# A Unified View of Monoids, Groups, Topological Groups, and Lie Groups

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#### Abstract

We'll review the definitions of monoids, groups, topological groups, and Lie groups, and see how writing them in terms of category-theoretic diagrams makes them magically unified and easy to understand.

## 1 Naive Definitions

First, let's define monoids, groups, topological groups, and Lie groups naively.

#### 1.1 Definition of a Monoid

A monoid is a semigroup with an identity element, satisfying associativity and having an identity. A set M with a binary operation  $\cdot: M \times M \to M$ 

- 1. Associativity: For any elements of M,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- 2. Existence of an identity element: There exists an element e in M such that for any element a in M,  $e \cdot a = a \cdot e = a$ .

## 1.2 Definition of a Group

A group is a monoid that also has an inverse element.

A set G with a binary operation  $\cdot: G \times G \to G$ 

- 1. Associativity: For any elements of G,  $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ .
- 2. Existence of an identity element: There exists an element e in M such that for any element a in G,  $e \cdot a = a \cdot e = a$ .
- 3. Existence of an inverse element: For any element a in G, there exists an element  $a^{-1}$  that satisfies  $a^{-1} \cdot a = a \cdot a^{-1} = e$ .

## 1.3 Definition of a Topological Group

A set G is a topological group if it is a group and satisfies the axioms of a topological space, and there exist continuous maps for the product and inverse.

- 1. G is a group.
- 2. G is a topological space.
- 3. The maps  $\mu: G \times G \to G$  and  $\nu: G \to G$ , defined as  $\mu(x,y) = xy$  and  $\nu(x) = x^{-1}$  respectively, are continuous.

## 1.4 Definition of a Lie Group

A set G is a Lie group if it is a differentiable manifold, and satisfies the axioms of a topological group, and the maps for the product and inverse are differentiable.

- 1. G is a differentiable manifold.
- 2. The maps  $\mu: G \times G \to G$  and  $\nu: G \to G$ , defined as  $\mu(x,y) = xy$  and  $\nu(x) = x^{-1}$  respectively, are differentiable.

# 2 A Unified View

#### 2.1 Monoid

Next, let's try to understand these concepts in a unified way using elementary category theory.

A one-point set  $1 = \{0\}$  acts as an identity element for the product (Cartesian product), considering the bijections

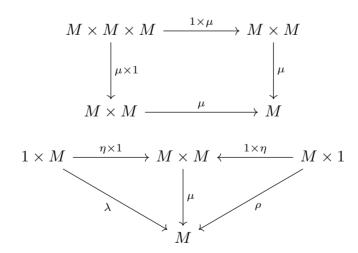
$$1 \times X \xrightarrow{\lambda} X \xleftarrow{\rho} X \times 1$$

given by  $\lambda < 0, x >= x$ ,  $\rho < x, 0 >= x$ .

A monoid M is a set M equipped with two functions

$$\mu: M \times M \to M , \quad \eta: 1 \to M$$

that make the following diagrams commute with respect to  $\mu$  and  $\eta$ .



## 2.2 Introducing a Map for the Inverse Element

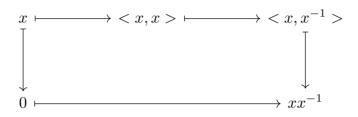
A group has an inverse element. A group is a monoid that makes the following diagram commute using a map  $\xi: M \to M$  where  $\xi: x \mapsto x^{-1}$ .

$$M \xrightarrow{\delta} M \times M \xrightarrow{1 \times \xi} M \times M$$

$$\downarrow^{\eta \circ !} \qquad \qquad \downarrow^{\mu}$$

$$1 \xrightarrow{\eta} M$$

where  $\eta$  is the arrow to the identity element and ! is the unique arrow from M to the one-point set 1.



Here,  $\delta: M \to M \times M$  is the diagonal map given by  $\delta: x \mapsto \langle x, x \rangle$ .

A group is what you get when M is a set and each morphism is just a function that makes the above diagrams commute.

A topological group is what you get when M is a topological space and each morphism is a continuous map that makes the diagrams commute.

A Lie group is what you get when M is a differentiable manifold and each morphism is a smooth map of manifolds that makes the diagrams commute.