

1. WEEK 7: DELIGNE–LUSZTIG CHARACTER FORMULA

Let G be a connected reductive group over $k = \mathbb{F}_q$ and F its associated Frobenius endomorphism. We fix a k -rational maximal torus T of G and a Borel subgroup B of G containing T . We also fix a character $\theta: T^F \rightarrow \mathbb{C}^\times$. Then we have the Deligne–Lusztig virtual representation $R_{T \subset B}^G(\theta)$ of G^F . By abuse of notation, we also write $R_{T \subset B}^G(\theta)$ for the Deligne–Lusztig virtual character, which is a class function $G^F \rightarrow \mathbb{C}$ defined to be the trace of the Deligne–Lusztig virtual representation. Today’s aim is to establish a character formula for $R_{T \subset B}^G(\theta)$.

1.1. Deligne–Lusztig character formula. We write G_{ss}^F and G_{unip}^F for the set of semisimple (equivalently, prime-to- p order) and unipotent elements of G^F (equivalently, p -power order), respectively. In the following, for any $g \in G$ and $h \in G$, we write ${}^g h = ghg^{-1}$. Similarly, for any $g \in G$ and a subgroup $H \subset G$, we write ${}^g H = gHg^{-1}$.

Definition 1.1. We define a function $Q_T^G: G_{\text{unip}}^F \rightarrow \mathbb{C}$ by $Q_T^G := R_{T \subset B}^G(\mathbb{1})|_{G_{\text{unip}}^F}$. We call Q_T^G the *Green function (of G associated to T)*.

We note that, for notational convenience, we simply write “ Q_T^G ” although a priori Q_T^G depends on the choice of a Borel subgroup B containing T . (But, in fact, later it will turn out that Q_T^G does not depend on B !)

To state the Deligne–Lusztig character formula, let us recall that any element $g \in G^F$ has the Jordan decomposition $g = su$, where $s \in G^F$ is a semisimple element and $u \in G^F$ is a unipotent element such that $su = us$.

Theorem 1.2 (Deligne–Lusztig character formula). *Let $g \in G^F$ with Jordan decomposition $g = su$. We shortly write G_s for the centralizer of s in G , i.e., $G_s = Z_G(s) = \{x \in G \mid xs = sx\}$. Then we have*

$$R_T^G(\theta)(g) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u).$$

Let us explain why the right-hand side of this formula makes sense. We first note the following result (see [Car85, 1.14]).

Lemma 1.3. (1) *For any $s \in G_{\text{ss}}^F$, the identity component G_s° of its centralizer G_s is a connected reductive group defined over k .*
 (2) *Any unipotent element of G_s lies in G_s° . In particular, when $g \in G^F$ has the Jordan decomposition $g = su$, its unipotent part u belongs to $(G_s^\circ)^F$.*

Let us look at the index set of the sum in the Deligne–Lusztig character formula. When $x^{-1}sx \in T$, we necessarily have the opposite inclusion $Z_G(x^{-1}sx) \supset Z_G(T)$. Here, it is easy to check that $Z_G(x^{-1}sx) = x^{-1}Z_G(s)x$. On the other hand, it is known that the centralizer of a maximal torus in a connected reductive group is the maximal torus itself, i.e., $Z_G(T) = T$ (see [Spr09, 7.6.4]). Hence, we have $x^{-1}Z_G(s)x \supset T$, or equivalently, ${}^x T = xTx^{-1} \subset Z_G(s) = G_s$. Since T is connected, this furthermore implies that ${}^x T \subset G_s^\circ$. Furthermore, it is known that $(B \cap G_s^\circ)^\circ$ is a Borel subgroup of G_s° and $U \cap G_s^\circ$ is its unipotent radical.¹

¹Here, note that $U \cap G_s^\circ$ is already connected!

In summary, when $x^{-1}sx \in T$, we obtain a k -rational maximal torus xT of a connected reductive group G_s° . Thus it makes sense to consider the Green function $Q_{xT}^{G_s^\circ}$ of G_s° associated to xT and $(B \cap G_s^\circ)^\circ$. Since u belongs to $(G_s^\circ)_{\text{unip}}^F$, it also makes sense to look at the value of $Q_{xT}^{G_s^\circ}$ at u .

Thus the Deligne–Lusztig character formula reflects an inductive nature of the theory of reductive groups. The contribution of the semisimple part s is given just by θ , which is very simple. On the other hand, the contribution of the unipotent part u is given by the Green function, which is independent of θ and taken in G_s° . Hence, ultimately, the Deligne–Lusztig characters of G are governed by the Green functions for G and all its “smaller” reductive subgroups.

1.2. Outline of the proof of DL character formula. The key of the proof of the Deligne–Lusztig character formula is the following general result, which is called *Deligne–Lusztig’s fixed point formula*:

Theorem 1.4 (Deligne–Lusztig fixed point formula). *Let X be an algebraic variety over k and g is an automorphism of X of finite order. Let s and u be automorphisms of X such that s is of prime-to- p order, u is of p -power order, and $g = su = us$. Then we have $\mathcal{L}(g, X) = \mathcal{L}(u, X^s)$, where $X^s := \{x \in X \mid s(x) = x\}$.*

Unfortunately, I cannot explain the proof of this theorem in this course. Please look at [DL76, Theorem 3.2] if you have an interest.

Now suppose that $g \in G^F$ has the Jordan decomposition $g = su = us$. As discussed in the last week, we have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

Let us compute each $\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G)$ using the Deligne–Lusztig fixed point formula.

Recall that the action of (g, t) on $\mathcal{X}_{T \subset B}^G = \{x \in G \mid x^{-1}F(x) \in F(U)\}$ is given by $x \mapsto gxt$. We note that the order of T^F is prime-to- p . (Indeed, if we suppose that T splits over \mathbb{F}_{q^n} , i.e., $T_{\mathbb{F}_{q^n}} = \mathbb{G}_m^r$ for some r , we have $T^F = T(\mathbb{F}_q) \subset T_{\mathbb{F}_{q^n}}(\mathbb{F}_{q^n}) \cong (\mathbb{F}_{q^n}^\times)^r$.) Hence the order of t is also prime-to- p . Thus, the decomposition $(g, t) = (s, t) \circ (u, 1)$ satisfies the assumption of the Deligne–Lusztig fixed point formula.

We determine $(\mathcal{X}_{T \subset B}^G)^{(s, t)}$. In the following, we simply write $\mathcal{X} := \mathcal{X}_{T \subset B}^G$.

Proposition 1.5. *We have*

$$\mathcal{X}^{(s, t)} = \bigsqcup_{\substack{x \in G^F / (G_t^\circ)^F \\ xt = s^{-1}}} \mathcal{X}^{(s, t)}(x),$$

where we put $\mathcal{X}^{(s, t)}(x) := \mathcal{X}^{(s, t)} \cap xG_t^\circ$.

Proof. Suppose that $y \in \mathcal{X}^{(s, t)}$, i.e., $y \in G$ is an element satisfying $syt = y$ and $y^{-1}F(y) \in F(U)$ (say $y^{-1}F(y) = v$). By applying F to $syt = y$, we get $sF(y)t = F(y)$, thus $syvt = yv$. Combining $syvt = yv$ with $syt = y$, we get $yt^{-1}vt = yv$, hence $t^{-1}vt = v$. This means that u belongs to $G_t = Z_G(t)$. As u is unipotent, u furthermore lies in G_t° . Let us apply Lang’s theorem to G_t° , which asserts that the map

$$G_t^\circ \rightarrow G_t^\circ: z \mapsto z^{-1}F(z)$$

is surjective; we can find an element $z \in G_t^\circ$ satisfying $z^{-1}F(z) = v$.

We put $x := yz^{-1}$. Then $F(x) = F(y)F(z)^{-1} = yvv^{-1}z^{-1} = yz^{-1} = x$, i.e., $x \in G^F$. Note that we have $y \in xG_t^\circ$. Furthermore, we have

$$xz = y = syt = s(xz)t = sxtz$$

(use that $z \in G_t$ in the last equality), hence ${}^xt = s^{-1}$.

From the discussion so far, we have obtained

$$\mathcal{X}^{(s,t)} = \bigcup_{\substack{x \in G^F / (G_t^\circ)^F \\ {}^xt = s^{-1}}} \mathcal{X}^{(s,t)}(x).$$

It is obvious that the union is disjoint. \square

Let us investigate each summand $\mathcal{X}^{(s,t)}(x)$. Note that, since $t \in T^F$, we have $T \subset G_t^\circ$. Moreover, $B_t^\circ := (B \cap G_t^\circ)^\circ$ is a Borel subgroup of G_t° with unipotent radical $U \cap G_t^\circ$ (see the paragraph after Lemma 1.3). Therefore, it makes sense to consider the Deligne–Lusztig variety for G_t° associated to $T \subset B_t^\circ$:

$$\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} = \{y' \in G_t^\circ \mid y'^{-1}F(y') \in U \cap G_t^\circ\}.$$

This is a variety equipped with an action of $(G_t^\circ)^F \times T^F$. On the other hand, $\mathcal{X}^{(s,t)}(x)$ is stable under the action of the subgroup $(G_s^\circ)^F \times T^F$ of $G^F \times T^F$ on \mathcal{X} .

Proposition 1.6. *Let $x \in G^F$ be an element satisfying ${}^xt = s^{-1}$. Then have an isomorphism of varieties*

$$\varphi_x : \mathcal{X}^{(s,t)}(x) \xrightarrow{\cong} \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ} : y \mapsto x^{-1}y,$$

which is equivariant with respect to the actions of $(G_s^\circ)^F \times T^F$ on $\mathcal{X}^{(s,t)}(x)$ and $(G_t^\circ)^F \times T^F$ on $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$. Here, $(G_s^\circ)^F \times T^F$ and $(G_t^\circ)^F \times T^F$ are identified by $(z, t') \mapsto (x^{-1}zx, t')$.

Proof. Suppose that $y \in \mathcal{X}^{(s,t)}(x)$, i.e., $y \in xG_t^\circ$ is an element satisfying $syt = y$ and $y^{-1}F(y) \in F(U)$. Then we have $x^{-1}y \in G_t^\circ$ and thus

$$(x^{-1}y)^{-1}F(x^{-1}y) = y^{-1}F(y) \in F(U) \cap G_t^\circ = F(U \cap G_t^\circ).$$

In other words, $\varphi_x(y) = x^{-1}y$ belongs to $\mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$. Conversely, for any element $y' \in \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}$, we can check that $\varphi_x^{-1}(y') = xy' \in \mathcal{X}^{(s,t)}(x)$.

Let us check the assertion on the equivariance. What we have to prove is that, for any $(z, t') \in (G_s^\circ)^F \times T^F$ and $y \in \mathcal{X}^{(s,t)}(x)$, we have

$$\varphi_x((z, t') \cdot y) = (x^{-1}zx, t') \cdot \varphi_x(y).$$

The left-hand side is given by $\varphi_x((z, t') \cdot y) = \varphi_x(zyt') = x^{-1}zyt'$. The right-hand side is given by $(x^{-1}zx, t') \cdot \varphi_x(y) = (x^{-1}zx, t') \cdot (x^{-1}y) = x^{-1}zx(x^{-1}y)t' = x^{-1}zyt'$. So these indeed coincide. \square

Now let us start the proof of the Deligne–Lusztig character formula:

Proof of Theorem 1.2. We have

$$R_{T \subset B}^G(\theta)(g) = \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G).$$

By applying the Deligne–Lusztig fixed point theorem to $(g, t) = (s, t) \circ (u, 1)$, we get

$$\mathcal{L}((g, t), \mathcal{X}_{T \subset B}^G) = \mathcal{L}((u, 1), (\mathcal{X}_{T \subset B}^G)^{(s, t)}).$$

By combining the above propositions, we get

$$\mathcal{L}(u, (\mathcal{X}_{T \subset B}^G)^{(s, t)}) = \sum_{\substack{x \in G^F / (G_t^\circ)^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}).$$

Hence we get

$$\begin{aligned} R_{T \subset B}^G(\theta)(g) &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \sum_{\substack{x \in G^F / (G_t^\circ)^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}) \\ &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t)^{-1} \cdot \frac{1}{|(G_t^\circ)^F|} \sum_{\substack{x \in G^F \\ xt = s^{-1}}} \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}). \end{aligned}$$

Note that the internal sum is nonzero only when there exists an element $x \in G^F$ satisfying $t = x^{-1}s^{-1}x$. In this case, $|(G_t^\circ)^F| = |(G_s^\circ)^F|$, hence the above equals

$$\frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{t \in T^F} \sum_{\substack{x \in G^F \\ xt = s^{-1}}} \theta(t)^{-1} \cdot \mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}).$$

We note that the set $\{(t, x) \in T^F \times G^F \mid xt = s^{-1}\}$ is bijective to $\{x \in G^F \mid x^{-1}sx \in T^F\}$ by $(t, x) \mapsto x$. By also noting that $\mathcal{L}(x^{-1}ux, \mathcal{X}_{T \subset B_t^\circ}^{G_t^\circ}) = \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ})$, we rewrite the above double sum:

$$\begin{aligned} &\frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}s^{-1}x)^{-1} \cdot \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ}) \\ &= \frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot \mathcal{L}(u, \mathcal{X}_{xT \subset B_s^\circ}^{G_s^\circ}). \end{aligned}$$

Here, in general, we have

$$Q_T^G(u) = \frac{1}{|T^F|} \cdot \mathcal{L}(u, \mathcal{X}_{T \subset B}^G).$$

Indeed, by definition, we have

$$Q_T^G(u) = \frac{1}{|T^F|} \sum_{t \in T^F} \mathcal{L}((u, t), \mathcal{X}_{T \subset B}^G).$$

By using the Deligne–Lusztig fixed point formula to $(u, t) = (u, 1) \cdot (1, t)$, we have $\mathcal{L}((u, t), \mathcal{X}_{T \subset B}^G) = \mathcal{L}((u, 1), (\mathcal{X}_{T \subset B}^G)^{(1, t)})$. However, $(\mathcal{X}_{T \subset B}^G)^{(1, t)}$ is nonempty only when $t = 1$ (indeed, $x \in \mathcal{X}_{T \subset B}^G$ is fixed by $(1, t)$ if and only if $xt = x$). Thus we get

$$Q_T^G(u) = \frac{1}{|T^F|} \mathcal{L}(u, \mathcal{X}_{T \subset B}^G).$$

Therefore, we finally obtain

$$R_{T \subset B}^G(g) = \frac{1}{|T^F| \cdot |(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u).$$

□

Corollary 1.7. *We have $R_{T \subset B}^G(\theta)|_{G_{\text{unip}}^F} = Q_T^G$ for any character $\theta: T^F \rightarrow \mathbb{C}^\times$.*

Proof. Let $g \in G_{\text{unip}}^F$ (hence its semisimple part s is 1 and unipotent part u is g). Then, by applying the Deligne–Lusztig character formula to g , we get

$$\begin{aligned} R_{T \subset B}^G(\theta)(g) &= \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(u) \\ &= \frac{1}{|G^F|} \sum_{x \in G^F} Q_{xT}^G(u). \end{aligned}$$

It is not difficult to check that, in general, we have $R_{T \subset B}^G(\theta)(g) = R_{xT \subset xB}^G({}^x\theta)({}^xg)$, where ${}^x\theta$ denotes the character of ${}^xT^F$ defined by ${}^x\theta({}^xt) = \theta(t)$. In particular, when $\theta = \mathbb{1}$, hence get $Q_T^G(u) = Q_{xT}^G({}^xu)$. By also noting that the Green function is invariant under conjugation (since it is the restriction of a Deligne–Lusztig character, which is a class function), we get $Q_T^G(u) = Q_{xT}^G({}^xu) = Q_{xT}^G(u)$. Hence the most right-hand side of the above equalities is $Q_T^G(u)$. □

Definition 1.8. We say that a semisimple element $s \in G$ is *regular* if G_s° is a maximal torus of G .

Example 1.9. Let $G = \text{GL}_2$. Let T be the diagonal maximal torus of G . We consider an element $s = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in T$. Then, since s is already diagonalized, s is semisimple. Let us compute the centralizer $G_s = Z_G(s)$ of s in G .

- When $a = b$, s commutes with any element of G . Thus $G_s = G$, hence $G_s^\circ = G^\circ = G$. Hence s is not regular in this case.
- Suppose that $a \neq b$. If $g = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in Z_G(s)$, we have $sgs^{-1} = g$. Since

$$sgs^{-1} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}^{-1} = \begin{pmatrix} x & aby^{-1} \\ a^{-1}zb & w \end{pmatrix},$$

we necessarily have $y = z = 0$, i.e., $g \in T$. Conversely, we obviously have $T \subset Z_G(s)$. Hence we get $G_s = T$, so $G_s^\circ = T$, which means that s is regular.

Exercise 1.10. Let $G = \text{GL}_n$ and $g \in G$. Prove that g is regular semisimple if and only if the characteristic polynomial of g has n distinct roots. (Hint: compute the centralizer of g in G by looking at the Jordan normal form of g .)

Exercise 1.11. Let $G = \text{GL}_n$. Recall that G^F -conjugacy classes of k -rational maximal tori of G correspond bijectively to the conjugacy classes of \mathfrak{S}_n (see Week 5 notes). Let S be a maximal torus of G corresponding to the cyclic permutation $(12 \cdots n) \in \mathfrak{S}_n$. Count the number of regular semisimple elements in $S^F = \mathbb{F}_{q^n}^\times$.

Corollary 1.12. *Suppose that $s \in G^F$ is a regular semisimple element. If s is not conjugate to any element of T^F , then we have $R_{T \subset B}^G(\theta)(s) = 0$. If s is conjugate to any element of T^F (suppose that s itself belongs to T^F), then we have*

$$R_{T \subset B}^G(\theta)(s) = \sum_{x \in W_{G^F}(T)} \theta(x^{-1}sx),$$

where $W_{G^F}(T) := N_{G^F}(T)/T^F$.

Proof. By the Deligne–Lusztig character formula, we have

$$R_{T \subset B}^G(\theta)(s) = \frac{1}{|(G_s^\circ)^F|} \sum_{\substack{x \in G^F \\ x^{-1}sx \in T^F}} \theta(x^{-1}sx) \cdot Q_{xT}^{G_s^\circ}(1).$$

Since the index set is empty if s is not conjugate to any element of T^F , we get the first assertion.

To show the second assertion, let us suppose that $s \in T^F$. Then, we must have $G_s^\circ = Z_G(s)^\circ \supset Z_G(T) = T$. Since G_s° is a maximal torus of G , this implies that $G_s^\circ = T$. By the same argument, the condition $x^{-1}sx \in T^F$ of the index set implies that $T = x^{-1}Tx$. In other words, $x \in N_{G^F}(T)$. Conversely, any element $x \in N_{G^F}(T)$ satisfies $x^{-1}sx \in T^F$. Thus, by noting that $Q_T^T(1) = 1$ (this can be checked by going back to the definition), we get

$$R_{T \subset B}^G(\theta)(s) = \frac{1}{|T^F|} \sum_{x \in N_{G^F}(T)} \theta(x^{-1}sx) = \sum_{x \in W_{G^F}(T)} \theta(x^{-1}sx).$$

□

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