INTRODUCTION TO THE LOCAL LANGLANDS CORRESPONDENCE

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Contents

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1. WEEK 1: COURSE OVERVIEW

1.1. **Class field theory.**

Let us begin with the following very famous and classical theorem in elementary number theory.

Theorem 1.1. *The number of the solutions to the equation $x^2 - 2 = 0$ in $\mathbb{F}_p$ is given as follows:*

$$|\{x \in \mathbb{F}_p \mid x^2 - 2 = 0\}| = \begin{cases} 2 & \text{if } p \equiv 1, 7 \pmod 8, \\ 0 & \text{if } p \equiv 3, 5 \pmod 8, \\ 1 & \text{if } p = 2. \end{cases}$$

This theorem is called *the second supplement to the quadratic reciprocity law* (see, e.g., [Ser73, Chapter I, §3]). In fact, more generally, the general quadratic reciprocity law implies the following:

Theorem 1.2. *Let $a \in \mathbb{Z}$ be an integer. Then there exists a positive integer $N \in \mathbb{Z}_{>0}$ such that the number $|\{x \in \mathbb{F}_p \mid x^2 - a = 0\}|$ depends only on the modulo N of p.*

For example, Theorem 1.1 says that N can be taken to be 8 when $a = 2$.

Exercise 1.3. (1) Explain the statement of the quadratic reciprocity law.
(2) Determine the number N in Theorem 1.2 using the quadratic reciprocity law.

Next let us consider the equation $x^3 - 2 = 0$. Can we find a simple description of the numbers of the solutions to this equation in $\mathbb{F}_p$ like above? In fact, the answer is NO! More precisely, there does not exist a positive integer $N \in \mathbb{Z}_{>0}$ such that the number $\{x \in \mathbb{F}_p \mid x^3 - 2 = 0\}$ depends only on the modulo N of p.

What causes such a difference between the quadratic and the cubic cases? To explain it, let us think about how to prove Theorem 1.1 from a modern viewpoint based on algebraic number theory. (In the following, we appeal to some basics of algebraic number theory. But it's not a material necessary for this course. If you are not familiar with them, please try to feel just its flavor.)

Since the equality $|\{x \in \mathbb{F}_2 \mid x^2 - 2 = 0\}| = 1$ is obvious, let us suppose that p is an odd prime number. Then Theorem 1.1 is rephrased as follows:

$$\mathbb{F}_p \text{ has a square root of 2 if and only if } p \equiv \pm 1 \pmod 8.$$

Noting this, let us introduce the quadratic extension $K := \mathbb{Q}(\sqrt{2})$ of $\mathbb{Q}$ obtained by adding a square root $\sqrt{2}$ of 2. The ring of integer $\mathcal{O}_K$ in K is given by $\mathbb{Z}[\sqrt{2}]$. Because the quadratic extension $K/\mathbb{Q}$ is unramified outside 2, any odd prime number p has only the following two possibilities about the ideal $p\mathcal{O}_K$ of $\mathcal{O}_K$ generated by p:

- $p\mathcal{O}_K$ is a prime (maximal) ideal of $\mathcal{O}_K$ (p "inerts" in K), or
- $p\mathcal{O}_K$ is the product $\mathfrak{p}_1\mathfrak{p}_2$ of two different prime ideals $\mathfrak{p}_1$ and $\mathfrak{p}_2$ of $\mathcal{O}_K$ (p "splits completely" in K).

Let us look at the quotient ring $\mathcal{O}_K/p\mathcal{O}_K$. This ring is

- a field if p inerts in K, and
- the product of two fields ($\mathcal{O}_K/\mathfrak{p}_1$ and $\mathcal{O}_K/\mathfrak{p}_2$) if p splits completely in K.

On the other hand,

$$\begin{aligned} \mathcal{O}_K/p\mathcal{O}_K = \mathbb{Z}[\sqrt{2}]/p\mathbb{Z}[\sqrt{2}] &\cong \big(\mathbb{Z}[x]/(x^2-2)\big)/p\big(\mathbb{Z}[x]/(x^2-2)\big) \\ &\cong \mathbb{F}_p[x]/(x^2-2). \end{aligned}$$

The right-hand side is

- a field (a quadratic extension of $\mathbb{F}_p$) if $\mathbb{F}_p$ does not have a square root of 2, and
- the product of two fields (both $\mathbb{F}_p$) if $\mathbb{F}_p$ has a square root of 2.

Hence, in summary, we see that

$$\mathbb{F}_p \text{ has a square root of 2 if and only if } p \text{ splits completely in } K.$$

Recall that each odd prime number p gives rise to a special element Frob_p of $\mathrm{Gal}(K/\mathbb{Q})$, called *Frobenius element* (again note that $K/\mathbb{Q}$ is unramified outside 2). The important property of the Frobenius is that it knows whether p splits completely or not. More precisely,

$$p \text{ splits completely in } K \text{ if } \mathrm{Frob}_p = \mathrm{id}.$$

So, our task is now reduced to investigate when $\mathrm{Frob}_p = \mathrm{id}$.

In fact, the argument so far can be carried out in general (e.g., for $x^3 - 2 = 0$ by replacing K with the smallest factorization field of $x^3 - 2 = 0$) more or less. But here we reach the stage where a special nature of the equation $x^2 - 2 = 0$ comes into play. The point is that the quadratic extension $K/\mathbb{Q}$ is abelian, i.e., its Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is abelian. In general, by the Kronecker–Weber theorem, any abelian extension of $\mathbb{Q}$ is contained in a cyclotomic field $\mathbb{Q}(\mu_N)$ (μ_N denotes the set of N-th roots of unity). The Galois group of $\mathbb{Q}(\mu_N)/\mathbb{Q}$ is given by $(\mathbb{Z}/N\mathbb{Z})^\times$; by choosing a primitive N-th root ζ_N of unity, it is described as follows:

$$\mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times : \quad [\zeta_N \mapsto \zeta_N^i] \mapsto i.$$

Under this identification, the Frobenius element Frob_p on the left-hand side is mapped to $p \in (\mathbb{Z}/N\mathbb{Z})^\times$ on the right-hand side (as long as p is unramified, which is equivalent to that p does not divide N).

In our situation, actually we have $\mathbb{Q}(\sqrt{-2}) \subset \mathbb{Q}(\mu_8)$. More precisely, under the Galois theory, $\mathbb{Q}(\sqrt{-2})$ is the subfield of $\mathbb{Q}(\mu_8)$ corresponding to the subgroup $\{\pm 1\}$ of $\mathrm{Gal}(\mathbb{Q}(\mu_8)/\mathbb{Q}) \cong (\mathbb{Z}/8\mathbb{Z})^\times$. Hence the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is identified with the quotient of $(\mathbb{Z}/8\mathbb{Z})^\times$ by $\{\pm 1\}$. Thus we conclude that

$$\mathrm{Frob}_p = \mathrm{id} \text{ if and only if } p \equiv \pm 1 \pmod 8.$$

Hence this completes the proof of Theorem 1.1.

The classical class field theory enables us to do a similar thing for more general number fields (finite extensions of $\mathbb{Q}$).

Theorem 1.4 (class field theory)**.** *Let F be a number field. Let F^{ab} be the maximal abelian extension of F. Then there exists a natural surjective continuous homomorphism*

$$\mathrm{Art}_F \colon \mathbb{A}_F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F),$$

which kernel is explicitly described.

Here, I do not explain the meaning of "natural" (it is formulated as the compatibility with the local class field theory, which will be explained later) nor even what "$\mathbb{A}_F$" on the source of the map is. But I just want to emphasize that this "$\mathbb{A}_F$" (which is called the adele ring of F) is defined only using the intrinsic data of the original object F. So, class field theory describes how the field F extends to a larger abelian field only by appealing to the internal data of F, which is much easier to grasp. For example, when $F = \mathbb{Q}$, the map Art_F exactly gives rise to the above-mentioned isomorphism $(\mathbb{Z}/N\mathbb{Z})^\times \cong \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q})$ by taking an appropriate finite quotient.

If we try to imitate the above discussion in the case of the equation $x^3 - 2 = 0$, we immediately notice that the last part does not work because the smallest splitting field $\mathbb{Q}(\sqrt[3]{2}, \mu_3)$ of the equation $x^3 - 2 = 0$ is not abelian over $\mathbb{Q}$; its Galois group is given by $\mathfrak{S}_3$.

1.2. **What is the Langlands correspondence?** Then, is it impossible to find any beautiful law on the behavior of the number $|\{x \in \mathbb{F}_p \mid x^3 - 2 = 0\}|$ over prime numbers p? In fact, the following holds:

Theorem 1.5. *We let $\sum_{n=1}^{\infty} a_n q^n$ be the infinite series given by the following infinite product:*

$$q \cdot \prod_{n=1}^{\infty} (1 - q^{6n}) \cdot (1 - q^{18n}) = \sum_{n=1}^{\infty} a_n q^n.$$

Then, for any prime number $p \neq 2, 3$, we have

$$|\{x \in \mathbb{F}_p \mid x^3 - 2 = 0\}| = 1 + a_p.$$

(See, e.g., [DS05, Section 4.11] for the more general case of $x^3 - a = 0$.)

Let us also introduce a different, but similar, phenomenon. We consider the following equation:

$$E\colon y^2 + y = x^3 - x^2.$$

The set of solutions of this equation forms a curve, which is called an *elliptic curve*. Let us think about the solutions in $\mathbb{F}_p$:

$$E(\mathbb{F}_p) := \{(x, y) \in \mathbb{F}_p \times \mathbb{F}_p \mid y^2 + y = x^3 - x^2\}.$$

Note that, in this case, the equation is not one-variable. So we do not even have a simple interpretation of the set $E(\mathbb{F}_p)$ in terms of field extensions of $\mathbb{Q}$. (In the case of $x^3 - 2 = 0$, although we cannot apply the class field theory, we can still relate the number $|\{x \in \mathbb{F}_p \mid x^3 - 2 = 0\}|$ to how p decomposes into prime ideals in the smallest splitting field of $x^3 - 2 = 0$.) Nevertheless, we have the following:

Theorem 1.6. *We let $\sum_{n=1}^{\infty} a_n q^n$ be the infinite series given by the following infinite product:*

$$\sum_{n=1}^{\infty} a_n q^n = q \cdot \prod_{n=1}^{\infty} (1 - q^n)^2 \cdot (1 - q^{11n})^2.$$

Then, for any prime number $p \neq 11$, we have

$$|E(\mathbb{F}_p)| = 1 + p - a_p.$$

In Theorems 1.5 and 1.6, by putting $q := \exp(2\pi i z)$ (for $z \in \mathbb{C}$), we may regard the infinite serieses as functions on the complex upper-half plane. In fact, they are examples of so-called "modular forms", which is a holomorphic function on the complex upper-half plane equipped with a lot of symmetry. Both elliptic curves and modular forms have been investigated in the context of number theory for a long time. A priori, they are totally different objects; elliptic curves are purely-algebraic while modular forms are purely-analytic, at least from the above descriptions. However, they are actually related in a surprising way as above.

All the phenomena introduced so far (Theorems 1.1, 1.5, 1.6) can be thought of as special cases of the *Langlands correspondence*. The Langlands correspondence is a vast, but conjectural, framework which connects two completely different mathematical objects: on the one hand are *automorphic representations* and on the other hand are *Galois representations*:

$$(\text{automorphic representations}) \quad \overset{\text{Langlands correspondence}}{\leftrightsquigarrow} \quad (\text{Galois representations})$$

Roughly speaking, an automorphic representation is an irreducible representation of $\mathrm{GL}_n(\mathbb{A}_F)$ realized in the space of automorphic forms, which are generalization of modular forms, and a Galois repersentaion is an n-dimensional continuous[1] representation of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$.

The important viewpoint here is not to look at the Galois group itself, but to consider representations of the Galois group. Recall that representation theory is a very strong tool (or even a modern "formulation") for studying non-abelian groups. For example, when $n = 1$, we have $\mathrm{GL}_1(\mathbb{A}_F) = \mathbb{A}_F^\times$; this implies an automorphic representation of $\mathrm{GL}_1(\mathbb{A}_F)$ is just a character of $\mathbb{A}_F^\times$. On the other hand, when the dimension of a Galois representation is 1, it must be a character, hence it necessarily factors through the maximal abelian quotient of $\mathrm{Gal}(\overline{F}/F)$, i.e., $\mathrm{Gal}(\overline{F}^{\mathrm{ab}}/F)$. Thus the Langlands correspondence in this case says that the characters of $\mathbb{A}_F^\times$ and $\mathrm{Gal}(\overline{F}^{\mathrm{ab}}/F)$ nicely correspond. This is exactly implied by the isomorphism $\mathbb{A}_F^\times \cong \mathrm{Gal}(\overline{F}^{\mathrm{ab}}/F)$ of class field theory.

When $n = 2$, the Shimura–Taniyama conjecture, which plays a crucial role in the proof of Fermart's conjecture, is also regarded as a special case of the Langlands correspondence. Theorem 1.6 is an example of the Shimura–Taniyama conjecture.

Other than these examples, It is known that various phenomena in number theory can be explained in a sophisticated way by appealing to the prediction of the Langlands correspondence. Therefore, one of the most important objectives in the modern number theory is to establish the Langlands correspondence.

Exercise 1.7. By looking at "LMFDB" (which is an online database of modular forms, elliptic curves, and so on), we can find a lot of examples of elliptic curves and modular forms which "correspond". For example, the elliptic curve and the modular form considered in Theorem 1.6 are labelled by "11.a3" and "11.2.a.a", respectively. I just randomly chose the following elliptic curve from this database: $y^2 + xy + y = x^3 - x$. Try to find the modular form corresponding to this elliptic curve using LMFDB (please explain how you arrive at it).

1.3. **Local-global principle in number theory.** Then, what is the "local" Langlands correspondence in the course title? To explain this, let us briefly talk about the philosophy of the local-global principle in number theory. Recall that the real number field $\mathbb{R}$ is the completion of the rational number field $\mathbb{Q}$ with respect to the normal metric on $\mathbb{Q}$. We note that $\mathbb{R}$ is not the only field obtained by such a procedure from $\mathbb{Q}$. Indeed, $\mathbb{Q}$ possesses non-trivial metrics other than the normal metric. For each fixed prime number p, if we put $|p^r \cdot \frac{n}{m}|_p := p^{-r}$ (here, n and m are integers prime to p), then $|\cdot|_p$ gives a well-defined metric on $\mathbb{Q}$ called the p-adic metric. If we complete $\mathbb{Q}$ with respect to the p-adic metric, we obtain a locally compact field different to $\mathbb{R}$, which is called the *p-adic number field* and denoted by $\mathbb{Q}_p$. The fundamental philosophy in number theory is that any problem on the rational number field $\mathbb{Q}$ should be able to be understood through its analog for $\mathbb{R}$ and $\mathbb{Q}_p$ for all prime numbers p; this is the idea of "local-global" in number theory.

$$\text{problem on } \mathbb{Q} \quad \xleftrightarrow{\text{local-global principle}} \quad \text{problems on } \mathbb{R} \text{ and } \mathbb{Q}_p \text{ (for all } p)$$

For example, the local analog of the class field theory is the *local class field theory*, which says that, for any p-adic field F (i.e., a finite extension of $\mathbb{Q}_p$), we have a natural injective

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[1] It is very important which kind of coefficient field/topology we adopt when we consider a representation of $\mathrm{Gal}(\overline{F}/F)$. But let us just ignore this subtlety here.

homomorphism

$$\mathrm{Art}_F\colon F^\times \to \mathrm{Gal}(F^{\mathrm{ab}}/F)$$

with dense image.

Both automorphic representations and Galois representations are objects related to the rational number field $\mathbb{Q}$ (or, more generally, any number field F). Thus it is natural to think about the analog of the Langlands correspondence for $\mathbb{R}$ or $\mathbb{Q}_p$ (or, more generally, any local field of characteristic zero, which means a finite extension of $\mathbb{R}$ or $\mathbb{Q}_p$); this is what is called the *local Langlands correspondence (LLC)*. This also generalized the local class field theory.

1.4. **What is the local Langlands correspondence?** Let us explain the LLC a bit more precisely. In the following, we let F be any p-adic field, i.e., a finite extension of $\mathbb{Q}_p$. The LLC is a natural correspondence between the set of "irreducible admissible representations" of $\mathrm{GL}_n(F)$ and the set of "n-dimensional Weil–Deligne representations":

$$(\text{irred. adm. repns. of } \mathrm{GL}_n(F)) \quad \overset{\text{LLC}}{\leftrightsquigarrow} \quad (n\text{-dim. WD repns.})$$

Here, roughly speaking,

- an *irreducible admissible representation* of $\mathrm{GL}_n(F)$ means an irreducible representation of the group $\mathrm{GL}_n(F)$ on a $\mathbb{C}$-vector space equipped with a certain finiteness condition (this can be thought of as the local version of an automorphic representation);
- a *Weil–Deligne representation* is a modified version of the notion of a continuous representation of $\mathrm{Gal}(\overline{F}/F)$.

Now recall that the starting point of our discussion was how to understand the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$. The point of class field theory is that it can be understood through a much easier object $F^\times$. However, at this point, we notice the following:

- The automorphic side of LLC is not so obvious to understand as in the case of $F^\times$. So we may also think that LLC enables us to investigate irreducible admissible representations of $\mathrm{GL}_n(F)$ through the Galois side, which consists of arithmetic objects.
- The automorphic side of LLC makes sense even if we replace GL_n with more general groups.

Keeping these observations in mind, let us present a naive formulation of LLC in general:

Conjecture 1.8 (local Langlands conjecture, naive form)**.** *Let G be a reductive group defined over F. Then there exists a natural map from the set of irreducible admissible representations of $G(F)$ to the set of "L-parameters" of G.*

For general G, we can no longer say that one of the automorphic or Galois sides is particularly easier than the other side. Therefore the local Langlands correspondence is very important not only from number-theoretic viewpoint, but also representation-theoretic viewpoint (representation theory of p-adic reductive groups).

At present, LLC is still conjectural in general, but has been constructed for several specific groups. For example,

- GL_n by Harris–Taylor [HT01], Henniart [Hen00],
- SO_n and Sp_{2n} (quasi-split) by Arthur [Art13],
- U_n (quasi-split) by Mok [Mok15],
- and so on...

On the other hand, there are also approaches for specific classes of irreducible admissible representations. For example,

- the classical construction by Satake for unramified representations,
- regular depth-zero supercuspidal representations by DeBacker–Reeder [DR09],
- regular (positive-depth) supercuspidal representations by Kaletha [Kal19],
- and so on...

The aims of this course to understand the following:

- A naive formulation of LLC in general. For this, I will explain some basics of representation theory of p-adic reductive groups (such as the notion of admissible representations) and also representations theory of local Galois groups (especially, Weil–Deligne representations etc).
- The precise formulation (characterization) of LLC for GL_n given by [HT01] and [Hen00]. For this, I will explain more details of representaiton theory of p-adic reductive groups by focusing on the case of GL_n (so-called "Bernstein–Zelevinsky classification"). It is far beyond my ability to explain the construction of LLC, so I'm not going to touch it.
- The precise formulation (characterization) of LLC for quasi-split classical groups given by [Art13] and [Mok15]. For this, I will explain basics about harmonic analysis on p-adic reductive groups including the Harish–Chandra characters of representaitons etc.
- Recent developments on explicit construction of LLC for certain supercuspidal representations by [DR09], [Kal19], etc.

Of course, this plan must be too ambitious. Let's see how much I can achieve...

2. Week 2: Overview of local class field theory

2.1. **Local fields and CDVR.** We briefly review some basic facts about local fields (see, e.g., [Ser79, Chapters 1, 2] or [Wei74, Chapter I]).

We first introduce the *p-adic number field* $\mathbb{Q}_p$. Recall that the real number field $\mathbb{R}$ is the completion of the rational number field $\mathbb{Q}$ with respect to the normal metric on $\mathbb{Q}$. In fact, there is a different way of completing $\mathbb{Q}$; for each prime number p, we put

$$|p^r \cdot \tfrac{n}{m}|_p := p^{-r}$$

(here, n and m are integers prime to p). Then $|\cdot|_p$ gives a well-defined metric on $\mathbb{Q}$ called the *p-adic metric*. If we complete $\mathbb{Q}$ with respect to the p-adic metric, we obtain a locally compact field different to $\mathbb{R}$, which is called the *p-adic number field* and denoted by $\mathbb{Q}_p$.

Local fields are generalizations of these fields.

Definition 2.1 (local field)**.** We say that a field F is a *local field* if it is a nondiscrete locally compact topological field.

Fact 2.2. *Any local field is isomorphic to one of the following:*

- $\mathbb{R}$ *or* $\mathbb{C}$ *(archimedean);*
- *a finite extension of* $\mathbb{Q}_p$ *(nonarchimedean, characteristic* 0*);*
- *a finite extension of* $\mathbb{F}_p((t))$ *(nonarchimedean, characteristic p).*

One notable characterization of a local field is that it is the completion of a *global field* (i.e., a finite extension of $\mathbb{Q}$ or $\mathbb{F}_p(t)$) with respect to a nontrivial metric. Thus, from the viewpoint of "global" number theory, both archimedean and nonarchimedean local fields have equal importance. However, in this course, we focus only on nonarhimedean local fields (and often assume even that characteristic is zero).

Let us introduce more ring-theoretic description of nonarchimedean local fields.

Definition 2.3 (DVR (discrete valuation ring))**.** Let F be a field. We say that a group homomorphism $v\colon F^\times \to \mathbb{Z}^\times$ is a *discrete valuation* of F if it is surjective and satisfies the following condition:

$$\text{For any } x, y \in F, \text{ we have } v(x+y) \geq \min\{v(x), v(y)\},$$

where we put $v(0) := \infty$. When F is equipped with a discrete valuation v, the set

$$\{x \in F \mid v(x) \geq 0\}$$

forms a subring of F, called the *valuation ring F (with respect to v)*. If a ring $\mathcal{O}$ is obtained as the valuation ring of a field with respect to its discrete valuation, we call it a *discrete valuation ring (DVR)*.

Fact 2.4. *Let $\mathcal{O}$ be a ring. Then $\mathcal{O}$ is a DVR if and only if it is a PID with unique nonzero prime (hence maximal) ideal.*

When $\mathcal{O}$ is a DVR with discrete valuation v, its subset

$$\{x \in F \mid v(x) = 0\}$$

forms the multiplicative group of units $\mathcal{O}^\times$. The maximal ideal of $\mathcal{O}$ is given by

$$\mathfrak{p} = \{x \in F \mid v(x) \geq 1\}.$$

Any generator of the maximal ideal $\mathfrak{p}$ is often referred to as a *uniformizer* of $\mathfrak{p}$. If we fix a uniformizer ϖ of $\mathfrak{p}$, then any nonzero ideal of $\mathcal{O}$ is expressed as [2]

$$\mathfrak{p}^n = \{x \in F \mid v(x) \geq n\} = \varpi^n \mathcal{O}.$$

We call $\mathcal{O}/\mathfrak{p}$ the *residue field* of $\mathcal{O}$.

Now let F be a fractional field of a DVR $\mathcal{O}$ with discrete valuation v. Then we can equip F with a metric $|x| := r^{v(x)}$ ($|0| := 0$) by choosing any real number $r \in (0,1)$. If we let $\hat{F}$ be the completion of F with respect to this metric, $\hat{F}$ naturally has a structure of a topological field. Moreover, we can equip $\hat{F}$ with a discrete valuation which extends v; the valuation ring of $\hat{F}$ is given by the closure of $\mathcal{O}$ in $\hat{F}$. By noting that $\{\mathfrak{p}^n\}_{n \in \mathbb{Z}_{\geq 0}}$ forms a fundamental system of open neighborhoods of 0 in $\mathcal{O}$, we can see that the closure of $\mathcal{O}$ in $\hat{F}$ is nothing but

$$\hat{\mathcal{O}} := \varprojlim_n \mathcal{O}/\mathfrak{p}^n,$$

where the transition map $\mathcal{O}/\mathfrak{p}^{n+1} \twoheadrightarrow \mathcal{O}/\mathfrak{p}^n$ is given by the natural surjection.

We say that a DVR $\mathcal{O}$ is *complete* (and say $\mathcal{O}$ is a *CDVR*) if $\hat{\mathcal{O}} = \mathcal{O}$.

Fact 2.5. *Let F be a field. Then F is a nonarchimedean local field if and only if F is a fractional field of CDVR ("CDVF") with finite residue field.*

Remark 2.6. When F is a nonarchimedean local field with valuation ring $\mathcal{O}$ and maximal ideal $\mathfrak{p}$, the characteristics of $(F, \mathcal{O}/\mathfrak{p})$ must be either $(0,p)$ (called *mixed characteristic*) or (p,p) (called *equal characteristic*). According to a classification result mentioned above, F is mixed characteristic if and only if it is a finite extension of $\mathbb{Q}_p$. In this case, we often say that F is a *p-adic field* (but this terminology depends on people).

Let F be a nonarcimedean local field. Recall that the absolute Galois group of F is, by definition, the Galois group $\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$ of the separable closure F^{sep} of F. [3] The separable closure F^{sep} is given by the direct limit (union) of all finite separable (Galois) extensions of F. We define F^{ab} to be the *maximal abelian extension* of F in F^{sep}, i.e., the direct limit (union) of all finite abelian extensions of F. (Note that this makes sense since the composite field of any two finite abelian extensions is again a finite abelian extension.) Then the Galois group $\mathrm{Gal}(F^{\mathrm{ab}}/F)$ is identified with the maximal abelian quotient of Γ_F, i.e., $\Gamma_F/\overline{[\Gamma_F, \Gamma_F]}$.

2.2. Extension of local fields.

Fact 2.7. *Let $\mathcal{O}_F$ be a CDVR with fractional field F. Let E/F be a finite separable extension of rank n. Then the integral closure of $\mathcal{O}_F$ in E (write $\mathcal{O}_E$) is a CDVR. Moreover, $\mathcal{O}_E$ is a free $\mathcal{O}_F$-module of rank $[E:F]$.*

By this fact, it makes sense to refer to $\mathcal{O}_F$ as the *ring of integers in F*.

Let E/F be a finite separable extension of non-archimedean local fields of degree n. Let $\mathcal{O}_F$ be the ring of integers in F, $\mathfrak{p}_F$ the maximal ideal of $\mathcal{O}_F$, $k_F := \mathcal{O}_F/\mathfrak{p}_F$ the residue field. Also for E, we define $\mathcal{O}_E$, $\mathfrak{p}_E$, and k_E in a similar way. We introduce two invariants of the extension E/F:

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[2] Another popular symbol for a uniformizer is π, but we often use ϖ in our area (representation theory of p-adic groups) in order to reserve π to denote a representation.

[3] Another standard symbol for the absolute Galois group is "G_F", but we avoid it because we want to use "G" for a reductive group over F.

- The ideal $\mathfrak{p}_F\mathcal{O}_E$ of $\mathcal{O}_E$ is of the form $\mathfrak{p}_E^e$, where $e \in \mathbb{Z}_{>0}$. We call e the *ramification index* of E/F.
- Noting that $k_F = \mathcal{O}_F/\mathfrak{p}_F$ is regarded as a subfield $k_E = \mathcal{O}_E/\mathfrak{p}_E$, we let f be the degree of the finite extension k_E/k_F. We call f the *residue degree* of E/F.

Note that these invariants satisfies the chain rule: if E/F is a finite separable extension with ramification index e and residue degree f and L/E is a finite separable extension with ramification index e' and residue degree f', then L/F is a finite separable extension with ramification index ee' and residue degree ff',

Fact 2.8. *We have $n = ef$.*

Definition 2.9. (1) We say that E/F is *unramified* if $e = 1$ and (so, equivalently, $n = f$) and the residual extension k_E/k_F is separable.
(2) We say that E/F is *ramified* if it is not unramified.
(3) We say that E/F is *totally ramified* if $e = n$ (so, equivalently, $f = 1$).

Note we don't have to be worried about the second condition of the unramifiedness (separability of k_E/k_F) for local field since k_F is finite, hence perfect. Also, in this case, the ramifiedness is equivalen to that $e > 1$.

Example 2.10. Let p be an odd prime number such that $p \equiv -1 \pmod 4$. Note that this condition is equivalent to that $\sqrt{-1} \notin \mathbb{F}_p$, which is furthermore equivalent to that $\sqrt{-1} \notin \mathbb{Q}_p$ by Hensel's lemma (explained later). We put $F_0 := \mathbb{Q}_p(\sqrt{-1})$ and $F_1 := \mathbb{Q}_p(\sqrt{p})$.

- The quadratic extension $F_0/\mathbb{Q}_p$ is unramified since the residue field of F_0 must contain $\sqrt{-1}$, hence be a quadratic extension of $\mathbb{F}_p$.
- The quadratic extension $F_1/\mathbb{Q}_p$ is ramified since the ring of integers $\mathcal{O}_{F_1}$ contains $\sqrt{p}$ and the ideal $\mathfrak{p}_{E_1}$ generate by $\sqrt{p}$ satisfies $\mathfrak{p}_{E_1}^2 = p\mathcal{O}_{F_1}$ (so $\mathfrak{p}_{E_1}$ must be the maximal ideal).

In fact, unramified extensions are much easier to understand than ramified extensions. The fundamental reason for this lies in the following theorem:

Fact 2.11 (Hensel's lemma)**.** *Let $\mathcal{O}$ be a CDVR with maximal ideal $\mathfrak{p}$ and residue field k. Let $f(X) \in \mathcal{O}[X]$ be a polynomial with mod $\mathfrak{p}$ reduction $\bar{f}(X) \in k[X]$. If $\bar{\alpha} \in k$ is a simple root of $\bar{f}(X)$, then there uniquely exists a root $\alpha \in \mathcal{O}$ of $f(X)$ such that $\alpha \equiv \bar{\alpha} \pmod{\mathfrak{p}}$.*

Example 2.12. Let p be an odd prime number. Then $\mathbb{Q}_p$ contains $\sqrt{-1}$ if and only if $p \equiv 1 \pmod 4$. Indeed, note that the monic $X^2 + 1$ has a root in $\mathbb{Q}_p$ is and only if it has a root in $\mathbb{Z}_p$ since $\mathbb{Z}_p$ is integrally closed. By Hensel's lemma, the latter condition is equivalent to that $X^2 + 1$ has a root in $\mathbb{F}_p$. Since $\sqrt{-1}$ is a primitive 4th root of unity (this is nothing but the definition of the symbol "$\sqrt{-1}$") and $\mathbb{F}_p^\times$ is cyclic of order $p-1$, we have $\sqrt{-1} \in \mathbb{F}_p^\times$ if and only if $4 \mid (p-1)$, which means that $p \equiv 1 \pmod 4$.

Proposition 2.13. *Let F be a CDVF with residue field k_F. The association $E \mapsto k_E$ for any finite unramified extension E/F gives a bijective map between the set of finite unramified extensions of F (in $\overline{F}$) and the set of finite separable extensions of k_F (in $\overline{k}_F$). Moreover, E/F is Galois if and only if so is k_E/k_F; in this case the Galois groups are identified.*

Proof. We just give a sketch here. For checking the surjectivity, we take a finite separable extension k' of k_F. We write $k' = k_f[X]/(\bar{f}(X))$ with $\bar{f}(X) \in k[X]$ and choose a lift $f(X) \in \mathcal{O}_F[X]$ of $\bar{f}(X)$. Then we can show that $F[X]/(f(X))$ is a finite unramified extension whose residue field is isomorphic to k'.

To show the remaining part, we take a finite unramified extension E of F. For the residual extension k_E/k_F, we choose $\bar{f}(X) \in k_F[X]$ as in the previous paragraph and lift it to $f(X) \in \mathcal{O}_F[X]$. Then, for any finite unramified extension E', we have

$$\mathrm{Hom}_F(E, E') \xleftrightarrow{1:1} \mathrm{Hom}_{\mathcal{O}_F}(\mathcal{O}_E, \mathcal{O}_{E'}) \xleftrightarrow{1:1} \{\text{roots of } f(X) \text{ in } \mathcal{O}_{E'}\}$$

(if $\alpha' \in \mathcal{O}_{E'}$ is a root of $f(X)$, then the corresponding $\mathcal{O}_F$-algebra homomorphism is determined by $\alpha \mapsto \alpha'$). On the other hand, we also have

$$\mathrm{Hom}_{k_F}(k_E, k_{E'}) \xleftrightarrow{1:1} \{\text{roots of } \bar{f}(X) \text{ in } k_{E'}\}$$

By Hensel's lemma, the right-hand sides of these are naturally bijective. Thus we get a natural bijection $\mathrm{Hom}_F(E, E') \cong \mathrm{Hom}_{k_F}(k_E, k_{E'})$. This shows the injection of the map in the assertion. Also, being Galois is preserved between E/F and k_E/k_F. $\square$

Note that, in particular, when E and E' are finite unramified extensions of F, their composite field EE' is also a finite unramified extension of F; this is the field corresponding to $k_E k_{E'}$ in the above proposition. Hence it makes sense to think about the *maximal unramified extension* of F, which is the direct limit (union) of all finite unramified extensions of F and denoted by F^{ur}. Then F^{ur} is a Galois extension of F whose Galois group $\mathrm{Gal}(F^{\mathrm{ur}}/F)$ is isomorphic to $\mathrm{Gal}(k_F^{\mathrm{sep}}/k_F)$. We remark that, for any finite extension E/F, the intersection $E \cap F^{\mathrm{ur}}$ gives the maximal unramified (over F) subextension of F in E; in other words, $E/E \cap F^{\mathrm{ur}}$ is totally ramified and $E \cap F^{\mathrm{ur}}/F$ is unramified.

Let us apply this to the case of nonarchimedean local field. Let F be a nonarchimedean local field, hence k_F is a finite field, say $\mathbb{F}_q$ (a field of q elements). As long as we fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$, there uniquely exists a degree n extension of $\mathbb{F}_q$ in $\overline{\mathbb{F}}_q$ for each $n \in \mathbb{Z}_{>0}$; it is $\mathbb{F}_{q^n}$, which is realized as the set of solutions of $x^{q^n} - x = 0$. This degree n extension $\mathbb{F}_{q^n}/\mathbb{F}_q$ is cyclic; $\mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q)$ has a natural generator called the *arithmetic Frobenius* element

$$\mathbb{F}_{q^n} \xrightarrow{\cong} \mathbb{F}_{q^n}; \quad x \mapsto x^q.$$

Note that the inverse to the arithmetic Frobenius element is also a generator. We call it the *geometric Frobenius* element and write $\mathrm{Frob}_{\mathbb{F}_q}$ for it[4]. Therefore, the Galois group of the infinite Galois extension $\overline{\mathbb{F}}_q/\mathbb{F}_q$ is isomorphic to the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$:

$$\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \cong \varprojlim_n \mathrm{Gal}(\mathbb{F}_{q^n}/\mathbb{F}_q) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z} =: \hat{\mathbb{Z}}.$$

Here, the topological generator 1 of $\hat{\mathbb{Z}}$ on the right-hand side corresponds to the arithmetic Frobenius element $\overline{\mathbb{F}}_q \xrightarrow{\cong} \overline{\mathbb{F}}_q \colon x \mapsto x^q$ on the left-hand side.

Now, by Proposition 2.13, for each $n \in \mathbb{Z}_{>0}$, there uniquely exists a degree n unramified extension F_n of F; it is generated by the solutions to the equation $x^{q^n} - x = 0$. In other words, F_n is obtained by adjoining all $(q^n - 1)$-th roots of unity to F.

Exercise 2.14. Let F be a nonarchimedean local field with residue field k_F of characteristic p. Prove that the maximal unramified extension F^{ur} is generated over F by roots of unity whose orders are prime-to-p.

We next consider ramified extensions. As mentioned before, ramified extensions are not so easy compared with unramified extension. For example, totally ramified extensions are not closed under the composition. Thus it does not make sense to think about something

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[4] Here we have some conflict of notations: in Week 1, I used this symbol for denoting (a lift of) the arithmetic Frobenius.

like “maximal totally ramified extension”. Related to this, there is also no canonical way of associating a “maximal totally ramified subextension” to a given extension E/F.

Example 2.15. Let p, $F_0 = \mathbb{Q}_p(\sqrt{-1})$, and $F_1 := \mathbb{Q}_p(\sqrt{p})$ be as in Example 2.10. We furthermore introduce another quadratic extension $F_2 := \mathbb{Q}_p(\sqrt{-p})$, which is ramified for the same reason as F_1. If we let E be the quartic extension $\mathbb{Q}_p(\sqrt{p}, \sqrt{-1})$ of $\mathbb{Q}_p$, then we have $E = F_0F_1 = F_0F_2 = F_1F_2$. The situation is summarized as follows:

$$\begin{tikzcd}
 & \mathbb{Q}_p(\sqrt{p}, \sqrt{-1}) \arrow[dl, dash, "\mathrm{ram}"'] \arrow[d, dash, "\mathrm{ur}"] \arrow[dr, dash, "\mathrm{ur}"] & \\
\mathbb{Q}_p(\sqrt{-1}) \arrow[dr, dash, "\mathrm{ur}"'] & \mathbb{Q}_p(\sqrt{p}) \arrow[d, dash, "\mathrm{ram}"] & \mathbb{Q}_p(\sqrt{-p}) \arrow[dl, dash, "\mathrm{ram}"] \\
 & \mathbb{Q}_p &
\end{tikzcd}$$

In particular, note that the composite of two totally ramified extensions F_1 and F_2 contains an unramified extension.

Definition 2.16. Let $\mathcal{O}$ be a CDVR with discrete valuation v. Let $f(X) \in \mathcal{O}[X]$ be a monic polynomial $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$. We say that $f(X)$ is an *Eisenstein polynomial* if $v(a_i) \geq 1$ for any $1 \leq i \leq n-1$ and $v(a_0) = 1$.

Fact 2.17. *Let $\mathcal{O}$ be a CDVR with fractional field F. Let $f(X) \in \mathcal{O}[X]$ be an Eisenstein polynomial of degree n. Then $f(X)$ is irreducible and the field $F[X]/(f(X))$ is a totally ramified extension of F of degree n.*

Exercise 2.18. Let $M_n(\mathbb{Q}_p)$ be the $\mathbb{Q}_p$-algebra of n-by-n matrices whose entries are in $\mathbb{Q}_p$. We consider the following element

$$\varphi := \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ p & & & 0 \end{pmatrix} \in M_n(\mathbb{Q}_p).$$

More precisely, $(i, i+1)$-entry of φ is 1 for $1 \leq i \leq n-1$, $(n,1)$-entry of φ is p, and all the other entries are 0. We consider the $\mathbb{Q}_p$-subalgebra $\mathbb{Q}_p[\varphi]$ of $M_n(\mathbb{Q}_p)$ generated by φ. Prove that $\mathbb{Q}_p[\varphi]$ is a finite extension of $\mathbb{Q}_p$ (in particular, $\mathbb{Q}_p[\varphi]$ is a field). Also, determine the extension degree, the ramification index, and the residue degree of $\mathbb{Q}_p[\varphi]/\mathbb{Q}_p$.

2.3. **Galois groups and Weil groups of local fields.** Let E/F be a finite Galois extension of nonarchmedean local fields. Then, any element of $\mathrm{Gal}(E/F)$ induces an element of the extension of residue fields k_E/k_F. In other words, we have a natural surjection $\mathrm{Gal}(E/F) \twoheadrightarrow \mathrm{Gal}(k_E/k_F)$. By letting E run over all finite Galois extensions of F, we also get a natural surjection $\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}/F) \twoheadrightarrow \mathrm{Gal}(\overline{k}_F/k_F)$.

Definition 2.19. We let I_F be the kernel of the map $\Gamma_F \twoheadrightarrow \mathrm{Gal}(\overline{k}_F/k_F)$ and call it the *inertia subgroup* of Γ_F.

Recall that we have $\mathrm{Gal}(F^{\mathrm{ur}}/F) \cong \mathrm{Gal}(\overline{k}_F/k_F)$. Hence the inertia subgroup I_F is nothing but the closed subgroup of Γ_F corresponding to the subextension F^{ur}, i.e., $I_F = \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{ur}})$.

Definition 2.20. We define a subgroup W_F of Γ_F to be the preimage of $\langle \mathrm{Frob}_{k_F} \rangle$ under the map $\Gamma_F \twoheadrightarrow \mathrm{Gal}(\overline{k}_F/k_F)$ and call it the *Weil group* of F.

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] \arrow[d, equal] & \Gamma_F \arrow[r] \arrow[d, phantom, "\cup"] & \mathrm{Gal}(\overline{k}_F/k_F) \cong \hat{\mathbb{Z}} \arrow[r] \arrow[d, phantom, "\cup"] & 1 \\
1 \arrow[r] & I_F \arrow[r] & W_F \arrow[r] & \langle \mathrm{Frob}_{k_F} \rangle \cong \mathbb{Z} \arrow[r] & 1
\end{tikzcd}$$

Note that the Weil group is not the Galois group for any Galois extension, hence there is no intrinsic topology on W_F. We equip W_F with the topology such that I_F is open in W_F and the induced topology on I_F coincides with the natural topology of I_F (as the Galois group of $F^{\mathrm{sep}}/F^{\mathrm{ur}}$). The natural inclusion $W_F \hookrightarrow \Gamma_F$ induces an inclusion between their maximal abelian quotients $W_F^{\mathrm{ab}} \hookrightarrow \Gamma_F^{\mathrm{ab}}$.

2.4. **Local class field theory.**

Theorem 2.21 (local class field theory)**.** *Let F be a non-archimedean local field with residue field k. Then there uniquely exists an isomorphism*

$$\mathrm{Art}_F \colon F^{\times} \xrightarrow{\cong} W_F^{\mathrm{ab}}$$

as topological groups satisfying the following properties:

1. *For any uniformizer $\varpi \in F^{\times}$, its image $\mathrm{Art}_F(\varpi) \in W_F^{\mathrm{ab}}$ is a lift of the geometric* Frobenius $\mathrm{Frob}_k \in \mathrm{Gal}(\overline{k}/k)$.
2. *For any finite separable extension E/F, the following diagram commutes:*

$$\begin{tikzcd}
E^{\times} \arrow[r, "\mathrm{Art}_E"] \arrow[d, "\mathrm{Nr}_{E/F}"'] & W_E^{\mathrm{ab}} \arrow[d, "\mathrm{res}"] \\
F^{\times} \arrow[r, "\mathrm{Art}_F"] & W_F^{\mathrm{ab}}
\end{tikzcd}$$

3. *For any finite abelian extension E/F, Art_F induces an isomorphism*

$$F^{\times}/\,\mathrm{Nr}_{E/F}(E^{\times}) \xrightarrow{\cong} \mathrm{Gal}(E/F).$$

Because of this theorem, it is important to know the structure of $F^{\times}$. So let us explain how $F^{\times}$ can be understood.

We first note the exact sequence

$$1 \to \mathcal{O}_F^{\times} \to F^{\times} \xrightarrow{v} \mathbb{Z} \to 1.$$

This splits by choosing a uniformizer $\varpi \in F^{\times}$, i.e., we have $F^{\times} \cong \mathcal{O}_F \times \langle \varpi \rangle$. Secondly, we have the exact sequence

$$1 \to (1 + \mathfrak{p}_F) \to \mathcal{O}_F^{\times} \to k_F^{\times} \to 1.$$

This splits by Hensel's lemma; elements of $k_F^{\times}$ are identified with $(q-1)$-roots of unity, where $q = |k_F|$. Finally, we consider the exponential/logarithm map between F and $F^{\times}$. Here, for simplicity, we suppose that $F = \mathbb{Q}_p$:

$$\exp \colon \mathbb{Q}_p \to \mathbb{Q}_p^{\times}; \quad x \mapsto \sum_{n=0}^{\infty} \frac{1}{n!} x^n,$$

$$\log \colon \mathbb{Q}_p^{\times} \to \mathbb{Q}_p; \quad x \mapsto \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (x-1)^n.$$

These maps do not converge on the whole domain, but gives group isomorphisms between

$$\begin{cases} p\mathbb{Z}_p \text{ and } 1+p\mathbb{Z}_p & \text{if } p \neq 2, \\ 4\mathbb{Z}_2 \text{ and } 1+4\mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

In the case where $p = 2$, we have $(1 + 2\mathbb{Z}_2) \cong \{\pm 1\} \times (1 + 4\mathbb{Z}_2)$. Thus, in conclusion, we have

$$\mathbb{Q}_p^\times \cong \begin{cases} \mathbb{Z} \times \mathbb{Z}/(p-1)\mathbb{Z} \times \mathbb{Z}_p & \text{if } p \neq 2, \\ \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}_2 & \text{if } p = 2. \end{cases}$$

Exercise 2.22. Count the number of (isomorphism classes of) quadratic extensions of $\mathbb{Q}_p$.

Exercise 2.23. For any finite abelian group G, can we always find a finite abelian extension of nonarchimedean local fields E/F whose Galois group is isomorphic to G?

3. WEEK 3: REPRESENTATIONS OF LOCALLY PROFINITE GROUPS

3.1. Locally profinite groups.

The reference for this section is [BH06, §1].

Definition 3.1. (1) We say that a topological group G is *profinite* if G is compact and the set of open subgroups of G forms a fundamental system of neighborhood of $1 \in G$ (i.e., any open neighborhood of $1 \in G$ contains an open subgroup of G).

(2) We say that a topological group G is *locally profinite* if it contains an open subgroup which is a profinite group.

Fact 3.2. *A topological group G is profinite if and only if it is written as the inverse limit $G = \varprojlim_n G_n$ with respect to a projective system $\{G_n\}_n$ of finite groups.*

We don't review here the fundamental properties of (locally) profinite groups, but just mention the following one, which will be used implicitly many many times.

Lemma 3.3. *Let G be a profinite group. Then any open subgroup of H is of finite index.*

Proof. Let us write $G = \bigsqcup_{g \in G/H} gH$. Then the g-translation $G \to G\colon x \mapsto gx$ is a homeomorphism, gH is also open in G. Thus, by the compactness of G, we conclude that the disjoint open covering $\{gH\}_{g \in G/H}$ must be a finite covering. Hence G/H is finite. □

Example 3.4. (1) Any non-archimedean local field F is a locally profinite group as an additive group. Indeed, by the definition of its metric, the descending filtration

$$\mathcal{O}_F \supset \mathfrak{p}_F \supset \mathfrak{p}_F^2 \supset \cdots \supset \{0\}$$

consisting of open subgroups $\mathfrak{p}^n$ forms a fundamental system of neighborhoods of 0. Since $\mathcal{O}_F$ is closed (and also open) in F and bounded with respect to the metric, $\mathcal{O}_F$ is compact (hence profinite). Note that we can write $\mathcal{O}_F \cong \varprojlim_n \mathcal{O}_F/\mathfrak{p}_F^n$.

(2) For any non-archimedean local field F, its multiplicative group $F^\times$ is a locally profinite group. Indeed, by the definition of its metric, the descending filtration

$$\mathcal{O}_F^\times \supset (1+\mathfrak{p}_F) \supset (1+\mathfrak{p}_F^2) \supset \cdots \supset \{0\}$$

consisting of open subgroups $(1+\mathfrak{p}^n)$ forms a fundamental system of neighborhoods of 1. Since $\mathcal{O}_F^\times$ is closed (and also open) in F and bounded with respect to the metric, $\mathcal{O}_F^\times$ is compact (hence profinite). Note that we can write $\mathcal{O}_F^\times \cong \varprojlim_n \mathcal{O}_F^\times/(1+\mathfrak{p}_F^n)$.

(3) The previous examples can be generalized as follows.

For any non-archimedean local field F, the additive group $M_n(F)$ of n-by-n matrices is a locally profinite group. Here, we just regard $M_n(F)$ as $F^{\oplus n^2}$ and equipped it with the product topology. A fundamental system of its open neighborhood (of the zero matrix) can be taken to be

$$M_n(\mathcal{O}_F) \supset M_n(\mathfrak{p}_F) \supset M_n(\mathfrak{p}_F^2) \supset \cdots \supset \{0\}.$$

Next, we consider $G = \mathrm{GL}_n(F)$ for a non-archimedean local field F. Then, with respect to the induced topology from $M_n(F)$, G is a locally profinite group. A fundamental system of its open neighborhood (of the identity matrix) can be taken to be

$$\mathrm{GL}_n(\mathcal{O}_F) \supset 1 + M_n(\mathfrak{p}_F) \supset 1 + M_n(\mathfrak{p}_F^2) \supset \cdots \supset \{I_n\}.$$

(4) The previous example can be furthermore generalized as follows. Let $\mathbf{G}$ be a linear algebraic group over F. Then, by definition, we can find an embedding (Zariski closed immersion) $\iota\colon \mathbf{G} \hookrightarrow \mathrm{GL}_n$ into some GL_n over F. Hence we may regard $\mathbf{G}$ as a Zariski closed subgroup of GL_n via ι. Here, recall that "Zariski closed"

means that $\mathbf{G}$ can be defined to be the subset of GL_n consisting of zeros of some polynomials. Thus any element $g \in \mathrm{GL}_n(F) \smallsetminus \mathbf{G}(F)$ is not a solution to those polynomials; then any element $h \in \mathrm{GL}_n(F) \smallsetminus \mathbf{G}(F)$ which is "sufficiently" close to g in the locally profinite topology (p-adic topology) cannot be a solution. In other words, the completment $\mathrm{GL}_n(F) \smallsetminus \mathbf{G}(F)$ is open, hence $\mathbf{G}(F)$ is closed in $\mathrm{GL}_n(F)$. In general, any closed subgroup of a locally profinite group is again a locally profinite group, hence so is $\mathbf{G}(F)$.

3.2. **Smooth representations of locally profinite groups.** The reference for this section is [BH06, §2].

Let G be a locally profinite group. In the following, by "a representation (π, V)" of G, we mean a $\mathbb{C}$-vector space V equipped with an action π of G, i.e., a group homomorphism $\pi\colon G \to \mathrm{GL}_{\mathbb{C}}(V)$. (Sometimes we just say "π is a representation of G".)

Definition 3.5. Let (π, V) be a representation of G. We say that (π, V) is *smooth* if any $v \in V$ is contained in V^K for some open compact subgroup K of G. Here, V^K denotes the subspace of K-fixed vectors, i.e.,

$$V^K := \{w \in V \mid \pi(k)(w) = w \text{ for any } k \in K\}.$$

In other words, (π, V) is smooth if and only if we have

$$V = \bigcup_{K \subset G} V^K,$$

where the index set is over all open compact subgroups K of G.

We want to examine examples of smooth representations. In representation theory of finite groups, an operation called *induction* plays a very important role in constructing representations of a given group. So let us consider whether the same technique is available in this context.

Let $H \subset G$ be a subgroup. What we want to do here is to construct a smooth representation of a "bigger" locally profinite group using a smooth representation of a "smaller" locally profinite group. So, firstly, let us assume that H is closed because this guarantees that H is again locally profinite. Let (σ, W) be a smooth representation of H. Let (π, V) be the induction of (σ, W) in the usual sense. More precisely, the underlying space V is

$$\{f\colon G \to W \mid f(hg) = \sigma(h)(f(g)) \text{ for any } h \in H\}$$

and the action π of G on V is given by the right-translation on the functions, i.e.,

$$(\pi(x)f)(g) := f(gx).$$

Then, is (π, V) smooth? In fact, NO in general. So that (π, V) is smooth, for any f, there must be an open compact subgroup $K \subset G$ satisfying $f(gK) = f(g)$ (for any $g \in G$). However, this property is not formally deduced from the definition of V in general.

The idea is to modify the definition of V so that this condition is satisfied. In other words, if we put

$$V^\infty := \{f\colon G \to W \mid \exists K \text{ s.t. } f(hgk) = \sigma(h)(f(g)) \text{ for any } h \in H,\, k \in K\},$$

then (π, V^∞) is smooth (with respect to the same right-translation action π).

Definition 3.6. Let H be a closed subgroup of G and (σ, W) be a smooth representation of H. We call the smooth representation (π, V) defined as in the previous paragraph the *smooth induction* of (σ, W) from H to G. In our context, we always only consider the smooth induction, so we just say the *induction* of (σ, W) and write $(\mathrm{Ind}_H^G \sigma, \mathrm{Ind}_H^G W)$ for it.

Before we think about examples, let us introduce one guiding fact:

Fact 3.7. *Any irreducible smooth representation of* $\mathrm{GL}_n(F)$ *is either one-dimensional (i.e., a character) or infinite dimensional.*

This fact says that, in representation theory of locally profinite groups, we seriously have to face infinite dimensional representations. However, it is still possible to formulate some finiteness condition for smooth representations; it is called the *admissibility*.

Definition 3.8. Let (π, V) be a representation of G. We say that (π, V) is *admissible* if it is smooth and $\dim_{\mathbb{C}}(V^K)$ is finite-dimensional for any open compact subgroup K of G.

Example 3.9. Let $G = \mathrm{GL}_2(F)$, where F is a nonarchimedean local field.

(1) Let $\chi\colon G \to \mathbb{C}^\times$ be a character, or equivalently, one-dimensional representation $(\chi, \mathbb{C})$. Then, by definition, $(\chi, \mathbb{C})$ is smooth if and only if χ is trivial on some open compact subgroup of G. (This is equivalent to that χ is continuous with respect to the discrete topology of $\mathbb{C}^\times$.) Any smooth character of G is of course admissible.

(2) Let $B \subset G$ be the subgroup of upper-triangular matrices (this is a closed subgroup). Let (π, V) be the (smooth) induction $(\mathrm{Ind}_B^G \mathbb{1}, \mathrm{Ind}_B^G \mathbb{C})$ of the trivial representation $(\mathbb{1}, \mathbb{C})$ of B to G. By definition, we can explicitly write

$$V = \{f\colon B\backslash G \to \mathbb{C} \mid \exists K \text{ s.t. } f(gk) = f(g) \text{ for any } k \in K\}.$$

To check the admissibility of (π, V), let us fix any open compact subgroup K of G. Then we have

$$V^K \cong \{f\colon B\backslash G/K \to \mathbb{C}\}.$$

This is finite-dimensional since $B\backslash G/K$ is finite. Indeed, to check it, we may replace K with any its open subgroup freely (recall that such a subgroup must be of finite index in K). Especially, we may assume that K is contained in $\mathrm{GL}_2(\mathcal{O}_F)$. Since K must be also open, hence of finite index, in $\mathrm{GL}_2(\mathcal{O}_F)$, it is enough to show that $B\backslash G/\mathrm{GL}_2(\mathcal{O}_F)$ is a finite set. It is a well-known fact that $G = B\,\mathrm{GL}_2(\mathcal{O}_F)$ (so-called the *Iwasawa decomposition*), hence $B\backslash G/\mathrm{GL}_2(\mathcal{O}_F)$ is a singleton.

(3) Next consider the subgroup $T \subset G$ of diagonal matrices (this is a closed subgroup). Let (π, V) be the (smooth) induction $(\mathrm{Ind}_T^G \mathbb{1}, \mathrm{Ind}_T^G \mathbb{C})$ of the trivial representation $(\mathbb{1}, \mathbb{C})$ of T to G. By definition, we can explicitly write

$$V = \{f\colon T\backslash G \to \mathbb{C} \mid \exists K \text{ s.t. } f(gk) = f(g) \text{ for any } k \in K\}.$$

To check the admissibility of (π, V), let us fix any open compact subgroup K of G. Then we have

$$V^K \cong \{f\colon T\backslash G/K \to \mathbb{C}\}.$$

However, this space is infinite dimensional (Exercise below). Hence (π, V) is smooth but not admissible.

Note that this example shows that B is large enough so that the admissibility is preserved by the induction to G, but T is too small. This idea will be elaborated as the "parabolic induction" later.

Exercise 3.10. Prove that the set $T\backslash G/K$ in the above example is infinite.

Fact 3.11. *Let* $G = \mathbf{G}(F)$ *for any connected reductive group* $\mathbf{G}$ *over a nonarchimedean local field* F*. Then any irreducible smooth representation of* G *is admissible.*

3.3. **Frobenius reciprocity.** Recall that, in representation theory of finite groups, we have so-called the *Frobenius reciprocity*, which is the adjunction between the induction functor and the restriction functor. In fact, the Frobenius reciprocity holds also for the smooth induction as well.

Theorem 3.12 (Frobenius reciprocity for Ind)**.** *Let H be a closed subgroup of G and (σ, W) a smooth representation of G. Let (π, V) be a smooth representation of G. Then we have an isomorphism*

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_H^G \sigma) \cong \mathrm{Hom}_H(\pi|_H, \sigma).$$

given by $\phi \mapsto \alpha \circ \phi$, where $\alpha\colon \mathrm{Ind}_H^G \sigma \to \sigma$ is $f \mapsto f(1)$.

For the proof, see [BH06, §2.4].

Here, we caution that the smooth induction is put on the target in $\mathrm{Hom}(-,-)$. In other words, the smooth induction is the right adjoint to the restriction. In contrast to the case of finite groups, representations may not be semisimple. Thus we cannot swap the source and target in $\mathrm{Hom}(-,-)$ freely in general.

Then, does the restriction have a left adjoint? In fact, the answer is YES when H is open; it is given by the following variant of a smooth induction:

Definition 3.13. Let H be an open subgroup of G and (σ, W) be a smooth representation of H. We put

$$V_c^\infty := \left\{ f\colon G \to W \;\middle|\; \begin{array}{l} \bullet\ f \text{ is compactly supported modulo } H \\ \bullet\ \exists K \text{ s.t. } f(hgk) = \sigma(h)(f(g)) \text{ for any } h \in H,\ k \in K \end{array} \right\}$$

and consider the right-translation action π of G on V_c^∞. Then (π, V_c^∞) is a smooth representation of G. We call it the *compact induction* of (σ, W) from H to G and write $(\text{c-}\mathrm{Ind}_H^G \sigma, \text{c-}\mathrm{Ind}_H^G W)$.

Theorem 3.14 (Frobenius reciprocity for c-Ind)**.** *Let H be a closed subgroup of G and (σ, W) a smooth representation of G. Let (π, V) be a smooth representation of G. Then we have an isomorphism*

$$\mathrm{Hom}_G(\text{c-}\mathrm{Ind}_H^G \sigma, \pi) \cong \mathrm{Hom}_H(\sigma, \pi|_H).$$

given by $\phi \mapsto \phi \circ \beta$. Here, $\beta\colon \sigma \mapsto \text{c-}\mathrm{Ind}_H^G \sigma$ is $w \mapsto f_w$, where $f_w\colon H \to W$ is such that $f_w(h) = \sigma(h)w$ for $h \in H$ and $f_w(g) = 0$ for $g \in G \smallsetminus H$.

3.4. **Representations of profinite groups.** We define the notion of a subrepresentation and a direct sum of smooth representations in the usual way. The following proposition is a simple consequence of Zorn's lemma (see [BH06, §2.2, Proposition]).

Proposition 3.15. *For any smooth representation (π, V), the following are equivalent:*

(1) (π, V) is the direct sum of irreducible subrepresentations.
(2) (π, V) is the sum of irreducible subrepresentations.
(3) for any subrepresentation (π_1, V_1), there exists a complement, i.e., another subrepresentation (π_2, V_2) such that $(\pi, V) \cong (\pi_1, V_1) \oplus (\pi_2, V_2)$.

Definition 3.16. When a smooth representation satisfies the conditions in the previous proposition, we say it is *semisimple*.

Note that, in contrast to the case of finite groups, there exist plenty non-semisimple smooth representations in general.

Exercise 3.17. Show that $\operatorname{Ind}_B^G \mathbb{1}$ in Example 3.9 (2) is non-semisimple. (Hint: check that $\operatorname{Ind}_B^G \mathbb{1}$ is not irreducible by finding a proper subrepresentation and show that $\operatorname{End}_G(\operatorname{Ind}_B^G \mathbb{1}) = 1$ using Frobenius reciprocity and Schu'r lemma, which will be explained later.)

Proposition 3.18. *Any smooth representation of a profinite group K is semisimple.*

Proof. Let (π, V) be a smooth representation of K. Then, for each $v \in V$, there exists an open (hence of finite index) subgroup K' of K such that $v \in V^{K'}$. Here, by shrinking K' if necessary, we may assume that K' is normal in K. (Indeed, by writing $K = \bigcup_{l \in K/K'} lK'$, it is enough to replace K' with $\bigcap_{l \in K/K'} lK'l^{-1}$.) Note that v is contained in the subrepresentation $\operatorname{Span}_{\mathbb{C}}\{\pi(k)v \mid k \in K\}$ generated by v. However, since the action of K' on this subrepresentation is trivial and K/K' is a finite group, this subrepresentation must be semisimple. Thus, (π, V) can be written as a sum of semisimple representations, hence so is itself. □

Definition 3.19. Let (π, V) be a smooth representation of G. For any open compact subgroup K of G, by the previous proposition, we can write

$$V = \bigoplus_{\sigma \in \operatorname{Irr}(K)} V[\sigma].$$

Here, the both hand sides are regarded as representations of K and $V[\sigma]$ denotes the sum of irreducible K-subrepresentations of V isomorphic to σ ($\operatorname{Irr}(K)$ is the set of isomorphism classes of irreducible smooth representations of K). We call $V[\sigma]$ the *σ-isotypic part* of V. Note that $V^K = V[\mathbb{1}]$.

3.5. **Contragredient representation.** Recall that, for any representation (π, V) of a finite group G, its dual (*contragredient*) representation (π^*, V^*) is defined by $V^* := \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$\langle \pi^*(g)(v^*), v \rangle = \langle v^*, \pi(g^{-1})(v) \rangle$$

for any g, $v \in V$, $v^* \in V^*$. In the context of locally profinite groups, this definition contains the issue as in the definition of Ind; i.e., the resulting representation may not be smooth. So, again, we consider smoothening V^*.

Definition 3.20. For a smooth representation (π, V) of G, we define its *contragredient* representation $(\pi^\vee, V^\vee)$ by

$$V^\vee := \bigcup_{K \subset G} (V^*)^K,$$

where K runs open compact subgroups of G and $\pi^\vee = \pi^*|_{V^\vee}$.

Exercise 3.21. Show that, for any open compact subgroup K of G, we have $(V^\vee)^K \cong (V^K)^*$.

Proposition 3.22. *For a smooth representation (π, V) of G, we consider the natural map $\pi \to (\pi^\vee)^\vee$ given by $v \mapsto [v^\vee \mapsto \langle v^\vee, v \rangle]$. This map is G-equivariant. Moreover, it is isomorphic if and only if (π, V) is admissible.*

Proof. The first statement can be checked easily. Then, for any open compact subgroup K of G, we get $\pi^K \to ((\pi^\vee)^\vee)^K$. Since $\pi = \bigcup_K \pi^K$ and $(\pi^\vee)^\vee = \bigcup_K ((\pi^\vee)^\vee)^K$, it is enough to discuss when this map is bijective (for all K). By applying the previous exercise twice, we see that this map is identified with the natural map $\pi^K \to ((\pi^K)^*)^*$. It is well-known fact in linear algebra that this natural map is bijective if and only if π^K is finite-dimensional. □

3.6. **Irreducible representations and Schur's lemma.**

Definition 3.23. Let (π, V) be a smooth representation of G. We say that (π, V) is *irreducible* if (π, V) has no G-subspace (subrepresentation) other than $\{0\}$ and V.

Lemma 3.24. *Let (π, V) be an irreducible smooth representation of G. Then V is generated by any nonzero vector $v \in V$, i.e., we have $V = \mathrm{Span}_{\mathbb{C}}(\{\pi(g)v \mid g \in G\})$.*

Proof. The $\mathbb{C}$-vector subspace $\mathrm{Span}_{\mathbb{C}}(\{\pi(g)v \mid g \in G\})$ of V is stable under G-action. Thus the irreducibility of V implies that it equals $\{0\}$ or V. Since $v \neq 0$, it must be V. $\square$

Definition 3.25. For smooth representations (π, V) and (π, V'), we define the set $\mathrm{Hom}_G(\pi, \pi')$ of G-equivariant homomorphisms by

$$\mathrm{Hom}_G(\pi, \pi') := \{\phi \in \mathrm{Hom}_{\mathbb{C}}(V, V') \mid \phi(\pi(g)v) = \pi'(g)\phi(v)\, \forall g \in G, \forall v \in V\}.$$

When $(\pi, V) = (\pi, V')$, we simply write $\mathrm{End}_G(\pi)$ for $\mathrm{Hom}_G(\pi, \pi)$.

Theorem 3.26 (Schur's lemma)**.** *Let (π, V) be an irreducible smooth representation of G. Assume that $\dim_{\mathbb{C}}(V)$ is countable. Then we have $\mathrm{End}_G(\pi) = \mathbb{C}$.*

Proof. Suppose that $\phi \in \mathrm{End}_G(\pi)$ is a nonzero G-equivariant endomorphism of (π, V). Then both $\mathrm{Ker}(\phi)$ and $\mathrm{Im}(\phi)$ are G-stable subspaces of V. Hence, by the irreducibility of V, they must be $\{0\}$ or V. Since ϕ is supposed to be nonzero, we necessarily have $\mathrm{Ker}(\phi) = 0$ and $\mathrm{Im}(\phi) = V$; in other words, ϕ is an isomorphism. Therefore, $\mathrm{End}_G(\pi)$ is a division $\mathbb{C}$-algebra (i.e., possibly non-commutative $\mathbb{C}$-algebra whose any nonzero element is invertible).

By Lemma 3.24, if we fix any nonzero vector $v \in V$, then v generates V. Hence, any G-equivariant endomorphism $\phi \in \mathrm{End}_G(\pi)$ is determined uniquely by the image $\phi(v)$ of v. If $\phi(v) \in V$ is equal to $\phi'(v) \in V$ up to scalar, then $\phi\, \mathrm{End}_G(\pi)$ equals $\phi' \in \mathrm{End}_G(\pi)$ up to scalar. In particular, the dimension of $\mathrm{End}_G(\pi)$ as a $\mathbb{C}$-vector space is bounded by the dimension of V. Since $\dim_{\mathbb{C}}(V)$ is countable, so is $\dim_{\mathbb{C}}(\mathrm{End}_G(\pi))$.

Now suppose that $\dim_{\mathbb{C}}(\mathrm{End}_G(\pi))$ is bigger than $\mathbb{C}$; then we can choose $\phi \in \mathrm{End}_G(\pi) \smallsetminus \mathbb{C}$. Then the division $\mathbb{C}$-algebra $\mathrm{End}_G(\pi)$ contains the rational function field $\mathbb{C}(\phi)$. However, the dimension of $\mathbb{C}(\phi)$ as a $\mathbb{C}$-vector space is uncountable. (For example, it can be easily checked that the subset $\{(\phi - a)^{-1} \mid a \in \mathbb{C}\}$ is linear independent.) Thus we get a contradiction. $\square$

A reasonable sufficient condition for that the assumption of Theorem 3.26 is satisfied is the following:

Lemma 3.27. *If there exists an open compact subgroup K_0 of G such that G/K_0 is countable, then any irreducible representation of G has countable dimension.*

Proof. Note that if K_0 is an open compact subgroup whose G/K_0 is countable, then any open compact subgroup K satisfies that G/K is countable. (Indeed, since $K \cap K_0$ is also open subgroup of K_0, it is compact and of finite index in K_0. Thus $G/(K \cap K_0)$ is countable. As $K \cap K_0$ is also of finite index in K, G/K is countable.) By Lemma 3.24, any nonzero vector $v \in V$ generates V. If we let K be an open compact subgroup of G satisfying $v \in V^K$, then $\dim_{\mathbb{C}}(V)$ is bounded by the cardinality of G/K, which is countable. $\square$

Example 3.28. When $G = \mathrm{GL}_n(F)$, G satisfies the countability assumption in the above lemma. Indeed, if we put $K_0 := \mathrm{GL}_n(\mathcal{O}_F)$, then K_0 is an open compact subgroup of G. Moreover, we have the following decomposition (so-called "*Cartan decomposition*", which is a consequence of elementary divisor theory):

$$G = \bigsqcup_{\substack{a,b\in\mathbb{Z} \\ a\le b}} K_0 \begin{pmatrix} \varpi^a & 0 \\ 0 & \varpi^b \end{pmatrix} K_0.$$

Each summand is compact, hence its right-K_0-cosets are finite since K_0 is open. As the index set is countable, we only have countable many right-K_0-cosets in G.

More generally, for any linear algebraic group $\mathbf{G}$ over F, $G := \mathbf{G}(F)$ satisfies the countability assumption. (take an embedding $\mathbf{G} \hookrightarrow \mathrm{GL}_n$ and put $K_0 := G \cap \mathrm{GL}_n(\mathcal{O}_F)$).

In the following, we always assume that there exists an open compact subgroup K_0 whose G/K_0 is countable. Let us suppse that (π, V) is an irreducible representation of G. Let Z be the center of G. Then, for any $z \in Z$, the automorphism $\pi(z) \in \mathrm{GL}_{\mathbb{C}}(V)$ is G-equivariant. Indeed, for any $g \in G$ and $v \in V$, we have

$$\pi(z)(\pi(g)v) = \pi(zg)v = \pi(gz)v = \pi(g)(\pi(z)v).$$

By Schur's lemma, $\pi(z)$ must be a (nonzero) scalar multiplication. Therefore, we get a map $Z \to \mathbb{C}^\times$. It is easy to check that this map is a smooth character.

Definition 3.29. For any irreducible representation (π, V) of G, we call the smooth character of Z defined in this way *the central character of* (π, V) and write ω_π.

4. WEEK 4: IRREDUCIBLE SMOOTH REPRESENTATIONS OF $\mathrm{GL}_2(F)$

Let F be a non-archimedean local field of residual characteristic p, hence a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$. The aim of this week is to give an overview on a classification of irreducible smooth representations of group $\mathrm{GL}_2(F)$.

4.1. Recap on irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

Let $\mathbb{F}_q$ be a finite field of order q and characteristic p. We first review how the irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are classified.

Let us simply write $G = \mathrm{GL}_2$ in the following. We introduce several subgroups of $\mathrm{GL}_2(\mathbb{F}_q)$ as follows:

$$B(\mathbb{F}_q) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\},$$

$$T(\mathbb{F}_q) := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times \right\},$$

$$U(\mathbb{F}_q) := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, b \in \mathbb{F}_q \right\}.$$

Note that $U(\mathbb{F}_q)$ is a normal subgroup in $B(\mathbb{F}_q)$ and that we have the semi-direct decomposition $B(\mathbb{F}_q) = T(\mathbb{F}_q) \ltimes U(\mathbb{F}_q)$. In particular, we have a natural surjection $B(\mathbb{F}_q) \twoheadrightarrow T(\mathbb{F}_q)$ by quotienting by $U(\mathbb{F}_q) \lhd B(\mathbb{F}_q)$.

Let us take two characters χ_1, χ_2 of $\mathbb{F}_q^\times$. Then we get a character of $T(\mathbb{F}_q)$

$$\chi := \chi_1 \boxtimes \chi_2 \colon T(\mathbb{F}_q) \to \mathbb{C}^\times; \quad \mathrm{diag}(t_1, t_2) \mapsto \chi_1(t_1) \cdot \chi_2(t_2).$$

By pulling back χ to $B(\mathbb{F}_q)$, we may regard χ as a character of $B(\mathbb{F}_q)$. Finally, by taking the induction to $G(\mathbb{F}_q)$, we get a representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi$ of $G(\mathbb{F}_q)$. We call the representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi$ a *principal series representation* (associated to χ).

The decomposition rule of $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi$ is described as follows.

Proposition 4.1 ([BH06, Section 6])**.** *(1) When $\chi_1 \neq \chi_2$, $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi$ is irreducible.*

(2) When $\chi_1 = \chi_2$ (say χ_0), we have $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi \cong (\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \mathbb{1}) \otimes (\chi_0 \circ \det)$ and $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \mathbb{1})$ decomposes into the sum of two irreducible representations; the one is the trivial representatation $\mathbb{1}$ of $G(\mathbb{F}_q)$ and the other one is called the Steinberg reperesentation St. *In summary, we have $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi = (\chi_0 \circ \det) \oplus \mathrm{St} \otimes (\chi_0 \circ \det)$.*

(3) Two principal series representations $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi_1 \boxtimes \chi_2$ and $\mathrm{Ind}_{B(\mathbb{F}_q)}^{G(\mathbb{F}_q)} \chi_1' \boxtimes \chi_2'$ contains a common irreducible representation if and only if $(\chi_1, \chi_2) = (\chi_1', \chi_2'), (\chi_2', \chi_1')$.

Can this construction exhaust all irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$? In fact, not; the missing representations are called *cuspidal* representations. In my course of the previous semester [Oi24], I introduced two ways of constructing all cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$. The one is a purely algebraic construction; we first define a virtual representation as a linear combination of several induced representations, and then show that it is in fact an irreducible representation which is not contained in any principal series representation. The details of this construction can be found in [BH06, Section 6]. The other one is a geometric construction; we first define an algebraic variety on which $\mathrm{GL}_2(\mathbb{F}_q)$ acts and then take its ℓ-adic étale cohomology. Then the resulting cohomology realizes cuspidal representations (and even also principal series representations); this is what is called *Deligne–Lusztig theory* [DL76].

4.2. **Principal series representations of** $\mathrm{GL}_2(F)$**.** Now let us consider the group $\mathrm{GL}_2(F)$. Recall that, in the last week, we have introduced the notion of the (smooth) induction in the context of smooth representation theory of locally profinite groups. Thus it is natural to try to imitate the construction of principal series also for $\mathrm{GL}_2(F)$.

We introduce several subgroups of $\mathrm{GL}_2(F)$ in the same way as before, just by replacing $\mathbb{F}_q$ with F:

$$B(F) := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(F) \,\middle|\, a, d \in F^\times, \, b \in F \right\},$$

$$T(F) := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(F) \,\middle|\, a, d \in F^\times \right\},$$

$$U(F) := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in M_2(F) \,\middle|\, b \in F \right\}.$$

Then these groups satisfy the same properties as before. Especially, $U(F)$ is a normal subgroup in $B(F)$ and we have the semi-direct decomposition $B(F) = T(F) \ltimes U(F)$. We have a natural surjection $B(F) \twoheadrightarrow T(F)$ by quotienting by $U(F) \lhd B(F)$.

Let us take two smooth characters χ_1, χ_2 of $F^\times$. Then we can define the repersentation $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ of $\mathrm{GL}_2(F)$ in exactly the same manner as before. We call $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ a *principal series representation* (associated to $\chi_1 \boxtimes \chi_2$). However, the decomposition rule of $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ is a bit more difficult than the case of $\mathrm{GL}_2(\mathbb{F}_q)$. The point is that representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are allways semisimple, but those of $\mathrm{GL}_2(F)$ are not. To be more precise, the situation is summarized as follows.

Let $\mathbb{F}_q$ be the residue field of F. Let $|-| \colon F^\times \to \mathbb{C}^\times$ denote the absolute value character, i.e., $|x| = q^{-v(x)}$, where v is the normalized valuation of F.

Proposition 4.2 ([BH06, Section 9])**.** *(1) The representation* $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ *is reducible if and only if* $\chi_1\chi_2^{-1}$ *equals either* $\mathbb{1}$ *or* $|-|^2$*. Moreover,* $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ *and* $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1' \boxtimes \chi_2'$ *are isomorphic if and only if* (χ_1, χ_2) *equals* (χ_1', χ_2') *or* $(\chi_2' \cdot |-|^{-\frac{1}{2}}, \chi_1' \cdot |-|^{\frac{1}{2}})$

(2) Suppose that $\chi_1\chi_2^{-1} = \mathbb{1}$*, hence* $\chi_1 = \chi_2 = \chi_0$ *for some smooth character* $\chi_0 \colon F^\times \to \mathbb{C}^\times$*. Then* $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2 \cong \mathrm{Ind}_{B(F)}^{G(F)} \mathbb{1} \otimes (\chi_0 \circ \det)$ *and we have the following non-split exact sequence of smooth representations of* $\mathrm{GL}_2(F)$

$$0 \to \mathbb{1} \to \mathrm{Ind}_{B(F)}^{G(F)} \mathbb{1} \to \mathrm{St} \to 0,$$

where $\mathbb{1}$ *is the trivial representation of* $\mathrm{GL}_2(F)$ *and* St *is an infinite-dimensional irreducible smooth representation of* $\mathrm{GL}_2(F)$ *(called the Steinberg representation). In other words,* $\mathbb{1}$ *is the unique irreducible subrepresentation of* $\mathrm{Ind}_{B(F)}^{G(F)} \mathbb{1}$ *which cannot be a quotient, and* St *is the unique irreducible quotient representation of* $\mathrm{Ind}_{B(F)}^{G(F)} \mathbb{1}$ *which cannot be a subrepresentation.*

(3) Suppose that $\chi_1\chi_2^{-1} = |-|^2$*, hence* $\chi_1 = \chi_0 \cdot |-|$ *and* $\chi_2 = \chi_0 \cdot |-|^{-1}$ *for some smooth character* $\chi_0 \colon F^\times \to \mathbb{C}^\times$*. Then* $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2 \cong \mathrm{Ind}_{B(F)}^{G(F)} (|-| \boxtimes |-|^{-1}) \otimes (\chi_0 \circ \det)$ *and we have the following non-split exact sequence of smooth representations of* $\mathrm{GL}_2(F)$

$$0 \to \mathrm{St} \to \mathrm{Ind}_{B(F)}^{G(F)} (|-| \boxtimes |-|^{-1}) \to \mathbb{1} \to 0.$$

In other words, $\mathbb{1}$ *is the unique irreducible quotient representation which cannot be a subrepresentation, and* St *is the unique irreducible subrepresentation which cannot be a quotient representation.*

As in the case of $\mathrm{GL}_2(\mathbb{F}_q)$, this construction does not produce all irreducible smooth representation of $\mathrm{GL}_2(F)$. If an irreducible smooth representation of $\mathrm{GL}_2(F)$ is not contained in any principal series representation, we call it a *supercuspidal representation.*

4.3. **Depth-zero supercuspidal representations of** $\mathrm{GL}_2(F)$**.** The question is: how to construct (all) irreducible supercuspidal representations of $\mathrm{GL}_2(F)$? For principal series representations, we just imitated the construction in the case of $\mathrm{GL}_2(\mathbb{F}_q)$. However, for supercuspidal representations, we can immediately see that the construction in the finite field case no longer works.

One idea is, instead of imitating, "reducing" the construction to the finite field case. The point is that $\mathrm{GL}_2(F)$ has the following open compact subgroup:

$$\mathrm{GL}_2(\mathcal{O}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(F) \,\middle|\, a, b, c, d \in \mathcal{O},\, ad - bc \in \mathcal{O}^\times \right\},$$

where $\mathcal{O}$ denotes the ring of integers in F. When each entry of a 2-by-2 matrix belongs to $\mathcal{O}$, it makes sense to take its mod-$\mathfrak{p}$ reduction for the maximal ideal $\mathfrak{p}$ of $\mathcal{O}$. Then we get a 2-by-2 matrix with entries in $\mathbb{F}_q = \mathcal{O}/\mathfrak{p}$:

$$\mathrm{GL}_2(\mathcal{O}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q) \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}.$$

Exercise 4.3. Prove that the map $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathbb{F}_q)$ is also surjective.

Now let κ be an irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$. By pulling back it along the above surjection, we get an irreducible smooth representation of $\mathrm{GL}_2(\mathcal{O})$ (let's again write κ). Recall that, in the last week, we introduced a variant of the usual (smooth) induction, which is called the compact induction "c-Ind". The basic strategy is to construct a smooth representation of $\mathrm{GL}_2(F)$ by applying the compact induction to this representation κ of $\mathrm{GL}_2(\mathcal{O})$.

However, here we have an ovbious obstruction. Let $Z(F)$ be the center of $\mathrm{GL}_2(F)$, i.e., the subgroup of non-zero scalar matrices. Then we have

$$\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O})}^{\mathrm{GL}_2(F)} \kappa = \text{c-Ind}_{Z(F)\,\mathrm{GL}_2(\mathcal{O})}^{\mathrm{GL}_2(F)} \left(\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O})}^{Z(F)\,\mathrm{GL}_2(\mathcal{O})} \kappa \right).$$

Since the quotient $Z(F)\,\mathrm{GL}_2(\mathcal{O})/\,\mathrm{GL}_2(\mathcal{O})$ is isomorphic to $Z(F)/Z(\mathcal{O}) \cong F^\times/\mathcal{O}^\times$, which is an infinite abelian group, the internal induction $\text{c-Ind}_{\mathrm{GL}_2(\mathcal{O})}^{Z(F)\,\mathrm{GL}_2(\mathcal{O})} \kappa$ breaks into infinitely many pieces of irreducible representations of $Z(F)\,\mathrm{GL}_2(\mathcal{O})$. So the whole compact induction cannot be irreducible.

To remedy this issue, we first extend the representation κ from $\mathrm{GL}_2(\mathcal{O})$ to $Z(F)\,\mathrm{GL}_2(\mathcal{O})$ by fixing a character of $Z(F)$. Note that, as κ is irreducible, the restriction of κ to $Z(\mathcal{O})$ is given by a character ("central character"). Let ω be a character of $Z(F)$ such that $\omega|_{Z(\mathcal{O})}$ coincides with the central character of κ. We define a representation $\tilde{\kappa}$ of $Z(F)\,\mathrm{GL}_2(\mathcal{O})$ by

$$\tilde{\kappa}(z) := \begin{cases} \omega(z) & \text{if } z \in Z(F), \\ \kappa(g) & \text{if } g \in \mathrm{GL}_2(\mathcal{O}). \end{cases}$$

We put

$$\pi_{\tilde{\kappa}} := \text{c-Ind}_{Z(F)\,\mathrm{GL}_2(\mathcal{O})}^{\mathrm{GL}_2(F)} \tilde{\kappa}.$$

Fact 4.4. *The representation $\pi_{\tilde{\kappa}}$ is an irreducible supercuspidal representation of* $\mathrm{GL}_2(F)$. *Moreover, for any other κ' and ω', the representations $\pi_{\tilde{\kappa}}$ and $\pi_{\tilde{\kappa}'}$ are isomorphic if and only if $\kappa \cong \kappa'$ and $\omega = \omega'$.*

Now we come up with the next question: does this construction exhaust all irreducible supercuspidal representations of $\mathrm{GL}_2(F)$? The answer is NO! In fact, rather, only very few supercuspidal representations are realized in this way. The supercuspidal representations constructed here are called *depth-zero supercuspidal representations.*

4.4. **Depth of representations.** Let us describe a general picture in the following. We first consider the structure of $\mathrm{GL}_1(F) = F^\times$. As reviewed in Week 2, we have the following isomorphism depending on the choice of a uniformizer ϖ of $F^\times$:

$$\begin{aligned} F^\times &\cong \langle\varpi\rangle \times \mathcal{O}^\times \\ &\cong \langle\varpi\rangle \times \mathbb{F}_q^\times \times (1+\mathfrak{p}). \end{aligned}$$

The last part $1+\mathfrak{p}$ is a profinite group having a descending filtration

$$(1+\mathfrak{p}) \supset (1+\mathfrak{p}^2) \supset \cdots \supset \{1\}.$$

This filtration gives a fundamental system of neighborhood. In particular, if a character χ of $F^\times$ is smooth, then it must be trivial on $1+\mathfrak{p}^r$ for some $r \in \mathbb{Z}_{\geq 0}$. By noting this, we introduce a numerical invariant for smooth characters of $F^\times$ as follows:

Definition 4.5. Suppose that $r \in \mathbb{Z}_{\geq 0}$ is the smallest integer such that χ is trivial on $1+\mathfrak{p}^{r+1}$ but nontrivial on $1+\mathfrak{p}^r$. (For convenience, we put $1+\mathfrak{p}^0 := \mathcal{O}^\times$.) We call this number r the *depth* (or *level*) of χ.

The idea is to generalize this argument to $\mathrm{GL}_2(F)$ (or even more general p-adic reductive groups). The following descending filtration gives a fundamental system of neighborhood of 1:

$$\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix} \supset \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \supset \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p}^2 \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{pmatrix} \supset \cdots \supset \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let us put $K_r := \begin{pmatrix} 1+\mathfrak{p}^r & \mathfrak{p}^r \\ \mathfrak{p}^r & 1+\mathfrak{p}^r \end{pmatrix}$ ($K_0 := \mathrm{GL}_2(\mathcal{O})$). The difference between GL_1 and GL_2 is that, for GL_2, an irreducible representation (π, V) may not be trivial on K_r for any $r \in \mathbb{Z}_{\geq 0}$. (This is because V could be infinite-dimensional; in fact, it happens as long as V is not 1-dimensional.) However, it is still true that V^{K_r} is nonzero for some $r \in \mathbb{Z}_{\geq 0}$ due to the definition of smoothness. Therefore, it still makes sense to look at the smallest integer $r \in \mathbb{Z}_{\geq 0}$ satisfying $V^{K_r} = 0$ but $V^{K_{r+1}} \neq 0$.

Then is it reasonable to define the "depth" of an irreducible smooth representation (π, V) to be this number r? Actually, NOT! The reason is that $\mathrm{GL}_2(\mathcal{O})$ has many interesting/impotant open compact subgroups other than K_r's. Let us consider the following descending chain:

$$\begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} \supset \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \supset \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix} \supset \begin{pmatrix} 1+\mathfrak{p}^2 & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p}^2 \end{pmatrix} \supset \cdots \supset \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We write I_0 for the most left-hand side subgroup and call it the *Iwahori* subgroup of $\mathrm{GL}_2(F)$. We give a half-integer numbering on each subgroup of this filtration as follows:

$$I_r := \begin{cases} \begin{pmatrix} 1+\mathfrak{p}^s & \mathfrak{p}^s \\ \mathfrak{p}^{s+1} & 1+\mathfrak{p}^s \end{pmatrix} & \text{if } r = s, \\ \begin{pmatrix} 1+\mathfrak{p}^{s+1} & \mathfrak{p}^s \\ \mathfrak{p}^{s+1} & 1+\mathfrak{p}^{s+1} \end{pmatrix} & \text{if } r = s + \frac{1}{2}, \end{cases}$$

where $s \in \mathbb{Z}_{\geq 0}$. Why this way of numbering is reasonable? The point is the second step subgroup is the pro-p-radical of I_0, i.e., maximal normal pro-p subgroup of I_0. In this sense, it is an analogue of K_1 for K_0. However, even if we raise the level of each entry of I_0, we do not get this second step subgroup; what we get is the third step subgroup. So it is fair to call the third step one "I_1" and the second step one "$I_{\frac{1}{2}}$".

Definition 4.6. Let (π, V) be an irreducible smooth representation of $\mathrm{GL}_2(F)$. Suppose that $r \in \mathbb{Z}_{\geq 0}$ is the smallest integer such that $V^{P_r} = 0$ but $V^{P_{r+1}} \neq 0$ for $P = K$ or $P = I$. We call the number r the *depth* of the representation (π, V).

The notion of depth can be generalized for any irreducible smooth representation of any p-adic reductive group; it was introduced by Moy–Prasad [MP94, MP96]. The subgroups K_0 and I_0 are generalized to so-called *parahoric subgroups* of p-adic redcutive groups, which can be classified by *Bruhat–Tits theory* [BT72, BT84]. Roughly speaking, Bruhat–Tits classified maximal open compact subgroups of a p-adic reductive group by introducing a geometric object equipped with an action of the p-adic group, which is called the *Bruhat–Tits building*. In the papers of Moy–Prasad [MP94, MP96], they introduced a descending filtration to each such maximal open compact subgroup, which is called the *Moy–Prasad filtration*. (The above filtrations $\{K_r\}_r$ and $\{I_r\}_r$ are nothing but the Moy–Prasad filtrations for K_0 and I_0.) Then, Moy–Prasad defined the notion of a depth using the all possible Moy–Prasad filtrations.

In general, the depth is known to be a non-negative rational number. Moreover, its possible denominator is determined by the given p-adic reductive group. For example, in the case of GL_2, the denominator can be only 1 or 2.

4.5. **Simple supercuspidal representations.** Now let us go back to how to think about supercuspidal representations of $\mathrm{GL}_2(F)$. The representation $\pi_{\tilde{\kappa}}$ constructed above has a non-zero K_1-fixed vector, thus its depth is zero. In fact, it is known that for any positive half-integer r, there exists an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ of depth r.

In contrast to the case of finite fields, classifying all positive-depth irreducible supercuspidal representations of $\mathrm{GL}_2(F)$ is not easy nor elementary at all. It's doable, but based on very subtle and deep analysis of the group structure of $\mathrm{GL}_2(F)$. Because I'm not going to go into its details in this course, here let's just cite Chapter 4 of Bushnell–Henniart's book [BH06]. The construction/classification given there can be thought of as a special case of Bushnell–Kutzko's *type theory* for GL_n [BK93].

But, instead, I just would like to explain how the minimal positive depth (i.e., depth $\frac{1}{2}$) supercuspidal representations can be constructed because it's fairly easy.

Recall that we have

$$I_0 = \begin{pmatrix} \mathcal{O}^\times & \mathcal{O} \\ \mathfrak{p} & \mathcal{O}^\times \end{pmatrix} \supset I_{\frac{1}{2}} = \begin{pmatrix} 1+\mathfrak{p} & \mathcal{O} \\ \mathfrak{p} & 1+\mathfrak{p} \end{pmatrix} \supset I_1 = \begin{pmatrix} 1+\mathfrak{p} & \mathfrak{p} \\ \mathfrak{p}^2 & 1+\mathfrak{p} \end{pmatrix}.$$

Note that the quotient $I_{\frac{1}{2}}/I_1$ is isomorphic to the abelian group $\mathbb{F}_q^{\oplus 2}$ by looking at the mod-$\mathfrak{p}$ reduction of $(1,2)$ and $(2,1)$ entries:

$$I_{\frac{1}{2}}/I_1 \xrightarrow{\cong} \mathbb{F}_q^{\oplus 2} \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (\bar{b}, \overline{c\varpi^{-1}}).$$

We choose a nontrivial additive character $\psi \colon \mathbb{F}_p \to \mathbb{C}^\times$ and define a smooth character of $I_{\frac{1}{2}}$ by

$$I_{\frac{1}{2}} \twoheadrightarrow I_{\frac{1}{2}}/I_1 \cong \mathbb{F}_q^{\oplus 2} \xrightarrow{\text{sum}} \mathbb{F}_q \xrightarrow{\psi} \mathbb{C}^\times \colon \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \psi(\bar{b} + \overline{c\varpi^{-1}}).$$

By abuse of notation, let us again write ψ for this character.

Basically we want to get an irreducible supercuspidal representation of $\mathrm{GL}_2(F)$ by applying the compact induction to this representation of $I_{\frac{1}{2}}$. As in the depth-zero case, we extend ψ to a bit bigger subgroup. The intermediate group we need is the following:

$$\mathrm{GL}_2(F) \supset Z(F) \cdot I_{\frac{1}{2}} \cdot \langle \varphi \rangle \supset I_{\frac{1}{2}}, \quad \varphi = \begin{pmatrix} 0 & 1 \\ \varpi & 0 \end{pmatrix}.$$

We choose any extension $\tilde{\psi}$ of ψ to this subgroup and put

$$\pi_{\tilde{\psi}} := \text{c-Ind}_{Z(F)I_{\frac{1}{2}}\langle\varphi\rangle}^{\mathrm{GL}_2(F)} \tilde{\psi}.$$

Fact 4.7. *The representation $\pi_{\tilde{\psi}}$ is an irreducible supercuspidal representation of* $\mathrm{GL}_2(F)$ *of depth $\frac{1}{2}$. Conversely, any such representation is of the form $\pi_{\tilde{\psi}}$.*

The representations obtained in this way are called *simple supercuspidal representations* and have discovered firstly by Gross–Reeder [GR10].

Exercise 4.8. Prove that the normalizer group of I_0 in $\mathrm{GL}_2(F)$ is given by $Z(F) \cdot I_0 \cdot \langle \varphi \rangle$.

Exercise 4.9. Describe all the possible extensions of ψ from $I_{\frac{1}{2}}$ to $Z(F)I_{\frac{1}{2}}\langle\varphi\rangle$.

5. WEEK 5: REPRESENTATION OF WEIL GROUPS

For this week's discussion, we follow [BH06, Chapter 7],

5.1. Representations absolute Galois groups.

Let us start with investigating continuous representations of profinite groups on $\mathbb{C}$-vector spaces.

Proposition 5.1. *Let G be a profinite group. Let (π, V) be a continuous representation of G on a finite-dimensional $\mathbb{C}$-vector space V. Then the image of $\pi\colon G \to \mathrm{Aut}_{\mathbb{C}}(V)$ is finite.*

Proof. By choosing a $\mathbb{C}$-basis of V, we identify $\mathrm{Aut}_{\mathbb{C}}(V)$ with $\mathrm{GL}_n(\mathbb{C})$, where n is the dimension of V. For any positive real number ε, we define an open subset U_ε of $M_n(\mathbb{C})$ by

$$U_\varepsilon := \{g = (g_{ij}) \in M_n(\mathbb{C}) \mid |g_{ij}| < \varepsilon\}.$$

If we choose ε to be sufficiently small, then $I_n + U_\varepsilon$ is a subset of $\mathrm{GL}_n(\mathbb{C})$. Moreover, it is an open neighborhood of I_n in $\mathrm{GL}_n(\mathbb{C})$. Let us write K_ε for this open subset of $\mathrm{GL}_n(\mathbb{C})$. Since π is continuous, the preimage $\pi^{-1}(K_\varepsilon)$ is also an open neighborhood of $1 \in G$.

Recall that G is profinite, hence it has a fundamental system of open neighborhood of $1 \in G$ consisting of open compact subgroups. In fact, even stronger, we can choose such a system so that each subgroup is normal (see Excercise below). Thus let us take an open normal compact subgroup K of G such that $K \subset \pi^{-1}(K_\varepsilon)$, or equivalently, $\pi(K) \subset K_\varepsilon$. Then $\pi(K)$ is a subgroup contained in K_ε.

However, as long as ε is sufficiently small, K_ε does not contain any nontrivial subgroup. Indeed, for the sake of contradiction, let us assume that K_ε contains a nontrivial subgroup K' and choose $k \in K' \smallsetminus \{I_n\}$. Let $\alpha_1, \ldots, \alpha_n$ be the generalized eigenvalues of k. Then $\alpha_1^r, \ldots, \alpha_n^r$ are the generalized eigenvalues of k^r for any $r \in \mathbb{Z}$. Note that all the eigenvalues of any element of K_ε must be sufficiently close to 1. Since K' is a subgroup contained in K_ε, k^r belongs to K_ε for any $r \in \mathbb{Z}$. Hence $\alpha_1^r, \ldots, \alpha_n^r$ are sufficiently close to 1 for any $r \in \mathbb{Z}$. This can happen only when $\alpha_1 = \cdots = \alpha_r = 1$. In other words, k must be a unipotent matrix. However, if k is not equal to I_n, k^r cannot belong to K_ε for sufficiently large $r \in \mathbb{Z}_{>0}$. (This can be easily seen by, e.g., taking the Jordan normal form of k).

Hence we obtained that $\pi(K)$ must be $\{I_n\}$. Thus π factors through the quotient G/K, which is a finite group, hence $\pi(G)$ is finite. □

Exercise 5.2. For any profinite group, prove that there exists a fundamental system of open neighborhoods of 1 consisting of open normal subgroups.

Our fundamental interest lies in understanding the absolute Galois groups of a non-archimedean local field (or even a global field). We approach to this by investigating representations of the absolute Galois group. Since the absolute Galois group is a topological group equipped with a profinite topology, it is natural to impose some topological constraints on the representations. However, the above proposition is saying that "as long as we consider continuous representations on $\mathbb{C}$-vector spaces, we cannot construct any interesting (nontrivial) examples beyond those coming from finite Galois groups".

Then, what should be the next candidates for the coefficients and the class of representations to be studied? In our context, one natural idea is to consider continuous representations on $\overline{\mathbb{Q}}_\ell$-vector spaces. This is because, for example, theory of étale cohomology provides machinery to systematically construct such a class of representations. Also, indeed, there exist plenty of continuous representations with infinite images.

Example 5.3. Let $G = \mathbb{Z}_\ell$. If we define a 2-dimensional representation $\rho\colon \mathbb{Z}_\ell \to \mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ by $x \mapsto \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right)$, then π is obviously continuous and has infinite image.

Now we consider the case where G is the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ of a non-archimedean local field F. So we investigate continuous representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector spaces, where ℓ is a prime number. If we let p be the residual characteristic of F, the situation changes according to whether $\ell = p$ or $\ell \neq p$. In fact, the case where $\ell = p$ is much more complicated and difficult. Although the continuous representations of $\mathrm{Gal}(F^{\mathrm{sep}}/F)$ on $\overline{\mathbb{Q}}_p$-vector spaces (often referred to as *p-adic Galois representations*) are very important object to be studied, we focus only on the case where $\ell \neq p$ in this course. In the following, when we say "*an ℓ-adic representation*", it means a continuous reprepresentation on a finite-dimensional $\overline{\mathbb{Q}}_\ell$ where ℓ is not equal to p.

But then we come up with another question: when $\ell \neq p$, how does the situation depend on the choice of ℓ? Recall that the other side of the Langlands correspondence (globally, automorphic representations; locally, irreducible admissible representations of a p-adic reductive group) does not involve such a choice of a prime number ℓ. In fact, *Grothendieck's monodromy theorem* provides an answer to this question.

5.2. **Galois group vs. Weil group.** Let F be a non-archimedean local field with residual characteristic $p > 0$. Recall that we have a natural surjection

$$\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}/F) \twoheadrightarrow \mathrm{Gal}(\overline{k}_F/k_F) \cong \hat{\mathbb{Z}}.$$

The kernel of this surjection is referred to as the *inertia subgroup* and denoted by I_F; this is nothing but the absolute Galois group $\mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{ur}})$ of the maximal unramified extension F^{ur} of F (see Week 2 notes). The *Weil group* W_F, which is a subgroup of Γ_F, is defined to be the preimage of $\langle \mathrm{Frob}_{k_F} \rangle$ under the map $\Gamma_F \twoheadrightarrow \mathrm{Gal}(\overline{k}_F/k_F)$:

$$\begin{tikzcd}
1 \arrow[r] & I_F \arrow[r] & \Gamma_F \arrow[r] & \mathrm{Gal}(\overline{k}_F/k_F) \cong \hat{\mathbb{Z}} \arrow[r] & 1 \\
1 \arrow[r] & I_F \arrow[r] \arrow[u, equal] & W_F \arrow[r] \arrow[u, phantom, "\cup"] & \langle \mathrm{Frob}_{k_F} \rangle \cong \mathbb{Z} \arrow[r] \arrow[u, phantom, "\cup"] & 1
\end{tikzcd}$$

We write v for the map $W_F \twoheadrightarrow \mathbb{Z}$.

In the following, we investigate ℓ-adic representation of W_F rather than Γ_F. Note that W_F is dense in Γ_F, hence any ℓ-adic representation ρ of Γ_F is uniquely determined by its restriction to W_F. To be more precise, we let

- $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_F)$ be the set of isomorphism classes of ℓ-adic representations of Γ_F, and
- $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F)$ be the set of isomorphism classes of ℓ-adic representations of W_F.

Then the natural restriction map gives an injection:

$$\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_F) \hookrightarrow \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F).$$

Thus at least restricting to W_F does not lose information of the original ℓ-adic representations of Γ_F. However, be careful that there are more ℓ-adic representations of W_F than those of Γ_F, i.e., the above map is not surjective.

Then, why do we work with $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F)$ rather than $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_F)$? This is because $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F)$ can be shown to be independent of ℓ in a certain sense as we will see in the following.

Exercise 5.4. Find an example of an ℓ-adic representation of W_F which cannot be extended to an ℓ-adic representation of Γ_F.

5.3. **More about Weil groups.** Let us explain a bit more about the group I_F. But, before it, recall that a finite extension of non-archimedean local fields E/F is called ramified when its ramification index is greater than 1.

Definition 5.5. Let E/F be a finite extension of non-archimedean local fields with residual characteristic $p > 0$. We say that E/F is *tamely ramified* when its ramification index is prime to p. Otherwise, we say that E/F is *wildly ramified*.

For any integer $n \in \mathbb{Z}_{>0}$ prime to p, there uniquely exists a degree n extension of F^{ur}. Explicitly, this extension is given by adjoining an(y) n-th root of a(ny) uniformizer ϖ of F. This extension is Galois and cyclic; we have an isomorphism

$$\mathrm{Gal}(F^{\mathrm{ur}}(\sqrt[n]{\varpi})/F^{\mathrm{ur}}) \cong \mu_n \colon \sigma \mapsto \sigma(\sqrt[n]{\varpi})/\sqrt[n]{\varpi},$$

where μ_n denotes the set of n-th roots of unity (in F^{sep}). We put F^{tame} to be the composite of all finite extensions of F^{ur} whose degree is prime to p. Then F^{tame} is a Galois extension of F. In fact, this gives the maximal tamely ramified extension of F. By the above description of the Galois group at each finite level, we have

$$\mathrm{Gal}(F^{\mathrm{tame}}/F^{\mathrm{ur}}) \cong \varprojlim_{(n,p)=1} \mu_n.$$

Note that, by fixing a system of generators of μ_n (i.e., a topological generator of the right-hand side), we also have

$$\varprojlim_{(n,p)=1} \mu_n \cong \varprojlim_{(n,p)=1} \mathbb{Z}/n\mathbb{Z} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell.$$

We let $P_F := \mathrm{Gal}(F^{\mathrm{sep}}/F^{\mathrm{tame}})$ and call it the *wild inertia subgroup*. In fact, P_F is the unique pro-p-Sylow subgroup of I_F. The quotient $I_F/P_F \cong \mathrm{Gal}(F^{\mathrm{tame}}/F^{\mathrm{ur}})$ is often referred to as the *tame inertia* group.

So we obtained the following chains:

$$\begin{array}{rcccl}
 & \Gamma_F & & F & \\
\Gamma_F/I_F \cong \hat{\mathbb{Z}} & \triangledown & & \cap & \text{add } n\text{-th roots of } 1\ (p \nmid n) \\
 & I_F & & F^{\mathrm{ur}} & \\
I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell & \triangledown & & \cap & \text{add } n\text{-th roots of } \varpi\ (p \nmid n) \\
 & P_F & & F^{\mathrm{tame}} & \\
\text{pro-}p \text{ subgroup} & \triangledown & & \cap & ??? \\
 & \{1\} & & F^{\mathrm{sep}} &
\end{array}$$

Since the conjugate action of Γ_F on itself preserves I_F and P_F, Γ_F also acts on the tame inertia group I_F/P_F. As the tame inertia is abelian, this action factors through the quotient Γ_F/I_F. The action of the subgroup W_F/I_F on I_F/P_F is described as follows:

Lemma 5.6. *For any $\tau \in W_F$ and $\sigma \in I_F/P_F$, we have*

$$\tau\sigma\tau^{-1} = \sigma^{q^{-v(\tau)}}.$$

Proof. By the above description of the tame inertia group ($I_F/P_F \cong \mathrm{Gal}(F^{\mathrm{tame}}/F^{\mathrm{ur}}) \cong \varprojlim_{(n,p)=1} \mu_n$), it is enough to check that

$$\tau\sigma\tau^{-1}(\sqrt[n]{\varpi}) = \sigma^{q^{-v(\tau)}}(\sqrt[n]{\varpi}),$$

for each $n \in \mathbb{Z}_0$ prime to p with the above notation.

Since τ preserves ϖ, $\tau(\sqrt[n]{\varpi})$ is again an n-th root of ϖ; let us write $\tau(\sqrt[n]{\varpi}) = \zeta \cdot \sqrt[n]{\varpi}$ with some $\zeta \in \mu_n$. We also write $\sigma(\sqrt[n]{\varpi}) = \xi \cdot \sqrt[n]{\varpi}$ with $\xi \in \mu_n$. Here, note that F^{ur} contains all roots of unity with prime-to-p order. In particular,

- σ acts on such roots of unity via identity, and
- τ acts on such roots of unity via $q^{-v(\tau)}$-power (recall that any lift Φ of the *geometric* Frobenius is supposed to have $v(\Phi) = 1$).

Hence we get

$$\tau\sigma\tau^{-1}(\sqrt[n]{\varpi}) = \tau\sigma(\zeta^{-q^{v(\tau)}} \cdot \sqrt[n]{\varpi}) = \tau(\xi \cdot \zeta^{-q^{v(\tau)}} \cdot \sqrt[n]{\varpi}) = \xi^{q^{-v(\tau)}} \cdot \zeta^{-1} \cdot \zeta \cdot \sqrt[n]{\varpi} = \xi^{q^{-v(\tau)}} \cdot \sqrt[n]{\varpi}.$$

On the other hand, we have

$$\sigma^{q^{-v(\tau)}}(\sqrt[n]{\varpi}) = \xi^{q^{-v(\tau)}} \cdot \sqrt[n]{\varpi}.$$

This completes the proof. □

5.4. **Grothendieck's monodromy theorem.** Recall that $I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$. In the following, we fix a prime number ℓ (supposed to be the "ℓ" of "ℓ-adic representations") and also fix a surjective homomorphism

$$t\colon I_F/P_F \cong \prod_{\ell \neq p} \mathbb{Z}_\ell \twoheadrightarrow \mathbb{Z}_\ell.$$

Note that hence we have $I_F \supset \mathrm{Ker}(t) \supset P_F$.

For any finite-dimensional C-vector space V (where C is any field of characteristic zero) and its nilpotent endomoprhism $N \in \mathrm{End}_C(V)$, we put

$$\exp(N) := \sum_{n=0}^{\infty} \frac{N^n}{n!}.$$

Note that, since $N^n = 0$ for sufficiently large n (at least for n greater than $\dim(V)$), this infinite sum is actually a finite sum. Moreover, $\exp(N)$ is a unipotent automorphism of V. Conversely, for any unipotent automorphism $u \in \mathrm{Aut}_C(V)$, we put

$$\log(u) := \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(u-1)^n}{n}.$$

Then this defines a nilpotent endomorphism of V. These operations give the inverse to each other.

Grothendieck's monodromy theorem says that any ℓ-adic representation of W_F is "quasi-unipotent":

Theorem 5.7 (Grothendieck's monodromy theorem)**.** *Let ρ be an ℓ-adic representation of W_F. Then there exists an open subgroup H of I_F and a unique nilpotent endomorphism $N \in \mathrm{End}_{\overline{\mathbb{Q}}_\ell}(V)$ satisfying*

$$\rho(\sigma) = \exp(t(\sigma) \cdot N)$$

for any element $\sigma \in H$.

Proof. Let us first check the uniqueness. Suppose that we have two pairs (H, N) and (H', N') as in the assertion. Then, for any $\sigma \in H \cap H'$, we have

$$\exp(t(\sigma) \cdot N) = \rho(\sigma) = \exp(t(\sigma) \cdot N').$$

Thus, by applying log, we get $t(\sigma) \cdot N = t(\sigma) \cdot N'$. Since $H \cap H'$ is open and of finite index in I_F, the restriction of t on $H \cap H'$ cannot be trivial. Hence we can find $\sigma \in H \cap H'$ such that $t(\sigma) \neq 0$, which implies that $N = N'$.

Let us show the existence. In the following, by fixing a $\overline{\mathbb{Q}}_\ell$-basis of V, we identify $\mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$ with $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Hence ρ is regarded as a continuous homomorphism

$$\rho\colon W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell).$$

For $r \in \mathbb{Z}_{>0}$, we define an open subgroup K_r of $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ by

$$K_r := I_n + \ell^r \cdot M_n(\overline{\mathbb{Z}}_\ell).$$

We define a subgroup J of I_F by $J := \rho^{-1}(K_2) \cap \mathrm{Ker}(t)$. We claim that $\rho(J) = \{I_n\}$. Indeed, note that $\mathrm{Ker}(t)$ is a profinite group whose pro-order is prime-to-ℓ, that is, $\mathrm{Ker}(t)$ does not have a finite quotient whose order is divided by ℓ. As ρ is continuous and I_F is compact (hence so is J), $\rho(J)$ must be a compact subgroup of K_2. Since K_2/K_3 is isomorphic to $M_n(\overline{\mathbb{Z}}_\ell/\ell\overline{\mathbb{Z}}_\ell)$, which is a discrete abelian group of exponent ℓ, the image of $\rho(J)$ in the quotient K_2/K_3 is discrete and compact, hence finite. But then its order must be ℓ-power, thus $\rho(J)$ is necessarily trivial. In other words, $\rho(J)$ is contained in K_3. By repeating this argument for K_3, K_4, and so on, eventually, we get $\rho(J) = \bigcap_{r>0} K_r = \{I_n\}$.

$$\begin{tikzcd}
I_F \arrow[r, "\rho|_{I_F}"] & \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \\
\triangledown & \cup \\
\mathrm{Ker}(t) & K_1 \\
\cup & \triangledown \\
J \arrow[r] \arrow[rdd] \arrow[rdddd] & K_2 \\
 & \triangledown \\
 & K_3 \\
\vdots & \triangledown \\
 & K_4 \\
 & \triangledown \\
 & \vdots
\end{tikzcd}$$

Since $J = \rho^{-1}(K_2) \cap \mathrm{Ker}(t)$, we have $(\rho^{-1}(K_2) \cap I_F)/J \subset I_F/\mathrm{Ker}(t)$. Hence the restriction of ρ to $\rho^{-1}(K_2) \cap I_F$ factors through the homomorphism $t\colon I_F/\mathrm{Ker}(t) \xrightarrow{\cong} \mathbb{Z}_\ell$. We let $\phi\colon t(\rho^{-1}(K_2) \cap I_F) \to K_2$ be the induced homomorphism:

$$\begin{tikzcd}
\rho^{-1}(K_2) \cap I_F \arrow[r, "\rho"] \arrow[d, two heads] & K_2 \\
(\rho^{-1}(K_2) \cap I_F)/J \arrow[r, "t", "\cong"'] & t(\rho^{-1}(K_2) \cap I_F) \arrow[u, "\phi"'] \\
\cap & \cap \\
I_F/\mathrm{Ker}(t) \arrow[r, "t", "\cong"'] & \mathbb{Z}_\ell
\end{tikzcd}$$

Let $\Phi \in W_F$ be any lift of the geometric Frobenius, i.e., an element such that $v(\Phi) = 1$. By Lemma 5.6, we have $\Phi\sigma\Phi^{-1} = \sigma^{-q}$ for any $\sigma \in I_F/P_F$. Hence, for any $\sigma \in \rho^{-1}(K_2) \cap I_F$, we have

$$\rho(\Phi)\rho(\sigma)\rho(\Phi)^{-1} = \rho(\Phi\sigma\Phi^{-1}) = \rho(\sigma^{-q}) = \rho(\sigma)^{-q}.$$

This equality implies that the set of eigenvalues of $\rho(\sigma)$ is stable under taking the q-power. Note that this only happens when every eigenvalue of $\rho(\sigma)$ is a root of unity. On the other hand, since $\rho(\sigma) \in K_2$, every eigenvalue of $\rho(\sigma)$ belongs to $1 + \ell^2\overline{\mathbb{Z}}_\ell$. A fun fact here is that these imply that any eigenvalue must be 1 (see exercise below; the reason why we are looking at "K_2" (not "K_1") is coming from here). Therefore, $\rho(\sigma)$ must be unipotent for any $\sigma \in \rho^{-1}(K_2) \cap I_F$.

Now we note that $\rho^{-1}(K_2)$ is open in W_F by the continuity of ρ, hence $(\rho^{-1}(K_2) \cap I_F)/J$ is also open (thus of finite index) in $I_F/\operatorname{Ker}(t)$. In particular, $t(\rho^{-1}(K_2) \cap I_F)$ cannot be zero. Hence we can choose $\sigma_0 \in \rho^{-1}(K_2) \cap I_F$ such that $t(\sigma_0) \neq 0$. Let us define a nilpotent endomorphism $N \in M_n(\overline{\mathbb{Q}}_\ell)$ by

$$N := t(\sigma_0)^{-1} \cdot \log(\rho(\sigma_0)).$$

Then obviously we have $\exp(t(\sigma_0) \cdot N) = \rho(\sigma_0) = \phi(t(\sigma_0))$. Let us consider an open (hence of finite index) subgroup $A := t(\sigma_0) \cdot \mathbb{Z}_\ell$ of $\mathbb{Z}_\ell$. By definition, we have

$$A \subset t(\rho^{-1}(K_2) \cap I_F) \subset \mathbb{Z}_\ell.$$

We claim that, for any $x \in A$,

$$\exp(x \cdot N) = \phi(x) \in K_2.$$

Indeed, as remarked above, this identity holds for $x = t(\sigma_0)$. Since both exp and ϕ are multiplicative, then the identity holds for any $x \in t(\sigma_0) \cdot \mathbb{Z} \subset t(\sigma_0) \cdot \mathbb{Z}_\ell = A$. As both exp and ϕ are continuous, the identity holds for any $x \in t(\sigma_0) \cdot \mathbb{Z}_\ell = A$ (simply because $\mathbb{Z}$ is dense in $\mathbb{Z}_\ell$).

We finally put H to be the preimage of A under the map

$$\rho^{-1}(K_2) \cap I_F \twoheadrightarrow (\rho^{-1}(K_2) \cap I_F)/J \xrightarrow{t} t(\rho^{-1}(K_2) \cap I_F).$$

Then H is open in $\rho^{-1}(K_2) \cap I_F$, hence also in I_F. By the observation in the previous paragraph, we have

$$\rho(\sigma) = \phi(t(\sigma)) = \exp(t(\sigma) \cdot N)$$

for any $\sigma \in H$. □

Exercise 5.8. Let $\alpha \in \overline{\mathbb{Q}}_\ell$ be a root of unity such that $\alpha \in 1 + \ell^2\overline{\mathbb{Z}}_\ell$. Prove that $\alpha = 1$.

5.5. **Weil–Deligne representations.**

Definition 5.9. Let C be any algebraically closed field of characteristic 0. An n-dimensional *Weil–Deligne representation of* W_F with C-coefficient is a triple (r, V, N) consisting of

(1) an n-dimensional smooth C-representation (r, V) of W_F,
(2) a nilpotent endomorphism $N \in \operatorname{End}_C(V)$ ("monodromy operator") satisfying

$$r(\sigma) \cdot N \cdot r(\sigma)^{-1} = q^{-v(\sigma)} \cdot N \quad \text{for any } \sigma \in W_F.$$

We can define the notion of a homomorphism (and so on) for Weil–Deligne representations in a natural way. We write WD_C for the set of isomorphism classes of finite-dimensional Weil–Deligne representations with C-coefficients.

Remark 5.10. Recall that the smoothness is equivalent to the continuity with respect to the *discrete* topology of the coefficient field. Thus the choice of C does not matter so much in the above definition. To be more precise, if we have an isomorphism $C \cong C'$ (as abstract fields), then we have $\mathrm{WD}_C \cong \mathrm{WD}_{C'}$.

Exercise 5.11. In fact, any endomorphism $N \in \operatorname{End}_{\mathbb{C}}(V)$ satisfying the condition as in Definition 5.9 (2) is necessarily nilpotent; prove this.

Let (ρ, V) be an ℓ-adic repersentation of W_F. Let N be the nilpotent endomorphism associated to ρ by Grothendieck's monodromy theorem. We fix a lift $\Phi \in W_F$ of the geometric Frobenius and define a map

$$r\colon W_F \to \mathrm{Aut}_{\overline{\mathbb{Q}}_\ell}(V)$$

by

$$r(\Phi^k \cdot \sigma) := \rho(\Phi^k \cdot \sigma) \cdot \exp(t(\sigma) \cdot N)^{-1}$$

for $k \in \mathbb{Z}$ and $\sigma \in I_F$. Then, it is not difficult to see that r is a homomorphism. By the monodromy theorem, r is trivial on an open subgroup of I_F. In other words, (r, V) is a smooth representation of W_F. Furthermore, it can be also checked that (r, V, N) is a Weil–Deligne representation.

Conversely, for any Weil–Deligne representation, we can define an ℓ-adic representation by reversing the above procedure.

Theorem 5.12 ("Second form" of Grothendieck's monodromy theorem)**.** *The above association $\rho \mapsto (r, V, N)$ gives an equivalence between*

- *the category of ℓ-adic representations of W_F, and*
- *the caegory of finite-dimensional Weil–Deligne representations.*

In particular, we obtain a bijective map

$$\mathrm{WD}\colon \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F) \xrightarrow{1:1} \mathrm{WD}_{\overline{\mathbb{Q}}_\ell}.$$

In fact, it is not difficult to prove that the "converse direction" association $(r, V, N) \mapsto \rho$ gives a well-defined functor and also that it is a fully faithful. So the nontrivial point of the above theorem is that this association can indeed exhausts all ℓ-adic representation; this is nothing but the content of Grothendieck's monodromy theorem. We omit the details of the proof of Theorem 5.12, but it is a routine work as long as we admit Grothendieck's monodromy theorem, which we already proved. See, e.g., [BH06, 32.6].

As mentioned above, the point here is that $\mathrm{WD}_{\overline{\mathbb{Q}}_\ell}$ is essentially independent of ℓ; for any distinct ℓ' (not equal to p), we have an abstract field isomorphism $\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}$, hence $\mathrm{WD}_{\overline{\mathbb{Q}}_\ell} \cong \mathrm{WD}_{\overline{\mathbb{Q}}_{\ell'}}$. Thus, now we arrived at the following picture.

$$\begin{tikzcd}
\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\Gamma_F) \arrow[r, hook] & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(W_F) \arrow[r, "1:1", "\mathrm{rest}"'] & \mathrm{WD}_{\overline{\mathbb{Q}}_\ell} \arrow[d, "1:1"', "\overline{\mathbb{Q}}_\ell \cong \overline{\mathbb{Q}}_{\ell'}"] \\
\mathrm{Rep}_{\overline{\mathbb{Q}}_{\ell'}}(\Gamma_F) \arrow[r, hook] & \mathrm{Rep}_{\overline{\mathbb{Q}}_{\ell'}}(W_F) \arrow[r, "1:1", "\mathrm{rest}"'] & \mathrm{WD}_{\overline{\mathbb{Q}}_{\ell'}}
\end{tikzcd}$$

6. Week 6: Local Langlands correspondence for GL_n

This week we discuss the statement of the local Langlands correspondence for $\mathrm{GL}(n)$, especially, its characterization.

6.1. Local Langlands correspondence for GL_n.

Let F be a non-archimedean local field with residue field $k_F = \mathbb{F}_q$, which is of characteristic p.

Recall that, last week we discussed the notion of Weil–Deligne representation of W_F. In the following, when we talk about a smooth representation of W_F, we always assume that it is finite-dimensional without particularly declaring.

Lemma 6.1. *For any Weil–Deligne representation (r, V, N), the following are equivalent:*

1. *The image of $r(\Phi)$ is semisimple for some lift Φ of the geometric Frobenius.*
2. *The image of $r(\Phi)$ is semisimple for any lift Φ of the geometric Frobenius.*
3. *The smooth representation (r, V) of W_F is semisimple.*

Proof. Here we omit the proof; see, e.g., [BH06, 32.7]. (Basically the idea is to go back to the proof of monodromy theorem.) ☐

Definition 6.2. Let (ρ, V, N) be a Weil–Deligne representation.

1. We say that (r, V, N) is *Frobenius-semisimple* if the image of $r(\Phi)$ is semisimple for a lift Φ of the geometric Frobenius.
2. We say that (r, V, N) is *semisimple* if it is Frobenius-semisimple and $N = 0$.

Remark 6.3. Note that our terminology is a bit confusing; when a Weil–Deligne representation (r, V, N) is Frobenius-semisimple and N is nonzero, (r, V, N) is not semisimple in our sense, but its underlying smooth representation (r, V) of W_F is semisimple.

We let

- $\Pi(\mathrm{GL}_n)$ be the set of irreducible admissible representations of $\mathrm{GL}_n(F)$, and
- $\mathrm{WD}^{\mathrm{Frob}}_{\mathbb{C},n}$ be the set of isomorphism classes of 2-dimensional Frobenius-semisimple Weil–Deligne representations.

The local Langlands correspondence for GL_n, which was established by Harris–Taylor and Henniart, asserts that there is a natural bijection between these two sets.

Theorem 6.4 (LLC for GL_n, [HT01, Hen00]). *There exists a unique bijection*

$$\mathrm{LLC}_{\mathrm{GL}_n} \colon \Pi(\mathrm{GL}_n) \xrightarrow{1:1} \mathrm{WD}^{\mathrm{Frob}}_{\mathbb{C},n}$$

satisfying the following properties:

1. *(compatibility with LCFT) For any $\chi \in \Pi(\mathrm{GL}_1)$, we have*
$$\mathrm{LLC}_{\mathrm{GL}_1}(\chi) = \chi \circ \mathrm{Art}_F^{-1},$$
where $\mathrm{Art}_F \colon F^\times \cong W_F^{\mathrm{ab}}$ denotes the local Artin map of the local class field theory.
2. *(compatibility with character twist) For any $\pi \in \Pi(\mathrm{GL}_n)$ and $\chi \in \Pi(\mathrm{GL}_1)$, we have*
$$\mathrm{LLC}_{\mathrm{GL}_n}(\pi \otimes (\chi \circ \det)) = \mathrm{LLC}_{\mathrm{GL}_n}(\pi) \otimes \mathrm{LLC}_{\mathrm{GL}_1}(\chi).$$
3. *(compatibility with central characters) For any $\pi \in \Pi(\mathrm{GL}_n)$ with central character $\omega_\pi \in \Pi(\mathrm{GL}_1)$, we have*
$$\mathrm{LLC}_{\mathrm{GL}_1}(\omega_\pi) = \det \circ \mathrm{LLC}_{\mathrm{GL}_n}(\pi).$$
4. *(compatibility with duality) For any $\pi \in \Pi(\mathrm{GL}_n)$, we have*
$$\mathrm{LLC}_{\mathrm{GL}_n}(\pi^\vee) = \mathrm{LLC}_{\mathrm{GL}_n}(\pi)^\vee.$$

(5) (preservation of local factors) For any $\pi_1 \in \Pi(\mathrm{GL}_{n_1})$ and $\pi_2 \in \Pi(\mathrm{GL}_{n_2})$, we have

$$L(s, \pi_1 \times \pi_2) = L(s, \mathrm{LLC}_{\mathrm{GL}_{n_1}}(\pi_1) \otimes \mathrm{LLC}_{\mathrm{GL}_{n_2}}(\pi_2)),$$
$$\varepsilon(s, \pi_1 \times \pi_2) = \varepsilon(s, \mathrm{LLC}_{\mathrm{GL}_{n_1}}(\pi_1) \otimes \mathrm{LLC}_{\mathrm{GL}_{n_2}}(\pi_2)).$$

Here, the left-hand sides are the automorphic local factors of Jacquet–Piatetski-Shapiro–Shalika [JPSS83] *and the right-hand sides are the Galois-theoretic local factors of Deligne–Langlands* [Del73].

Note that although the properties (1)–(4) are quite important, they do not determine the map $\mathrm{LLC}_{\mathrm{GL}_n}$ uniquely at all. For the unique characterization, the propery (5) is really essential.

6.2. **Example: the case of** GL_2**.** Before we discuss the property (5) of Theorem 6.4, we consider the case of GL_2.

Recall that irreducible admissible representations of $\mathrm{GL}_2(F)$ are classified as follows (Week 4):

1. Irreducible principal series representations. The representation $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ is irreducible if and only if $\chi_1\chi_2^{-1} \neq \mathbb{1}, |-|^2$.
2. Character twistes of Steinberg/trivial representations. If $\chi_1\chi_2^{-1} = \mathbb{1}$, hence $\chi_1 = \chi_2 = \chi_0$ for some smooth character $\chi_0 \colon F^\times \to \mathbb{C}^\times$, then $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2 \cong \mathrm{Ind}_{B(F)}^{G(F)} \mathbb{1} \otimes (\chi_0 \circ \det)$ has two irreducible subquotients $\chi_0 \circ \det$ and $\mathrm{St}_{\mathrm{GL}_2} \otimes (\chi_0 \circ \det)$.
3. Irreducible supercuspidal representations. The representations which are not of the above two types are called supercuspidal representations.
   - Depth-zero supercuspidal representations.
   - Simple supercuspidal representations (depth $\frac{1}{2}$).
   - Deeper-depth supercuspidal representations.

Let us also classify 2-dimensional semisimple Weil–Deligne representations. Let (r, V, N) be such a representation.

When $N = 0$, we only have two possibilities; (r, V) is an irreducible 2-dimensional smooth representation of W_F or the sum of two smooth 1-dimensional representations (characters) of W_F. Here, we do not talk about how to further classify irreducible 2-dimensional smooth representations of W_F.

We consider the case where $N \neq 0$. In this case, we may choose a basis of V to regard $V \cong \mathbb{C}^{\oplus 2}$ such that the matrix representation of N is given by $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. Then, for any $\sigma \in W_F$, we can check that the form of $r(\sigma)$ is very limited as follows.

Exercise 6.5. Prove that the conditions

- $r(\sigma)$ is semisimple,
- $r(\sigma) \cdot N \cdot r(\sigma) = q^{-v(\sigma)} \cdot N$

implies the following[5]: by replacing a $\mathbb{C}$-basis of V if necessary, the representation matrix of $r(\sigma)$ is given by

$$r(\sigma) = \begin{pmatrix} z \cdot q^{-\frac{v(\sigma)}{2}} & 0 \\ 0 & z \cdot q^{\frac{v(\sigma)}{2}} \end{pmatrix}$$

for some $z \in \mathbb{C}^\times$.

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[5] Hao-An Wu pointed out during my lecture that the initial version of the statement written here was wrong, so I upldated. Thank you very much for pointing out!

Let $|\cdot|\colon W_F \to \mathbb{C}^\times$ be the absolute value character, i.e., $|\sigma| := q^{-v(\sigma)}$. Then the above observation implies that we must have

$$r = (\chi \otimes |\cdot|^{\frac{1}{2}}) \oplus (\chi \otimes |\cdot|^{-\frac{1}{2}}),$$

where χ is a smooth character of W_F. In other words, if we define a 2-dimensional Frobenius-semisimple Weil–Deligne representation "Sp(2)" by

$$\mathrm{Sp}(2) := (|\cdot|^{\frac{1}{2}} \oplus |\cdot|^{-\frac{1}{2}}, \mathbb{C}^{\oplus 2}, N),$$

then we have $(r, V, N) = \mathrm{Sp}(2) \otimes \chi$.

Now the local Langlands correspondence for GL_2 is stated more precisely as follows:

(1) An irreducible principal series representation $\mathrm{Ind}_{B(F)}^{G(F)} \chi_1 \boxtimes \chi_2$ corresponds to

$$(\chi_1 \otimes |\cdot|^{\frac{1}{2}} \oplus \chi_2 \otimes |\cdot|^{-\frac{1}{2}}, \mathbb{C}^{\oplus 2}, 0).$$

(2) A character $\chi \circ \det$ corresponds to

$$(\chi \otimes |\cdot|^{\frac{1}{2}} \oplus \chi \otimes |\cdot|^{-\frac{1}{2}}, \mathbb{C}^{\oplus 2}, 0).$$

(3) A character twist of the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2} \otimes (\chi \circ \det)$ corresponds to

$$\mathrm{Sp}(2) \otimes \chi = (\chi \otimes |\cdot|^{\frac{1}{2}} \oplus \chi \otimes |\cdot|^{-\frac{1}{2}}, \mathbb{C}^{\oplus 2}, N).$$

(4) An irreducilbe supercuspidal representation corresponds to

$$(r, V, 0),$$

where (r, V) is a 2-dimensional irreducible smooth representation of W_F.

6.3. Idea of the characterization of LLC for GL_n.

The fundamental philosophy in number theory is:

> *we should be able to attach a ζ-function or L-function to any number-theoretic object.*

Because this is just a slogan, the meaning of "ζ/L-function" or "number-theoretic object" are not clear. Please just keep in mind that the most basic example is the Riemann ζ-function $\zeta(s)$. So a ζ/L-function in general is something expected to satisfy various nice properties similarly to $\zeta(s)$, e.g., meromorphic continuation to the whole plane $\mathbb{C}$, functional equaltion, Euler product decomposition into local factors, and so on. If you have studied theory of modular forms, please remember that we can associate the L-function to any nice modular form and that they indeed satisfy such properties.

Recall that, in Week 1, we looked at an example of the global Langlands correspondence, which relates a modular form (say f) and an elliptic curve (say E). In fact, the mysterious connection between them explained there can be stated in a cleaner way by appealing to their L-functions, i.e., $L(s, f) = L(s, E)$. The point here is that the relation between f and E can be uniquely characterized by this equation (this is a consequence of so-called "strong multiplicity one theorem" on the automorphic side and "Chebotarev density theorem" on the Galois side).

So the idea of formulating the local Langlands correspondence for GL_n is to introduce a local version of L-functions (called "local L-factors") for irreducible admissible representations of $\mathrm{GL}_n(F)$ and Weil–Deligne representations and then characterize the correspondence using them. However, in fact, using only local L-factors is not enough for the unique characterization. We additionally need "local ε-factors" and also their further variants for "pairs" of representations.

Theory of local factors is too deep to be explained within just one week, so please let me first declare that my explanation below is very naive.

6.4. **Local L-factors and ε-factors.** Let us start with the following well-known elementary lemma.

Lemma 6.6. *Let G be a group and (ρ, V) be a representation of G. For any normal subgroup H of G, the subspace V^H of H-fixed vectors is G-stable, i.e., a G-subrepresentation. Moreover, the action of G on V^H factors through G/H.*

Proof. Let $v \in V^H$. Our task is to show that, for any $g \in G$, $\rho(g)(v)$ again belongs to V^H. For any $h \in H$, we have

$$\rho(h)\big(\rho(g)(v)\big) = \rho(g)\big(\rho(g^{-1}hg)(v)\big) = \rho(g)(v),$$

hence $\rho(g)(v)$ is fixed by $\rho(h)$ (in the second equality, we used that H is normal in G). The second assertion is obvious. □

We fisrt define the local L-factor of a smooth representation of W_F. Recall that the inertia subgroup is a normal subgroup of W_F such that W_F/I_F is isomorphic to the subgroup of $\mathrm{Gal}(\overline{k}_F/k_F)$ which is generated by the geometric Frobenius element Frob_{k_F} (inverse to $x \mapsto x^q$). We fix a lift $\Phi \in W_F$ of Frob_{k_F}.

Definition 6.7. Let (r, V) be a semisimple smooth representation of W_F. We define a complex function $L(s, r)$ on $s \in \mathbb{C}$ by

$$L(s, r) := \det\big(1 - r(\Phi) \cdot q^{-s} \mid V^{I_F}\big)^{-1}.$$

We call $L(s, r)$ the *local L-factor* of (r, V).

Remark 6.8. Note that $W_F/I_F \cong \mathbb{Z}$, hence any its semisimple representation decomposes into the sum of 1-dimensional characters of W_F/I_F. We say that such a character (i.e., a character of W_F trivial on I_F) an *unramified character*. If we write $V^{I_F} = \bigoplus_{i=1}^{r} \chi_i$, where each χ_i is an unramified character of W_F, then we get

$$L(s, r) = \prod_{i=1}^{r}\big(1 - \chi_i(\Phi) \cdot q^{-s}\big)^{-1}.$$

Example 6.9. Let us give two extremal examples of local L-factors.

(1) If (r, V) is the trivial representation of W_F, then $L(s, r) = (1 - q^{-s})^{-1}$. Note that, when $q = p$, this is nothing but the local factor of the Riemann ζ-function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} (1 - p^{-s})^{-1}.$$

(2) If (r, V) is an irreducible smooth representation of W_F, then V^{I_F} must be either 0 or V since it is a W_F-subrepresentation of V. If $V^{I_F} = V$, then it means that (r, V) is a 1-dimensional unramified character. Especially, if the dimension of (r, V) is greater than 1, its L-factor is always trivial ($L(s, r) = 1$).

These examples shows that the local L-factor only knows the unramified part of the given representation. From global perspective, this is enough because any "nice" ℓ-adic Galois representation of a global field is unramified at almost all places and uniquely determined by its behavior there by Cheobtarev density theorem. However, from local perspective, the local L-factor is not enough for the unique characterization.

We next define the local L-factor for a Weil–Deligne representation.

Lemma 6.10. *Let (r, V, N) be a Frobenius-semisimple Weil–Deligne representation. Let $V^{N=0}$ denote $\mathrm{Ker}(N\colon V \to V)$. Then $V^{N=0}$ is stable under the W_F-action, hence is a semisimple smooth representation of W_F.*

Proof. Suppose $v \in V^{N=0}$. Our task is to show that, for any $\sigma \in W_F$, $r(\sigma)(v)$ again belongs to $V^{N=0}$. We have

$$N\bigl(r(\sigma)(v)\bigr) = r(\sigma)\bigl(r(\sigma)^{-1} \cdot N \cdot r(\sigma)(v)\bigr) = r(\sigma)\bigl(q \cdot N(v)\bigr) = r(\sigma)(0) = 0.$$

□

Definition 6.11. Let (ρ, V, N) be a Frobenius-semisimple Weil–Deligne representation. We define a complex function $L(s, (r, V, N))$ on $s \in \mathbb{C}$ by

$$L(s, (r, V, N)) := L(s, V^{N=0}).$$

We call $L(s, (r, V, N))$ the *local L-factor* of (r, V, N).

Exercise 6.12. Compute $L(s, \mathrm{Sp}(2))$.

We just give a brief comment on "local ε-factors". Recall that the (completed) Riemann ζ-function $\hat{\zeta}(s)$ satisfies the functional equation $\hat{\zeta}(s) = \hat{\zeta}(1-s)$. Then, should we expect that such a symmetric equation can be satisfied in general by any L-function associated to a sufficiently nice global object? In fact, it's not literally so in general, but it is expected that the functional equation holds after adding a correction term called the ε-factor. The typical form of the functional equation for a global object X is like

$$L(s, X) = \varepsilon(s, X) \cdot L(1-s, X^\vee).$$

It is expected that $\varepsilon(s, X)$ also decomposes into the product of local factors, and those local factors are called local ε-factors[6].

This is just a philosophical explanation of the role of local ε-factors. In our context (i.e., smooth representations of W_F and also Weil–Deligne representations), there are axiomatic properties of the local ε-factor $\varepsilon(s, \rho)$, which is a complex function on $s \in C$. It is proved that the function $\varepsilon(s, \rho)$ always exists and is uniquely characterized by those properties. In the case of smooth characters, its definion was given by Tate (so-called "Tate's thesis"). In the general case, it is as follows (see, [BH06, Section 29]).

Theorem 6.13. *For any semisimple smooth representation r of W_F, there uniquely exists a complex function $\varepsilon(s, r) \in \mathbb{C}[q^{\pm s}]^\times$ satisfying the following properties:*

1. *If r is 1-dimensional, then $\varepsilon(s, r)$ coincides with Tate's one.*
2. *For any two semisimple smooth representations r_1 and r_2 of W_F, we have $\varepsilon(s, r_1 \oplus r_2) = \varepsilon(s, r_1) \oplus \varepsilon(s, r_2)$.*
3. *For any finite separable extensions $E \supset K \supset F$ and a semisimple smooth representation r of W_E, we have*

$$\frac{\varepsilon(s, \mathrm{Ind}_{W_E}^{W_K} r)}{\varepsilon(s, r)} = \frac{\varepsilon(s, \mathrm{Ind}_{W_E}^{W_K} \mathbb{1}_{W_E})^{\dim r}}{\varepsilon(s, \mathbb{1}_{W_E})^{\dim r}}.$$

---

[6] But note that the local L-factors and the local ε-factors are NOT expected to satisfy the local functional equation. For example, when (ρ, V) is a ramified irreducible representation, the L-factors $L(s, \rho)$ and $L(s, \rho^\vee)$ are trivial, but $\varepsilon(s, \rho)$ is nontrivial and knows how "deep" the ramification of ρ is.

Once we define the local ε-factor for semisimple smooth representations of W_F in this way, we can also extend it to any Frobenius-semisimple Weil–Deligne representations; see [BH06, Section 31] for details.

Remark 6.14. For a smooth representation ρ of W_F, its local ε-factor is defined by choosing a nontrivial additive character ψ_F of F. For this reason, it is usually denoted by $\varepsilon(s, \rho, \psi_F)$, but here we omit it from the notation. Note that, in (3) of the above theorem, we choose such a character to be $\psi \circ \mathrm{Tr}_{E/F}$ for any finite separable extension E/F by fixing one ψ_F. (The same is true for Weil–Deligne represetations.)

Exercise 6.15. Let (r, V) be a semisimple smooth representation of W_E for a finite separable extension E/F. Prove that $L(s, \mathrm{Ind}_{W_E}^{W_F} r) = L(s, r)$.

So far, we have only talked about the local factors on the Galois side. In fact, there is also a parallel picture established on the automorphic side. It was initiated by Tate in the case of GL_1 (the above-mentioned local factors for 1-dimensional characters of W_F are nothing but the "transfer" of Tate's factors on the automoprhic side via local class field theory) and then far generalizes to GL_n by Godement–Jacquet.

In this course, I do not explain anything about its definition; actually, it is not easy at all even to state the definition of the local factors on the automoprhic side. In some sense, this difficulty of providing a definition and the consequences derived from it are in a "trade-off" relationship. On the Galois side, it is quite easy to define the local L-factor. But it is typically so nontrivial to show that those factors indeed satisfy nice properties, especially, global properties such as meromorphic continuation, functional equation, etc. On the automorphic side, it is already a highly nontrivial task to give its definition. But, once the definition is given, we can prove a lot about its properties by appealing to the well-established general theory of automoprhic representations.

6.5. **Local L-factors and ε-factors for pairs.** I finally also give some comments about pairs.

As mentioned above, the local ε-factor enables us to get more information of the given irreducible admissible representation of $\mathrm{GL}_n(F)$ or Frobenius semisimple Weil–Deligne representation. For example, we can read off the depth (a.k.a., conductor/slope on the Galois side) from the local ε-factor. However, it is still not enough to uniquely determine the given representation. In other words, it really happens that two non-isomorphic representations π_1 and π_2 satisfy $L(s, \pi_1) = L(s, \pi_2)$ and $\varepsilon(s, \pi_1) = \varepsilon(s, \pi_2)$.

The idea is to consider "pairs". Let us first look at the Galois side. Suppose that an irreducible smooth representation (r, V) of W_F whose dimension is greater than 1 is given. Then it is impossible to recover r from $L(s, r)$ because $L(s, r) = 1$ as explained above. However, what will happen if we consider $L(s, r \otimes r')$ for "all" semisimple smooth representations r' of W_F? By definition of the local L-factor, $L(s, r)$ has a pole at $s = 0$ if and only if r contains the trivial representation of W_F. Hence, $L(s, r \otimes r')$ contains a pole at $s = 0$ if and only if $r \otimes r'$ contains the trivial representation. Note that

$$\mathrm{Hom}_{W_F}(\mathbb{1}, r \otimes r') = \mathrm{Hom}_{W_F}(r^\vee, r').$$

In particular, when r' is irreducible, we see that $L(s, r \otimes r')$ has a pole at $s = 0$ if and only if $r^\vee$ is isomorphic to r'. Therefore, when two irreducible smooth representations r_1 and r_2 of W_F are given, we can distinguish them by looking at the poles of $L(s, r_1 \otimes r)$ and $L(s, r_2 \otimes r)$.

Note that, so that this idea works, we need the notion of "the tensor product". On the Galois side, for a given semisimple representations r_1 and r_2 of W_F whose dimensions are

n_1 and n_2, we can construct their tensor product representation $r_1 \otimes r_2$, whose dimension is $n_1 n_2$. Thus what we need on the automorphic side is a way of associating an irreducible smooth representation "$\pi_1 \otimes \pi_2$" of $\mathrm{GL}_{n_1 n_2}(F)$ to any pair of irreducible admissible representations π_1 of $\mathrm{GL}_{n_1}(F)$ and π_2 of $\mathrm{GL}_{n_2}(F)$. Such an a-priori-hypothetical object is called the *Rankin–Selberg product* of π_1 and π_2. In fact, the Rankin–Selberg product can only make sense after we prove the local Langlands correspondence for GL_n. (Such an operation on the automorphic side which can be defined by appealing to the local Langlands correspondence is in general referred to as the *Langlands functoriality*). However, the point is that it is possible to establish the definition of $L(s, \pi_1 \otimes \pi_2)$ and $\varepsilon(s, \pi_1 \otimes \pi_2)$ without defining $\pi_1 \otimes \pi_2$; they are called *Rankin–Selberg local factors* of Jacquet–Piatetski-Shapiro–Shalika [JPSS83].

7. WEEK 7: LOCAL LANGLANDS CORRESPONDENCE FOR GENERAL GROUPS

Let F be a non-archimedean local field. Recall that the local Langlands correspondence for GL_n over F is a natural bijective map

$$\mathrm{LLC}_{\mathrm{GL}_n}\colon \Pi(\mathrm{GL}_n) \xrightarrow{1:1} \mathrm{WD}^{\mathrm{Frob}}_{\mathbb{C},n},$$

where

- $\Pi(\mathrm{GL}_n)$ is the set of isomorphism classes of irreducible smooth (or, equivalently, admissible) representations of $\mathrm{GL}_n(F)$, and
- $\mathrm{WD}^{\mathrm{Frob}}_{\mathbb{C},n}$ is the set of isomorphism classes of Frobenius-semisimple Weil–Deligne representations.

It is conjectured that this correspondence can be generalized to any connected reductive group G over F. The aim of this week is to understand the rough statement of the local Langlands correspondence for general G.

7.1. Reductive groups and Langlands dual groups.

Recall that a *linear algebraic group* over F is an algebraic group over F (i.e., an algebraic variety over F equipped with a group structure whose morphisms are algebraic) which can be embedded into some GL_n as a closed subgroup.

Proposition/Definition 7.1. Let G be a connected linear algebraic group over F. There uniquely exists a maximal closed connected normal unipotent subgroup of G defined over F; we call it the *unipotent radical* of G and write $R_u(G)$. We say that G is *reductive* if $R_u(G)$ is trivial.

Example 7.2. The general linear group GL_n over F is connected and reductive;

$$\mathrm{GL}_n(F) = \{g \in M_n(F) \mid \det(g) \in F^\times\}.$$

The special linear group

$$\mathrm{SL}_n(F) = \{g \in M_n(F) \mid \det(g) = 1\}$$

and the projective linear group

$$\mathrm{PGL}_n(F) = \mathrm{GL}_n(F)/F^\times$$

are also connected reductive groups[7].

Example 7.3. Let $J \in M_{2n}(F)$ be any skew-symmetric (i.e., ${}^tJ = -J$) non-degenerate matrix. Then its associated symplectic group is connected and reductive;

$$\mathrm{Sp}(J)(F) := \{g \in \mathrm{GL}_{2n}(F) \mid {}^tgJg = J\}.$$

Note that any skew-symmetric non-degenerate matrices J and J' of $M_{2n}(F)$ are conjugate over F, which implies that their associated symplectic groups $\mathrm{Sp}(J)$ and $\mathrm{Sp}(J')$ are isomorphic over F. For this reason, we often fix a symplectic form J and write Sp_{2n} instead of $\mathrm{Sp}(J)$. The typical choices of the symplectic matrices are, for example,

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[7] Note that here and from now on, we only describe the group of F-rational points. For example, precisely speaking, PGL_n is an algebraic group over F whose group of R-valued points is given by $(\mathrm{GL}_n/\mathbb{G}_{\mathrm{m}})(R)$ for any F-algebra R, where $\mathbb{G}_{\mathrm{m}}$ is embedded in GL_n as the subgroup of scalar matrices.

- $\begin{pmatrix} & & & & 1 \\ & & & -1 & \\ & & 1 & & \\ & \iddots & & & \\ -1 & & & & \end{pmatrix}$ (+1 and -1 are put alternatively on the anti-diagonals)[8],
- $\begin{pmatrix} & I_n \\ -I_n & \end{pmatrix}$ (I_n denotes the identity matrix of size n).

Example 7.4. Let $J \in M_n(F)$ be a symmetric non-degenerate matrix, i.e., ${}^tJ = J$. We consider its associated orthogonal group;

$$\mathrm{O}(J)(F) := \{g \in \mathrm{GL}_n(F) \mid {}^tgJg = J\}.$$

This group is not connected and its identity component is of index two in $\mathrm{O}(J)$. We write $\mathrm{SO}(J)$ for it and call the special orthogonal group associated to J; this is a reductive group. Note that, in contrast to the symplectic case, symmetric non-degenerate matrices J and J' of $M_n(F)$ may not necessarily conjugate over F, which means that their associated special orthogonal groups $\mathrm{SO}(J)$ and $\mathrm{SO}(J')$ may not isomorphic over F. Hence it is really essential to specify which symmetric matrix (or at least its conjugacy class) is chosen in its definition.

Example 7.5. Let E/F be a quadratic extension. Let $J \in M_n(E)$ be any hermitian matrix, i.e., ${}^t\overline{J} = -J$, where $\overline{(-)}$ denotes the Galois conjugate of a matrix. Then its associated unitary group is connected and reductive;

$$\mathrm{U}(J)(F) := \{g \in \mathrm{GL}_n(E) \mid {}^t\overline{g}Jg = J\}.$$

Simialrly to the even orthogonal case, hermitian matrices are not unique up to conjugacy, hence we must specify the choice of J.

When G is a connected reductive group over F, by fixing an embedding of G into some general linear group GL_n, we equip $G(F)$ with the topology induced from that of $\mathrm{GL}_n(F)$. Since G is Zariski-closed in GL_n, $G(F)$ is also closed with respect to the locally-profinite topology on $\mathrm{GL}_n(F)$. Hence $G(F)$ is a locally profinite group. We often call a group obtained in this way (i.e., of the form $G(F)$ for some connected reductive group G over F) a *p-adic reductive group*[9]. As stated in Week 3, it is known that any irreducible smooth representation of $G(F)$ is automatically admissible.

Now we can explain how to modify the automorphic side of the local Langlands correspondence in general. For a connected reductive group G over F, we just replace $\Pi(\mathrm{GL}_n)$ with $\Pi(G)$, which is the set of isomorphism classes of irreducible smooth (or, admissible) representations of $G(F)$.

Then, how about the Galois side? For general connected reductive G, the notion of a Weil–Deligne representation is replaced with the notion of an "L-parameter". To introduce the definition of an L-parameter, we first have to review the notion of the *Langlands dual group.*

It is known that the isomorphism classes of connected reductive groups over $\overline{F}$ can be classified by combinatorial/linear-algebraic objects called *(reduced) root data.* For any reduced root datum, we can naturally define its "dual". Thus, again using the classification

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[8] I prefere this one!

[9] Sometimes this terminology is a bit confusing because we say "p-adic" even when the characteristic of F is not zero. (Recall that F is a non-archimedean local field with any characteristic and that we say F is a p-adic field when it's of characteristic 0.)

theorem in the reverse direction, we get another connected reductive group; this is called the Langlands dual group and denoted by $\widehat{G}$. Here, the classification theorem works even if we replace $\overline{F}$ with any algebraically closed field, so let us choose $\mathbb{C}$ in the definition of $\widehat{G}$. If G is defined over F, then the corresponding root datum Ψ has an action of $\Gamma_F := \mathrm{Gal}(F^{\mathrm{sep}}/F)$, thus so does $\Psi^\vee$. This furthermore induces an action of Γ_F on $\widehat{G}$.

$$\begin{tikzcd}
\{\text{conn. red. gps over } \overline{F}\} \arrow[r, leftrightarrow, "1:1"] & \{\text{(reduced) root data}\} \arrow[d, leftrightarrow, "\text{dual}"] & G \arrow[r, mapsto] & \Psi \arrow[d, mapsto] \\
\{\text{conn. red. gps over } \mathbb{C}\} \arrow[r, leftrightarrow, "1:1"] & \{\text{(reduced) root data}\} & \widehat{G} & \Psi^\vee \arrow[l, mapsto]
\end{tikzcd}$$

For the details of the discussion so far, see, for example, [Bor79]. Here, we just list examples of the Langlands dual groups: Note that the dual group of the unitary group U_n (with respect to some hermitian form) is $\mathrm{GL}_n(\mathbb{C})$, which is the same as the dual group of $\mathrm{GL}_n(\mathbb{C})$. This is because U_n is isomorphic to GL_n over $\overline{F}$. However, there are not isomoprhic over F, hence the Galois actions induced on their root data are different; trivial for GL_n, but non-trivial (involutive) for U_n. Consequently, the actions of Γ_F on $\widehat{\mathrm{GL}_n}$ and $\widehat{\mathrm{U}_n}$ are different; the former is trivial, but the latter is not.

G	GL_n	SL_n	PGL_n	U_n	SO_{2n+1}	Sp_{2n}	SO_{2n}
$\widehat{G}$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{PGL}_n(\mathbb{C})$	$\mathrm{SL}_n(\mathbb{C})$	$\mathrm{GL}_n(\mathbb{C})$	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\mathrm{SO}_{2n}(\mathbb{C})$

Keeping this in mind, let us define the *L-group* of G to be ${}^LG := \widehat{G} \rtimes W_F$.

7.2. ***L*-parameters and rough form of LLC.** Let G be a connected reductive group over F.

Definition 7.6. An *L-parameter of* G is a homomorphism $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG$ satisfying the following conditions:

(1) $W_F \xrightarrow{\phi|_{W_F}} {}^LG = \widehat{G} \rtimes W_F \xrightarrow{\mathrm{pr}_2} W_F$ is the identity;

(2) $W_F \xrightarrow{\phi|_{W_F}} {}^LG = \widehat{G} \rtimes W_F \xrightarrow{\mathrm{pr}_1} \widehat{G}$ is smooth (i.e., trivial on an open subgroup $H \subset I_F$) and semisimple (i.e., any element of $\mathrm{pr}_1 \circ \phi(W_F)$ is a semisimple element of $\widehat{G}$);

(3) the image of $\mathrm{SL}_2(\mathbb{C}) \xrightarrow{\phi|_{\mathrm{SL}_2(\mathbb{C})}} {}^LG$ lies in $\widehat{G}$ and induces an algebraic homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$.

Remark 7.7. When the action of Γ_F (hence also W_F) on $\widehat{G}$ is trivial, the L-group is just the direct product $\widehat{G} \times W_F$. Any homomorphism $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG = \widehat{G} \times W_F$ is of the form $\phi = (\phi_1, \phi_2)$, where both $\phi_1\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ and $\phi_2\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to W_F$ are group homomorphisms[10]. Note that the above conditions (1) and (3) implies that ϕ_2 is necessarily equal to the first projection. Thus, in this case, we can define an L-parameter to be a homomorphism $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ satisfying the following:

(1) $\phi|_{W_F}$ is a smooth and semisimple homomorphism;
(2) $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is an algebraic homomorphism.

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[10] If the action of Γ_F on $\widehat{G}$ is not trivial, the first factor is a 1-cocycle valued in $\widehat{G}$.

Definition 7.8. We say that L-parameters ϕ and ϕ' are $\widehat{G}$-conjugate if there exists $g \in \widehat{G}$ such that $\phi'(\sigma, x) = g \cdot \phi(\sigma, x) \cdot g^{-1}$ for any $(\sigma, x) \in W_F \times \mathrm{SL}_2(\mathbb{C})$. We let $\Phi(G)$ denote the set of $\widehat{G}$-conjugacy classes of L-parameters of G.

Recall that the Galois side of the local Langlands correspondence for GL_n is formulated in terms of Weil–Deligne representations. The set $\Phi(G)$ exactly generalizes $\mathrm{WD}_{\mathbb{C},n}^{\mathrm{Frob}}$. To see this, let us take G to be GL_n.

We first take an L-parameter $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$. By putting $V := \mathbb{C}^{\oplus n}$ and

$$r(\sigma) := \phi\left(\sigma, \begin{pmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{pmatrix}\right)$$

for $\sigma \in W_F$. Since $\phi|_{W_F}$ is smooth, r is also a smooth homomorphism. Moreover, as $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic, the image of $(1, \left(\begin{smallmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{smallmatrix}\right))$ is semisimple. As $(\sigma, 1) \in W_F \times \mathrm{SL}_2(\mathbb{C})$ and $(1, \left(\begin{smallmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{smallmatrix}\right)) \in W_F \times \mathrm{SL}_2(\mathbb{C})$ commute, so are their images under ϕ; in particular, $r(\sigma)$ is the product of two commuting semisimple elements, hence semisimple. On the other hand, we put

$$N := \log \phi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right).$$

(Again noting that $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic, the image of $(1, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right))$ in $\widehat{G}$ is unipotent, hence its "log" makes sense.) Let us check that (r, V, N) is a Frobenius-semisimple Weil–Deligne representation. For this, it is enough to verify that

$$r(\sigma) \cdot N \cdot r(\sigma)^{-1} = q^{-v(\sigma)} \cdot N$$

for any $\sigma \in W_F$. The left-hand side equals

$$\begin{aligned}
&\phi\left(\sigma, \begin{pmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{pmatrix}\right) \log \phi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) \phi\left(\sigma^{-1}, \begin{pmatrix} q^{v(\sigma)/2} & 0 \\ 0 & q^{-v(\sigma)/2} \end{pmatrix}\right) \\
&= \sum_{m=1} \frac{(-1)^m}{m} \phi\left(\sigma, \begin{pmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{pmatrix}\right) \left(\phi\left(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) - 1\right)^m \phi\left(\sigma, \begin{pmatrix} q^{-v(\sigma)/2} & 0 \\ 0 & q^{v(\sigma)/2} \end{pmatrix}\right) \\
&= \sum_{m=1} \frac{(-1)^m}{m} \left(\phi\left(1, \begin{pmatrix} 1 & q^{-v(\sigma)} \\ 0 & 1 \end{pmatrix}\right) - 1\right)^m \\
&= \log \phi\left(1, \begin{pmatrix} 1 & q^{-v(\sigma)} \\ 0 & 1 \end{pmatrix}\right).
\end{aligned}$$

Finally, by the multiplicativity of log, the most-right-hand side of the above computation equals $q^{-v(\sigma)} \cdot N$.

We next consider the converse direction, i.e., start with taking a Frobenius semisimple Weil–Deligne representation (r, V, N). By choosing a $\mathbb{C}$-basis of V, we may regard $V = \mathbb{C}^{\oplus n}$. Then the monodromy operator N defines a nilpotent element of $\mathrm{End}_{\mathbb{C}}(V)$. Roughly speaking, the idea of associating an L-parameter to (r, V, N) is to apply the "Jacobson–Morosov theorem" to N, which claims that there exists an embedding of $\mathfrak{sl}_2(\mathbb{C})$ (Lie algebra of $\mathrm{SL}_2(\mathbb{C})$) into $\mathfrak{gl}_n(\mathbb{C})$ as Lie algebras such that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ maps to N. Then, by the simply-connectedness of $\mathrm{SL}_2(\mathbb{C})$, we can find an algebraic homomorphism $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ whose derivative coincides with the above one. Then it is not very difficult to see that this homomorphism extends to an L-parameter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$. (See, e.g., [GR10, Section 2.1] for the details).

We can also check that $\Phi(\mathrm{GL}_n)$ is bijective to $\mathrm{WD}_{\mathbb{C},n}^{\mathrm{Frob}}$ by this correspondence.

Remark 7.9. (1) Recall that we say that a Weil–Deligne representation is semisimple if it is Frobenius-semisimple and $N = 0$. By the above construction, we can easily see that this condition is equivalent to that the corresponding L-parameter is trivial on $\mathrm{SL}_2(\mathbb{C})$. Keeping this observation in mind, for general connected reductive group G, let us say that an L-parameter for G is *semisimple* if it is trivial on $\mathrm{SL}_2(\mathbb{C})$.

(2) When $G = \mathrm{GL}_n$, we may regard any L-parameter of G as an n-dimensional representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$. Please be careful that, as a representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$, it is always semisimple in the sense that it decomposes into the direct of irreducible subrepresentations. So the term "semisimple" in a "semisimple L-parameter" should be understood as referring to the semisimplicity of the corresponding Weil–Deligne representation.

So the situation for GL_n can be summarized as follows:

$$\begin{tikzcd}
 & \mathrm{WD}_{\mathbb{C},n} \arrow[r, equal] & \mathrm{WD}_{\overline{\mathbb{Q}}_\ell,n} \arrow[r, leftrightarrow, "\text{Groth.}"] & \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell,n}(W_F) \\
 & \cup & \cup & \cup \\
\Phi(\mathrm{GL}_n) \arrow[r, leftrightarrow, "\text{JM}"] & \mathrm{WD}^{\text{Frob-ss}}_{\mathbb{C},n} \arrow[r, equal] & \mathrm{WD}^{\text{Frob-ss}}_{\overline{\mathbb{Q}}_\ell,n} \arrow[r, leftrightarrow, "\text{Groth.}"] & \mathrm{Rep}^{\text{Frob-ss}}_{\overline{\mathbb{Q}}_\ell,n}(W_F) \\
\cup & \cup & \cup & \cup \\
\Phi^{\mathrm{ss}}(\mathrm{GL}_n) \arrow[r, leftrightarrow, "\text{JM}"] & \mathrm{WD}^{\mathrm{ss}}_{\mathbb{C},n} \arrow[r, equal] & \mathrm{WD}^{\mathrm{ss}}_{\overline{\mathbb{Q}}_\ell,n} \arrow[r, leftrightarrow, "\text{Groth.}"] & \mathrm{Rep}^{\mathrm{ss}}_{\overline{\mathbb{Q}}_\ell,n}(W_F)
\end{tikzcd}$$

Remark 7.10. For Frobenius-semisimple Weil–Deligne representations, we can naturally define the notion of the *semisimplification* by associating $(r, V, 0)$ to (r, V, N). On $\Phi(\mathrm{GL}_n)$, this operation is

$$\Phi(\mathrm{GL}_n) \to \Phi^{\mathrm{ss}}(\mathrm{GL}_n) \colon \phi \mapsto \phi^{\mathrm{ss}},$$

where

$$\phi^{\mathrm{ss}}(\sigma, x) := \phi\Bigl(\sigma, \begin{pmatrix} q^{-|\sigma|/2} & 0 \\ 0 & q^{|\sigma|/2} \end{pmatrix}\Bigr)$$

for $(\sigma, x) \in W_F \times \mathrm{SL}_2(\mathbb{C})$. (In particular, be careful that ϕ_{ss} is not defined by just forgetting the $\mathrm{SL}_2(\mathbb{C})$-part).

7.3. Rough form of LLC for general groups.

Conjecture 7.11 (LLC; the most rough form)**.** *There exists a natural map*

$$\mathrm{LLC}_G \colon \Pi(G) \to \Phi(G).$$

This conjecture is not rigorously stated at all because the meaning of "natural" is not clear in any sense. Remember that the local Langlands correspondence for GL_n is characterized by several axiomatic properties. So, what we desire to do here is to list properties of the map LLC_G which are considered appropriate to be satisfied, and use them to formulate the "naturality", i.e., characterize the map LLC_G.

In fact, at present, there is no characterization which can be uniformly formulated for arbitrary groups. For example, in the case of GL_n, a characterization is given via local factors, but it has not been known how to extend theory of local factors to arbitrary irreducible admissible representations of arbitrary p-adic reductive groups. However, at least there is a general consensus on the standard properties of LLC which should be satisifed even though they cannot determine the map LLC_G uniquely in general. Moreover, for some specific groups such as, e.g., Sp_{2n}, SO_n, or U_n (so-called "classical groups"), there is an ad hoc way to characterize the map LLC_G uniquely. It is one of the aims of this course to understand the statements of such expected properties of LLC and also a characterization for some particular class of p-adic reductive groups.

For today, let's just discuss only the notion of an *L-packet*, which serves the first step towards those general stories. It is NOT expected that the map LLC_G is bijective in general, but still expected to have finite fibers, i.e., the set $\mathrm{LLC}_G^{-1}(\phi)$ is a finite subset of $\Pi(G)$ for each $\phi \in \Phi(G)$. This finite set is referred to as the *L-packet* for ϕ; here let us write Π_ϕ for it. With this terminology and symbol, we may think of the local Langlands correspondence for G as a "natural" decomposition of the set $\Pi(G)$ into the disjoint union of finite-subsets of irreducible admissible representations of $G(F)$:

$$\Pi(G) = \bigsqcup_{\phi \in \Phi(G)} \Pi_\phi.$$

Unlike the case of GL_n, this is not quite enough from the viewpoint of the classification of irreducible admissible representations of $G(F)$. We also want to know the "structure" of each finite set Π_ϕ. In fact, it is also expected that the membere of the finite set Π_ϕ are labelled by Galois-theoretic information.

In the following, for simplicity, we assume that G is split[11]. For any L-parameter $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$, we define S_ϕ to be the centralizer group of the image of ϕ in $\widehat{G}$:

$$S_\phi := \mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi)).$$

Then we put

$$\mathcal{S}_\phi := \pi_0(S_\phi/Z(\widehat{G})),$$

where $Z(\widehat{G})$ denotes the center of $\widehat{G}$ and π_0 denotes the group of connected components. Note that, since $S_\phi/Z(\widehat{G})$ is a linear algebraic group, $\mathcal{S}_\phi$ is necessarily a finite group.

Conjecture 7.12. *For each $\phi \in \Phi(G)$, there exists a natural bijection $\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$.*

In other words, members of Π_ϕ are labelled by irreducible representations of $\mathcal{S}_\phi$. So it is often said that, for $\pi \in \Pi(G)$, you can think of $\phi := \mathrm{LLC}_G(\pi)$ as the "family name" of π and its corresponding element in $\mathrm{Irr}(\mathcal{S}_\phi)$ as the "first name" of π.

Again, note that the meaning of "natural" here is not quite clear. So that it makes sense, we have to understand the theory of *endoscopy*. Hopefully, we can encounter it a few weeks later.

Remark 7.13. We also remark that the bijection $\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$ is not canonical. It is supposed to depend on the choice of a Whittaker datum of G.

Exercise 7.14. Prove that $\mathcal{S}_\phi$ is always trivial for any $\phi \in \Phi(G)$ when $G = \mathrm{GL}_n$.

Exercise 7.15. Let E/F be a quadratic extension and $\chi\colon W_E \to \mathbb{C}^\times$ be a smooth character. Then $\mathrm{Ind}_{W_E}^{W_F} \chi$ is a 2-dimensional semisimple representation of W_F. Hence we may regard it as an L-parameter of GL_2 with trivial $\mathrm{SL}_2(\mathbb{C})$-part. Recalling that the Langlands dual group of SL_2 is PGL_2, we furthermore regard it as an L-parameter of SL_2 (by projecting along $\mathrm{GL}_2(\mathbb{C}) \twoheadrightarrow \mathrm{PGL}_2(\mathbb{C})$). Compute the order of $\mathcal{S}_\phi$. (If it is too difficult, you can freely choose χ to be some particular character of W_E.)

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[11] But actually, this assumption is essential rather than just "for simplicity". The conjectural description of each L-packet has been formulated also for non-split groups, but it is very deep and has a long history.

8. WEEK 8: PHILOSOPHY OF LANGLANDS FUNCTORIALITY

8.1. **Philosophy of Langlands functoriality.** In the last week, we discussed the conjectural local Langlands correspondence, which predicts the existence of a natural map

$$\mathrm{LLC}_G\colon \Pi(G) \to \Phi(G)$$

with fibers Π^G_ϕ called L-packets (we used the symbol Π_ϕ in the last week, but here we additionally put G). Recall that we consider the Langlands dual group $\widehat{G}$, not the group G, on the Galois side $\Phi(G)$. The important observation here is that two different connected reductive groups G and H could be related through their duals (even though G and H themselves are not directly related!).

To be more precise, we say that a map $\xi\colon {}^LH \to {}^LG$ is an *L-homomorphism* if it is compatible with projections to W_F and restricts to an algebraic homomorphism $\widehat{H} \to \widehat{G}$. If we have such a homomorphism, then we can associate an L-parameter $\xi \circ \phi$ of G to an L-parameter ϕ of H. In other words, we have a map $\Phi(H) \to \Phi(G)\colon \phi \mapsto \xi \circ \phi$. Therefore, if we have the local Langlands correspondences for both G and H, we obtain an association between L-packets $\Pi^H_\phi \mapsto \Pi^G_{\xi\circ\phi}$. This operation is called the *functorial lifting* associated to ξ. The L-packet $\Pi^G_{\xi\circ\phi}$ is called the *functorial lift* of Π^H_ϕ associated to ξ.

$$\begin{tikzcd}
\Pi^G_{\xi\circ\phi} \arrow[r, leftrightsquigarrow, "\text{LLC for } G"] & {} & & {}^LG \\
\Pi^H_\phi \arrow[u, rightsquigarrow, "\text{functorial lifting}"] \arrow[r, leftrightsquigarrow, "\text{LLC for } H"] & {} & W_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[r, "\phi"] \arrow[ur, "\xi\circ\phi"] & {}^LH \arrow[u, "\xi"']
\end{tikzcd}$$

So the definition of the functorial lifting depends on the existence of the local Langlands correspondence for both G and H. The point is that, however, sometimes it is possible to construct "a natural association" $\Pi^H_\phi \mapsto \Pi^G_{\xi\circ\phi}$ without appealing to the local Langlands correspondences for G and H. The meaning of "without appealing to" is that the definition of $\Pi^H_\phi \mapsto \Pi^G_{\xi\circ\phi}$ is given in terms of purely-representation theoretic language. The meaning of "a natural association" is that it is expected to be proved to coincide with the correct one once we establish the local Langlands correspondence for G and H.

Example 8.1 (Rankin–Selberg product)**.** Let $G = \mathrm{GL}_{nm}$ and $H = \mathrm{GL}_n \times \mathrm{GL}_m$. Then $\widehat{G} = \mathrm{GL}_{nm}(\mathbb{C})$ and $\widehat{H} = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C})$, hence we have a homomorphism ${}^LH \to {}^LG$ defined by the tensor product $(g, g') \mapsto g \otimes g'$. The resulting functorial lifting is called the *Rankin–Selberg lifting.*

Example 8.2 (Symmetric/Exterior power)**.** Let $G = \mathrm{GL}_N$ and $H = \mathrm{GL}_n$, where $N = n(n+1)/2$. Then $\widehat{G} = \mathrm{GL}_N(\mathbb{C})$ and $\widehat{H} = \mathrm{GL}_n(\mathbb{C})$. We consider the symmetric square representation $\mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_N(\mathbb{C})$, that is, any element $g \in \mathrm{GL}_n(\mathbb{C})$ acts on $\mathrm{Sym}^2(\mathbb{C}^{\oplus n}) \cong \mathbb{C}^N$ via $g \otimes g$. This induces an L-homomorphism ${}^LH \to {}^LG$. The expected functorial lifting is called the *symmetric square power lifting.* Similarly, we can consider the functorial lifting associated to any power of symmetric product or exterior product.

Example 8.3 (Local Jacquet–Langlands correspondence)**.** Let $G = \mathrm{GL}_2$ and $H = D^\times$. Here, D denotes the division quaternion algebra over F which is regarded as an algebraic group over F by $R \mapsto (D \otimes_F R)^\times$ for any F-algebra R. Since $D \otimes \overline{F}$ is isomorphic to $M_2(\overline{F})$, H is isomoprhic to G over $\overline{F}$. Furthermore, the Galois actions on $\widehat{G}$ and $\widehat{H}$ are both trivial.[12] Therefore, we have an identification ${}^{L}H \cong {}^{L}G$. The expected lifting is called the *local Jacquet–Langlands correspondence.*

Example 8.4 (Base change lifting)**.** Let E/F be a finite extension. Let $G = \mathrm{GL}_n$ over E and $H = \mathrm{GL}_n$ over F. Then, on the Galois side, by restricting an L-paramter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ of H to $W_E \times \mathrm{SL}_2(\mathbb{C})$, we get an L-parameter of G. The resulting lifting $\Pi(\mathrm{GL}_n(F)) \to \Pi(\mathrm{GL}_n(E))$ is called the *base change lifting.*[13]

Example 8.5 (Automorphic induction)**.** Let E/F be a finite extension of degree r. Let $G = \mathrm{GL}_{nr}$ over F and $H = \mathrm{GL}_n$ over E. Then, on the Galois side, by inducing an L-paramter of H (regarded as an n-dimensional representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$) to $W_F \times \mathrm{SL}_2(\mathbb{C})$, we get an L-parameter of G (regarded as an nr-dimensional representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$). The resulting lifting $\Pi(\mathrm{GL}_n(E)) \to \Pi(\mathrm{GL}_{nr}(F))$ is called the *automorphic induction.*[14]

Example 8.6 (Endoscopic lifting)**.** Let $G = \mathrm{GL}_{2n}$ and $H = \mathrm{SO}_{2n+1}$. Then $\widehat{G} = \mathrm{GL}_{2n}(\mathbb{C})$ and $\widehat{H} = \mathrm{Sp}_{2n}(\mathbb{C})$, hence we have an inclusion ${}^{L}H \hookrightarrow {}^{L}G$. The resulting functorial lifting is a special case of the *endoscopic lifting.*

Remark 8.7. In fact, Examples 8.1 and 8.3 are also special cases of the endoscopic lifting. Examples 8.4 and 8.5 are special cases of the endoscopic lifting when E/F is cyclic. In these cases, it is known how to characterize the functorial lifting in a purely-representation theoretic way. We will discuss the case of the local Jacquet–Langlands correspondence later. However, it has not been known in other cases. For example, we certainly have the Sym^r lifting for any r because the local Lanlgands correspondence for general linear groups has been established; but we don't know how to characterize the lifting without mentioning the local Langlands correspondence!

8.2. **Hecke algebra.** In the following discussion, we do not explain any proof; we only state basic properties of Hecke algebras. For the details, see [BH06, Section 3].

Proposition/Definition 8.8. Any locally compact group G has a right-invariant (resp. left-invariant) Radon measure μ, which we call a *right (resp. left) Haar measure.* Here, "right-invariant" (resp. "left-invariant") means that we have $\mu(Xg) = \mu(X)$ (resp. $\mu(gX) = \mu(X)$) for any $g \in G$ and Borel subset $X \subset G$. Any nonzero Haar measure is unique up to nonzero constant.

We do not review the definition of a Radon measure; see, for example, [RV99, Section 1.2].

Fact 8.9. *Any p-adic reductive group is unimodular, i.e., a right Haar measure is also a left Haar measure.*

From now on, let G be a p-adic reductive group. (So, we have a connected reductive group G over F and write G for $G(F)$ by abuse of notation.) We fix a Haar measure μ on G; then we can consider the integration on the group G.

We let $\mathcal{H} := C_c^\infty(G)$, i.e., the set of $\mathbb{C}$-valued locally constant and compactly supported functions on G. Note that any such function $f \in \mathcal{H}$ is integrable. More explicitly, f can

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[12] G and H are inner forms to each other. In general, two groups have the same L-groups if and only if they are inner forms to each other.

[13] By regarding G as the Weil restriction of GL_n from E to F, we can also understand that this lifting is associated to an L-homomorphism from H to G.

[14] By regarding H as the Weil restriction of GL_n from E to F, we can also understand that this lifting is associated to an L-homomorphism from H to G.

be written as $f = \sum_{i=1}^{r} c_i \cdot \mathbb{1}_{K_i}$ with $c_i \in \mathbb{C}$ and open compact subsets $K_i \subset G$, where $\mathbb{1}_{K_i}$ denotes the characteristic function of K_i; then we have

$$\int_G f \, d\mu = \sum_{i=1}^{r} c_i \cdot \mu(K_i).$$

We define a product $*$ on $\mathcal{H}$ by

$$f_1 * f_2(g) := \int_G f_1(gh) \cdot f_2(h^{-1}) \, d\mu.$$

Then we can check that $\mathcal{H}$ is an associative $\mathbb{C}$-algebra with respect to $*$ (non-commutative and non-unital in general). We call $\mathcal{H}$ the *Hecke algebra of G*.

For each open compact subgroup $K \subset G$, we define $\mathcal{H}_K$ to be a subset of $\mathcal{H}$ consisting of compactly supported bi-K-invariant functions on G. Then $\mathcal{H}_K$ is a sub-$\mathbb{C}$-algebra of $\mathcal{H}$. Moreover, we have $\mathcal{H} = \bigcup_K \mathcal{H}_K$, where K runs over all open and compact subsets of G.

Exercise 8.10. Show that $\mu(K)^{-1} \cdot \mathbb{1}_K$ is the unit element of the $\mathbb{C}$-algebra $\mathcal{H}_K$.

Now let (π, V) be a smooth representation of G. We define an action of $\mathcal{H}$ on (π, V) by

$$\pi(f)(v) := \int_G f(g) \cdot \pi(g)(v) \, d\mu$$

for any $f \in \mathcal{H}$. More explicitly, for each $v \in V$, we can find an open compact subgroup $K \subset G$ such that $v \in V^K$ by the smoothness of (π, V). By choosing K to be sufficiently small so that $f \in \mathcal{H}_K$, we may write $f = \sum_{i=1}^{r} c_i \cdot \mathbb{1}_{g_i \cdot K}$. Then

$$\begin{aligned}\pi(f)(v) &= \sum_{i=1}^{r} c_i \cdot \int_{g_i \cdot K} \pi(g)(v) \, d\mu \\ &= \sum_{i=1}^{r} c_i \cdot \mu(K) \cdot \pi(g_i)(v).\end{aligned}$$

For $f \in \mathcal{H}_K$, we define an operator

$$\pi(f) \colon V^K \to V^K$$

We can check that this gives a well-defined $\mathcal{H}$-module structure on V.

Note that, in general, an $\mathcal{H}$-module W may not satisfy the equality $\mathcal{H} \cdot W = W$ because $\mathcal{H}$ does not have the unit element. However, the $\mathcal{H}$-module V constructed from a smooth representation (π, V) of G satisfies this condition. Indeed, for any $v \in V$, the smoothness implies that $v \in V^K$ for some K. Then we have $\pi(\mu(K)^{-1} \cdot \mathbb{1}_K)(v) = v$; in particular, $\mathcal{H}_K \cdot V \ni v$. We say that an $\mathcal{H}$-module W is *smooth* if $\mathcal{H} \cdot W = W$.

The following fact says that investigating (irreducible) smooth representations is equivalent to investigating (simple) smooth modules over Hecke algebras.

Fact 8.11. *(1) The above construction $(\pi, V) \mapsto V$ induces an equivalence between*
  - *the category of smooth representations of G and*
  - *the category of smooth $\mathcal{H}$-modules.*

(2) For any open compact subgroup $K \subset G$, the association $(\pi, V) \mapsto V^K$ induces a bijection between
  - *the set of irreducible smooth representations (π, V) of G satisfying $V^K \neq 0$ and*
  - *the set of simple $\mathcal{H}_K$-modules.*

8.3. **Characters of representations.** Recall that any representation (π, V) of a finite group G is completely determined by its *trace character* Θ_π, which is a function

$$\Theta_\pi \colon G \to \mathbb{C}; \quad g \mapsto \mathrm{tr}(\pi(g)).$$

It is natural to ask if the same thing makes sense in representation theory of p-adic reductive groups. But we immediately notice that at least the above definition does not work because most of smooth representations of a p-adic group is infinite-dimensional. The idea to resolve this issue is to rephrase the notion of a smooth representation in terms of the Hecke algebra (as performed in the previous subsection) and then discuss at each V^K by assuming the admissibility of (π, V).

Let (π, V) be an admissible representation of G. Then, as explained above, we may regard V as a smooth $\mathcal{H}$-module. Note that if $f \in \mathcal{H}_K$, then the image of the associated operator $\pi(f)$ lies in V^K. In particular, we get $\pi(f) \colon V^K \to V^K$. In other words, $\pi(f)$ induces an endomorphism of the finite-dimensional $\mathbb{C}$-vector space V^K for each open compact subgroup $K \subset G$. Hence it makes sense to talk about the trace of $\pi(f)$ on V^K. In fact, we can check that $\mathrm{tr}(\pi(f) \mid V^K)$ is independent of the choice of K (Excercise below). Thus let us simply write $\mathrm{tr}(\pi(f))$ for $\mathrm{tr}(\pi(f) \mid V^K)$.

Exercise 8.12. Let (π, V) be an admissible representation of G. Let $f \in \mathcal{H}$. We choose an open compact subgroup K of G such that $f \in \mathcal{H}_K$. Show that $\mathrm{tr}(\pi(f) \mid V^K)$ is independent of the choice of K.

Definition 8.13. We call the $\mathbb{C}$-linear functional

$$C_c^\infty(G) \to \mathbb{C}; \quad f \mapsto \mathrm{tr}(\pi(f))$$

the *character distribution*[15] of (π, V). We write $\Theta_\pi^{\mathrm{dist}}$ for the character distribution of (π, V), i.e., $\Theta_\pi^{\mathrm{dist}}(f) = \mathrm{tr}(\pi(f))$.

Remark 8.14. Although it is not indicated in the symbol, $\Theta_\pi^{\mathrm{dist}}$ depends on the choice of a Haar measure on G.

Here, let us consider the character distribution in the case where G is finite. Note that the notion of the Hecke algbera and the equivalence between the categories of representations of G and $\mathcal{H}$-modules still make sense in the finite group setting (rather, it's easier). If we choose a Haar measure on G to be the counting measure, then we have

$$\pi(f) = \sum_{g \in G} f(g) \cdot \pi(g),$$

hence

$$\Theta_\pi^{\mathrm{dist}}(f) := \mathrm{tr}(\pi(f)) = \sum_{g \in G} f(g) \cdot \Theta_\pi(g). \tag{1}$$

Especially, the character distribution$\Theta_\pi^{\mathrm{dist}}$ is determined by the character Θ_π and vice versa by the equality (1).

The point is that, also in the context of representations of p-adic reductive groups, the existence of a function "Θ_π" satisfying the equality (1) can be shown (although it cannot be defined naively by $\Theta_\pi(g) = \mathrm{tr}(\pi(g))$ as in the finite group case)!

Definition 8.15. We say that a semisimple element g of a connected reductive group G is *regular* if its connected centralizer $Z_G(g)^\circ$ is a maximal torus of G. We let G_{rs} denote the set of regular semisimple elements of G.

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[15] In our context, the word "distribution" just means a $\mathbb{C}$-linear functional on $C_c^\infty(G)$.

Theorem 8.16 (Harish-Chandra, [HC70]). *There exists a unique locally constant function* $\Theta_\pi \colon G_{\mathrm{rs}}(F) \to \mathbb{C}$ *satisfying*

$$\Theta_\pi^{\mathrm{dist}}(f) = \int_G f(g) \cdot \Theta_\pi(g)\, d\mu$$

for any $f \in C_c^\infty(G)$.

We call Θ_π the *(Harish-Chandra) character of* (π, V).

Theorem 8.17 (linear independence of characters). *For any finite sets of irreducible admissible representations* $\pi_1, \ldots, \pi_r$ *of* G *non-isomorphic to each other, their characters* $\Theta_{\pi_1}, \ldots, \Theta_{\pi_r}$ *are linear independent as* $\mathbb{C}$*-valued functions on* $G_{\mathrm{rs}}(F)$. *In particular, for any irreducible admissible representations* π_1 *and* π_2, *we have* $\pi_1 \cong \pi_2$ *if and only if* $\Theta_{\pi_1} = \Theta_{\pi_2}$.

Remark 8.18. Note that Θ_π is independent of the choice of a Haar measure μ.

8.4. **Local Jacquet–Langlands correspondence.** Now let $G = \mathrm{GL}_2$ and $H = D^\times$ over F as in Example 8.3. Note that these groups are isomorphic over $\overline{F}$, furthremore, are inner forms to each other. More precisely, there exists an isomorphism $\psi \colon G \to H$ over $\overline{F}$ such that, for any $\sigma \in \Gamma_F$, $\psi^{-1} \circ \sigma(\psi)$ is an inner automorphism of G, i.e., $\mathrm{Int}(g_\sigma)$ for some $g_\sigma \in G$. By fixing such ψ, we define the notion of "matching conjugacy classes" as follows:

Definition 8.19. We say that $g \in G(F)$ and $h \in H(F)$ *match* if $\psi(g)$ is conjugate to h over $\overline{F}$. (Note that the choice of ψ is unique up to inner automorphism, so this notion is well-defined.)

Theorem 8.20 (Local Jacquet–Langlands correspondence for GL_2). *Let* π_H *be any irreducible admissible representation of* $H(F)$. *Then there exists a unique irreducible "discrete series" representation* π_G *of* $G(F)$ *satisfying the identity*

$$\Theta_{\pi_G}(g) = -\Theta_{\pi_H}(h)$$

for any matching $g \in G(F)$ *and* $h \in H(F)$.

Remark 8.21. This theorem was proved by Jacquet–Langlands [JL70] and then generalized to any GL_n and central simple algebras by Rogawski [Rog83] and Deligne–Kazhdan–Vigneras [DKV84].

9. WEEK 9: MORE ABOUT LOCAL LANGLANDS CORRESPONDENCE FOR GENERAL GROUPS

9.1. **Hierarchy of irreducible admissible representations.** Let G be a p-adic reductive group, so $G = \mathbf{G}(F)$ for a connected reductive group $\mathbf{G}$ over a non-archimedean local field F. We write Z for the center of G. Recall that any irreducible admissible representation (π, V) of G has its central character, for which we write $\omega_\pi \colon Z \to \mathbb{C}^\times$; i.e., for any $z \in Z$ and $v \in V$, we have $\pi(z)(v) = \omega_\pi(z) \cdot v$.

Definition 9.1. Let (π, V) be an irreducible admissible representation of G. We let $(\pi^\vee, V^\vee)$ be its contragredient representation. A *matrix coefficient of* π is a function $f \colon G \to \mathbb{C}$ of the form

$$f(g) = \langle \pi(g)(v), v^\vee \rangle,$$

where $v \in V$ and $v^\vee \in V^\vee$.

Note that, for any $z \in Z$, we have $f(zg) = \omega_\pi(z) \cdot f(g)$. Also note that, by the smoothness of π, any matrix coefficient is a locally constant function.

Definition 9.2. Let (π, V) be an irreducible admissible representation of G.

(1) We say that π is *supercuspidal* if some nonzero matrix coefficient has compact-modulo-Z support.

(2) We say that π is *square-integrable* if ω_π is unitary (i.e., $|\omega_\pi| = 1$) and some nonzero matrix coefficient f is square-integrable modulo Z, i.e.,

$$\int_{G/Z} |f(g)|^2 \, dg < \infty,$$

where dg is any Haar measure on G/Z.

(2') We say that π is *discrete series* (or *essentially square-integrable*) if there exists a character $\chi \colon G \to \mathbb{C}^\times$ such that $\pi \otimes \chi$ is square-integrable.

(3) We say that π is *tempered* if ω_π is unitary and some nonzero matrix coefficient f is $(2 + \varepsilon)$-integrable modulo Z, i.e.,

$$\int_{G/Z} |f(g)|^{2+\varepsilon} \, dg < \infty$$

for a positive real number $\varepsilon > 0$, where dg is any Haar measure on G/Z.

(3') We say that π is *essentially tempered* if there exists a character $\chi \colon G \to \mathbb{C}^\times$ such that $\pi \otimes \chi$ is tempered.

Remark 9.3. If $\mathbf{G}$ is semisimple (equivalently, its center if finite), then any character of Z is unitary. In particular, "square-integrable" (resp. tempered) and "essentially square-integrable" (resp. essentially tempered) are equivalent.

Lemma 9.4. *The condition "some nonzero matrix coefficient..." in the above definition can be replaced with "any nonzero matrix coefficient...".*

Proof. Let us only consider the definition of a supercuspidal representation since the same argument can be applied to the others.

Suppose that $f(g) = \langle \pi(g)(v), v^\vee \rangle$ is a nonzero matrix coefficient of π which has compact-mod-Z support. Let $f'(g) = \langle \pi(g)(w), w^\vee \rangle$ be any other matrix coefficient. Our task is to show that f' has also compact-mod-Z support. Since π is irreducible, we have $V = \mathrm{Span}_\mathbb{C}\{\pi(g)(v) \mid g \in G\}$. Hence we may write $w = \sum_{i=1}^{r} \pi(g_i)(v)$ with some $g_1, \ldots, g_r \in G$.

Similarly, by the irreducibility of $\pi^\vee$, we may write $w^\vee = \sum_{j=1}^{s} \pi^\vee(h_j)(v^\vee)$ with some $h_1, \ldots, h_s \in G$. Then we have

$$\begin{aligned} f'(g) = \langle \pi(g)(w), w^\vee \rangle &= \sum_{i=1}^{r}\sum_{j=1}^{s} \langle \pi(gg_i)(v), \pi^\vee(h_j)(v^\vee)\rangle \\ &= \sum_{i=1}^{r}\sum_{j=1}^{s} \langle \pi(h_j^{-1}gg_i)(v), v^\vee\rangle = \sum_{i=1}^{r}\sum_{j=1}^{s} f(h_j^{-1}gg_i). \end{aligned}$$

In patricular, f' is written as a sum of translations of f', hence must have compact-mod-Z support. ☐

Lemma 9.5. *If π is a supercuspidal representation of G, then π is discrete series.*

Proof. Note that any supercuspidal representation π with unitary central character ω_π is obviously square-integrable. Also note that any character twist of a supercuspidal representation is again supercuspidal. Hence, it is enough to show that, for any supercuspidal representation π of G, there exists a character $\chi\colon G \to \mathbb{C}^\times$ such that $\pi \otimes \chi$ has unitary central character.

For simplicity, we explain the argument only in the case of GL_n. We identify $|\omega_\pi|\colon Z \to \mathbb{R}_>0$ with a character of $F^\times$ and define $\chi := |\omega_\pi|^{-\frac{1}{n}} \circ \det\colon G \to \mathbb{R}_{>0}$. Then $\pi \otimes \chi$ has unitary central character. Indeed, we have

$$\omega_{\pi\otimes\chi} = \omega_\pi \otimes (\chi|_Z) = \omega_\pi \otimes |\omega_\pi|^{-1},$$

hence $|\omega_{\pi\otimes\chi}| = 1$. ☐

Exercise 9.6. Generalize the above proof for general p-adic reductive group.

Definition 9.7. We say that an irreducible admissible representation (π, V) of G is *unitary* if there exists a Hermitian inner product $(-,-)\colon V \times V \to \mathbb{C}$ which is G-invariant, i.e., $(\pi(g)v, \pi(g)w) = (v, w)$ for any $g \in G$ and $v, w \in V$.

Remark 9.8. In the classical context, a "unitary representation" means a representation realized on a Hilbert space, i.e., a $\mathbb{C}$-vector space which is complete with respect to the metric determined by a Hermitian inner product. The dimension of such a representation as a $\mathbb{C}$-vector space[16] is necessarily uncountable unless it is not a finite-dimensional. However, any irreducible admissible representation of our interest has a countable dimension (recall: Week 3), hence cannot be a unitary representation in the classical sense. In fact, the unitarity defined in Definition 9.7 and the unitarity in the classical sense can be translated into each other by taking the completion/taking the smooth part; see, e.g., [BZ76, 4.21].

Proposition 9.9. *Let π be a discrete series (resp. essentially tempered) representation of G. Then π is square-integrable (resp. tempered) if and only if π is unitary.*

Proof. We only explain the outline. Suppose that π is square-integrable representation. We fix a non-zero element $v^\vee \in V^\vee$ and define a $\mathbb{C}$-valued function $(-,-)\colon V \times V \to \mathbb{C}$ by

$$(v, v') := \int_{G/Z} \langle \pi(g)(v), v^\vee\rangle \cdot \overline{\langle \pi(g)(v'), v^\vee\rangle}\, dg.$$

Once we can prove the well-definedness (see [Wal03, Lemma III.1.3]), it is not difficult to see that $(-,-)$ defines a G-invariant Hermitian inner product.

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[16] caution: not the dimension as a Hilbert space

It is much more involved to show that any tempered representation is unitary; see, e.g., [GH24, Section 4.8].

Conversely, we suupose that π is unitary discrete series. As π is unitary, the central character ω_π is also unitary. By definition, there exists a character $\chi\colon G \to \mathbb{C}^\times$ such that $\pi \otimes \chi$ is square-integrable. Hence $\omega_{\pi\otimes\chi} = \omega_\pi \otimes \chi$ is also unitary. Thus χ must be unitary. This implies that π is square-integrable.

It can be checked in the same way that any unitary essentially tempered representation is tempered. ☐

The "hierarchy" we have discussed so far can be summarized as follows.

$$\begin{array}{ccccccc}
\text{(supercuspidal)} & \subset & \text{(discrete series)} & \subset & \text{(ess. tempered)} & \subset & \text{(irr. admissible)} \\
\cup & & \cup & & \cup & & \cup \\
\text{(unitary s.c.)} & \subset & \text{(square-integrable)} & \subset & \text{(tempered)} & \subset & \text{(unitary)}
\end{array}$$

We finally note that supercuspidal representations are thought of as "atoms" in representation theory of p-adic reductive groups by the following fact:

Fact 9.10. *Let π be an irreducible admissible representation of G. Then π is supercuspidal if and only if there does not exists a proper F-rational parabolic subgroup $P \subsetneq G$ with Levi subgroup L and an irreducible admissible representation ρ of L such that π is realized as a subquotient of $\operatorname{Ind}_P^G \rho$.*

Example 9.11. Let us consider the case where $G = \mathrm{GL}_2(F)$.

(1) The Steinberg repersentation $\mathrm{St}_{\mathrm{GL}_2}$ is a square-integrable representation which is not supercuspidal. Hence any its twist $\mathrm{St}_{\mathrm{GL}_2} \otimes \chi$ is discrete series. Conversely, any discrete series representation is of this form.

(2) Any character of G is not essentially-tempered.

(3) Any irreducible admissible reprsentation realized in a principal series representation is not discrete series; whether it is tempered or not depends on the inducing characters.

9.2. **Hierarchy of L-parameters.** Let us explain the (expected) relation between the above-mentioned classes of irreducible admissible representations and the local Langlands correspondence.

In the following, we assume that G is "quasi-split" for simplicity.

Recall that we write the (conjectural) local Langlands correspondence by

$$\mathrm{LLC}_G\colon \Pi(G) \to \Phi(G).$$

Also recall that, for each $\phi \in \Phi(G)$, its L-packet $\Pi_\phi := \mathrm{LLC}_G^{-1}(\phi)$ is (conjecturally) equipped with a bijective map

$$\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi),$$

where $\mathrm{Irr}(\mathcal{S}_\phi)$ denotes the set of isomorphism classes of irreducible representations of $\mathcal{S}_\phi := \pi_0(S_\phi/Z(\widehat{G}))$.

Definition 9.12. Let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ be an L-parameter of G.

(1) We say that ϕ is *tempered* if the image $\phi(W_F)$ is bounded with respect to the complex topology of $\widehat{G}$.

(2) We say that ϕ is *discrete* if the centralizer group $S_\phi := \mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))$ contains the center $Z(\widehat{G})$ of $\widehat{G}$ as its finite-index subgroup.

(3) We say that ϕ is *supercuspidal* if ϕ is discrete and trivial on $\mathrm{SL}_2(\mathbb{C})$.

Conjecture 9.13. *The local Langlands correspondence satisfies the following.*

(1) The following are equivalent:
  (a) ϕ is tempered,
  (b) some member of Π_ϕ is tempered,
  (c) all members of Π_ϕ are tempered.

(2) The following are equivalent:
  (a) ϕ is discrete,
  (b) some member of Π_ϕ is discrete series,
  (c) all members of Π_ϕ are discrete series.

(3) The following are equivalent:
  (a) ϕ is supercuspidal,
  (b) all members of Π_ϕ are supercuspidal.

Here, please be careful that the last condition is NOT equivalent to the condition that "some member of Π_ϕ is supercuspidal". In fact, it could happen that an L-packet contains both a supercuspidal representation and a non-supercuspidal but discrete series representation. It is expected that being supercuspidal or not for members of Π_ϕ can be read off from the corresponding irreducible representations of $\mathcal{S}_\phi$ under the bijective map $\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$, but it is a quite deep matter (called the Aubert–Moussaoui–Solleveld conjecture, which appeal to Lusztig's theory of cuspidal perverse sheaves; [AMS18]). We elaborate it in the next subsection, but only a little bit.

Remark 9.14. Conjecture 9.13 (1) and (2) are expected to hold also for non-quasi-split G, but the last statement (3) is only for quasi-split groups. This is also related to what we discuss next.

Remark 9.15. It is considered that describing the unitarity of irreducible admissible representations in terms of the Galois side is a quite difficult problem.

9.3. **Labeling of members of an L-packet.**

We continue to assume that G is "quasi-split". The bijective map $\Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$ is NOT canonical, but expected to depend only on the choice of an additional datum; let us elaborate it.

Definition 9.16. A *Whittaker datum* of G is a pair (B, ψ) such that

- B is an F-rational Borel subgroup of G (let us write $B = TU$, where T is an F-rational maximal torus contained in B and U is the unipotent radical of B),
- $\psi\colon U(F) \to \mathbb{C}^\times$ is a *generic* character in the sense that the stabilizer of ψ with respect to the action of $T(F)$ equals $Z(F)$.

Exercise 9.17. Let $G := \mathrm{GL}_n$. Let B be the upper-triangular Borel subgroup of G; hence U is the subgroup of upper-triangular unipotent matrices and we have $B = TU$, where T is the subgroup of diagonal matrices.

(1) We fix a nontrivial additive character $\psi_F\colon F \to \mathbb{C}^\times$ and define $\psi\colon U(F) \to \mathbb{C}^\times$ by

$$\psi((u_{i,j})_{i,j}) := \psi_F(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n}).$$

Show that (B, ψ) is a Whittaker datum of G.

(2) Determine all possible Whittaker data of G.

Definition 9.18. Let $\mathfrak{w} = (B, \psi)$ be a Whittaker datum of G. We say that an irreducible admissible representation (π, V) of G is $\mathfrak{w}$-*generic* if we have

$$\mathrm{Hom}_{U(F)}(\pi|_{U(F)}, \psi) \neq 0.$$

Remark 9.19. In fact, the dimension of the space $\mathrm{Hom}_{U(F)}(\pi|_{U(F)}, \psi)$ is at most one (known as Shalika's multiplicity one theorem [Sha74]). In other words, a $\mathbb{C}$-linear functional $\lambda\colon V \to \mathbb{C}$ satisfying $\lambda(\pi(u)v) = \psi(u) \cdot v$ for any $u \in U(F)$ and $v \in V$ (i.e., an element of $\mathrm{Hom}_{U(F)}(\pi|_{U(F)}, \psi)$) is at most unique by $\mathbb{C}$-multiplication. We refer to such a nonzero $\mathbb{C}$-linear functional as a *Whittaker functional.* If π is $\mathfrak{w}$-generic, by Frobenius reciprocity, we also have that the space

$$\mathrm{Hom}_G(\pi, \mathrm{Ind}_{U(F)}^G \psi)$$

is 1-dimensional. In other words, as π is irreducible, π is realized as the subspace of the "huge" representation $\mathrm{Ind}_{U(F)}^G \psi$ in a unique way. We refer to the embedded subspace of $\mathrm{Ind}_{U(F)}^G \psi$ as the *Whittaker model of* (π, V).

Now let us go back to the statement of the conjectural local Langlands correspondence. It is expected that the above-mentioned bijective map between Π_ϕ and $\mathrm{Irr}(\mathcal{S}_\phi)$ is determined canonically once we fix a Whittaker datum of G. For this reason, let us fix a Whitatker datum $\mathfrak{w}$ of G and write

$$\iota_{\mathfrak{w}} : \Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi).$$

Definition 9.20. We say that an L-parameter $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ is *generic* if the local L-factor[17] $L(s, \mathrm{Ad} \circ \phi)$ associated to $\mathrm{Ad} \circ \phi$ is holomorphic at $s = 1$, where Ad denotes the adjoint representation $\mathrm{Ad}\colon \widehat{G} \to \mathrm{Aut}_{\mathbb{C}}(\mathrm{Lie}(\widehat{G}))$.

The following is the (refinement of the) conjecture of Gross–Prasad [GP92]:

Conjecture 9.21. *The L-packet Π_ϕ contains a $\mathfrak{w}$-generic representation if and only if ϕ is generic. Moreover, in this case, a $\mathfrak{w}$-generic representation in Π_ϕ is the unique one which corresponds to the trivial representation $\mathbb{1} \in \mathrm{Irr}(\mathcal{S}_\phi)$ under the bijection $\iota_{\mathfrak{w}}$.*

Remark 9.22. In fact, for an L-parameter ϕ, being tempered implies being generic. Thus, the above conjecture in particular claims that "any tempered L-packet contains a $\mathfrak{w}$-generic representation." This has been known as Shahidi's tempered L-packet conjecture (proposed in [Sha90]).

Exercise 9.23. Prove that any tempered L-parameter is generic. (Hint: being tempered is preserved under composing the adjoint representation. Thus it suffices to verify the statement only for GL_n.)

Conjecture 9.13 (3) is a bit refined as follows:

Conjecture 9.24. *The following are equivalent:*

(1) ϕ is supercuspidal,
(2) the unique $\mathfrak{w}$-generic member of Π_ϕ is supercuspidal,
(3) all members of Π_ϕ are supercuspdial.

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[17] Recall that we defined a local L-factor associated to a Weil–Deligne representation in Week 6. By the dictionary between Weil–Deligne representations and L-parameters of GL_n explained in Week 7, we can also consider local L-factors for L-parameters of GL_n.

9.4. **The case of GL_n.**

Let us investigate the case of GL_n; recall that, in this case, every L-packet is a singleton.

Let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ be an L-parameter. Recall that $\phi|_{W_F}$ is semisimple (completely reducible) as a representation of W_F. Also, as $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is algebraic, $\phi|_{\mathrm{SL}_2(\mathbb{C})}$ is semisimple (see, e.g., [Mil17, Theorem 22.42]). This implies that ϕ can be described as an n-dimensional representation of $W_F \times \mathrm{SL}_2(\mathbb{C})$ as follows:

$$\phi = \bigoplus_{i=1}^{r} \rho_i \boxtimes S_{n_i},$$

where ρ_i is an irreducible smooth representation of W_F and S_{n_i} is the n_i-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$ for some $n_i \in \mathbb{Z}_{>0}$[18] such that $\sum_{i=1}^{r} \dim(\rho_i) \cdot n_i = n$.

With this description, being tempered is equivalent to that each ρ_i is bounded. Moreover, being discrete is actually equivalent to that $r = 1$ (Exercise below). Hence, being supercuspidal is equivalent to that $r = 1$ and $n_1 = 1$, i.e., ϕ is an irreducible n-dimensional smooth representation of W_F.

Exercise 9.25. Prove that $\phi = \bigoplus_{i=1}^{r} \rho_i \boxtimes S_{n_i}$ is discrete if and only if $r = 1$.

In the GL_n-case, all the above conjectures are theorems:

Theorem 9.26. *Let $\pi \in \Pi(\mathrm{GL}_n)$ and $\mathrm{LLC}_{\mathrm{GL}_n}(\pi) = \bigoplus_{i=1}^{r} \rho_i \boxtimes S_{n_i}$ be its L-parameter.*

1. *π is supercuspidal if and only if $r = 1$ and $n_1 = 1$.*
2. *π is discrete series if and only if $r = 1$.*
3. *π is tempered if and only if every ρ_i is bounded.*
4. *π is $\mathfrak{w}$-generic if and only if $\mathrm{LLC}_{\mathrm{GL}_n}(\pi)$ is generic.*

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[18] Note that an irreducible algebraic representation of $\mathrm{SL}_2(\mathbb{C})$ is determined only by its dimension; explicitly, for $m \in \mathbb{Z}_{>0}$, the unique m-dimensional irreducible representation S_m is given by $\mathrm{Sym}^{m-1}\,\mathrm{std}$, where std denotes the 2-dimensional standard representation of $\mathrm{SL}_2(\mathbb{C})$ on $\mathbb{C}^{\oplus 2}$.

10. WEEK 10: LOCAL JACQUET–LANGLANDS CORRESPONDENCE FOR GL_2

Recall that we the local Jacquet–Langlands correspondence for GL_2 gives a bijection between irreducible admissible representations of $D^\times$ and irreducible discrete series representations of $\mathrm{GL}_2(F)$ for a division quaternion algebra D over F:

Theorem 10.1 (LJLC for GL_2 (Week 8))**.** *Let π be any irreducible admissible representation of $D^\times$. Then there exists a unique irreducible discrete series representation π' of $\mathrm{GL}_2(F)$ satisfying the identity*

$$\Theta_{\pi'}(g') = -\Theta_\pi(g)$$

for any matching regular semisimple elements $g \in D^\times$ and $g' \in \mathrm{GL}_2(F)$.

Let $\Pi(D^\times)$ (resp. $\Pi(\mathrm{GL}_2)_{\mathrm{disc}}$) be the sets of irreducible admissible representations of $D^\times$ (resp. irreducible discrete series representations of $\mathrm{GL}_2(F)$). We write

$$\mathrm{LJLC}\colon \Pi(D^\times) \xrightarrow{1:1} \Pi(\mathrm{GL}_2)_{\mathrm{disc}}$$

for the bijective map of the local Jacquet–Langlands correspondence. The aim of this week is to explicate the map LJLC in more detail.

10.1. Division algebras.

We first review some basics of division quaternion algebras. (For more general theory, see [Wei74] for example.)

Proposition/Definition 10.2. Let F be a field. For any $a, b \in F^\times$, we consider a F-vector space

$$(a, b/F) := F \oplus Fi \oplus Fj \oplus Fk.$$

We define a multiplication law in the following way:

$$i^2 = a, \quad j^2 = b, \quad k^2 = -ab, \quad ij = k = -ji.$$

Then this defines a central simple F-algebra structure on $(a, b/F)$, i.e., $(a, b/F)$ is a simple[19] F-algebra whose center is $F \subset (a, b/F)$. We call $(a, b/F)$ a *quaternion algebra* (with respect to (F, a, b)).

Remark 10.3. By choosing (F, a, b) to be $(\mathbb{R}, -1, -1)$, we get Hamilton's quaternion algebra $\mathbb{H}$.

Exercise 10.4. Suppose that $a \in (F^\times)^2 := \{c^2 \mid c \in F^\times\}$ or $b \in (F^\times)^2$ Then show that $(a, b/F) \cong M_2(F)$ and that the reduced norm of $(a, b/F)$ is identified with the determinant of $M_2(F)$.

Now let F be a non-archimedean local field as usual. Then we have the following fact:

Fact 10.5. *Let Q be a quaternion algebra over F. Then Q is either*

1. *isomorphic to the matrix algebra $M_2(F)$ or*
2. *a division algebra (i.e., any nonzero element is invertible).*

Moreover, when Q is a division algebra, we have $Q \otimes_F E \cong M_2(E)$ as E-algebras, where E is the[20] *quadratic unramified extension of F.*

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[19] i.e., there is no nontrivial two-sided ideal

[20] Recall that, for a fixed degree, there uniquely exists an unramified extension of F with that degree in the fixed algebraic closure; Week 2

For a quaternion algebra Q over F, we consider a functor

$$(F\text{-algebras}) \to \text{Group}\colon R \mapsto (Q \otimes_F R)^\times.$$

In fact, this functor is representable by a linear algebraic group over F, which is isomoprhic to GL_2 over $\overline{F}$. In particular, $Q^\times$ is regarded as a p-adic reductive group. By abuse of notation, let us again write $Q^\times$ for the algebraic group, hence $Q^\times(F) = Q^\times$. The algebraic group $Q^\times$ is compatible with base change. To be more precise, for any quaternion algebra $(a, b/F)$ and an extension E/F of fields, the base change $Q_E^\times$ of $Q^\times$ from F to E is the algebraic group representing the functor

$$(E\text{-algebras}) \to \text{Group}\colon R \mapsto (Q' \otimes_E R)^\times,$$

where $Q' = (a, b/E)$. Therefore, by the previous fact, we see that the algebraic group $Q^\times$ associated to any division quaternion algebra Q is isomoprhic to GL_2 over $\overline{F}$ (or even the quadratic unramified extension E of F).

Remark 10.6. By the Skolem–Noether theorem in the theory of central simple algebras, we can furthermore show that $D^\times$ is an inner form of GL_2. (More generally, inner forms of GL_n are classified by central simple algebras; see [PRR23, Section 2.3], for example.) In particular, $D^\times$ and GL_2 have the same L-groups. This gives a philosophical explanation to why we should have the local Jacquet–Langlands correspondence.

Proposition/Definition 10.7. Let $Q = (a, b/F)$ be a quaternion algebra. The map

$$\iota\colon Q \to Q; \quad x + yi + zj + wk \mapsto x - yi - zj - wk$$

gives an F-algebra anti-automorhism of Q (i.e., F-vector space automorphism such that $\iota(\alpha\beta) = \iota(\beta)\iota(\alpha)$). By putting

$$\mathrm{Nrd}\colon Q \to F; \quad \alpha \mapsto \alpha\iota(\alpha),$$

we get a multiplicative map Nrd; explicitly, for $\alpha = x + yi + zj + wk$, we have $\mathrm{Nrd}(\alpha) = x^2 - ay^2 - bz^2 + abw^2$. We call Nrd the *reduced norm* of Q.

Let D be a division quaternion algebra over F. Let v denote the valuation of F normalized so that $v(\varpi) = 1$, where ϖ is a uniformizer. Then we can uniquely extend v to D by

$$v_D\colon D^\times \to \frac{1}{2}\mathbb{Z}; \quad v_D(x) := \frac{1}{2}v\bigl(\mathrm{Nrd}(x)\bigr)$$

for $x \in D$. Similarly to F, we can introduce a metric on D induced by v such that D is complete. We define the valutaion subring $\mathcal{O}_D$ of D and its maximal two-sided ideal $\mathfrak{P}_D$ by

$$\mathcal{O}_D := \{x \in D \mid v_D(x) \geq 0\} \quad \text{and} \quad \mathfrak{P}_D := \{x \in D \mid v_D(x) > 0\}.$$

Then we have $\mathcal{O}_D^\times = \mathcal{O}_D \smallsetminus \mathfrak{P}_D = \{x \in D \mid v_D(x) = 0\}$. Hence, by fixing an element ("uniformizer") ϖ_D of $D^\times$ such that $v_D(x) = \frac{1}{2}$, we have $D^\times = \mathcal{O}_D^\times\langle\varpi_D\rangle$. Since $v_D(\varpi) = 1$, this implies that $D^\times/F^\times$ is covered by the images of $\mathcal{O}_D^\times$ and $\mathcal{O}_D^\times\varpi_D$. In patricular, $D^\times/F^\times$ is compact ($D^\times$ is compact-modulo-center).

Lemma 10.8. *(1) Any irreducible admissible representation of $D^\times$ is supercuspidal.*

(2) Any irreducible admissible representation of $D^\times$ is finite-dimensional.

Proof. Since $D^\times$ is compact-modulo-center as explained above, the first assertion is clear by the definition of the supercuspidality. We leave the second assertion as an exercise. ☐

Exercise 10.9. Prove (2) of the above lemma. (Hint: use that $D^\times$ is compact-modulo-center.)

Lemma 10.10. *Let π be a one-dimensional irreducible admissible representation of $D^\times$. Then there exists a smooth character $\chi\colon F^\times \to \mathbb{C}^\times$ such that $\pi = \chi \circ \mathrm{Nrd}$.*

Proof. This follows from that the kernel of Nrd is exactly given by $[D^\times, D^\times]$; see [PRR23, Section 1.4.3]. ☐

10.2. **LJLC for Discrete series representations.** Recall that we say a regular semisimple element $g \in D^\times$ and $g' \in \mathrm{GL}_2(F)$ "match" if g and h have the same characteristic polynomial. To be more precise, we fix an isomorphism $\psi\colon D^\times_{\overline{F}} \xrightarrow{\cong} \mathrm{GL}_{2,\overline{F}}$ over $\overline{F}$. Then regular semisimple elements $g \in D^\times$ and $g' \in \mathrm{GL}_2(F)$ are said to match if g' and $\psi(g)$ are conjugate in $\mathrm{GL}_2(\overline{F})$.

Definition 10.11. We say a regular semisimple element of $\mathrm{GL}_2(F)$ is *elliptic* if its characteristic polynomial is irreducible over F.

Fact 10.12.
1. *For any regular semisimple element $g \in D^\times$, there exists an elliptic regular semisimple element $g' \in \mathrm{GL}_2(F)$ which match with g.*
2. *Conversely, for any elliptic regular semisimple element $g' \in \mathrm{GL}_2(F)$, there exists a regular semisimple element $g \in D^\times$ which match with g'.*

Example 10.13. Let $g' := \left(\begin{smallmatrix} 0 & 1 \\ \varpi & 0 \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$. Then the characteristic polynomial of g' is $X^2 - \varpi \in F[X]$. Note that this is Eisenstein, hence irreducible (Week 2). Hence g' is elliptic regular semisimple. Thus, the above lemma implies that there is a regular semisimple element $g \in D^\times$ which match with g'. Since $\psi(g)$ is conjugate to g' in $\mathrm{GL}_2(\overline{F})$, $\psi(g)^2$ is also conjugate to g'^2 in $\mathrm{GL}_2(\overline{F})$. Noting that $g'^2 = \left(\begin{smallmatrix} \varpi & 0 \\ 0 & \varpi \end{smallmatrix}\right)$, we see that $g^2 = \varpi \in F^\times \subset D^\times$. In particular, $g \in D^\times$ is a uniformizer of $D^\times$.

Proposition 10.14. *Let $\chi\colon F^\times \to \mathbb{C}^\times$ be a smooth character. Then we have*

$$\mathrm{LJLC}(\chi \circ \mathrm{Nrd}) = \mathrm{St}_{\mathrm{GL}_2} \otimes (\chi \circ \det).$$

Proof. Since $\mathrm{St}_{\mathrm{GL}_2} \otimes (\chi \circ \det)$ is an irreducible discrete series representation of $\mathrm{GL}_2(F)$, in order to show the claim, it is enough to verify the identity

$$\Theta_{\chi\circ\mathrm{Nrd}}(g) = -\Theta_{\mathrm{St}_{\mathrm{GL}_2}\otimes(\chi\circ\det)}(g')$$

for any matching regular semisimple elements $g \in D^\times$ and $g' \in \mathrm{GL}_2(F)$.

We have $\Theta_{\chi\circ\mathrm{Nrd}}(g) = \chi(\mathrm{Nrd}(g))$. As Nrd is identified with the determinant of GL_2 over $\overline{F}$, we have $\chi(\mathrm{Nrd}(g)) = \chi(\det(g'))$.

On the other hand, recall that we have the short exact sequence

$$0 \to \chi \circ \det \to \mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi \boxtimes \chi) \to \mathrm{St}_{\mathrm{GL}_2} \otimes (\chi \circ \det) \to 0.$$

The Harish-Chandra character is additive in the sense that we have

$$\Theta_{\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi\boxtimes\chi)} = \Theta_{\chi\circ\det} + \Theta_{\mathrm{St}_{\mathrm{GL}_2}\otimes(\chi\circ\det)}.$$

In fact, the Harish-Chandra character of a parabolically induced representation is something computable by "van Dijk's formula" [vD72]. In patricular, it is known that $\Theta_{\mathrm{Ind}_B^{\mathrm{GL}_2(F)}(\chi\boxtimes\chi)}(g') = 0$ for any elliptic regular semisimple element $g' \in \mathrm{GL}_2(F)$. (This is because any elliptic regular semisimple element of $\mathrm{GL}_2(F)$ cannot be conjugate to an element of the diagonal maximal torus.) Hence we have

$$\Theta_{\mathrm{St}_{\mathrm{GL}_2}\otimes(\chi\circ\det)}(g') = -\Theta_{\chi\circ\det}(g') = -\chi(\det(g'))$$

for any elliptic regular semisimple $g' \in \mathrm{GL}_2(F)$. ☐

10.3. **LJLC for supercuspidal representations.** We next want to investigate LJLC explicitly also for supercuspidal repesentations of GL_2. For this, recall that we can associate a non-negative rational number called "depth" to any irreducible admissible representation of a p-adic reductive group (Week 4). In the cases of $\mathrm{GL}_2(F)$, the possible depths are all non-negative half-integers $\frac{1}{2}\mathbb{Z}_{\geq 0}$.

In fact, the possible depths of irreducible admissible representations of $D^\times$ are also all non-negative half-integers $\frac{1}{2}\mathbb{Z}_{\geq 0}$. In the case of $D^\times$, the Bruhat–Tits building (essentially) consists only of a sigle point, which means that we have only one parahoric subgroup and its Moy–Prasad filtration. It is given by

$$\mathcal{O}_D^\times \supset 1+\mathfrak{P}_D \supset 1+\mathfrak{P}_D^2 \supset 1+\mathfrak{P}_D^3 \supset \cdots \supset 1.$$

If we wrtite $J = J_0$ for the unique parahoric subgroup $\mathcal{O}_D^\times$, then its $\frac{r}{2}$-th filtration ($r \in \frac{1}{2}\mathbb{Z}_{>0}$) is $1+\mathfrak{P}_D^{2r}$. Thus an irreducible admissible representation (π, V) of $D^\times$ is of depth $r \in \frac{1}{2}\mathbb{Z}_{>0}$ if $V^{J_r} = 0$ and $V^{J^{r+\frac{1}{2}}} \neq 0$ ("depth zero" simply means that $V^{J_{\frac{1}{2}}} \neq 0$). Note that J_r is normal in $D^\times$. Hence, if $V^{J_r} \neq 0$, then V is regarded as a representation of $D^\times/J_r$ by Lemma 6.6.

Theorem 10.15. *The local Jacquet–Langlands correspondence preserves the depth.*

This fact is deep; so we cannot go into the details of the proof (we need to appeal to theory of local factors and conductors; see [ABPS16]). Instead, we examine what we can deduce from this fact in the following.

10.4. **Depth-zero supercuspidal rerpesentations.** We first consider the case of depth-zero representations.

Recall that depth-zero supercuspidal representations of $\mathrm{GL}_2(F)$ can be constructed and classified as follows (Week 4). We put $K_0 := \mathrm{GL}_2(\mathcal{O})$ and $K_1 := 1 + M_2(\mathfrak{p})$, where $\mathcal{O}$ denotes the ring of integers in F and $\mathfrak{p}$ is the maximal ideal of $\mathcal{O}$. If we let $\mathbb{F}_q$ be the residue field of $\mathcal{O}$, then we have a natural reduction map $K_0 = \mathrm{GL}_2(\mathcal{O}) \twoheadrightarrow \mathrm{GL}_2(\mathbb{F}_q)$ whose kernel is exactly K_1. For any irreducible cuspidal representation κ of $\mathrm{GL}_2(\mathbb{F}_q)$, by pulling back it along the reduction map, we get an irreducible smooth representation of $\mathrm{GL}_2(\mathcal{O})$ (let's again write κ). We extend the representation κ from $\mathrm{GL}_2(\mathcal{O})$ to $Z(F)\,\mathrm{GL}_2(\mathcal{O})$ by fixing a character of $Z(F)$. More precisely, as κ is irreducible, the restriction of κ to $Z(\mathcal{O})$ is given by a character. Let ω be a character of $Z(F)$ such that $\omega|_{Z(\mathcal{O})}$ coincides with the central character of κ. We define a representation $\tilde{\kappa}$ of $Z(F)\,\mathrm{GL}_2(\mathcal{O})$ by

$$\tilde{\kappa}(z) := \begin{cases} \omega(z) & \text{if } z \in Z(F), \\ \kappa(g) & \text{if } g \in \mathrm{GL}_2(\mathcal{O}), \end{cases}$$

and put

$$\pi_{\tilde{\kappa}} := \text{c-Ind}_{Z(F)\,\mathrm{GL}_2(\mathcal{O})}^{\mathrm{GL}_2(F)} \tilde{\kappa}.$$

Then the representation $\pi_{\tilde{\kappa}}$ is an irreducible depth-zero supercuspidal representation of $\mathrm{GL}_2(F)$. Moreover, for any other κ' and ω', the representations $\pi_{\tilde{\kappa}}$ and $\pi_{\tilde{\kappa}'}$ are isomorphic if and only if $\kappa \cong \kappa'$ and $\omega = \omega'$.

We can completely imitate this construction for $D^\times$. We consider the reduction map

$$J_0 = \mathcal{O}_D^\times \to \mathcal{O}_D^\times/(1+\mathfrak{P}_D) = J_0/J_{\frac{1}{2}}.$$

Note that $\mathcal{O}_D/\mathfrak{P}_D$ is a division algebra over $\mathcal{O}/\mathfrak{p} = \mathbb{F}_q$; in fact, this is the quadratic extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$. The quotient group $J_0/J_{\frac{1}{2}}$ is then identified with $\mathbb{F}_{q^2}^\times$. So let us take a smooth character $\theta\colon \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ and regard it as a character of J_0. In a similar manner to above, we extend θ to $F^\times \mathcal{O}_D^\times$ by fixing a character ω of $F^\times$ such that $\omega|_{\mathcal{O}^\times} = \theta|_{\mathcal{O}^\times}$. Let us write $\tilde{\theta}\colon F^\times \mathcal{O}_D^\times \to \mathbb{C}^\times$ for this character. Now the question is whether this character extends to $D^\times$ or not. The subgroup $F^\times \mathcal{O}_D^\times$ is of index 2 in $D^\times$; if we fix a uniformizer ϖ_D of $D^\times$, then we have $D^\times = F^\times \mathcal{O}_D^\times \sqcup \varpi_D F^\times \mathcal{O}_D^\times$. In fact, the conjugate action of ϖ_D on $D^\times$ induces a nontrivial Galois action on $\mathbb{F}_{q^2}$, i.e., the Frobenius automorphism $x \mapsto x^q$.

Lemma 10.16. *Let G be a group and H its subgroup of index 2 (hence normal). Let ρ be an irreducible representation of H and ρ^g the representation of H given by $\rho^g(h) = (ghg^{-1})$ for $g \in G \smallsetminus H$. The following are equivalent:*

(1) ρ extends to a representation of G,

(2) $\operatorname{Ind}_H^G \rho$ is reducible (decomposes into two irreducible representations),

(3) $\rho^g \cong \rho$ for a(ny) $g \in G \smallsetminus H$.

Exercise 10.17. Prove this lemma.

By applying this general lemma to our setting, we see that

- $\tilde{\theta}$ extends to $D^\times$ (hence a one-dimensional depth-zero smooth representation) if $\theta^q = \theta$,
- $\operatorname{Ind}_{F^\times \mathcal{O}_D^\times}^{D^\times} \tilde{\theta}$ is an irreducible 2-dimensional depth-zero smooth representation of $D^\times$ when $\theta^q \neq \theta$. Let us write $\pi_{\tilde{\theta}}$ for this representation.

Because we are intereted in the case where the local Jacquet–Langlands image in $\Pi(\mathrm{GL}_2)_{\mathrm{disc}}$ is supercuspidal, let us only consider the latter case.

The point is that irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are classified exactly by characters θ of $\mathbb{F}_{q^2}^\times$ satisfying $\theta^q \neq \theta$ (see [BH06, Chapter 2] or my course notes of the last semester). To be more precise, for any such character θ, there exists an irreducible cuspidal representation κ_θ of $\mathrm{GL}_2(\mathbb{F}_q)$. Any irreducible cuspidal representation of $\mathrm{GL}_2(\mathbb{F}_q)$ is of the form κ_θ for some such θ. Finally, we have $\kappa_\theta \cong \kappa_{\theta'}$ if and only if $\theta' = \theta$ or $\theta' = \theta^q$.

Proposition 10.18. *Let $\theta\colon \mathbb{F}_{q^2}^\times \to \mathbb{C}^\times$ be a smooth character satisfying $\theta^q \neq \theta$. By fixing a character $\omega\colon F^\times \to \mathbb{C}^\times$ such that $\omega|_{\mathcal{O}^\times}$ equals (the inflation of) $\theta|_{\mathbb{F}_q^\times}$ and consider the representations $\pi_{\tilde{\theta}}$ and $\pi_{\tilde{\kappa}_\theta}$ as above. Then we have*

$$\mathrm{LJLC}(\pi_{\tilde{\theta}}) = \pi_{\tilde{\kappa}_\theta}.$$

Proof. By admitting the depth-preservation of LJLC, at least we see that $\mathrm{LJLC}(\pi_{\tilde{\theta}})$ is a depth-zero supercuspdial representation of $\mathrm{GL}_2(F)$, hence of the form $\pi_{\tilde{\kappa}_{\theta'}}$ for some $\theta'\colon \mathbb{F}_{q^2} \to \mathbb{C}^\times$ satisfying $(\theta')^q \neq \theta'$. It is easy to see that LJLC preserves the central characters, so our task is to show that $\theta' = \theta$ or $\theta' = \theta^q$.

Let $g \in \mathbb{F}_{q^2}^\times$ and regard it as an element of $\mathcal{O}_D^\times$ (by the Teichimuller lifting). Then, by the Frobenius character formula, we have

$$\Theta_{\pi_{\tilde{\theta}}}(g) = \sum_{\substack{x \in F^\times \mathcal{O}_D^\times \backslash D^\times \\ xgx^{-1} \in F^\times \mathcal{O}_D^\times}} \tilde{\theta}(xgx^{-1}) = \theta(g) + \theta^q(g).$$

On the other hand, whenever $g \in \mathbb{F}_{q^2}^\times \smallsetminus \mathbb{F}_q^\times$ (i.e., g is not central), g is regular semisimple, hence has a matching elliptic regular semisimple element $g' \in \mathrm{GL}_2(F)$. By the Frobenius character formula for supercuspidal representations of p-adic reductive groups, we have

$$\Theta_{\pi_{\tilde{\kappa}_{\theta'}}}(g') = \sum_{\substack{x \in Z(F)\,\mathrm{GL}_2(\mathcal{O}) \backslash \mathrm{GL}_2(F) \\ xg'x^{-1} \in Z(F)\,\mathrm{GL}_2(\mathcal{O})}} \Theta_{\tilde{\kappa}_{\theta'}}(xg'x^{-1})$$

as long as the sum is finite. In fact, we can check that the index set is represented by 1 (a consequence of the Cartan decomposition). Thus, by combining it with the character formula of $\kappa_{\theta'}$, we obtain

$$\Theta_{\pi_{\tilde{\kappa}_{\theta'}}}(g') = -\theta'(g) - \theta'^q(g).$$

Therefore, the character relation of LJLC, we have

$$\theta(g) + \theta^q(g) = \Theta_{\pi_{\tilde{\theta}}}(g) = -\Theta_{\pi_{\tilde{\kappa}_{\theta'}}}(g') = \theta'(g) + \theta'^q(g)$$

for any $g \in \mathbb{F}_{q^2}^\times \smallsetminus \mathbb{F}_q^\times$. Note that this identity holds even when $g \in \mathbb{F}_q^\times$ as LJLC preserves the central characters. Hence linear independence of characters (Artin's lemma) implies that $\theta' = \theta$ or $\theta' = \theta^q$. $\square$

11. Week 11: Unramified local Langlands correspondence

11.1. **Unramified representations.** For simplicity, we assume that G is a split connected reductive group over F in the following. We fix a split maximal torus T of G over F (i.e., $T \cong \mathbb{G}_{\mathrm{m}}^r$ for some r.) We also fix an F-rational Borel subgroup B of G containing T.

We fix a "hyperspecial" maximal open compact subgroup K of $G(F)$. Here, we do not go into the details of the definition of a hyperspecial maximal open compact subgroup. Typically, this group is chosen in the following way. When G is split, we can find its integral model $\mathcal{G}$ over $\mathcal{O}$ whose fibers are reductive. Then $\mathcal{G}(\mathcal{O})$ gives a hyperspecial maximal open compact subgroup of $G(F)$. The important property of a hyperspecial maximal open compact subgroup is that $G(F) = B(F)K$; this equality is often called the *Iwasawa decomposition*.

Example 11.1. Let $G = \mathrm{GL}_n$. We take T to be the diagonal maximal torus and B the upper-triangular Borel subgroup. The defining equation of GL_n can be taken such that only 0, ±1 are contained in its coefficient; for example,

$$\mathrm{GL}_n = \{((g_{ij})_{ij}, d) \in \mathbb{A}^{n^2} \times \mathbb{A}^1 \mid \det(g_{ij}) \cdot d = 1\}.$$

Hence we may naturally regard GL_n as a group scheme over $\mathcal{O}$, which gives a reductive model of G. Then we get a hyperspecial maximal open compact subgroup $K = \mathrm{GL}_n(\mathcal{O})$.

Definition 11.2. Let (π, V) be an irreducible admissible representation of $G(F)$. We say that (π, V) is *unramified* (or *K-spherical*) if $V^K \neq 0$.

Let $\Pi(G)_{\mathrm{ur}}$ denote the set of isomorphism classes of unramified representations of $G(F)$. Note that the notion of an unramified representation, hence also the definition of the set $\Pi(G)_{\mathrm{ur}}$, depends on the choice of K.

We write $\mathcal{H}_K := C_c^\infty(G(F)//K)$ for the $\mathbb{C}$-vector space of compactly supported bi-K-invariant $\mathbb{C}$-valued functions. Recall that this is a subalgebra of the Hecke algebra $\mathcal{H} = C_c^\infty(G(F))$ with respect to the convolution product. Here, we fix a Haar measure dg of $G(F)$ used in the definition of the convolution product such that $dg(K) = 1$. As mentioned in Week 8, the set of isomorphism classes of unramified representations corresponds to the set of isomorphism classes of simple $\mathcal{H}_K$-modules:

$$\Pi(G)_{\mathrm{ur}} \xrightarrow{1:1} \{\text{simple } \mathcal{H}_K\text{-modules}\}/\sim \colon (\pi, V) \mapsto V^K$$

To study the set $\Pi(G)_{\mathrm{ur}}$, it is crucial to understand the structure of the $\mathbb{C}$-algebra $\mathcal{H}_K$. We call $\mathcal{H}_K$ the *(K-)spherical Hecke algebra of* $G(F)$.

Let us put $T_0 := T(\mathcal{O})$ (by implicitly choosing a model of T over $\mathcal{O}$ contained in $\mathcal{G}$). Note that then we have a natural identification

$$T(F)/T_0 \xrightarrow{\cong} X_*(T) \colon t \mapsto \chi_t^\vee.$$

Here, $\chi_t^\vee$ denotes the cocharacter of T characterized by $\langle \chi, \chi_t^\vee \rangle := v(\chi(t))$ for any $\chi \in X^*(T)$, where v is the normalized valuation of F and $\langle -, - \rangle \colon X^*(T) \times X_*(T) \to \mathbb{Z}$ is a natural perfect pairing between the character and cocharacter groups. If we fix a uniformizer ϖ of F, then the inverse of this natural map is given by $X_*(T) \xrightarrow{\cong} T(F)/T_0 \colon \chi^\vee \mapsto \chi^\vee(\varpi)$. This identification induces a natural isomorphism of commutative $\mathbb{C}$-algebras

$$C_c^\infty(T(F)//T_0) \xrightarrow{\cong} \mathbb{C}[X_*(T)],$$

where we use the Haar measure dt of $T(F)$ such that $dt(T_0) = 1$ on the left-hand side and the righ-hand side denotes the group algebra associated to $X_*(T)$.

Let U be the unipotent radical of the fixed Borel subgroup B. We choose a Haar measure du of $U(F)$ such that $du(U(F) \cap K) = 1$. We define a character $\delta\colon T(F) \to \mathbb{C}^\times$ by

$$\delta(t) := |\det(\mathrm{Ad}(t) \mid \mathrm{Lie}(U)(F))|.$$

This character is called the *modulus character* of $T(F)$ (with respect to U).

Theorem 11.3 (Satake isomorphism)**.** *We define a map S from $\mathcal{H}_K$ to $\mathbb{C}[X_*(T)]$ by*

$$S\colon \mathcal{H}_K \to C_c^\infty(T(F)//T_0) \cong \mathbb{C}[X_*(T)]; \quad (Sf)(t) := \delta(t)^{\frac{1}{2}} \cdot \int_{U(F)} f(tu)\,du.$$

Then S is a $\mathbb{C}$-algebra homomorphism which induces an isomoprhism

$$S\colon \mathcal{H}_K \xrightarrow{\cong} \mathbb{C}[X_*(T)]^W,$$

where $\mathbb{C}[X_*(T)]^W$ denotes the subalgebra of W-invariant elements.

For the proof, see, e.g., [Car79, Theorem 4.1].

Example 11.4. When $G = \mathrm{GL}_n$ and T is its diagonal maximal torus, we can write $X_*(T) = \bigoplus_{i=1}^n \mathbb{Z}e_i^\vee$, where $e_i^\vee$ denotes the cocharacter of T defined by $x \mapsto \mathrm{diag}(1, \ldots, 1, x, 1, \ldots, 1)$ (x is put at the i-the entry). Hence we have $\mathbb{Z}[X_*(T)] \cong \mathbb{Z}[X_1^{\pm1}, \ldots, X_n^{\pm1}]$. The Weyl group $W \cong \mathfrak{S}_n$ acts on this algebra by permutation of variables $X_1, \ldots, X_n$.

Exercise 11.5. Determine the modulus character δ of GL_n.

Exercise 11.6. Let $G = \mathrm{GL}_2$ and $K = \mathrm{GL}_2(\mathcal{O})$.

(1) Let f be the characteristic function of $K\,\mathrm{diag}(\varpi, \varpi)K$. Compute $S(f)$.
(2) Let g be the characteristic function of $K\,\mathrm{diag}(\varpi, 1)K$. Compute $S(g)$.

Note that the Satake isomorphism especially tells us that the spherical Hecke algebra $\mathcal{H}_K$ is commutative. In patricular, this implies that any simple $\mathcal{H}_K$-module is 1-dimensional. For an unramified representation (π, V) of $G(F)$, a nonzero element of V^K (this is unique up to a scalar multiple since $\dim_{\mathbb{C}}(V^K) = 1$) is referred to as a *($K$-)spherical vector of V*.

Now let us think about $\Pi(G)_{\mathrm{ur}}$ again using the Satake isomorphism. Since $\mathbb{C}[X_*(T)]^W$ is a commutative $\mathbb{C}$-algebra, the isomorphism classes of its simple modules are classified by the maximal ideals of $\mathbb{C}[X_*(T)]^W$. (Indeed, if M is a simple $\mathbb{C}[X_*(T)]^W$-module, then it is generated by any nonzero vector $v \in M$. Then, as $\mathbb{C}[X_*(T)]^W$-modules, M is isomorphic to $\mathbb{C}[X_*(T)]^W/\mathrm{Ann}(v)$, where $\mathrm{Ann}(v)$ is the annihilator of v; an isomoprhism is given by $f \mapsto f \cdot v$. This annihilator ideal must be maximal so that M is simple. Conversely, for any maximal ideal $\mathfrak{m}$ of $\mathbb{C}[X_*(T)]^W$, $\mathbb{C}[X_*(T)]^W/\mathfrak{m}$ defines a simple $\mathbb{C}[X_*(T)]^W$-module.) In other words, $\Pi(G)_{\mathrm{ur}}$ corresponds to

$$\mathrm{Spm}(\mathbb{C}[X_*(T)^W]) \cong \mathrm{Spm}(\mathbb{C}[X_*(T)])/W,$$

where Spm denotes the set of maximal ideals. We let $\widehat{T}$ be the dual torus of T, i.e., a complex torus given by

$$\widehat{T} := X^*(T) \otimes \mathbb{C}^\times \cong \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C}^\times).$$

Note that $\mathrm{Spm}(\mathbb{C}[X_*(T)]) \cong \mathrm{Hom}_{\mathbb{C}\text{-alg}}(\mathbb{C}[X_*(T)], \mathbb{C}) \cong \mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C}^\times)$. Therefore, we have an identification

$$\mathrm{Spm}(\mathbb{C}[X_*(T)])/W \cong \widehat{T}/W.$$

In summary, we obtain

$$\Pi(G)_{\mathrm{ur}} \xrightarrow{1:1} \widehat{T}/W.$$

For an irreducible unramified representation $(\pi, V) \in \Pi(G)_{\mathrm{ur}}$, the corresponding element of $\widehat{T}/W$ is often referred to as the *Satake parameter* of (π, V).

Example 11.7. In the case of GL_n, the set of Satake parameters in this case is identified with $(\mathbb{C}^\times)^n/\mathfrak{S}_n$.

11.2. **Unramified principal series.** Let us also discuss how to explicitly construct unramified representations.

Definition 11.8. Let χ be a character of $T(F)$. We define the *normalized parabolic induction* $\text{n-Ind}_B^G \chi$ to be

$$\text{n-Ind}_B^G \chi := \mathrm{Ind}_{B(F)}^{G(F)}(\delta^{\frac{1}{2}}\chi),$$

where $\delta^{\frac{1}{2}}\chi$ is regarded as a character of $B(F)$ by pulling back along the natural quotient map $B(F) \twoheadrightarrow T(F)$. We also call $\text{n-Ind}_B^G \chi$ a *principal series* representation.

Lemma 11.9. *Let $\chi\colon T(F) \to \mathbb{C}^\times$ be a character trivial on T_0. Then there exists a unique unramified representation in the subquotients of* $\text{n-Ind}_B^G \chi$.

Proof. Taking the K-fixed part defines an exact functor from the category of smooth representations of $G(F)$ to the category of $\mathbb{C}$-vector space (see, e.g., [BH06, 2.3]). Hence, by realling that the subspace of K-fixed vectors in an irreducible unramified representation is 1-dimensional, if $\text{n-Ind}_B^G \chi$ contains r unramified representations in its subquotients, then $(\text{n-Ind}_B^G \chi)^K$ must be r-dimensional.

By definition, we have

$$(\text{n-Ind}_B^G \chi)^K = \left\{ f\colon G(F) \to \mathbb{C} \;\middle|\; \begin{array}{l} f(bg) = (\delta^{\frac{1}{2}}\chi)(b)f(g) \text{ for any } b \in B(F) \text{ and } g \in G(F), \\ f(gk) = f(g) \text{ for any } g \in G(F) \text{ and } k \in K \end{array} \right\}.$$

By the Iwasawa decomposition $G(F) = B(F)K$, this space is 1-dimensional. $\square$

Let χ be a character of $T(F)$ trivial on T_0. Then we get a natural map $\mathbb{C}[X_*(T)] \cong C_c^\infty(T(F)//T_0) \to \mathbb{C}$ induced by χ. For any $f_T \in \mathbb{C}[X_*]$, we write $\chi(f_T)$ for its image under this map.

Proposition 11.10. *Let $\chi\colon T(F) \to \mathbb{C}^\times$ be a character trivial on T_0. The action of $f \in \mathcal{H}_K$ on $(\text{n-Ind}_B^G \chi)^K$ is given by $\chi(Sf)$-multiplication.*

Proof. Let $f_K \in \mathcal{H}_K$ and $f \in (\text{n-Ind}_B^G \chi)^K$. Then, by the definition of the action of $\mathcal{H}_K$ on $(\text{n-Ind}_B^G \chi)^K$, the element $f_K \cdot f \in (\text{n-Ind}_B^G \chi)^K$ is given by

$$\begin{aligned}
(f_K \cdot f)(1) &= \int_{g \in G(F)} f_K(g) \cdot f(1 \cdot g)\, dg \\
&= \int_{b \in B(F)} \int_{k \in K} f_K(bk) \cdot f(bk)\, db\, dk \\
&= \int_{t \in T(F)} \int_{u \in U(F)} \int_{k \in K} f_K(tuk) \cdot f(tuk)\, dt\, du\, dk \\
&= \int_{t \in T(F)} \int_{u \in U(F)} \int_{k \in K} f_K(tu) \cdot \delta(t)^{\frac{1}{2}} \chi(t) f(1)\, dt\, du\, dk \\
&= \left(\int_{t \in T(F)} \int_{u \in U(F)} f_K(tu) \cdot \delta(t)^{\frac{1}{2}} \chi(t)\, dt\, du \right) \cdot \left(\int_{k \in K} f(1)\, dk \right).
\end{aligned}$$

Here, we used the following two kinds of integration formulas:

(1) We take the product measure db on $B(F)$ determined by dt and du. The, for any integrable function $\phi\colon B(F) \to \mathbb{C}$, we have
$$\int_{b\in B(F)} \phi(b)\, db = \int_{t\in T(F)} \int_{u\in U(F)} \phi(tu)\, dt\, du.$$

(2) for any integrable function $\phi\colon G(F) \to \mathbb{C}$, we have
$$\int_{g\in G(F)} \phi(g)\, dg = \int_{b\in B(F)} \int_{k\in K} \phi(bk)\, db\, dk.$$

On the other hand, recall that
$$(Sf_K)(t) := \delta(t)^{\frac{1}{2}} \cdot \int_{U(F)} f_K(tu)\, du.$$

Hence the above equals
$$\begin{aligned}(f_K \cdot f)(1) &= \int_{t\in T(F)} (Sf_K)(t) \cdot \chi(t)\, dt \cdot f(1) \\ &= \left(\sum_{t\in T(F)/T_0} (Sf_K)(t) \cdot \chi(t) \right) \cdot f(1) = \chi(Sf_K) \cdot f(1).\end{aligned}$$

In summary, we showed that $f_K \cdot f \in (\text{n-Ind}_B^G \chi)^K$ is an element satisfying $(f_K \cdot f)(1) = \chi(Sf_K) \cdot f(1)$. Since $(\text{n-Ind}_B^G \chi)^K$ is 1-dimensional, we conclude that $(f_K \cdot f) = \chi(Sf_K) \cdot f$. $\square$

Corollary 11.11. *Let $\chi\colon T(F) \to \mathbb{C}^\times$ be a character trivial on T_0. Let π_χ be the unique unramified subquotient of* $\text{n-Ind}_B^G \chi$. *Then the Satake parameter of π_χ is given by χ, where $\chi\colon T(F)/T_0 \to \mathbb{C}^\times$ is regarded as an element of* $\mathrm{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{C}^\times) =: \widehat{T}$.

11.3. Unramified L-parameters.

Definition 11.12. Let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ be an L-parameter. We say that ϕ is *unramified* if ϕ is trivial on $I_F \times \mathrm{SL}_2(\mathbb{C})$.

When $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ is an unramified L-parameter, ϕ factors through the projection to the quotient $W_F/I_F \cong \langle \mathrm{Frob} \rangle$, where Frob is the geometric Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ or its lift to $\mathrm{Gal}(F^{\mathrm{sep}}/F)$. Since the image of an L-parameter consists of semisimple elemenets by definition, an unramified L-parameter is determined only by the semisimple element of $\widehat{G}$ which is the image of Frob.

Recall that two L-parameters ϕ and ϕ' are said to be conjugate if there exists an element $g \in \widehat{G}$ satisfying $\phi' = \mathrm{Int}(g) \circ \phi$. In our notation, the set of conjugacy classes of L-parameters of G is denoted by $\Phi(G)$. Let $\Phi(G)_{\mathrm{ur}}$ be the subset of conjugacy classes of unramified L-parameters. Then we get a bijection
$$\widehat{G}_{\mathrm{ss}}/(\widehat{G}\text{-conj.}) \xrightarrow{1:1} \Phi(G)_{\mathrm{ur}} \colon s \mapsto [\mathrm{Frob} \mapsto s],$$
where $\widehat{G}_{\mathrm{ss}}$ denotes the subset of semisimple elements.

Let us fix a maximal torus $\widehat{T}$ of $\widehat{G}$. If we write $\widehat{W}$ for its Weyl group, i.e., $\widehat{W} := N_{\widehat{G}}(\widehat{T})/\widehat{T}$, we have a natural (conjugate) action of $\widehat{W}$ on $\widehat{T}$. Any element of $\widehat{T}$ is semisimple, hence we have a natural map
$$\widehat{T}/\widehat{W} \to \widehat{G}_{\mathrm{ss}}/(\widehat{G}\text{-conj.}).$$

Lemma 11.13. *The map $\widehat{T}/\widehat{W} \to \widehat{G}_{\mathrm{ss}}/(\widehat{G}$-conj.$)$ is bijective.*

Proof. It is a well-known fact that any semisimple element is conjugate to an element of $\widehat{T}$, which implies the surjectivity of the map. To check the injectivity, let us suppose that $t, t' \in \widehat{T}$ are conjugate in $\widehat{G}$; let $g \in \widehat{G}$ be an element such that $t' = gtg^{-1}$. Then, taking their connected centralizers, we get $Z_{\widehat{G}}(t')^\circ = Z_{\widehat{G}}(gtg^{-1})^\circ = gZ_{\widehat{G}}(t)^\circ g^{-1}$. As the centralizer of a maximal torus in a connected reductive group is itself, the relation $t \in \widehat{T}$ implies that $Z_{\widehat{G}}(t)^\circ \supset Z_{\widehat{G}}(\widehat{T})^\circ = \widehat{T}$. Similarly, we also have $Z_{\widehat{G}}(t')^\circ = gZ_{\widehat{G}}(t)^\circ g^{-1} \supset \widehat{T}$, or equivalently, $Z_{\widehat{G}}(t)^\circ \supset g^{-1}\widehat{T}g$. In particular, we get two maximal tori $\widehat{T}$ and $g^{-1}\widehat{T}g$ of $Z_{\widehat{G}}(t)^\circ$. Note that $Z_{\widehat{G}}(t)^\circ$ is again a connected reductive group because t is semisimple. Since all maximal tori of a connected reductive group are conjugate, we can find an element $h \in Z_{\widehat{G}}(t)^\circ$ satisfying $g^{-1}\widehat{T}g = h\widehat{T}h^{-1}$. In other words, $gh \in N_{\widehat{G}}(\widehat{T})$. Since h commutes with t, we get $t' = gtg^{-1} = (gh)t(gh)^{-1}$. □

In summary, we get a bijection

$$\widehat{T}/\widehat{W} \xrightarrow{1:1} \Phi(G)_{\mathrm{ur}} \colon s \mapsto [\mathrm{Frob} \mapsto s].$$

11.4. **Unramified local Langlands correspondence.** Recall that, in the classification of $\Pi(G)_{\mathrm{ur}}$ we have introduced the dual torus "$\widehat{T}$" of a split maximal torus T of G. On the other hand, in the classification of $\Phi(G)_{\mathrm{ur}}$ we used the same symbol "$\widehat{T}$" for a maximal torus of $\widehat{G}$. This is not an accident. The Langlands dual group $\widehat{G}$ is, by definition, a connected reductive group over $\mathbb{C}$ whose "root datum" is isomorphic to the "dual" of the root datum of G. The point is that we fix such an isomorphism of root data, i.e., the Langlands dual group $\widehat{G}$ is not just an abstract connected group over $\mathbb{C}$, but an abstract connected group over $\mathbb{C}$ equipped with the isomorphism of root data. This additional data contains, in particular, an isomorphism between $X^*(T)$ and $X_*(\widehat{T})$, where $\widehat{T}$ is a fixed maximal torus of $\widehat{G}$. In general, for a complex torus S, we have a natural identification $X_*(S) \otimes \mathbb{C}^\times \cong S \colon \chi^\vee \otimes z \mapsto \chi^\vee(z)$. Thus we get

$$X^*(T) \otimes \mathbb{C}^\times \cong X_*(\widehat{T}) \otimes \mathbb{C}^\times \cong \widehat{T}.$$

Note that the left-hand side is nothing but the definition of the dual torus of T. In summary, the notion of the dual torus itself is defined depending only on a torus over F. However, when the torus over F is a maximal torus of a connected reductive group G over F, the dual torus can be naturally identified with a maximal torus of $\widehat{G}$.

Moreover, the Weyl groups W and $\widehat{W}$ are also identified through the above-mentioned the isomorphism between root data. The action of W on $\widehat{T}$ (regarded as the dual torus) is then identified as the action of $\widehat{W}$ on $\widehat{T}$ (regarded as a maximal torus of $\widehat{G}$) through the identificaion $W \cong \widehat{W}$.

Now let us combine our discussions in the previous subsections. By composing the bijections $\Pi(G)_{\mathrm{ur}} \xrightarrow{1:1} \widehat{T}/W$ and $\Phi(G)_{\mathrm{ur}} \xrightarrow{1:1} \widehat{T}/\widehat{W}$, we obtain

$$\Pi(G)_{\mathrm{ur}} \xrightarrow{1:1} \widehat{T}/W \xrightarrow{1:1} \widehat{T}/\widehat{W} \xrightarrow{1:1} \Phi(G)_{\mathrm{ur}}.$$

This bijection is called the *unramified local Langlands correspondence*.

Conjecture 11.14. *The local Langlands correspondence* $\mathrm{LLC}_G \colon \Pi(G) \to \Phi(G)$ *extends the unramified local Langlands correspondence.*

Exercise 11.15. (1) Show that the Steinberg representation $\mathrm{St}_{\mathrm{GL}_2}$ of $\mathrm{GL}_2(F)$ is not unramified.

(2) Determine the Satake parameter of the trivial representation of $\mathrm{GL}_2(F)$.

11.5. **Structure of an unramified L-packet.** Recall that the map of the local Langlands correspondence $\mathrm{LLC}_G\colon \Pi(G) \to \Phi(G)$ is not expected to be bijective in general. Hence, for an unramified L-parameter $\phi \in \Phi(G)_{\mathrm{ur}}$, it can (and really does) happen that its L-packet Π_ϕ contains also non-unramified representations; the unramified member must be unique in the L-packet. Thus the natural question is how we can describe this special member in terms of the bijective map ("first name map") $\iota_{\mathfrak{w}}\colon \Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$. Here, remember that $S_\phi := \mathrm{Cent}_{\widehat{G}}(\mathrm{Im}(\phi))$ and $\mathcal{S}_\phi := \pi_0(S_\phi/Z(\widehat{G}))$. The map $\iota_{\mathfrak{w}}$ is (expected to be) defined depending on the choice of a Whittaker datum $\mathfrak{w}$ of G.

In the definition of a Whittaker datum, we consider a "generic character" of the unipotent radical $U(F)$ of the Borel subgroup. For example, in the case of GL_n, the character

$$\psi((u_{i,j})_{i,j}) := \psi_F(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$$

is generic, where ψ_F is a nontrivial additive character of F (this was an exercise in Week 9). In fact, for any non-zero elements $a_1, \ldots, a_{n-1} \in F^\times$, the character

$$\psi((u_{i,j})_{i,j}) := \psi_F(a_1u_{1,2} + a_2u_{2,3} + \cdots + a_{n-1}u_{n-1,n})$$

is also generic.[21] Using the language of algebraic groups, we can understand this as follows. If we let $\mathfrak{g}_{i,i+1}(F)$ be the subspace of $\mathfrak{gl}_n(F) = M_n(F)$ (the Lie algebra of $\mathrm{GL}_n(F)$) consisting of matrices such that the $(i, i+1)$-entry is nonzero and others are zero. Then $\mathfrak{g}_{i,i+1}(F)$ is the root subspace of $\mathfrak{gl}_n(F)$ associated to a simple root ("$e_i - e_{i+1}$"). Hence any nonzero element $a_i \in F^\times$ gives an F-basis of the 1-dimensional F-vector space $\mathfrak{g}_{i,i+1}(F)$. Hence, to any family of bases of simple root subspaces $(a_1, \ldots, a_{n-1})$, we may associate a generic character of $U(F)$. Such a family is called an "F-pinning" or "F-splitting".

On the other hand, recall that we have fixed a hyperspecial open compact subgroup of $G(F)$, which can be defined by fixing a reductive integral model $\mathcal{G}$ of G over $\mathcal{O}$. In fact, choosing an F-pinning also enables us to define a reductive integral model. For example, the model realizing $\mathrm{GL}_n(\mathcal{O})$ is nothing but the one determined by the "standard" F-splitting $(1, \ldots, 1)$. (This is a part of theory of Chevalley groups.)

In summary, for any F-splitting of G, we can define

- a Whittaker datum $\mathfrak{w}$ of G, and
- a hyperspecial open compact subgroup K of $G(F)$.

Let us choose $\mathfrak{w}$ and K consistently in this sense.

Conjecture 11.16. *Let $\phi \in \Phi(G)_{\mathrm{ur}}$ be an unramified L-parameter. Then its L-packet Π_ϕ contains a unique unramified (K-spherical) element, which corresponds to the trivial representation of $\mathcal{S}_\phi$ under the bijection $\iota_{\mathfrak{w}}\colon \Pi_\phi \xrightarrow{1:1} \mathrm{Irr}(\mathcal{S}_\phi)$. In patricular, when ϕ is generic, the unramified representation is the unique generic member of Π_ϕ.*

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[21] Actually this is a partial answer to the exercise.

12. WEEK 12: LOCAL LANGLANDS CORRESPONDENCE FOR CLASSICAL GROUPS

This week we discuss the local Langlands correspondence for split classical groups. Here, by "split classical groups", we mean one of the following groups:

(1) Odd special orthogonal group SO_{2n+1}:

$$\mathrm{SO}_{2n+1} := \{g \in \mathrm{GL}_{2n+1} \mid {}^t g J_{2n+1} g = J_{2n+1},\, \det(g) = 1\},$$

where we put J_N to be the anti-diagonal matrix of size N whose $(i, N+1-i)$-th entry is given by $(-1)^{i+1}$.

(2) Symplectic group Sp_{2n}:

$$\mathrm{Sp}_{2n} := \{g \in \mathrm{GL}_{2n} \mid {}^t g J_{2n} g = J_{2n}\}.$$

(3) Even special orthogonal group SO_{2n}:

$$\mathrm{SO}_{2n} := \{g \in \mathrm{GL}_{2n} \mid {}^t g J^+_{2n} g = J^+_{2n},\, \det(g) = 1\},$$

where we put J^+_N to be the anti-diagonal matrix of size N whose all anti-diagonal entries are given by 1.

These are one of the basic examples of connected reductive groups, hence it is expected that we have the local Langlands correspondence for them; it was established by Arthur [Art13]. The aim of this week is to understand its characterization.

Remark 12.1. (1) Arthur's work [Art13] also treats "quasi-split" (non-split) special even orthogonal groups, but here we only focus on the split case for simplicity. (In the odd special orthogonal case or the symplectic case, being quasi-split automatically implies being split.)

(2) Quasi-split unitary groups can be treated by the same idea. The local Langlands correspondence for quasi-split unitary groups was established by Mok [Mok15].

12.1. **Philosophy of the Langlands functoriality again.** Let F be a p-adic field, i.e., a non-archimedean local field of characteritic zero (hence a finite extension of $\mathbb{Q}_p$). Let G be a split classical group over F in the above sense.

We first go back to the idea of the Langlands functoriality. The Langlands dual group $\widehat{G}$ of G is a complex connected reductive group depending on G as follows:

$$\widehat{G} = \begin{cases} \mathrm{Sp}_{2n}(\mathbb{C}) & \text{if } G = \mathrm{SO}_{2n+1}, \\ \mathrm{SO}_{2n+1}(\mathbb{C}) & \text{if } G = \mathrm{Sp}_{2n}, \\ \mathrm{SO}_{2n}(\mathbb{C}) & \text{if } G = \mathrm{SO}_{2n}. \end{cases}$$

The point is that, in all the cases, $\widehat{G}$ is again naturally regarded as a subgroup of GL_N for some $N \in \mathbb{Z}_{>0}$. (For example, $N = 2n$ in the case where $G = \mathrm{SO}_{2n+1}$.) Let us write $\xi\colon \widehat{G} \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ for the natural embedding. In patricular, we have a map

$$\xi_*\colon \Phi(G) \to \Phi(\mathrm{GL}_N)$$

between the set of equivalence classes of L-parameters of G and GL_N given by composing the embedding $\xi\colon \widehat{G} \hookrightarrow \mathrm{GL}_N(\mathbb{C})$ (recall that the Langlands dual of GL_N is $\mathrm{GL}_N(\mathbb{C})$). Hence, if we believe the existence of the local Langlands correspondence for G, we should be able to attach an irreducible admissible representation π_ϕ of $\mathrm{GL}_N(F)$ to each irreducible admissible representation π of $G(F)$ (recall that the local Langlands correspondence for

GL_N is available by the work of Harris–Taylor and that it is bijective), where we let $\phi_G := \mathrm{LLC}_G(\pi)$ and put $\phi := \xi \circ \phi_G$.

$$\begin{tikzcd}
\Pi(\mathrm{GL}_N) \ni \pi_\phi & & \mathrm{GL}_N(\mathbb{C}) \arrow[ll, mapsto, "\mathrm{LLC}_{\mathrm{GL}_N}^{-1}"'] \\
\Pi(G) \supset \Pi_{\phi_G} \ni \pi \arrow[u, squiggly, "\text{lifting}"] \arrow[r, mapsto, "\mathrm{LLC}_G"] & W_F \times \mathrm{SL}_2(\mathbb{C}) \arrow[ur, "\phi"] \arrow[r, "\phi_G"] & \widehat{G} \arrow[u, hook, "\xi"']
\end{tikzcd}$$

This operation is referred to as the "lifting" in general. Note that, by this construction, any members of the fiber of the map LLC_G (the L-packet for ϕ_G, Π_{ϕ_G}) obviously should lift to the same representation π_ϕ. The philosophy of the Langlands functoriality is that

> *the lifting $\pi \mapsto \pi_\phi$ should be able to be characterized only in terms of representation theory of p-adic reductive groups (in patricular, without appealing to the local Langlands correspondence).*

This idea itself can be applicable to any pair of (split) connected reductive groups G_1 and G_2 equipped with a map between Langlands dual groups $\widehat{G}_1 \to \widehat{G}_2$. However, in the current setting, we have the following particular fact:

Proposition 12.2. *The map $\xi_* \colon \Phi(G) \to \Phi(\mathrm{GL}_N)$ is injective when $G = \mathrm{SO}_{2n+1}$ or Sp_{2n}.*

In other words, any L-parameter ϕ_G of G is determined by the representation π_ϕ of $\mathrm{GL}_N(F)$ uniquely. Keeping this in mind, we reverse our perspective as follows.

> *if we can define the lifting map $\Pi(G) \to \Pi(\mathrm{GL}_N)$ (say "*Lift*") only in terms of the representation theory of p-adic reductive groups, then we can define the local Langlands correspondence for G by $\mathrm{LLC}_G := \xi_*^{-1} \circ \mathrm{LLC}_{\mathrm{GL}_N} \circ \mathrm{Lift}$.*

Remark 12.3. The above proposition does not hold when $G = \mathrm{SO}_{2n}$; in this case, each fiber of the map $\xi_* \colon \Phi(\mathrm{SO}_{2n}) \to \Phi(\mathrm{GL}_{2n})$ exactly consists of the $\mathrm{O}_{2n}(\mathbb{C})$-orbit. The even special orthogonal group $\mathrm{SO}_{2n}(\mathbb{C})$ is an index two normal subgroup of the even full orthogonal group

$$\mathrm{O}_{2n} := \{g \in \mathrm{GL}_{2n} \mid {}^t g J_{2n}^{+} g = J_{2n}^{+}\}.$$

The conjugate action of $\mathrm{O}_{2n}(\mathbb{C})$ preserves $\mathrm{SO}_{2n}(\mathbb{C})$, hence induces an action on the set of equivalence classes of L-parameters $\Phi(\mathrm{SO}_{2n})$. Thus, in principle, it is impossible to remove this ambiguity as long as we insist on the above strategy. Indeed, precisely speaking, Arthur's result has been established only modulo the $\mathrm{O}_{2n}(\mathbb{C})$-action.

12.2. **Linear algebra.** Let us prove Proposition 12.2; the argument is purely linear-algebraic.

We first review some basic facts about bilinear forms. We put $V := \mathbb{C}^{\oplus n}$ and fix its standard basis. Then any $\mathbb{C}$-bilinear form $\langle -,- \rangle \colon V \times V \to \mathbb{C}$ corresponds bijectively to a matrix $J \in M_n(\mathbb{C})$ by the relation

$$\langle v, w \rangle = {}^t v J w$$

for any $v, w \in V$. A bilinear form $\langle -,- \rangle$ is non-degenerate if and only if its representation matrix J is invertible. We say that $\langle -,- \rangle$ is symmetric (resp. skew-symmetric) if it satisfies $\langle v, w \rangle = \langle w, v \rangle$ (resp. $\langle v, w \rangle = -\langle w, v \rangle$) for any $v, w \in V$. Note that, in terms of the representation matrix J, this condition is equivalent to ${}^t J = J$ (resp. ${}^t J = -J$).

When $V = \mathbb{C}^{\oplus n}$ is equipped with an action of a group G (i.e., a representation of G) and a bilinear form $\langle -,- \rangle \colon V \times V \to \mathbb{C}$ satisfies

$$\langle g \cdot v, g \cdot w \rangle = \langle v, w \rangle$$

for any $v, w \in V$ and $g \in G$, we say that $\langle -, - \rangle$ is G-invariant. Note that, if we let J be the representation matrix of $\langle -, - \rangle$, this condition is equivalent to that

$$ {}^{t}gJg = J. $$

Lemma 12.4. *Suppose that V is an irreducible representation of G. Then there exists a non-degenerate G-invariant bilinear form $\langle -, - \rangle \colon V \times V \to \mathbb{C}$ if and only if V is self-dual (i.e., $V^\vee \cong V$ as representations of G). Moreover, in this case, such bilinear form is necessarily symmetric or skew-symmetric.*

Proof. Note that we have $\mathrm{Hom}_G(V \otimes V, \mathbb{C}) \cong \mathrm{Hom}_G(V, V^\vee)$. Since V is irreducible, so is $V^\vee$. Thus, by Schur's lemma (which holds with some appropriate assumtion on G and V), the space $\mathrm{Hom}_G(V, V^\vee)$ is nonzero only when $V \cong V^\vee$ and is 1-dimensional in that case.

Assuming that V is self-dual, let us a G-invariant non-degenerate bilinear form $\langle -, - \rangle$ on V. Then its "transpose" ${}^{t}\langle -, - \rangle$, which is defined by

$$ {}^{t}\langle v, w \rangle = \langle w, v \rangle $$

is also a non-degenerate G-invariant bilinear form on V. Since such forms are unique up to scalar, there exists a constant $c \in \mathbb{C}^\times$ satisfying ${}^{t}\langle -, - \rangle = c \cdot \langle -, - \rangle$. Applying the transpose again to this equality, we get

$$ {}^{t}({}^{t}\langle -, - \rangle) = c \cdot {}^{t}\langle -, - \rangle = c^2 \cdot \langle -, - \rangle. $$

As ${}^{t}({}^{t}\langle -, - \rangle) = \langle -, - \rangle$, we see $c^2 = 1$. Hence $\langle -, - \rangle$ is symmetric or skew-symmetric depending on $c = +1$ or $c = -1$. ☐

We often call "c" in the above proof the *sign* of a nondegenerate G-invariant bilinear form.

Lemma 12.5. *Let V be a representation of G equipped with a non-degenerate G-invariant bilinear form $\langle -, - \rangle$. Suppose that we have a decomposition $V \cong V_1 \oplus V_2$ as representation of G. If $V_1 \not\cong V_2^\vee$ (equivalently, $V_2 \not\cong V_1^\vee$), then the restrictions of $\langle -, - \rangle$ to $V_1 \times V_1$ and $V_2 \times V_2$ are again non-degenerate and G-invariant.*

Proof. The G-invariant bilinear form $\langle -, - \rangle \colon V \otimes V \to \mathbb{C}$ induces a G-equivariant isomorphism $V \cong V^\vee$, hence $V_1 \oplus V_2 \cong V_1^\vee \oplus V_2^\vee$. Projecting to $V_1^\vee$, we get a G-equivariant surjection $V_1 \oplus V_2 \twoheadrightarrow V_1^\vee$. Since $V_2 \not\cong V_1^\vee$, its restriction to V_2 is zero, Thus its restriction to V_1 is surjective, hence isomorphic. Similarly, we also get a G-equivariant isomorphism $V_2 \twoheadrightarrow V_2^\vee$. ☐

Now let us prove Proposition 12.2. Because the case where $G = \mathrm{Sp}_{2n}$ can be treated in the same manner, we only consider the case where $G = \mathrm{SO}_{2n+1}$. In the following proof, we simply write $\mathrm{WD}_F := W_F \times \mathrm{SL}_2(\mathbb{C})$.

Proposition 12.6. *Let $V = \mathbb{C}^{\oplus n}$ be a representation of a group G. Let $\langle -, - \rangle$ and $\langle -, - \rangle'$ be G-invariant symplectic forms on G. Then there exists an element $x \in \mathrm{GL}_n(\mathbb{C})$ satisfying $\langle v, w \rangle' = \langle xv, xw \rangle$ for any $v, w \in V$ (equivalently, if we let J and J' be the representation matrices of $\langle -, - \rangle$ and $\langle -, - \rangle'$, then $J' = {}^{t}xJx$) and $g \cdot x = x \cdot g$ for any $g \in G$.*

Proof. We decompose V into the sum of irreducible representations:

$$ V = \bigoplus_i V_i^{\oplus n_i} = \bigoplus_i V_i \otimes M_i, $$

where V_i's are irreducible representations which are distinct to each other and $M_i := \mathbb{C}^{\oplus n_i}$ ("multiplicity space"; G acts on M_i trivially). Since V is self-dual, we have $\bigoplus_i V_i^\vee \otimes M_i \cong \bigoplus_i V_i \otimes M_i$. Hence each V_i satisfies one of the following:

(1) $V_i^\vee \cong V_i$,

(2) $V_i^\vee \cong V_j$ and $V_j^\vee \cong V_i$ for a unique j; we moreover have $n_i = n_j$.

Rearranging the above direct sum decomposition according to the types of V_i in this sense, we may write

$$V = \Bigl(\bigoplus_i V_i \otimes M_i\Bigr) \oplus \Bigl(\bigoplus_i (V_i \oplus V_i^\vee) \otimes M_i\Bigr),$$

where the first direct sum is over i's such that V_i is of type (1) and the second is over type (2). By Lemma 12.5, restrictions of $\langle -,- \rangle$ and $\langle -,- \rangle'$ to each piece is again a G-invariant symplectic form. Therefore, it is enough to show the proposition for each direct summand $V_i \otimes M_i$ or $(V_i \oplus V_i^\vee) \otimes M_i$.

Let us first consider the case (1). Because i is fixed, we temporarily write $W := V_i$, $M := M_i$, $m = n_i$ in short. Note that we have a canonical isomorphism

$$\mathrm{Hom}_{\mathbb{C}}(W \otimes W, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C}) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{C}}\bigl((W \otimes M) \otimes (W \otimes M), \mathbb{C}\bigr)$$

which is G-equivariant. Hence, taking the G-invariant part, we get

$$\begin{aligned}\bigl(\mathrm{Hom}_{\mathbb{C}}(W \otimes W, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C})\bigr)^G &\cong \mathrm{Hom}_{\mathbb{C}}\bigl((W \otimes M) \otimes (W \otimes M), \mathbb{C}\bigr)^G \\ &= \mathrm{Hom}_G\bigl((W \otimes M) \otimes (W \otimes M), \mathbb{C}\bigr).\end{aligned}$$

Since the action on $\mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C})$ is trivial, we have

$$\begin{aligned}\bigl(\mathrm{Hom}_{\mathbb{C}}(W \otimes W, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C})\bigr)^G &= \mathrm{Hom}_{\mathbb{C}}(W \otimes W, \mathbb{C})^G \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C}) \\ &= \mathrm{Hom}_G(W \otimes W, \mathbb{C}) \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C}).\end{aligned}$$

Note that $\mathrm{Hom}_G(W \otimes W, \mathbb{C})$ is 1-dimensional since V is irreducible self-dual; let c be the sign of a unique-up-to-scalar bilinear form. Then, so that the resulting (under the above identifications) bilinear form on $V \otimes M$ is symplectic, a bilinear form on M must be nondegenerate and have sign $-c$. Thus, in summary, giving a nondegenerate bilinear form on M with sign $-c$ is equivalent to giving a G-invariant symplectic form on $W \otimes M$. Let us write $\langle -,- \rangle = \langle -,- \rangle_W \otimes \langle -,- \rangle_M$ and $\langle -,- \rangle' = \langle -,- \rangle_W \otimes \langle -,- \rangle'_M$ according to this discussion. It is a well-known fact that any nondegenerate bilinear forms on M with the same sign are equivalent. In other words, there exists a $y \in \mathrm{GL}_m(\mathbb{C})$ such that $\langle m, m' \rangle'_M = \langle ym, ym' \rangle_M$ for any $m, m' \in M$. Hence, by putting $x := \mathrm{id}_W \otimes y \in \mathrm{GL}_{\mathbb{C}}(W \otimes M)$, we get a desired element.

The case (2) can be treated in a similar manner. Agaian in this case, we have a natural identification

$$\begin{aligned}&\mathrm{Hom}_G\bigl((W \oplus W^\vee) \otimes (W \oplus W^\vee), \mathbb{C}\bigr) \otimes \mathrm{Hom}_{\mathbb{C}}(M \otimes M, \mathbb{C}) \\ &\xrightarrow{\cong} \mathrm{Hom}_G\bigl(((W \oplus W^\vee) \otimes M) \otimes ((W \oplus W^\vee) \otimes M), \mathbb{C}\bigr).\end{aligned}$$

Thus we may write $\langle -,- \rangle = \langle -,- \rangle_{W \oplus W^\vee} \otimes \langle -,- \rangle_M$ and $\langle -,- \rangle' = \langle -,- \rangle'_{W \oplus W^\vee} \otimes \langle -,- \rangle'_M$. Note that, since $W \ncong W^\vee$, the restrictions of $\langle -,- \rangle_{W \oplus W^\vee}$ and $\langle -,- \rangle'_{W \oplus W^\vee}$ to $W \otimes W$ and $W^\vee \otimes W^\vee$ must be zero. Moreover, by the symplecticity, the restrictions of $\langle -,- \rangle_{W \oplus W^\vee}$ and $\langle -,- \rangle'_{W \oplus W^\vee}$ to $W \otimes W^\vee$ are automatically determined by the restrictions to $W^\vee \otimes W$. By also noting that the restriction to $W \otimes W^\vee$ must be unique up to scalar (by the irreducibility of W), we conclude that giving a nondegenerate bilinear form on M[22] is equivalent to giving a G-invariant symplectic form on $(W \oplus W^\vee) \otimes M$. It is a well-known fact that there exists elements $y, y' \in \mathrm{GL}_{\mathbb{C}}(M)$ such that $\langle m, m' \rangle'_M = \langle ym, y'm' \rangle_M$ for any $m, m' \in M$. By

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[22] Note that, in contrast to the previous case, we don't have any prescription on the sign of M

putting $x := (\mathrm{id}_W \otimes 0_{W^\vee}) \otimes y + (0_W \otimes \mathrm{id}_{W^\vee}) \otimes y' \in \mathrm{GL}_{\mathbb{C}}((W \oplus W^\vee) \otimes M)$, we get a desired element. ☐

Proof of Proposition 12.2. Let ϕ and ϕ' be L-parameters of SO_{2n+1}, hence these are homomorphism $\mathrm{WD}_F \to \mathrm{Sp}_{2n}(\mathbb{C})$. Recall that our Sp_{2n} is defined via symplectic form (i.e., non-degenerate skew-symmetric form) represented by J_{2n}; let $\langle -,- \rangle \colon V \otimes V \to \mathbb{C}$ denote this symplectic form, where $V = \mathbb{C}^{\oplus 2n}$. Thus, L-parameters ϕ and ϕ' can be thought of as representations of WD_F on $V = \mathbb{C}^{\oplus 2n}$ such that the symplectic form $\langle -,- \rangle$ is WD_F-invariant.

The assumption is that ϕ and ϕ' are $\mathrm{GL}_{2n}(\mathbb{C})$-conjugate, i.e., there exists an element $g \in \mathrm{GL}_{2n}(\mathbb{C})$ satisfying $\phi'(\sigma) = g\phi(\sigma)g^{-1}$ for any $\sigma \in \mathrm{WD}_F$. From this assumption, we want to deduce that there exists $h \in \mathrm{Sp}_{2n}(\mathbb{C})$ satisfying $\phi'(\sigma) = h\phi(\sigma)h^{-1}$ for any $\sigma \in \mathrm{WD}_F$.

Since the image of ϕ' is in $\mathrm{Sp}_{2n}(\mathbb{C})$, we have

$$ {}^t\phi'(\sigma) J_{2n} \phi'(\sigma) = J_{2n}. $$

As $\phi'(\sigma) = g\phi(\sigma)g^{-1}$, this is equivalent to

$$ {}^t(g\phi(\sigma)g^{-1}) J_{2n} (g\phi(\sigma)g^{-1}) = J_{2n}, $$

which is furthermore equivalent to

$$ {}^t\phi(\sigma) \cdot {}^t g J_{2n} g \cdot \phi(\sigma) = {}^t g J_{2n} g. $$

Note that ${}^t g J_{2n} g$ is also a non-degenerate skew-symmetric matrix (let us write $J'_{2n} := {}^t g J_{2n} g$). In other words, also the symplectic form on V realized by this matrix (say $\langle -,- \rangle'$) is WD_F-invariant with respect to the representation ϕ.

Applying Proposition 12.6 to $\langle -,- \rangle$ and $\langle -,- \rangle'$, we can find an element $x \in \mathrm{GL}_{\mathbb{C}}(V)$ such that $J'_{2n} = xJ_{2n}x^{-1}$ and $x\phi(\sigma) = \phi(\sigma)x$ for any $\sigma \in \mathrm{WD}_F$. Hence we get ${}^t g J_{2n} g = {}^t x J_{2n} x$, which implies that $gx^{-1} \in \mathrm{Sp}_{2n}(\mathbb{C})$. Therefore, by putting $h := gx^{-1} \in \mathrm{Sp}_{2n}(\mathbb{C})$, we obtain

$$ \phi'(\sigma) = g\phi(\sigma)g^{-1} = gx^{-1}\phi(\sigma)xg^{-1} = h\phi(\sigma)h^{-1} $$

for any $\sigma \in \mathrm{WD}_F$. ☐

12.3. **Appearance of twisted representation theory.** Now the problem is how to define the map

$$ \mathrm{Lift} \colon \Pi(G) \to \Pi(\mathrm{GL}_N). $$

In fact, we have already considered the same problem in a different, but much simpler, context; that is, the local Jacquet–Langlands correspondence for GL_2. Recall that we have a map

$$ \mathrm{LJLC} \colon \Pi(D^\times) \to \Pi(\mathrm{GL}_2)_{\mathrm{disc}}, $$

where $D^\times$ is the multiplicative group of the division quaternion algebra over F. The characterization of the map LJLC is given by the following identity (write $\pi' := \mathrm{LJLC}(\pi)$ for a $\pi \in \pi(D^\times)$)

$$ \Theta_\pi(g) = -\Theta_{\pi'}(g'). $$

Here, Θ_π and $\Theta_{\pi'}$ are Harish-Chandra characters of π and π', hence functions on the regular semisimple loci of $D^\times(F)$ and $\mathrm{GL}_2(F)$, respectively. Thus, so that this identity makes sense, it is necessary to clarify the meaning of "g" and "g'"; g is any regular semisimple element of $D^\times(F)$ and g' is its "matching" regular semisimple element of $\mathrm{GL}_2(F)$ in the sense that g and g' are conjugate over $\overline{F}$. Hence, the crucially important point here is that GL_2 and $D^\times$ are isomorphic over $\overline{F}$.

The strategy is to imitate this idea. So our first task is to define the notion of "matching elements" between GL_N and G. For simplicity, let us only discuss the case where $G = \mathrm{SO}_{2n+1}$, hence $\mathrm{GL}_N = \mathrm{GL}_{2n}$. Recall that any regular semisimple element of $\mathrm{GL}_{2n}(F)$ is conjugate to a diagonal elemeny of $\mathrm{GL}_{2n}(\overline{F})$ such that all diagonal entries are distinct. A similar fact for SO_{2n+1} is the following.

Lemma 12.7. *Let g be a regular semisimple element of $\mathrm{SO}_{2n+1}(F)$. Then g is conjugate to a diagonal element of $\mathrm{SO}_{2n+1}(\overline{F})$ of the form*

$$\mathrm{diag}(t_1, \ldots, t_n, 1, t_n^{-1}, \ldots, t_1^{-1}),$$

where $t_1, \ldots, t_n \in \overline{F}^{\times}$ are elements which are distinct to each other and not equal to 1.

From this description, we immediately notice that regular semisimple elements of GL_{2n} are much abundant compared with those of SO_{2n+1}. Especially, it seems impossible to define a nice relationship between them so that the resulting character identity characterizes a map $\mathrm{Lift}\colon \Pi(\mathrm{SO}_{2n+1}) \to \Pi(\mathrm{GL}_{2n})$.

The idea is to, instead, appeal to the "twisted" representation theory of GL_{2n}. The point is that $\widehat{G} = \mathrm{Sp}_{2n}(\mathbb{C})$ is regarded as the fixed point set of $\mathrm{GL}_{2n}(\mathbb{C})$ with respect to the involution

$$\theta\colon \mathrm{GL}_{2n} \xrightarrow{\cong} \mathrm{GL}_{2n}; \quad g \mapsto J_{2n}{}^{t}g^{-1}J_{2n}^{-1}.$$

In particular, an L-parameter ϕ of GL_{2n} is coming from an L-parameter of SO_{2n+1} if and only if $\theta \circ \phi = \phi$. The involution θ can be also defined for GL_{2n} over F on the automorphic side. It is known that the local Langlands correspondence for GL_{2n} preserves the θ-stability, i.e., $\pi \in \Pi(\mathrm{GL}_{2n})$ is *θ-stable* (in the sense that $\pi \cong \pi \circ \theta$) if and only if $\phi := \mathrm{LLC}_{\mathrm{GL}_{2n}}(\pi)$ is *θ-stable* (in the sense that $\theta \circ \phi \cong \phi$.). Therefore, the image of the map $\mathrm{Lift}\colon \Pi(\mathrm{SO}_{2n+1}) \to \Pi(\mathrm{GL}_{2n})$, which we want to define now, should consist of θ-stable representations.

Exercise 12.8. Let $\phi\colon W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_{2n}(\mathbb{C})$ be an L-parameter. Prove that the following are equivalent:

(1) $\theta \circ \phi \cong \phi$, i.e., there exists an element $g \in \mathrm{GL}_{2n}(\mathbb{C})$ satisfying $\theta \circ \phi = \mathrm{Int}(g) \circ \phi$.
(2) $\theta \circ \phi = \phi$ after replacing ϕ with its equivalent L-parameter (i.e., $\mathrm{Int}(g) \circ \phi$ for some $g \in \mathrm{GL}_{2n}(\mathbb{C})$) if necessary.

In fact, any θ-stable irreducible admissible representation of $\mathrm{GL}_N(F)$ can be also characterized in terms of the *twisted Harish-Chandra character*. Recall that, for any irreducible admissible representation (π, V) of $\mathrm{GL}_N(F)$, we call

$$\Theta_\pi^{\mathrm{dist}}\colon C_c^\infty(\mathrm{GL}_N) \to \mathbb{C}; \quad f \mapsto \mathrm{tr}(\pi(f))$$

the character distribution of (π, V), where $\mathrm{tr}(\pi(f))$ is an operator on V defined by

$$\pi(f)(v) := \int_G f(g) \cdot \pi(g)(v)\, dg$$

Then the Harish-Chandra character Θ_π of (π, V) is defined to be the unique locally constant function $\Theta_\pi\colon \mathrm{GL}_N(F)_{\mathrm{rs}} \to \mathbb{C}$ satisfying

$$\Theta_\pi^{\mathrm{dist}}(f) = \int_{\mathrm{GL}_N(F)} f(g) \cdot \Theta_\pi(g)\, dg$$

for any $f \in C_c^\infty(\mathrm{GL}_N)$ (the unique existence of such a function is a highly nontrivial theorem of Harish-Chandra). When (π, V) is θ-stable, we can consider a variant of Θ_π as follows. We first fix an isomorphism $I \colon \pi \xrightarrow{\cong} \pi \circ \theta$. If we consider the distribution

$$\tilde{\Theta}_\pi^{\mathrm{dist}} \colon C_c^\infty(\mathrm{GL}_N) \to \mathbb{C}; \quad f \mapsto \mathrm{tr}(\pi(f) \circ I),$$

then there exists a unique locally constant function $\Theta_\pi \colon \widetilde{\mathrm{GL}}_N(F)_{\mathrm{rs}} \to \mathbb{C}$ satisfying

$$\tilde{\Theta}_\pi^{\mathrm{dist}}(f) = \int_{\mathrm{GL}_N(F)} f(g) \cdot \tilde{\Theta}_\pi(g)\, dg$$

for any $f \in C_c^\infty(\mathrm{GL}_N)$. Here, $\widetilde{\mathrm{GL}}_N(F)_{\mathrm{rs}}$ denotes the set of "θ-regular θ-semisimple" elements. More precisely, we say that an element $g \in \mathrm{GL}_N(F)$ is *θ-semisimple* if there exists an element $x \in \mathrm{GL}_N(\overline{F})$ such that

$$xg\theta(x)^{-1} = \mathrm{diag}(t_1, \ldots, t_{2n}).$$

Note that, in this case, we have

$$\begin{aligned} xg\theta(x)^{-1} \cdot \theta(xg\theta(x)^{-1}) &= xg\theta(g)x^{-1} \\ &= \mathrm{diag}(t_1, \ldots, t_{2n}) \cdot \theta(\mathrm{diag}(t_1, \ldots, t_{2n})) \\ &= \mathrm{diag}(t_1, \ldots, t_{2n}) \cdot \mathrm{diag}(t_{2n}^{-1}, \ldots, t_1^{-1}) \\ &= \mathrm{diag}(t_1/t_{2n}, \ldots, t_{2n}/t_1). \end{aligned}$$

We call a θ-semisimple element g is *θ-regular* if all $t_1/t_{2n}, \ldots, t_{2n}/t_1$ are distinct. The function $\tilde{\Theta}_\pi$ is called the *twisted Harish-Chandra character of* (π, V). As in the "untwisted case", any θ-stable irreducible admissible representation of $G(F)$ is uniquely determined by its θ-twisted character. So the idea is to use the θ-twisted character in the formulation of the character relation.

Definition 12.9. Let $g \in \mathrm{SO}_{2n+1}(F)$ be a regular semisimple element; we let

$$\mathrm{diag}(s_1, \ldots, s_n, 1, s_n^{-1}, \ldots, s_1^{-1}),$$

be an element of $\mathrm{SO}_{2n+1}(\overline{F})$ conjugate to g. Let $\tilde{g} \in \mathrm{GL}_{2n}(F)$ be a θ-regular θ-semisimple element; we let

$$\mathrm{diag}(t_1/t_{2n}, \ldots, t_{2n}/t_1)$$

be an element of $\mathrm{GL}_{2n}(\overline{F})$ conjugate to $\tilde{g}\theta(\tilde{g})$. We say that g is a *norm* of $\tilde{g}$ if

$$\{s_1, \ldots, s_n, s_n^{-1}, \ldots, s_1^{-1}\} = \{t_1/t_{2n}, \ldots, t_{2n}/t_1\}.$$

12.4. **Twisted endoscopic character relation.** Now we are ready to state the definition of the lifting map.

Theorem 12.10. *There exists a unique map ("twisted endoscopic lifting")*

$$\mathrm{Lift} \colon \Pi(\mathrm{SO}_{2n+1})_{\mathrm{temp}} \to \Pi(\mathrm{GL}_{2n})_{\mathrm{temp}}$$

characterized by the following identity ("twisted endoscopic character relation"):

$$\tilde{\Theta}_{\mathrm{Lift}(\pi)}(\tilde{g}) = \sum_{\pi'} \Theta_{\pi'}(g),$$

where π' in the sum runs over elements of $\mathrm{Lift}^{-1} \circ \mathrm{Lift}(\pi)$ and g and $\tilde{g}$ are any matching (θ-)regular (θ-)semisimple elements of $\mathrm{SO}_{2n+1}(F)$ and $\mathrm{GL}_{2n}(F)$.

It is a subtle problem whether we can always find a matching element $\tilde{g}$ for any g (or also g for any $\tilde{g}$). But, at least modulo this subtlety, the uniqueness of the map Lift is just a consequence of the linear independence of (twisted) Harish-Chandra charcaters of representation. On the other hand, the existence of such a map Lift is much more nontrivial fact, which is nothing but the local main theorem in the theory of Arthur ([Art13, Theorem 2.2.1]).

Also note that the subscript "temp" denotes the subset of isomorphism classes of irreducible tempered representations. In fact, the Harish-Chandra characters of representations behave not so well for non-tempetered representation in some sense. Hence, so that Theorem 12.10 holds, it is really necessary to restrict ourselves to tempered representations.

Remark 12.11. The above formulation of the map Lift works also for other quasi-split classical groups or even more general pairs of groups G_1 and G_2 satisfying certain axioms (called *twisted endoscopic groups*). However, in general, the endoscopic character relation could be much more complicated. Especially, the endoscopic side (the side of the source of the map Lift) contains a very subtle correction terms called the *transfer factor* $\Delta(g, \tilde{g})$ depending on g and $\tilde{g}$. The general theory of twisted endoscopy was initiated by Kottwitz–Shelstad [KS99] based on the preceding work of Langlands–Shelstad [LS87].

Now we can "define" the local Langlands correspondence for G. As described above, for any $\pi \in \Pi(G)_{\mathrm{temp}}$, we let

$$\mathrm{LLC}_G(\pi) := \xi_*^{-1} \circ \mathrm{LLC}_{\mathrm{GL}_N} \circ \mathrm{Lift}(\pi).$$

Recall that $\mathrm{LLC}_{\mathrm{GL}_N} \colon \Pi(\mathrm{GL}_N) \to \Phi(\mathrm{GL}_N)$ maps $\Pi(\mathrm{GL}_N)_{\mathrm{temp}}$ to $\Phi(\mathrm{GL}_N)_{\mathrm{temp}}$. Hence $\mathrm{LLC}_{\mathrm{GL}_N} \circ \mathrm{Lift}(\pi)$ is a tempered L-parameter of GL_N. Moreover, also recall that the temperedness of an L-parameter is defined to be the boundedness of the image of W_F. In particular, the map ξ_* preserves the temperedness. Consequently, we obtained a map

$$\mathrm{LLC}_G \colon \Pi(G)_{\mathrm{temp}} \to \Phi(G)_{\mathrm{temp}}.$$

We finally comment on how to extend this map to $\Pi(G)$. For general p-adic reductive group G, it is known that any non-tempered representation of G can be constructed by the parabolic induction using a tempered representation of a Levi subgroup of G; it is so-called the *Langlands classification*; [BW00, Chapter IX]. On the other hand, also on the Galois side, we can construct the non-tempered L-parameters of G using tempered L-parameters of Levi subgroups of G [SZ18]. Hence, as long as the local Langlands correspondences for tempered representations of all Levi subgroups of G are established, we can formally extend them to the non-tempered representations of G. So the important observation here is that any Levi subgroup of a split classical group is again a split classical group of the same type; for example, any maximal Levi subgroup of SO_{2n+1} is of the form $\mathrm{GL}_r \times \mathrm{SO}_{2s+1}$, where $r + s = n$. From this, we can see that the construction of the map LLC for split classical groups can be given inductively on the size of the group.

References

[ABPS16] A.-M. Aubert, P. Baum, R. Plymen, and M. Solleveld, *Depth and the local Langlands correspondence*, Arbeitstagung Bonn 2013, Progr. Math., vol. 319, Birkhäuser/Springer, Cham, 2016, pp. 17–41.

[AMS18] A.-M. Aubert, A. Moussaoui, and M. Solleveld, *Generalizations of the Springer correspondence and cuspidal Langlands parameters*, Manuscripta Math. **157** (2018), no. 1-2, 121–192.

[Art13] J. Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[BH06] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* $\mathrm{GL}(2)$, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.

[BK93] C. J. Bushnell and P. C. Kutzko, *The admissible dual of* $\mathrm{GL}(N)$ *via compact open subgroups*, Annals of Mathematics Studies, vol. 129, Princeton University Press, Princeton, NJ, 1993.

[Bor79] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 27–61.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.

[BT84] ———, *Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée*, Inst. Hautes Études Sci. Publ. Math. (1984), no. 60, 197–376.

[BW00] A. Borel and N. Wallach, *Continuous cohomology, discrete subgroups, and representations of reductive groups*, second ed., Mathematical Surveys and Monographs, vol. 67, American Mathematical Society, Providence, RI, 2000.

[BZ76] I. N. Bernstein and A. V. Zelevinsky, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70.

[Car79] P. Cartier, *Representations of p-adic groups: a survey*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 111–155.

[Del73] P. Deligne, *Les constantes des équations fonctionnelles des fonctions L*, Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972), 1973, pp. 501–597. Lecture Notes in Math., Vol. 349.

[DKV84] P. Deligne, D. Kazhdan, and M.-F. Vignéras, *Représentations des algèbres centrales simples p-adiques*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, pp. 33–117.

[DL76] P. Deligne and G. Lusztig, *Representations of reductive groups over finite fields*, Ann. of Math. (2) **103** (1976), no. 1, 103–161.

[DR09] S. DeBacker and M. Reeder, *Depth-zero supercuspidal L-packets and their stability*, Ann. of Math. (2) **169** (2009), no. 3, 795–901.

[DS05] F. Diamond and J. Shurman, *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York, 2005.

[GH24] J. R. Getz and H. Hahn, *An introduction to automorphic representations—with a view toward trace formulae*, Graduate Texts in Mathematics, vol. 300, Springer, Cham, 2024.

[GP92] B. H. Gross and D. Prasad, *On the decomposition of a representation of* SO_n *when restricted to* SO_{n-1}, Canad. J. Math. **44** (1992), no. 5, 974–1002.

[GR10] B. H. Gross and M. Reeder, *Arithmetic invariants of discrete Langlands parameters*, Duke Math. J. **154** (2010), no. 3, 431–508.

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups*, Lecture Notes in Mathematics, Vol. 162, Springer-Verlag, Berlin-New York, 1970, Notes by G. van Dijk.

[Hen00] G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique*, Invent. Math. **139** (2000), no. 2, 439–455.

[HT01] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, Annals of Mathematics Studies, vol. 151, Princeton University Press, Princeton, NJ, 2001, With an appendix by Vladimir G. Berkovich.

[JL70] H. Jacquet and R. P. Langlands, *Automorphic forms on* $\mathrm{GL}(2)$, Lecture Notes in Mathematics, vol. Vol. 114, Springer-Verlag, Berlin-New York, 1970.

[JPSS83] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, *Rankin-Selberg convolutions*, Amer. J. Math. **105** (1983), no. 2, 367–464.

[Kal19] T. Kaletha, *Regular supercuspidal representations*, J. Amer. Math. Soc. **32** (2019), no. 4, 1071–1170.

[KS99] R. E. Kottwitz and D. Shelstad, *Foundations of twisted endoscopy*, Astérisque (1999), no. 255, vi+190.

[LS87] R. P. Langlands and D. Shelstad, *On the definition of transfer factors*, Math. Ann. **278** (1987), no. 1-4, 219–271.

[Mil17] J. S. Milne, *Algebraic groups*, Cambridge Studies in Advanced Mathematics, vol. 170, Cambridge University Press, Cambridge, 2017, The theory of group schemes of finite type over a field.

[Mok15] C. P. Mok, *Endoscopic classification of representations of quasi-split unitary groups*, Mem. Amer. Math. Soc. **235** (2015), no. 1108, vi+248.

[MP94] A. Moy and G. Prasad, *Unrefined minimal K-types for p-adic groups*, Invent. Math. **116** (1994), no. 1-3, 393–408.

[MP96] ———, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121.

[Oi24] M. Oi, *Representation theory of finite groups of Lie type*, lecture note, available at `https://masaooi.github.io/DL.pdf`, 2024.

[PRR23] V. Platonov, A. Rapinchuk, and I. Rapinchuk, *Algebraic groups and number theory. Vol. I*, russian ed., Cambridge Studies in Advanced Mathematics, vol. 205, Cambridge University Press, Cambridge, 2023.

[Rog83] J. D. Rogawski, *Representations of* $\mathrm{GL}(n)$ *and division algebras over a p-adic field*, Duke Math. J. **50** (1983), no. 1, 161–196.

[RV99] D. Ramakrishnan and R. J. Valenza, *Fourier analysis on number fields*, Graduate Texts in Mathematics, vol. 186, Springer-Verlag, New York, 1999.

[Ser73] J.-P. Serre, *A course in arithmetic*, Graduate Texts in Mathematics, vol. No. 7, Springer-Verlag, New York-Heidelberg, 1973, Translated from the French.

[Ser79] ———, *Local fields*, Graduate Texts in Mathematics, vol. 67, Springer-Verlag, New York-Berlin, 1979, Translated from the French by Marvin Jay Greenberg.

[Sha74] J. A. Shalika, *The multiplicity one theorem for* GL_n, Ann. of Math. (2) **100** (1974), 171–193.

[Sha90] F. Shahidi, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups*, Ann. of Math. (2) **132** (1990), no. 2, 273–330.

[SZ18] A. J. Silberger and E.-W. Zink, *Langlands classification for L-parameters*, J. Algebra **511** (2018), 299–357.

[vD72] G. van Dijk, *Computation of certain induced characters of* $\mathfrak{p}$*-adic groups*, Math. Ann. **199** (1972), 229–240.

[Wal03] J.-L. Waldspurger, *La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)*, J. Inst. Math. Jussieu **2** (2003), no. 2, 235–333.

[Wei74] A. Weil, *Basic number theory*, third ed., Die Grundlehren der mathematischen Wissenschaften, vol. Band 144, Springer-Verlag, New York-Berlin, 1974.

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