In [1]:
```
import numpy as np
import matplotlib.pyplot as plt
from scipy import stats
```

Bayesian estimate of μ and σ.

Let's go through some of the calculations in Gregory's Chapter 9 through a specific example. There are three cases for estimating μ discussed in that chapter -- A) known noise σ same for all e_i (Section 9.2.1), B) known noise σ_i unequal for all e_i (Section 9.2.2), and C) unknown noise σ same for all e_i (Section 9.2.3). Also discussed is a Bayesian estimate of σ (Section 9.2.4) for Case C.

Let's implement Cases A and C. Try to implement Case B yourself at home.

In [2]:
```
# set a seed so we always get same results
np.random.seed(12345)

mu  = 5.0   # parent population mean
sig = 2.0   # parent population standard deviation

n   = 10

# sample from Gaussian; compute sample mean, variance, and chi^2
d = np.random.normal(loc=mu, scale=sig, size=n)
smu   = np.mean(d)
sigmu = sig/np.sqrt(n)
chi2min = np.sum(((d-smu)/sig)**2)

# print summary
np.set_printoptions(formatter={'float': '{: 0.4f}'.format})
print("                Data  = ", d)
print("            MLE mean  = %8.4f" % smu)
print("   MLE error on mean  = %8.4f" % sigmu)
print(" minimum chi^2 value  = %8.4f" % chi2min)
```

```
                Data  =  [ 4.5906  5.9579  3.9611  3.8885  8.9316  7.7868  5.1
858  5.5635  6.5380
  7.4929]
            MLE mean  =   5.9897
   MLE error on mean  =   0.6325
 minimum chi^2 value  =   6.4402
```

In [3]:
```
# upper and lower bounds for mu and sigma priors
muL  = 0.0
muH  = 10.0
sigL = 0.1
sigH = 6.0

# generate grid of mu and sigma values that extend beyond
# the non-zero prior ranges for plotting purposes
# we will use them throughout this notebook
```

```
mugrid  = np.linspace(0, 20, num=1001, endpoint=True)
siggrid = np.linspace(0.1, 10, num=1001, endpoint=True)
```

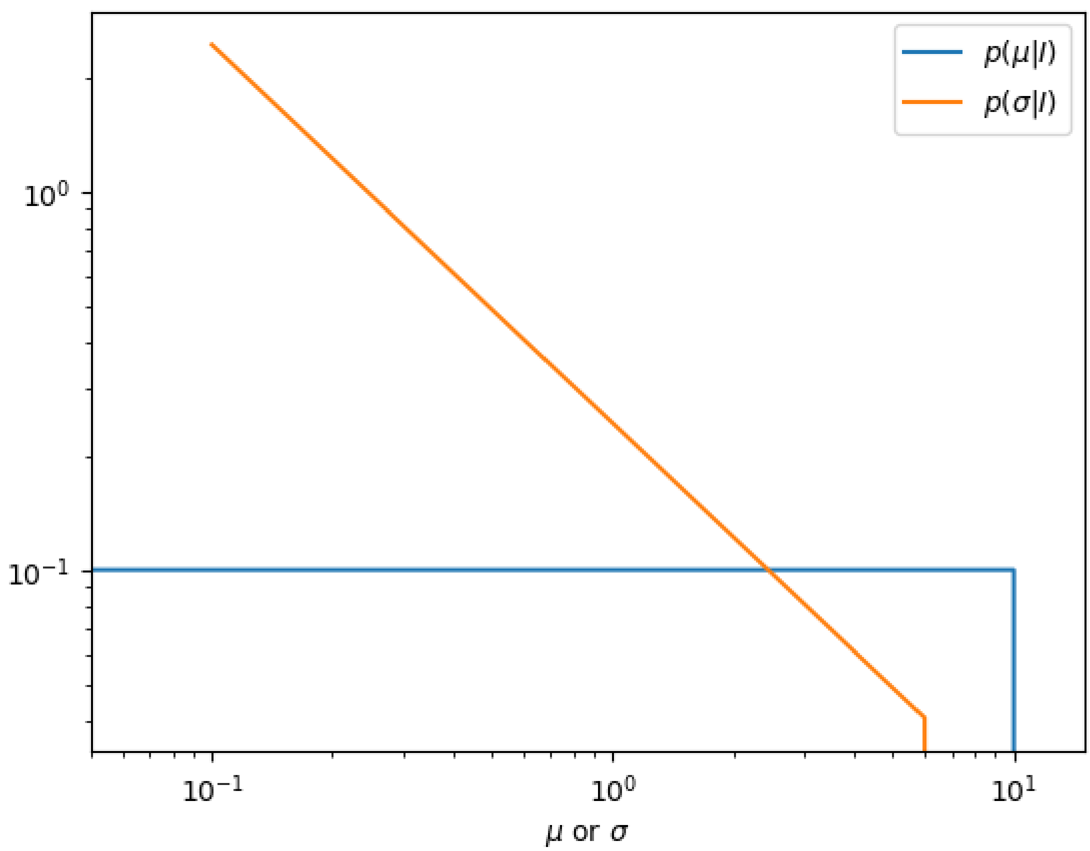
In [4]:
```
def priorUniform(mu, mumin=muL, mumax=muH):
    # uniform prior
    prior = np.zeros_like(mu)+1.0/(mumax-mumin)
    # make sure to set prior outside range to zero
    prior[mu<mumin] = prior[mu>mumax] = 0.0
    return prior
```

In [5]:
```
def priorJeffreys(sig, sigmin=sigL, sigmax=sigH):
    # Jeffreys prior
    prior = 1.0/(sig*np.log(sigmax/sigmin))
    # make sure to set prior outside range to zero
    prior[sig<sigmin] = prior[sig>sigmax] = 0.0
    return prior
```

In [6]:
```
priormu  = priorUniform(mugrid)
priorsig = priorJeffreys(siggrid)
```

In [7]:
```
plt.plot(mugrid, priormu, label='$p(\mu|I)$')
plt.plot(siggrid, priorsig, label='$p(\sigma|I)$')
plt.xscale('log')
plt.yscale('log')
plt.xlim([0.05, 15])
plt.xlabel('$\mu$ or $\sigma$')
plt.legend()
```

Out[7]: <matplotlib.legend.Legend at 0x1167602b0>



In [8]:
```
# check prior normalization
print("   prior(mu) integral = ", np.trapz(priormu, mugrid))
print("prior(sigma) integral = ", np.trapz(priorsig, siggrid))
# If you want to be precise, you can divide the prior values
# by these normalization factors.  Alternatively, you can
# properly account for the prior values at the bins that
# contain muL, muH, sigL, sigH.  But it will have only a
# tiny effect.
```

```
   prior(mu) integral =  1.001
prior(sigma) integral =  1.0000140305235403
```

Case A : $e_i = \sigma$, same for all i

The posterior is given by,

$$p(\mu|D,I) = \frac{e^{-Q/(2\sigma^2)}}{\int_{\mu_L}^{\mu_H} e^{-Q/(2\sigma^2)} d\mu} = \frac{\text{NUM}}{\text{DEN}}$$

where

$$Q = \sum_{i=1}^{n} (d_i - \mu)^2$$

The simplified form is Equation (9.8),

$$p(\mu|D,I) = \frac{\exp\left\{-\frac{(\mu-\bar{d})^2}{2\sigma^2/N}\right\}}{\int_{\mu_L}^{\mu_H} \exp\left\{-\frac{(\mu-\bar{d})^2}{2\sigma^2/N}\right\} d\mu} = \frac{\text{NUM}}{\text{DEN}}$$

where

$$\bar{d} = \frac{1}{N}\sum_{i=1}^{N} d_i$$

Let's implement both expressions; call them A1 and A2, respectively. Keep in mind that NUM is zero outside the prior range.

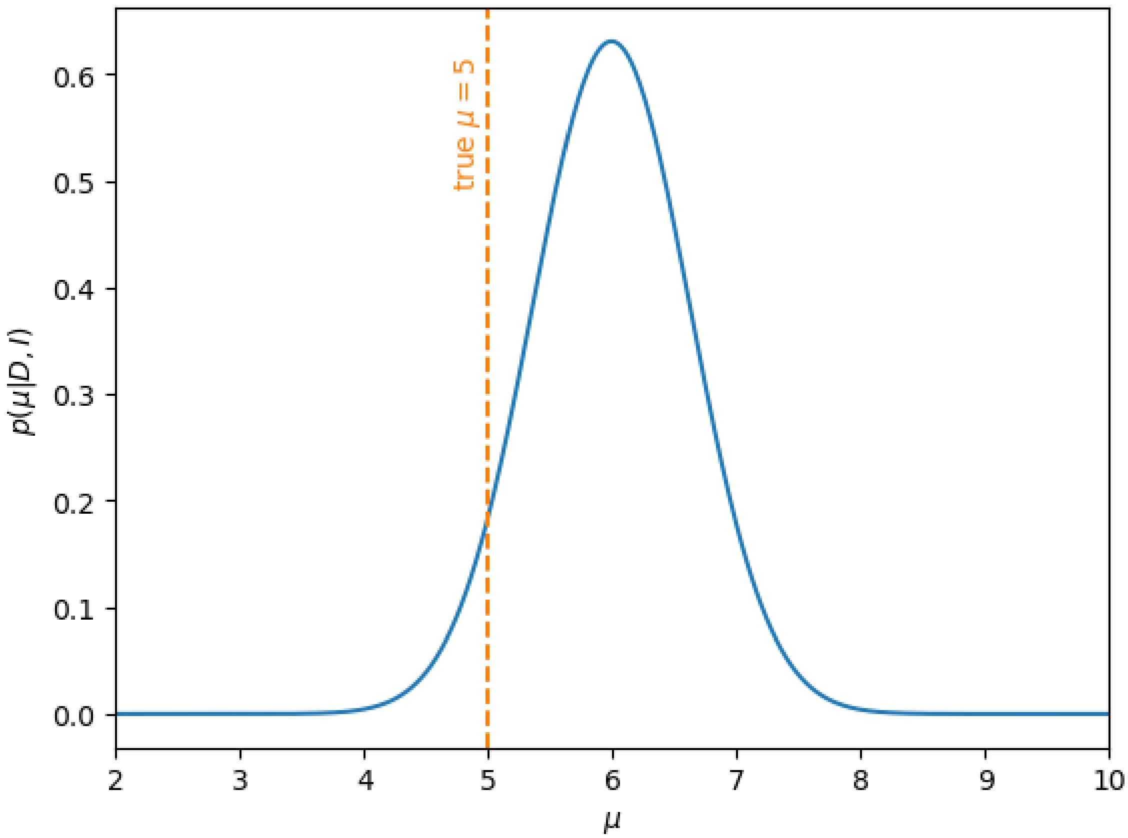
In [9]:
```
NUM = np.zeros_like(mugrid)

# we are computing p(mu), so loop over mugrid values
for i in range(len(mugrid)):
    Q = np.sum((d - mugrid[i])**2)
    NUM[i] = np.exp(-Q/(2*sig*sig))

# make sure you multiply this by the prior
NUM = priormu*NUM

DEN = np.trapz(NUM, mugrid)
posteriorA1 = NUM/DEN
```

In [10]:
```
# check normalization; this should come out to exactly 1.0
print(np.trapz(posteriorA1, mugrid))
```

```
1.0
```

In [11]:
```
plt.plot(mugrid, posteriorA1)
# you can overplot the following to see that the posterior
# is a gaussian.
#plt.plot(mugrid, stats.norm.pdf(mugrid, loc=smu, scale=sigmu))
plt.xlim([2,10])
plt.xlabel('$\mu$')
plt.ylabel('$p(\mu|D,I)$')
plt.axvline(x=mu, ls='--', c='#ff7f0e')
plt.text(4.7, 0.5, 'true $\mu=5$', rotation=90, c='#ff7f0e')
```

Out[11]:
```
Text(4.7, 0.5, 'true $\\mu=5$')
```



This is the implementation of the second expression, which will of course yield the exact same results.

In [12]:
```
dbar = d.mean() # = smu
NUM = priormu*np.exp(-n*(mugrid-dbar)**2/(2*sig*sig))
DEN = np.trapz(NUM, mugrid)
posteriorA2 = NUM/DEN
```

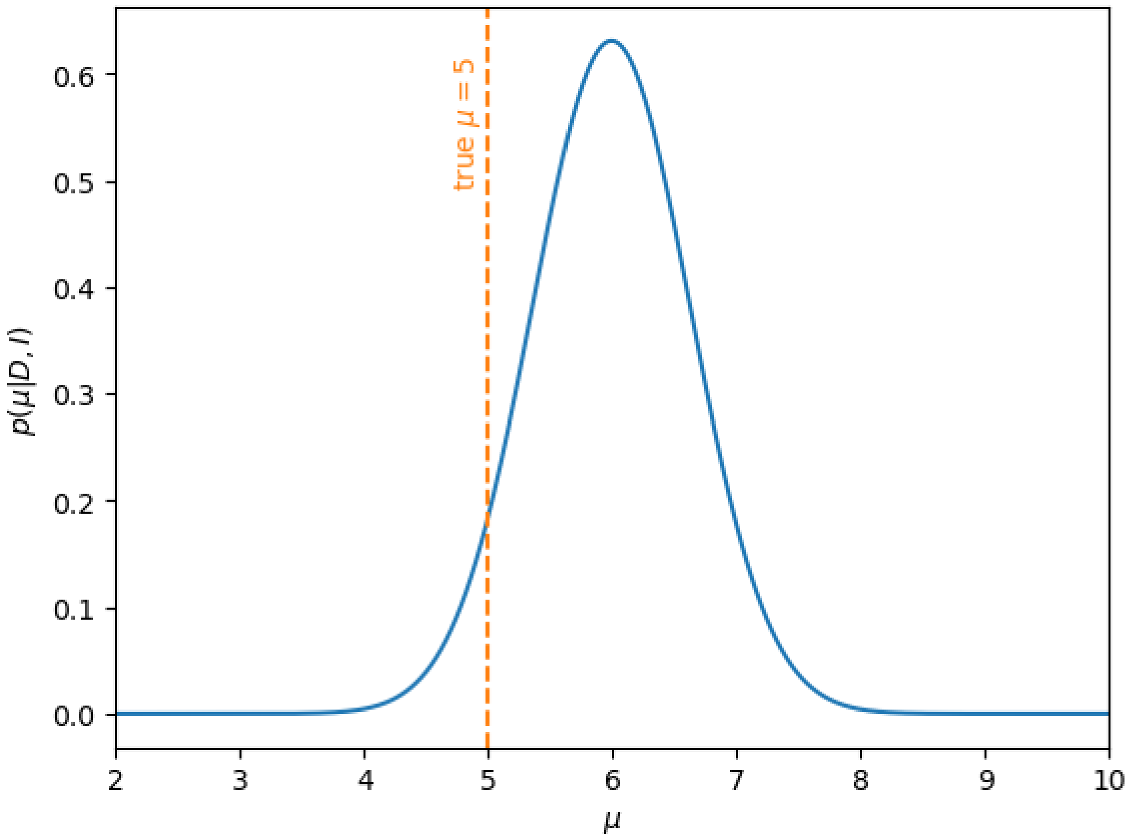
In [13]:
```
# check normalization
print(np.trapz(posteriorA2, mugrid))
```

```
1.0
```

In [14]:
```
plt.plot(mugrid, posteriorA2)
```

```
plt.xlim([2,10])
plt.xlabel('$\mu$')
plt.ylabel('$p(\mu|D,I)$')
plt.axvline(x=mu, ls='--', c='#ff7f0e')
plt.text(4.7, 0.5, 'true $\mu=5$', rotation=90, c='#ff7f0e')
```

Out[14]: Text(4.7, 0.5, 'true $\\mu=5$')



Case B : $e_i = \sigma_i$, unequal for all i

The posterior is given by,

$$p(\mu|D,I) = \frac{e^{-Q'/2}}{\int_{\mu_L}^{\mu_H} e^{-Q'/2} d\mu}$$

where

$$Q' = \sum_{i=1}^{N} \left(\frac{d_i - \mu}{\sigma_i} \right)^2$$

The posterior, after simplification, is given by Equation (9.19),

$$p(\mu|D,I) = \frac{\exp\left\{-\frac{(\mu-\bar{d}_w)^2}{2\sigma_w^2}\right\}}{\int_{\mu_L}^{\mu_H} \exp\left\{-\frac{(\mu-\bar{d}_w)^2}{2\sigma_w^2}\right\} d\mu} = \frac{\text{NUM}}{\text{DEN}}$$

where

$$\bar{d_w} = \frac{\sum_{i=1}^{N} w_i d_i}{\sum_{i=1}^{N} w_i}$$

$$\sigma_w^2 = \frac{1}{\sum_{i=1}^{N} w_i}$$

and

$$w_i = \frac{1}{\sigma_i^2}$$

SEE IF YOU CAN IMPLEMENT THIS YOURSELF!

Case C : $e_i = \sigma$, equal for all i but unknown.

The posterior for μ marginalized over σ is given by Equation (9.32),

$$p(\mu|D,I) = \frac{p(\mu|I)\int_{\sigma_L}^{\sigma_H} p(\sigma|I)\sigma^{-N}e^{-Q/2\sigma^2}d\sigma}{\int_{\mu_L}^{\mu_H} p(\mu|I)\int_{\sigma_L}^{\sigma_H} p(\sigma|I)\sigma^{-N}e^{-Q/2\sigma^2}d\sigma d\mu} = \frac{\int_{\sigma_L}^{\sigma_H} \sigma^{-(N+1)}e^{-Q/2\sigma^2}d\sigma}{\int_{\mu_L}^{\mu_H}\int_{\sigma_L}^{\sigma_H} \sigma^{-(N+1)}e^{-Q/2\sigma^2}d\sigma d\mu} =$$

where

$$Q = \sum_{i=1}^{N}(d_i - \mu)^2$$

Again, keep in mind that NUM is zero outside the prior range.

In [15]:
```
NUM = np.zeros_like(mugrid)

# we are computing p(mu), so loop over mugrid values
for i in range(len(mugrid)):
    Q = np.sum((d - mugrid[i])**2)
    integrand = priorsig/(siggrid**n) * np.exp(-Q/(2*siggrid*siggrid))
    NUM[i] = np.trapz(integrand, siggrid)

NUM = priormu*NUM

DEN = np.trapz(NUM, mugrid)
posteriorC = NUM/DEN
```

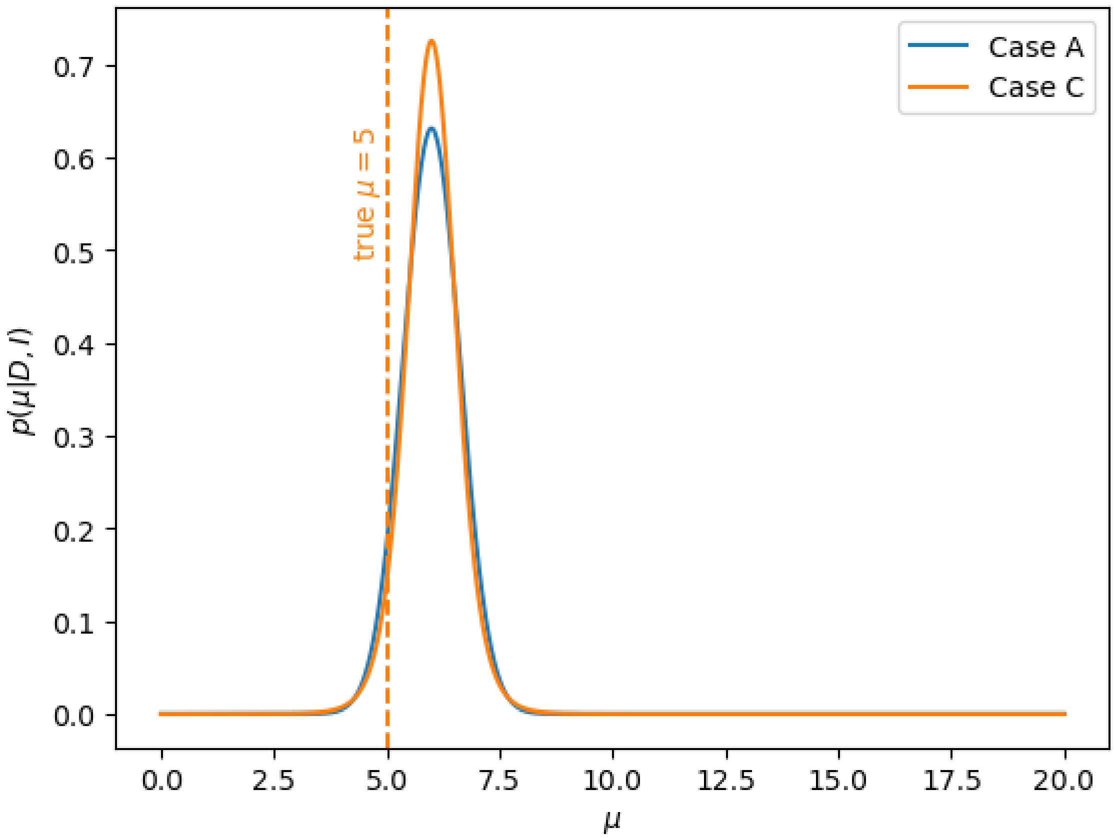
In [16]:
```
# always check normalization
print(np.trapz(posteriorC, mugrid))
```

```
1.0000000000000002
```

In [17]:
```
plt.plot(mugrid, posteriorA2, label='Case A')
plt.plot(mugrid, posteriorC, label='Case C')
plt.xlabel('$\mu$')
```

```
plt.ylabel('$p(\mu|D,I)$')
plt.axvline(x=mu, ls='--', c='#ff7f0e')
plt.text(4.2, 0.5, 'true $\mu=5$', rotation=90, c='#ff7f0e')
#plt.yscale('log')
plt.legend()
```

Out[17]: <matplotlib.legend.Legend at 0x116e63d00>



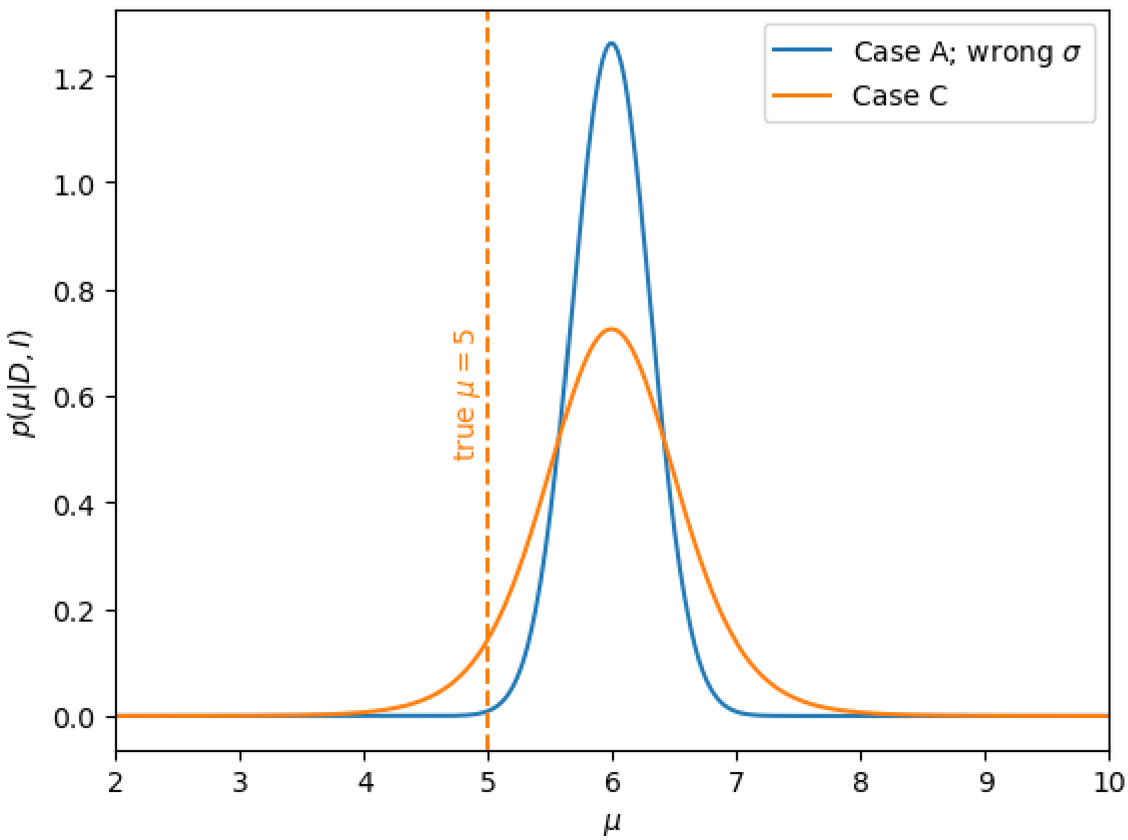
Note that the posterior PDF for Case C is similar to Case A (in fact, it is narrower than). This is because the value of $\sigma = 2$ given correctly describes the scatter in the data points.

If, however, you were given the same set of data points D but with, say, $\sigma = 1$ (which underestimates the scatter), the PDF for Case A would have been incorrectly narrower. Let's see this by recomputing the PDF for A assuming $\sigma = 1$. We are calling this posterior A3.

In [18]:
```
NUM = np.zeros_like(mugrid)

sigwrong = 1.0

# we are computing p(mu), so loop over mugrid values
for i in range(len(mugrid)):
    Q = np.sum((d - mugrid[i])**2)
    NUM[i] = np.exp(-Q/(2*sigwrong*sigwrong))

# make sure you multiply this by the prior
NUM = priormu*NUM

DEN = np.trapz(NUM, mugrid)
posteriorA3 = NUM/DEN
```

In [19]:
```
plt.plot(mugrid, posteriorA3, label='Case A; wrong $\sigma$')
plt.xlim([2,10])
plt.xlabel('$\mu$')
plt.ylabel('$p(\mu|D,I)$')
plt.axvline(x=mu, ls='--', c='#ff7f0e')
plt.text(4.7, 0.5, 'true $\mu=5$', rotation=90, c='#ff7f0e')
# overplot Case C
plt.plot(mugrid, posteriorC, label='Case C')
plt.legend()
```

Out[19]: <matplotlib.legend.Legend at 0x116e17160>



The book goes ahead and shows that the PDF for Case C is a Student's t distribution with

$$\frac{t^2}{\nu} = \frac{\mu - \bar{d}}{r^2}$$

with $\nu = N - 1$ degrees of freedom and

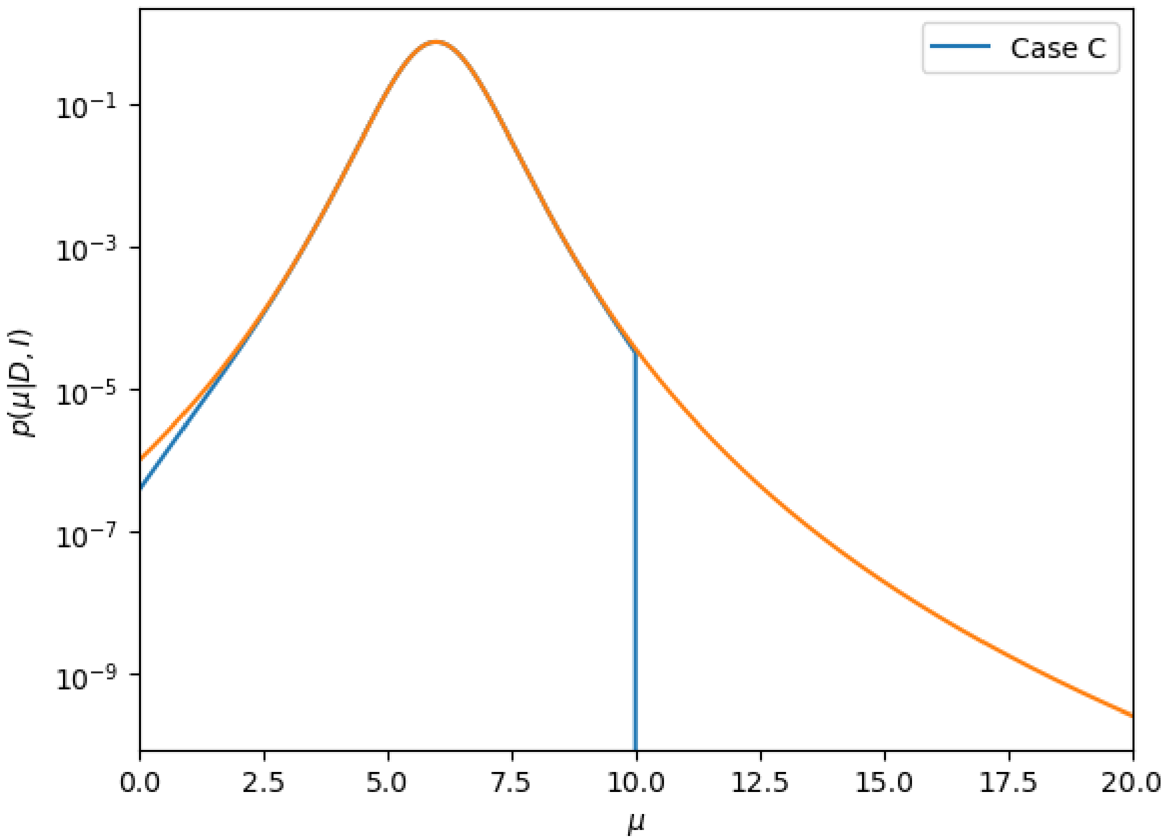
$$r^2 = \frac{1}{N}\sum_{i=1}^{N}(d_i - \bar{d}\,)^2$$

is the mean square deviation from $\bar{d}$. You can see the comparison here. Note that our PDF is truncated because of our choice of prior.

In [20]:
```
#plt.plot(mugrid, posteriorA1, label='Case A')
plt.plot(mugrid, posteriorC, label='Case C')
plt.xlim([0,20])
plt.xlabel('$\mu$')
```

```
plt.ylabel('$p(\mu|D,I)$')
#plt.axvline(x=mu, ls='--', c='#ff7f0e')
#plt.text(4.7, 0.5, 'true $\mu=5$', rotation=90, c='#ff7f0e')
r2  = (1/n)*np.sum((d-dbar)**2)
plt.plot(mugrid, stats.t.pdf(mugrid, df=n-1, loc=dbar, scale=np.sqrt(r2/(n-1)))
plt.yscale('log')
plt.legend()
```

Out[20]: <matplotlib.legend.Legend at 0x116d69d80>



In [ ]:

Calculation of $p(\sigma|D,I)$

We can also compute the posterior for σ marginalized over μ,

$$p(\sigma|D,I) = \frac{p(\sigma|I)\sigma^{-N}\int_{\mu_L}^{\mu_H} e^{-Q/2\sigma^2}d\mu}{\int_{\sigma_L}^{\sigma_H} p(\sigma|I)\sigma^{-N}\int_{\mu_L}^{\mu_H} e^{-Q/2\sigma^2}d\mu d\sigma} = \frac{\sigma^{-(N+1)}\int_{\mu_L}^{\mu_H} e^{-Q/2\sigma^2}d\mu}{\int_{\sigma_L}^{\sigma_H}\int_{\mu_L}^{\mu_H}\sigma^{-(N+1)}e^{-Q/2\sigma^2}d\mu d\sigma} = \frac{\mathrm{NUM}}{\mathrm{DEN}}$$

In [21]:
```
NUM = np.zeros_like(siggrid)
Q   = np.zeros_like(mugrid)

# we are computing p(sigma), so loop over siggrid values
for i in range(len(siggrid)):

    for j in range(len(mugrid)):
        Q[j] = np.sum((d - mugrid[j])**2)
```

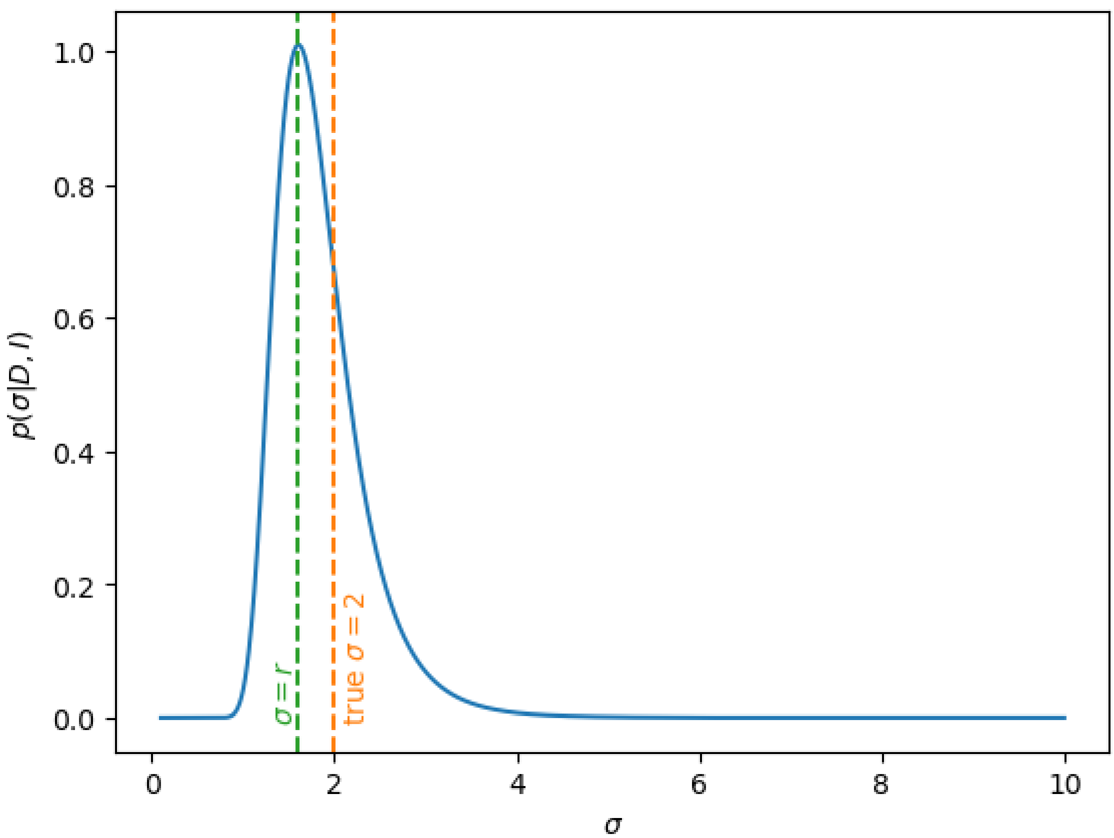
```
    integrand = priormu/(siggrid[i]**n) * np.exp(-Q/(2*siggrid[i]*siggrid[i]))
    NUM[i] = priorsig[i]*np.trapz(integrand, mugrid)

DEN = np.trapz(NUM, siggrid)
posteriorSig1 = NUM/DEN
```

In [22]:
```
# always check normalization
print(np.trapz(posteriorSig1, siggrid))
```

```
1.0
```

In [23]:
```
plt.plot(siggrid, posteriorSig1)
plt.xlabel('$\sigma$')
plt.ylabel('$p(\sigma|D,I)$')
plt.axvline(x=sig, ls='--', c='#ff7f0e')
plt.text(2.1, 0.0, 'true $\sigma=2$', rotation=90, c='#ff7f0e')
# the most-probable value is r=sqrt(r2)
r2  = (1/n)*np.sum((d-dbar)**2)
plt.axvline(x=np.sqrt(r2), ls='--', c='#2ca02c')
plt.text(1.3, 0.0, '$\sigma=r$', rotation=90, c='#2ca02c')
```

Out[23]:
```
Text(1.3, 0.0, '$\\sigma=r$')
```



The simplified form is Equation (9.43), which assumes that $p(\mu|I)$ cancels out.

$$p(\sigma|D,I) = \frac{\frac{1}{\sigma^N} e^{-Nr^2/2\sigma^2}}{\int_{\sigma_L}^{\sigma_H} \frac{1}{\sigma^N} e^{-Nr^2/2\sigma^2} d\sigma}$$

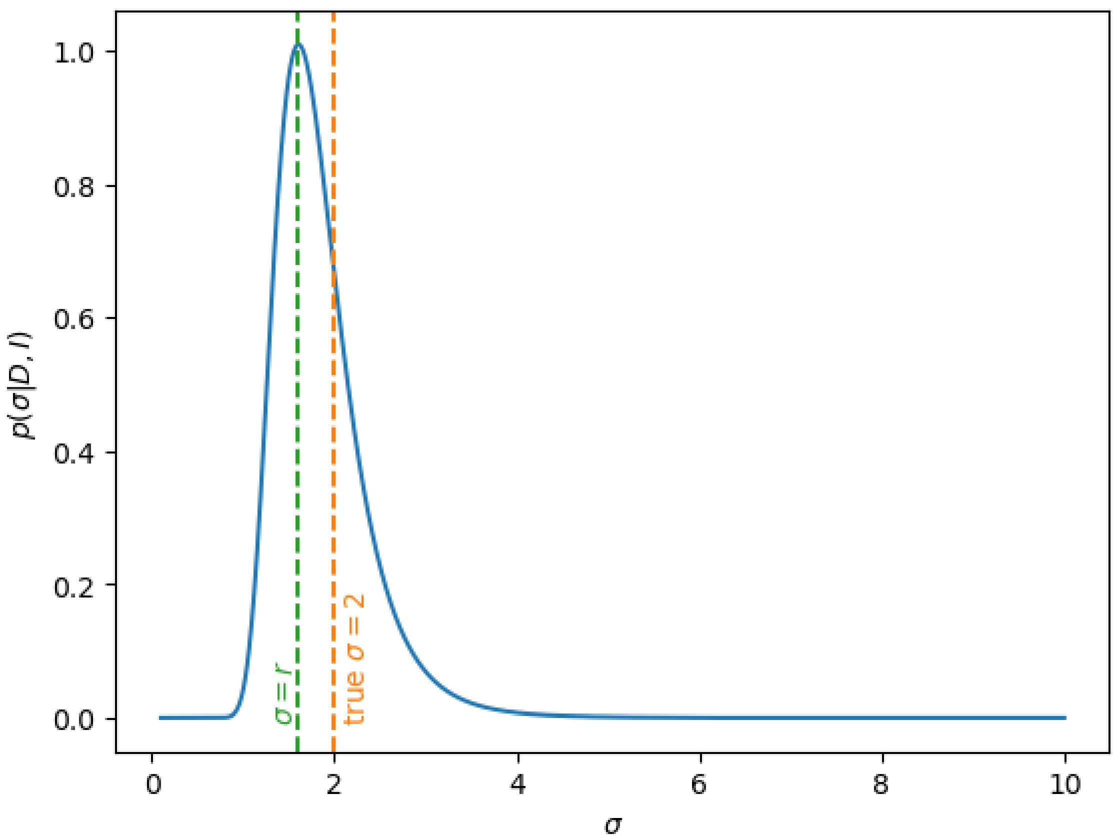
where

$$r^2 = \frac{1}{N}\sum_{i=1}^{N}(d_i - \bar{d})^2$$

In [24]:
```
r2  = (1/n)*np.sum((d-dbar)**2)
NUM = priorsig*np.exp(-n*r2/(2*siggrid*siggrid))/(siggrid**(n-1))
DEN = np.trapz(NUM, siggrid)
posteriorSig = NUM/DEN
```

In [25]:
```
plt.plot(siggrid, posteriorSig)
plt.xlabel('$\sigma$')
plt.ylabel('$p(\sigma|D,I)$')
plt.axvline(x=sig, ls='--', c='#ff7f0e')
plt.text(2.1, 0.0, 'true $\sigma=2$', rotation=90, c='#ff7f0e')
# the most-probable value is r=sqrt(r2)
r2  = (1/n)*np.sum((d-dbar)**2)
plt.axvline(x=np.sqrt(r2), ls='--', c='#2ca02c')
plt.text(1.3, 0.0, '$\sigma=r$', rotation=90, c='#2ca02c')
```

Out[25]:
```
Text(1.3, 0.0, '$\\sigma=r$')
```



Finally, let's also look at the joint likelihood $p(D|\mu, \sigma, I)$ ignoring the priors.

$$\mathcal{L}(\mu,\sigma) \propto \frac{1}{\sigma^n} e^{-Q/(2\sigma^2)}$$

In [26]:
```
def likelihood(mu, sig, d, n):
    dbar = d.mean()
    r2 = (1/n)*np.sum((d-dbar)**2)
```

```
    Q  = n*((mu-dbar)**2) + n*r2  # this is the alternate expression derived ir
    L  = (1/sig**n) * np.exp(-Q/(2*sig*sig))
    return L
```

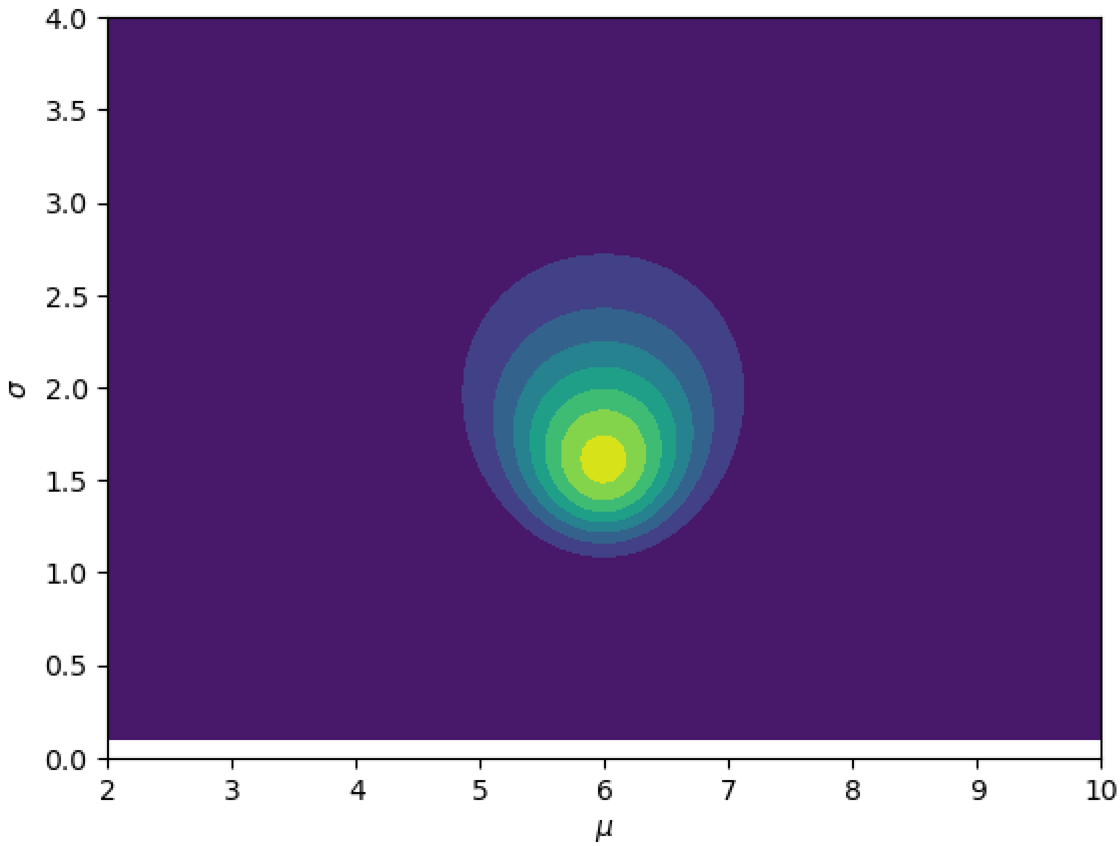
In [27]:
```
X, Y = np.meshgrid(mugrid, siggrid)
like = likelihood(X, Y, d, n)
```

In [28]:
```
like.shape
```

Out[28]: (1001, 1001)

In [29]:
```
fig = plt.figure()
ax  = fig.add_subplot()
ax.contourf(X, Y, like)
ax.set_xlabel('$\mu$')
ax.set_ylabel('$\sigma$')
ax.set_xlim([2, 10])
ax.set_ylim([0, 4])
```

Out[29]: (0.0, 4.0)



In [ ]: