# Runge-Kutta Method

Masaru Okada

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## 1 Euler method

To begin, let's introduce the simplest method for numerically solving an ordinary differential equation (ODE). This is also known as the (1st order) Runge-Kutta method. The equation to be solved is assumed to be:

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

Here, $f(x, y)$ is a given function. To simplify the notation, we set $y_i = y(x_i)$ and $x_{i+1} = x_i + h$. The initial condition (at $i = 0$) is given as $(x_0, y_0)$. If the $x$-mesh step $h$ is chosen such that $|h| \ll 1$, $y_{i+1}$ can be expanded by a Taylor series in $h$.

$$\begin{aligned} y_{i+1} &= y(x_i + h) \\ &= y(x_i) \quad + \left.\frac{dy}{dx}\right|_{x=x_i} h + O[h^2] \\ &= y_i \quad + f(x_i, y_i)h + O[h^2] \end{aligned} \tag{2}$$

Thus, we obtain the following expression:

$$f(x_i, y_i) = \frac{y_{i+1} - y_i}{h} \tag{3}$$

This is correct to the first order in $h$. The solution is given by the following recurrence relation:

$$\begin{cases} x_{i+1} & = & x_i + h \\ y(x_{i+1}) & = & y(x_i) + f[x_i, y(x_i)] \end{cases} \qquad (x_0 \text{ is given.}) \tag{4}$$

## 2 2nd order Runge-Kutta method (Heun method)

In a similar manner, $y_{i+1}$ is expanded with respect to $h$. Writing it out to the second order, we have:

$$\begin{aligned} y_{i+1} &= y(x_i + h) \\ &= y(x_i) \quad + y'(x_i)h + \frac{1}{2}y''(x_i)h^2 + O[h^3] \end{aligned} \tag{5}$$

$$\begin{aligned} \Delta y &= y_{i+1} - y_i \\ &= y_1 - y_0 \\ &= y'(x_0)h + \frac{1}{2}y''(x_0)h^2 \end{aligned} \tag{6}$$

While the value $y'(x_0)$ was used before, this time let's try selecting another value, $y'(x_1)$, in addition.

$$\Delta y = h[\alpha y'(x_0) + \beta y'(x_1)] \tag{7}$$

where $\alpha$ and $\beta$ are undetermined constants. Here, $\beta y'(x_1)$ can also be expanded:

$$\begin{aligned} \beta y'(x_1) &= \beta y'(x_0 + h) \\ &= \beta[y'(x_0) + hy''(x_0) + O(h^2)] \end{aligned} \tag{8}$$

Hence,

$$\Delta y = (\alpha + \beta)y'(x_0)h + \beta y''(x_0)h^2 \tag{9}$$

By comparing Eq. (6) and Eq. (9), the parameters are determined as $\alpha = \beta = \dfrac{1}{2}$. Finally, the second-order recurrence relation is as follows:

$$\left\{ \begin{array}{rcl} k_{1n} &=& hf(x_n, y_n) \\ k_{2n} &=& hf(x_n + h, y_n + k_{1n}) \\ y_{n+1} &=& y_n + \dfrac{1}{2}(k_{1n} + k_{2n}) \end{array} \right. \tag{10}$$

This is called the '**Heun method**' and provides a more accurate solution than the Euler method.

## 3 4th order Runge-Kutta method (RK4)

Following the same procedure, and considering terms up to the fourth order in $h$, the following recurrence relations are obtained:

$$\left\{ \begin{array}{rcl} k_{1n} &=& hf(x_n, y_n) \\ k_{2n} &=& hf(x_n + \dfrac{h}{2}, y_n + \dfrac{k_1}{2}) \\ k_{3n} &=& hf(x_n + \dfrac{h}{2}, y_n + \dfrac{k_2}{2}) \\ k_{4n} &=& hf(x_n + h, y_n + k_{3n}) \\ y_{n+1} &=& y_n + \dfrac{1}{6}(k_{1n} + 2k_{2n} + 2k_{3n} + k_{4n}) \end{array} \right. \tag{11}$$

RK4 is often considered the most reasonable method for solving ODEs numerically. This is because using even higher-order (e.g., 5th order or more) Runge-Kutta calculations may require more computation time than RK4 to achieve the same precision, sometimes making the extra computational cost unjustifiable.