1. Week 9: Semisimple character formula and exhaustion theorem

Recall that we proved the inner product formula for Deligne–Lusztig representations by assuming the orthogonality relation for Green functions. The aim of this week is to partially prove the orthogonality relation. More precisely, we introduce another result which we call the "disjointness theorem" and then deduce the orthogonality relation from the disjointness theorem.

Chart: Disjointness Theorem (Theorem 1.4, not proved this week)

$\overset{\text{this week}}{\Longrightarrow}$ Orthogonality relation for Green functions

$\overset{\text{last week}}{\Longrightarrow}$ Inner product formula for Deligne–Lusztig representaitons

(the second $\Longrightarrow$ is in fact $\Longleftrightarrow$).

After that, we also prove that any irreducible representation can be realized in some Deligne–Lusztig representation.

1.1. Geometric conjugacy and disjointness theorem.

Let G be a connected reductive group over $k = \mathbb{F}_q$ with associated Frobenius endomorphism F. Suppose that T is a k-rational maximal torus of G. Note that then we have $T^{F^r} = T(\mathbb{F}_{q^r})$ for any $r \in \mathbb{Z}_{>0}$. We define the *norm map* N_r from T^{F^r} to T^F by

$$N_r\colon T^{F^r} \to T^F; \quad t \mapsto t \cdot F(t) \cdots F^{r-1}(t).$$

Note that, if $T = \mathbb{G}_{\mathrm{m}}$, then N_r is nothing but the usual norm map from $T^{F^r} = \mathbb{F}_{q^r}^\times$ to $T^F = \mathbb{F}_q^\times$. Recall that the norm map from $\mathbb{F}_{q^r}^\times$ to $T^F = \mathbb{F}_q^\times$ is surjective. In fact, the same property holds for the norm map for any T:

Lemma 1.1. *The norm map $N_r\colon T^{F^r} \to T^F$ is surjective.*

Exercise 1.2. Prove this lemma.

Hint: Suppose $t \in T^F$. Apply Lang's theorem for $F^r\colon T \to T$ to t; then we get an $s \in T$ satisfying $F^r(s)s^{-1} = t$. Show that $F(s)s^{-1}$ belongs to T^{F^r} and maps to t under N_r.

Definition 1.3. Let T and T' be k-rational maximal tori of G. We say that characters θ of T^F and θ' of T'^F are *geometrically conjugate* if $(T, \theta \circ N_r)$ and $(T', \theta' \circ N_r)$ are G^{F^r}-conjugate for some $r \in \mathbb{Z}_{>0}$, i.e., there exists $x \in G^{F^r}$ satisfying $T' = {}^xT$ and $\theta' \circ N_r = {}^x(\theta \circ N_r)$.

Note that if θ and θ' are conjugate, then they are geometrically conjugate (r can be taken to be 1).

The following theorem is a key to the proof of the orthogonality relation (for convenience, let us call the following the "disjointness theorem"):

Theorem 1.4 (Disjointness theorem)**.** *Let T and T' be k-rational maximal tori of G. Suppose that characters θ of T^F and θ' of T'^F are not geometrically conjugate. Then $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain a common irreducible representation.*

Remark 1.5. (1) The precise meaning of "a virtual representation R contains an irreducible representation σ" is that "if we write R as the sum $\sum_\rho n_\rho \rho$ over all (isomorphism classes of) irreducible representations ($n_\rho \in \mathbb{Z}$), then $n_\sigma \neq 0$". Each coefficient n_ρ is often called the "multiplicity" of ρ in R.

(2) Recall that, as a corollary of the inner product formula, we obtained that "if θ and θ' are not G^F-conjugate, then $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = 0$". On the other hand, the statement of Theorem 1.4 is stronger than the equality $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle = 0$. Thus both the assumption and the conclusion of Theorem 1.4 are stronger than (a consequence of) the inner product formula.

1.2. **Orthogonality relation for Green functions.** Recall that, for any connected reductive group G with center Z, the quotient group $G_{\mathrm{ad}} := G/Z$ is of adjoint type (i.e., a connected reductive group with trivial center). Moreover, it is not difficult to see the following.

- For any k-rational maximal torus T of G, its image T_{ad} in G_{ad} is a k-rational maximal torus of G_{ad}.
- The natural quotient map $G \to G_{\mathrm{ad}}$ induces a bijection $G^F_{\mathrm{unip}} \xrightarrow{1:1} G^F_{\mathrm{ad,unip}}$.

Lemma 1.6. *For any $u \in G^F_{\mathrm{unip}}$, we have $Q^G_T(u) = Q^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}}(\bar{u})$, where $\bar{u} \in G^F_{\mathrm{ad,unip}}$ is the image of u.*

Sketch of Proof. This follows from an alternative description of the Green function in terms of the variants of the Deligne–Lusztig varieties. A bit more precisely, the Green function Q^G_T can be also interpreted as the Lefschetz number of the action of G^F_{unip} on the variety " $X^G_{T\subset B}$" (see Week 5 notes). We can easily check that $X^G_{T\subset B}$ is canonically isomorphic to $X^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}\subset B_{\mathrm{ad}}}$, which implies that $Q^G_T(u) = Q^{G_{\mathrm{ad}}}_{T_{\mathrm{ad}}}(\bar{u})$. See [**?**, Definition 1.9] and its preceding remark for more details. □

Theorem 1.7 (Orthogonality relation for Green functions)**.** *Let T and T' be k-rational maximal tori of G. Let B and B' be Borel subgroup of G containing T and Q^G_T and $Q^G_{T'}$ associated Green functions. Then we have*

$$\frac{1}{|G^F|}\sum_{u\in G^F_{\mathrm{unip}}} Q^G_T(u)\cdot Q^G_{T'}(u) = \frac{|N_{G^F}(T,T')|}{|T^F|\cdot|T'^F|}.$$

Proof. The asserted identity is trivial if G is a torus. We handle the general case by the induction on $\dim G$. (Note that any 1-dimensional connected reductive group is a torus.) We also note that the desired identity does not change even if we replace G with G_{ad}. (The Green functions do not change by the previous lemma; all other numbers are multiplied by the same number.) Thus we may suppose that G is of adjoint type in the following.

Here, let us remember the proof of the inner product formula. For any characters θ of T^F and θ' of T'^F, we first computed $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle$ as follows:

$$(*)\quad \langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\theta')\rangle$$
$$= \frac{1}{|G^F|}\sum_{s\in G^F_{\mathrm{ss}}}\frac{1}{|(G^\circ_s)^F|^2}\sum_{\substack{x,y\in G^F\\ x^{-1}sx\in T^F\\ y^{-1}sy\in T'^F}}\theta(x^{-1}sx)\overline{\theta'(y^{-1}sy)}\sum_{u\in(G^\circ_s)^F_{\mathrm{unip}}} Q^{G^\circ_s}_{{}^xT}(u)\overline{Q^{G^\circ_s}_{{}^yT'}(u)}.$$

Then we utilized the orthogonality relation to rewrite this as follows:

$$= \cdots = \frac{1}{|G^F| \cdot |T^F|^2} \sum_{s \in G^F_{\mathrm{ss}}} \sum_{\substack{x \in G^F \\ n' \in N_{G^F}(T,T') \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx)\overline{{}^{n'^{-1}}\theta'(x^{-1}sx)}$$

$$= \cdots = \sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} = \cdots .$$

The point here is that, in the current setting, the same computation works for any $s \neq 1$. This is because, since G has trivial center, any nontrivial semisimple element s satisfies $\dim G^\circ_s < \dim G$, hence we can apply the induction hypothesis to G°_s. (Note that the condition that $s \neq 1$ is rephrased as the condition that $t \neq 1$ in the last sum.) On the other hand, for $s = 1$, the contribution to $(*)$ is simply given by

$$\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)}.$$

Therefore, we see that $(*)$ is equal to

$$\underbrace{\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)}}_{s=1} + \underbrace{\sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t)\overline{{}^{w^{-1}}\theta'(t)}}_{s \neq 1}.$$

Here note that the second term for $s \neq 1$ can be computed as follows:

$$\begin{aligned} &\sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} \\ &= \sum_{w \in W_{G^F}(T,T')} \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)\overline{{}^{w^{-1}}\theta'(t)} - \frac{|W_{G^F}(T,T')|}{|T^F|} \\ &= |\{w \in W_{G^F}(T,T') \mid {}^w\theta = \theta'\}| - \frac{|W_{G^F}(T,T')|}{|T^F|}. \end{aligned}$$

In other words, we obtained

$$\begin{aligned} &\frac{1}{|G^F|} \sum_{u \in G^F_{\mathrm{unip}}} Q^G_T(u)\overline{Q^G_{T'}(u)} \\ &= \langle R^G_{T \subset B}(\theta), R^G_{T' \subset B'}(\theta') \rangle - |\{w \in W_{G^F}(T,T') \mid {}^w\theta = \theta'\}| + \frac{|W_{G^F}(T,T')|}{|T^F|}. \end{aligned}$$

This shows the following:

> To obtain the orthogonality relation for Q^G_T and $Q^G_{T'}$, it is enough to find just one example of a pair (θ, θ') satisfying the inner product formula for $\langle R^G_{T \subset B}(\theta), R^G_{T' \subset B'}(\theta') \rangle$ (of course, in a way which is not based on the orthogonality relation)!

We first consider the case where either T^F or T'^F has a nontrivial character; we may assume that T^F has a nontrivial character θ. Note that, since the norm map for a torus is surjective (Lemma 1.1), θ cannot be geometrically conjugate to the trivial character of T'^F. Thus, by the disjointness theorem (Theorem 1.4), we have $\langle R^G_{T\subset B}(\theta), R^G_{T'\subset B'}(\mathbb{1})\rangle = 0$. On the other hand, obviously we have $|\{w \in W_{G^F}(T,T') \mid {}^w\theta = \mathbb{1}\}| = 0$.

We next consider the case where T^F and T'^F do not have a nontrivial character; this is equivalent to that $|T^F| = |T'^F| = 1$. In this case, q must be 2 and T and T' must be split over k. (We leave this for an exercise below.) This implies that T and T' are G^F-conjugate to a split k-rational maximal torus T_0 and also that $R^G_{T\subset B}(\mathbb{1}) \cong R^G_{T'\subset B'}(\mathbb{1}) \cong \mathrm{Ind}^{G^F}_{B_0^F}\mathbb{1}$ (we proved this in Week 6), where B_0 is a k-rational Borel subgroup of G containing T_0. Then we can check that

$$\langle R^G_{T\subset B}(\mathbb{1}), R^G_{T'\subset B'}(\mathbb{1})\rangle = \langle \mathrm{Ind}^{G^F}_{B_0^F}\mathbb{1}, \mathrm{Ind}^{G^F}_{B_0^F}\mathbb{1}\rangle = |W_{G^F}(T)|$$

(let me also leave this for an exercise!). On the other hand, obviously we have $|\{w \in W_{G^F}(T,T') \mid {}^w\mathbb{1} = \mathbb{1}\}| = |W_{G^F}(T)|$.

Therefore, in both cases, we found an example of a pair (θ, θ') satisfying the inner product formula. This completes the proof. □

Exercise 1.8. Let T be a k-rational maximal torus of a connected reductive group G over k. Prove that $|T^F| = 1$ only when $q = 2$ and T is split over k.

Hint: utilize the formula of $|T^F|$ in terms of the character group of T; see Week 5 notes.

Exercise 1.9. Prove that

$$\langle \mathrm{Ind}^{G^F}_{B_0^F}\mathbb{1}, \mathrm{Ind}^{G^F}_{B_0^F}\mathbb{1}\rangle = |W_{G^F}(T)|.$$

Hint: Recall that we proved this in the GL_2 case in Week 2. In fact, the same argument works; combine (1) Frobenius reciprocity, (2) Mackey decomposition formula, and (3) Bruhat decomposition.

1.3. **Steinberg representations.** Recall that, for $G = \mathrm{GL}_2$, the principal series representation $\mathrm{Ind}^G_B\mathbb{1}$ is the sum of two irreducible representations; the trivial representation and the Steinberg representations. (In this subsection, we temporarily omit the symbol "F" in the induced representations to make the notation lighter.) In fact, the notion of the Steinberg representation can be generalized to any connected reductive group over k.

Instead of explaining its definition in general, let us present an example of GL_3. Let $G := \mathrm{GL}_3$ and B be the upper-triangular Borel subgroup of G. We consider the principal series representation $\mathrm{Ind}^G_B\mathbb{1}$. Then, as in the GL_2 case, $\mathrm{Ind}^G_B\mathbb{1}$ contains the trivial representation. However, the different point is that $\mathrm{Ind}^G_B\mathbb{1}$ contains further more irreducible representations. To see this, let us consider the following subgroup:

$$P_{2,1} := \begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{pmatrix} \subset G.$$

Since B is contained in $P_{2,1}$, the associativity of the induction implies that

$$\mathrm{Ind}^G_B\mathbb{1} = \mathrm{Ind}^G_{P_{2,1}}(\mathrm{Ind}^{P_{2,1}}_B\mathbb{1}) \supset \mathrm{Ind}^G_{P_{2,1}}\mathbb{1}.$$

Then, how about subtracting $\mathrm{Ind}^G_{P_{2,1}}\mathbb{1}$ from $\mathrm{Ind}^G_B\mathbb{1}$? In fact, the remaining representation is still not irreducible! So let us also consider the following subgroup:

$$P_{1,2} := \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \subset G.$$

Then, for the same reason as above, we have $\mathrm{Ind}_B^G \mathbb{1} \supset \mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$. In fact, $\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$ is a different subrepresentation from $\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}$. So, how about subtracting both $\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$ and $\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}$ from $\mathrm{Ind}_B^G \mathbb{1}$? This also does not work because both $\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}$ and $\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}$ contains the trivial representation, but the multiplicity of the trivial representation in $\mathrm{Ind}_B^G \mathbb{1}$ is exactly one! In other words, the trivial representation is subtracted doubly. We shouldn't give up here; how about considering the following representation:

$$(\mathrm{Ind}_B^G \mathbb{1}) - (\mathrm{Ind}_{P_{1,2}}^G \mathbb{1}) - (\mathrm{Ind}_{P_{2,1}}^G \mathbb{1}) + \mathbb{1}.$$

In fact, this gives an irreducible subrepresentation of $\mathrm{Ind}_B^G \mathbb{1}$! This is the definition of the Steinberg representation of $\mathrm{GL}_3(\mathbb{F}_q)$.

In general, the Steinberg representation is defined according to a similar idea. The subgroups $P_{1,2}$ and $P_{2,1}$ are examples of so-called *parabolic subgroups* of G. The idea is to consider a certain signed sum of the induced representations from all parabolic subgroups based on the "inclusion-exclusion principle" as in the GL_3 case. The Steinberg representation can be investigated independently of Deligne–Lusztig theory. The precise definition of the Steinberg representation of G^F (let us write St_G) and its basic properties are summarized in, for example, [Car85, Chapter 6].

Therefore, in this course, let us just believe the existence of the representation St_G of G^F satisfying the following properties.

Proposition 1.10 (Character formula for St_G). *For any $g \in G^F$, we have*

$$\mathrm{St}_G(g) = \begin{cases} (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1) & \text{if } g = s \text{ is semisimple,} \\ 0 & \text{otherwise.} \end{cases}$$

Here, for any connected reductive group G over k, we let r_G denote its k-split rank, i.e., the dimension of the maximal k-split torus of G.

Proposition 1.11 (Dimension formula). *We let B_0 be a k-rational Borel subgroup of G with unipotent radical U_0. Then we have $\mathrm{St}_G(1) = |U_0^F|$.*

Exercise 1.12. Show that the above propositions in the case where $G = \mathrm{GL}_2$.

1.4. Character formula for DL representations on semisimple elements.

Theorem 1.13 (Dimension formula). *We have*

$$R_T^G(1) = (-1)^{r_G - r_T} \cdot \frac{|G^F|}{|T^F| \cdot \mathrm{St}_G(1)} = (-1)^{r_G - r_T} \cdot |G^F/T^F|_{p'},$$

where $(-)_{p'}$ denotes the prime-to-p part.

Proof. In fact, $|U_0^F|$ is equal to the p-part of $|G^F|$. On the other hand, $|T^F|$ is prime-to-p for any k-rational maximal torus of G. Hence, by the dimension formula of the Steinberg representation, we have

$$|G^F/T^F|_{p'} = \frac{|G^F|}{|T^F| \cdot \mathrm{St}_G(1)}.$$

Thus our task is to show the first equality.

Recall that $R_T^G(1) = Q_T^G(1)$ by the definition of the Green function. By the same reasoning as in the proof of the orthogonality relation, we may assume that G is of adjoint type. Moreover, by induction on $\dim G$, we may assume that the identity holds for G_s° for any semisimple $s \neq 1$.

We first consider the case where T^F does not have a nontrivial character. Recall that, in this case, T must be a split maximal torus T_0. Thus we have

$$R_T^G(1) = \dim \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1} = \frac{|G^F|}{|B_0^F|} = \frac{|G^F|}{|T_0^F| \cdot |U_0^F|} = \frac{|G^F|}{|T^F| \cdot |\mathrm{St}_G(1)|}.$$

We next consider the case where T^F has a nontrivial character θ. Recall that the Steinberg representation St_G is contained in $R_{T_0}^G(\mathbb{1}) = \mathrm{Ind}_{B_0^F}^{G^F} \mathbb{1}$. Thus, by applying the disjointness theorem to $R_T^G(\theta)$ and $R_{T_0}^G(\mathbb{1})$, we get

$$\langle R_T^G(\theta), \mathrm{St}_G \rangle = 0.$$

On the other hand, we can also compute $\langle R_T^G(\theta), \mathrm{St}_G \rangle$ directly by using the Deligne–Lusztig character formula and the character formula for Steinberg representations as follows:

$$\begin{aligned}
\langle R_T^G(\theta), \mathrm{St}_G \rangle &= \frac{1}{|G^F|} \sum_{g \in G^F} R_T^G(\theta)(g) \cdot \overline{\mathrm{St}_G(g)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} \sum_{u \in (G_s^\circ)_{\mathrm{unip}}^F} R_T^G(\theta)(su) \cdot \overline{\mathrm{St}_G(su)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} R_T^G(\theta)(s) \cdot \overline{\mathrm{St}_G(s)} \\
&= \frac{1}{|G^F|} \sum_{s \in G_{\mathrm{ss}}^F} \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1).
\end{aligned}$$

The idea of the proof is similar to that of the orthogonality relation. We divide the above sum according to $s = 1$ or $s \neq 1$. For $s = 1$, the summand is $Q_T^G(1) \cdot \mathrm{St}_G(1)$. For $s \neq 1$, by the induction hypothesis, the summand is given by

$$\begin{aligned}
&\frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1) \\
&= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot (-1)^{r_{G_s^\circ} - r_{{}^xT}} \cdot \frac{|(G_s^\circ)^F|}{|{}^xT^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \cdot (-1)^{r_G - r_{G_s^\circ}} \cdot \mathrm{St}_{G_s^\circ}(1) \\
&= \frac{1}{|T^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} (-1)^{r_G - r_T} {}^x\theta(s).
\end{aligned}$$

Therefore, since $\langle R_T^G(\theta), \mathrm{St}_G \rangle = 0$, we get

$$\underbrace{Q_T^G(1) \cdot \mathrm{St}_G(1)}_{s=1} + \underbrace{\sum_{s \in G_{\mathrm{ss}}^F \smallsetminus \{1\}} \frac{(-1)^{r_G - r_T}}{|T^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s)}_{s \neq 1} = 0.$$

By the same trick as in the proof of the orthogonality relation, the second term (the $s \neq 1$ part) on the left-hand side is equal to

$$\frac{|G^F|}{|T^F|} \cdot (-1)^{r_G - r_T} \sum_{t \in T^F \smallsetminus \{1\}} \theta(t) = -\frac{|G^F|}{|T^F|} \cdot (-1)^{r_G - r_T}$$

(we used that θ is a nontrivial character). Hence we get

$$Q_T^G(1) = (-1)^{r_G - r_T} \cdot \frac{|G^F|}{|T^F| \cdot \mathrm{St}_G(1)}.$$

□

Corollary 1.14 (Deligne–Lusztig character formula on semisimple elements)**.** *For any $s \in G^F_{\mathrm{ss}}$, we have*

$$R_T^G(\theta)(s) = \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s).$$

Proof. By the Deligne–Lusztig character formula and the dimension formula, we have

$$\begin{aligned} R_T^G(\theta)(s) &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot Q_{{}^xT}^{G_s^\circ}(1) \\ &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s) \cdot (-1)^{r_{G_s^\circ} - r_T} \cdot \frac{|(G_s^\circ)^F|}{|{}^xT^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \\ &= \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} {}^x\theta(s). \end{aligned}$$

□

Exercise 1.15. Show that the above corollary implies that $R_T^G(\theta) \otimes \mathrm{St}_G \cong \mathrm{Ind}_{T^F}^{G^F} \theta$.

Hint: Use the Frobenius character formula for induced representations.

1.5. **Exhaustion theorem.** For any $s \in G^F$, we let $\mathbb{1}_{[s]}$ denote the characteristic function of the G^F-conjugacy class $G^F \cdot s := \{xsx^{-1} \mid x \in G^F\}$ of s, i.e., $\mathbb{1}_{[s]} \colon G^F \to \mathbb{C}$ is a class function such that

$$\mathbb{1}_{[s]}(g) = \begin{cases} 1 & g \in G^F \cdot s, \\ 0 & g \notin G^F \cdot s. \end{cases}$$

We write $\mathcal{T}_G$ for the set of k-rational maximal tori of G (literally, all such tori; not G^F-conjugacy classes). For any $T \in \mathcal{T}_G$, we write $(T^F)^\vee$ for the set of characters of T^F.

Proposition 1.16. *For any $s \in G^F_{\mathrm{ss}}$, we have*

$$\frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot R_T^G(\theta) = |(G^F)_s| \cdot \mathbb{1}_{[s]}.$$

Note that $(G^F)_s$ denotes the centralizer of s in G^F (this might differ from $(G_s)^F$ when G_s is disconnected!).

Proof. We put $\mu := \text{LHS}$ and $\mu' := \text{RHS}$. To show that $\mu = \mu'$, it is enough to check that $\langle \mu - \mu', \mu - \mu' \rangle = 0$. For this, it suffices to show that all of $\langle \mu, \mu \rangle$, $\langle \mu, \mu' \rangle$, and $\langle \mu', \mu' \rangle$ are equal.

Let us first compute $\langle \mu, \mu \rangle$. By using the inner product formula, we get

$$\begin{aligned}\langle \mu, \mu \rangle &= \frac{1}{\mathrm{St}_G(s)^2} \sum_{\substack{s \in T \in \mathcal{T}_G \\ s \in T' \in \mathcal{T}_G}} \sum_{\substack{\theta \in (T^F)^\vee \\ \theta' \in (T'^F)^\vee}} \theta(s)^{-1} \overline{\theta'(s)^{-1}} \cdot \langle R_T^G(\theta), R_{T'}^G(\theta') \rangle \\ &= \frac{1}{\mathrm{St}_G(s)^2} \sum_{\substack{s \in T \in \mathcal{T}_G \\ s \in T' \in \mathcal{T}_G}} \sum_{\substack{\theta \in (T^F)^\vee \\ \theta' \in (T'^F)^\vee}} \theta(s)^{-1} \overline{\theta'(s)^{-1}} \cdot |\{ w \in W_{G^F}(T, T') \mid \theta' = {}^w\theta \}|.\end{aligned}$$

Here, we change the index sets by noting the following bijection:

$$\begin{aligned}&\{ ((T, \theta), n) \in \mathcal{I} \times G^F \mid s \in T, s \in {}^nT \} \\ &\xrightarrow{1:1} \{ ((T, \theta), (T'\theta'), n) \in \mathcal{I} \times \mathcal{I} \times G^F \mid s \in T, s \in T', n \in N_{G^F}(T, T'), \theta' = {}^n\theta \} \\ &\qquad\qquad : ((T, \theta), n) \mapsto ((T, \theta), ({}^nT, {}^n\theta), n)\end{aligned}$$

where we put $\mathcal{I}$ to be the set of pairs (T, θ) of $T \in \mathcal{T}_G$ and $\theta \in (T^F)^\vee$. Then the above sum equals

$$\frac{1}{\mathrm{St}_G(s)^2 \cdot |T^F|} \sum_{\substack{((T,\theta),n) \in \mathcal{I} \times G^F \\ s \in T \\ s \in {}^nT}} \theta(s)^{-1} \cdot {}^n\theta(s).$$

We note that the sum of $\theta(s)^{-1n} \cdot \theta(s) = \theta(s^{-1}n^{-1}sn)$ over all characters θ of T^F is zero when $s^{-1}n^{-1}sn \neq 1$ and equal to $|T^F|$ when $s^{-1}n^{-1}sn = 1$ (equivalently, $n \in (G^F)_s$). Therefore, the above equals

$$\frac{1}{\mathrm{St}_G(s)^2 \cdot |T^F|} \sum_{\substack{(T,n) \in \mathcal{T}_G \times (G^F)_s \\ s \in T}} |T^F| = \frac{|(G^F)_s|}{\mathrm{St}_G(s)^2} \cdot |\{ T \in \mathcal{T}_G \mid s \in T \}|.$$

We note that $\mathrm{St}_G(s) = (-1)^{r_G - r_{(G_s)^\circ}} \mathrm{St}_{G_s^\circ}(1)$ and also that $s \in T$ if and only if $T \subset G_s^\circ$ (hence $\{ T \in \mathcal{T}_G \mid s \in T \}$ is nothing but $\mathcal{T}_{G_s^\circ}$). Then, by using the fact that $|\mathcal{T}_{G_s^\circ}| = \mathrm{St}_{G_s^\circ}(1)^2$ (see [Car85, Theorem 3.4.1]), we arrive at

$$\langle \mu, \mu \rangle = |(G^F)_s|.$$

Let us next compute $\langle \mu, \mu' \rangle$. We note that, for any class function $f \in C(G)$, we have $\langle f, \mathbb{1}_{[s]} \rangle = f(s)$. Indeed,

$$\langle f, \mathbb{1}_{[s]} \rangle = \frac{1}{|G^F|} \sum_{g \in G^F \cdot s} f(g) \cdot |(G^F)_s| = \frac{|G^F \cdot s| \cdot |(G^F)_s|}{|G^F|} f(s) = f(s).$$

Keeping this in mind, by using the Deligne–Lusztig character formula on semisimple elements, we get

$$
\begin{aligned}
\langle \mu, \mu' \rangle &= \mu(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot R_T^G(\theta)(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot \frac{(-1)^{r_{G_s^\circ} - r_T}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \sum_{\substack{x \in G^F \\ x^{-1} s x \in T^F}} \theta^x(s) \\
&= \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} \sum_{\substack{x \in G^F \\ x^{-1} s x \in T^F}} \frac{(-1)^{r_G - r_{G_s^\circ}}}{|T^F| \cdot \mathrm{St}_{G_s^\circ}(1)} \theta(s^{-1} x^{-1} s x).
\end{aligned}
$$

Here we carry out a similar argument to the previous computation; the sum of $\theta(s^{-1}x^{-1}sx)$ over all characters θ of T^F is zero when $s^{-1}x^{-1}sx = 1$ and equal to $|T^F|$ when $s^{-1}x^{-1}sx = 1$ (equivalently, $x \in (G^F)_s$). Hence the above equals

$$
\frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \frac{(-1)^{r_G - r_{G_s^\circ}}}{\mathrm{St}_{G_s^\circ}(1)} \cdot |(G^F)_s|.
$$

Again noting that the index set is equal to $\mathcal{T}_{G_s^\circ}$ and that $\mathrm{St}_G(s) = (-1)^{r_G - r_{(G_s)^\circ}} \mathrm{St}_{G_s^\circ}(1)$, we conclude

$$
\langle \mu, \mu' \rangle = |(G^F)_s|
$$

by using [Car85, Theorem 3.4.1].

Let us finally compute $\langle \mu', \mu' \rangle$:

$$
\langle \mu', \mu' \rangle = \mu'(s) = |(G^F)_s|.
$$

□

Corollary 1.17. *Let ρ be an irreducible representation of G^F. For any $s \in G^F_{\mathrm{ss}}$, we have*

$$
\Theta_\rho(s) = \frac{1}{\mathrm{St}_G(s)} \sum_{s \in T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \theta(s)^{-1} \cdot \langle \rho, R_T^G(\theta) \rangle.
$$

Proof. As noted in the proof of the previous proposition, we have $\Theta_\rho(s) = \langle \Theta_\rho, \mu' \rangle$ with the notation as there. By the proposition, we get $\Theta_\rho(s) = \langle \Theta_\rho, \mu \rangle$; this is nothing but the right-hand side of the asserted equality. □

Theorem 1.18 (Exhaustion theorem)**.** *For any irreducible representation ρ of G^F, there exists a k-rational maximal torus T of G and its character θ such that ρ is contained in $R_T^G(\theta)$.*

Proof. Apply the previous corollary to $s = 1$; then we get

$$
\Theta_\rho(1) = \frac{1}{\mathrm{St}_G(1)} \sum_{T \in \mathcal{T}_G} \sum_{\theta \in (T^F)^\vee} (-1)^{r_G - r_T} \cdot \langle \rho, R_T^G(\theta) \rangle.
$$

The left-hand side is the dimension of ρ, hence not zero. Thus the right-hand side is also not zero. In particular, $\langle \rho, R_T^G(\theta) \rangle$ must be nonzero for at least one (T, θ). □

References

[Car85] R. W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication.

7:59am, November 5, 2024