1. Week 4: Reductive groups

1.1. Definition of a reductive group.

Proposition/Definition 1.1 ([Spr09, 6.4.14])**.** Let G be a connected linear algebraic group over k.

(1) There uniquely exists a maximal closed connected normal solvable[1] subgroup of G defined over k, which is called the *radical* of G. We write $R(G)$ for the radical of G.
(2) There uniquely exists a maximal closed connected normal unipotent[2] subgroup of G defined over k, which is called the *unipotent radical* of G. We write $R_u(G)$ for the unipotent radical of G.

Definition 1.2 (semisimple/reductive groups)**.** Let G be a connected linear algebraic group over k.

(1) We say that G is *semisimple* if $R(G)$ is trivial.
(2) We say that G is *reductive* if $R_u(G)$ is trivial.

Remark 1.3. In general, any unipotent group is nilpotent, hence solvable (see [Spr09, 2.4.13]). In particular, $R_u(G)$ is contained in $R(G)$. This means that if G is semisimple, then G is reductive.

Remark 1.4. In general, $R_u(G_{\overline{k}})$ could be different from the base change of $R_u(G)$ from k to $\overline{k}$. This means that the condition that a connected linear algebraic G group over k is reductive in the above sense is not equivalent to the condition that $G_{\overline{k}}$ is reductive. However, such a phenomenon does not happen as long as k is perfect, i.e., we have $R_u(G)_{\overline{k}} = R_u(G_{\overline{k}})$ for any perfect k. In the situation where k is not perfect, a connected linear algebraic group over k with trivial $R_u(G)$ is called a *pseudo-reductive* group. See [CGP15, Section 1.1] for details.

The following proposition basically follows from the definition of being solvable/unipotent.

Proposition 1.5. *The unipotent radical $R_u(G)$ is the set of unipotent elements of $R(G)$.*

Proposition 1.6. *Let G be a connected reductive group over k.*

(1) The center $Z(G)$ of G is finite if and only if G is semisimple.
(2) The derived subgroup $G_{\mathrm{der}} := [G,G]$ is a connected semisimple group over k. Moreover, we have $G = Z(G) \cdot G_{\mathrm{der}}$.

Proof. See [Spr09, 7.3.1 and 8.1.6]. □

Now, let us introduce several practical propositions to determine the unipotent radical of a given connected reductive group. As mentioned above, the unipotent radical behaves consistently with the base change of the field k as long as it is perfect. Thus, in the rest of this section, let us assume that k is algebraically closed and omit the word "over k".

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[1] Solvability is defined in the same way as in the usual group theory, i.e., an algebraic group G is said to be solvable when $G_n = \{1\}$ for sufficiently large n, where $G_n := [G_{n-1}, G_{n-1}]$ and $G_1 := G$.

[2] i.e., all elements are unipotent

Definition 1.7 (Borel subgroup)**.** Let G be a linear algebraic group. A subgroup B of G is called a *Borel subgroup of G* if it is a maximal connected solvable closed subgroup of G.

Theorem 1.8 (Lie–Kolchin's theorem, [Spr09, 6.3.1])**.** *Let B be a connected solvable closed subgroup of GL_n. Let B_n be the group of upper triangular matrices of GL_n. Then B is conjugate to a subgroup of B_n.*

Note that, in particular, B_n is a Borel subgroup of GL_n by Lie–Kolchin's theorem.

Proposition 1.9. *Let G be a connected linear algebraic group. All Borel subgroups of G are conjugate.*

Proof. See [Spr09, 6.2.7]. □

Corollary 1.10. *Let G be a connected linear algebraic group. Then its radical $R(G)$ equals the identity component of the intersection of all Borel subgroups of G.*

Proof. By definition, $R(G)$ is contained in a Borel subgroup. Since $R(G)$ is normal in G and all Borel subgroups of G are conjugate, $R(G)$ is contained in the intersection of all Borel subgroups of G. As $R(G)$ is connected, it must be contained in the identity component of the intersection. Since the identity component of the intersection of all Borel subgroups of G is closed, connected, normal, and solvable, it must be equal to $R(G)$ by the maximality of $R(G)$. □

1.2. **Examples of reductive groups.**

Example 1.11 (tori)**.** Any torus T is reductive. Indeed, since T is commutative, hence solvable, $R(T)$ is T itself. Since all elements of T are semi-simple, $R_u(T)$ is trivial.

Example 1.12 (general linear group)**.** The general linear group GL_n is reductive. To check this, note that B_n is a Borel subgroup of GL_n, hence its any conjugate is also a Borel subgroup of GL_n. In particular, the opposite $\overline{B}_n$ (i.e., the subgroup of lower triangular matrices) is also Borel. Hence their intersection, which is the diagonal subgroup T of GL_n, must contain $R(\mathrm{GL}_n)$. This implies that all elements of $R(\mathrm{GL}_n)$ is semisimple, hence $R_u(\mathrm{GL}_n)$ is trivial.

Exercise 1.13. Prove that $R(\mathrm{GL}_n) = Z(\mathrm{GL}_n)$.

Example 1.14 (symplectic group)**.** The symplectic group Sp_{2n} is reductive. Indeed, if we put B to be $B_n \cap \mathrm{Sp}_{2n}$ (i.e., the subgroup of Sp_{2n} consisting of matrices of the upper-triangular form), then we can show that B is a Borel subgroup of Sp_{2n}. (See the following exercise.) Similarly, its opposite $\overline{B} := \overline{B}_n \cap \mathrm{Sp}_{2n}$ is also a Borel subgroup of Sp_{2n}, Thus the same argument as in the case of GL_n implies that $R_u(\mathrm{SO}(J))$ is trivial.

Example 1.15 (orthogonal group)**.** Let us assume that the characteristic of k is not 2. Let $J'_n \in \mathrm{GL}_n(k)$ be the anti-diagonal matrix whose anti-diagonal entries are given by 1. Then, by the same argument as in the previous case, we can show that the special orthogonal group $\mathrm{SO}_{2n} = \mathrm{SO}(J'_n)$ is reductive. (Note that, for any symmetric matrix J, the special orthogonal group $\mathrm{SO}(J)$ is reductive. But an explicit description of its Borel subgroups depends on the choice of J and more complicated.)

Example 1.16 (unitary group)**.** Let k' be a quadratic extension of k. Let σ be the nontrivial element of $\mathrm{Gal}(k'/k)$. Let $J \in \mathrm{GL}_n(k')$ be a hermitian matrix, i.e., ${}^t\sigma(J) = J$. We define the *unitary group* $\mathrm{U}(J)$ by

$$U(J)(R) := \{g \in \mathrm{GL}_n(R \otimes_k k') \mid {}^t\sigma(g)Jg = J\}.$$

(In particular, we have $U(J)(k) := \{g \in \mathrm{GL}_n(k') \mid {}^t\sigma(g)Jg = J\}$.) Then, by the same argument as in the previous cases, we can show that the special orthogonal group $\mathrm{U}(J)$ is reductive.

Exercise 1.17. Determine a Borel subgroup of Sp_{2n}.

1.3. **Classification of connected reductive groups via root data.** In the following (of this section), let k be an algebraically closed field. Over an algebraically closed field, isomorphism classes of connected reductive groups can be classified in terms of linear algebraic data called *root data*.

Theorem 1.18 ([Spr09, 9.6.2, 10.1.1])**.** *There exists a bijection between*

- *the set of isomorphism classes of connected reductive groups and*
- *the set of isomorphism classes of reduced root data.*

Let us introduce the definition of a root datum.

Definition 1.19 (root datum)**.** A *root datum* is a quadruple $(X, R, X^\vee, R^\vee)$, where

- X and $X^\vee$ are free abelian groups of finite rank equipped with a perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$ and
- R and $R^\vee$ are finite subsets of X and $X^\vee$ (called the sets of *roots* and *coroots*) equipped with a bijection $R \leftrightarrow R^\vee \colon \alpha \mapsto \alpha^\vee$

satisfying

1. for any $\alpha \in R$, we have $\langle \alpha, \alpha^\vee \rangle = 2$,
2. for any $\alpha \in R$, we have $s_\alpha(R) = R$ and $s_\alpha^\vee(R^\vee) = R^\vee$.

Here, s_α and $s_\alpha^\vee$ denote the automorphisms of X and $X^\vee$ given by

$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad s_\alpha^\vee(x^\vee) = x^\vee - \langle \alpha, x^\vee \rangle \alpha^\vee.$$

We say that a root datum $(X, R, X^\vee, R^\vee)$ is *reduced* if for any $\alpha \in R$, we have $R \cap \mathbb{Q}\alpha = \{\pm\alpha\}$.

In the following, we explain how to construct the map in Theorem 1.18. Thus our aim is to construct a root datum from a given connected reductive group G over k. There are several ways of explaining this procedure. Here, we follow [Car85, Section 1.9].

We first take a maximal torus T of G. We put $X := X^*(T)$ and $X^\vee := X_*(T)$. Note that then X and $X^\vee$ have a natural perfect pairing $\langle -, - \rangle \colon X \times X^\vee \to \mathbb{Z}$.

Suppose that U is a minimal nontrivial closed unipotent subgroup of G normalized by T. Then, in fact, U is isomorphic to $\mathbb{G}_\mathrm{a}$. By fixing an isomorphism $\iota \colon \mathbb{G}_\mathrm{a} \xrightarrow{\cong} U$, we get an element $\alpha \in X$ satisfying

$$t \cdot \iota(x) \cdot t^{-1} = \iota(\alpha(t) \cdot x)$$

for any $x \in \mathbb{G}_\mathrm{a}$. This element α is independent of the choice of ι. Furthermore, if U' is another (different to U) minimal nontrivial closed unipotent subgroup of G normalized by T, then the associated element of X is also different. Thus it makes sense to write U_α for U. We call α a *root of T in G* and U_α its *root subgroup*. We put R to be the set of roots of T in G.

It can be proved that $-\alpha$ is also a root when α is a root. Moreover, the subgroup $\langle U_\alpha, U_{-\alpha}\rangle$ generated by U_α and $U_{-\alpha}$ is isomorphic to SL_2 or $\mathrm{PGL}_2 := \mathrm{SL}_2/\{\pm 1\}$. Furthermore, in any case, there exists a homomorphism $\phi\colon \mathrm{SL}_2 \to \langle U_\alpha, U_{-\alpha}\rangle$ satisfying

$$\phi\left(\left(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix}\right)\right) = U_\alpha \quad \text{and} \quad \phi\left(\left(\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right)\right) = U_{-\alpha}.$$

This homomorphism ϕ maps any diagonal element of SL_2 into T. Thus, we can define a cocharacter $\alpha^\vee \in X^\vee$ by

$$\alpha^\vee(y) := \phi\left(\left(\begin{smallmatrix} y & 0 \\ 0 & y^{-1} \end{smallmatrix}\right)\right).$$

We call $\alpha^\vee$ the *coroot associated to* α. We put $R^\vee$ to be the set of all coroots obtained in this way.

Proposition 1.20. *For any connected reductive group G, the quadruple $(X, R, X^\vee, R^\vee)$ forms a reduced root datum.*

1.4. **Classification of reductive groups: more concrete version.**

Definition 1.21 (isogeny)**.** We say that a homomorphism $f\colon G \to G'$ of algebraic groups is an *isogeny* if it is surjective and has finite kernel. We say that two algebraic groups G and G' are *isogenous* if there exists an isogeny between G and G'.

Recall that, any connected reductive group G can be written as $G = Z(G) \cdot G_{\mathrm{der}}$, where G_{der} is semisimple. Especially, we have a surjective homomorphism $f\colon Z(G) \times G_{\mathrm{der}} \to G\colon (z, g) \mapsto zg$. Since $Z(G) \cap G_{\mathrm{der}}$ is contained in $Z(G_{\mathrm{der}})$, which is finite, f is an isogeny. In other words, any connected reductive group is realized as the quotient of $Z(G) \times G_{\mathrm{der}}$ by its finite subgroup. Thus, let us discuss how to classify semisimple groups in the following. (Being semisimple can be expressed in terms of root data: a connected reductive group G is semisimple if and only if R spans $X_{\mathbb{Q}} := X \otimes_{\mathbb{Z}} \mathbb{Q}$ as a $\mathbb{Q}$-vector space.)

We say that a semisimple group G is *adjoint* if its center $Z(G)$ is trivial. In fact, for any semisimple group G, its quotient $G/Z(G)$ is the unique adjoint group isogenous to G; this is denoted by G_{ad}. The adjoint quotient G_{ad} is a semisimple group whose center is minimal (trivial) among all semisimple groups isogenous to G.

On the other hand, for any semisimple group G, there uniquely exists a semisimple group "G_{sc}" such that any isogeny to G can be lifted to an isogeny from G_{sc} to G; this group is called *the simply-connected cover of* G. The simply-connected cover G_{sc} is a semisimple group whose center is maximal among all semisimple groups isogenous to G.

$$\begin{tikzcd}
G_{\mathrm{sc}} \arrow[d, two heads] \arrow[dr, two heads] & & \\
G' \arrow[r, two heads] & G \arrow[r, two heads] & G_{\mathrm{ad}}
\end{tikzcd}$$

Proposition 1.22. *Let G be a semisimple group.*

1. *We say that G is simply-connected if $R^\vee$ spans $X^\vee$ over $\mathbb{Z}$.*
2. *We say that G is adjoint if R spans X over $\mathbb{Z}$.*

Example 1.23. Let $G := \mathrm{GL}_n$ and Z be its center. We put $\mathrm{SL}_n := \{g \in G \mid \det(g) = 1\}$ and $\mathrm{PGL}_n := \mathrm{GL}_n / Z$.[3] Then we obviously have a natural map $\mathrm{SL}_n \to \mathrm{PGL}_n$, which is surjective. Moreover, this map has finite kernel; it is given by $\{z \in Z \mid \det(z) = 1\}$, which is isomorphic to the group of n-th roots of unity. Hence $\mathrm{SL}_n \to \mathrm{PGL}_n$ is an isogeny. On the other hand, the quotient map $\mathrm{GL}_n \twoheadrightarrow \mathrm{PGL}_n$ is not an isogeny since its kernel is given by Z, which is not finite. In fact, SL_n is simply-connected and PGL_n is adjoint.

Definition 1.24 (almost simple group)**.** We say that a semisimple group G is *almost simple* if it does not contain any nontrivial closed normal subgroup of positive dimension.

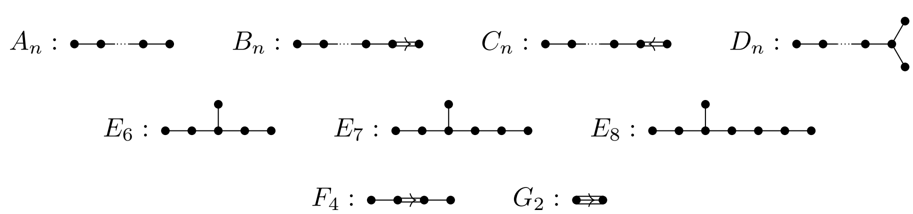
Proposition 1.25. *Let G be a simply-connected (resp. adjoint) group. Then G is written as a product of almost simple simply-connected (resp. adjoint) subgroups.*

Definition 1.26. We say that a root datum $\Psi = (X, R, X^\vee, R^\vee)$ is *reducible* if there exist nonzero root data $\Psi_1 = (X_1, R_1, X_1^\vee, R_1^\vee)$ and $\Psi_2 = (X_2, R_2, X_2^\vee, R_2^\vee)$ such that $\Psi = \Psi_1 \oplus \Psi_2$ (in the obvious sense) and Ψ_1 and Ψ_2 are orthogonal. We say that Ψ is *irreducible* if it is not reducible.

Proposition 1.27. *Let G be an almost simple simply-connected (or adjoint) group with root data Ψ. Then G is almost simple if and only if Ψ is irreducible.*

By the discussion so far, the classification problem of semisimple groups is now reduced ("modulo isogeny") to classifying all almost simple simply-connected subgroups. Moreover, it is equivalent to classifying all irreducible reducible root data such that $R^\vee$ spans $X^\vee$.

The miraculous fact is that there are very limited number of such groups! Such groups can be parametrized by combinatorial objects called *Dynkin diagrams*. Among them, the types A_n, B_n, C_n, and D_n are called *classical types*, and the types E_6, E_7, E_8, F_4, and G_2 are called *exceptional types*.



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[CGP15] B. Conrad, O. Gabber, and G. Prasad, *Pseudo-reductive groups*, second ed., New Mathematical Monographs, vol. 26, Cambridge University Press, Cambridge, 2015.

[3]Here, the quotient is taken as an algebraic group. In general, for any linear algebraic group G and its closed subgroup H over k, we can define and prove the existence of the quotient of G by H (see [Spr09, 5.5]). One difficult point to care about is that $(G/H)(R)$ might not be equal to $G(R)/H(R)$. (But at least we have the equality for $R = \overline{k}$. Thus, in this example, we may think of $\mathrm{PGL}_n(\overline{k})$ as the quotient of $\mathrm{GL}_n(\overline{k})$ by its center.)

[Spr09] T. A. Springer, *Linear algebraic groups*, second ed., Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2009.

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