta(s) & \text{if } s \in T^F. \end{cases}$$

Here, the subscript "rs" denotes the subset of regular semisimple elements.

Before we proceed, let us give some comments. First, the inequality in the assumption basically says that we have "many" regular semisimple elements. Thus the intuitive meaning of this theorem is that "if we have sufficiently many regular semisimple elements, the Deligne–Lusztig character formula on regular semisimple elements is enough to determine a regular Deligne–Lustig representation". Because this inequality is first considered in the work of Henniart for $G = \mathrm{GL}_n$, let us call it the *Henniart inequality*.

Second, recall that $|T^F|$ can be described by looking at the characteristic polynomial of a Weyl element which defines the k-rational maximal torus T. In fact, it is also possible to determine $|T^F_{\mathrm{rs}}|$ as long as G and the Weyl element are explicitly specified. Thus, in principle, we can explicate the Henniart inequality. In particular, we can show that the Henniart inequality always holds whenever q is sufficiently large; we will present some examples later.

Now let us prove the above theorem. In the following, we put $G_{\mathrm{nrs}} := G \smallsetminus G_{\mathrm{rs}}$ and $T_{\mathrm{nrs}} := T \smallsetminus T_{\mathrm{rs}}$. According to the disjoint union decomposition $G^F = G^F_{\mathrm{rs}} \sqcup G^F_{\mathrm{nrs}}$, we divide the inner product $\langle -, - \rangle$ on the space of class function on G^F as follows:

$$\langle f_1, f_2 \rangle_\bullet := \frac{1}{|G^F|} \sum_{g \in G^F_\bullet} f_1(g) \cdot \overline{f_2(g)},$$

where $\bullet \in \{\mathrm{rs}, \mathrm{nrs}\}$. Hence we have $\langle f_1, f_2 \rangle = \langle f_1, f_2 \rangle_{\mathrm{rs}} + \langle f_1, f_2 \rangle_{\mathrm{nrs}}$.

Proof. Suppose that ρ is another irreducible virtual representation of G^F satisfying the same character formula as $R^G_T(\theta)$ on G^F_{rs}. We put $R := R^G_T(\theta)$ Then our task is to show that $\langle \rho, R \rangle \neq 0$.

We have

$$\langle \rho, \rho \rangle = \langle \rho, \rho \rangle_{\mathrm{rs}} + \langle \rho, \rho \rangle_{\mathrm{nrs}}$$

and

$$\langle R, R \rangle = \langle R, R \rangle_{\mathrm{rs}} + \langle R, R \rangle_{\mathrm{nrs}}.$$

Since both ρ and R are irreducible (the latter is due to that θ is regular), we have $\langle \rho, \rho \rangle = \langle R, R \rangle = 1$. On the other hand, by the assumption on ρ, we also have $\langle \rho, \rho \rangle_{\mathrm{rs}} = \langle R, R \rangle_{\mathrm{rs}}$. Hence we get $\langle \rho, \rho \rangle_{\mathrm{nrs}} = \langle R, R \rangle_{\mathrm{nrs}}$. Let us put

$$X := \langle \rho, \rho \rangle_{\mathrm{rs}} = \langle R, R \rangle_{\mathrm{rs}}, \quad Y := \langle \rho, \rho \rangle_{\mathrm{nrs}} = \langle R, R \rangle_{\mathrm{nrs}}$$

(thus X and Y are non-negative numbers satisfying $X + Y = 1$).

We have

$$\langle \rho, R \rangle = \langle \rho, R \rangle_{\mathrm{rs}} + \langle \rho, R \rangle_{\mathrm{nrs}}.$$

Again by the assumption on ρ, we have $\langle \rho, R\rangle_{\mathrm{rs}} = X$. On the other hand, by the Cauchy–Schwarz inequality, we have

$$|\langle \rho, R\rangle_{\mathrm{nrs}}| \leq \langle \rho, \rho\rangle_{\mathrm{nrs}}^{\frac{1}{2}} \cdot \langle R, R\rangle_{\mathrm{nrs}}^{\frac{1}{2}} = Y.$$

Therefore, if we have $X > Y$, then $\langle \rho, R\rangle$ cannot be equal to 0. Since $X + Y = 1$, the condition $X > Y$ is equivalent to that $X > \frac{1}{2}$.

Let us evaluate X:

$$X = \langle R, R\rangle_{\mathrm{rs}} = \frac{1}{|G^F|} \sum_{g \in G^F_{\mathrm{rs}}} R_T^G(\theta)(g) \cdot \overline{R_T^G(\theta)(g)}.$$

By the regular semisimple Deligne–Lusztig character formula, $R_T^G(\theta)(g) = 0$ for any $g \in G^F_{\mathrm{rs}}$ which is not conjugate to an element of T^F. Note that we have

$$\begin{aligned} (G^F/N_{G^F}(T)) \times T^F_{\mathrm{rs}} &\xrightarrow{1:1} \{g \in G^F_{\mathrm{rs}} \mid g \text{ is conjugate to an element of } T^F\} \\ (x, t) &\mapsto xtx^{-1}. \end{aligned}$$

Thus, again by using the regular semisimple Deligne–Lusztig character formula, we have

$$\begin{aligned} X &= \frac{1}{|G^F|} \sum_{x \in G^F/N_{G^F}(T)} \sum_{t \in T^F_{\mathrm{rs}}} R_T^G(\theta)(xtx^{-1}) \cdot \overline{R_T^G(\theta)(xtx^{-1})} \\ &= \frac{1}{|G^F|} \sum_{x \in G^F/N_{G^F}(T)} \sum_{t \in T^F_{\mathrm{rs}}} R_T^G(\theta)(t) \cdot \overline{R_T^G(\theta)(t)} \\ &= \frac{1}{|N_{G^F}(T)|} \sum_{t \in T^F_{\mathrm{rs}}} \sum_{w, w' \in W_{G^F}(T)} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)}. \end{aligned}$$

By noting that $T^F_{\mathrm{rs}} = T^F - T^F_{\mathrm{nrs}}$, we get

$$X = \frac{1}{|N_{G^F}(T)|} \sum_{w, w' \in W_{G^F}(T)} \left(\sum_{t \in T^F} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)} - \sum_{t \in T^F_{\mathrm{nrs}}} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)} \right).$$

Here, since θ is regular, the orthogonality relation of characters implies that

$$\sum_{t \in T^F} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)} = \begin{cases} |T^F| & \text{if } w = w', \\ 0 & \text{otherwise.} \end{cases}$$

Thus we get

$$\begin{aligned} X &= \frac{1}{|N_{G^F}(T)|} \cdot |W_{G^F}(T)| \cdot |T^F| - \frac{1}{|N_{G^F}(T)|} \cdot \sum_{w, w' \in W_{G^F}(T)} \sum_{t \in T^F_{\mathrm{nrs}}} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)} \\ &= 1 - \frac{1}{|N_{G^F}(T)|} \cdot \sum_{w, w' \in W_{G^F}(T)} \sum_{t \in T^F_{\mathrm{nrs}}} {}^{w}\theta(t) \cdot \overline{{}^{w'}\theta(t)}. \end{aligned}$$

Hence, the triangle inequality implies that

$$X \geq 1 - \frac{1}{|N_{G^F}(T)|} \cdot |W_{G^F}(T)|^2 \cdot |T^F_{\mathrm{nrs}}| = 1 - \frac{|T^F_{\mathrm{nrs}}|}{|T^F|} \cdot |W_{G^F}(T)|.$$

Note that the Henniart inequality is equivalent to that

$$\frac{|T^F_{\mathrm{nrs}}|}{|T^F|} \cdot |W_{G^F}(T)| < \frac{1}{2}.$$

Hence, if the Henniart inequality holds, we obtain $X > \frac{1}{2}$. $\square$

1.2. **Henniart inequality for Coxeter tori of exceptional groups.** As mentioned above, as long as the group G and its maximal torus T are specified, it is possible to explicate the Henniart inequality. For example, for any split adjoint simple group of exceptional type, the Henniart inequality for a k-rational maximal torus S of "Coxeter type"[2] is as in the following table:

TABLE 1. Henniart inequalities for Coxeter tori of exceptional groups

G	$\lvert S^F\rvert$	$\lvert S^F_{\mathrm{nrs}}\rvert$	condition on q
E_6	$(q^4-q^2+1)(q^2+q+1)$	q^2+q+1	$q>2$
E_7	$(q^6-q^3+1)(q+1)$	$\begin{cases} 3(q+1) & q\equiv -1 \mod 3 \\ q+1 & q\not\equiv -1 \mod 3 \end{cases}$	$\begin{cases} q>2 \\ q\text{: any} \end{cases}$
E_8	$q^8+q^7-q^5-q^4-q^3+q+1$	1	q: any
F_4	q^4-q^2+1	1	$q>2$
G_2	q^2-q+1	$\begin{cases} 3 & q\equiv -1 \mod 3 \\ 1 & q\not\equiv -1 \mod 3 \end{cases}$	$\begin{cases} q>6 \\ q>3 \end{cases}$

Therefore, only the cases which do not satisfy the Henniart inequality are

- G is of type E_6, $q=2$;
- G is of type F_4, $q=2$;
- G is of type G_2, $q=2,3,5$.

1.3. **The case of $G_2(\mathbb{F}_3)$.** In the following, let us investigate what is happening in the case where $G=G_2$ over $\mathbb{F}_3$. In fact, in this case, our characterization theorem for regular Deligne–Lusztig representations does not hold!

First, again recall that G^F-conjugacy classes of k-rational maximal tori of G are parametrized by the conjugacy classes in the absolute Weyl group W_0 of G. The group G_2 has 6 conjugacy classes; they are named "$\varnothing$", "A_1", "$\tilde{A}_1$", "$A_1\times\tilde{A}_1$", "A_2", and "G_2" (see [Car72, Table 7]). For any such conjugacy class Γ, let us write T_Γ for a k-rational maximal torus corresponding to Γ. Then the orders of T_Γ^F and $W_{G^F}(T_\Gamma)$ are given as follows (see also [Car72, Table 3 and Lemma 26]):

TABLE 2. Maximal tori of $G_2(\mathbb{F}_q)$

Γ	$\lvert T_\Gamma^F\rvert$	$\lvert T_\Gamma^F\rvert$ $(q=3)$	$\lvert W_{G^F}(T_\Gamma)\rvert$	split rank
$\varnothing$	$(q-1)^2$	4	12	2 (split)
A_1	$(q-1)(q+1)$	8	4	1
$\tilde{A}_1$	$(q-1)(q+1)$	8	4	1
$A_1\times\tilde{A}_1$	$(q+1)^2$	16	12	0 (elliptic)
A_2	q^2+q+1	13	6	0 (elliptic)
G_2	q^2-q+1	7	6	0 (elliptic, Coxeter)

These are actually contained in GAP3. To see it, first put:

```
gap> W:=CoxeterGroup("G",2);
```

[2]The Weyl group has a particular conjugacy class consisting of elements called "Coxeter elements". The maximal torus S corresponds to the Coxeter conjugacy class.

Then, by putting

```
gap> CharTable(W).classnames;
```

GAP3 gives the following output:

```
[ "A0", "~A1", "A1", "G2", "A2", "A1+~A1" ]
```

Moreover, the following gives the list of rational maximal tori corresponding to the above conjugacy classes:

```
gap> Twistings(W,[]);
```

```
[ (q-1)^2, (q-1)(q+1), (q-1)(q+1), (q^2-q+1), (q^2+q+1), (q+1)^2 ]
```

Note that "G_2" is the conjugacy class of Coxeter elements. Hence T_{G_2} is our maximal torus S. We can check that S^F has a non-regular semisimple element other than unit if and only if $q \equiv -1 \pmod 3$; in this case, the number of non-regular semisimple elements is 3. Also note that the rational Weyl group $W_{G^F}(S)$ is cyclic of order 6.

From now on, we focus on the case where $k = \mathbb{F}_3$.

The group $G_2(\mathbb{F}_3)$ has 23 conjugacy classes, hence has 23 irreducible representations. Table 3 is the list of 23 conjugacy classes; if a conjugacy class has name "nx", then it means that the order of any representative of the class is given by n. The last column of Table 3 expresses which tori contain semisimple elements within the conjugacy classes.

The character table of $G_2(\mathbb{F}_3)$ is as in Table 6. The 23 irreducible representations are named "Xn" in the decreasing order according to their dimensions. This table is cited from GAP3 ([S+97]); if you are familiar with GAP3, Table 6 can be output just by typing:

```
>gap DisplayCharTable( CharTable( "G2(3)" ) );
```

(see `https://webusers.imj-prg.fr/~jean.michel/gap3/htm/chap049.htm#SECT037` for the details). In the following, we write X_n for the irreducible representation Xn.

We remark that among the 23 irreducible representations, the unipotent representations are

$$X_1, X_2, X_3, X_4, X_5, X_7, X_8, X_9, X_{10}, X_{21}$$

(X_2, X_3, X_4, and X_5 are cuspidal unipotent representations). This can be also seen by using GAP3:

```
gap> Display(UnipotentCharacters(CoxeterGroup("G",2)));
```

In the GAP3 output, the above unipotent representations are expressed as `phi{1,0}`, `G2[1]`, `G2[E3]`, `G2[E3^2]`, `G2[-1]`, `phi{1,3}'`, `phi{1,3}''`, `phi{2,1}`, `phi{2,2}`, `phi{1,6}`. (See `https://webusers.imj-prg.fr/~jean.michel/gap3/htm/chap098.htm` and also [Lus84, 372 page].)

We also remark that each unipotent representation is realized in $R^G_{T_\Gamma}(\mathbb{1})$ as in Table 5. To see this via GAP3, type the following:

```
gap> DeligneLusztigCharacter(CoxeterGroup("G",2),n);
```

Here, `n` means the nth conjugacy class of the Weyl group of G_2, where the conjugacy classes are arranged in the following order (`gap> PrintRec(CoxeterGroup("G",2));`):

$$\varnothing, \tilde{A}_1, A_1, G_2, A_2, A_1 \times \tilde{A}_1.$$

Table 3. Conjugacy classes of $G_2(\mathbb{F}_3)$

conjugacy class	order	order of centralizer	type	tori
1a	1	4245696	unit	all
2a	2	576	ss.	$\varnothing, A_1, \tilde{A}_1, A_1 \times \tilde{A}_1$
3a	3	5832	unip.	–
3b	3	5832	unip.	–
3c	3	729	unip.	–
3d	3	162	unip.	–
3e	3	162	unip.	–
4a	4	96	ss.	$\tilde{A}_1, A_1 \times \tilde{A}_1$
4b	4	96	ss.	$A_1, A_1 \times \tilde{A}_1$
6a	6	72	–	–
6b	6	72	–	–
6c	6	18	–	–
6d	6	18	–	–
7a	7	7	reg. ss.	G_2
8a	8	8	reg. ss.	$\tilde{A}_1$
8b	8	8	reg. ss.	A_1
9a	9	27	unip.	–
9b	9	27	unip.	–
9c	9	27	unip.	–
12a	12	12	–	–
12b	12	12	–	–
13a	13	13	reg. ss.	A_2
13b	13	13	reg. ss.	A_2

Now we discuss a counterexample to our characterization theorem for regular Deligne–Lusztig representations. We have $S^F \cong \mathbb{Z}/7\mathbb{Z}$ and $S^F_{\text{nrs}} = \{1\}$. Moreover, we can check that $W_{G^F}(S)$ acts on the set of regular semisimple elements of S^F

Table 4. Unipotent representations of $G_2(\mathbb{F}_q)$

GAP3 label	dimension	label ($q=3$)	dim ($q=3$)	label (Lusztig)
`phi{1,0}`	1	X_1	1	–
`phi{1,6}`	q^6	X_{21}	729	–
`phi{1,3}'`	$q\phi_3(q)\phi_6(q)/3$	X_7	91	$(1, r)$
`phi{1,3}''`	$q\phi_3(q)\phi_6(q)/3$	X_8	91	$(g_3, 1)$
`phi{2,1}`	$q\phi_2^2(q)\phi_3(q)/6$	X_9	104	$(1, 1)$
`phi{2,2}`	$q\phi_2^2(q)\phi_6(q)/2$	X_{10}	168	$(g_2, 1)$
`G2[-1]`	$q\phi_1^2(q)\phi_3(q)/2$	X_5	78	(g_2, ε)
`G2[1]`	$q\phi_1^2(q)\phi_6(q)/6$	X_2	14	$(1, \varepsilon)$
`G2[E3]`	$q\phi_1^2(q)\phi_2^2(q)/3$	X_3	64	(g_3, θ)
`G2[E3^2]`	$q\phi_1^2(q)\phi_2^2(q)/3$	X_4	64	(g_3, θ^2)

$\phi_1(q) = q - 1$, $\phi_2(q) = q + 1$, $\phi_3(q) = q^2 + q + 1$, $\phi_6(q) = q^2 - q + 1$.

Table 5. Unipotent Deligne–Lusztig representations $G_2(\mathbb{F}_q)$

Γ	$R^G_{T_\Gamma}(\mathbb{1})$
$\varnothing$	$X_1 + X_7 + X_8 + 2X_9 + 2X_{10} + X_{21}$
A_1	$X_1 - X_7 + X_8 - X_{21}$
$\tilde{A}_1$	$X_1 + X_7 - X_8 - X_{21}$
$A_1 \times \tilde{A}_1$	$X_1 - 2X_2 - 2X_5 - X_7 - X_8 + X_{21}$
A_2	$X_1 + X_2 - X_3 - X_4 - X_{10} + X_{21}$
G_2	$X_1 + X_3 + X_4 + X_5 - X_9 + X_{21}$

simply-transitively. Thus we see that there exists only one regular character θ of S^F up to conjugation.

By the dimension formula of Deligne–Lusztig representations, we have

$$\dim R^G_S(\theta) = \frac{|G^F|}{\dim \mathrm{St}_G \cdot |S^F|}.$$

Note that $r_G = 2$ and $r_S = 0$, hence the sign appearing in the dimension formula is trivial. In other words, $R^G_S(\theta)$ is a genuine representation. Since we have

- $|G^F| = q^6 \cdot (q^2 - 1) \cdot (q^6 - 1)$ (see [Car85, Section 2.9]),
- $\dim \mathrm{St}_G = q^6$ (see [Car85, Proposition 6.4.4]),
- $|S^F| = q^2 - q + 1$,

we have

$$\dim R^G_S(\theta) = (q-1)^2 \cdot (q+1)^2 \cdot (q^2 + q + 1) = 832.$$

Thus we conclude that $R^G_S(\theta)$ is the irreducible representation X_{23}.

By the above description of the group S^F and the action of $W_{G^F}(S)$ on S^F, we see that

$$\sum_{w \in W_{G^F}(S)} \theta^w(s) = \sum_{i=1}^{6} \zeta_7^i = -1$$

for any regular semisimple element $s \in S^F$, where ζ_7 is a primitive 7th root of unity. Therefore, our characterization theorem in this case is asking whether an irreducible virtual representation of G^F such that

- $\Theta_\rho(s) = 0$ if the conjugacy class of s is one of "8a", "8b", "13a", and "13b" (see Table 3) and
- $\Theta_\rho(s) = \pm 1$ if the conjugacy class of s is "7a" (see Table 3)

is necessarily equal to $\pm R^G_S(\theta)$ or not.

By looking at the character table (Table 6), we can easily find that X_5 and X_9 satisfy these assumptions!

TABLE 6. Character table of $G_2(\mathbb{F}_3)$

	1a	2a	3a	3b	3c	3d	3e	4a	4b	6a	6b	6c	6d	7a	8a	8b	9a	9b	9c	12a	12b	13a	13b
X1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
X2	14	−2	5	5	−4	2	−1	2	2	1	1	−2	1	·	·	·	2	−1	−1	−1	−1	1	1
X3	64	·	−8	−8	1	4	−2	·	·	·	·	·	·	1	·	·	1	α	β	·	·	−1	−1
X4	64	·	−8	−8	1	4	−2	·	·	·	·	·	·	1	·	·	1	β	α	·	·	−1	−1
X5	78	−2	−3	−3	−3	−3	6	2	2	1	1	1	−2	1	·	·	·	·	·	−1	−1	·	·
X6	91	−5	10	10	10	1	1	3	3	−2	−2	1	1	·	−1	−1	1	1	1	·	·	·	·
X7	91	3	−8	19	1	4	−2	3	−1	·	3	·	·	·	1	−1	−2	1	1	·	−1	·	·
X8	91	3	19	−8	1	4	−2	−1	3	3	·	·	·	·	−1	1	−2	1	1	−1	·	·	·
X9	104	8	14	14	5	2	−1	·	·	2	2	2	−1	−1	·	·	2	−1	−1	·	·	·	·
X10	168	8	6	6	6	−3	6	·	·	2	2	−1	2	·	·	·	·	·	·	·	·	−1	−1
X11	182	6	20	−7	−7	2	2	2	2	·	−3	·	·	·	·	·	−1	−1	−1	2	−1	·	·
X12	182	6	−7	20	−7	2	2	2	2	−3	·	·	·	·	·	·	−1	−1	−1	−1	2	·	·
X13	273	−7	30	3	3	3	3	−3	1	2	−1	−1	−1	·	1	−1	·	·	·	·	1	·	·
X14	273	−7	3	30	3	3	3	1	−3	−1	2	−1	−1	·	−1	1	·	·	·	1	·	·	·
X15	448	·	16	16	−11	−2	−2	·	·	·	·	·	·	·	·	·	1	1	1	·	·	γ	δ
X16	448	·	16	16	−11	−2	−2	·	·	·	·	·	·	·	·	·	1	1	1	·	·	δ	γ
X17	546	2	−21	6	6	−3	−3	−2	6	−1	2	−1	−1	·	·	·	·	·	·	1	·	·	·
X18	546	2	6	−21	6	−3	−3	6	−2	2	−1	−1	−1	·	·	·	·	·	·	·	1	·	·
X19	728	−8	26	−28	−1	−1	−1	·	·	−2	4	1	1	·	·	·	−1	−1	−1	·	·	·	·
X20	728	−8	−28	26	−1	−1	−1	·	·	4	−2	1	1	·	·	·	−1	−1	−1	·	·	·	·
X21	729	9	·	·	·	·	·	−3	−3	·	·	·	·	1	−1	−1	·	·	·	·	·	1	1
X22	819	3	9	9	9	·	·	−1	−1	−3	−3	·	·	·	1	1	·	·	·	−1	−1	·	·
X23	832	·	−32	−32	−5	4	4	·	·	·	·	·	·	−1	·	·	1	1	1	·	·	·	·

every dot denotes 0, $\alpha := \frac{-1+3\sqrt{-3}}{2}$, $\beta := \frac{-1-3\sqrt{-3}}{2}$, $\gamma := \frac{-1+\sqrt{13}}{2}$, $\delta := \frac{-1-\sqrt{13}}{2}$.

1.4. **Unipotent representations.** One may notice that the above counterexample is given by unipotent representations. In fact, this is not an accident.

Let G be a connected reductive group over $k = \mathbb{F}_q$, T a k-rational maximal torus of G, and θ a regular character of T^F. We suppose that ρ is an irreducible representation of G^F having the same regular semisimple character values as $R_T^G(\theta)$.

Lemma 1.4. *Suppose that there exists a character $\theta'\colon T^F \to \mathbb{C}^\times$ such that*

(i) θ and θ' are not $W_{G^F}(T)$-conjugate, and

(ii) $\theta|_{T^F_{\mathrm{nrs}}} \equiv \theta'|_{T^F_{\mathrm{nrs}}}$.

Then we have either $\langle \rho, R_T^G(\theta)\rangle \neq 0$ or $\langle \rho, R_T^G(\theta')\rangle \neq 0$.

Proof. By the regularity of θ, we have

$$1 = \langle R_T^G(\theta), R_T^G(\theta)\rangle = \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} + \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{nrs}}. \tag{1}$$

On the other hand, by the assumption (i) and the inner product formula, we have

$$0 = \langle R_T^G(\theta), R_T^G(\theta')\rangle = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}} + \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{nrs}}. \tag{2}$$

Recall that, by the Deligne–Lusztig character formula

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x\in G^F \\ x^{-1}sx\in T^F}} \theta(x^{-1}sx)\cdot Q^{G_s^\circ}_{{}^xT}(u).$$

we see that the character of $R_T^G(\theta)$ on G^F_{nrs} depends only on $\theta|_{T^F_{\mathrm{nrs}}}$. Thus assumption (ii) implies that $R_T^G(\theta)$ equals $R_T^G(\theta')$ on G^F_{nrs}. In particular, we have $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{nrs}} = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{nrs}}$. Thus, by the equalities (1) and (2), we get $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$.

We next look at the following two equalities:

$$\langle \rho, R_T^G(\theta)\rangle = \langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} + \langle \rho, R_T^G(\theta)\rangle_{\mathrm{nrs}}, \tag{3}$$

$$\langle \rho, R_T^G(\theta')\rangle = \langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}} + \langle \rho, R_T^G(\theta')\rangle_{\mathrm{nrs}}. \tag{4}$$

Again by the same observation as above, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{nrs}} = \langle \rho, R_T^G(\theta')\rangle_{\mathrm{nrs}}$. Moreover, by the assumption on ρ, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} = \langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}}$ and $\langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}} = \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$. Since we obtained $\langle R_T^G(\theta), R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle R_T^G(\theta), R_T^G(\theta')\rangle_{\mathrm{rs}}$ in the previous paragraph, we have $\langle \rho, R_T^G(\theta)\rangle_{\mathrm{rs}} \neq \langle \rho, R_T^G(\theta')\rangle_{\mathrm{rs}}$. Therefore, by combining these equalities with (3) and (4), we get $\langle \rho, R_T^G(\theta)\rangle \neq \langle \rho, R_T^G(\theta')\rangle$. In particular, at least one of $\langle \rho, R_T^G(\theta)\rangle$ and $\langle \rho, R_T^G(\theta')\rangle$ is not zero. □

Note that Lemma 1.4 has the following immediate consequence (choose θ' to be the trivial character $\mathbb{1}$ of T^F):

Lemma 1.5. *If $\theta|_{T^F_{\mathrm{nrs}}} \equiv \mathbb{1}$, then we have either $\langle \rho, R_T^G(\theta)\rangle \neq 0$ or $\langle \rho, R_T^G(\mathbb{1})\rangle \neq 0$.*

Hence we get the following theorem (note that this result requires NO assumption on q):

Theorem 1.6. *Suppose that θ is a regular character of T^F whose restriction to T^F_{nrs} is trivial. Suppose that ρ is an irreducible representation of G^F equipped with a sign ε such that, for any regular semisimple element $g \in G^F$,*

$$\Theta_\rho(g) = \varepsilon \cdot \Theta_{R_T^G(\theta)}(g).$$

If ρ is not unipotent, then we necessarily have $\rho \cong \varepsilon R_T^G(\theta)$.

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8:48am, December 17, 2024