1. Week 10: Proof of the orthogonality relation for Green functions

Recall that we proved the inner product formula for Deligne–Lusztig representations by assuming the following:

Theorem 1.1 (Disjointness theorem)**.** *Let T and T' be k-rational maximal tori of G. Suppose that characters θ of T^F and θ' of T'^F are not geometrically conjugate. Then $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain a common irreducible representation.*

The aim of this week is to prove the disjointness theorem.

1.1. Preliminary reduction.

Before we prove the disjointness theorem, let us introduce some purely-algebraic lemmas. Recall that, for any representation (ρ, V) of G^F, its dual (*contragredient*) representation $(\rho^\vee, V^\vee)$ is defined by $V^\vee := \mathrm{Hom}_{\mathbb{C}}(V, \mathbb{C})$ and

$$\langle \rho^\vee(g)(v^\vee), v\rangle = \langle v^\vee, \rho(g^{-1})(v)\rangle$$

for any $g \in G^F$, $v \in V$, $v^\vee \in V^\vee$.

Lemma 1.2. *For any representation ρ of G^F, we have $\Theta_{\pi^\vee}(g) = \Theta_\pi(g^{-1}) = \overline{\Theta_\pi(g)}$.*

Exercise 1.3. Prove Lemma 1.2.

Lemma 1.4. *We have $R^G_{T\subset B}(\theta)^\vee \cong R^G_{T\subset B}(\theta^{-1})$.*

Proof. By Lemma 1.2, to prove the assertion, it suffices to check that $\overline{R^G_{T\subset B}(\theta)(g)} = R^G_{T\subset B}(\theta^{-1})(g)$ for any $g \in G^F$. If we write $g = su$ for the Jordan decomposition of g, then, by the Deligne–Lustig character formula, we have

$$\begin{aligned}
\overline{R^G_{T\subset B}(\theta)(g)} &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \overline{\theta(x^{-1}sx)} \cdot \overline{Q^{G_s^\circ}_{{}^xT}(u)} \\
&= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x\in G^F\\ x^{-1}sx\in T^F}} \theta^{-1}(x^{-1}sx) \cdot Q^{G_s^\circ}_{{}^xT}(u) = R^G_{T\subset B}(\theta^{-1})(g).
\end{aligned}$$

(Recall that the Green function is $\mathbb{Z}$-valued and that $\overline{\theta} = \theta^{-1}$). □

Lemma 1.5. *Let R and R' be representations of G^F. Then R and R' contain a common irreducible representation if and only if $R \otimes R'^\vee$ contains the trivial representation of G^F.*

Proof. Let us write $R = \sum_\rho n_\rho \rho$ and $R' = \sum_\rho n'_\rho \rho$. Here, note that $n_\rho, n'_\rho \in \mathbb{Z}_{\geq 0}$ since R and R' are "genuine" (not "virtual") representations of G^F. Then we have

$$R \otimes R'^\vee = \sum_{\rho,\rho'} n_\rho n'_{\rho'} \rho \otimes \rho'^\vee,$$

where ρ and ρ' run all (isomorphism classes of) irreducible representations of G^F. Note that $\rho \otimes \rho'^\vee$ contains $\mathbb{1}$ if and only if $\mathrm{Hom}_{G^F}(\mathbb{1}, \rho\otimes\rho'^\vee) \neq 0$. Since we have

$$\mathrm{Hom}_{G^F}(\mathbb{1}, \rho\otimes\rho'^\vee) \cong \mathrm{Hom}_{G^F}(\rho', \rho)$$

(so-called the Hom $-\otimes$ adjunction), it is furthermore equivalent to that $\rho \cong \rho'$ since ρ and ρ' are irreducible. Moreover, in this case, $\mathrm{Hom}_{G^F}(\rho', \rho)$ is 1-dimensional by Schur's lemma. In other words, $\rho\otimes\rho'^\vee$ contains $\mathbb{1}$ with multiplicity one. Therefore, the multiplicity of the trivial representation $\mathbb{1}$ in $R \otimes R'^{\vee}$ is given by $\sum_{\rho} n_\rho n'_\rho$. Since $n_\rho, n'_\rho \in \mathbb{Z}_{\geq 0}$, we have $\sum_\rho n_\rho n'_\rho \neq 0$ if and only if there exists ρ satisfying $n_\rho n'_\rho \neq 0$, i.e., both R and R' contains ρ. $\square$

Now let us start to prove Theorem **??**. Suppose that θ of T^F and θ' of T'^F are characters not geometrically conjugate. Our goal is to show that $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ have no common irreducible constituent. To show this, it is enough to show the following:

Proposition 1.6. *If θ of T^F and θ' of T'^F are characters not geometrically conjugate, then $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the trivial representation for any $i, j \in \mathbb{Z}_{\geq 0}$.*

Indeed, since we have

$$R^G_{T\subset B}(\theta^{-1}) \otimes R^G_{T'\subset B'}(\theta') \cong \sum_{i,j\in\mathbb{Z}_{\geq 0}} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta'],$$

Proposition 1.6 implies that $R^G_{T\subset B}(\theta^{-1}) \otimes R^G_{T'\subset B'}(\theta')$ do not contain the trivial representation. Then, by Lemmas 1.5 and 1.4, we see that $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain the same irreducible representation.

Remark 1.7. Here is a "dangerous bend". To show that $R^G_T(\theta)^\vee \cong R^G_T(\theta^{-1})$ in Lemma 1.4, we utilized the Deligne–Lusztig character formula; taking the alternating sum is crucially important for this. In other words, it could be possible that each individual $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta]^\vee$ is **not** isomorphic to $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}]$. Therefore, we **cannot** discuss in the following way: [1]

> If $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the trivial representation, then $H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta]$ and $H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta']$ do not contain the same irreducible representation (this part is **wrong** for the above reason). Hence, in particular, $R^G_{T\subset B}(\theta)$ and $R^G_{T'\subset B'}(\theta')$ do not contain the trivial representation.

By the "Künneth formula", we have

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell) \cong \bigoplus_{i+j=k} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell) \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)$$

(this is a general fact about ℓ-adic cohomology, which holds for any product $X_1 \times X_2$ of algebraic varieties X_1 and X_2; see [Car85, Property 7.1.9]). This isomorphism is $G^F \times T^F \times T'^F$-equivariant. Here, on the left-hand side, we consider the action of $G^F \times T^F \times T'^F$ on $\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}$ given by $(g, t, t') \cdot (x, x') := (gxt, gx't')$. Therefore, we get

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta^{-1} \boxtimes \theta'] \cong \bigoplus_{i+j=k} H^i_c(\mathcal{X}^G_{T\subset B}, \overline{\mathbb{Q}}_\ell)[\theta^{-1}] \otimes H^j_c(\mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\theta'].$$

Hence, by putting $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$, it is enough to show that

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$$

does not contain the trivial representation for any k, or equivalently,

$$H^k_c(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}, \overline{\mathbb{Q}}_\ell)^{G^F}[\boldsymbol{\theta}] = 0$$

[1] I have to confess that I was enough stupid to try this at the beginning.

for any k (the upper G^F denotes the G^F-invariant part).

Now we appeal to another fact on the ℓ-adic cohomology (see [Car85, Property 7.1.8]):

$$H_c^k(\mathcal{X}_{T\subset B}^G \times \mathcal{X}_{T'\subset B'}^G, \overline{\mathbb{Q}}_\ell)^{G^F} \cong H_c^k((\mathcal{X}_{T\subset B}^G \times \mathcal{X}_{T'\subset B'}^G)/G^F, \overline{\mathbb{Q}}_\ell),$$

where $(\mathcal{X}_{T\subset B}^G \times \mathcal{X}_{T'\subset B'}^G)/G^F$ denotes the quotient of $\mathcal{X}_{T\subset B}^G \times \mathcal{X}_{T'\subset B'}^G$ by the action of the finite group G^F (given by $g\cdot(x,x')=(gx,gx')$).

We summarize our discussion so far. The disjoint theorem for $R_{T\subset B}^G(\theta)$ and $R_{T'\subset B'}^G(\theta')$ is now reduced to the following:

Claim. If θ and θ' are characters of T^F and T'^F not geometrically conjugate, then

$$H_c^k((\mathcal{X}_{T\subset B}^G \times \mathcal{X}_{T'\subset B'}^G)/G^F, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] = 0$$

for any $k\in\mathbb{Z}_{\geq 0}$, where we put $\boldsymbol{\theta} := \theta^{-1}\boxtimes\theta'$.

1.2. **Reformulation of geometric conjugacy.** Let $\mathbb{Z}_{(p)}$ be the localization of $\mathbb{Z}$ with respect to the prime ideal (p), i.e.,

$$\mathbb{Z}_{(p)} := \{a/b\in\mathbb{Q} \mid a,b\in\mathbb{Z},\, p\nmid b\}\subset\mathbb{Q}.$$

Note that the groups $\overline{\mathbb{F}}_p^\times$ and $\mathbb{Z}_{(p)}/\mathbb{Z}$ are isomorphic. A naive explanation of this fact is as follows. Recall that, for any $n\in\mathbb{Z}_{>0}$, $\mathbb{F}_{p^n}$ is generated over $\mathbb{F}_p$ by the solutions to the equation $x^{p^n}-x=0$. Hence $\mathbb{F}_{p^n}^\times$ is a subset of $\overline{\mathbb{F}}_p^\times$ consisting of the solutions to $x^{p^n-1}-1=0$, i.e., the subset of (p^n-1)-th roots of unity. Thus, if we fix its generator ζ_{p^n-1}, then we can define an isomorphism

$$\mathbb{F}_{p^n}^\times \xrightarrow{\cong} \tfrac{1}{p^n-1}\mathbb{Z}/\mathbb{Z}\colon \zeta_{p^n-1}^k\mapsto k.$$

Since $\overline{\mathbb{F}}_p = \bigcup_{n\in\mathbb{Z}_{>0}}\mathbb{F}_{p^n}$, by choosing the generators ζ_{p^n-1} in a "coherent way", we can extend the above isomorphism to

$$\overline{\mathbb{F}}_p^\times \xrightarrow{\cong} \varinjlim_{n\in\mathbb{Z}_{>0}} \tfrac{1}{p^n-1}\mathbb{Z}/\mathbb{Z}.$$

The right-hand side is nothing but $\mathbb{Z}_{(p)}/\mathbb{Z}$ (note that any prime-to-p positive integer divides p^n-1 for some $n\in\mathbb{Z}_{>0}$).

As we can see from this construction, we do **not** have a canonical choice of an isomorphism $\overline{\mathbb{F}}_p^\times\cong\mathbb{Z}_{(p)}/\mathbb{Z}$. In the following, let us fix such an isomorphism.

Now let T be a k-rational maximal torus of a connected reductive group G over k. Recall that its cocharacter group $X_*(T)=\mathrm{Hom}(\mathbb{G}_{\mathrm{m}},T)$ has an action of the Frobenius F, which is given by $\gamma\mapsto F\circ\gamma$. We write $X_*(T)_{(p)} := X_*(T)\otimes_{\mathbb{Z}}\mathbb{Z}_{(p)}$. Let us consider the following short exact sequence:

$$0\to\mathbb{Z}\to\mathbb{Z}_{(p)}\to\mathbb{Z}_{(p)}/\mathbb{Z}\to 0.$$

Since $X_*(T)$ is a free $\mathbb{Z}$-module, this induces

$$0\to X_*(T)\to X_*(T)_{(p)}\to X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z})\to 0.$$

Since the Frobenius action preserves each term, we get a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X_*(T) \arrow[r] \arrow[d, "F-1"] & X_*(T)_{(p)} \arrow[r] \arrow[d, "F-1"] & X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}) \arrow[r] \arrow[d, "F-1"] & 0 \\
0 \arrow[r] & X_*(T) \arrow[r] & X_*(T)_{(p)} \arrow[r] & X_*(T)\otimes_{\mathbb{Z}}(\mathbb{Z}_{(p)}/\mathbb{Z}) \arrow[r] & 0.
\end{tikzcd}$$

Therefore, by applying the snake lemma, we get an exact sequence

$$\begin{aligned}\operatorname{Ker}(F-1 \mid X_*(T)_{(p)}) &\to \operatorname{Ker}(F-1 \mid X_*(T)\otimes_{\mathbb{Z}} (\mathbb{Z}_{(p)}/\mathbb{Z})) \\ &\to \operatorname{Cok}(F-1 \mid X_*(T)) \to \operatorname{Cok}(F-1 \mid X_*(T)_{(p)}).\end{aligned}$$

Lemma 1.8. *The kernel of the endomorphism $F-1$ of $X_*(T)\otimes_{\mathbb{Z}} (\mathbb{Z}_{(p)}/\mathbb{Z})$ is isomorphic to T^F.*

Proof. Recall that we have fixed an isomorphism $\overline{\mathbb{F}}_q^\times \cong \mathbb{Z}_{(p)}/\mathbb{Z}$, hence we have $X_*(T)\otimes_{\mathbb{Z}} (\mathbb{Z}_{(p)}/\mathbb{Z}) \cong X_*(T)\otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q^\times$. We consider the following map:

$$X_*(T)\otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q^\times \to T(\overline{\mathbb{F}}_q) = T\colon \gamma\otimes x \mapsto \gamma(x).$$

Then this is a well-defined homomorphism, which is consistent with the Frobenius actions on the both sides. Moreover, this is a bijection (for example, we can easily check it by fixing an isomorphism $T\cong \mathbb{G}_{\mathrm{m}}^r$). Hence the kernel of the endomorphism $F-1$ of $X_*(T)\otimes_{\mathbb{Z}} \overline{\mathbb{F}}_q^\times$ is identified with T^F on the right-hand side. $\square$

Lemma 1.9. *The endomorphism $F-1$ of $X_*(T)_{(p)}$ is an isomorphism. In particular, the connecting homomorphism*

$$T^F \to \operatorname{Cok}(F-1 \mid X_*(T)) = X_*(T)/(F-1)X_*(T).$$

constructed above is an isomorpshim.

Proof. Note that $X_*(T)_{(p)}$ is contained in $X_*(T)_{\mathbb{Q}} := X_*(T)\otimes_{\mathbb{Z}}\mathbb{Q}$. To show that $F-1$ is an isomorphism, it is enough to check that the determinant of $F-1$ is a prime-to-p integer. (Then, the inverse matrix to $F-1$, which is taken in $X_*(T)_{\mathbb{Q}}$, has its entries in $X_*(T)_{(p)}$).

Recall (from Week 5) that the endomorphism F of $X_*(T)_{\mathbb{Q}}$ is equal to qF_0, where q denotes the q-multiplication map and F_0 is an endomorphism of $X_*(T)_{\mathbb{Q}}$ of finite order. This means that $\det(F-1)$ is expressed as $\prod_{i=1}^r (q\zeta_i - 1)$, where $r=\dim T$ and ζ_i is a root of unity. Let $K := \mathbb{Q}(\zeta_i \mid i=1,\ldots,r)$; then each $q\zeta_i-1$ belongs to the ring of intergers $\mathcal{O}_K$ of K. It suffices to check that $q\zeta_i - 1$ is not contained in $p\mathcal{O}_K$, but this is clear because $q\zeta_i-1$ is equivalent to -1 modulo $p\mathcal{O}_K$. $\square$

We have obtained an identification

$$X_*(T)/(F-1)X_*(T) \cong T^F.$$

In particular, if a character θ of T^F is given, then we can regard it as a character of $X_*(T)$.

Proposition 1.10. *Let T and T' be k-rational maximal tori of G. Let θ and θ' be characters of T^F and T'^F. Then θ and θ' are geometrically conjugate if and only if there exists $g\in G$ such that $T' = {}^gT$ and the induced map $\operatorname{Int}(g)\colon X_*(T)\cong X_*(T')$ transfers θ to θ'.*

The proof of this proposition is not difficult, but we omit; see [Car85, Propositions 4.1.2 and 4.1.3]. When the latter condition of the above proposition is satisfied, let us say "the characters of $X_*(T)$ and $X_*(T')$ induced by θ and θ' are geometrically conjugate".

1.3. **Structure of the quotient of Deligne–Lusztig varieties.** Let us investigate the structure of the quotient variety $(\mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'})/G^F$. We write $\mathcal{S}$ for this quotient variety. We put

$$\mathcal{S}' := \{(u, u', z) \in F(U) \times F(U') \times G \mid uF(z) = zu'\}.$$

Proposition 1.11. *The following map is bijective and $T^F \times T'^F$-equivariant:*

$$\varphi\colon \mathcal{S} \to \mathcal{S}'\colon (x, x') \mapsto (x^{-1}F(x), x'^{-1}F(x'), x^{-1}x').$$

Here, $T^F \times T'^F$ acts on the left-hand side by $(t, t') \cdot (x, x') = (xt, xt')$ and on the right-hand side by $(t, t') \cdot (u, u', z) = (t^{-1}ut, t'^{-1}u't', t^{-1}zt')$. Furthermore, this bijection is an isomorphism of algebraic varieties.

Proof. The well-definedness of the map can be easily checked by recalling the definition of the Deligne–Lusztig variety:

$$\mathcal{X}^G_{T\subset B} := \{x \in G \mid x^{-1}F(x) \in F(U)\}.$$

The equivariance is also clear.

Let us check the injectivity of the map. Suppose that $(x, x'), (y, y') \in \mathcal{X}^G_{T\subset B} \times \mathcal{X}^G_{T'\subset B'}$ map to the same element, i.e,

$$(x^{-1}F(x), x'^{-1}F(x'), x^{-1}x') = (y^{-1}F(y), y'^{-1}F(y'), y^{-1}y').$$

By comparing the first entries, we see that $yx^{-1} \in G^F$; in other words, there exists an element $g \in G^F$ satisfying $y = gx$. Similarly, by comparing the second entries, there exists an element $g' \in G^F$ satisfying $y' = g'x'$. Finally, by looking at the third entries, we obtain $g = g'$. This means that (x, x') and (y, y') are in the same G^F-orbit.

Let us next check the surjectivity. Suppose that $(u, u', z) \in \mathcal{S}$, i.e., $u \in F(U)$, $u' \in F(U')$, $z \in G$ satisfy $uF(z) = zu'$. By applying Lang's theorem to u and u', we can find an element $x, x' \in G$ satisfying $x^{-1}F(x) = u$ and $x'^{-1}F(x') = u'$, respectively. Note that then $xzx'^{-1} \in G^F$. Indeed, we have

$$F(xzx'^{-1}) = F(x)F(z)F(x')^{-1} = (xu) \cdot (u^{-1}zu') \cdot (x'u')^{-1} = xzx'^{-1}.$$

Hence, if we put $g := xzx'^{-1} \in G^F$, then we have $\varphi(x, gx') = (u, u', z)$.

To show that this bijection is in fact an isomorphism of algebraic varieties, we need more about algebraic geometry. We do not explain the details in this course; please see [Car85, Proof of Theorem 7.3.8, 221-222 pages]. $\square$

By this proposition, our task is furthermore reduced to show the vanishing of $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$ for each $i \in \mathbb{Z}_{\geq 0}$. The idea of computing the cohomology of $\mathcal{S}'$ is to divide $\mathcal{S}'$ into "cells", where the cohomologies are more computable. The key is the following general fact, which is a generalization of the decomposition $\mathrm{GL}_2 = B \sqcup B\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)B$ used in Week2:

Theorem 1.12 (Bruhat decomposition)**.** *We have the following disjoint union decomposition:*

$$G = \bigsqcup_{w\in W_G(T)} B\dot{w}B,$$

where $\dot{w} \in N_G(T)$ is any representative of $w \in W_G(T)$. Here, each $B\dot{w}B$ is locally closed and equal to $UT\dot{w}U_w$, where $U_w := U \cap w^{-1}\overline{U}w$.[2] *Moreover, for any $w' \in$*

[2]The symbol $\overline{U}$ denotes the unipotent radical of the "opposite" Borel subgroup $\overline{B}$. You can just think of it as a generalization of the lower-triangular Borel subgroup of GL_n.

$W_G(T)$, the union $\bigsqcup_{w\leq w'} B\dot{w}B$ is closed, where "$\leq$" denotes the "Bruhat order" on the Weyl group.

Let us first rewrite the Bruhat decomposition in a way more useful for our purpose. Recall that B be a Borel subgroup of G containing T with unipotent radical U. Since T is k-rational, $F^{-1}(B)$ is also a Borel subgroup of G containing T; its unipotent radical is given by $F^{-1}(U)$. The same statement holds for $B' = T'U'$. We fix $g \in G$ satisfying ${}^{g}T' = T$ and ${}^{g}F^{-1}(B') = F^{-1}(B)$ (hence ${}^{g}F^{-1}(U') = F^{-1}(U)$). For each $w \in W_G(T)$, we fix its representative $\dot{w} \in N_G(T)$ and put

$$G_w := (U \cap {}^{w}\overline{U})T\dot{w}gU'.$$

Lemma 1.13. *We have $G = \bigsqcup_{w\in W_G(T)} G_w$. Moreover, each G_w is locally closed in G and satisfies the same closure relation as the Bruhat decomposition $G = \bigsqcup_{w\in W_G(T)} B\dot{w}B$.*

Proof. By the Bruhat decomposition, we have

$$G = \bigsqcup_{w\in W_G(T)} UT\dot{w}U_w = \bigsqcup_{w\in W_G(T)} UT\dot{w}(U \cap w^{-1}(\overline{U})w)$$

By inverting the both side, we get

$$G = \bigsqcup_{w\in W_G(T)} (U \cap w^{-1}(\overline{U})w)\dot{w}^{-1}TU = \bigsqcup_{w\in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}TU.$$

(Here, in the second equality, we replaced w with w^{-1}.) Since we have ${}^{g}U' = U$, we get

$$G = \bigsqcup_{w\in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}T{}^{g}U'.$$

By multiplying both sides by g from the right, we get

$$G = \bigsqcup_{w\in W_G(T)} (U \cap {}^{w}\overline{U})\dot{w}TgU' = \bigsqcup_{w\in W_G(T)} G_w$$

(note that $T\dot{w} = \dot{w}T$).

The assertion on the topology follows from by the above proof (we just rewrote each cell). $\square$

Recall that

$$\mathcal{S}' := \{(u, u', z) \in F(U) \times F(U') \times G \mid uF(z) = zu'\}.$$

For each $w \in W$, we put

$$\mathcal{S}'_w := \{(u, u', z) \in F(U) \times F(U') \times G_w \mid uF(z) = zu'\}.$$

Then we obviously have $\mathcal{S}' = \bigsqcup_{w\in W_G(T)} \mathcal{S}'_w$ and each cell $\mathcal{S}'_w$ is locally closed in $\mathcal{S}'$. Moreover, it can be easily checked that each G_w is stable under the left T-multiplication and the right T'-multiplication. This implies that $\mathcal{S}'_w$ is stable under the action of $T^F \times T'^F$ on $\mathcal{S}'$. Therefore, by a property of ℓ-adic cohomology (see [Car85, Property 7.1.6]), we have the following:

> If $H^i_c(\mathcal{S}'_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] = 0$ for each $i \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, then we have $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}]$ for each $i \in \mathbb{Z}_{\geq 0}$.

Note that, by a property of the Bruhat decomposition, the natural product map

$$(U \cap {}^w\overline{U}) \times T\dot{w}g \times U' \to (U \cap {}^w\overline{U})T\dot{w}gU' =: G_w$$

is bijective Thus we have

$\mathcal{S}'_w = \{(u, u', v, a, v') \in F(U) \times F(U') \times (U \cap {}^w\overline{U}) \times T\dot{w}g \times U' \mid uF(vav') = vav'u'\}.$

We finally introduce the following variety for each $w \in W_G(T)$:

$\mathcal{S}''_w := \{(\xi, \xi', v, a, v') \in F(U) \times F(U') \times (U \cap {}^w\overline{U}) \times T\dot{w}g \times U' \mid \xi F(a) = vav'\xi'\}.$

Then it is easy to verify that the map

$$(u, u', v, a, v') \mapsto (uF(v), u'F(v')^{-1}, v, a, v')$$

gives an isomorphism of varieties $\mathcal{S}'_w \cong \mathcal{S}''_w$. Moreover, under this isomorphism, the action of $T^F \times T'^F$ on $\mathcal{S}'_w$ is transformed into an action on $\mathcal{S}''_w$ given by

$$(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't').$$

Let us summarize our discussion so far. Now the proof of the disjointness theorem is reduced to the following:

Claim. If θ and θ' are characters of T^F and T'^F not geometrically conjugate, then

$$H^k_c(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] = 0$$

for any $k \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, where we put $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$.

1.4. **Proof of the disjointness theorem.** We introduce a subgroup H_w of $T \times T'$ as follows:

$$H_w := \{(t, t') \in T \times T' \mid F(t')t'^{-1} = F(\dot{w}g)^{-1}(F(t)t^{-1})F(\dot{w}g)\}.$$

Thus is a closed subgroup of $T \times T'$ contains $T^F \times T'^F$. The crucially important property of this subgroup is the following:

Lemma 1.14. *The action of $T^F \times T'^F$ on $\mathcal{S}''_w$ extends to an action of H_w which is given by the same formula.*

Proof. For any $(t, t') \in H_w$ and $(\xi, \xi', v, a, v') \in \mathcal{S}''_w$, let us check that $(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't')$ belongs to $\mathcal{S}''_w$. Recall that

$$(t, t') \cdot (\xi, \xi', v, a, v') = (t^{-1}\xi t, t'^{-1}\xi' t', t^{-1}vt, t^{-1}at', t'^{-1}v't').$$

Thus the right-hand side of the defining equation of $\mathcal{S}''_w$ (i.e., "$\xi F(a)$") is given by

$$(t^{-1}\xi t) \cdot F(t^{-1}at') = t^{-1}\xi tF(t)^{-1}F(a)F(t').$$

On the other hand, the left-hand side of the defining equation of $\mathcal{S}''_w$ (i.e., "$vav'\xi'$") is given by

$$(t^{-1}vt) \cdot (t^{-1}at') \cdot (t'^{-1}v't') \cdot (t'^{-1}\xi' t') = t^{-1}vav'\xi' t' = t^{-1}\xi F(a)t'$$

(we used the defining equation of $\mathcal{S}''_w$ in the second equality). Hence these coincide if and only if we have

$$tF(t)^{-1}F(a)F(t') = F(a)t'.$$

By putting $a = s\dot{w}g$ for some $s \in T$, this is equivalent to

$$tF(t)^{-1}F(\dot{w}g)F(t') = F(\dot{w}g)t'$$

(we used that $F(s)$ commutes with $tF(t)^{-1}$). This is nothing but the defining equation of H_w. $\square$

Proposition 1.15. *Let X be an algebraic variety with an action of a connected algebraic group H. Then the action of H on $H^i_c(X, \overline{\mathbb{Q}}_\ell)$ is trivial.*

By this proposition, the action of H_w° on $H^i_c(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)$ is trivial. In particular, the action of $(T^F \times T'^F) \cap H_w^\circ$ on $H^i_c(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)$ is trivial.

Now let us complete the proof of the disjointness theorem. We write $\tilde{\theta}$ and $\tilde{\theta}'$ for the characters of $X_*(T)$ and $X_*(T')$ induced by θ and θ', respectively. By the characterization of the geometric conjugacy, our task is to show the following:

Claim. Suppose that

$$H^i_c(\mathcal{S}''_w, \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] \neq 0$$

for some $i \in \mathbb{Z}_{\geq 0}$ and $w \in W_G(T)$, where we put $\boldsymbol{\theta} := \theta^{-1} \boxtimes \theta'$. Then $\tilde{\theta}$ and $\tilde{\theta}'$ are geometrically conjugate.

We suppose that $H^i_c(\mathcal{S}', \overline{\mathbb{Q}}_\ell)[\boldsymbol{\theta}] \neq 0$. Then, since $(T^F \times T'^F) \cap H_w^\circ$ acts on this space trivially, we have that $\boldsymbol{\theta} = \theta^{-1} \boxtimes \theta'$ is trivial on $(T^F \times T'^F) \cap H_w^\circ$.

We define a group homomorphism

$$\phi\colon T \times T' \to T; \quad (t, t') \mapsto F(\dot{w}g)t'F(\dot{w}g)^{-1}t.$$

We consider the "Lang map" of $T \times T'$ (note that this is a group homomorphism since $T \times T'$ is abelian):

$$L\colon T \times T' \to T \times T'; \quad (t, t') \mapsto (F(t)t^{-1}, F(t')t'^{-1}).$$

Then, by definition, we see that $H_w \subset T \times T'$ is nothing but the kernel of $\phi \circ L$.

We look at the maps on cocharacter groups induced by ϕ and L.

Lemma 1.16. *Let S be a k-rational subtorus of T. Let $X_*(T) \twoheadrightarrow X_*(T)/(F-1)X_*(T) \cong T^F$ be the surjective homomorphism constructed above. Then the image of $X_*(T) \cap (F-1)X_*(S)_{p'}$ is contained in $T^F \cap S$.*

Exercise 1.17. Prove this lemma. Hint: Go back to the construction of the identification $X_*(T)/(F-1)X_*(T) \cong T^F$ in Section 1.2 (the connecting homomorphism of the snake lemma).

We apply this lemma to $H_w^\circ \subset T \times T'$. Then we see that, under the homomorphism

$$X_*(T) \oplus X_*(T') \to T^F \times T'^F,$$

the subgroup $(X_*(T) \oplus X_*(T')) \cap (F-1)X_*(H_w^\circ)_{(p)}$ is mapped into $(T^F \times T'^F) \cap H_w^\circ$. In other words, the character $(\tilde{\theta}^{-1}, \tilde{\theta}')$ of $X_*(T) \oplus X_*(T')$ is trivial on $(X_*(T) \oplus X_*(T')) \cap (F-1)X_*(H_w^\circ)_{(p)}$.

Lemma 1.18. *We put $M := \mathrm{Ker}(\phi\colon X_*(T) \oplus X_*(T') \to X_*(T))$. Then M is contained in the kernel of $(\tilde{\theta}^{-1}, \tilde{\theta}')$.*

Proof. Let $m \in M$. Since $(X_*(T) \oplus X_*(T'))/(F-1)(X_*(T) \oplus X_*(T'))$ is isomorphic to $T^F \times T'^F$, its order is finite and prime-to-p. Thus there exists a prime-to-p integer $n \in \mathbb{Z}$ such that $nm = (F-1)\xi$ for some $\xi \in X_*(T) \oplus X_*(T')$. As $(F-1)\xi = nm \in M$, we have that $\xi \in \mathrm{Ker}(\phi \circ L) = X_*(H_w) = X_*(H_w^\circ)$. Hence m belongs to $(F-1)X_*(H_w^\circ)_{(p)}$, which means that m lies in the kernel of $(\tilde{\theta}^{-1}, \tilde{\theta}')$ by the remark in the paragraph above Lemma. $\square$

Let $\gamma \in X_*(T)$. Then, by the definition of M, $(\gamma, \mathrm{Int}(F(\dot{w}g)) \circ \gamma) \in X_*(T) \oplus X_*(T')$ belongs to M. Hence, by the above lemma, $(\tilde{\theta}^{-1}, \tilde{\theta}')$ maps $(\gamma, \mathrm{Int}(F(\dot{w}g)) \circ \gamma)$ to 1. In other words, we have

$$\tilde{\theta}^{-1}(\gamma) \cdot \tilde{\theta}'(\mathrm{Int}(F(\dot{w}g)) \circ \gamma) = 1.$$

Equivalently, we have

$$\tilde{\theta}(\gamma) = \tilde{\theta}'(\mathrm{Int}(F(\dot{w}g)) \circ \gamma).$$

This means that the characters $\tilde{\theta}$ and $\tilde{\theta}'$ are geometrically conjugate.

References

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9:05am, November 19, 2024