1. Week 12: Unipotent representations and Lusztig's Jordan decomposition

1.1. **Langlands dual and geometric conjugacy.** Let G be a connected reductive group over $k = \mathbb{F}_q$ as usual (F denotes its geometric Frobenius endomorphism). For simplicity, in the following discussion, we assume that G is split.

Recall that split connected reductive groups over k are classified by root data. Let $(X, R, X^\vee, R^\vee)$ the root datum determined by G (if we take a k-rational split maximal torus T_0 of G, then X and $X^\vee$ can be taken to be $X^*(T_0)$ and $X_*(T_0)$, respectively). We note that the swapped quadruple $(X^\vee, R^\vee, X, R)$ also satisfies the axioms of a root datum. We call this root datum the *dual root datum* of $(X, R, X^\vee, R^\vee)$. Again by the classification theorem of reductive groups, there exists a split connected reductive group over k whose root datum is given by $(X^\vee, R^\vee, X, R)$. We call this reductive group the *Langlands dual group* of G. Let $\hat{G}$ denote it (we use the same symbol "F" for the geometric Frobenius of $\hat{G}$). Hence, if we take a k-rational split maximal torus $\hat{T}_0$ of $\hat{G}$, then we have $X^\vee \cong X^*(\hat{T}_0)$ and $X \cong X_*(\hat{T}_0)$.

Remark 1.1. (1) The Dynkin diagram of $\hat{G}$ is the dual diagram of that of G in the sense that the underlying diagram is the same and the directions of arrows are reversed. In particular, among A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2, only B_n and C_n are swapped under taking the dual; all other diagrams are self-dual.

(2) The Langlands dual group $\hat{G}$ is simply-connected (resp. adjoint) if and only if G is adjoint (resp. simply-connected).

<table>
<tr><td>type of G</td><td colspan="3">type A_{n-1}</td><td colspan="2">type B_n</td></tr>
<tr><td>G</td><td>GL_n</td><td>SL_n</td><td>PGL_n</td><td>Spin_{2n+1}</td><td>SO_{2n+1}</td></tr>
<tr><td>$\hat{G}$</td><td>GL_n</td><td>PGL_n</td><td>SL_n</td><td>PSp_{2n}</td><td>Sp_{2n}</td></tr>
<tr><td>type of $\hat{G}$</td><td colspan="3">type A_{n-1}</td><td colspan="2">type C_n</td></tr>
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<table>
<tr><td>type of G</td><td colspan="2">type C_n</td><td colspan="3">type D_n</td></tr>
<tr><td>G</td><td>Sp_{2n}</td><td>PSp_{2n}</td><td>Spin_{2n}</td><td>SO_{2n}</td><td>PSO_{2n}</td></tr>
<tr><td>$\hat{G}$</td><td>SO_{2n+1}</td><td>Spin_{2n+1}</td><td>PSO_{2n}</td><td>SO_{2n}</td><td>Spin_{2n}</td></tr>
<tr><td>type of $\hat{G}$</td><td colspan="2">type B_n</td><td colspan="3">type D_n</td></tr>
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Now we reinterpret the notion of the geometric conjugacy in terms of the Langlands dual group. Recall that G^F-conjugacy classes of k-rational maximal tori of G are classified by the conjugacy classes of $W_0 := W_G(T_0)$. Let T be a k-rational maximal torus of G whose conjugacy class is represented by $w \in W_0$. In fact, the Weyl group of the Langlands dual group $\hat{W}_0 := W_{\hat{G}}(\hat{T}_0)$ is isomorphic to W_0. Thus, by regarding w as an element of $\hat{W}_0$, we can find a k-rational maximal torus $\hat{T}$ of $\hat{G}_0$ whose conjugacy class is represented by $w \in \hat{W}_0$.

We note that $X_*(T_0) \cong X^\vee \cong X^*(\hat{T}_0)$. This isomorphism is equivariant with respect to the action of the Frobenius (in fact, since we are assuming that G is split, the Frobenius actions on $X_*(T_0)$ and $X^*(\hat{T}_0)$ are trivial). Since any maximal tori are conjugate, by fixing $g \in G$ such that $T = {}^gT_0$, we obtain an isomorphism $X_*(T_0) \cong X_*(T)$ (given by the pull-back via g-conjugation). Similarly, we also have an isomorphism $X^*(\hat{T}_0) \cong X^*(\hat{T})$. Consequently, we obtain

$$X_*(T) \cong X_*(T_0) \cong X^\vee \cong X^*(\hat{T}_0) \cong X^*(\hat{T}).$$

By chasing the above construction of $\hat{T}$ carefully, we can check the following:

> we may find $\hat{T}$ such that the resulting isomorphism $X_*(T) \cong X^*(\hat{T})$ is equivariant with respect to the Frobenius actions.

Now recall that we have an isomorphism

$$T^F \cong X_*(T)/(F-1)X_*(T).$$

(Week 10). In fact, we also have

$$(T^F)^\vee \cong X^*(T)/(F-1)X^*(T),$$

where $(T^F)^\vee := \mathrm{Hom}(T^F, \mathbb{C}^\times)$ (see [Car85, Proposition 3.2.3]). Therefore, by also using the previous Frobenius-equivariant identification $X_*(T) \cong X^*(\hat{T})$, we finally obtain an identification

$$(T^F)^\vee \cong X^*(T)/(F-1)X^*(T) \cong X_*(\hat{T})/(F-1)X_*(\hat{T}) \cong \hat{T}^F.$$

Hence, any character of T^F can be regarded as an element of $\hat{T}^F \subset \hat{G}^F$.

Let us summarize our discussion. We put $\mathcal{T}_G$ to be the set of k-rational maximal tori of G. We put $\mathcal{I}_G$ to be the set of pairs (T, θ) such that $T \in \mathcal{T}_G$ and $\theta \in (T^F)^\vee$. Similarly, we put $\mathcal{J}_{\hat{G}}$ to be the set of pairs $(\hat{T}, s)$ such that $\hat{T} \in \mathcal{T}_{\hat{G}}$ and $s \in \hat{T}^F$. We constructed an element $(\hat{T}, s) \in \mathcal{J}_{\hat{G}}$ from a pair $(T, \theta) \in \mathcal{I}_G$.

Note that both sets $\mathcal{I}_G$ and $\mathcal{J}_{\hat{G}}$ are equipped with the actions of G^F and $\hat{G}^F$ by conjugation, respectively. We denote the sets of their G^F-conjugacy classes by the symbol $\mathcal{I}_G/{\sim_{G^F}}$ and $\mathcal{J}_G/{\sim_{\hat{G}^F}}$.

On the other hand, we also have an equivalence relation on $\mathcal{I}_G$ given by $(T_1, \theta_1) \sim (T_2, \theta_2)$ if and only if $R^G_{T_1}(\theta_1)$ and $R^G_{T_2}(\theta_2)$ contains a common irreducible constituent.

Theorem 1.2. *The previous construction induces the following diagram*

$$\begin{tikzcd}
\mathcal{I}_G/{\sim_{G^F}} \arrow[r, "1:1"] \arrow[d, two heads] & \mathcal{J}_G/{\sim_{\hat{G}^F}} \arrow[d, two heads] & (T,\theta) \arrow[r, mapsto] \arrow[d, mapsto] & (\hat{T}, s) \arrow[d, mapsto] \\
\mathcal{I}_G/{\sim} \arrow[r, "1:1"] & \hat{G}^F_{\mathrm{ss}}/{\sim_{\hat{G}^F}} & (T,\theta) \arrow[r, mapsto] & s
\end{tikzcd}$$

Proof. We omit the proof; see, for example, [GM20, Corollary 2.5.14]. □

1.2. **Lusztig's Jordan decomposition.**

Definition 1.3. Let $s \in \hat{G}^F_{\mathrm{ss}}$. We let $\mathcal{E}(G^F, s)$ be the set of isomorphism classes of irreducible representations ρ of G^F such that $\langle \rho, R^G_T(\theta) \rangle$ for some $(T, \theta) \in \mathcal{I}_G$ whose G^F-conjugacy class (associated as in the previous section) corresponds to s. We call the set $\mathcal{E}(G^F, s)$ the *Lustig series of irreducible representations* associated to $s \in \hat{G}^F_{\mathrm{ss}}$.

Remark 1.4. Recall that we say an irreducible representation ρ of G^F is unipotent if there exists a k-rational maximal torus T of G satisfying $\langle \rho, R_T^G(\mathbb{1})\rangle$. Then the associated semisimple element of G^F is 1. Hence, $\mathcal{E}(G^F, 1)$ is nothing but the set of irreducible unipotent representations of G^F.

Let us write $\mathrm{Irr}(G^F)$ for the set of isomorphism classes of irreducible representations of G^F.

Theorem 1.5. *We have a decomposition*

$$\mathrm{Irr}(G^F) = \bigsqcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F, s),$$

where the sum is over $\hat{G}^F$-conjugacy classes of semisimple elements of $\hat{G}^F$.

Proof. We first utilize the exhaustion theorem. The exhaustion theorem tells us that, for any $\rho \in \mathrm{Irr}(G)$, we can find a pair $(T, \theta) \in \mathcal{I}_G$ such that the associated Deligne–Lusztig representation $R_T^G(\theta)$ contains ρ, i.e., $\langle \rho, R_T^G(\theta)\rangle \neq 0$. Hence, by putting $s \in \hat{G}^F_{\mathrm{ss}}$ to be an element corresponding to (T, θ), we have $\rho \in \mathcal{E}(G^F, s)$. In other words, we get $\mathrm{Irr}(G^F) = \bigcup_{s \in \hat{G}^F_{\mathrm{ss}}} \mathcal{E}(G^F, s)$. Moreover, by definition, $\mathcal{E}(G^F, s)$ depends only on the G^F-conjugacy class of s. Hence $\mathrm{Irr}(G^F) = \bigcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F, s)$.

We next use the disjointness theorem. Suppose that $\mathcal{E}(G^F, s_1)$ and $\mathcal{E}(G^F, s_2)$ has nonempty intersection ($s_1, s_2 \in G^F_{\mathrm{ss}}$); let ρ be any element of $\mathcal{E}(G^F, s_1) \cap \mathcal{E}(G^F, s_2)$. Then there exists $(T_i, \theta_i) \in \mathcal{I}_G$ whose geometric conjugacy class corresponds to the G^F-conjugacy class of s_i for each $i = 1, 2$. By the disjointness theorem, the geometric conjugacy classes of (T_2, θ_1) and (T_2, θ_2) must coincide. In other words, G^F-conjugacy classes of s_1 and s_2 are the same. □

By the above theorem, to classify the irreducible representations of G^F, it is enough to determine $\mathcal{E}(G^F, s)$ for each $s \in G^F_{\mathrm{ss}}$.

Theorem 1.6 (Lusztig)**.** *Suppose that the center of G is connencted. Then, for each $s \in G^F_{\mathrm{ss}}$, there exists a bijection*

$$\mathcal{E}(G^F, s) \xrightarrow{1:1} \mathcal{E}(G^F_s, 1) \colon \rho \mapsto \rho_0$$

such that, for any $(T, \theta) \in \mathcal{I}_{G_s} \subset \mathcal{I}_G$ which corresponds to s, we have

$$(-1)^{r_G} \langle \rho, R_T^G(\theta)\rangle_{G^F} = (-1)^{r_{G_s}} \langle \rho_0, R_T^{G_s}(\theta)\rangle_{G_s^F}.$$

In particular, by combining this theorem with the previous one, we get

$$\mathrm{Irr}(G^F) \cong \bigsqcup_{s \in \hat{G}^F_{\mathrm{ss}}/\sim} \mathcal{E}(G^F_s, 1).$$

This decomposition is called Lusztig's Jordan decomposition. By Lusztig's Jordan decomposition, in order to classify irreducible representations of G^F, we are reduced to classify all irreducible unipotent representations of G^F and its smaller reductive subgroups.

Here let us compare Lusztig's Jordan decomposition with the normal Jordan decomposition:

$$G^F = \bigsqcup_{s \in G^F_{\mathrm{ss}}} (G^F_s)_{\mathrm{unip}},$$

which induces a decomposition of the rational conjugacy classes:

$$G^F/{\sim_{G^F}} = \bigsqcup_{s\in G^F_{\mathrm{ss}}/\sim_{G^F}} (G^F_s)_{\mathrm{unip}}/{\sim_{G^F_s}}.$$

(Here, we are still assuming that the center of G is connected. In fact, this implies that the centralizer group $Z_G(s)$ of any element $s \in G^F_{\mathrm{ss}}$ is connected.)

Recall that, for any finite group G, the number of the isomorphism classes of irreducible representations of G is equal to the number of the G^F-conjugacy classes of G. Then, does this suggests that there is an explicit relationship (in particular, a bijection) between them? In general, the answer is NO (although sometimes it is possible; for example, when $G = \mathfrak{S}_n$, both the sets of irreducible representations and conjugacy classes are parametrized by Young diagrams.) Nevertheless, we can often find parallel phenomena in these two different worlds; the phenomena on representations and conjugacy classes are often referred to as *spectral* and *geometric* counterparts of the group theory of G, respectively. In this sense, Lusztig's Jordan decomposition can be thought of as a spectral analogue of the usual Jordan decomposition.

1.3. **Representations of Weyl groups.** In Lusztig's classification of irreducible unipotent representations of G^F, irreducible representations of the Weyl group W_0 play a crucial rule. Here we introduce some ingredients needed to state Lusztig's results.

Recall that the dimension of $\mathrm{End}_{G^F}(\mathrm{Ind}_{B^F}^{G^F}\mathbb{1})$ is given by $|W_0|$. In fact, we furthermore have that $\mathrm{End}_{G^F}(\mathrm{Ind}_{B^F}^{G^F}\mathbb{1})$ and $\mathbb{C}[W_0]$ are isomorphic as $\mathbb{C}$-algebras. This implies that the irreducible representations of G^F contained in $\mathrm{Ind}_{B^F}^{G^F}\mathbb{1}$ bijectively correspond to irreducible representations of W_0. Let ρ_χ denote the irreducible constituent of $\mathrm{Ind}_{B^F}^{G^F}\mathbb{1}$ corresponding to $\chi \in \mathrm{Irr}(W_0)$.

By the theory of *Iwahori–Hecke algebra*, we can explicitly describe the dimension of ρ_χ as a polynomial in q (the cardinality of $k = \mathbb{F}_q$). We let $d_\chi(t) \in \mathbb{Q}[t]$ be the polynomial obtained by replacing q in the explicit dimension formula of ρ_χ with "t", which is a formal variable. We call this polynomial *generic degree* or *formal dimension* of $\chi \in \mathrm{Irr}(W_0)$. We define a non-negative integer $a_\chi \in \mathbb{Z}_{\geq 0}$ to be the greatest integer such that t^{a_χ} divides $d_\chi(t)$.

On the other hand, we introduce the *coinvariant ring* $R(W_0)$ of W_0 in the following way. Let S be the symmetric algebra associated to the real vector space $X^*(T_0)_{\mathbb{R}}$. Since $X^*(T_0)$ has an action of W_0, this is a graded $\mathbb{R}$-algebra equipped with an action of W_0. Let J_+ be the ideal of S generated by all W-invariant homogeneous vectors of positive degree. Then we define $R(W_0) := S/J_+$. It is known that $R(W_0)$ is a finite-dimensional graded algebra $R(W_0) = \bigoplus_{i\geq 0} R_i$ such that each R_i has an action of W_0. We define a non-negative integer $b_\chi \in \mathbb{Z}_{\geq 0}$ for $\chi \in \mathrm{Irr}(W_0)$ to be the smallest integer such that R_{b_χ} contains χ as a representation of W_0.

Proposition/Definition 1.7. In general, it is known that we have $a_\chi \leq b_\chi$. We say that $\chi \in \mathrm{Irr}(W_0)$ is *special* when $a_\chi = b_\chi$.

1.4. **Unipotent representations.** Let us still keep assuming that G is split. Again recall that the G^F-conjugacy classes of k-rational maximal tori of G are parametrized by the conjugacy classes of the Weyl group W_0. Now our aim is to classify all irreducible unipotent representations of G. In other words, we want to determine the irreducible decompositions of $R^G_{T_w}(\mathbb{1})$ for $w \in W_0$, where T_w denotes any k-rational maximal torus of G^F corresponding to w.

For any $\chi \in \mathrm{Irr}(W_0)$, we define a virtual representation R_χ of G^F by

$$R_\chi := \frac{1}{|W_0|} \sum_{w \in W_0} \Theta_\chi(w) \cdot R^G_{T_w}(\mathbb{1}).$$

Then determining the irreducible decompositions of $R^G_{T_w}(\mathbb{1})$ for $w \in W_0$ is equivalent to determining the irreducible decompositions of R_χ for $\chi \in \mathrm{Irr}(W_0)$. Indeed, suppose that we know "all" about R_χ for any $\chi \in \mathrm{Irr}(W_0)$. Then we can extract the information of $R^G_{T_{w_0}}(\mathbb{1})$ for a given $w_0 \in W_0$ in the following way:

$$\begin{aligned}\sum_{\chi \in \mathrm{Irr}(W_0)} R_\chi \cdot \overline{\Theta_\chi(w_0)} &= \sum_{\chi \in \mathrm{Irr}(W_0)} \frac{1}{|W_0|} \sum_{w \in W_0} \Theta_\chi(w) \cdot R^G_{T_w}(\mathbb{1}) \cdot \overline{\Theta_\chi(w_0)} \\ &= \sum_{w \in W_0} \frac{1}{|W_0|} \sum_{\chi \in \mathrm{Irr}(W_0)} \Theta_\chi(w) \cdot \overline{\Theta_\chi(w_0)} \cdot R^G_{T_w}(\mathbb{1}) = R^G_{T_{w_0}}(\mathbb{1}).\end{aligned}$$

Here, in the last equality, we used the fact that

$$\sum_{\chi \in \mathrm{Irr}(W_0)} \Theta_\chi(w) \cdot \overline{\Theta_\chi(w_0)} = \begin{cases} \frac{|W_0|}{|W_0 \cdot w_0|} & \text{if } w \text{ is conjugate to } w_0, \\ 0 & \text{otherwise,} \end{cases}$$

where $W_0 \cdot w_0$ denotes the conjugacy class of w_0 (the orthogonality relation of irreducible characters of a finite group; for example, see [Ser77, Chapter 2, Proposition 7]).

For any finite group Γ, we put

$$\mathcal{M}(\Gamma) := \{(x, \sigma) \mid x \in \Gamma/{\sim_\Gamma}, \sigma \in \mathrm{Irr}(\Gamma_x)\},$$

where $\Gamma/{\sim_\Gamma}$ is the set of conjugacy classes and $\Gamma_x := Z_\Gamma(x)$. We define a pairing $\{-,-\}\colon \mathcal{M}(\Gamma) \times \mathcal{M}(\Gamma) \to \mathbb{C}$ by

$$\{(x,\sigma),(y,\tau)\} := \sum_{\substack{g \in \Gamma \\ xgyg^{-1} = gyg^{-1}x}} |\Gamma_x|^{-1} \cdot |\Gamma_y|^{-1} \cdot \Theta_\sigma(gyg^{-1}) \cdot \overline{\Theta_\tau(g^{-1}xg)}.$$

For any function $f\colon \mathcal{M}(\Gamma) \to \mathbb{C}$, we define a function $\hat{f}\colon \mathcal{M}(\Gamma) \to \mathbb{C}$ by

$$\hat{f}((y,\tau)) := \sum_{(x,\sigma) \in \mathcal{M}(\Gamma)} \{(x,\sigma),(y,\tau)\} \cdot f((x,\sigma)).$$

We call the function $\hat{f}$ the *non-abelian Fourier transform* of f.

Now we explain Lusztig's result. For each family $\mathcal{F} \subset \mathrm{Irr}(W_0)$, Lusztig constructed a finite group $\Gamma_\mathcal{F}$ equipped with an embedding $\mathcal{F} \subset \mathcal{M}(\Gamma_\mathcal{F})$. We define

$$X(W_0) := \bigsqcup_{\mathcal{F}} \mathcal{M}(\Gamma_\mathcal{F}),$$

where the sum is over all families of $\mathrm{Irr}(W_0)$. For each $\chi \in \mathcal{F}$, we let z_χ denote its image in $\mathcal{M}(\Gamma_\mathcal{F}) \subset X(W_0)$. Recall that each $\mathcal{M}(\Gamma_\mathcal{F})$ is equipped with a pairing $\{-,-\}$. We extend them to $X(W_0)$ in an obvious way, i.e., for any distinct families $\mathcal{F} \neq \mathcal{F}'$, the extended pairing $\{-,-\}$ is zero on $\mathcal{M}(\Gamma_\mathcal{F}) \times \mathcal{M}(\Gamma_{\mathcal{F}'})$.

Theorem 1.8. *There exists a bijection*

$$X(W_0) \to \mathcal{E}(G^F, 1) \colon z \mapsto \rho_z$$

satisfying

$$R_\chi = \sum_{z' \in X(W_0)} \{z', z_\chi\} \cdot \rho_{z'}.$$

Remark 1.9. (1) The above theorem says that, in particular, the number of irreducible unipotent representations of G^F is independent of q. It is governed by the Weyl group W_0, which is only determined by G.

(2) In fact, when G is of type E_7 or E_8, we have to modify the definition of the pairing $\{-,-\}$ a bit for some particular families $\mathcal{F}$ called *exceptional families.*

(3) When G is simple, only possibilities of a finite group $\Gamma_{\mathcal{F}}$ for a family $\mathcal{F}$ are $(\mathbb{Z}/2\mathbb{Z})^m$ (for some $m \in \mathbb{Z}_{>0}$), $\mathfrak{S}_3$, $\mathfrak{S}_4$, $\mathfrak{S}_5$.

(4) By noting the above description of R_χ, we define a virtual representation R_z for any $z \in X(W_0)$ by

$$R_z = \sum_{z' \in X(W_0)} \{z', z\} \cdot \rho_{z'}.$$

This virtual representation (or its character) is called an *almost character* of G^F.

(5) By looking at the book [Lus84] (or also [Car85, Sections 13.8 and 13.9]), we can find tables of all irreducible unipotent representations of G^F.

References

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9:06am, December 10, 2024