1. Week 6: Deligne–Lusztig representations

1.1. **Quick overview of étale cohomology.** In the following, we quickly introduce the basic properties of the étale cohomology for algebraic varieties. (Here, we do not even give the definition of the étale cohomology. Carter's book [Car85, Appendix] has a beautiful summary of the étale cohomology theory, so please look at it if you want to know more about some details.)

Let us briefly recall the notion of *ℓ-adic numbers*. Let ℓ be a prime number. We consider the inverse system of finite rings

$$\cdots \to \mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^{n}\mathbb{Z} \to \cdots \to \mathbb{Z}/\ell^{2}\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z},$$

where the transition map $\mathbb{Z}/\ell^{n+1}\mathbb{Z} \to \mathbb{Z}/\ell^{n}\mathbb{Z}$ is given by the natural surjection. The inverse limit of this system forms a ring, which is called the *ring of ℓ-adic integers* and denoted by $\mathbb{Z}_\ell$:

$$\mathbb{Z}_\ell := \varprojlim_{n} \mathbb{Z}/\ell^{n}\mathbb{Z} := \{(x_n)_n \in \prod_{n\geq 1} \mathbb{Z}/\ell^{n}\mathbb{Z} \mid \overline{x_{n+1}} = x_n\}.$$

Since $\mathbb{Z}_\ell$ is an integral domain, it makes sense to consider its fractional field; it is called the *field of ℓ-adic numbers* and denoted by $\mathbb{Q}_\ell$. [1]

Lemma 1.1. *Let $\overline{\mathbb{Q}}_\ell$ be an algebraic closure of $\mathbb{Q}_\ell$. Then $\overline{\mathbb{Q}}_\ell$ is isomorphic to the complex number field $\mathbb{C}$ as an abstract field.* [2]

Exercise 1.2. Prove this lemma. Hint: note that both $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ are algebraically closed fields of characteristic 0 and the same cardinality.

Now let k be a finite field $\mathbb{F}_q$ of characteristic $p > 0$. In the following, let ℓ be a prime number distinct to p. For any algebraic variety X over $\overline{k} = \overline{\mathbb{F}}_p$ and for each $i \in \mathbb{Z}_{\geq 0}$, we can associate a $\overline{\mathbb{Q}}_\ell$-vector space $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ called the *compactly supported (i-th) étale cohomology of X with $\overline{\mathbb{Q}}_\ell$-coefficient.* In this course, we simply refer to it by the *ℓ-adic cohomology of X*. [3]

It is known that $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ satisfies various "basic" properties. For a moment, let us introduce only the following:

Theorem 1.3. *(1) For any X, $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is finite-dimensional.*
(2) For any X, $H^i_c(X, \overline{\mathbb{Q}}_\ell) \neq 0$ only for $0 \leq i \leq 2\dim(X)$.
(3) For any morphism of algebraic varieties $f\colon X \to Y$ over $\overline{k}$, a $\overline{\mathbb{Q}}_\ell$-vector space homomorphism $f^*\colon H^i_c(Y, \overline{\mathbb{Q}}_\ell) \to H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is canonically (functorially) associated (for each i).

For references on these facts, see [Car85, Section 7.1].

Now suppose that X is an algebraic variety over k. Then we have the Frobenius endomorphism $F\colon X_{\overline{k}} \to X_{\overline{k}}$. Thus, by the functoriality, we also have an endomorphism F^* of $H^i_c(X, \overline{\mathbb{Q}}_\ell)$.

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[1] Another equivalent way of defining $\mathbb{Q}_\ell$ is to complete the rational number field $\mathbb{Q}$ with respect to the ℓ-adic distance. But the above definition seems better in this context because the ℓ-adic cohomology is defined by taking the limit of torsion coefficient ($\mathbb{Z}/\ell^n\mathbb{Z}$-coefficient) cohomologies.

[2] Note that, however, $\overline{\mathbb{Q}}_\ell$ and $\mathbb{C}$ cannot be topologically isomorphic.

[3] There is also the "(i-th) étale cohomology of X with $\overline{\mathbb{Q}}_\ell$-coefficient", so this terminology is a bit too abbreviated. But we do not mind because we only use the compactly supported one in this course.

Theorem 1.4 (Grothendieck–Lefschetz fixed point theorem)**.** *We have*

$$|X^F| = \sum_{i\geq 0}(-1)^i \operatorname{Tr}(F^* \mid H^i_c(X, \overline{\mathbb{Q}}_\ell)).$$

One of the important application of the fixed point theorem is the following ℓ-independence result: Suppose that X is furthermore equipped with an action of a finite group G. Then, by the functoriality of ℓ-adic cohomology, we obtain a representation of G on a finite-dimensional $\overline{\mathbb{Q}}_\ell$-vector space $g \mapsto (g^{-1})^*$. (Here it is better to take the inverse of g since $(-)^*$ is contravariant.) By abuse of notation, let us simply write "g" for the action $(g^{-1})^*$ on $H^i_c(X, \overline{\mathbb{Q}}_\ell)$.

Theorem 1.5. *Suppose that an element $g \in G$ satisfies $g \circ F = F \circ g$ as an endomorphism of $X_{\overline{k}}$. Then the number*

$$\sum_{i\geq 0}(-1)^i \operatorname{Tr}(g \mid H^i_c(X, \overline{\mathbb{Q}}_\ell))$$

is an integer independent of ℓ (called the "Lefschetz number" of g).

Proof. Here we need the fact that, for any $n \geq 1$, the endomorphism $g \circ F^n$ of $X_{\overline{k}}$ associated to another $\mathbb{F}_{q^n}$-rational structure of $X_{\overline{k}}$. Let us write X_n for the algebraic variety over $\mathbb{F}_{q^n}$ determined by this rational structure. Then $X^{g\circ F^n}$ is the set of $\mathbb{F}_{q^n}$-rational points of X_n, hence finite.

We first investigate the following formal series:

$$R(t) := -\sum_{n=1}^{\infty} |X^{g\circ F^n}| \cdot t^n \in \mathbb{Z}[\![t]\!] \subset \overline{\mathbb{Q}}_\ell(\!(t)\!).$$

Since g and F^* are commuting endomorphism of $V := \bigoplus_{i\geq 0} H^i_c(X, \overline{\mathbb{Q}}_\ell)$ (note that this is finite-dimensional), we can simultaneously triangulate g and F^*. Let $v_1, \ldots, v_k$ be a set of simultaneous eigenvectors ($d := \dim V$) with eigenvalues $\alpha_1, \ldots, \alpha_d \in \overline{\mathbb{Q}}_\ell$ for g^* and $\beta_1, \ldots, \beta_d \in \overline{\mathbb{Q}}_\ell$ for F^*. Here, we may assume that each v_j is contained in $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ for some i. For each $j = 1, \ldots, k$, we define a sign ϵ_j by

$$\epsilon_j := \begin{cases} 1 & \text{if } v_j \text{ is contained in an even degree cohomology,} \\ -1 & \text{if } v_j \text{ is contained in an odd degree cohomology.} \end{cases}$$

Then, by applying the fixed point formula to X_n over $\mathbb{F}_{q^n}$, we get

$$|X^{g\circ F^n}| = \sum_{j=1}^{d} \epsilon_j \alpha_j \beta_j^n.$$

Therefore, we get

$$\begin{aligned} R(t) = -\sum_{n=1}^{\infty} |X^{g\circ F^n}| \cdot t^n &= -\sum_{n=1}^{\infty}\sum_{j=1}^{d} \epsilon_j \alpha_j \beta_j^n \cdot t^n \\ &= -\sum_{j=1}^{d} \epsilon_j \alpha_j \sum_{n=1}^{\infty} \beta_j^n \cdot t^n \\ &= -\sum_{j=1}^{d} \epsilon_j \alpha_j \frac{\beta_j t}{1-\beta_j t} \in \overline{\mathbb{Q}}_\ell(t). \end{aligned}$$

In particular, $R(t)$ is a rational function which does not have a pole at $t=\infty$. Let us write $R(t)=p(t)/q(t)$ with polynomials $p(t),q(t)\in\overline{\mathbb{Q}}_\ell[t]$; then, by noting that $R(t)$ is initially given by a formal series with $\mathbb{Z}$-coefficients, we can easily check that the coefficients of $p(t)$ and $q(t)$ can be taken to be in $\mathbb{Q}$. In other words, we have $R(t)\in\mathbb{Q}(t)$.

On the other hand, we note that $R(\infty)$ is given by $\sum_{j=1}^{d}\epsilon_j\alpha_j$, which is nothing but $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))$. Since $R(t)$ is independent of ℓ (by its definition) and belongs to $\mathbb{Q}(t)$, we have that $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))$ is a rational number which is independent of ℓ. Moreover, since g is of finite order, $\alpha_j\in\overline{\mathbb{Q}}_\ell$ also must be of finite order. In particular, $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))=\sum_{j=1}^{d}\epsilon_j\alpha_j$ is an algebraic integer. As $\mathbb{Q}\cap\overline{\mathbb{Z}}=\mathbb{Z}$, we get $\sum_{i\geq 0}(-1)^i\operatorname{Tr}(g\mid H_c^i(X,\overline{\mathbb{Q}}_\ell))\in\mathbb{Z}$. □

We let $\mathcal{L}(g,X)$ denote the Lefschetz number of g.

1.2. **Deligne–Lusztig representation.** In the following, we let k be a finite field $\mathbb{F}_q$ of characteristic $p>0$. We fix a prime number $\ell\neq p$ and also fix an isomorphism $\iota\colon\overline{\mathbb{Q}}_\ell\xrightarrow{\cong}\mathbb{C}$. Let G be a connected reductive group over k.

Recall that, for any k-rational maximal torus T of G and a Borel subgroup B containing T[4], the Deligne–Lusztig variety $\mathcal{X}^G_{T\subset B}$ is defined; this is an algebraic variety over $\overline{k}$ equipped with an action of $G^F\times T^F$. Therefore, its ℓ-adic cohomology $H_c^i(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)$ is a finite-dimensional representation (on a $\overline{\mathbb{Q}}_\ell$-vector space) of $G^F\times T^F$.

Now suppose that $\theta\colon T^F\to\mathbb{C}^\times$ is a character. Then, through the fixed isomorphism ι, we may regard θ as a $\overline{\mathbb{Q}}_\ell^\times$-valued character of T^F. Let us write $\theta_\iota:=\iota^{-1}\circ\theta\colon T^F\to\overline{\mathbb{Q}}_\ell^\times$. Then it makes sense to consider the θ_ι-isotypic part $H_c^i(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ of $H_c^i(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)$, which is a finite-dimensional representation of G^F on a $\overline{\mathbb{Q}}_\ell$-vector space.

Definition 1.6. We call the alternating sum of $H_c^i(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ the *Deligne–Lusztig (virtual) representation of G^F associated to* (T,θ_ι) and write $R^G_T(\theta_\iota)$ for it:

$$R^G_{T\subset B}(\theta_\iota):=\sum_{i\geq 0}(-1)^iH_c^i(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)[\theta_\iota].$$

By abuse of notation, we also write $R^G_{T\subset B}(\theta_\iota)$ for the character of the Deligne–Lusztig (virtual) representation (called *Deligne–Lusztig (virtual) character*).

Remark 1.7. Let us say a bit more about the notion of the θ_ι-isotypic part $H_c^i(X,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$. By definition, it is the maximal subspace of $H_c^i(X,\overline{\mathbb{Q}}_\ell)$ whose action of T^F is given by θ_ι, i.e., $t\cdot v=\theta_\iota(t)v$ for any $t\in T^F$ and $v\in H_c^i(X,\overline{\mathbb{Q}}_\ell)$ (such a subspace always uniquely exists since any representation of T^F on a finite-dimensional vector space is semisimple). More explicitly, $H_c^i(X,\overline{\mathbb{Q}}_\ell)[\theta_\iota]$ is realized as the image of the following endomorphism of $H_c^i(X,\overline{\mathbb{Q}}_\ell)$:

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot t.$$

[4]Here, B is a subgroup of $G_{\overline{k}}$ which may not defined over $\overline{k}$. So, precisely speaking, it might be better to write " a Borel subgroup B containing $T_{\overline{k}}$".

Now let us discuss the ℓ-independence of the Deligne–Lusztig representation. At this point, the coefficients of the Deligne–Lusztig representation is taken to be $\overline{\mathbb{Q}}_\ell$ and its construction depends on $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$. Hence, the Deligne–Lusztig character is also a class function on G^F valued in $\overline{\mathbb{Q}}_\ell$.

We note that the Deligne–Lusztig variety $\mathcal{X}^G_{T\subset B}$ might not be defined over k. However, there exists a finite extension k' of k such that $\mathcal{X}^G_{T\subset B}$ is defined over k. Indeed, suppose that T splits over $k' = \mathbb{F}_{q^n}$. Then we can choose a Borel subgroup B containing T so that it is defined over k'. This is equivalent to that U satisfies $F^n(U) = U$. Hence, if $g \in G$ satisfies $g^{-1}F(g) \in F(U)$, then we have $F^n(g)^{-1}F(F^n(g)) = F^n(g^{-1}F(g)) \in F^n(F(U)) = F(U)$. In other words, $\mathcal{X}^G_{T\subset B}$ is a subset of G which is stable under F^n. Thus, by the Galois descent, $\mathcal{X}^G_{T\subset B}$ is defined over k'. Note that the Frobenius endomorphism of $\mathcal{X}^G_{T\subset B}$ associated to this k'-rational structure is given by F^n.

Now let us apply Theorem 1.5 to the action of $G^F \times T^F$ on $\mathcal{X}^G_{T\subset B}$. Any $(g,t) \in G^F \times T^F$ satisfies $(g,t)\circ F^n = F^n \circ (g,t)$. Indeed, for any $x \in \mathcal{X}^G_{T\subset B}$, we have

$$(g,t)\circ F^n(x) = gF^n(x)t = F^n(gxt) = F^n \circ (g,t)(x)$$

(note that g and t are fixed by F). In other words, the (g,t)-action on $\mathcal{X}^G_{T\subset B}$ satisfies the assumption of Theorem 1.5. Hence the Lefschetz number of (g,t) is an integer independent of ℓ:

$$\mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}) := \sum_{i\geq 0}(-1)^i \operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \in \mathbb{Z}.$$

Proposition 1.8. *For any $g \in G^F$, we have*

$$R^G_{T\subset B}(\theta_\iota)(g) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot \mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}).$$

Proof. By Remark 1.7, we have

$$\begin{aligned}
R^G_{T\subset B}(\theta_\iota)(g) &= \sum_{i\geq 0}(-1)^i \operatorname{Tr}(g \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)[\theta_\iota]) \\
&= \sum_{i\geq 0}(-1)^i \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \\
&= \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\sum_{i\geq 0}(-1)^i \operatorname{Tr}((g,t) \mid H^i_c(\mathcal{X}^G_{T\subset B},\overline{\mathbb{Q}}_\ell)) \\
&= \frac{1}{|T^F|}\sum_{t\in T^F}\theta_\iota(t)^{-1}\cdot \mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}).
\end{aligned}$$

□

Note that, though the isomorphism $\iota\colon \overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we can regard $R^G_{T\subset B}(\theta_\iota)$ as a $\mathbb{C}$-valued class function on G^F. By the above proposition, then its values is given by

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t)^{-1}\cdot \mathcal{L}((g,t),\mathcal{X}^G_{T\subset B}),$$

which is independent of ℓ (and also of ι). Let us write $R^G_{T\subset B}(\theta)$ for the virtual representation/character of G^F with $\mathbb{C}$-coefficients obtained in this way.

Example 1.9. Let us present an example in the GL_2-case without any justification. Recall that (Week 2) irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ are constructed by two different kinds of inductions:

(1) To any character $\boldsymbol{\chi}$ of $\mathbb{F}_q^\times \times \mathbb{F}_q^\times$, we can associate a principal series representation $\mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \boldsymbol{\chi}$.
(2) To any character θ of $\mathbb{F}_{q^2}^\times$ satisfying $\theta^{q-1} \neq \mathbb{1}$, we can associate a cuspidal representation π_θ.

Also recall that (Week 5) G^F-conjugacy classes of k-rational maximal tori of a connected reductive group G over k can be classified by the F-conjugacy classes of Weyl group of G. When $G = \mathrm{GL}_2$, its Weyl group W is equal to $\mathfrak{S}_2 = \{1, s\}$ with trivial F-action. So there exist exactly two G^F-conjugacy classes of k-rational maximal tori of GL_2:

(1) The one T_1 corresponding to the trivial element $1 \in W$ is split; $T_1(\mathbb{F}_q) \cong (\mathbb{F}_q^\times)^2$. For any character $\boldsymbol{\chi}$ of $T_1(\mathbb{F}_q)$, we have $R_{T_1 \subset B}^G(\boldsymbol{\chi}) \cong \mathrm{Ind}_{B(\mathbb{F}_q)}^{\mathrm{GL}_2(\mathbb{F}_q)} \boldsymbol{\chi}$.
(2) The other one T_s corresponding to the non-trivial element $s \in W$ is non-split; $T_s(\mathbb{F}_q) \cong \mathbb{F}_{q^2}^\times$. If we take a character θ of $T_s(\mathbb{F}_q)$ satisfying $\theta^{q-1} \neq \mathbb{1}$, then we have $R_{T_s \subset B}^G(\theta) \cong -\pi_\theta$.[5][6]

1.3. **Split case: principal series.** Let us first investigate the Deligne–Lusztig representation in the case where G is split and T is a split maximal torus ("base torus") T_0. Then we can find a Borel subgroup B of G containing T which is defined over k. Let $\theta\colon T^F \to \mathbb{C}^\times$ be any character. Since B is equal to the semi-direct product of its unipotent radical U and T (T normalizes U), we have a natural surjective homomorphism $B \twoheadrightarrow B/U = T$. By inflating through this homomorphism, we can regard θ as a character of B^F. We define the *principal series representation of G^F (associated to θ)* to be $\mathrm{Ind}_{B^F}^{G^F} \theta$.

Proposition 1.10. *We have $R_{T\subset B}^G(\theta) \cong \mathrm{Ind}_{B^F}^{G^F} \theta$.*

Proof. We let $\mathcal{B}^F$ denote the set of k-rational Borel subgroups of G. We note that any two k-rational Borel subgroups of G are G^F-conjugate; in particular, $\mathcal{B}^F$ is a finite set. We define a morphism π from $\mathcal{X}_{T\subset B}^G$ to $\mathcal{B}^F$ by

$$\pi\colon \mathcal{X}_{T\subset B}^G = \{g \in G \mid g^{-1}F(g) \in U\} \to \mathcal{B}^F; \quad g \mapsto gBg^{-1}$$

(note that $F(U)$ in the definition of $\mathcal{X}_{T\subset B}^G$ is equal to U since U is k-rational). This morphism is well-defined; indeed, if $g \in G$ satisfies $g^{-1}F(g) \in U$ (say $g^{-1}F(g) = u$), then we have

$$F(gBg^{-1}) = F(g)BF(g)^{-1} = guBu^{-1}g^{-1} = gBg^{-1}.$$

Hence gBg^{-1} is a k-rational Borel subgroup of G. Moreover, π is surjective. To check this, let us take a k-rational Borel subgroup B' of G. Then there exists an element $g \in G^F$ satisfying $B' = gBg^{-1}$. since $g^{-1}F(g) = 1 \in U$, g belongs to

[5]Note that the Deligne-Lusztig representation itself can be defined even if θ does not satisfy the condition $\theta^{q-1} \neq \mathbb{1}$.

[6]Here, a Borel subgroup B containing T_s cannot be taken to be the standard upoper-triangular one.

$\mathcal{X}^G_{T\subset B}$ and satisfies $\pi(g) = B'$. Therefore, we obtain a disjoint union decomposition $\mathcal{X}^G_{T\subset B}$ into finite number of closed subvarieties:

$$\mathcal{X}^G_{T\subset B} = \bigsqcup_{B'\in\mathcal{B}^F} \pi^{-1}(B').$$

Recall that, $\mathcal{X}^G_{T\subset B}$ has an action of $G^F \times T^F$ given by $(x,t)\colon g \mapsto xgt$. We introduce an action of $G^F \times T^F$ on $\mathcal{B}^F$ by $(x,t)\colon B' \mapsto xB'x^{-1}$. Then π is $G^F\times T^F$-equivariant, i.e., $\pi((x,t)\cdot g) = (x,t)\cdot\pi(g)$. Note that the action of $G^F\times T^F$ permutes the closed subvarieties $\pi^{-1}(B')$ (for $B' \in \mathcal{B}^F$). The resulting action $G^F \times T^F$ of on the finite set $\{\pi^{-1}(B') \mid B' \in \mathcal{B}^F\}$ is transitive and the stabilizer of $\pi^{-1}(B)$ is given by $B^F \times T^F$. In this setting, we have that the class function

$$G^F \times T^F \to \mathbb{Z}\colon (g,t) \mapsto \mathcal{L}((g,t), \mathcal{X}^G_{T\subset B})$$

is given by the induction of

$$B^F \times T^F \to \mathbb{Z}\colon (b,t) \mapsto \mathcal{L}((b,t), \pi^{-1}(B))$$

(This is a general fact which holds for the Lefschetz number of a variety equipped with a finite group action; see [Car85, Property 7.1.7]).

Hence, by Proposition 1.8, the Deligne–Lusztig character $R^G_{T\subset B}(\theta)$ is given by the induction of the following class function from B^F to G^F:

$$b \mapsto \frac{1}{|T^F|}\sum_{t\in T^F} \theta(t)^{-1}\cdot\mathcal{L}((b,t),\pi^{-1}(B)).$$

Let us compute $\mathcal{L}((b,t),\pi^{-1}(B))$. By recalling that $N_G(B) = B$, we see that $\pi^{-1}(B)$ is given by

$$\mathcal{X}^G_{T\subset B} \cap N_G(B) = \mathcal{X}^G_{T\subset B} \cap B = T^F U.$$

Note that each fiber of the quotient map $T^F U \twoheadrightarrow T^F U/U$ is isomorphic to U, which is furthermore isomorphic to an affine space $\mathbb{A}^{\dim U}$ (this is a general property of an unipotent group). In fact, it is known that such a map ("affine fibration") does not change the Lefschetz number, i.e., $\mathcal{L}((b,t),\pi^{-1}(B)) = \mathcal{L}((b,t), T^F U/U)$ (see [Car85, Property 7.1.5]). Here, $B^F \times T^F$ acts on $T^F U/U$ in an obvious way, that is, $(b,t)\cdot sU = bstU$.

Now note that $T^F U/U = T^F U^F/U^F$ is a finite set. Thus $\mathcal{L}((b,t), T^F U^F/U^F)$ is equal to the cardinality of the set $(T^F U^F/U^F)^{(b,t)}$ of points of $T^F U^F/U^F$ fixed by (b,t) (see the exercise below). For any $sU^F \in T^F U^F/U^F$, we have $(b,t)\cdot sU^F = sU^F$ if and only if $bstU^F = sU^F$, which is equivalent to $b \in t^{-1}U^F$. This implies that the fixed points set $(T^F U^F/U^F)^{(b,t)}$ is empty if $b \notin t^{-1}U^F$ and equal to $T^F U^F/U^F$ if $b \in t^{-1}U^F$. Since $|T^F U^F/U^F| = |T^F|$, we get

$$\mathcal{L}((b,t),\pi^{-1}(B)) = \begin{cases} |T^F| & \text{if } b \in t^{-1}U^F, \\ 0 & \text{if } b \notin t^{-1}U^F. \end{cases}$$

Therefore, $R^G_{T\subset B}(\theta)$ is given by the induction of

$$b = su \mapsto \frac{1}{|T^F|}\sum_{t\in T^F} \theta(t)^{-1}\cdot\mathcal{L}((su,t),\pi^{-1}(B)) = \theta(s).$$

This means that $R^G_{T\subset B}(\theta)$ is the induction of the inflation of θ, i.e., $\mathrm{Ind}^{G^F}_{B^F}\theta$. $\square$

Exercise 1.11. Prove the following claim:

Let X be a finite set (this can be regarded as a 0-dimensional algebraic variety $\bigsqcup_{x\in X} \operatorname{Spec} \overline{k}$). Suppose that g is an automorphism of X. Then we have $\mathcal{L}(g, X) = |X^g|$.

Hint:

(1) Show that the Frobenius endomorphism F induced from the obvious k-rational structure $\bigsqcup_{x\in X} \operatorname{Spec} k$ is the identity of X.

(2) Define a formal power series $R(t)$ in the same way as the proof of Theorem 1.5 and do the same argument.

References

[Car85] R. W. Carter, *Finite groups of Lie type*, Pure and Applied Mathematics (New York), John Wiley & Sons, Inc., New York, 1985, Conjugacy classes and complex characters, A Wiley-Interscience Publication.

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