1. Week 5: Deligne-Lusztig varieties

1.1. **Frobenius endomorphism.** In the following, we let $k = \mathbb{F}_q$. Note that then the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is a pro-cyclic group isomorphic to $\hat{\mathbb{Z}}$. This group has the Frobenius automorphism $F\colon \overline{\mathbb{F}}_q \to \overline{\mathbb{F}}_q; x \mapsto x^q$ as its (topological) generator.

Now let us suppose that X is an affine algebraic variety over k. Recall that, in our sense, this means that X is a scheme equipped with a morphism to $\operatorname{Spec} k$ such that its base change $X_{\overline{k}}$ to $\operatorname{Spec} \overline{k}$ corresponds to an algebraic variety in the classical sense. Let $k[X]$ be the coordinate ring of X, i.e., $X = \operatorname{Spec} k[X]$ (hence $X_{\overline{k}} = \operatorname{Spec} \overline{k}[X]$, where $\overline{k}[X] = k[X] \otimes_k \overline{k}$). We define a ring endomorphism F^* of $\overline{k}[X]$ by

$$F^*\colon k[X] \otimes_k \overline{k} \to k[X] \otimes_k \overline{k}; \quad f \otimes a \mapsto f^q \otimes a.$$

(Note that this is a well-defined ring homomorphism since k is of characteristic p!) By abuse of notation, we write $F\colon X_{\overline{k}} \to X_{\overline{k}}$ for the endomorphism of $X_{\overline{k}}$ induced by F^*. Naively, F can be thought of as the entry-wise q-th power map.

In the following (and actually, so far in this course), we often simply write "$g \in X$" to mean that $g \in X(\overline{k}) = X_{\overline{k}}(\overline{k})$. Then it makes sense to talk about the image $F(g)$ of g under the Frobenius morphism. Following the definition, we can easily check that the set of fixed points $X^F = X_{\overline{k}}(\overline{k})^F$ is nothing but $X(k)$.

We finally note that a closed subvariety $Y_{\overline{k}}$ of $X_{\overline{k}}$ is k-rational if and only if $Y_{\overline{k}}$ is stable under F; this fact is a special case of so-called the *Galois descent* (see [Spr09, 11.2]).

1.2. **Definition of a Deligne–Lusztig variety.** Let G be a connected reductive group over $k = \mathbb{F}_q$. Let $F\colon G_{\overline{k}} \to G_{\overline{k}}$ be the Frobenius endomorphism of G. (Note that F is compatible with the Hopf algebra structure of the coordinate ring of $G_{\overline{k}}$, hence F is a group endomorphism of $G_{\overline{k}}$.)

Definition 1.1 (Deligne–Lusztig variety)**.** Let T be a k-rational maximal torus of G. We take a Borel subgroup B of G containing T. Let U be the unipotent radical of B. We define an algebraic variety $\mathcal{X}^G_{T\subset B}$ (over $\overline{k}$) by

$$\mathcal{X}^G_{T\subset B} := \{g \in G \mid g^{-1}F(g) \in F(U)\}.$$

We call $\mathcal{X}^G_{T\subset B}$ the *Deligne–Lusztig variety associated to T (and B).*

Remark 1.2. Recall that a Borel subgroup of G is a maximal connected solvable closed subgroup of G. Since any subtorus of G is connected solvable and closed, we can always find a Borel subgroup B of G containing a given maximal torus T of G. But be careful that B might not be taken to be k-rational even when T is k-rational (hence U also may not be k-rational).

Let us fix a T in the following and shortly write $\mathcal{X}$ for $\mathcal{X}^G_{T\subset B}$.

First suppose that $g \in G^F$ and $x \in \mathcal{X}$. Then we have

$$(gx)^{-1}F(gx) = x^{-1}g^{-1}F(g)F(x) = x^{-1}g^{-1}gF(x) = x^{-1}F(x) \in F(U).$$

In other words, the element $gx \in G$ again belongs to $\mathcal{X}$. Thus we get an action of G^F on $\mathcal{X}$ by left multiplication.

Next suppose that $t \in T^F$ and $x \in \mathcal{X}$. Then we have

$$(xt)^{-1}F(xt) = t^{-1}x^{-1}F(x)F(t) = t^{-1}x^{-1}F(x)t \in t^{-1}F(U)t = F(U),$$

where we used that T normalizes $F(U)$ in the last equality. In other words, the element $xt \in G$ again belongs to $\mathcal{X}$. Thus we get an action of T^F on $\mathcal{X}$ by right multiplication.

Note that the actions of G^F and T^F on $\mathcal{X}$ obviously commute. Hence we get an action of the direct product group $G^F \times T^F$ on $\mathcal{X}$.

This observation is very important; by the functoriality, the étale cohomology of $\mathcal{X}$ also has an action of $G^F \times T^F$. In other words, we can construct a representation of $G^F \times T^F$. The aim of this course (Deligne–Lusztig theory) is to investigate the representations of G^F realized in this way through the geometry of $\mathcal{X}$.

1.3. **Classification of maximal tori.** Deligne–Lusztig varieties are determined by the choice of a k-rational maximal torus of G. Then, how many k-rational maximal tori does G have (up to k-conjugacy)? Let us investigate it (following [Car85, 3.3]).

We first note the following fact:

Proposition 1.3. *Any connected reductive group G over k possesses a k-rational Borel subgroup.* [1]

Let us fix a k-rational Borel subgroup B_0 of G. Let T_0 be a k-rational maximal torus of G contained in B_0. We call this maximal torus T_0 the "base torus" (this is our temporary terminology). We write $N_G(T_0)/T_0$ for the normalizer group of T_0 in G and $W_G(T_0) := N_G(T_0)/T_0$ for the *Weyl group* of T_0 in G. We often write W_0 for $W_G(T_0) := N_G(T_0)/T_0$ in short. Note that, since T_0 is k-rational, so is $N_G(T_0)$. Hence we have a natural action of F on W_0. We say that two elements $w_1, w_2 \in W_0$ are *F-conjugate* if there exists an element $v \in W_0$ satisfying $w_2 = vw_1F(v)^{-1}$. Note that this is an equivalence relation on W_0.

Now let T be a k-rational maximal torus of G. Recall that all maximal tori of G are conjugate (over $\overline{k}$). Thus let us choose an element $g \in G$ satisfying $T = {}^gT_0$, where ${}^g(-) := g(-)g^{-1}$. Since both T and T_0 are k-rational subgroups of G, T and T_0 are stable under F. Hence we get

$$ {}^{F(g)}T_0 = F({}^gT_0) = F(T) = T = {}^gT_0. $$

In particular, we have ${}^{g^{-1}F(g)}T_0 = T_0$. In other words, the element $g^{-1}F(g)$ belongs to the normalizer $N_G(T_0)$ of T_0 in G. We let w be the image of $g^{-1}F(g) \in N_G(T_0)$ in the Weyl group $W_G(T_0)$.

Lemma 1.4. *The F-conjugacy class of $w \in W_0$ is well-defined, i.e., independent of the choice of $g \in G$ satisfying ${}^gT_0 = T$. Moreover, two G^F-conjugate k-rational maximal tori of G give rise to the same F-conjugacy class of W_0.*

Proof. Suppose that $g_1, g_2 \in G$ are elements satisfying ${}^{g_1}T_0 = T$ and ${}^{g_2}T_0 = T$. Let w_1 and w_2 be the images of $g_1^{-1}F(g_1)$ and $g_2^{-1}F(g_2)$ in W_0, respectively.

As we have ${}^{g_1}T_0 = T = {}^{g_2}T_0$, we have $g_1^{-1}g_2 \in N_G(T_0)$. Hence, if we put v to be the image of $g_1^{-1}g_2$ in W_0, we get $w_2 = v^{-1}w_1F(v)$.

It is easy to check the latter assertion. $\square$

[1] In general, a connected reductive group G over k (any field) is said to be "quasi-split" if it has a k-rational Borel subgroup. The proposition says that any connected reductive group over $\mathbb{F}_q$ is quasi-split.

By this lemma, we see that the above procedure $T \mapsto w$ induces a well-defined map

$$\{k\text{-rational maximal tori of } G\}/G^F\text{-conj.} \to W_0/F\text{-conj.}$$

Proposition 1.5. *This map is bijective.*

To show this proposition, we introduce following famous fact, which is known as *Lang's theorem*.

Theorem 1.6 ([Spr09, 4.4.17])**.** *Let G be a connected algebraic group over $k = \mathbb{F}_q$. Then the map $G_{\overline{k}} \to G_{\overline{k}}\colon g \mapsto g^{-1}F(g)$ is surjective.*

Proof of Proposition 1.5. Let us first show the surjectivity. Let $w \in W_0$ and $n \in N_G(T_0)$ be any its representative. By Lang's theorem for G, we can find an element $g \in G$ satisfying $g^{-1}F(g) = n$. If we put $T := {}^gT_0$, then the condition $g^{-1}F(g) = n \in N_G(T_0)$ implies that T is F-stable. Hence T is k-rational.

Let us next show the injectivity. Suppose that T_1 and T_2 are k-rational maximal tori of G which give rise to the same F-conjugacy class of W_0. If we write $T_1 = {}^{g_1}T_0$ and $T_2 = {}^{g_2}T_0$, then we have $g_1^{-1}F(g_1) = n^{-1}g_2^{-1}F(g_2)F(n)t_0$ for some elements $n \in N_G(T_0)$ and $t_0 \in T_0$. By noting that $F(g_2)F(n)t_0 = tF(g_2)F(n)$ for an element t of T_2 and applying Lang's theorem for T_2 to t, we can find an element $s \in T_2$ satisfying $s^{-1}F(s) = t$. Hence we get

$$g_1^{-1}F(g_1) = n^{-1}g_2^{-1}s^{-1}F(s)F(g_2)F(n),$$

which implies that $F(sg_2ng_1^{-1}) = sg_2ng_1^{-1}$, i.e., $sg_2ng_1^{-1} \in G^F$. If we put g to be this element, then we have

$${}^gT_1 = {}^{gg_1}T_0 = {}^{sg_2n}T_0 = {}^sT_2 = T_2.$$

Hence T_1 and T_2 are G^F-conjugate. $\square$

In the following, for any element $w \in W_0$, let T_w denote a k-rational maximal torus of G corresponding to the F-conjugacy class of w. Let us describe the rational structure of T_w in terms of the base torus T_0. Let $g \in G$ be an element satisfying $T_w = {}^gT_0$. By replacing g with an element of $gN_G(T_0)$ if necessary, we may assume that the image of $g^{-1}F(g) \in N_G(T_0)$ in W_0 is exactly w. Then, the action of F on T_w is transferred to the composition of $\mathrm{Int}(w)$ and F on T_0 through the isomorphism $\mathrm{Int}(g)^{-1}\colon T_w \to T_0$:

$$\begin{tikzcd}
T_w \arrow[d, "F"'] & T_0 \arrow[l, "\mathrm{Int}(g)"'] & gtg^{-1} \arrow[d, mapsto] & & t \arrow[ll, mapsto] \\
T_w \arrow[r, "\mathrm{Int}(g)^{-1}"'] & T_0 & F(g)F(t)F(g)^{-1} \arrow[rr, mapsto] & & g^{-1}F(g)F(t)F(g)^{-1}g = \mathrm{Int}(w)\circ F(t)
\end{tikzcd}$$

Example 1.7. Let $G = \mathrm{GL}_n$. In this case, the base torus T_0 can be taken to be the diagonal maximal torus. Thus we have $T_0 \cong (\overline{\mathbb{F}}_q^{\times})^n$ (if we loosely identify T_0 with $T_0(\overline{\mathbb{F}}_q)$) and the action F on T_0 is given by

$$(t_1, t_2, \ldots, t_n) \mapsto (t_1^q, t_2^q, \ldots, t_n^q).$$

The Weyl group W_0 can be naturally identified with the subgroup of permutation matrices of GL_n, hence isomorphic to $\mathfrak{S}_n$.

(1) We first consider the case where $w \in \mathfrak{S}_n$ is trivial. In this case, T_w is nothing but T_0 itself. Hence $T_w^F = T_0^F \cong (\mathbb{F}_q^\times)^n$.
(2) We next consider the case where $w \in \mathfrak{S}_n$ is the cyclic permutation $(1\,2\,\ldots\,n)$ of length n (this element is so-called a "Coxeter element"). The action $\mathrm{Int}(w) \circ F$ on T_0 is explicitly written by
$$(t_1, t_2, \ldots, t_n) \mapsto (t_n^q, t_1^q \ldots, t_{n-1}^q).$$
Thus $(t_1, t_2, \ldots, t_n) \in T_0$ is fixed by $\mathrm{Int}(w) \circ F$ if and only if $(t_1, t_2, \ldots, t_n) = (t_n^q, t_1^q \ldots, t_{n-1}^q)$, which is equivalent to
$$(t_1, t_2, \ldots, t_n) = (t_1, t_1^q, \ldots, t_1^{q^{n-1}}) \quad \text{and} \quad t_1^{q^n} = t_1.$$
In other words, T_w^F is identified with $\mathbb{F}_{q^n}^\times$, hence is of order $q^n - 1$.
(3) We finally consider the general case. The Frobenius F acts on W_0 trivially, thus the F-conjugacy of W_0 is nothing but the usual conjugacy. Recall that the conjugacy classes of $\mathfrak{S}_n$ correspond to the partitions of n bijectively. Suppose that the conjugacy class of $w \in \mathfrak{S}_n$ corresponds to a partition $(n_1, n_2, \ldots, n_r)$ of n, where $n_1 \geq \cdots \geq n_r > 0$ and $n_1 + \cdots + n_r = n$. [2] Then, by a similar argument to (2), we can check that T_w^F is identified with $\mathbb{F}_{q^{n_1}}^\times \times \cdots \times \mathbb{F}_{q^{n_r}}^\times$. Hence the order of T_w^F is given by $(q^{n_1} - 1) \cdots (q^{n_r} - 1)$.

As demonstrated in the above example, it is not very difficult to describe k-rational maximal tori of G as long as the descriptions of the base torus T_0 and its Weyl group explicitly.

Let us finally mention a general proposition on the order of T_w. We first note that the actions of F and W_0 on $X^*(T_0)$ are induced as follows:
$$F(\chi)(t) := \chi(F(t)) \qquad \text{for any } \chi \in X^*(T_0),\ t \in T_0,$$
$$w(\chi)(t) := \chi(w^{-1}tw) \qquad \text{for any } \chi \in X^*(T_0),\ t \in T_0.$$
Similarly, the actions of F and W_0 on $X_*(T_0)$ are induced as follows:
$$F(\chi^\vee)(t) := F(\chi^\vee(t)) \qquad \text{for any } \chi^\vee \in X_*(T_0),\ t \in T_0,$$
$$w(\chi^\vee)(t) := w^{-1}\chi^\vee(t)w \qquad \text{for any } \chi^\vee \in X_*(T_0),\ t \in T_0.$$
Then it is a routine task to check that the maps on $X^*(T)$ and $X_*(T)$ induced by F in a similar way are identified with $F \circ w^{-1}$ and $w^{-1} \circ F$ on $X^*(T_0)$ and $X_*(T_0)$, respectively (see [Car85, Proposition 3.3.4]). This leads to the following (see [Car85, Proposition 3.3.5]):

Proposition 1.8. *The order of T_w^F is given by $|\det(w^{-1} \circ F - \mathrm{id} \mid X_*(T_0)_\mathbb{R})|$. More explicitly, if we write $F = qF_0$ (then F_0 is an automorphism of $X_*(T_0)_\mathbb{R}$ of finite order) and let $\chi(-)$ be the characteristic polynomial of $F_0^{-1} \circ w$ on $X_*(T_0)_\mathbb{R}$, then we have $T_w^F = \chi(q)$.*

Remark 1.9. Note that F_0 is the identity when G is split.

Exercise 1.10. Compute the order of T^F for all k-rational maximal tori T of Sp_{2n}.

[2] For example, the trivial permutation corresponds to $(1, \ldots, 1)$ and the cyclic permutation $(1\,2\,\ldots\,n)$ of length n corresponds to (n).

1.4. **Some variants.** Now we introduce of several variants of the Deligne–Lustig variety. Later (after next weeks), it will turn out that all of these variants are technically convenient. (The description given here follows [DL76, 1.18–1.20] and [Car85, 7.7].)

Let T be a k-rational maximal torus of G. As before, we take a Borel subgroup B of G containing T. Let U be the unipotent radical of B. Recall that

$$\mathcal{X}^G_{T\subset B} := \{g \in G \mid g^{-1}F(g) \in F(U)\}.$$

Note that $\mathcal{X}^G_{T\subset B}$ is also stable under the right multiplication by $U \cap F(U)$. We define algebraic varieties $\tilde{X}^G_{T\subset B}$ and $X^G_{T\subset B}$ (over $\overline{k}$) by

$$\begin{aligned} \tilde{X}^G_{T\subset B} &:= \{g \in G \mid g^{-1}F(g) \in F(U)\}/(U \cap F(U)) \\ X^G_{T\subset B} &:= \{g \in G \mid g^{-1}F(g) \in F(U)\}/T^F(U \cap F(U)). \end{aligned}$$

Then $\mathcal{X}^G_{T\subset B}$ is a G^F-equivariant $U \cap F(U)$-torsor over $\tilde{X}^G_{T\subset B}$ and $\tilde{X}^G_{T\subset B}$ is a G^F-equivariant T^F-torsor over $X^G_{T\subset B}$.

$$\mathcal{X}^G_{T\subset B} \xrightarrow{(U\cap F(U))\text{-torsor}} \tilde{X}^G_{T\subset B} \xrightarrow{T^F\text{-torsor}} X^G_{T\subset B}.$$

Now assume that T corresponds to $w \in W$. What we want to do in the following is to understand the above varieties in a more concrete language based on flag varieties. For this, again let us fix a k-rational Borel subgroup B_0 of G and a base torus $T_0 \subset B_0$. We define the variety $\mathcal{B}$ to be the quotient G/B_0 of G by B_0. (By a fundamental fact in the theory of algebraic groups, this is a projective variety.) Note that the $\overline{k}$-rational points of $\mathcal{B}_0$ can be identified with the set of all Borel subgroups of G via map $g \mapsto {}^gB_0$. This can be checked by using the following facts:

(1) all Borel subgroups of G are conjugate, and
(2) for any Borel subgroup B of G, we have $N_G(B) = B$.

We call $\mathcal{B} = G/B_0$ the *flag variety* of G.

Proposition 1.11. *We have bijections*

$$W_0 = N_G(T_0)/T_0 \overset{1:1}{\longleftrightarrow} B_0\backslash G/B_0 \overset{1:1}{\longleftrightarrow} G\backslash(\mathcal{B}\times\mathcal{B}).$$

Here, the first map is $n \mapsto BnB$ and the second map is $g \mapsto G(B_0, {}^gB_0)$. (The action of G on $\mathcal{B}\times\mathcal{B}$ is given by a diagonal conjugation, i.e., $g(B_1, B_2) = ({}^gB_1, {}^gB_2)$).

Proof. The bijectivity of the first map is known as the “Bruhat decomposition”. See, for example, [Spr09, 8.3]. The bijectivity of the second map can be checked again by the above-mentioned fundamental properties (1) and (2) of Borel subgroups. □

Let $O(w)$ denote the cell of $\mathcal{B}\times\mathcal{B}$ corresponding to $w \in W_0$ under the above identification; explicitly, this is given by $O(w) = G(B_0, {}^wB_0)$. When a pair of two Borel subgroup (B_1, B_2) belongs to $O(w)$, we say that B_1 and B_2 are *in relative position w*.

We define a set $X(w)$ to be the subset of $\mathcal{B}$ consisting of all Borel subgroups B of G such that B and $F(B)$ are in relative position w:

$$\begin{aligned} X(w) &:= \{gB_0 \in G/B_0 \mid ({}^gB_0, F({}^gB_0)) \in O(w)\} \\ &= \{gB_0 \in G/B_0 \mid g^{-1}F(g) \in B_0wB_0\}. \end{aligned}$$

Since $X(w)$ is locally closed in $\mathcal{B}$, $X(w)$ has a variety structure. We put $\tilde{\mathcal{B}} := G/U_0$; hence $\tilde{\mathcal{B}}$ is a T_0-torsor over $\mathcal{B}$. By choosing a representative $\dot{w} \in N_G(T_0)$ of $w \in W_0$, we define a similar subset $\tilde{X}(\dot{w})$ of $\tilde{\mathcal{B}}$ as follows:

$$\begin{aligned}\tilde{X}(\dot{w}) &:= \{gU_0 \in G/U_0 \mid F(gU_0) = gU_0\dot{w}\} \\ &= \{gU_0 \in G/U_0 \mid g^{-1}F(g) \in U_0\dot{w}U_0\}.\end{aligned}$$

Then the covering $\tilde{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ restricts to a covering $\tilde{X}(\dot{w}) \twoheadrightarrow X(w)$, which is G^F-equivariant. Let us compute the fiber of this map. Suppose that $gU_0 \in \tilde{X}(\dot{w})$, hence $gB_0 \in X(w)$. The fiber of $\tilde{\mathcal{B}} \twoheadrightarrow \mathcal{B}$ at gB_0 is simply given by $\{gtU_0 \mid t \in T_0\}$. It is not difficult to check that gtU_0 belongs to $\tilde{X}(\dot{w})$ if and only if $wF(t)w^{-1} \in U_0t$. By noting that both $wF(t)w^{-1}$ and t belong to T_0, this is furthermore equivalent to that $wF(t)w^{-1} = t$, i.e., $t \in T_0^{\mathrm{Int}(w)\circ F}$. (Indeed, $wF(t)w^{-1}t^{-1}$ must be an element of $T_0 \cap U_0 = \{1\}$.) Therefore, we conclude that

$$\tilde{X}(\dot{w}) \twoheadrightarrow X(w)$$

is a G^F-equivariant $T_0^{\mathrm{Int}(w)\circ F}$-torsor. We note that $T_0^{\mathrm{Int}(w)\circ F}$ is identified with T_w^F by the map $T_0^{\mathrm{Int}(w)\circ F} \to T_w^F : t \mapsto gtg^{-1}$.

All the relations between the varieties we introduced so far are summarized as follows:

Proposition 1.12. *Suppose that $T = T_w$ for a $w \in W$. Let $x \in G$ be an element such that $\dot{w} := x^{-1}F(x)$ belongs to $N_G(T_0)$ and lifts w (hence $T = {}^{x}T_0$). We take B to be ${}^{x}B_0$, hence $U = {}^{x}U_0$. Then the map $g \mapsto gx$ induces a bijection from the G^F-equivariant T^F-torsor $\tilde{X}^G_{T\subset B} \to X^G_{T\subset B}$ to the G^F-equivariant $T_0^{\mathrm{Int}(w)\circ F}$-torsor $\tilde{X}(\dot{w}) \to X(w)$ (T^F and $T_0^{\mathrm{Int}(w)\circ F}$ are identified under the map $t \mapsto g^{-1}tg$).*

$$\begin{tikzcd}
\mathcal{X}^G_{T\subset B} \arrow[d, two heads, "/(U\cap F(U))"'] & & \\
\tilde{X}^G_{T\subset B} \arrow[r, "\sim", "g\mapsto gx"'] \arrow[d, two heads, "/T^F"'] & \tilde{X}(\dot{w}) \arrow[r, hook] \arrow[d, two heads, "/T_0^{\mathrm{Int}(w)\circ F}"] & \tilde{\mathcal{B}} \arrow[d, two heads] \\
X^G_{T\subset B} \arrow[r, "\sim", "g\mapsto gx"'] & X(w) \arrow[r, hook] & \mathcal{B}
\end{tikzcd}$$

1.5. **Example: GL_n case.** Let us investigate the variety $\tilde{X}(\dot{w})$ in the case where $G = \mathrm{GL}_n$ and $w = (1\,2\,\ldots\,n) \in \mathfrak{S}_n$. Let T_0 be the diagonal maximal torus of G and B_0 the upper-triangular Borel subgroup of G.

Definition 1.13. Let V be a finite dimensional k-vector space. A *flag* of V is a sequence of subspaces $\mathcal{F} = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V)$. We say that a flag $\mathcal{F}$ is *complete* if $\dim V_i = i$.

Let $V := \mathbb{F}_q^{\oplus n}$ and $\{e_i\}_{i=1}^n$ be the standard basis of V (i.e., $e_1 = (1, 0, \ldots, 0)$ and so on). Let $\mathcal{F}_{\mathrm{std}}$ be the complete flag of V given by $V_i = \bigoplus_{j=1}^i \mathbb{F}_q e_j$. We call $\mathcal{F}_{\mathrm{std}}$ the *standard flag* of V. Note that the set of points of $\mathcal{B} = G/B_0$ parametrizes the complete flags of V. Indeed, G acts on the set of complete flags via natural multiplication, i.e., $g \cdot (V_0 \subsetneq \cdots \subsetneq V_n) := (g(V_0) \subsetneq \cdots \subsetneq g(V_n))$. It is easy to see that this action is transitive and that the stabilizer of $\mathcal{F}_{\mathrm{std}}$ is nothing but B_0.

Definition 1.14. Let V be a finite dimensional k-vector space. A *marked flag* of V is a flag $(0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_r = V)$ equipped with nonzero element $v_i \in V_i/V_{i-1}$ for each $1 \leq i \leq r$.

Note that the standard flag $\mathcal{F}_{\mathrm{std}}$ can be upgraded to a marked complete flag with mark $\{e_i \in V_i/V_{i-1}\}_{i=1}^n$. Then, similarly to above, we see that the set of points of $\tilde{\mathcal{B}} = G/U_0$ parametrizes the marked complete flags of V.

Recall that $O(w)$ parametrizes pairs of Borel subgroups of G whose relative position is w. Let (B, B') be a pair of Borel subgroups of G. Let $\mathcal{F}^{(\prime)} = (0 = V_0^{(\prime)} \subsetneq V_1^{(\prime)} \subsetneq \cdots \subsetneq V_n^{(\prime)} = V^{(\prime)})$ be the complete flag of V corresponding to $B^{(\prime)}$.

Exercise 1.15. Check that $(\mathcal{F}, \mathcal{F}')$ is in relative position w if and only if $(\mathcal{F}, \mathcal{F}')$ satisfies the following conditions:

$$\begin{cases} V_i + V_i' = V_{i+1} \quad (1 \leq i \leq n-1), \\ V_1 + V_{n-1}' = V. \end{cases}$$

Next recall that $X(w)$ parametrizes Borel subgroups B of G such that $(B, F(B))$ belongs to $O(w)$. By the above exercise, this is equivalent to that a complete flag $\mathcal{F} = (0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V)$ corresponding to B satisfies the following:

$$\begin{cases} V_i + F(V_i) = V_{i+1} \quad (1 \leq i \leq n-1), \\ V_1 + F(V_{n-1}) = V. \end{cases}$$

We now consider $\tilde{X}(\dot{w})$. Similarly to above, we can check that $\tilde{X}(\dot{w})$ parametrizes marked complete flags $(\mathcal{F}, \{v_1\}_{i=1}^n)$ satisfying

$$\begin{cases} v_{i+1} \equiv F(v_i) \pmod{V_i} \quad (1 \leq i \leq n-1), \\ v_1 \equiv F^n(v_1) \pmod{F(v_1), \ldots, F(v_{n-1})}. \end{cases}$$

Exercise 1.16. Check that this condition is equivalent to that

$$v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1) = F^n(v_1) \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1)$$

(and both sides are nonzero), which can be also written as

$$F(v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1)) = (-1)^{n-1} \cdot v_1 \wedge F(v_1) \wedge \cdots \wedge F^{n-1}(v_1).$$

Let us explicate this equality by writing $v_1 \in V$ via the standard basis as $v_1 = \sum_{i=1}^n x_i e_i$. Since F acts on V via q-th power on the coefficients, we have that $F^i(v_1) = \sum_{i=1}^n x_i^{q^i} e_i$. Therefore, the above equality is equivalent to that

$$\left(\det(x_i^{q^{j-1}})_{1 \leq i,j \leq n}\right)^q = (-1)^{n-1} \cdot \det(x_i^{q^{j-1}})_{1 \leq i,j \leq n}.$$

Since both sides are necessarily nonzero, this is equivalent to

$$(-1)^{n-1} \cdot \left(\det(x_i^{q^{j-1}})_{1 \leq i,j \leq n}\right)^{q-1} = 1.$$

This is quite close to (and more complicated than) the Drinfeld curve! In fact, $\tilde{X}(\dot{w})$ exactly generalizes the Drinfeld curve.

Exercise 1.17. Verify that $\tilde{X}(\dot{w})$ exactly coincides with the Drinfeld curve $\{(x, y \in \mathbb{A}^2_{\overline{F}_p}) \mid xy^q - x^q y = 1\}$ when $G = \mathrm{SL}_2$ and w is the Coxeter element, i.e., the unique nontrivial element of the Weyl group. (CAUTION: In the case of special linear groups, we cannot simply take the representatives of the Weyl group elements to be permutation matrices because of the determinant condition. In particular, $\dot{w}$ cannot taken to be $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Instead, for example, we can use $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. But then we get a nontrivial sign contribution to the defining equation of $\tilde{X}(\dot{w})$).

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