

THEORY OF ALGEBRAIC GROUPS

(2025 FALL @ NTU)

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1. WEEK 1: COURSE OVERVIEW

1.1. Why algebraic groups? If you have ever studied the theory of manifolds, you might have encountered the notion of a Lie group. A Lie group is a mathematical object equipped with two different kinds of mathematical structures in a consistent way; the one is a manifold structure, and the other is a group structure. An “algebraic group” is an algebraic version of the notion of a Lie group, where a “manifold structure” is replaced with an “algebraic variety structure”.

The theory of algebraic groups is interesting in its own right, but it also plays a very important role in applications. For example, much of modern representation

theory is founded on the theory of algebraic groups. Nowadays, theory of algebraic groups has become an indispensable “language” for developing representation theory.

The aim of this course is to learn basics of the theory of algebraic groups, mainly following the textbooks [Bor91, Spr09, Mil17].

1.2. Algebraic varieties. Before introducing the definition of an algebraic group, we briefly review the notion of schemes. See any textbook on algebraic geometry for more details, for example, [Har77], [Liu02], etc...

Definition 1.1. For a ring¹ R , we put $\text{Spec } R$ to be the set of all prime ideals of R . We call $\text{Spec } R$ the *spectrum* of R .

Let R be a ring. For any ideal $I \subset R$, we define a subset $V(I)$ of $\text{Spec } R$ by

$$V(I) := \{\mathfrak{p} \in \text{Spec } R \mid I \subset \mathfrak{p}\}.$$

When I is a principal ideal (f) generated by an element $f \in R$, we simply write $V(f)$ instead of $V((f))$. Also, we put $D(f) := \text{Spec } R \setminus V(f)$.

Lemma 1.2. (1) For any ideals $I, J \subset R$, we have $V(I) \cup V(J) = V(I \cap J)$.
 (2) For any family of ideals $\{I_\lambda\}_{\lambda \in \Lambda}$ of R , we have $\bigcap_{\lambda \in \Lambda} V(I_\lambda) = V(\sum_{\lambda \in \Lambda} I_\lambda)$.
 (3) We have $V(R) = \emptyset$ and $V(0) = \text{Spec } R$.

Exercise 1.3. Prove this lemma.

The above lemma shows that the family $\{V(I) \mid I \subset R: \text{ideal}\}$ defines a topology on $\text{Spec } R$ such that the closed subsets are the sets of the form $V(I)$. We call the topology on $\text{Spec } R$ defined in this way the *Zariski topology*.

Note that, from the above definition, the closed points of $\text{Spec } R$ are nothing but the maximal ideals of R .

Example 1.4. Let k be an algebraically closed field. We put $\mathbb{A}_k^n := \text{Spec } k[x_1, \dots, x_n]$ (where $k[x_1, \dots, x_n]$ is the polynomial ring with n variables over k). Then \mathbb{A}_k^n is called the *n -dimensional affine space* over k .

- (1) Let us first consider the subset of closed points of \mathbb{A}_k^n . By the Hilbert’s Nullstellensatz, any maximal ideal of $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$ (note that, for this, it is needed that k is algebraically closed).
- (2) Let us next consider a closed subset $V(I) \subset \mathbb{A}_k^n$ for an ideal $I = (f_1, \dots, f_r)$ of R generated by $f_1, \dots, f_r \in k[x_1, \dots, x_n]$. Let $x \in \mathbb{A}_k^n$ be a closed point corresponding to a maximal ideal $\mathfrak{m} = (x_1 - a_1, \dots, x_n - a_n)$. Then $x \in V(I)$ if and only if $\mathfrak{m} \supset I$, which is furthermore equivalent to $f_1(a_1, \dots, a_n) = \dots = f_r(a_1, \dots, a_n) = 0$. In other words, the subset of closed points of $V(I)$ is identified with the set of simultaneous solutions to polynomial equations $f_1 = \dots = f_r = 0$ in k^n .

Definition 1.5. Let X be a topological space. A *presheaf* \mathcal{F} of abelian groups (resp. rings) on X is a contravariant functor from the category of open sets of X to the category of abelian groups (resp. rings). More precisely, \mathcal{F} associates an abelian group (resp. a ring) $\mathcal{F}(U)$ to each open set $U \subset X$ such that

- (1) $\mathcal{F}(\emptyset) = 0$,

¹In this lecture, the word “ring” always means a commutative ring with unit.

- (2) for any open subsets $V \subset U \subset X$, we have a group homomorphism (resp. ring homomorphism) $\rho_{U,V}: \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ (called the *restriction* homomorphism) satisfying
- $\rho_{U,U} = \text{id}_U$ for any open subset $U \subset X$,
 - $\rho_{U,W} = \rho_{V,W} \circ \rho_{U,V}$ for any open subsets $W \subset V \subset U \subset X$.

For each open set $U \subset X$, we call an element $s \in \mathcal{F}(U)$ a *section* of \mathcal{F} over U . We write $s|_V$ in short for $\rho_{V,U}(s)$.

Definition 1.6. We say that a presheaf \mathcal{F} on X is a *sheaf* if it satisfies the following conditions:

- (1) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i \in I}$, if a section $s \in \mathcal{F}(U)$ satisfies $s|_{U_i} = 0$ for every $i \in I$, then $s = 0$.
- (2) For any open subset $U \subset X$ and its open covering $\{U_i\}_{i \in I}$, if a family of sections $\{s_i \in \mathcal{F}(U_i)\}_{i \in I}$ satisfies $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for every $i, j \in I$, then there exists $s \in \mathcal{F}(U)$ satisfying $s|_{U_i} = s_i$.

Now we let $X = \text{Spec } R$ for a ring R . Then we can construct a (unique) sheaf of rings \mathcal{O}_X on X such that

- $\mathcal{O}_X(D(f)) = R_f$ for any $f \in R$ (R_f denotes the localization of R with respect to f), and
- for any $f, g \in R$ such that $D(g) \subset D(f)$, the restriction $\rho_{D(f), D(g)}: \mathcal{O}_X(D(f)) \rightarrow \mathcal{O}_X(D(g))$ is given by the natural homomorphism $R_f \rightarrow R_g$ (note that f is invertible in R_g when $D(g) \subset D(f)$).

We call the sheaf \mathcal{O}_X the *structure sheaf* of X .

Definition 1.7. We call the pair $(X = \text{Spec } R, \mathcal{O}_X)$ the *affine scheme* associated to the ring R . We refer to R as the *coordinate ring* of X .

In general, a topological space equipped with a sheaf of rings is called a “ringed space”. For ringed spaces, we can define the notion of a morphism. When a ringed space (X, \mathcal{O}_X) is locally isomorphic to affine schemes (more precisely, there exists an open covering $\{U_i\}_{i \in I}$ of X such that each $(X, \mathcal{O}_X)|_{U_i}$ is isomorphic to an affine scheme), we call (X, \mathcal{O}_X) a *scheme*. (We often omit the symbol \mathcal{O}_X of the structure sheaf and simply write “ X ” for a scheme (X, \mathcal{O}_X) .)

Note that, when we have a ring homomorphism $\varphi: R \rightarrow S$, we can naturally define a continuous map $\varphi^\#: \text{Spec } S \rightarrow \text{Spec } R$ by $\varphi^\#(\mathfrak{p}) := \varphi^{-1}(\mathfrak{p})$ for any $\mathfrak{p} \in \text{Spec } S$. This map furthermore naturally induces a morphism between ringed spaces $(X := \text{Spec } S, \mathcal{O}_X) \rightarrow (Y := \text{Spec } R, \mathcal{O}_Y)$.

Fact 1.8. The association $R \mapsto (X = \text{Spec } R, \mathcal{O}_X)$ gives a contravariant equivalence between

- the category of rings and
- the category of affine schemes.

The inverse is given by $(X, \mathcal{O}_X) \mapsto \mathcal{O}_X(X)$.

When a ring R is a k -algebra, we say that the affine scheme $\text{Spec } R$ is “over k ”. When X is an affine scheme over k , its coordinate ring (i.e., the ring R when $X = \text{Spec } R$) is often denoted by $k[X]$.

When a scheme is made from affine schemes over k (such that any restriction morphism is a k -algebra homomorphism), we say that the scheme is over k . Any scheme X over k is equipped with a morphism $X \rightarrow \text{Spec } k$; locally, this is a

morphism of affine schemes corresponding to the structure morphism $k \rightarrow R$ of a k -algebra R . We call $X \rightarrow \text{Spec } k$ the “structure morphism” of X .

Definition 1.9. Let k be an algebraically closed field.²

- (1) When R is a reduced finitely generated k -algebra, we call $\text{Spec } R$ an *affine algebraic variety* over k .
- (2) When a scheme X over k has a finite open covering $\{U_i\}_{i \in I}$ such that each U_i is an affine algebraic variety, we call X an *algebraic variety* over k .

As long as k is fixed and there is no confusion, we often omit the word “over k ”.

1.3. Definition and examples of algebraic groups. For any schemes X and Y over k , there uniquely (up to a unique isomorphism) exists their “fibered product” $X \times_k Y$, which is a scheme over k equipped with morphisms $p_1: X \times_k Y \rightarrow X$ and $p_2: X \times_k Y \rightarrow Y$ over k satisfying the following “universal property”:

for any scheme Z over k equipped with morphisms $q_1: Z \rightarrow X$ and $q_2: Z \rightarrow Y$ over k , there uniquely exists a morphism $f: Z \rightarrow X \times_k Y$ over k such that $q_1 = p_1 \circ f$ and $q_2 = p_2 \circ f$.

Note that, when $X = \text{Spec } R$ and $Y = \text{Spec } S$ for k -algebras R and S , their fibered product is simply given by $\text{Spec}(R \otimes_k S)$ (the morphisms p_1 and p_2 are given by the natural k -algebra homomorphisms $R \rightarrow R \otimes_k S$ and $S \rightarrow R \otimes_k S$).

Definition 1.10 (algebraic group). Let G be an algebraic variety over k . We say that G is an *algebraic group over k* if G is equipped with a group structure, i.e., morphisms of schemes over k

- $m: G \times_k G \rightarrow G$ (“multiplication morphism”),
- $i: G \rightarrow G$ (“inversion morphism”), and
- $e: \text{Spec } k \rightarrow G$ (“unit element”)

satisfying the axioms of groups. More precisely, the following diagrams are commutative:

$$\begin{array}{ccc}
 G \times_k G \times_k G & \xrightarrow{m \times \text{id}} & G \times_k G \\
 \text{id} \times m \downarrow & \circlearrowleft & \downarrow m \\
 G \times_k G & \xrightarrow{m} & G
 \end{array}
 \qquad
 \begin{array}{ccc}
 G & \xrightarrow{\text{id} \times e} & G \times_k G \\
 e \times \text{id} \downarrow & \searrow \text{id} \circlearrowleft & \downarrow m \\
 G \times_k G & \xrightarrow{m} & G
 \end{array}$$

$$\begin{array}{ccccc}
 G \times_k G & \xleftarrow{\Delta} & G & \xrightarrow{\Delta} & G \times_k G \\
 \text{id} \times i \downarrow & \circlearrowleft & \downarrow \epsilon & \circlearrowleft & \downarrow i \times \text{id} \\
 G \times_k G & \xrightarrow{m} & G & \xleftarrow{m} & G \times_k G
 \end{array}$$

Here, ϵ denotes the composition of the structure morphism $G \rightarrow \text{Spec } k$ and $e: \text{Spec } k \rightarrow G$.

Definition 1.11. Let G and H be algebraic group over k . We say that a morphism $f: G \rightarrow H$ over k is a *homomorphism* of algebraic groups if the following diagram

²In this lecture, for the definition of an algebraic variety, we always assume that k is an algebraically closed field. Also, please be careful that the definition of the word “algebraic variety” heavily depends on textbooks. The definition given here may not be very universal.

is commutative:

$$\begin{array}{ccc} G \times_k G & \xrightarrow{f \times f} & H \times_k H \\ m \downarrow & \circlearrowleft & \downarrow f \\ G & \xrightarrow{m} & H \end{array}$$

Here, the left vertical arrow denotes the multiplication morphism for G and the right one denotes that for H .

Remark 1.12. Suppose that G is an affine algebraic variety with coordinate ring $k[G]$ (i.e., $G = \operatorname{Spec} k[G]$). Recall that the category of affine schemes is equivalent to the category of rings. Thus giving G an algebraic group structure is equivalent to defining k -algebra homomorphisms

- $m: k[G] \rightarrow k[G] \otimes_k k[G]$,
- $i: k[G] \rightarrow k[G]$,
- $e: k[G] \rightarrow k$.

In general, a commutative ring equipped with such an additional structure is called a *Hopf algebra*.

Example 1.13. (1) We put $\mathbb{G}_a := \operatorname{Spec} k[x]$ and define m , i , and e at the level of rings as follows:

- $m: k[x] \rightarrow k[x] \otimes_k k[x]; \quad x \mapsto x \otimes 1 + 1 \otimes x$,
- $i: k[x] \rightarrow k[x]; \quad x \mapsto -x$,
- $e: k[x] \rightarrow k; \quad x \mapsto 0$.

Then \mathbb{G}_a is an algebraic group over k with respect to the corresponding morphisms. We call \mathbb{G}_a the *additive group* over k .

(2) We put $\mathbb{G}_m := \operatorname{Spec} k[x, x^{-1}]$ and define m , i , and e at the level of rings as follows:

- $m: k[x, x^{-1}] \rightarrow k[x, x^{-1}] \otimes_k k[x, x^{-1}]; \quad x \mapsto x \otimes x$,
- $i: k[x, x^{-1}] \rightarrow k[x, x^{-1}]; \quad x \mapsto x^{-1}$,
- $e: k[x, x^{-1}] \rightarrow k; \quad x \mapsto 1$.

Then \mathbb{G}_m is an algebraic group over k with respect to the corresponding morphisms. We call \mathbb{G}_m the *multiplicative group* over k .

(3) We put $\operatorname{GL}_n := \operatorname{Spec} k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n]$, where $D := \det(x_{ij})_{1 \leq i, j \leq n}$. We define m , i , and e at the level of rings as follows:

- $m(x_{ij}) := \sum_{k=1}^n x_{ik} \otimes x_{kj}$,
- $i(x_{ij}) :=$ the (i, j) -entry of the inverse of the matrix $(x_{ij})_{1 \leq i, j \leq n}$,
- $e(x_{ij}) := \delta_{ij}$ (Kronecker's delta).

Then GL_n is an algebraic group over k with respect to the corresponding morphisms. We call GL_n the *general linear group (of rank n)* over k . (Note that $\operatorname{GL}_1 \cong \mathbb{G}_m$.)

Now we explain a “functorial” viewpoint of algebraic groups, which is more practical.

Let $X = \operatorname{Spec} k[X]$ be an affine scheme over k . We consider a functor $X(-)$ from the category of k -algebras to the category of sets given by

$$X(R) := \operatorname{Hom}_k(\operatorname{Spec} R, X)$$

for any k -algebra R , where $\operatorname{Hom}_k(-, -)$ denotes the set of morphisms of affine schemes over k . Since the category of affine schemes is equivalent to the category

of rings, we have

$$\mathrm{Hom}_k(\mathrm{Spec} R, X) \cong \mathrm{Hom}_k(k[X], R),$$

where the latter $\mathrm{Hom}_k(-, -)$ denotes the set of k -algebra homomorphisms. In fact, the affine scheme X is determined by the functor $X(-)$. Therefore, we may regard the affine scheme X as a “machine” which associate to each k -algebra R a set $X(R)$ in a functorial way. (More precisely, the association $X \mapsto X(-)$ gives a fully faithful functor from the category of affine schemes over k to the category of functors from the category of affine schemes over k to the category of sets; this is so-called “Yoneda’s lemma”.)

We call an element of $X(R)$ an *R -valued point* or an *R -rational point* of X .

Example 1.14. Let $X = \mathrm{Spec} k[x, y]/(y^2 - x^3)$. Then, for any k -algebra R , we have

$$X(R) \cong \mathrm{Hom}_k(k[x, y]/(y^2 - x^3), R).$$

Note that, any k -algebra homomorphism f from $k[x, y]/(y^2 - x^3)$ to R is uniquely determined by the images $f(x), f(y) \in R$ of x, y . Since x and y satisfies the equation $y^2 - x^3 = 0$ in the coordinate ring $k[x, y]/(y^2 - x^3)$, their images must satisfy $f(y)^2 - f(x)^3 = 0$. Conversely, for any elements $(a, b) \in R^2$ satisfying the equation $b^2 - a^3 = 0$, we can define a k -algebra homomorphism $f: k[x, y]/(y^2 - x^3) \rightarrow R$ by $f(x) = a$ and $f(y) = b$. Therefore, we get

$$X(R) \cong \mathrm{Hom}_k(k[x, y]/(y^2 - x^3), R) \cong \{(a, b) \in R^2 \mid b^2 - a^3 = 0\}.$$

In other words, we can think of X as a machine which associates to each R the set of solutions to the equation $y^2 - x^3 = 0$ in R^2 .

Now let G be an algebraic group over k . Then the multiplication morphism $m: G \times_k G \rightarrow G$ induces a map $m_R: G(R) \times G(R) \rightarrow G(R)$ for each k -algebra R . Indeed, let $g_1, g_2 \in G(R) = \mathrm{Hom}_k(\mathrm{Spec} R, G)$. Then we can define an element $m_R(g_1, g_2) \in G(R)$ to be

$$m_R(g_1, g_2): \mathrm{Spec} R \xrightarrow{(g_1, g_2)} G \times_k G \xrightarrow{m} G.$$

(Here, (g_1, g_2) denotes the morphism induced from g_1 and g_2 by the universal property of the fibered product $G \times_k G$.) Similarly, we also have a map $i_R: G(R) \rightarrow G(R)$ induced by i . Furthermore, the unit morphism $e: \mathrm{Spec} k \rightarrow G$ induces an element $e_R \in G(R)$ given by $e_R: \mathrm{Spec} R \rightarrow \mathrm{Spec} k \xrightarrow{e} G$, where the first arrow is the structure morphism for $\mathrm{Spec} R$. Then, it can be easily checked that the axiom of an algebraic group implies that $G(R)$ is a group in the usual sense with respect to the map m_R with inversion map i_R and unit element e_R . As a result, $G(-)$ gives a functor from the category of k -algebras to the category of groups.

Example 1.15. (1) For a k -algebra R , we have $\mathbb{G}_a(R) \cong R$, where the group structure on R is given by the additive structure of R . Indeed, we have

$$\mathbb{G}_a(R) = \mathrm{Hom}_k(\mathrm{Spec} R, \mathbb{G}_a) \cong \mathrm{Hom}_k(k[x], R) \cong R,$$

where the last map is given by $f \mapsto f(x)$. The multiplication map m_R induced on $\mathbb{G}_a(R)$ corresponds to the addition on R . Indeed, let us take any elements $g_1, g_2 \in \mathbb{G}_a(R)$, hence $m_R(-, -)$ is given by the composition

$$m_R(g_1, g_2): \mathrm{Spec} R \xrightarrow{(g_1, g_2)} G \times_k G \xrightarrow{m} G.$$

At the level of rings, this amounts to the composition

$$k[x] \xrightarrow{m} k[x] \otimes_k k[x] \xrightarrow{g_1 \otimes g_2} R.$$

Since $m(x) = x \otimes 1 + 1 \otimes x$ by definition, we get

$$(g_1 \otimes g_2) \circ m(x) = (g_1 \otimes g_2)(x \otimes 1 + 1 \otimes x) = g_1(x) + g_2(x).$$

This is why \mathbb{G}_a is called the “additive group”.

- (2) For a k -algebra R , we have $\mathbb{G}_m(R) \cong R^\times$, where R^\times denotes the unit group of R with respect to the multiplicative structure of R . Indeed, we have

$$\mathbb{G}_m(R) = \text{Hom}_k(\text{Spec } R, \mathbb{G}_m) \cong \text{Hom}_k(k[x, x^{-1}], R) \cong R^\times,$$

where the last map is given by $f \mapsto f(x)$. In a similar manner to above, we can check that the multiplication map m_R on $\mathbb{G}_m(R)$ corresponds to the multiplication on R^\times . This is why \mathbb{G}_m is called the “multiplicative group”.

- (3) For a k -algebra R , we have

$$\text{GL}_n(R) \cong \{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}.$$

Indeed, by definition, we have

$$\text{GL}_n(R) = \text{Hom}_k(\text{Spec } R, \text{GL}_n) \cong \text{Hom}_k(k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n], R).$$

The right-hand side is isomorphic to (at least as sets) $\{g = (g_{ij})_{i,j} \in M_n(R) \mid \det(g) \in R^\times\}$ by the map $f \mapsto (f(x_{ij}))_{i,j}$. It is a routine work to check that this bijection is indeed a group isomorphism.

- (4) The *symplectic group* Sp_{2n} is an affine algebraic group such that the group of its R -valued points is given as follows:

$$\text{Sp}_{2n}(R) \cong \{g = (g_{ij})_{i,j} \in \text{GL}_{2n}(R) \mid {}^t g J_{2n} g = J_{2n}\},$$

where J_{2n} denotes the antidiagonal matrix whose antidiagonal entries are given by 1 and -1 alternatively:

$$J_{2n} := \begin{pmatrix} & & & 1 \\ & & -1 & \\ & 1 & & \\ \ddots & & & \end{pmatrix}.$$

Here, we don't explain how to define the coordinate ring of Sp_{2n} and also how to introduce the group structure at the level of the coordinate ring. Only the important viewpoint here is what kind of groups are associated as the groups of R -valued points! So, in this course, let us just believe that the functor Sp_{2n} is indeed *representable*, i.e., realized as the functor of points of some affine algebraic groups. This remark is always applied to any affine algebraic group which we will encounter in the future.

2. WEEK 2: VERY BASIC PROPERTIES OF GENERAL ALGEBRAIC GROUPS

Recall that, in general, a *scheme* X is a topological space equipped with a sheaf of rings \mathcal{O}_X (“structure sheaf”) which is locally isomorphic to affine schemes (“ $\text{Spec } A$ ” for a commutative ring A).

In the following, we let k be an algebraically closed field. Also, when we say “an algebraic variety”, it always means “an algebraic variety over k ”. Here, recall that we say that a scheme X is an algebraic variety over k if it is locally isomorphic to $\text{Spec } A$ for a finitely generated reduced k -algebra (hence, in particular, A is of the form $k[x_1, \dots, x_n]/I$ for an ideal I of $k[x_1, \dots, x_n]$).

For any algebraic variety X over k , the subset of closed points of X can be identified with the set $X(k)$ of k -rational points of X ; for any k -rational point $\text{Spec } k \rightarrow X$, the image of the unique point of $\text{Spec } k$ is a closed point of X , and vice versa. From now on, we freely identify the set of closed points of X with $X(k)$. Moreover, the subset of closed points of X is dense in X because k is algebraically closed. (Both these facts are consequences of Hilbert’s “nullstellensatz”, which asserts that any maximal ideal of $k[x_1, \dots, x_n]$ is of the form $(x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$; this fact assumes that k is algebraically closed.)

2.1. Identity component subgroup. Let G be an algebraic group over k . Recall that, in particular, G is equipped with a unit element $e \in G(k)$. Let G° denote the connected component of G containing the closed point e .

Proposition 2.1. *The subset G° is a subgroup of G . Moreover, G° is normal of finite index in G .*

Proof. We have to show that G° is closed under the multiplication morphism $m: G \times G \rightarrow G$ and the inversion morphism $i: G \rightarrow G$. More precisely, our task is to check that $m(G^\circ, G^\circ) \subset G^\circ$ and $i(G^\circ) \subset G^\circ$. But both statements follow by combining a general fact that the image of a connected set under a continuous map is again connected with that $m(e, e) = e$ and $i(e) = e$.

To show the second assertion, let us take $g \in G(k)$. (By definition, being normal means that $gG^\circ g^{-1} \subset G^\circ$ for any $g \in G(k)$.) Then it can be easily checked that $gG^\circ g^{-1}$ is a subgroup of G which is connected and contains the unit element. Hence we get $gG^\circ g^{-1} \subset G^\circ$. The finite-index property follows from that the set of connected components of an algebraic variety is finite. \square

Definition 2.2. We call the algebraic subgroup G° of G the *identity component* of G .

2.2. Smoothness of algebraic groups. Let us first look at the following example: we consider an affine algebraic variety $X := \text{Spec } k[x, y]/(y^2 - x^3)$, i.e., X is the spectrum of the quotient ring of $k[x, y]$ by the ideal generated by $(y^2 - x^3)$. Recall that, X represents the space of solutions to the equation $y^2 - x^3 = 0$. More precisely, for any k -algebra R , the set $X(R)$ of R -rational points of X is equal to the set of solutions to $y^2 - x^3 = 0$ in R . If we try to draw a picture of the set $X(\mathbb{R}) \subset \mathbb{R}^2$, then we can immediately notice that the resulting curve is “smooth” except for the origin $(0, 0)$; at the origin, the curve has a “singular point”³.

³Because we assume k is algebraically closed in this lecture, it’s not actually allowed to take R to be \mathbb{R} . If you want to be rigorous please take the coefficient k to be any smaller field, for example, \mathbb{Q} .

In fact, the difference between the point $(0,0)$ and the other points in this example can be explained in terms of ring-theoretic properties of the coordinate ring $k[x, y]/(y^2 - x^3)$. Let us explain how to introduce the notion of a “smooth point” and also a “singular point” for general schemes in the following.

Let X be a scheme. For any point $x \in X$, we define a ring $\mathcal{O}_{X,x}$ by

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U} \mathcal{O}_X(U),$$

where the inductive limit is over open sets U of X containing $x \in X$ (the structure morphisms are given by the restriction maps $\mathcal{O}_X(V) \rightarrow \mathcal{O}_X(U)$ for any $x \in V \subset U$). This ring is a local ring and called the *stalk* of X at $x \in X$. If $x \in X$ is contained in an affine open subscheme $U \subset X$ isomorphic to $\text{Spec } A$, where x is identified with a prime ideal \mathfrak{p} of A , then the stalk $\mathcal{O}_{X,x}$ is nothing but the localization $A_{\mathfrak{p}}$ of A with respect to \mathfrak{p} .

For any $x \in X$, we write \mathfrak{m}_x for the unique maximal ideal of the stalk $\mathcal{O}_{X,x}$. We put $\kappa(x) := \mathcal{O}_{X,x}/\mathfrak{m}_x$ and call $\kappa(x)$ the *residue field* of X at $x \in X$.

Definition 2.3. Let X be an algebraic variety over k .

- (1) We say that a point $x \in X$ is *smooth* if the local ring $\mathcal{O}_{X,x}$ of X at x is a regular local ring, i.e., we have

$$\dim(\mathcal{O}_{X,x}) = \dim_{\kappa(x)}(\mathfrak{m}_x/\mathfrak{m}_x^2).$$

Here, the left-hand side denotes the Krull dimension of the ring $\mathcal{O}_{X,x}$ and the right-hand side denotes the dimension of $\mathfrak{m}_x/\mathfrak{m}_x^2$ as a $\kappa(x)$ -vector space.

- (2) We say that X is *smooth* if every point of X is smooth.

Fact 2.4. Let X be an algebraic variety over k . Then the subset of smooth points of X is open dense in X .

The subset of smooth point of X is often referred to as the *smooth locus* of X .

Proposition 2.5. Let G be an algebraic group over k . Then G is smooth.

Proof. Let U be the smooth locus of G , which is open dense in G by the above fact. Let us show that any closed point g of G is contained in U . If we can show this, then the assertion follows. Indeed, the complement $G \setminus U$ is a closed subset of G ; if this is not empty, then it contains at least one closed point of G , hence a contradiction.

Firstly, U contains at least one closed point g_0 of G because, otherwise, $G \setminus U$ is a closed subset of G containing all closed points, hence equal to G by the density of closed points. Next, for any closed point g of G , we consider the (gg_0^{-1}) -multiplication morphism

$$G \rightarrow G: x \mapsto gg_0^{-1}x.$$

(Precisely speaking, for any $h \in G(k)$, the h -multiplication morphism is defined to be the composition $G \cong \text{Spec } k \times_k G \rightarrow G \times_k G \rightarrow G$, where the second arrow is the fibered product of $h: \text{Spec } k \rightarrow G$ and id_G and the last arrow is the multiplication morphism of G . At the level of k -rational points, this realizes the intuitive map $x \mapsto hx$.) Then, because this is an isomorphism of algebraic varieties, any smooth point is mapped to a smooth point. In particular, g , which is the image of the smooth point g_0 , is also smooth. Thus U contains g . \square

Remark 2.6. The word “smooth” usually means a property of a morphism of schemes $f: X \rightarrow Y$; the definition introduced above is usually referred to as the regularity (non-singularity) of X (at x), which is an “absolute” notion depending only on X . When $Y = \operatorname{Spec} k$ (where k is an algebraically closed field), the smoothness for the morphism f is equivalent to the regularity (non-singularity) of X . In general, we must be careful about the difference between the regularity and the smoothness; see, e.g., [Mil17, §1.b].

2.3. Homomorphism between algebraic groups. Let us investigate a homomorphism between algebraic groups over k .

Proposition 2.7. *Let $\alpha: G \rightarrow G'$ be a homomorphism between algebraic groups over k . Then the image $\alpha(G)$ is a closed subgroup of G' .*

To show this proposition, let us first review some general notions for topological spaces.

Definition 2.8. Let X be a topological spaces.

- (1) We say that a subset Z of X is *locally closed* if Z is an intersection of an open subset of X and a closed subset of X .
- (2) We say that a subset Z of X is *constructible* if Z is a finite union of locally closed subsets of X .
- (3) We say that X is *noetherian* if any open subset of X is quasi-compact.

Remark 2.9. In the above definition, the word “quasi-compact” just means “compact”, i.e., any open covering has a finite subcovering. This is because, sometimes (depending on areas), the word “compact” is used to mean “Hausdorff and compact”. In the context of algebraic geometry, we often use the word “quasi-compact” to emphasize that the Hausdorff property is not assumed.

The following fact is a general nonsense on topological spaces:

Lemma 2.10. *Let X be a noetherian topological space. Let Y be a constructible subset of X . Then Y contains an open dense subset of its closure \bar{Y} in X .*

Exercise 2.11. Prove the above lemma.

Note that, any algebraic variety over k is a noetherian topological space, hence the above lemma can be applied.

On the other hand, the following fact is much deeper:

Fact 2.12. *Let $f: X \rightarrow Y$ be a morphism between algebraic varieties over k . Then the image of any constructible subset under f is a constructible subset of Y .*

Let us utilize these facts to deduce some useful facts on algebraic groups.

Lemma 2.13. *Let G be an algebraic group over k . Then, for any open dense sets U and V of G , we have $U \cdot V = G$, where we put $U \cdot V := \{u \cdot v \in G \mid u \in U, v \in V\}$.*

Proof. It is enough to show that the open subset $U \cdot V$ contains every closed point g of G . Let $g \in G$ be a closed point. Then both U and $g \cdot V^{-1}$ are dense open subsets of G , hence have a nonempty open intersection. By the density of closed points, there exists a closed point in $U \cap (g \cdot V^{-1})$. In other words, there exists closed points $u \in U$ and $v \in V$ satisfying $u = gv^{-1}$, hence $h = uv \in U \cdot V$. \square

Proposition 2.14. *Let G be an algebraic group over k . Then any constructible subgroup H of G is closed.*

Proof. By Lemma 2.10, H contains an open dense subset U of its closure \overline{H} in G . Since H is a subgroup of G , we obtain

$$U \cdot U \subset H \cdot H \subset H.$$

By the above lemma, we have $U \cdot U = \overline{H}$, hence $H = \overline{H}$. \square

Corollary 2.15. *Let $\alpha: G \rightarrow G'$ be a homomorphism between algebraic groups over k . Then the image $\alpha(G)$ is a closed subgroup of G' .*

Proof. By Fact 2.12, $\alpha(G)$ is a constructible subset of G' . Since $\alpha(G)$ is a subgroup of G' , the above proposition implies that $\alpha(G)$ is closed. \square

Remark 2.16. The notion of a “kernel” in the context of algebraic groups is quite subtle. Scheme-theoretically, the kernel of α is defined to be the fibered product of $\alpha: G \rightarrow G'$ and $e': \operatorname{Spec} k \rightarrow G'$, where e' denotes the unit element of G' . However, the problem is that this fibered product is not necessarily reduced, hence not necessarily an algebraic variety in our sense. For example, consider the morphism $\mathbb{G}_m \rightarrow \mathbb{G}_m: x \mapsto x^p$ for the multiplicative group defined over an algebraically closed field k of characteristic $p > 0$. Then, as “points”, its kernel is equal to $\mu_p(k) := \{x \in k \mid x^p = 1\} = \{1\}$. However, the fibered product is isomorphic to $\operatorname{Spec} k[x]/(x-1)^p$, which is not reduced. This observation suggests that, for a better treatment of algebraic groups, we should work with more general notion of group schemes.

2.4. Dimension of algebraic groups.

Definition 2.17. Let X be an algebraic variety. We say that a closed subset Y of X is *irreducible* if Y is non-empty and cannot be written as $Y = Z_1 \cup Z_2$ for non-empty proper closed subsets $Z_1, Z_2 \subsetneq Y$. We call a maximal irreducible subset of X an *irreducible component* of X .

Definition 2.18. For an algebraic variety X , we define the *dimension* $\dim X$ of X to be the maximum of the length d of a chain

$$Y_0 \subsetneq Y_1 \subsetneq \cdots \subsetneq Y_d$$

of irreducible subsets of Y_d .

In fact, the dimension of an algebraic variety is related to the Krull dimension of its stalks in the following sense: let Y be an irreducible component of X . Then, for any $x \in Y$, we have $\dim \mathcal{O}_{X,x} = \dim Y$.

Fact 2.19. *Let $\alpha: G \rightarrow G'$ be a homomorphism between algebraic groups over k . Then we have*

$$\dim G = \dim \operatorname{Ker}(\alpha) + \dim \alpha(G).$$

Here, as noted above, $\alpha(G)$ is a closed subgroup of G while $\operatorname{Ker}(\alpha)$ is not in general because it might not be reduced. So the (ad hoc) meaning of “ $\operatorname{Ker}(\alpha)$ ” is that it is the set-theoretic preimage of the unit element $e' \in G'$ under α . Since α is continuous and e' is closed, the preimage is closed in G , hence it makes sense to talk about its dimension.

For the proof of this fact, see [Mil17, Proposition 1.63].

2.5. Algebraic group action on algebraic varieties.

Definition 2.20. Let G be an algebraic group over k and X an algebraic variety over k . We say that G *acts on* X if there exists a morphism of algebraic varieties $\alpha: G \times X \rightarrow X$ satisfying the usual axioms of group actions, i.e., the following diagrams are commutative:

$$\begin{array}{ccc} G \times_k G \times_k X & \xrightarrow{m \times \text{id}} & G \times_k X \\ \text{id} \times \alpha \downarrow & \circlearrowleft & \downarrow \alpha \\ G \times_k X & \xrightarrow{\alpha} & G \end{array} \quad \begin{array}{ccc} X & \xrightarrow{e \times \text{id}} & G \times_k X \\ \text{id} \searrow & \circlearrowleft & \downarrow m \\ & & X \end{array}$$

We can also consider the usual notion on the group action such as normalizer, stabilizer, and so on, in the context of algebraic groups.

Proposition-Definition 2.21. Suppose that an algebraic group G acts on an algebraic variety X .

- (1) For any closed subvarieties Y and Z of X , there exists a closed subvariety $N_G(Y, Z)$ satisfying
$$N_G(Y, Z)(R) = N_{G(R)}(Y(R), Z(R)) := \{n \in G(R) \mid nY(R) \subset Z(R)\}$$
for any k -algebra R . We call $N_G(Y, Z)(R)$ the transporter from Y to Z in G .
- (2) When $Y = Z$, we call the transporter $N_G(Y, Y)$ the normalizer of Y in Z and write $N_G(Y) := N_G(Y, Y)$. Note that the normalizer is a subgroup of G .
- (3) When Y consists of a single closed point $x \in X$, we call the normalizer group $N_G(\{x\})$ the stabilizer group of x in G and write $G_x := N_G(\{x\})$. More generally, for any closed subvariety $Y \subset X$, we put $G_Y := \bigcap_{x \in Y} G_x$.⁴

The subtle point of the above definition is that, so that the resulting “subfunctor” $N_G(Y, Z)$ is indeed given by a “subvariety” (more naively speaking, the subset $\{n \in G \mid nY \subset Z\}$ has a natural subscheme structure), we need to assume that the subsets Y and Z are *closed* subvarieties of G . See [Mil17, 1.79] for the details.

Proposition 2.22 (“Closed orbit lemma”). Let G be an algebraic group acting on an algebraic variety X . For any closed point $x \in X$, let Gx denote its orbit.

- (1) Each Gx is a smooth variety which is open in its closure \overline{Gx} in X .
- (2) The boundary $\overline{Gx} \setminus Gx$ is a union of orbits of strictly smaller dimension.

Proof. Note that $G \cdot x$ is (by definition) the image of the morphism $G \rightarrow X: g \mapsto gx$. Using the fact that the image of any constructible set is again constructible (Fact 2.12), we see that Gx contains a dense open subset U of its closure \overline{Gx} . Here note that both Gx and \overline{Gx} are stable under the G -action. In particular, we have

$$Gx = \bigcup_{g \in G(k)} gU.$$

(Precisely speaking, we first see that the closed points contained in $\bigcup_{g \in G(k)} gU$ are the same as those of Gx . Then, by the density of closed points, we get the equality

⁴When $X = G$ and the action of G on X is the conjugation, we call the stabilizer G_X the *centralizer* of X in G .

as subvarieties.) Each gU is open in \overline{Gx} , hence this equality implies that Gx is open in \overline{Gx} . The smoothness follows from the same argument as in the proof of the smoothness of algebraic groups, i.e., use the open-density of the smooth locus and that G acts on Gx transitively.

It can be easily checked that any dense open subset of a noetherian space intersects every irreducible component. In particular, the boundary $\overline{Gx} \setminus Gx$ does not contain any irreducible component \overline{Gx} . In other words, $\overline{Gx} \setminus Gx$ is a closed subset of \overline{Gx} of strictly smaller dimension. Since $\overline{Gx} \setminus Gx$ is G -stable, it can be written as the union of its G -orbits. \square

Corollary 2.23. *Let G be an algebraic group acting on an algebraic variety X . Then any G -orbit of minimal dimension is closed. In particular, X always has a closed G -orbit.*

Proof. If the dimension of a G -orbit Gx is minimal, then the boundary $\overline{Gx} \setminus Gx$ must be empty by the above proposition. Hence Gx is closed. \square

Example 2.24. A typical example of the application of the above proposition is the following. Let $G = \mathrm{GL}_n$. We consider $\mathcal{N} := \{N \in M_n \mid (N - I_n)^r = 0 \text{ for some } r \in \mathbb{Z}_{\leq 0}\}$. In other words, \mathcal{N} is an algebraic subvariety of $M_n \cong \mathbb{A}_k^{n^2}$ (the affine space of n -by- n matrices) consisting of nilpotent matrices. Then G acts on \mathcal{N} via conjugation. By the theory of Jordan normal form, each nilpotent G -orbit corresponds to a partition of n . It is known that the “closure relation” on \mathcal{N} (i.e., when a G -orbit Gx is contained in the closure of another G -orbit \overline{Gy}) can be described in terms of the combinatorics on the partition of n .

3. WEEK 3: JORDAN DECOMPOSITION

As before, we let k be an algebraically closed field. When we say “an algebraic variety”, it always means “an algebraic variety over k ”.

3.1. Linear algebraic groups. The first goal of this week is to show the following.

Proposition 3.1. *Let G be an algebraic group. Then G is affine if and only if there exists a closed immersion $G \hookrightarrow \mathrm{GL}_n$ for some $n \in \mathbb{Z}_{>0}$.*

Here let us give some comments on the notion of a “closed immersion” of schemes. When we say that a map between topological spaces $f: X \rightarrow Y$ is a “closed immersion”, we usually mean that f is a homeomorphism onto a closed subset of Y . In the context of schemes, a “morphism $f: X \rightarrow Y$ between schemes” is defined to be a pair of a continuous map $f: X \rightarrow Y$ between the underlying topological spaces and a morphism of sheaves $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ on Y . Here, $f_*\mathcal{O}_X$ is the sheaf of rings on Y defined by $f_*\mathcal{O}_X(V) := \mathcal{O}_X(f^{-1}(V))$ for any open subset V of Y (called the direct image of \mathcal{O}_X by f). Such a morphism f is said to be a closed immersion if it is a closed immersion as a map between topological spaces and also the map $f^\#: \mathcal{O}_Y \rightarrow f_*\mathcal{O}_X$ of sheaves is surjective (i.e., surjective on each open set $V \subset Y$).

Recall that, when schemes X and Y are affine (say $X = \mathrm{Spec} S$ and $Y = \mathrm{Spec} R$), any morphism $f: X \rightarrow Y$ corresponds to a ring homomorphism $f^*: R \rightarrow S$ (although I have not explained how to associate f to f^* in this lecture). In fact, in this case, f is a closed immersion of schemes if and only if the ring homomorphism is surjective. See, for example, [Liu02, §2.3.2] for more details.

Note that the general linear group GL_n is affine by definition. In general, any closed subscheme of an affine scheme is again affine (see [Liu02, Proposition 3.20]). So what is really nontrivial in Proposition 3.1 is that its converse (any affine algebraic group G can be embedded into GL_n for some $n \in \mathbb{Z}_{>0}$) also holds once the group structure is given. For this reason, we call any affine algebraic group a *linear algebraic group*.

3.2. Translation on coordinate rings. Before we start proving the above proposition, we investigate the notion of the “translation” action induced by an action of an algebraic group. Let G be an affine algebraic group with coordinate ring $k[G]$. Suppose that G acts on an affine algebraic variety X with coordinate ring $k[X]$. Let $\alpha: G \times_k X \rightarrow X$ be the morphism defining the action and

$$\alpha^*: k[X] \rightarrow k[G] \otimes_k k[X]$$

the corresponding k -algebra homomorphism.

Let $g \in G(k)$; recall that, scheme-theoretically, this is a morphism $g: \mathrm{Spec} k \rightarrow G$, which corresponds to a k -algebra homomorphism $g^*: k[G] \rightarrow k$. We define a k -algebra endomorphism λ_g on $k[X]$ to be $((g^{-1})^* \otimes \mathrm{id}) \circ \alpha^*$:

$$\lambda_g: k[X] \xrightarrow{\alpha^*} k[G] \otimes_k k[X] \xrightarrow{(g^{-1})^* \otimes \mathrm{id}} k[X]$$

Then this defines an action $\lambda_{(-)}$ of the group $G(k)$ on the k -vector space $k[X]$. We call this action the *left translation action* of $G(k)$ on $k[X]$.

Remark 3.2. (1) Note that the “action” of $G(k)$ on $k[X]$ is not in the sense of algebraic group actions defined in the last week since $k[X]$ is an infinite-dimensional k -vector space. (But, of course, if you look at a finite-dimensional

$G(k)$ -stable subspace, then you can ask if the action is algebraic or not when the finite-dimensional subspace is regarded as an affine space.)

- (2) The morphism $X \rightarrow X$ induced by the k -algebra endomorphism λ_g is nothing but the left-translation by g^{-1} , i.e., at the level of R -rational points for any k -algebra,

$$X(R) \rightarrow X(R); \quad x \mapsto g^{-1} \cdot x,$$

where the dot denotes the induced action of $G(k) \subset G(R)$ on $X(R)$.

- (3) One important perspective is to look at the coordinate ring $k[X]$ of an affine algebraic variety X as a set of functions on X . To be more precise, for any $f \in k[X]$ and a closed point $x \in X(k)$ which corresponds to $x^*: k[X] \rightarrow k$, we put $f(x) := x^*(f)$. Then f is regarded as a k -valued function on $X(k)$. With this viewpoint, the left-translation action of $G(k)$ on $k[X]$ can be literally thought of as the “left-translation”, i.e., for any $f \in k[X]$, we have

$$(\lambda_g(f))(x) = f(g^{-1} \cdot x)$$

for any $x \in X(k)$.

Lemma 3.3. *Let V be a finite-dimensional subspace of $k[X]$. Then,*

- (1) *V is $G(k)$ -stable if and only if $\alpha^*(V) \subset k[G] \otimes_k V$;*
- (2) *there exists a V' be a $G(k)$ -stable finite-dimensional subspace of $k[X]$ such that $V \subset V'$.*

Proof. Let us first show (1). We fix a k -basis $\{f_1, \dots, f_r\}$ of V and extend it to a k -basis $\{f_1, \dots, f_r, f_{r+1}, \dots\}$ of $k[X]$. For any $f \in k[X]$, we write $\alpha^*(f) \in k[G] \otimes_k k[X]$ as

$$\alpha^*(f) = \sum_i h_i \otimes f_i,$$

where $h_i \in k[G]$. Then we have

$$\lambda_g(f) = ((g^{-1})^* \otimes \text{id}) \circ \alpha^*(f) = \sum_i (g^{-1})^*(h_i) \cdot f_i = \sum_i h_i(g^{-1}) \cdot f_i$$

Now suppose that V is $G(k)$ -stable. Then, for any $f \in V$ and any $g \in G(k)$, we have $\sum_i h_i(g^{-1}) \cdot f_i \in V$. In particular, we have $h_i(g^{-1}) = 0$ for any $i > r$. Since this holds for any $g \in G(k)$, we get $h_i = 0$ for any $i > r$, which means that $\alpha^*(f) \in k[G] \otimes_k V$. Conversely, let us suppose that $\alpha^*(V) \subset k[G] \otimes_k V$. Then, for any $f \in V$, we have $\alpha^*(f) = \sum_{i=1}^r h_i \otimes f_i$. Hence, again by the computation as above, we get that $\lambda_g(f) \in V$.

We next show (2). It is enough to consider only the case where V is one-dimensional, which is generated by $f \in k[X]$. Let us write

$$\alpha^*(f) = \sum_{i=1}^s h_i \otimes f_i,$$

where $h_i \in k[G]$ and $f_i \in k[X]$, hence

$$\lambda_g(f) = \sum_{i=1}^s h_i(g^{-1}) \cdot f_i$$

for any $g \in G(k)$. This implies that the subspace

$$\text{Span}_k \{ \lambda_g(f) \in k[X] \mid g \in G(k) \}$$

of $k[X]$ is at least contained in $\text{Span}_k\{f_1, \dots, f_s\}$, hence must be finite-dimensional. Since $\text{Span}_k\{\lambda_g(f) \in k[X] \mid g \in G(k)\}$ is obviously $G(k)$, we can choose V' to be this subspace. \square

Now we consider the case where $X = G$. In this case, we can consider two kinds on actions of G on $X = G$; $g \cdot x = gx$ and $g \cdot x = xg^{-1}$. As above, we call the action of $G(k)$ on $k[G]$ induced by the former action ($g \cdot x = gx$) the *left-translation* and write λ_g . We call the action $G(k)$ on $k[G]$ induced by the latter one ($g \cdot x = xg^{-1}$) the *right-translation* and write ρ_g . If we regard elements of $k[G]$ as functions on $G(k)$, these actions can be described as

$$\lambda_g(f)(x) = f(g^{-1}x), \quad \rho_g(f)(x) = f(xg).$$

Proposition 3.4. *Any finite-dimensional subspace of $k[G]$ is contained in a finite-dimensional subspace of $k[G]$ stable under both left and right translations.*

Proof. This immediately follows by applying Lemma 3.3 to the action of $G \times G$ on G given by $(g, h) \cdot x = gxh^{-1}$. \square

Remark 3.5. In general, we say that a representation $G(k)$ on a k -vector space is *locally finite* if any element $v \in V$ is contained in a $G(k)$ -stable finite-dimensional subspace of V . By the discussion so far, for any affine algebraic group G , the representation (either left-translation or right-translation) of $G(k)$ on the coordinate ring $k[G]$ is locally finite.

3.3. Proof of the linearity of affine algebraic groups. Now let us prove Proposition 3.1.

Proof of Proposition 3.1. Let G be an affine algebraic group with coordinate ring $k[G]$. We let α be the action of G on G given by $g \cdot x := xg^{-1}$.

Since $k[G]$ is finitely generated, we can write $k[G] = k[f_1, \dots, f_n]$ with $f_i \in k[G]$. Here, by Lemma 3.3, we may assume that $V := \text{Span}_k\{f_1, \dots, f_n\}$ is $G(k)$ -stable. (Indeed, otherwise, by applying Lemma 3.3 to the subspace V , we can find a $G(k)$ -stable finite-dimensional subspace V' which contains V , hence generates $k[G]$ as a k -algebra. Hence, it is enough to replace f_1, \dots, f_n with any k -basis of V' .)

As V is $G(k)$ -stable, $\alpha^*(V) \subset k[G] \otimes_k V$ (Lemma 3.3). Note that, this is equivalent to that $m^*(V) \subset V \otimes_k k[G]$, where m denotes the multiplication morphism $G \times_k G \rightarrow G$. Indeed, this can be checked by noting that the following diagram is commutative, where i is the inversion morphism and sw denotes the morphism swapping the first and second entries:

$$\begin{array}{ccc} G \times G & \xrightarrow{\alpha} & G \\ i \times \text{id} \downarrow & & \uparrow m \\ G \times G & \xrightarrow{\text{sw}} & G \times G \end{array} \quad \begin{array}{ccc} (g, x) & \longmapsto & xg^{-1} \\ \downarrow & & \uparrow \\ (g^{-1}, x) & \longmapsto & (x, g^{-1}) \end{array}$$

Hence, for each $1 \leq i \leq n$, we can write

$$m^*(f_i) = \sum_{j=1}^n f_j \otimes h_{ji} \in V \otimes_k k[G]$$

with $h_{ji} \in k[G]$ ($j = 1, \dots, n$). Note that then, for each $g \in G(k)$, we have

$$\begin{aligned}\rho_g(f_i) &= ((g^{-1})^* \otimes \text{id}) \circ \alpha^*(f_i) \\ &= (g^* \otimes \text{id}) \circ (i^* \otimes \text{id}) \circ \alpha^*(f_i) \\ &= (g^* \otimes \text{id}) \circ \text{sw}^* \circ m^*(f_i) = \sum_{j=1}^n h_{ji}(g) \cdot f_j.\end{aligned}$$

In other words, for any $g \in G(k)$, the matrix $(h_{ji}(g))_{ji}$ represents the k -linear automorphism ρ_g of V with respect to the basis $\{f_1, \dots, f_n\}$. In particular, this shows that the resulting map

$$G(k) \rightarrow \text{GL}_n(k): g \mapsto (h_{ji}(g))_{ji}$$

is a group homomorphism.

Recall that the coordinate ring of GL_n is $k[x_{ij}, D^{-1} \mid 1 \leq i, j \leq n]$, where $D := \det(x_{ij})_{1 \leq i, j \leq n}$. Using the coefficients h_{ij} 's, we define a k -algebra homomorphism

$$\iota^*: k[\text{GL}_n] \rightarrow k[G]$$

by $\iota^*(x_{ji}) := h_{ji}$. Then the corresponding morphism of algebraic varieties $G \rightarrow \text{GL}_n$ obviously realizes the above map $G(k) \rightarrow \text{GL}_n(k): g \mapsto (h_{ji}(g))_{ji}$ at the level of k -rational points. Since ι is thus a group homomorphism on k -rational points, the density of k -rational points implies that ι is a homomorphism between algebraic groups (see Exercise below).

Hence, to complete the proof, it suffices to check that ι^* is surjective. For this, let us evaluate the function $\rho_g(f_i) = \sum_{j=1}^n h_{ji}(g) \cdot f_j \in k[G]$ at the unit element $e \in G(k)$. From the left-hand side, we get

$$\rho_g(f_i)(e) = f_i(e \cdot g) = f_i(g).$$

From the right-hand side, we get

$$\sum_{j=1}^n h_{ji}(g) \cdot f_j(e).$$

In other words, as a function on $G(k)$, we can express f_i as k -linear combination of h_{ji} 's. Since f_j 's generate $k[G]$ as a k -algebra, this means that ι^* is surjective. \square

Exercise 3.6. Let G and H be affine algebraic groups. Let $f: G \rightarrow H$ be a morphism as algebraic varieties. Prove that, if the induced map $G(k) \rightarrow H(k)$ on k -rational points is a group homomorphism, then f is a homomorphism of algebraic groups. (Hint: Use that the diagonal subset $\Delta X := \{(x, x) \in X \times_k X\}$ of $X \times_k X$ is closed for any affine scheme X over k (any affine scheme is “separated”).)

Remark 3.7. From the above proof, we can see that the infinite-dimensional representation $\rho_{(-)}$ of $G(k)$ on $k[G]$ is faithful, i.e., $\rho_{(g)}$ is not trivial for any $g \neq e$. Indeed, if ρ_g is trivial, then its restriction to $V \subset k[G]$ is also trivial, where V is as in the above proof. However, this means that the image of g under the closed immersion $\text{GL}_n \hookrightarrow \text{GL}_n$ constructed above is trivial, hence $g = e$.

3.4. Jordan decomposition for GL_n .

Definition 3.8. Let g be an element of $\mathrm{GL}_n(k)$. We say that g is

- (1) *semisimple* if g is diagonalizable in $\mathrm{GL}_n(\bar{k})$;
- (2) *nilpotent* if all the eigenvalues of g are 0 (equivalently, g^n is zero),
- (3) *unipotent* if all the eigenvalues of g are 1 (equivalently, $g - I_n$ is nilpotent).

Proposition 3.9. For any $g \in \mathrm{GL}_n(k)$, there exists a unique decomposition $g = g_s + g_n$ such that

- $g_s g_n = g_n g_s$,
- $g_s \in \mathrm{GL}_n(k)$ is semisimple, and
- $g_n \in \mathrm{GL}_n(k)$ is nilpotent.

Proof. We regard $g \in \mathrm{GL}_n(k)$ as an endomorphism of $V := k^{\oplus n}$. We let $\{\alpha_1, \dots, \alpha_r\}$ be the set of eigenvalues of g . Recall that the generalized eigenspace of g with respect to its eigenvalue α_i is defined by

$$V_i := \mathrm{Ker}(g - \alpha_i \cdot I_n)^{n_i},$$

Then, by the theorem of Cayley–Hamilton, we have $V = \bigoplus_{i=1}^r V_i$.

We put $g_i := g|_{V_i} \in \mathrm{End}_k(V_i)$. If we put $g_{i,s} := \alpha_i \cdot I_{\dim V_i}$ and $g_{i,n} := g_i - g_{i,s}$, then we have

- $g_{i,s}$ is semisimple,
- $g_{i,n}$ is nilpotent, and
- $g_{i,s} g_{i,n} = g_{i,n} g_{i,s}$.

Thus, by putting $g_s := \bigoplus_{i=1}^r g_{i,s}$ and $g_n := \bigoplus_{i=1}^r g_{i,n}$, we get a decomposition $g = g_s + g_n$ satisfying the desired conditions.

To check the uniqueness of such a decomposition, suppose that we have another such decomposition $g = g'_s + g'_n$. Then, since g'_s commutes with g , g'_s preserves each V_i . By noting that $g_i - (g'_s)|_{V_i} = (g'_n)|_{V_i}$, which is nilpotent, we see that g and g'_s have the same eigenvalues on V_i . As g'_s is semisimple, this implies that g'_s must be equal to $\alpha_i \cdot I_{\dim V_i}$. Hence we also get $g_n = g'_n$. \square

We call the decomposition $g = g_s + g_n$ the *additive Jordan decomposition* of g .

Corollary 3.10. For any $g \in \mathrm{GL}_n(k)$, there exists a unique decomposition $g = g_s g_u$ such that

- $g_s g_u = g_u g_s$,
- g_s is semisimple, and
- g_u is unipotent.

Proof. Let $g = g_s + g_n$ be the additive Jordan decomposition of g . Then we have $g = g_s(I_n + g_s^{-1}g_n)$. Since $g_s^{-1}g_n$ is nilpotent (use that g_s and g_n commute), $I_n + g_s^{-1}g_n$ is unipotent. Let us put $g_u := I_n + g_s^{-1}g_n$. As g_s commutes with g_u , $g = g_s g_u$ is a desired decomposition.

To check the uniqueness, let us assume that $g = g'_s g'_u$ is another such decomposition. Then, by putting $g'_n := g'_s(g'_u - I_n)$, we get the additive Jordan decomposition $g = g'_s + g'_n$. By the uniqueness of the additive Jordan decomposition, we have $g'_s = g_s$ and $g'_u = g_u$. \square

We call the decomposition $g = g_s g_u$ the *Jordan decomposition* of g .

3.5. Jordan decomposition for linear algebraic groups. We next investigate the Jordan decomposition for linear algebraic groups. The key is to consider the right-translation action of $G(k)$ on $k[G]$.

Let G be a linear algebraic group. Recall that the right-translation action $\rho_{(-)}$ of $G(k)$ on $k[G]$ is locally finite. Hence, we can write $V = \sum_i V_i$ with $G(k)$ -stable finite-dimensional subspaces V_i of $k[G]$. For any $g \in G(k)$, since the restriction of ρ_g to V_i is an element of $\mathrm{GL}_k(V_i)$, we can consider its Jordan decomposition $\rho_g|_{V_i} = (\rho_g|_{V_i})_s (\rho_g|_{V_i})_u$. Since the Jordan decomposition is unique, these decompositions are consistent on any intersection $V_i \cap V_j$, hence defines a decomposition $\rho_g = (\rho_g)_s (\rho_g)_u$ of ρ_g (i.e., $(\rho_g)_s$ and $(\rho_g)_u$ are unique automorphisms of $k[G]$ such that $(\rho_g)_s|_{V_i} = (\rho_g|_{V_i})_s$ and $(\rho_g)_u|_{V_i} = (\rho_g|_{V_i})_u$ for each i). Moreover, again by the uniqueness of the Jordan decomposition, we also see that this definition of $(\rho_g)_s$ and $(\rho_g)_u$ does not depend on the choice of the decomposition $k[G] = \sum_i V_i$. We also call this decomposition the *Jordan decomposition* of ρ_g .

Before we proceed, let us note the following:

Lemma 3.11. *The right-translation representation is faithful, i.e., the homomorphism $\rho_{(-)}: G(k) \rightarrow \mathrm{Aut}_k(k[G])$ is injective.*

Proof. Let us show the kernel of ρ is trivial. For this, we suppose that $g \in G(k)$ satisfies $\rho_g = \mathrm{id}$. Then, for any $f \in k[G]$, we have $f = \rho_g(f)$. In other words, we have $f(x) = f(xg)$ for any $x \in G(k)$. Since this holds for any $f \in k[G]$, we necessarily have $x = xg$. Hence $g = e$. \square

Also, note that the Jordan decomposition for GL_n is consistent with the Jordan decomposition of the corresponding right-translation action in the following sense:

Proposition 3.12. *Let $G = \mathrm{GL}_n$. Let g be an element of $G(k)$ with Jordan decomposition $g = g_s g_u$. Let $\rho_g = (\rho_g)_s (\rho_g)_u$ be the Jordan decomposition of $\rho_g \in \mathrm{Aut}_k(k[G])$. Then $(\rho_g)_s = \rho_{g_s}$ and $(\rho_g)_u = \rho_{g_u}$. Moreover, g_s and g_u are characterized as the unique elements of $G(k)$ satisfying this condition.*

Proof. Since $\rho_{(-)}$ is a group homomorphism, we have $\rho_g = \rho_{g_s} \rho_{g_u} = \rho_{g_u} \rho_{g_s}$. Thus, by the uniqueness of the Jordan decomposition of ρ_g , the task is to show that ρ_{g_s} is semisimple and ρ_{g_u} is unipotent. This is actually not quite obvious, but please allow me to omit in this course; see, e.g., [Bor91, §4.3]. \square

Now let us discuss the Jordan decomposition for general linear algebraic groups.

Theorem 3.13. *Let G be a linear algebraic group. For any $g \in G(k)$, let $\rho_g = (\rho_g)_s (\rho_g)_u$ be the Jordan decomposition of $\rho_g \in \mathrm{Aut}_k(k[G])$. There exist unique elements $g_s, g_u \in G(k)$ such that $g = g_s g_u = g_u g_s$, $(\rho_g)_s = \rho_{g_s}$, and $(\rho_g)_u = \rho_{g_u}$.*

Proof. We first note that, if such elements g_s and g_u really exist, then they must be unique by the faithfulness of $\rho_{(-)}$ and the conditions $(\rho_g)_s = \rho_{g_s}$ and $(\rho_g)_u = \rho_{g_u}$.

Let us take a closed immersion $\iota: G \hookrightarrow \mathrm{GL}_n$ for some $n \in \mathbb{Z}_{>0}$. We let $\iota^*: k[\mathrm{GL}_n] \twoheadrightarrow k[G]$ be the corresponding surjective k -algebra homomorphism. Let $I \subset k[\mathrm{GL}_n]$ be the kernel of ι^* . Then, via ι , $G(k)$ identified with the subgroup $\{g \in \mathrm{GL}_n(k) \mid f(g) = 0 \text{ for any } f \in I\}$ of $\mathrm{GL}_n(k)$. Conversely, $I = \{f \in k[\mathrm{GL}_n] \mid f(G(k)) = 0\}$. This observation implies that $G(k)$ can be also thought of as the subgroup $\{g \in \mathrm{GL}_n(k) \mid \rho_g(I) = I\}$. Moreover, we see that the right-translation actions for $G(k)$ and $\mathrm{GL}_n(k)$ are consistent under the maps ι and ι^* in the sense

that the following holds for any $g \in G(k)$:

$$\rho_{\iota(g)} = \rho_g.$$

Here, the right-hand side is an element of $\text{Aut}_k(k[G])$ while the left-hand side is a priori an element of $\text{Aut}_k(k[\text{GL}_n])$ but regarded as an element of $\text{Aut}_k(k[G])$ by noting that $\rho_g(I) = I$.

$$\begin{array}{ccc} G(k) & \xrightarrow{\iota} & \text{GL}_n(k) \\ \rho \downarrow & \circlearrowleft & \downarrow \rho \\ \text{Aut}_k(k[G]) & \xleftarrow{\iota^*} & \text{Aut}_k(k[\text{GL}_n]) \end{array}$$

We take the Jordan decomposition $\iota(g) = \iota(g)_s \iota(g)_u$ of $\iota(g) \in \text{GL}_n(k)$. Then, $\iota(g)_s$ and $\iota(g)_u$ belong to $G(k)$. Indeed, by the discussion in the previous paragraph, $\rho_{\iota(g)}$ preserves I . This implies that $\rho_{\iota(g)}|_I$ also has a Jordan decomposition and its nothing but the restrictions of $\rho_{\iota(g)_s}$ and $\rho_{\iota(g)_u}$ to I (recall the construction of the Jordan decomposition for locally finite automorphisms). In particular, $\rho_{\iota(g)_s}$ and $\rho_{\iota(g)_u}$ also preserve I , hence $\iota(g)_s$ and $\iota(g)_u$ belong to $G(k)$.

Let us put $g_s := \iota^{-1}(\iota(g)_s)$ and $g_u := \iota^{-1}(\iota(g)_u)$. The remaining thing is to check that $(\rho_g)_s = \rho_{g_s} \in \text{Aut}_k(k[G])$, and $(\rho_g)_u = \rho_{g_u} \in \text{Aut}_k(k[G])$. By this definition, $\rho_{g_s} \in \text{Aut}_k(k[G])$ is nothing but the automorphism induced by $\rho_{\iota(g_s)} = \rho_{\iota(g)_s} \in \text{Aut}_k(k[\text{GL}_n])$ (recall the discussion above the diagram). On the other hand, $(\rho_g)_s$ is the automorphism induced by $(\rho_{\iota(g)})_s \in \text{Aut}_k(k[\text{GL}_n])$. Since we have $\rho_{\iota(g)_s} = (\rho_{\iota(g)})_s$ by the previous proposition, this completes the proof. \square

We call the decomposition $g = g_s g_u$ as in the above theorem the *Jordan decomposition* of g .

Proposition 3.14. *The Jordan decomposition is preserved under any homomorphism of linear algebraic groups. To be more precise, let G and H be linear algebraic groups and $f: G \rightarrow H$ a homomorphism of algebraic groups. Then, for any $g \in G(k)$ with Jordan decomposition $g = g_s g_u$, the Jordan decomposition of $f(g) \in H(k)$ is given by $f(g) = f(g_s) f(g_u)$.*

Proof. Recall that the image of any homomorphism of algebraic groups is closed (Corollary 2.15). Hence, we may decompose any homomorphism into a surjective homomorphism and a closed immersion. Thus it suffices to show the claim only for the homomorphisms of these two types. In the case where f is a closed immersion, we can check the claim by a routine argument. In the case where f is surjective, we only need one additional fact that the corresponding k -algebra homomorphism $f^*: k[H] \rightarrow k[G]$ is injective (this is a general property of “dominant” morphisms between reduced affine schemes), but the remaining part is the same. \square

Exercise 3.15. We consider the following matrix:

$$g := \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}.$$

- (1) Compute the Jordan decomposition of g when g is regarded as an element of $\text{GL}_2(\mathbb{C})$.
- (2) Compute the Jordan decomposition of g when g is regarded as an element of $\text{GL}_2(\overline{\mathbb{F}}_p)$.

4. WEEK 4: UNIPOTENT GROUPS, SOLVABLE GROUPS, NILPOTENT GROUPS

4.1. Simultaneous Jordan decomposition for commutative groups. As usual, let k be an algebraically closed field. We consider a linear algebraic group G (over k) with coordinate ring $k[G]$. Let $\rho(-)$ denote the right-translation action of $G(k)$ on $k[G]$. Then, as ρ_g is a locally finite automorphism of $k[G]$, we can take its Jordan decomposition $\rho_g = (\rho_g)_s(\rho_g)_u$.

In the last week, we proved that any element $g \in G(k)$ has a unique decomposition $g = g_s g_u$ called the Jordan decomposition of g . Here, $g_u \in G(k)$ and $g_s \in G(k)$ are characterized by the following properties:

- $g = g_u g_s = g_s g_u$,
- $\rho_{g_s} = (\rho_g)_s$,
- $\rho_{g_u} = (\rho_g)_u$.

Definition 4.1. For $g \in G(k)$, we say that g is

- *semisimple* if $g = g_s$, and
- *unipotent* if $g = g_u$.

Lemma 4.2. For $g \in G(k)$, g is semisimple (resp. unipotent) if and only if $\iota(g)$ is semisimple (resp. unipotent) for some embedding $G \hookrightarrow \mathrm{GL}_n$. This is furthermore equivalent to that $\iota(g)$ is semisimple (resp. unipotent) for any embedding $G \hookrightarrow \mathrm{GL}_n$.

Proof. Recall that the Jordan decomposition is preserved under any group homomorphism. In particular, for any embedding ι , g is semisimple (resp. unipotent) if and only if so is $\iota(g)$. \square

Let us consider the subsets

$$G(k)_s := \{g \in G(k) \mid g = g_s\} \quad \text{and} \quad G(k)_u := \{g \in G(k) \mid g = g_u\}.$$

We call $G(k)_s$ and $G(k)_u$ the *semisimple locus* (resp. *unipotent locus*) of $G(k)$.

Note that if we fix an embedding $\iota: G \hookrightarrow \mathrm{GL}_n$, then, for $g \in G(k)$, g belongs to $G(k)_u$ if and only if $(\iota(g) - I_n)^n = 0$. The equation $(\iota(g) - I_n)^n = 0$ can be expressed via polynomial in the coordinates of GL_n (i.e., entries of n -by- n matrices). In particular, we see that $G(k)_u$ is the set of k -rational points of a closed subset G_u of G . We also call G_u the unipotent locus of G . On the other hand, note that $G_s(k)$ cannot be thought of as k -rational points of a closed subset in general. Also, note that $G(k)_s$ and $G(k)_u$ are not subgroups of $G(k)$ in general.

However, when G is commutative, the following particular fact holds:

Theorem 4.3. Let G be a commutative linear algebraic group. Then G_u is a closed subgroup of G and $G_s(k)$ is also regarded as the group of k -rational points of a closed subgroup G_s of G . Furthermore, the multiplication map $m: G_s \times G_u \rightarrow G$ is an isomorphism of algebraic groups.

Proof. The commutativity implies that $G(k)_u$ is a subgroup of $G(k)$, hence G_u is a subgroup of G as an algebraic group. (In general, it can be easily checked that a closed subvariety of an algebraic group is closed under the multiplication and inversion at the level of k -rational points, then it is closed under the multiplication and inversion; this follows from the density of closed points.)

On the other hand, it is a basic fact in linear algebra that any commutative family of semisimple (diagonalizable) elements of $\mathrm{GL}_n(k)$ can be simultaneously diagonalized. Thus, we can find an embedding $\iota: G \hookrightarrow \mathrm{GL}_n$ such that the image

of $G(k)_s$ is contained in the diagonal subgroup $T_n(k) \subset \mathrm{GL}_n(k)$ by replacing ι by $\mathrm{GL}_n(k)$ -conjugation if necessary. Since any element of $T_n(k)$ is semisimple, we then have $G(k)_s = \iota^{-1}(T_n(k)) = \iota^{-1}(T_n)(k)$. In other words, if we define $G_s := \iota^{-1}(T_n)$, then G_s is a closed subgroup of G whose k -rational points realize $G(k)_s$.

Now let us check that $G_s \times G_u \rightarrow G$ is an isomorphism of algebraic groups. (At least we already know that this is bijective at the level of k -rational points; but we have to show that this homomorphism has an inverse homomorphism as algebraic groups.) For this, we again appeal to a more genral fact in linear algebra, that is, any commutative family of elements of $\mathrm{GL}_n(k)$ can be simultaneously triangulated. In other words, if we let B_n be the closed subgroup of GL_n consisting of upper-triangular matrices, then we may suppose that the image of ι is contained in B_n . We define a closed subgroup U_n of B_n to be $(B_n)_u$, i.e., upper-triangular unipotent matrices. Then G_s and G_u are nothing but the preimages of T_n and U_n under ι . For the morphism $T_n \times U_n \rightarrow B_n$ we can check that it is an isomorphism of algebraic varieties, hence we have its inverse. Under ι , this inverse induces the inverse of $G_s \times G_u \rightarrow G$. \square

4.2. Simultaneous triangulation for unipotent groups.

Definition 4.4. We say that a linear algebraic group G is *unipotent* if $G = G_u$.

Remark 4.5 (CAUTION). Later, we will introduce the notion of a “semisimple group”, but it does not mean $G = G_s$. In fact, the condition $G = G_s$ is quite strong so that it implies G is commutative.

Theorem 4.6. *Let G be a closed unipotent subgroup of GL_n . Then G is conjugate to a subgroup of U_n .*

Proof. It is enough to show the assertion at the level of k -rational points. Let $V = k^{\oplus n}$ and identify $\mathrm{GL}_n(k) \cong \mathrm{Aut}_k(V)$. We appeal to the induction on dimension of V . The assertion is clear when $\dim_k V = 1$, thus let us consider the case where $\dim_k V > 1$.

It is enough to find an element $v \in V \setminus \{0\}$ which is a simultaneous eigenvector for all elements of G (i.e., $g \cdot v = v$ for any $g \in G(k)$). Indeed, if we can do this, then we can apply the induction hypothesis to V/kv .

If V is not irreducible as a representation of $G(k)$ (i.e., there exists a nonzero proper subspace $0 \subsetneq W \subsetneq V$), then we can apply the induction hypothesis of W . Especially, we can find a simultaneous eigenvector in W . Thus, the essential case is when the action of $G(k)$ on V is irreducible.

The key fact is the following (so-called “Burnside’s theorem”):

Let V be a finite dimensional k -vector space over an algebraically closed field k . Let $\rho: G \rightarrow \mathrm{Aut}_k(V)$ be an irreducible representation of a group G (in the abstract sense). Then we have $\mathrm{End}_k(V) = \mathrm{Span}_k\{\rho(g) \mid g \in G\}$.

Now we claim that $G(k) = \{I_n\}$. For the sake of contradiction, we assume that $g \in G(k)$ such that $g \neq I_n$ and write $g = I_n + h$ (hence h is a nilpotent matrix). For any $x \in G(k)$, we have

$$\mathrm{Tr}(hx) = \mathrm{Tr}((g - I_n)x) = \mathrm{Tr}(gx) - \mathrm{Tr}(x) = 0$$

(in the last equality, note that gx and x are elements of $G(k)$, hence unipotent; in particular, their traces are $\dim_k(V)$). Since $\mathrm{End}_k(V) = \mathrm{Span}_k\{\rho(g) \mid g \in G\}$ by

the Burnside's theorem, this implies that $\text{Tr}(hx) = 0$ for any $x \in \text{End}_k(V)$. But this forces that $x = 0$.

As $G(k) = \{I_n\}$, the dimension of its irreducible representation V must be 1. This completes the proof. \square

Corollary 4.7. *Let G be a unipotent algebraic group. Then G is isomorphic to a closed subgroup of U_n for some n .*

Proof. We choose any embedding ι of G into GL_n . Then apply the previous theorem to $\iota(G) \subset \text{GL}_n$ and replace ι with the suitably conjugated one. \square

4.3. Solvable and Nilpotent groups. We next introduce the notions of “solvable” and “nilpotent” algebraic groups. Let us first recall their definitions in the context of abstract group theory.

Definition 4.8. Let G be an abstract group. For any subgroups H_1 and H_2 of G , we let $[H_1, H_2]$ denote the subgroup of G generated by (i.e., the smallest subgroup containing) $\{h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2\}$.

- (1) We put $D^0 G = G$ and $D^{n+1} G := [D^n G, D^n G]$. If $D^n G = \{e\}$ for $n \gg 0$, we say that G is *solvable*.
- (2) We put $C^0 G = G$ and $C^{n+1} G := [G, C^n G]$. If $C^n G = \{e\}$ for $n \gg 0$, we say that G is *nilpotent*.

(Note that $D^n G \subset C^n G$, in particular, any nilpotent group is solvable.)

Now let G be a linear algebraic group. We define the solvability and the nilpotency for G in the exactly same way as above by replacing “the smallest subgroup containing” with “the smallest algebraic subgroup containing” and also “ $\{h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2\}$ ” with “the image of $H_1 \times H_2$ under the commutator morphism $G \times G \rightarrow G: (x, y) \mapsto xyx^{-1}y^{-1}$ ”.

Note that $[H_1(k), H_2(k)]$ is dense in $[H_1, H_2]$. This implies that $D^n(G(k))$ and $C^n(G(k))$ are dense in $D^n(G)$ and $C^n(G)$, respectively. In particular, in order to check if a given algebraic group is solvable or nilpotent, it suffices to look at only k -rational points.

- Example 4.9.**
- (1) Any commutative linear algebraic group G satisfies $[G, G] = \{e\}$, hence must be nilpotent. In particular, \mathbb{G}_a and \mathbb{G}_m are nilpotent.
 - (2) The closed subgroup B_n of GL_n consisting of upper-triangular matrices is solvable. This is well-known at the level of k -rational points. Note that B_n is not nilpotent.
 - (3) The closed subgroup U_n of B_n consisting of upper-triangular unipotent matrices is nilpotent. This is well-known at the level of k -rational points.

Proposition 4.10. *Any unipotent linear algebraic group is nilpotent.*

Proof. Recall that any unipotent linear algebraic group can be embedded into $U_n \subset \text{GL}_n$. Then we have $C^m G \subset C^m U_n$. As U_n is nilpotent, we have $C^m U_n = \{e\}$ for $e \gg 0$, hence $C^m G = \{e\}$ for $e \gg 0$. \square

The following are generalizations of the results for unipotent groups which we introduced this week. (We will explain the proofs in the future.)

Theorem 4.11 (Lie–Kolchin’s theorem). *Let G be a connected closed subgroup of GL_n . If G is solvable, then G is conjugate to a subgroup of B_n .*

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