Notes on the Derivation of the Chern-Simons Form

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Abstract

A memorandum. My goal is to be able to derive the Chern-Simons 3-form $\omega_3 = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)$ from scratch.

1 Covariant Derivative of a p-form

We write the exterior derivative as d, the connection as A, the covariant derivative as $D = d + A$, and the curvature as $F = D^2 = dA + A^2$.

For a p-form C and an appropriate differential form ϕ, we have:

$$d(C\phi) = (dC)\phi + (-1)^p C d\phi$$

Now, if we introduce the connection A and replace the exterior derivative with the covariant derivative, $d \to D$, we get:

$$D(C\phi) = (DC)\phi + (-1)^p C D\phi$$

Rearranging the terms, we get:

$$(DC)\phi = D(C\phi) - (-1)^p C D\phi$$

$$= (dC)\phi + (AC)\phi - (-1)^p CA\phi$$

$$= \big(dC + [A, C]\big)\phi$$

where we have defined

$$[A, C] = AC - (-1)^p CA$$

Therefore, the covariant derivative D acting on a p-form C is

$$DC = dC + [A, C]$$

This is the result.

2 Introduction of a Parameterized Connection

Here, we introduce a parameter $s \in \mathbb{R}$ and set $A_s = sA$.

Correspondingly, we have:

$$\begin{cases} D_s &= d + sA \\ F_s = (D_s)^2 &= sdA + s^2A^2 \end{cases}$$

In this case, we also get

$$\begin{cases} \dfrac{dF_s}{ds} = dA + 2sA^2 = D_sA \\ D_sF_s = 0 \end{cases}$$

and so on.

3 Derivation of the Chern-Simons $2n-1$ Form

The Chern-Simons $2n-1$ form ω_{2n-1} is defined by

$$d\omega_{2n-1} = \mathrm{tr}F^n$$

Here we use the fact that

$$\mathrm{tr}F^n = \int_0^1 ds \frac{d}{ds}\mathrm{tr}F_s^n$$

The integrand of the right-hand side is

$$\begin{aligned} \frac{d}{ds}\mathrm{tr}F_s^n &= \mathrm{tr}\frac{dF_s^n}{ds}nF_s^{n-1} \\ &= n\ \mathrm{tr}(D_sA)F_s^{n-1} \\ &= n\ \mathrm{tr}D_s(AF_s^{n-1}) \end{aligned}$$

Here we used

$$D_sF_s^{n-1} = 0$$

Furthermore, using

$$D_sC = dC + [A_s, C]$$

we get

$$n\ \mathrm{tr}D_s(AF_s^{n-1}) = n\ \mathrm{tr}D_s(AF_s^{n-1}) + [D_s, (AF_s^{n-1})]$$

The second term is zero, so we finally get

$$\frac{d}{ds}\ \mathrm{tr}F_s^n = d\Big(n\ \mathrm{tr}(AF_s^{n-1})\Big)$$

Returning to the definition of the Chern-Simons form,

$$d\omega_{2n-1} = d\Big(\int_0^1 ds\ n\ \mathrm{tr}(AF_s^{n-1})\Big)$$

$$\omega_{2n-1} = \int_0^1 ds\, n\, \mathrm{tr}(AF_s^{n-1}) \quad + \text{exact form}$$

This can be written.

When $n = 2$,

$$\omega_3 = \int_0^1 ds\ 2\ \mathrm{tr}A(sdA + s^2A^2) = \mathrm{tr}\left(AdA + \frac{2}{3}A^3\right)$$