

## 1. WEEK 7: DELIGNE–LUSZTIG CHARACTER FORMULA

Let  $G$  be a connected reductive group over  $k = \mathbb{F}_q$  and  $F$  its associated Frobenius endomorphism. We fix a  $k$ -rational maximal torus  $T$  of  $G$  and a Borel subgroup  $B$  of  $G$  containing  $T$ . We also fix a character  $\theta: T^F \rightarrow \mathbb{C}^\times$ . Then we have the Deligne–Lusztig virtual representation  $R_{T \subset B}^G(\theta)$  of  $G^F$ . By abuse of notation, we also write  $R_{T \subset B}^G(\theta)$  for the Deligne–Lusztig virtual character, which is a class function  $G^F \rightarrow \mathbb{C}$  defined to be the trace of the Deligne–Lusztig virtual representation. Today’s aim is to establish a character formula for  $R_{T \subset B}^G(\theta)$ .

**1.1. Deligne–Lusztig character formula.** We write  $G_{\text{ss}}^F$  and  $G_{\text{unip}}^F$  for the set of semisimple (equivalently, prime-to- $p$  order) and unipotent elements of  $G^F$  (equivalently,  $p$ -power order), respectively. In the following, for any  $g \in G$  and  $h \in G$ , we write  ${}^g h = ghg^{-1}$ . Similarly, for any  $g \in G$  and a subgroup  $H \subset G$ , we write  ${}^g H = gHg^{-1}$ .

**Definition 1.1.** We define a function  $Q_T^G: G_{\text{unip}}^F \rightarrow \mathbb{C}$  by  $Q_T^G := R_{T \subset B}^G(\mathbb{1})|_{G_{\text{unip}}^F}$ . We call  $Q_T^G$  the *Green function (of  $G$  associated to  $T$ )*.

We note that, for notational convenience, we simply write “ $Q_T^G$ ” although a priori  $Q_T^G$  depends on the choice of a Borel subgroup  $B$  containing  $T$ . (But, in fact, later it will turn out that  $Q_T^G$  does not depend on  $B$ !)

To state the Deligne–Lusztig character formula, let us recall that any element  $g \in G^F$  has the Jordan decomposition  $g = su$ , where  $s \in G^F$  is a semisimple element and  $u \in G^F$  is a unipotent element such that  $su = us$ .

**Theorem 1.2** (Deligne–Lusztig character formula). *Let  $g \in G^F$  with Jordan decomposition  $g = su$ . We shortly write  $G_s$  for the centralizer of  $s$  in  $G$ , i.e.,  $G_s = Z_G(s) = \{x \in G \mid xs = sx\}$ . Then we have*

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u).$$

Let us explain why the right-hand side of this formula makes sense. We first note the following result (see [Car85, 1.14]).

**Lemma 1.3.** (1) *For any  $s \in G_{\text{ss}}^F$ , the identity component  $G_s^\circ$  of its centralizer  $G_s$  is a connected reductive group defined over  $k$ .*  
 (2) *Any unipotent element of  $G_s$  lies in  $G_s^\circ$ . In particular, when  $g \in G^F$  has the Jordan decomposition  $g = su$ , its unipotent part  $u$  belongs to  $(G_s^\circ)^F$ .*

Let us look at the index set of the sum in the Deligne–Lusztig character formula. When  $x^{-1}sx \in T$ , we necessarily have the opposite inclusion  $Z_G(x^{-1}sx) \supset Z_G(T)$ . Here, it is easy to check that  $Z_G(x^{-1}sx) = x^{-1}Z_G(s)x$ . On the other hand, it is known that the centralizer of a maximal torus in a connected reductive group is the maximal torus itself, i.e.,  $Z_G(T) = T$  (see [Spr09, 7.6.4]). Hence, we have  $x^{-1}Z_G(s)x \supset T$ , or equivalently,  ${}^x T = xTx^{-1} \subset Z_G(s) = G_s$ . Since  $T$  is connected, this furthermore implies that  ${}^x T \subset G_s^\circ$ . Furthermore, it is known that  $(B \cap G_s^\circ)^\circ$  is a Borel subgroup of  $G_s^\circ$  and  $U \cap G_s^\circ$  is its unipotent radical.<sup>1</sup>

<sup>1</sup>Here, note that  $U \cap G_s^\circ$  is already connected!

In summary, when  $x^{-1}sx \in T$ , we obtain a  $k$ -rational maximal torus  ${}^xT$  of a connected reductive group  $G_s^\circ$ . Thus it makes sense to consider the Green function  $Q_{xT}^{G_s^\circ}$  of  $G_s^\circ$  associated to  ${}^xT$  and  $(B \cap G_s^\circ)^\circ$ . Since  $u$  belongs to  $(G_s^\circ)_{\text{unip}}^F$ , it also makes sense to look at the value of  $Q_{xT}^{G_s^\circ}$  at  $u$ .

Thus the Deligne–Lusztig character formula reflects an inductive nature of the theory of reductive groups. The contribution of the semisimple part  $s$  is given just by  $\theta$ , which is very simple. On the other hand, the contribution of the unipotent part  $u$  is given by the Green function, which is independent of  $\theta$  and taken in  $G_s^\circ$ . Hence, ultimately, the Deligne–Lusztig characters of  $G$  are governed by the Green functions for  $G$  and all its “smaller” reductive subgroups.

**1.2. Outline of the proof of DL character formula.** The key of the proof of the Deligne–Lusztig character formula is the following general result, which is called *Deligne–Lusztig’s fixed point formula*:

**Theorem 1.4** (Deligne–Lusztig fixed point formula). *Let  $X$  be an algebraic variety over  $k$  and  $g$  is an automorphism of  $X$  of finite order. Let  $s$  and  $u$  be automorphisms of  $X$  such that  $s$  is of prime-to- $p$  order,  $u$  is of  $p$ -power order, and  $g = su = us$ . Then we have  $\mathcal{L}(g, X) = \mathcal{L}(u, X^s)$ , where  $X^s := \{x \in X \mid s(x) = x\}$ .*

Now suppose that  $g \in G^F$  has the Jordan decomposition  $g = su = us$ . As disxussed in the last week, we have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

Let us compute each  $\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G)$  using the Deligne–Lusztig fixed point formula.

Recall that the action of  $(g, t)$  on  $\mathcal{X}_{T \subset B}^G = \{x \in G \mid x^{-1}F(x) \in F(U)\}$  is given by  $x \mapsto gxt$ . We note that the order of  $T^F$  is prime-to- $p$ . (Indeed, if we suppose that  $T$  splits over  $\mathbb{F}_{q^n}$ , i.e.,  $T_{\mathbb{F}_{q^n}} = \mathbb{G}_m^r$  for some  $r$ , we have  $T^F = T(\mathbb{F}_q) \subset T_{\mathbb{F}_{q^n}}(\mathbb{F}_{q^n}) \cong (\mathbb{F}_{q^n}^\times)^r$ .) Hence the order of  $t$  is also prime-to- $p$ . Thus, the decomposition  $(g, t) = (s, t) \circ (u, 1)$  satisfies the assumption of the Deligne–Lusztig fixed point formula.

We determine  $(\mathcal{X}_{T \subset B}^G)^{(s, t)}$ . In the following, we simply write  $\mathcal{X} := \mathcal{X}_{T \subset B}^G$ .

**Proposition 1.5.** *We have*

$$\mathcal{X}^{(s, t)} = \bigsqcup_{\substack{x \in G^F / (G_t^\circ)^F \\ x_t = s^{-1}}} \mathcal{X}^{(s, t)}(x),$$

where we put  $\mathcal{X}^{(s, t)}(x) := \mathcal{X}^{(s, t)} \cap xG_t^\circ$ .

*Proof.* Suppose that  $y \in \mathcal{X}^{(s, t)}$ , i.e.,  $y \in G$  is an element satisfying  $syt = y$  and  $y^{-1}F(y) \in F(U)$  (say  $y^{-1}F(y) = v$ ). By applying  $F$  to  $syt = y$ , we get  $sF(y)t = F(y)$ , thus  $syvt = yv$ . Combining  $syvt = yv$  with  $syt = y$ , we get  $yt^{-1}vt = yv$ , hence  $t^{-1}vt = v$ . This means that  $u$  belongs to  $G_t = Z_G(t)$ . As  $u$  is unipotent,  $u$  furthermore lies in  $G_t^\circ$ . Let us apply Lang’s theorem to  $G_t^\circ$ , which asserts that the map

$$G_t^\circ \rightarrow G_t^\circ: z \mapsto z^{-1}F(z)$$

is surjective; we can find an element  $z \in G_t^\circ$  satisfying  $z^{-1}F(z) = v$ .

We put  $x := yz^{-1}$ . Then  $F(x) = F(y)F(z)^{-1} = yvv^{-1}z^{-1} = yz^{-1} = x$ , i.e.,  $x \in G^F$ . Note that we have  $y \in xG_t^\circ$ . Furthermore, we have

$$xz = y = syt = s(xz)t = sxtz$$

(use that  $z \in G_t$  in the last equality), hence  ${}^xt = s^{-1}$ .

From the discussion so far, we have obtained

$$\mathcal{X}^{(s,t)} = \bigcup_{\substack{x \in G^F / (G_t^\circ)^F \\ {}^xt = s^{-1}}} \mathcal{X}^{(s,t)}(x).$$

It is obvious that the union is disjoint.  $\square$

Let us investigate each summand  $\mathcal{X}^{(s,t)}(x)$ . Note that, since  $t \in T^F$ , we have  $T \subset G_t^\circ$ . Moreover,  $B_t^\circ := (B \cap G_t^\circ)^\circ$  is a Borel subgroup of  $G_t^\circ$  with unipotent radical  $U \cap G_t^\circ$  (see the paragraph after Lemma 1.3). Therefore, it makes sense to consider the Deligne–Lusztig variety for  $G_t^\circ$  associated to  $T \subset B_t^\circ$ :

$$\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} = \{y' \in G_t^\circ \mid y'^{-1}F(y') \in U \cap G_t^\circ\}.$$

This is a variety equipped with an action of  $(G_t^\circ)^F \times T^F$ . On the other hand,  $\mathcal{X}^{(s,t)}(x)$  is stable under the action of the subgroup  $(G_s^\circ)^F \times T^F$  of  $G^F \times T^F$  on  $\mathcal{X}$ .

**Proposition 1.6.** *Let  $x \in G^F$  be an element satisfying  ${}^xt = s^{-1}$ . Then have an isomorphism of varieties*

$$\varphi_x : \mathcal{X}^{(s,t)}(x) \xrightarrow{\cong} \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} : y \mapsto x^{-1}y,$$

which is equivariant with respect to the actions of  $(G_s^\circ)^F \times T^F$  on  $\mathcal{X}^{(s,t)}(x)$  and  $(G_t^\circ)^F \times T^F$  on  $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$ . Here,  $(G_s^\circ)^F \times T^F$  and  $(G_t^\circ)^F \times T^F$  are identified by  $(z, t') \mapsto (x^{-1}zx, t')$ .

*Proof.* Suppose that  $y \in \mathcal{X}^{(s,t)}(x)$ , i.e.,  $y \in xG_t^\circ$  is an element satisfying  $syt = y$  and  $y^{-1}F(y) \in F(U)$ . Then we have  $x^{-1}y \in G_t^\circ$  and thus

$$(x^{-1}y)^{-1}F(x^{-1}y) = y^{-1}F(y) \in F(U) \cap G_t^\circ = F(U \cap G_t^\circ).$$

In other words,  $\varphi_x(y) = x^{-1}y$  belongs to  $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$ . Conversely, for any element  $y' \in \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$ , we can check that  $\varphi_x^{-1}(y') = xy' \in \mathcal{X}^{(s,t)}(x)$ .

Let us check the assertion on the equivariance. What we have to prove is that, for any  $(z, t') \in (G_s^\circ)^F \times T^F$  and  $y \in \mathcal{X}^{(s,t)}(x)$ , we have

$$\varphi_x((z, t') \cdot y) = (x^{-1}zx, t') \cdot \varphi_x(y).$$

The left-hand side is given by  $\varphi_x((z, t') \cdot y) = \varphi_x(zyt') = x^{-1}zyt'$ . The right-hand side is given by  $(x^{-1}zx, t') \cdot \varphi_x(y) = (x^{-1}zx, t') \cdot (x^{-1}y) = x^{-1}zx(x^{-1}y)t' = x^{-1}zyt'$ . So these indeed coincide.  $\square$

Now let us start the proof of the Deligne–Lusztig character formula:

*Proof of Theorem 1.2.* We have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

By applying the Deligne–Lusztig fixed point theorem to  $(g, t) = (s, t) \circ (u, 1)$ , we get

$$\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G) = \mathcal{L}((u, 1), (\mathcal{X}_{T \subset B}^G)^{(s, t)}).$$

By combining the above propositions, we get

$$\mathcal{L}(u, (\mathcal{X}_{T \subset B}^G)^{(s, t)}) = \sum_{\substack{x \in G^F / (G_t^\circ)^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}).$$

Hence we get

$$\begin{aligned} R_{T \subset B}^G(\theta)(g) &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \sum_{\substack{x \in G^F / (G_t^\circ)^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \frac{1}{|(G_t^\circ)^F|} \sum_{\substack{x \in G^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}). \end{aligned}$$

Note that the internal sum is nonzero only when there exists an element  $x \in G^F$  satisfying  $t = x^{-1}s^{-1}x$ . In this case,  $|(G_t^\circ)^F| = |(G_s^\circ)^F|$ , hence the above equals

$$\frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{t \in T^F} \sum_{\substack{x \in G^F \\ xt = s^{-1}}} \theta(t)^{-1} \cdot \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}).$$

We note that the set  $\{(t, x) \in T^F \times G^F \mid xt = s^{-1}\}$  is bijective to  $\{x \in G^F \mid x^{-1}sx \in T^F\}$  by  $(t, x) \mapsto x$ . By also noting that  $\mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}) = \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ})$ , we rewrite the above double sum:

$$\begin{aligned} &\frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}s^{-1}x)^{-1} \cdot \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ}) \\ &= \frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ}). \end{aligned}$$

Here, in general, we have

$$Q_T^G(u) = \frac{1}{|T^F|} \cdot \mathcal{L}(u, \mathcal{X}_{T \subset B}^G).$$

Indeed, by definition, we have

$$Q_T^G(u) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathcal{L}((u, t), \mathcal{X}_{T \subset B}^G).$$

By using the Deligne–Lusztig fixed point formula to  $(u, t) = (u, 1) \cdot (1, t)$ , we have  $\mathcal{L}((u, t), \mathcal{X}_{T \subset B}^G) = \mathcal{L}((u, 1), (\mathcal{X}_{T \subset B}^G)^{(1, t)})$ . However,  $(\mathcal{X}_{T \subset B}^G)^{(1, t)}$  is nonempty only when  $t = 1$  (indeed,  $x \in \mathcal{X}_{T \subset B}^G$  is fixed by  $(1, t)$  if and only if  $xt = x$ ). Thus we get

$$Q_T^G(u) = \frac{1}{|T^F|} \mathcal{L}(u, \mathcal{X}_{T \subset B}^G).$$

Therefore, we finally obtain

$$R_{T \subset B}^G(g) = \frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u).$$

□

**Corollary 1.7.** *We have  $R_{T \subset B}^G(\theta)|_{G_{\text{unip}}^F} = Q_T^G$  for any character  $\theta: T^F \rightarrow \mathbb{C}^\times$ .*

*Proof.* Let  $g \in G_{\text{unip}}^F$  (hence its semisimple part  $s$  is 1 and unipotent part  $u$  is  $g$ ). Then, by applying the Deligne–Lusztig character formula to  $g$ , we get

$$\begin{aligned} R_{T \subset B}^G(\theta)(g) &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u) \\ &= \frac{1}{|G^F|} \sum_{x \in G^F} Q_{xT}^G(u). \end{aligned}$$

It is not difficult to check that, in general, we have  $R_{T \subset B}^G(\theta)(g) = R_{xT \subset xB}^G({}^x\theta)({}^xg)$ , where  ${}^x\theta$  denotes the character of  ${}^xT^F$  defined by  ${}^x\theta({}^xt) = \theta(t)$ . In particular, when  $\theta = \mathbb{1}$ , hence get  $Q_T^G(u) = Q_{xT}^G({}^xu)$ . By also noting that the Green function is invariant under conjugation (since it is the restriction of a Deligne–Lusztig character, which is a class function), we get  $Q_T^G(u) = Q_{xT}^G({}^xu) = Q_{xT}^G(u)$ . Hence the most right-hand side of the above equalities is  $Q_T^G(u)$ . □

**Definition 1.8.** We say that a semisimple element  $s \in G$  is *regular* if  $G_s^\circ$  is a maximal torus of  $G$ .

**Example 1.9.** Let  $G = \text{GL}_2$ . Let  $T$  be the diagonal maximal torus of  $G$ . We consider an element  $s = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T$ . Then, since  $s$  is already diagonalized,  $s$  is semisimple. Let us compute the centralizer  $G_s = Z_G(s)$  of  $s$  in  $G$ .

- When  $a = b$ ,  $s$  commutes with any element of  $G$ . Thus  $G_s = G$ , hence  $G_s^\circ = G^\circ = G$ . Hence  $s$  is not regular in this case.
- Suppose that  $a \neq b$ . If  $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in Z_G(s)$ , we have  $sgs^{-1} = g$ . Since

$$sgs^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} x & ayb^{-1} \\ a^{-1}zb & w \end{pmatrix},$$

we necessarily have  $y = z = 0$ , i.e.,  $g \in T$ . Conversely, we obviously have  $T \subset Z_G(s)$ . Hence we get  $G_s = T$ , so  $G_s^\circ = T$ , which means that  $s$  is regular.

**Corollary 1.10.** *Suppose that  $s \in G^F$  is a regular semisimple element. If  $s$  is not conjugate to any element of  $T^F$ , then we have  $R_{T \subset B}^G(\theta)(s) = 0$ . If  $s$  is conjugate to any element of  $T^F$  (suppose that  $s$  itself belongs to  $T^F$ ), then we have*

$$R_{T \subset B}^G(\theta)(s) = \sum_{x \in W_{G^F}(T)} \theta(x^{-1}sx),$$

where  $W_{G^F}(T) := N_{G^F}(T)/T^F$ .

*Proof.* By the Deligne–Lusztig character formula, we have

$$R_{T \subset B}^G(\theta)(s) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(1).$$

Since the index set is empty if  $s$  is not conjugate to any element of  $T^F$ , we get the first assertion.

To show the second assertion, let us suppose that  $s \in T^F$ . Then, we must have  $G_s^\circ = Z_G(s)^\circ \supset Z_G(T) = T$ . Since  $G_s^\circ$  is a maximal torus of  $G$ , this implies that  $G_s^\circ = T$ . By the same argument, the condition  $x^{-1}sx \in T^F$  of the index set implies that  $T = x^{-1}Tx$ . In other words,  $x \in N_{G^F}(T)$ . Conversely, any element  $x \in N_{G^F}(T)$  satisfies  $x^{-1}sx \in T^F$ . Thus, by noting that  $Q_T^T(1) = 1$  (this can be checked by going back to the definition), we get

$$R_{T \subset B}^G(\theta)(s) = \frac{1}{|T^F|} \sum_{x \in N_{G^F}(T)} \theta(x^{-1}sx) = \sum_{x \in W_{G^F}(T)} \theta(x^{-1}sx).$$

□

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10:10pm, October 21, 2024