# Adjunction

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# 1 Definition

### 1.1 Definition of Adjunction

Given categories C and D, we have functors  $F:C\to D$  and  $G:D\to C$ . We say that 'F and G are adjoint', denoted  $F\dashv G:C\to D$ ,

- Adjunction —

if for every object c in C and every object d in D, there exists a natural bijection

$$\phi_{cd}: \operatorname{Hom}_D(Fc,d) \to \operatorname{Hom}_C(c,Gd)$$

### 1.2 Meaning of 'Natural'

The meaning of 'natural' here is that it is:

- 1. natural in c when d is fixed, and
- 2. natural in d when c is fixed.

In other words, there are two conditions.

First, 'natural in c' means:

Natural in c

$$\theta_c: \operatorname{Hom}_D(F-,d) \to \operatorname{Hom}_C(-,Gd)$$

that there exists a natural transformation  $\theta_c$ .

Since this is a natural transformation, both its domain and codomain must be functors.

Recall that Hom is a functor.

In general, for an object  $a \in C$  in a category C, the functor  $\operatorname{Hom}_C(-,a)$  is a functor from  $C^{op}$  to **Set**.

Similarly,  $\operatorname{Hom}_C(-,Gd)$ , for the object  $Gd \in C$ , is a functor  $C^{op} \to \mathbf{Set}$ .

Let's consider the meaning of the other functor,  $\operatorname{Hom}_D(F-,d)$ .

First, without F,  $\operatorname{Hom}_D(-,d)$  is a functor  $D^{op} \to \mathbf{Set}$ .

Furthermore, we have the functor  $F: C \to D$ ,

$$C^{op} \xrightarrow{F} D^{op} \xrightarrow{\operatorname{Hom}_D(-,d)} \mathbf{Set}$$

The composition of these two functors is  $\operatorname{Hom}_D(F-,d)$ .

The existence of a natural transformation  $\theta_c$  between these functors is the meaning of 'natural in c'.

'Natural in d' is similar,

$$\theta_d: \operatorname{Hom}_D(Fc, -) \to \operatorname{Hom}_C(c, G-)$$

meaning that there exists a natural transformation  $\theta_d$ .

#### 1.3 Left and Right

When  $F \dashv G : C \to D$ , that is,

for every object c in C and every object d in D, there exists a natural bijection

$$\phi_{cd}: \operatorname{Hom}_D(Fc, d) \to \operatorname{Hom}_C(c, Gd)$$

F is called the **left adjoint functor** of G, and G is called the **right adjoint functor** of F.

The functor applied to the object on the left (in  $\operatorname{Hom}_D(Fc,d)$ ) is the left adjoint, and the functor applied to the object on the right (in  $\operatorname{Hom}_C(c,Gd)$ ) is the right adjoint. Apparently, the left/right convention is sometimes reversed depending on the literature.

This is tough for those with left-right confusion, but when writing  $F \dashv G : C \to D$ , we say 'F is left adjoint' and, meaning the same thing, 'F has a right adjoint'.

## 2 Examples

#### 2.1 Adjunction between the Category of Vector Spaces and the Category of Sets

Let V be an object from the category of vector spaces, **Vect**.

V is equipped with axioms, such as scalar multiplication and addition preserving linearity. We can take a functor U that 'forgets' these conditions (a **forgetful functor**).

$$\begin{array}{cccc} U: & \mathbf{Vect} & \to & \mathbf{Set} \\ & & & & & \psi \\ & V & \mapsto & U(V) \end{array}$$

A morphism in **Vect** is a linear map  $f: V \to W$ . Forgetting that it preserves the linear structure and regarding it as merely a map, we get  $U(f): U(V) \to U(W)$ , which is a morphism in **Set**.

Consider an adjoint for this U. That is, we seek a functor F such that for any set X, F(X) becomes a vector space.

This is obtained by taking the vector space with X as its basis.

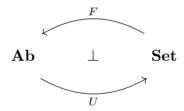
In this case, F is a functor  $F : \mathbf{Set} \to \mathbf{Vect}$ .

Moreover, F is adjoint to U. This can be shown by explicitly constructing the natural bijection

$$\phi_{cd}: \operatorname{Hom}_{\mathbf{Vect}}(Fc, d) \to \operatorname{Hom}_{\mathbf{Set}}(c, Ud)$$

### 2.2 Adjunction between the Category of Abelian Groups and the Category of Sets

There is an adjunction between the forgetful functor U from the category of abelian groups  $\mathbf{Ab}$  to the category of sets  $\mathbf{Set}$  (forgetting the structure), and the functor F, where F(X) forms the  $\mathbf{free}$  abelian group generated by the set X.



#### 2.3 Adjunction between the Category of Topological Spaces and the Category of Sets

Between the forgetful functor U from the category of topological spaces **Top** to the category of sets **Set** (forgetting the structure), and the functor F, where F(X) equips the set X with the **discrete topology**, there is an adjunction.

The functor G that equips X with the **indiscrete topology** is also an adjoint of U; it is the right adjoint of U.

 $F\dashv U\dashv G$ 

$$\begin{array}{c}
F \\
\hline
Top \\
\hline
U
\end{array}$$
 Set