1. Week 2: Representations of $\mathrm{GL}_2(\mathbb{F}_q)$

Aim of this week. The aim of this week is to construct/classify all irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$, especially, write the character table. Through this example, we should be able to encounter various basic notions on reductive groups and representation theory of finite groups of Lie type. The explanation given here follows [BH06, Section 6].

1.1. Group structure of $\mathrm{GL}_2(\mathbb{F}_q)$.

Let $\mathbb{F}_q$ be a finite field of order q and characteristic $p > 0$ (hence q is a power of p). Let $\mathrm{GL}_2(\mathbb{F}_q)$ denote the general linear group of size 2 with $\mathbb{F}_q$-coefficients, i.e.,

$$\mathrm{GL}_2(\mathbb{F}_q) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, ad - bc \in \mathbb{F}_q^\times \right\}.$$

In the following, we simply write G for $\mathrm{GL}_2(\mathbb{F}_q)$. It is a basic fact that the order of $\mathrm{GL}_2(\mathbb{F}_q)$ is given by $(q^2-1)(q^2-q)$.

Exercise 1.1. More generally, it is known that the order of $\mathrm{GL}_n(\mathbb{F}_q)$ is given by $\prod_{i=0}^{n-1}(q^n - q^i)$. Prove this.

We can classify the conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$ by looking at the characteristic polynomials as follows. For an element $g \in \mathrm{GL}_2(\mathbb{F}_q)$, let $\phi_g(x) \in \mathbb{F}_q[x]$ denote its characteristic polynomial. Then we have the following three possibilities:

(1) $\phi_g(x)$ is of the form $(x-a)^2$ for some $a \in \mathbb{F}_q^\times$.
(2) $\phi_g(x)$ is of the form $(x-a)(x-b)$ for some distinct $a, b \in \mathbb{F}_q^\times$.
(3) $\phi_g(x)$ is an irreducible monic of degree 2.

We first consider the case (1). If $\phi_g(x) = (x-a)^2$, then the minimal polynomial of g is either $x - a$ or $(x-a)^2$. In the former case, g is equal to

$$z_a := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

This element is central in G. Thus the conjugacy class of g is simply given by $\{z_a\}$.

In the latter case, by theory of Jordan normal form, g is conjugate to

$$u_a := \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}.$$

By a simple computation, we can check that the centralizer of u_g in G is given by ZU (see Section 1.3 for the notation), which is of order $q(q-1)$. Hence the conjugacy class of u_g is of order $|G|/q(q-1) = q^2 - 1$.

We next consider the case (2). In this case, g is necessarily conjugate to

$$t_{a,b} := \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

The centralizer of $t_{a,b}$ is given by T (see Section 1.3 for the notation), which is of order $(q-1)^2$. Hence the conjugacy class of $t_{a,b}$ is of order $|G|/(q-1)^2 = q^2 + q$. We caution that $t_{a,b}$ and $t_{a',b'}$ are conjugate if and only if $(a', b') = (a, b), (b, a)$. In particular, there are $\binom{q-1}{2} = \frac{(q-1)(q-2)}{2}$ conjugacy classes of this type.

We finally consider the case (3). Suppose that ϕ_g is an irreducible monic of degree 2. The subring $\mathbb{F}_q[g]$ of $M_2(\mathbb{F}_q)$ is a degree 2 extension of $\mathbb{F}_q$ (given by the minimal polynomial ϕ_g), hence isomorphic to $\mathbb{F}_{q^2}$. The centralizer of g in G is given by $\mathbb{F}_q[g]^\times$, which is of order $q^2 - 1$. (Indeed, the centralizer of g in $M_2(\mathbb{F}_q)$ (let us write E) must be a commutative subring containing $\mathbb{F}_q[g]$. Since it can be regarded as a $\mathbb{F}_q[g]$-vector space, by counting the dimensions, we see that E must be $\mathbb{F}_q[g]$ or $M_2(\mathbb{F}_q)$. However, the latter case is impossible as $M_2(\mathbb{F}_q)$ is not commutative. Thus $E = \mathbb{F}_q[g]$, hence the centralizer of g in G is given by $E^\times = \mathbb{F}_q[g]^\times$.) Hence the conjugacy class of g is of order $|G|/(q^2-1) = q^2 - q$. Note that, by choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_q[g]$ to be $\{1, g\}$, then the g-multiplication action on $\mathbb{F}_q[g]$ is represented by

$$s_{a,b} := \begin{pmatrix} 0 & -b \\ 1 & -a \end{pmatrix},$$

where we write $\phi_g(x) = x^2 + ax + b$. This matrix represents the conjugacy class of g. An easy computation shows that there are $\frac{q^2-q}{2}$ irreducible degree 2 monics in total. Hence the number of conjugacy classes of this type is also given by $\frac{q^2-q}{2}$.

Now we see that there are

$$(q-1) + (q-1) + \frac{(q-1)(q-2)}{2} + \frac{q^2-q}{2} = q^2 - 1$$

conjugacy classes of G in total. Hence, the number of irreducible representations of G must be $q^2 - 1$.

Table 1. Conjugacy classes of $\mathrm{GL}_2(\mathbb{F}_q)$

representative	order of the conjugacy class	parameter	# of parameters
z_a	1	$a \in \mathbb{F}_q^\times$	$q-1$
u_a	q^2-1	$a \in \mathbb{F}_q^\times$	$q-1$
$t_{a,b}$	q^2+q	$a, b \in \mathbb{F}_q^\times$, $a \neq b$	$\frac{(q-1)(q-2)}{2}$
$s_{a,b}$	q^2-q	irr. deg. 2 monic	$\frac{q^2-q}{2}$

1.2. **Philosophy of induction.** We next give some explanation on a general strategy to construct irreducible representations. For this, here let G temporarily denote any finite group.

Definition 1.2. For a representation (σ, W) of a subgroup H of G, its *induction* to G is defined by

$$\mathrm{Ind}_H^G \sigma := \{f \colon G \to W \mid f(hg) = \sigma(h)(f(g)) \text{ for any } h \in H \text{ and } g \in G\},$$

where G acts via right translation, i.e.,

$$(x \cdot f)(g) := f(gx)$$

for any $x \in G$ and $g \in G$.

Recall that the character of the induced representation $\mathrm{Ind}_H^G \sigma$ can be expressed in terms of the character of σ and the group-theoretic relation between G and H as follows:

Theorem 1.3 (Frobenius formula)**.** *For any $g \in G$, we have*

$$\Theta_{\mathrm{Ind}_H^G \sigma}(g) = \sum_{\substack{x \in H\backslash G \\ xgx^{-1} \in H}} \Theta_\sigma(xgx^{-1}).$$

So, in principle, we should be able to know all about the induced representation $\operatorname{Ind}_H^G \sigma$ as long as the subgroup H and its representation σ are "well-understood". Based on this idea, one can try to construct irreducible representations of G using "well-understood" irreducible representations of subgroups of G. Note that the dimension of $\operatorname{Ind}_H^G \sigma$ is given by $[G : H] \cdot \dim \sigma$. Especially, if H is smaller, then the dimension of $\operatorname{Ind}_H^G \sigma$ is larger. Thus it is possible to expect that we can find more irreducible representations in $\operatorname{Ind}_H^G \sigma$ for small H. Indeed, we have the following fundamental theorem:

Theorem 1.4. *Let G be a finite group. Then the induction of the trivial representation of the trivial subgroup to G decomposes as follows:*

$$\operatorname{Ind}_{\{1\}}^G \mathbb{1} \cong \bigoplus_{\rho} \rho^{\oplus \dim \rho},$$

where the direct sum is over the isomorphism classes of all irreducible representations of G.

It is beautiful that every irreducible representation is realized in the induction of the trivial representation. However, we can also think that here too many irreducible representations are mixed together, hence it's difficult to distinguish them. So, for example, it would be great if we could find a subgroup H of G which is simultaneously

- small enough that the induction to G can produce various irreducible representations and
- large enough that the inductions are irreducible (or "almost" irreducible).

What we will see in the next section is an example of such a nice subgroup for $\mathrm{GL}_2(\mathbb{F}_q)$, which is called a "Borel subgroup". (In fact, we can also find a family of such nice subgroups for any finite group of Lie type, called "parabolic subgroups".)

1.3. **Principal representations of $\mathrm{GL}_2(\mathbb{F}_q)$.** We introduce the subgroups B, T, U of $\mathrm{GL}_2(\mathbb{F}_q)$ as follows:

$$B := \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times,\, b \in \mathbb{F}_q \right\},$$

$$T := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a, d \in \mathbb{F}_q^\times \right\},$$

$$U := \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, b \in \mathbb{F}_q \right\}.$$

Note that U is a normal subgroup in B and that we have the semi-direct decomposition $B = T \ltimes U$. In particular, we have a natural surjection $B \twoheadrightarrow T$ by quotienting by $U \triangleleft B$. We let Z denote the center of G, which consists of scalar matrices:

$$Z := \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \in M_2(\mathbb{F}_q) \,\middle|\, a \in \mathbb{F}_q^\times \right\}.$$

Remark 1.5. In the context of theory of reductive groups, the subgroups B, T, and U are called a *Borel subgroup*, a *maximal torus*, and the *unipotent radical (of B)*, respectively.

Definition 1.6 (Principal series representation)**.** Suppose that $\boldsymbol{\chi}\colon T\to\mathbb{C}^\times$ is a character. Then, by pulling back $\boldsymbol{\chi}$ via $B\twoheadrightarrow T$, we may regard it as a character of B (this procedure is called the *inflation*). We call the induction $\mathrm{Ind}_B^G\boldsymbol{\chi}$ of $\boldsymbol{\chi}$ from B to G a *principal series representation* (for $\boldsymbol{\chi}$).

Note that we have $T\cong\mathbb{F}_q^\times\times\mathbb{F}_q^\times$, hence any character $\boldsymbol{\chi}$ of T can be expressed as $\boldsymbol{\chi}=\chi_1\boxtimes\chi_2$ with some characters χ_1 and χ_2 of $\mathbb{F}_q^\times$, i.e., for any $(t_1,t_2)\in T$, we have $\boldsymbol{\chi}(t_1,t_2)=\chi_1(t_1)\cdot\chi_2(t_2)$. We shortly write $\chi_1\times\chi_2$ for $\mathrm{Ind}_B^G\boldsymbol{\chi}=\mathrm{Ind}_B^G(\chi_1\boxtimes\chi_2)$. Since the dimension of $\chi_1\times\chi_2$ is equal to the index of B in G, we have

$$\dim(\chi_1\times\chi_2)=\frac{|G|}{|B|}=\frac{(q^2-1)(q^2-q)}{(q-1)^2q}=q+1.$$

Let us first investigate the principal series representations for $\boldsymbol{\chi}=\chi_1\boxtimes\chi_2$ such that $\chi_1\neq\chi_2$.

Proposition 1.7. *If $\chi_1\neq\chi_2$, then $\chi_1\times\chi_2$ is an irreducible representation of G of dimension $q+1$. Moreover, for two characters $\chi_1\boxtimes\chi_2$ and $\chi_1'\boxtimes\chi_2'$ of T,*

$$\chi_1\times\chi_2\cong\chi_1'\times\chi_2'\iff\chi_1'\boxtimes\chi_2'=\chi_1\boxtimes\chi_2\ \text{or}\ \chi_2\boxtimes\chi_1.$$

We next consider the case where $\chi_1=\chi_2$.

Proposition 1.8.
1. *The principal series representation $\mathbb{1}\times\mathbb{1}=\mathrm{Ind}_B^G\mathbb{1}$ associated to the trivial character of T is the sum of two irreducible representations of G:*
   - *one is the trivial representation of G;*
   - *the other is a q-dimensional irreducible representation of G, for which we write St_G (we call the "Steinberg representation" of G).*
2. *For any character χ of $\mathbb{F}_q^\times$, we have $\chi\times\chi\cong(\mathbb{1}\times\mathbb{1})\otimes(\chi\circ\det)$. In particular, we have*
$$\chi\times\chi\cong(\chi\circ\det)\oplus\mathrm{St}_G\otimes(\chi\circ\det).$$

We prove Propositions 1.7 and 1.8 simultaneously.

Proof. Fisrt, by Frobenius reciprocity (the adjunction formula between the induction and restriction), we have

$$\mathrm{Hom}_G(\chi_1\times\chi_2,\chi_1'\times\chi_2')\cong\mathrm{Hom}_B\bigl(\mathrm{Res}_B^G(\chi_1\times\chi_2),\chi_1'\boxtimes\chi_2'\bigr).$$

By applying the Mackey decomposition formula, we have

$$\mathrm{Res}_B^G(\chi_1\times\chi_2)\cong\bigoplus_{s\in B\backslash G/B}\mathrm{Ind}_{B\cap s^{-1}Bs}^B\mathrm{Res}_{B\cap s^{-1}Bs}^{s^{-1}Bs}(\chi_1\boxtimes\chi_2)^s,$$

where $(\chi_1\boxtimes\chi_2)^s$ denotes the character of $s^{-1}Bs$ defined by $(\chi_1\boxtimes\chi_2)^s(s^{-1}bs)=(\chi_1\boxtimes\chi_2)(b)$. Now we use the *Bruhat decomposition*:

$$G=B\sqcup BwB,\quad w:=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

Since w maps B to its transpose $\overline{B}$ and swaps the first and second factors of $T\cong\mathbb{F}_q^\times\times\mathbb{F}_q^\times$, we get

$$\bigoplus_{s\in B\backslash G/B}\mathrm{Ind}_{B\cap s^{-1}Bs}^B\mathrm{Res}_{B\cap s^{-1}Bs}^{s^{-1}Bs}(\chi_1\boxtimes\chi_2)^s\cong\underbrace{(\chi_1\boxtimes\chi_2)}_{s=1}\oplus\underbrace{\mathrm{Ind}_T^B\mathrm{Res}_T^{\overline{B}}(\chi_2\boxtimes\chi_1)}_{s=w}.$$

(Note that, on the right-hand side, $\chi_1 \boxtimes \chi_2$ and $\chi_2 \boxtimes \chi_1$ are regarded as characters of B and $\overline{B}$ by inflation, respectively.) This implies that

$$\begin{aligned}&\mathrm{Hom}_B\big(\mathrm{Res}^G_B(\chi_1 \times \chi_2), \chi_1' \boxtimes \chi_2'\big)\\ &\qquad\cong \mathrm{Hom}_B(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2') \oplus \mathrm{Hom}_B\big(\mathrm{Ind}^B_T \mathrm{Res}^{\overline{B}}_T(\chi_2 \boxtimes \chi_1), \chi_1' \boxtimes \chi_2'\big).\end{aligned}$$

The first summand on the right-hand side is equal to $\mathrm{Hom}_T(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2')$. By Frobenius reciprocity, the second summand is equal to $\mathrm{Hom}_T\big(\chi_2 \boxtimes \chi_1, \chi_1' \boxtimes \chi_2'\big)$. So we conclude that

$$(\star)\quad \mathrm{Hom}_G(\chi_1 \times \chi_2, \chi_1' \times \chi_2') \cong \mathrm{Hom}_T\big(\chi_1 \boxtimes \chi_2, \chi_1' \boxtimes \chi_2'\big) \oplus \mathrm{Hom}_T\big(\chi_2 \boxtimes \chi_1, \chi_1' \boxtimes \chi_2'\big).$$

In particular, this implies that

$$\mathrm{End}_G(\chi_1 \times \chi_2) \cong \begin{cases} \mathbb{C} & \chi_1 \neq \chi_2, \\ \mathbb{C} \oplus \mathbb{C} & \chi_1 = \chi_2. \end{cases}$$

By Schur's lemma, this says that $\chi_1 \times \chi_2$ is irreducible when $\chi_1 \neq \chi_2$ and decomposes into a sum of two irreducible representations when $\chi_1 = \chi_2$. So we obtained Proposition 1.7. (The latter assertion of Proposition 1.7 can be checked by the formula $(\star)$). When $\chi_1 = \chi_2 = \chi$, we can easily check that $\chi \times \chi$ contains $\chi \circ \det$. It's also not difficult to check that $\chi \times \chi$ is isomorphic to $(\mathbb{1} \times \mathbb{1}) \otimes (\chi \circ \det)$. (For example, again use Frobenius reciprocity.) Then Proposition 1.8 follows. □

So, how many irreducible representations have we obtained so far? Since there are $(q-1)$ characters of $\mathbb{F}_q^\times$, the principal series construction produces

$$\underbrace{\binom{q-1}{2}}_{\chi_1 \neq \chi_2} + \underbrace{2 \cdot (q-1)}_{\chi_1 = \chi_2} = \frac{q^2+q-2}{2}$$

irreducible representations of $\mathrm{GL}_2(\mathbb{F}_q)$ in total. Thus there should be exactly

$$(q^2-1) - \frac{q^2+q-2}{2} = \frac{q^2-q}{2}$$

more irreducible representations! These are called "cuspidal" representations.

1.4. Cuspidal representations of $\mathrm{GL}_2(\mathbb{F}_q)$.

Definition 1.9 (Cuspidal representations)**.** Let ρ be an irreducible representation of G. We say that ρ is *cuspidal* if ρ is not contained in any principal series representation.

Remark 1.10. We caution that this definition is somehow misleading for understanding the definition of a cuspidal representation in general. In general, there is a notion of a "parabolic subgroup" of a finite group of Lie type. When G is a finite group of Lie type, we say that its irreducible representation is cuspidal if it is not contained in the induction of any representation of the "reductive part" of any nontrivial parabolic subgroup of G (so-called "parabolic induction"). A Borel subgroup is a minimal parabolic subgroup. Because any nontrivial parabolic subgroup is Borel when $G = \mathrm{GL}_2(\mathbb{F}_q)$, we only have to care about principal series representations in the above definition.

Lemma 1.11. *Suppose that ρ is an irreducible representation of G. The following are equivalent:*

(1) ρ is cuspidal.
(2) The U-coinvariant ρ_U of ρ is zero.
(3) The U-invariant ρ^U of ρ is zero.
(4) $\langle \mathrm{Res}_U^G \rho, \mathbb{1}_U \rangle = 0$.

Proof. We first note that

$$\mathrm{Ind}_U^G \mathbb{1}_U = \mathrm{Ind}_B^G(\mathrm{Ind}_U^B \mathbb{1}_U) = \mathrm{Ind}_B^G\Big(\bigoplus_{\boldsymbol{\chi}\colon T\to\mathbb{C}^\times} \boldsymbol{\chi}\Big). = \bigoplus_{\boldsymbol{\chi}\colon T\to\mathbb{C}^\times} (\mathrm{Ind}_B^G \boldsymbol{\chi}).$$

Thus, by definition, ρ is cuspidal if and only if $\mathrm{Hom}_G(\rho, \mathrm{Ind}_U^G \mathbb{1}_U) = 0$. By Frobenius reciprocity, this is equivalent to that $\mathrm{Hom}_U(\mathrm{Res}_U^G \rho, \mathbb{1}_U) = 0$. As ρ is semisimple as representation of U, this is also equivalent to $\mathrm{Hom}_U(\mathbb{1}_U, \mathrm{Res}_U^G \rho) = 0$. The equivalences between (1)–(4) all follows from these observations. □

Now we construct all cuspidal irreducible representations of G "by hand". By regarding $\mathbb{F}_q^2$ as a 2-dimensional $\mathbb{F}_q$-vector space, we embed $\mathbb{F}_{q^2}$ into $M_2(\mathbb{F}_q)$. To be more precise, by choosing an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$ (hence get $\mathbb{F}_{q^2} \cong \mathbb{F}_q^{\oplus 2}$, which is regarded as the space of rank 2 column vectors), the multiplication of $\alpha \in \mathbb{F}_{q^2}$ on $\mathbb{F}_{q^2} \cong \mathbb{F}_q^{\oplus 2}$ itself can be written by

$$\alpha \cdot \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then the image of $\alpha \in \mathbb{F}_{q^2}$ in $M_2(\mathbb{F}_q)$ is given by $\left(\begin{smallmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{smallmatrix}\right)$. Note that this embedding depends on the choice of an $\mathbb{F}_q$-basis of $\mathbb{F}_{q^2}$, hence not canonical. We fix such an embedding and define a subgroup $S \subset G$ to be the image of $\mathbb{F}_{q^2}^\times$ (note that S contains Z; it is nothing but $\mathbb{F}_q^\times$ contained in $\mathbb{F}_{q^2}^\times$). We also fix a nontrivial character $\psi\colon U \to \mathbb{C}^\times$.

Definition 1.12. For any character $\theta\colon S \to \mathbb{C}^\times$ satisfying $\theta^{q-1} \neq \mathbb{1}$, we define a virtual representation π_θ of G by

$$\pi_\theta := \mathrm{Ind}_{ZU}^G(\theta|_Z \boxtimes \psi) - \mathrm{Ind}_S^G \theta.$$

Here, the right-hand side is considered in the Grothendieck group of representations of G (or, π_θ can be simply regarded as a class function on G).

Proposition 1.13. *The virtual representation π_θ is a $(q-1)$-dimensional irreducible cuspidal representation.*

To prove this proposition, let us first investigate the characters of π_θ.

Lemma 1.14. *The character values of π_θ are given as follows:*
(1) $\Theta_{\pi_\theta}(z_a) = (q-1)\theta(a)$ for $a \in \mathbb{F}_q^\times$,
(2) $\Theta_{\pi_\theta}(u_a) = -\theta(a)$ for $a \in \mathbb{F}_q^\times$,
(3) $\Theta_{\pi_\theta}(t_{a,b}) = 0$ for distinct $a, b \in \mathbb{F}_q^\times$,
(4) $\Theta_{\pi_\theta}(s) = -\theta(s) - \theta(s)^q$ for $s \in S \smallsetminus Z$.

Proof. The idea is to apply the Frobenius formula. Here let us only check (4).

First recall that $S \subset G$ is defined by the multiplication action of $\mathbb{F}_{q^2}$ on $\mathbb{F}_{q^2}$ itself. This implies that if $s \in S$ does not lie in $Z \subset S$, then the characteristic polynomial of s is an irreducible monic of degree 2. Conversely, for any irreducible monic of degree 2, there exists an $s \in S$ having the monic as its characteristic polynomial.

(The point of this argument is that any irreducible monic of degree 2 generates the degree 2 extension $\mathbb{F}_{q^2}$ of $\mathbb{F}_q$ in $\overline{\mathbb{F}}_q$.)

Now let $s \in S$ be an element with irreducible characteristic polynomial x^2+ax+b, hence conjugate to $s_{a,b}$. We first compute the character of $\mathrm{Ind}_{ZU}^{G}(\theta|_Z \boxtimes \psi)$ at s. By Frobenius formula, we have

$$\Theta_{\mathrm{Ind}_{ZU}^{G}(\theta|_Z\boxtimes\psi)}(s) = \sum_{\substack{x\in ZU\backslash G\\ xsx^{-1}\in ZU}} (\theta|_Z \boxtimes \psi)(xsx^{-1}).$$

However, since any element of ZU cannot have $x^2 + ax + b$ as its characteristic polynomial, s cannot be conjugate to an element of ZU. In other words, the index set of the above sum must be empty, hence $\Theta_{\mathrm{Ind}_{ZU}^{G}(\theta|_Z\boxtimes\psi)}(s) = 0$.

We next compute the character of $\mathrm{Ind}_S^G \theta$ at s. Again by Frobenius formula, we have

$$\Theta_{\mathrm{Ind}_S^G \theta}(s) = \sum_{\substack{x\in S\backslash G\\ xsx^{-1}\in S}} \theta(xsx^{-1}).$$

Let us determine the index set. Note that $S = \mathbb{F}_q[s]^\times$. In particular, if $xsx^{-1} \in S$, then we have $xSx^{-1} \subset S$, which furthermore implies that $xSx^{-1} = S$, i.e., $x \in N_G(S)$. Suppose that we have an element $x \in N_G(S) \smallsetminus S$. Then the conjugation via x should induce a nontrivial $\mathbb{F}_q$-automorphism of $\mathbb{F}_q[s]$ ($\cong \mathbb{F}_{q^2}$) (otherwise, x must be in $Z_G(S)$, which equals S). From this, we see that the index set can be regarded as a subset of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$. In fact, there indeed exists an element $x \in N_G(S) \smallsetminus S$. To see this, let us note that s and s^q have the same characteristic polynomials. Especially, there exists an element $x \in G$ satisfying $xsx^{-1} = s^q$. Since both s and s^q generate $\mathbb{F}_q[s]$, this implies that $xSx^{-1} = S$, hence $x \in N_G(S)$. Of course, this element x cannot be in S. In summary, we get

$$\sum_{\substack{x\in S\backslash G\\ xsx^{-1}\in S}} \Theta_\theta(xsx^{-1}) = \theta(s) + \theta(s^q).$$

Finally, recalling that π_θ is defined to be $\mathrm{Ind}_{ZU}^{G}(\theta|_Z \boxtimes \psi) - \mathrm{Ind}_S^G \theta$, we get the result. □

Exercise 1.15. Check (1), (2), and (3).

Now let us prove Proposition 1.13.

Proof of Proposition 1.13. To show that π_θ, it suffices to check that $\langle \pi_\theta, \pi_\theta \rangle = 1$. Note that, even if we can show this, there is a possibility that π_θ is the "minus" of an irreducible representation. However, this possibility is excluded since the character value of π_θ at the unit element z_1 is given by $(q-1)$. (Also, we see that the dimension is $(q-1)$ from this.)

Recall that

$$\langle \pi_\theta, \pi_\theta \rangle = \frac{1}{|G|} \sum_{g\in G} \Theta_{\pi_\theta}(g) \cdot \overline{\Theta_{\pi_\theta}(g)}.$$

(1) The sum (not divided by $|G|$) over the conjugacy classes of z_a is

$$\sum_{a\in\mathbb{F}_q^\times} 1 \cdot (q-1)\theta(a) \cdot \overline{(q-1)\theta(a)} = \sum_{a\in\mathbb{F}_q^\times} 1 \cdot (q-1)^2 = (q-1)^3.$$

(2) The sum (not divided by $|G|$) over the conjugacy classes of u_a is
$$\sum_{a\in\mathbb{F}_q^\times}(q^2-1)\cdot(-\theta(a))\cdot\overline{(-\theta(a))}=\sum_{a\in\mathbb{F}_q^\times}(q^2-1)=(q^2-1)(q-1).$$
(3) The sum (not divided by $|G|$) over the conjugacy classes of $t_{a,b}$ is zero since each character value is zero.
(4) By noting that the orbits of $(S\smallsetminus Z)$ by the action of $\mathrm{Gal}(\mathbb{F}_{q^2}/\mathbb{F}_q)$ bijectively correspond to the conjugacy classes of elements of the form $s_{a,b}$. Hence, the sum (note divided by $|G|$) over the conjugacy classes of $s_{a,b}$ is
$$\begin{aligned}&\frac{1}{2}\sum_{s\in S\smallsetminus Z}(q^2-q)\cdot(-\theta(s)-\theta(s)^q)\cdot\overline{(-\theta(s)-\theta(s)^q)}\\&=\frac{q^2-q}{2}\sum_{s\in\mathbb{F}_{q^2}^\times\smallsetminus\mathbb{F}_q^\times}(-\theta(s)-\theta(s)^q)\cdot\overline{(-\theta(s)-\theta(s)^q)}.\end{aligned}$$
By an elementary computation, we can check that this equals $(q^2-q)(q-1)^2$.

Therefore, we get
$$\langle\pi_\theta,\pi_\theta\rangle=\frac{1}{|G|}\cdot\Big((q-1)^3+(q^2-1)(q-1)+0+(q^2-q)(q-1)^2\Big)=1.$$
Finally, let us check the cuspidality of π_θ. It suffices to show that
$$\langle\mathrm{Res}^G_U\pi_\theta,\mathbb{1}_U\rangle=\frac{1}{|U|}\sum_{u\in U}\Theta_{\pi_\theta}=0.$$
Any element $\left(\begin{smallmatrix}1&a\\0&1\end{smallmatrix}\right)$ with $a\in\mathbb{F}_q^\times$ is conjugate to $u_1=\left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$. Hence,
$$\sum_{u\in U}\Theta_{\pi_\theta}=\Theta_{\pi_\theta}(z_1)+(q-1)\Theta_{\pi_\theta}(u_1)=(q-1)\theta(1)-(q-1)\theta(1)=0.$$
□

Exercise 1.16. Complete the computation skipped in the above proof.

Proposition 1.17. *For any $\theta_i\colon S\to\mathbb{C}^\times$ satisfying $\theta_i^{q-1}\neq\mathbb{1}$ $(i=1,2)$, we have $\pi_{\theta_1}\cong\pi_{\theta_2}$ if and only if $\theta_1=\theta_2$ or $\theta_1=\theta_2^q$.*

Proof. By the character formulas of Θ_{π_θ}, we see that $\pi_{\theta_1}\cong\pi_{\theta_2}$ only if
$$\theta_1(s)+\theta_1(s)^q=\theta_2(s)+\theta_2(s)^q$$
for any $s\in S$. Recall that *Artin's lemma* says that distinct characters of any finite group are linear independent. Hence, by noting that $\theta^q\neq\theta$, the above condition is equivalent to that $\theta_1=\theta_2$ or $\theta_1=\theta_2^q$. Conversely, if this is satisfied, then we have $\pi_{\theta_1}\cong\pi_{\theta_2}$ by the character formula of Θ_{π_θ}. □

Here, note that S is of order q^2-1, hence any character θ of S satisfies $\theta^{q^2}=\theta$. Thus the condition $\theta_1=\theta_2^q$ is also equivalent to $\theta_2=\theta_1^q$. Proposition 1.17 enables us to count the number of irreducible cuspidal representations obtained in this way. The group S is cyclic of order q^2-1, thus there exactly $(q-1)$ characters of S satisfying $\theta^{q-1}=\mathbb{1}$. In other words, there exactly q^2-q characters of S satisfying $\theta^{q-1}\neq\mathbb{1}$. Therefore the above construction provides $\frac{q^2-q}{2}$ irreducible cuspidal representations, hence all!

1.5. **What is Deligne–Lusztig theory? (a bit more precisely).** The construction of π_θ presented above is somehow mysterious and seems difficult to generalize. So, we want a more conceptual construction of π_θ, which could work in a more general setting. We can find a hint in Drinfeld's observation.

Before we talk about "the curve" of Drinfeld, let us introduce the groups $G' := \mathrm{SL}_2(\mathbb{F}_q)$ and $S' := S \cap G'$. Note that S' is identified with the norm 1 subgroup of $\mathbb{F}_{q^2}^\times$, i.e.,

$$S' \cong \mathrm{Ker}(\mathrm{Nr}\colon \mathbb{F}_{q^2}^\times \to \mathbb{F}_q^\times).$$

In particular, S' is cyclic of order $(q+1)$. We can also introduce the notions of principal series or cuspidal representations for G' in a similar way. Basically, the representation theory of G' can be "derived" from that of G. Especially, the cuspidal representations of G' can be constructed by restricting those of G to G'. Thus let's talk about how to understand cuspidal representations of G' in the following.

Drinfeld investigated the following curve (see [Bon11, Chapter 2]).

Definition 1.18 (Drinfeld curve)**.** Let X be the curve defined by

$$X := \{(x,y) \in \mathbb{A}^2_{\overline{\mathbb{F}}_q} \mid xy^q - x^q y = 1\}.$$

The curve X has the following properties:

- G' acts on X by $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot (x,y) = (ax+by, cx+dy)$;
- S' acts on X by $s \cdot (x,y) = (sx, sy)$;
- the actions of G' and S' commute.

Because the étale cohomology has the functoriality in spaces, we can obtain a representation of $G' \times S'$ on the étale cohomology of X. By cutting it along any character θ of S', we get a representation of G'. In fact, this resulting representation is nothing but "π_θ". In other words, Drinfeld's curve gives a geometric realization of the cuspidal representation π_θ which was constructed in a mysterious way previously!

Deligne–Lusztig theory exactly generalizes this idea. Let G be a finite group of Lie type. The input/output of Deligne–Lusztig theory are as follows:

Input: a pair (S,θ) of
- a "maximal torus" S of G and
- a character $\theta\colon S(\mathbb{F}_q) \to \mathbb{C}^\times$.

Output: a virtual representation $R_S^G(\theta)$ of $G(\mathbb{F}_q)$ ("Deligne–Lusztig virtual representaiton").

For a given input (S,θ), Deligne–Lusztig first defined an algebraic variety X_S^G over $\overline{\mathbb{F}}_q$ equipped with an action of $G(\mathbb{F}_q) \times S(\mathbb{F}_q)$. This is called the Deligne–Lusztig variety (associated to (G,S)); this is a far generalization of the Drinfeld curve. Deligne–Lusztig considered its ℓ-adic étale cohomology $H_c^i(X_S^G, \overline{\mathbb{Q}}_\ell)$. Then, as explained above, we obtain a representation of $G(\mathbb{F}_q) \times S(\mathbb{F}_q)$ on $H_c^i(X_S^G, \overline{\mathbb{Q}}_\ell)$. By taking the alternating sum of the θ-isotypic part of each degree, we get the "output":

$$R_S^G(\theta) := \sum_{i \geq 0} (-1)^i H_c^i(X_S^G, \overline{\mathbb{Q}}_\ell)[\theta]$$

Remark 1.19. (1) At this point, you do not have to be able to understand the meaning of the terminologies such as "finite group of Lie type" or "maximal torus". It is also one of the purposes of this course to get familiar with these notions (through various examples).[1]

(2) As its symbol suggests, $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$ is a $\overline{\mathbb{Q}}_\ell$-vector space; not a $\mathbb{C}$-vector space. However, by choosing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we may convert $H^i_c(X^G_S, \overline{\mathbb{Q}}_\ell)$ to a $\mathbb{C}$-vector space. In fact, the resulting representation $R^G_S(\theta)$ with $\mathbb{C}$-coefficients is independent of the choice of such an isomorphism ("ℓ-independence", which is an important part of Deligne–Lusztig theory).

(3) The subgroup S of $\mathrm{GL}_2(\mathbb{F}_q)$ introduced in the previous section (or S' of $\mathrm{SL}_2(\mathbb{F}_q)$) is an example of a "maximal torus". With the above notation, we have $\pi_\theta \cong R^G_S(\theta)$ for $G = \mathrm{GL}_2(\mathbb{F}_q)$[2]. Recall that $T \cong \mathbb{F}_q^\times \times \mathbb{F}_q^\times$ is another example of a maximal torus of $\mathrm{GL}_2(\mathbb{F}_q)$. One surprising point is that Deligne–Lusztig theory also naturally generalizes the principal series construction. Namely, for any character $\boldsymbol{\chi}$ of $T \subset \mathrm{GL}_2$, we have $\mathrm{Ind}^G_B\, \boldsymbol{\chi} \cong R^G_T(\boldsymbol{\chi})$.

One of the highlights of the theory is that there exists an explicit formula of the Deligne–Lusztig virtual representaiton $R^G_S(\theta)$ called "Deligne–Lusztig character formula". We can analyze the representation $R^G_S(\theta)$ through that formula; for example, we can prove that any irreducible representation ρ of $G(\mathbb{F}_q)$ can be realized in $R^G_S(\theta)$ for some pair (S, θ).

References

[BH06] C. J. Bushnell and G. Henniart, *The local Langlands conjecture for* GL(2), Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 335, Springer-Verlag, Berlin, 2006.

[Bon11] C. Bonnafé, *Representations of* $\mathrm{SL}_2(\mathbb{F}_q)$, Algebra and Applications, Volume 13, 2011.

8:47pm, September 9, 2024

[1] On the other hand, I have to confess that I will only give a few words about the theory of étale cohomology.

[2] Precisely speaking, we need "up to sign" here