Massless Real Scalar Field in 5D Minkowski Space Gaining Mass Upon Compactification of One Spatial Dimension

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Abstract

These are notes from a chat I had with a friend.

I'll make a note of how a massless real scalar field acquires mass when one spatial dimension is compactified in 5D Minkowski spacetime.

1 Set Up

Let $\mu, \nu = 0, 1, 2, 3$ and $M, N = 0, 1, 2, 3, 4$. We define the metric $\eta_{\mu\nu}$ for 4D Minkowski space M_4 and the metric η_{MN} for 5D Minkowski space with the following signatures:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \eta_{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

We compactify the 5th component (the 4th spatial dimension) into a circle with radius a. That is, we parameterize the 5th component, which has become a circle, with an angle θ. Then the coordinates become:

$$(x^0, x^1, x^2, x^3, x^4) \to (x^0, x^1, x^2, x^3, \theta)$$

The infinitesimal element of the 5th component is $dx^4 = ad\theta$, so the line element ds of this space $M_4 \times S^1$ is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + a^2 d\theta^2$$

The metric g_{MN} of this space is

$$g_{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & a^2 \end{pmatrix}, \quad g^{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \frac{1}{a^2} \end{pmatrix}$$

2 How does a Massless Scalar Field ϕ Change When Compactified from $M_5 \to M_4 \times S^1$?

For simplicity, we'll write the spacetime coordinates as x in 4D and X in 5D.

The action for a massless real scalar field $\phi(x)$ on M_4 is as follows:

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi(x) \partial^\nu \phi(x) \right)$$

For the Lagrangian to be invariant in the general case of $g_{\mu\nu}$, it is necessary to write it as:

$$S=\int d^4x\sqrt{-g}\left(-\frac{1}{2}\partial_\mu\phi(x)\partial^\nu\phi(x)\right)\quad \text{where}\quad g=\det(g_{\mu\nu})$$

We will now consider the action by extending this to the 5D case:

$$S=\int d^5X\sqrt{-g}\left(-\frac{1}{2}g^{MN}\partial_M\phi(X)\partial_N\phi(X)\right)\quad \text{where}\quad g=\det(g_{MN})$$

The 5th component of the scalar field is compactified, and since its period is 2π, we can perform a Fourier series expansion.

$$\begin{aligned}\phi(X)&=\phi(x,\theta)\\ &=\frac{1}{\sqrt{2\pi}}\sum_n\phi_n(x)e^{in\theta}\end{aligned}$$

Substituting this into the action:

$$S=-\frac{1}{2}\int d^5X\sqrt{-\det(g_{MN})}g^{MN}\partial_M\left(\frac{1}{\sqrt{2\pi}}\sum_m\phi_m(x)e^{im\theta}\right)\partial_N\left(\frac{1}{\sqrt{2\pi}}\sum_n\phi_n(x)e^{in\theta}\right)$$

Here,

$$\det(g_{MN})=\begin{vmatrix}-1&&&&\\&1&&&\\&&1&&\\&&&1&\\&&&&a^2\end{vmatrix}=-a^2$$

Furthermore, when we separate the calculation into 4D Minkowski space and S^1 components, we get:

$$\begin{aligned}=-\frac{1}{2}a\int d^4xd\theta\Bigg\{&g_{\mu\nu}\partial_\mu\left(\frac{1}{\sqrt{2\pi}}\sum_m\phi_m(x)e^{im\theta}\right)\partial_\nu\phi(x)\left(\frac{1}{\sqrt{2\pi}}\sum_n\phi_n(x)e^{in\theta}\right)\\ &+g^{\theta\theta}\partial_\theta\left(\frac{1}{\sqrt{2\pi}}\sum_m\phi_m(x)e^{im\theta}\right)\partial_\theta\left(\frac{1}{\sqrt{2\pi}}\sum_n\phi_n(x)e^{in\theta}\right)\Bigg\}\end{aligned}$$

2.1 Calculation of the First Term of the Action

The first term is:

$$-\frac{1}{2}a\int d^4xd\theta\sum_{n,m}\frac{1}{2\pi}e^{i(n+m)\theta}\eta^{\mu\nu}\partial_\mu\phi_m(x)\partial_\nu\phi_n(x)$$

Now, let's evaluate the θ integral.

$$\int d\theta e^{i(n+m)\theta}=2\pi\delta_{n,-m}$$

So we can also perform the summation over m to get:

$$-\frac{1}{2}a\sum_n\int d^4x\eta^{\mu\nu}\partial_\mu\phi_m(x)\partial_\nu\phi_{-n}(x)$$

Since ϕ is a real scalar field,

$$\phi(x)=\phi^*(x)$$

Therefore,

$$\sum_n\frac{1}{\sqrt{2\pi}}\phi_n(x)e^{in\theta}=\sum_n\frac{1}{\sqrt{2\pi}}\phi_n^*(x)e^{-in\theta}$$

$$= \sum_n \frac{1}{\sqrt{2\pi}} \phi^*_{-n}(x) e^{in\theta}$$

In the last equality, we substituted $n \to -n$.

The set $\left\{e^{in\theta}\right\}_{n=0,\pm1,\pm2,\cdots}$ is an orthonormal basis, so:

$$\phi_n(x) = \phi^*_{-n}(x)$$

In particular, when $n = 0$:

$$\phi_0(x) = \phi^*_0(x)$$

so $\phi_0(x) \in \mathbb{R}$

Thus, the first term of the action is:

$$-\frac{1}{2} a \sum_{n \in \mathbb{Z}} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi^*_n(x)$$

$$= -a \sum_{n=1}^{\infty} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi^*_n(x) - \frac{1}{2} a \int d^4x \partial_\mu \phi_0(x) \partial^\nu \phi^*_0(x)$$

2.2 Calculation of the Second Term of the Action

Let's proceed with the calculation of the second term of the action.

Here, $g^{\theta\theta} = \frac{1}{a^2}$, and the θ derivative is:

$$\partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) = \frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x)(im) e^{im\theta}$$

So,

$$-\frac{1}{2} a \int d^4x d\theta g^{\theta\theta} \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \right)$$

$$= -\frac{1}{2} a \int d^4x \frac{1}{a^2} \frac{1}{2\pi} \sum_{m,n} \left\{ \int d\theta e^{i(n+m)\theta} \right\} (-nm) \phi_n(x) \phi_m(x)$$

$$= -\frac{1}{2} a \int d^4x \frac{1}{a^2} \frac{1}{2\pi} \sum_{m,n} \left\{ 2\pi \delta_{n,-m} \right\} (-nm) \phi_n(x) \phi_m(x)$$

Performing the summation over m, we get:

$$= -\frac{1}{2} a \int d^4x \frac{1}{a^2} \sum_n n^2 \phi_n(x) \phi^*_n(x)$$

$$= -\frac{1}{2} a \sum_{n=1}^{\infty} \int d^4x \frac{n^2}{a^2} \phi_n(x) \phi^*_n(x)$$

From the above, the action for a massless real scalar field on $M_4 \times S^1$ is:

$$S = -a \sum_{n=1}^{\infty} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi^*_n(x) - \frac{1}{2} a \int d^4x \partial_\mu \phi_0(x) \partial^\nu \phi^*_0(x) - \frac{1}{2} a \sum_{n=1}^{\infty} \int d^4x \frac{n^2}{a^2} \phi_n(x) \phi^*_n(x)$$

$$= a \int d^4x \left\{ -\frac{1}{2} \partial_\mu \phi_0(x) \partial^\nu \phi^*_0(x) - \sum_{n=1}^{\infty} \left[\partial_\mu \phi_m(x) \partial^\nu \phi^*_n(x) + \frac{n^2}{a^2} \phi_n(x) \phi^*_n(x) \right] \right\}$$

3 Interpretation

Let's write the integrand of the 4D coordinate x in the action as L_4:

$$
\begin{aligned}
L_4 &= -\frac{1}{2}\partial_\mu\phi_0(x)\partial^\nu\phi_0^*(x) - \sum_{n=1}^{\infty}\left[\partial_\mu\phi_m(x)\partial^\nu\phi_n^*(x) + \frac{n^2}{a^2}\phi_n(x)\phi_n^*(x)\right] \\
&= L_{\text{free}} + L_{\text{mass}}
\end{aligned}
$$

The first term,

$$L_{\text{free}} = -\frac{1}{2}\partial_\mu\phi_0(x)\partial^\nu\phi_0^*(x)$$

is a massless real scalar field on 4D Minkowski space, and the second term,

$$L_{\text{mass}} = -\sum_{n=1}^{\infty}\left[\partial_\mu\phi_m(x)\partial^\nu\phi_n^*(x) + \frac{n^2}{a^2}\phi_n(x)\phi_n^*(x)\right]$$

is the sum of complex scalar fields $\phi_n(x)$ $(n = 1, 2, 3, \cdots)$ with an effective mass of $M_n = \left|\frac{n}{a}\right|$.

In the end, a massless real scalar field on the compactified space $M_4 \times S^1$ can be seen, from a perspective that ignores the S^1, as a superposition on M_4 of:

$$
\begin{cases}
\phi_0(x) & \cdots \ \text{a massless real scalar field} \\
\phi_n(x) & \cdots \ \text{a complex scalar field with mass } \left|\dfrac{n}{a}\right|
\end{cases}
$$

In conclusion, it has been shown that when one spatial dimension of a 5D spacetime is compactified, a massless real scalar field from the original theory gives rise to complex scalar fields that have mass.

If the radius a of the compactified dimension is very small, the mass $M_n = \left|\frac{n}{a}\right|$ becomes very large, so the complex scalar fields $\phi_n(x)$ do not affect physical observations, and only the massless real scalar field $\phi_0(x)$ is observed.

It is known that 5D general relativity, when viewed as a theory on 4D spacetime with a compactified S^1 like this, gives rise to 4D general relativity and 4D electromagnetism.

It was thought that by extending this to higher dimensions, it might be possible to unify

$$
\begin{cases}
\cdot \ \text{gravity} \\
\cdot \ \text{electromagnetism} \\
\cdot \ \text{the weak force} \\
\cdot \ \text{the strong force}
\end{cases}
$$

This idea led to attempts at supergravity theory, but the theory was found to contradict observations. Currently, string theory is considered the most promising candidate for a unified theory of the four forces.

It is known that supergravity theory is the low-energy limit of superstring theory.