

1. WEEK 6: DELIGNE–LUSZTIG REPRESENTATIONS

1.1. Quick overview of étale cohomology. In the following, we quickly introduce the basic properties of the étale cohomology for algebraic varieties. (Here, we do not even give the definition of the étale cohomology. Carter’s book [Car85, Appendix] has a beautiful summary of the étale cohomology theory, so please look at it if you want to know more about some details.)

Let us briefly recall the notion of ℓ -adic numbers. Let ℓ be a prime number. We consider the inverse system of finite rings

$$\cdots \rightarrow \mathbb{Z}/\ell^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/\ell^n\mathbb{Z} \rightarrow \cdots \rightarrow \mathbb{Z}/\ell^2\mathbb{Z} \rightarrow \mathbb{Z}/\ell\mathbb{Z},$$

where the transition map $\mathbb{Z}/\ell^{n+1}\mathbb{Z} \rightarrow \mathbb{Z}/\ell^n\mathbb{Z}$ is given by the natural surjection. The inverse limit of this system forms a ring, which is called the *ring of ℓ -adic integers* and denoted by \mathbb{Z}_ℓ :

$$\mathbb{Z}_\ell := \varprojlim_n \mathbb{Z}/\ell^n\mathbb{Z} := \{(x_n)_n \in \prod_{n \geq 1} \mathbb{Z}/\ell^n\mathbb{Z} \mid \overline{x_{n+1}} = x_n\}.$$

Since \mathbb{Z}_ℓ is an integral domain, it makes sense to consider its fractional field; it is called the *field of ℓ -adic numbers* and denoted by \mathbb{Q}_ℓ .¹

Lemma 1.1. *Let $\overline{\mathbb{Q}_\ell}$ be an algebraic closure of \mathbb{Q}_ℓ . Then $\overline{\mathbb{Q}_\ell}$ is isomorphic to the complex number field \mathbb{C} as an abstract field.*²

Exercise 1.2. Prove this lemma. Hint: note that both $\overline{\mathbb{Q}_\ell}$ and \mathbb{C} are algebraically closed fields of characteristic 0 and the same cardinality.

Now let k be a finite field \mathbb{F}_q of characteristic $p > 0$. In the following, let ℓ be a prime number distinct to p . For any algebraic variety X over $\bar{k} = \overline{\mathbb{F}_p}$ and for each $i \in \mathbb{Z}_{\geq 0}$, we can associate a $\overline{\mathbb{Q}_\ell}$ -vector space $H_c^i(X, \overline{\mathbb{Q}_\ell})$ called the *compactly supported (i -th) étale cohomology of X with $\overline{\mathbb{Q}_\ell}$ -coefficient*. In this course, we simply refer to it by the *ℓ -adic cohomology of X* .³

It is known that $H_c^i(X, \overline{\mathbb{Q}_\ell})$ satisfies various “basic” properties. For a moment, let us introduce only the following:

Theorem 1.3. (1) *For any X , $H_c^i(X, \overline{\mathbb{Q}_\ell})$ is finite-dimensional.*
(2) *For any X , $H_c^i(X, \overline{\mathbb{Q}_\ell}) \neq 0$ only for $0 \leq i \leq 2 \dim(X)$.*
(3) *For any proper⁴ morphism of algebraic varieties $f: X \rightarrow Y$ over \bar{k} , a $\overline{\mathbb{Q}_\ell}$ -vector space homomorphism $f^*: H_c^i(Y, \overline{\mathbb{Q}_\ell}) \rightarrow H_c^i(X, \overline{\mathbb{Q}_\ell})$ is canonically (functorially) associated (for each i).*

For references on these facts, see [Car85, Section 7.1].

Now suppose that X is an algebraic variety over k . Then we have the Frobenius endomorphism $F: X_{\bar{k}} \rightarrow X_{\bar{k}}$. Thus, by the functoriality, we also have an endomorphism F^* of $H_c^i(X, \overline{\mathbb{Q}_\ell})$.

¹Another equivalent way of defining \mathbb{Q}_ℓ is to complete the rational number field \mathbb{Q} with respect to the ℓ -adic distance. But the above definition seems better in this context because the ℓ -adic cohomology is defined by taking the limit of torsion coefficient ($\mathbb{Z}/\ell^n\mathbb{Z}$ -coefficient) cohomologies.

²Note that, however, $\overline{\mathbb{Q}_\ell}$ and \mathbb{C} cannot be topologically isomorphic.

³There is also the “(i -th) étale cohomology of X with $\overline{\mathbb{Q}_\ell}$ -coefficient”, so this terminology is a bit too abbreviated. But we do not mind because we only use the compactly supported one in this course.

⁴Here, we don’t give the definition of the properness of a morphism of algebraic varieties. We only note that any isomorphism is proper and also a Frobenius endomorphism is also proper.

Theorem 1.4 (Grothendieck–Lefschetz fixed point theorem). *We have*

$$|X^F| = \sum_{i \geq 0} (-1)^i \operatorname{Tr}(F^* | H_c^i(X, \overline{\mathbb{Q}}_\ell)).$$

One of the important application of the fixed point theorem is the following ℓ -independence result: Suppose that X is furthermore equipped with an action of a finite group G . Then, by the functoriality of ℓ -adic cohomology, we obtain a representation of G on a finite-dimensional $\overline{\mathbb{Q}}_\ell$ -vector space $g \mapsto (g^{-1})^*$. (Here it is better to take the inverse of g since $(-)^*$ is contravariant.) By abuse of notation, let us simply write “ g ” for the action $(g^{-1})^*$ on $H_c^i(X, \overline{\mathbb{Q}}_\ell)$.

Theorem 1.5. *Suppose that an element $g \in G$ satisfies $g \circ F = F \circ g$ as an endomorphism of $X_{\overline{k}}$. Then the number*

$$\sum_{i \geq 0} (-1)^i \operatorname{Tr}(g | H_c^i(X, \overline{\mathbb{Q}}_\ell))$$

is an integer independent of ℓ (called the “Lefschetz number” of g).

Proof. Here we need the fact that, for any $n \geq 1$, the endomorphism $g \circ F^n$ of $X_{\overline{k}}$ associated to another \mathbb{F}_{q^n} -rational structure of $X_{\overline{k}}$. Let us write X_n for the algebraic variety over \mathbb{F}_{q^n} determined by this rational structure. Then $X^{g \circ F^n}$ is the set of \mathbb{F}_{q^n} -rational points of X_n , hence finite.

We first investigate the following formal series:

$$R(t) := - \sum_{n=1}^{\infty} |X^{g \circ F^n}| \cdot t^n \in \mathbb{Z}[[t]] \subset \overline{\mathbb{Q}}_\ell((t)).$$

Since g and F^* are commuting endomorphism of $V := \bigoplus_{i \geq 0} H_c^i(X, \overline{\mathbb{Q}}_\ell)$ (note that this is finite-dimensional), we can simultaneously triangulate g and F^* . Let v_1, \dots, v_k be a set of simultaneous eigenvectors ($d := \dim V$) with eigenvalues $\alpha_1, \dots, \alpha_d \in \overline{\mathbb{Q}}_\ell$ for g^* and $\beta_1, \dots, \beta_d \in \overline{\mathbb{Q}}_\ell$ for F^* . Here, we may assume that each v_j is contained in $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ for some i . For each $j = 1, \dots, k$, we define a sign ϵ_j by

$$\epsilon_j := \begin{cases} 1 & \text{if } v_j \text{ is contained in an even degree cohomology,} \\ -1 & \text{if } v_j \text{ is contained in an odd degree cohomology.} \end{cases}$$

Then, by applying the fixed point formula to X_n over \mathbb{F}_{q^n} , we get

$$|X^{g \circ F^n}| = \sum_{j=1}^d \epsilon_j \alpha_j \beta_j^n.$$

Therefore, we get

$$\begin{aligned} R(t) &= - \sum_{n=1}^{\infty} |X^{g \circ F^n}| \cdot t^n = - \sum_{n=1}^{\infty} \sum_{j=1}^d \epsilon_j \alpha_j \beta_j^n \cdot t^n \\ &= - \sum_{j=1}^d \epsilon_j \alpha_j \sum_{n=1}^{\infty} \beta_j^n \cdot t^n \\ &= - \sum_{j=1}^d \epsilon_j \alpha_j \frac{\beta_j t}{1 - \beta_j t} \in \overline{\mathbb{Q}}_\ell(t). \end{aligned}$$

In particular, $R(t)$ is a rational function which does not have a pole at $t = \infty$. Let us write $R(t) = p(t)/q(t)$ with polynomials $p(t), q(t) \in \overline{\mathbb{Q}}_\ell[t]$; then, by noting that $R(t)$ is initially given by a formal series with \mathbb{Z} -coefficients, we can easily check that the coefficients of $p(t)$ and $q(t)$ can be taken to be in \mathbb{Q} . In other words, we have $R(t) \in \mathbb{Q}(t)$.

On the other hand, we note that $R(\infty)$ is given by $\sum_{j=1}^d \epsilon_j \alpha_j$, which is nothing but $\sum_{i \geq 0} (-1)^i \text{Tr}(g \mid H_c^i(X, \overline{\mathbb{Q}}_\ell))$. Since $R(t)$ is independent of ℓ (by its definition) and belongs to $\mathbb{Q}(t)$, we have that $\sum_{i \geq 0} (-1)^i \text{Tr}(g \mid H_c^i(X, \overline{\mathbb{Q}}_\ell))$ is a rational number which is independent of ℓ . Moreover, since g is of finite order, $\alpha_j \in \overline{\mathbb{Q}}_\ell$ also must be of finite order. In particular, $\sum_{i \geq 0} (-1)^i \text{Tr}(g \mid H_c^i(X, \overline{\mathbb{Q}}_\ell)) = \sum_{j=1}^d \epsilon_j \alpha_j$ is an algebraic integer. As $\mathbb{Q} \cap \overline{\mathbb{Z}} = \mathbb{Z}$, we get $\sum_{i \geq 0} (-1)^i \text{Tr}(g \mid H_c^i(X, \overline{\mathbb{Q}}_\ell)) \in \mathbb{Z}$. \square

We let $\mathcal{L}(g, X)$ denote the Lefschetz number of g .

1.2. Deligne–Lusztig representation. In the following, we let k be a finite field \mathbb{F}_q of characteristic $p > 0$. We fix a prime number $\ell \neq p$ and also fix an isomorphism $\iota: \overline{\mathbb{Q}}_\ell \xrightarrow{\cong} \mathbb{C}$. Let G be a connected reductive group over k .

Recall that, for any k -rational maximal torus T of G and a Borel subgroup B containing T^5 , the Deligne–Lusztig variety $\mathcal{X}_{T \subset B}^G$ is defined; this is an algebraic variety over \overline{k} equipped with an action of $G^F \times T^F$. Therefore, its ℓ -adic cohomology $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)$ is a finite-dimensional representation (on a $\overline{\mathbb{Q}}_\ell$ -vector space) of $G^F \times T^F$.

Now suppose that $\theta: T^F \rightarrow \mathbb{C}^\times$ is a character. Then, through the fixed isomorphism ι , we may regard θ as a $\overline{\mathbb{Q}}_\ell^\times$ -valued character of T^F . Let us write $\theta_\iota := \iota^{-1} \circ \theta: T^F \rightarrow \overline{\mathbb{Q}}_\ell^\times$. Then it makes sense to consider the θ_ι -isotypic part $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota]$ of $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)$, which is a finite-dimensional representation of G^F on a $\overline{\mathbb{Q}}_\ell$ -vector space.

Definition 1.6. We call the alternating sum of $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota]$ the *Deligne–Lusztig (virtual) representation of G^F associated to (T, θ_ι)* and write $R_T^G(\theta_\iota)$ for it:

$$R_{T \subset B}^G(\theta_\iota) := \sum_{i \geq 0} (-1)^i H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota].$$

By abuse of notation, we also write $R_{T \subset B}^G(\theta_\iota)$ for the character of the Deligne–Lusztig (virtual) representation (called *Deligne–Lusztig (virtual) character*).

Remark 1.7. Let us say a bit more about the notion of the θ_ι -isotypic part $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota]$. By definition, it is the maximal subspace of $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)$ whose action of T^F is given by θ_ι , i.e., $t \cdot v = \theta_\iota(t)v$ for any $t \in T^F$ and $v \in H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)$ (such a subspace always uniquely exists since any representation of T^F on a finite-dimensional vector space is semisimple). More explicitly, $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota]$ is realized as the image of the following endomorphism of $H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)$:

$$\frac{1}{|T^F|} \sum_{t \in T^F} \theta_\iota(t)^{-1} \cdot t.$$

⁵Here, B is a subgroup of $G_{\overline{k}}$ which may not be defined over \overline{k} . So, precisely speaking, it might be better to write “a Borel subgroup B containing T_k^- ”.

Now let us discuss the ℓ -independence of the Deligne–Lusztig representation. At this point, the coefficients of the Deligne–Lusztig representation is taken to be $\overline{\mathbb{Q}}_\ell$ and its construction depends on $\iota: \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Hence, the Deligne–Lusztig character is also a class function on G^F valued in $\overline{\mathbb{Q}}_\ell$.

We note that the Deligne–Lusztig variety $\mathcal{X}_{T \subset B}^G$ might not be defined over k . However, there exists a finite extension k' of k such that $\mathcal{X}_{T \subset B}^G$ is defined over k' . Indeed, suppose that T splits over $k' = \mathbb{F}_{q^n}$. Then we can choose a Borel subgroup B containing T so that it is defined over k' . This is equivalent to that U satisfies $F^n(U) = U$. Hence, if $g \in G$ satisfies $g^{-1}F(g) \in F(U)$, then we have $F^n(g)^{-1}F(F^n(g)) = F^n(g^{-1}F(g)) \in F^n(F(U)) = F(U)$. In other words, $\mathcal{X}_{T \subset B}^G$ is a subset of G which is stable under F^n . Thus, by the Galois descent, $\mathcal{X}_{T \subset B}^G$ is defined over k' . Note that the Frobenius endomorphism of $\mathcal{X}_{T \subset B}^G$ associated to this k' -rational structure is given by F^n .

Now let us apply Theorem 1.5 to the action of $G^F \times T^F$ on $\mathcal{X}_{T \subset B}^G$. Any $(g, t) \in G^F \times T^F$ satisfies $(g, t) \circ F^n = F^n \circ (g, t)$. Indeed, for any $x \in \mathcal{X}_{T \subset B}^G$, we have

$$(g, t) \circ F^n(x) = gF^n(x)t = F^n(gxt) = F^n \circ (g, t)(x)$$

(note that g and t are fixed by F). In other words, the (g, t) -action on $\mathcal{X}_{T \subset B}^G$ satisfies the assumption of Theorem 1.5. Hence the Lefschetz number of (g, t) is an integer independent of ℓ :

$$\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G) := \sum_{i \geq 0} (-1)^i \operatorname{Tr}((g, t) \mid H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)) \in \mathbb{Z}.$$

Proposition 1.8. *For any $g \in G^F$, we have*

$$R_{T \subset B}^G(\theta_\iota)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta_\iota(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

Proof. By Remark 1.7, we have

$$\begin{aligned} R_{T \subset B}^G(\theta_\iota)(g) &= \sum_{i \geq 0} (-1)^i \operatorname{Tr}(g \mid H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)[\theta_\iota]) \\ &= \sum_{i \geq 0} (-1)^i \frac{1}{|T^F|} \sum_{t \in T^F} \theta_\iota(t)^{-1} \operatorname{Tr}((g, t) \mid H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta_\iota(t)^{-1} \sum_{i \geq 0} (-1)^i \operatorname{Tr}((g, t) \mid H_c^i(\mathcal{X}_{T \subset B}^G, \overline{\mathbb{Q}}_\ell)) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta_\iota(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G). \end{aligned}$$

□

Note that, though the isomorphism $\iota: \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can regard $R_{T \subset B}^G(\theta_\iota)$ as a \mathbb{C} -valued class function on G^F . By the above proposition, then its values is given by

$$\frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G),$$

which is independent of ℓ (and also of ι). Let us write $R_{T \subset B}^G(\theta)$ for the virtual representation/character of G^F with \mathbb{C} -coefficients obtained in this way.

Example 1.9. Let us present an example in the GL_2 -case without any justification. Recall that (Week 2) irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are constructed by two different kinds of inductions:

- (1) To any character χ of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, we can associate a principal series representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$.
- (2) To any character θ of $\mathbb{F}_{q^2}^\times$ satisfying $\theta^{q-1} \neq 1$, we can associate a cuspidal representation π_θ .

Also recall that (Week 5) G^F -conjugacy classes of k -rational maximal tori of a connected reductive group G over k can be classified by the F -conjugacy classes of Weyl group of G . When $G = \mathrm{GL}_2$, its Weyl group W is equal to $\mathfrak{S}_2 = \{1, s\}$ with trivial F -action. So there exist exactly two G^F -conjugacy classes of k -rational maximal tori of GL_2 :

- (1) The one T_1 corresponding to the trivial element $1 \in W$ is split; $T_1(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^2$. For any character χ of $T_1(\mathbb{F}_q)$, we have $R_{T_1 \subset B}^G(\chi) \cong \mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \chi$.
- (2) The other one T_s corresponding to the non-trivial element $s \in W$ is non-split; $T_s(\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times$. If we take a character θ of $T_s(\mathbb{F}_q)$ satisfying $\theta^{q-1} \neq 1$, then we have $R_{T_s \subset B}^G(\theta) \cong -\pi_\theta$.⁶⁷

1.3. Split case: principal series. Let us first investigate the Deligne–Lusztig representation in the case where G is split and T is a split maximal torus (“base torus”) T_0 . Then we can find a Borel subgroup B of G containing T which is defined over k . Let $\theta: T^F \rightarrow \mathbb{C}^\times$ be any character. Since B is equal to the semi-direct product of its unipotent radical U and T (T normalizes U), we have a natural surjective homomorphism $B \twoheadrightarrow B/U = T$. By inflating through this homomorphism, we can regard θ as a character of B^F . We define the *principal series representation of G^F (associated to θ)* to be $\mathrm{Ind}_{B^F}^{G^F} \theta$.

Proposition 1.10. *We have $R_{T \subset B}^G(\theta) \cong \mathrm{Ind}_{B^F}^{G^F} \theta$.*

Proof. We let \mathcal{B}^F denote the set of k -rational Borel subgroups of G . We note that any two k -rational Borel subgroups of G are G^F -conjugate; in particular, \mathcal{B}^F is a finite set. We define a morphism π from $\mathcal{X}_{T \subset B}^G$ to \mathcal{B}^F by

$$\pi: \mathcal{X}_{T \subset B}^G = \{g \in G \mid g^{-1}F(g) \in U\} \rightarrow \mathcal{B}^F; \quad g \mapsto gBg^{-1}$$

(note that $F(U)$ in the definition of $\mathcal{X}_{T \subset B}^G$ is equal to U since U is k -rational). This morphism is well-defined; indeed, if $g \in G$ satisfies $g^{-1}F(g) \in U$ (say $g^{-1}F(g) = u$), then we have

$$F(gBg^{-1}) = F(g)BF(g)^{-1} = guBu^{-1}g^{-1} = gBg^{-1}.$$

Hence gBg^{-1} is a k -rational Borel subgroup of G . Moreover, π is surjective. To check this, let us take a k -rational Borel subgroup B' of G . Then there exists an element $g \in G^F$ satisfying $B' = gBg^{-1}$. since $g^{-1}F(g) = 1 \in U$, g belongs to

⁶Note that the Deligne–Lusztig representation itself can be defined even if θ does not satisfy the condition $\theta^{q-1} \neq 1$.

⁷Here, a Borel subgroup B containing T_s cannot be taken to be the standard upper-triangular one.

$\mathcal{X}_{T \subset B}^G$ and satisfies $\pi(g) = B'$. Therefore, we obtain a disjoint union decomposition $\mathcal{X}_{T \subset B}^G$ into finite number of closed subvarieties:

$$\mathcal{X}_{T \subset B}^G = \bigsqcup_{B' \in \mathcal{B}^F} \pi^{-1}(B').$$

Recall that, $\mathcal{X}_{T \subset B}^G$ has an action of $G^F \times T^F$ given by $(x, t): g \mapsto xgt$. We introduce an action of $G^F \times T^F$ on \mathcal{B}^F by $(x, t): B' \mapsto xB'x^{-1}$. Then π is $G^F \times T^F$ -equivariant, i.e., $\pi((x, t) \cdot g) = (x, t) \cdot \pi(g)$. Note that the action of $G^F \times T^F$ permutes the closed subvarieties $\pi^{-1}(B')$ (for $B' \in \mathcal{B}^F$). The resulting action $G^F \times T^F$ of on the finite set $\{\pi^{-1}(B') \mid B' \in \mathcal{B}^F\}$ is transitive and the stabilizer of $\pi^{-1}(B)$ is given by $B^F \times T^F$. In this setting, we have that the class function

$$G^F \times T^F \rightarrow \mathbb{Z}: (g, t) \mapsto \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G)$$

is given by the induction of

$$B^F \times T^F \rightarrow \mathbb{Z}: (b, t) \mapsto \mathcal{L}((b, t), \pi^{-1}(B))$$

(This is a general fact which holds for the Lefschetz number of a variety equipped with a finite group action; see [Car85, Property 7.1.7]).

Hence, by Proposition 1.8, the Deligne–Lusztig character $R_{T \subset B}^G(\theta)$ is given by the induction of the following class function from B^F to G^F :

$$b \mapsto \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((b, t), \pi^{-1}(B)).$$

Let us compute $\mathcal{L}((b, t), \pi^{-1}(B))$. By recalling that $N_G(B) = B$, we see that $\pi^{-1}(B)$ is given by

$$\mathcal{X}_{T \subset B}^G \cap N_G(B) = \mathcal{X}_{T \subset B}^G \cap B = T^F U.$$

Note that each fiber of the quotient map $T^F U \twoheadrightarrow T^F U/U$ is isomorphic to U , which is furthermore isomorphic to an affine space $\mathbb{A}^{\dim U}$ (this is a general property of a unipotent group). In fact, it is known that such a map (“affine fibration”) does not change the Lefschetz number, i.e., $\mathcal{L}((b, t), \pi^{-1}(B)) = \mathcal{L}((b, t), T^F U/U)$ (see [Car85, Property 7.1.5]). Here, $B^F \times T^F$ acts on $T^F U/U$ in an obvious way, that is, $(b, t) \cdot sU = bstU$.

Now note that $T^F U/U = T^F U^F/U^F$ is a finite set. Thus $\mathcal{L}((b, t), T^F U^F/U^F)$ is equal to the cardinality of the set $(T^F U^F/U^F)^{(b, t)}$ of points of $T^F U^F/U^F$ fixed by (b, t) (see the exercise below). For any $sU^F \in T^F U^F/U^F$, we have $(b, t) \cdot sU^F = sU^F$ if and only if $bstU^F = sU^F$, which is equivalent to $b \in t^{-1}U^F$. This implies that the fixed points set $(T^F U^F/U^F)^{(b, t)}$ is empty if $b \notin t^{-1}U^F$ and equal to $T^F U^F/U^F$ if $b \in t^{-1}U^F$. Since $|T^F U^F/U^F| = |T^F|$, we get

$$\mathcal{L}((b, t), \pi^{-1}(B)) = \begin{cases} |T^F| & \text{if } b \in t^{-1}U^F, \\ 0 & \text{if } b \notin t^{-1}U^F. \end{cases}$$

Therefore, $R_{T \subset B}^G(\theta)$ is given by the induction of

$$b = su \mapsto \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((su, t), \pi^{-1}(B)) = \theta(s).$$

This means that $R_{T \subset B}^G(\theta)$ is the induction of the inflation of θ , i.e., $\text{Ind}_{B^F}^{G^F} \theta$. \square

Exercise 1.11. Prove the following claim:

Let X be a finite set (this can be regarded as a 0-dimensional algebraic variety $\bigsqcup_{x \in X} \operatorname{Spec} \bar{k}$). Suppose that g is an automorphism of X . Then we have $\mathcal{L}(g, X) = |X^g|$.

Hint:

- (1) Show that the Frobenius endomorphism F induced from the obvious k -rational structure $\bigsqcup_{x \in X} \operatorname{Spec} k$ is the identity of X .
- (2) Define a formal power series $R(t)$ in the same way as the proof of Theorem 1.5 and do the same argument.

REFERENCES

- [Car85] R. W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication.

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