Summary Notes on *Quanto Derivatives*

Reading from `Baxter & Rennie`, Chapter 4.5

Masaru Okada

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Contents

1 Quanto Contracts

A contract in which the currency of the underlying asset differs from the currency of the contract's payoff.

For instance, a contract that pays S_T yen when a product listed on a U.S. market is priced at S_T dollars.

2 The Idea of a Quanto (The Model We Will Consider)

Using two non-independent $\mathbb{P}$-Brownian motions, $W_1(t)$ and $W_2(t)$, we consider the following model:

$$\begin{array}{rclrcl} S_t & = & S_0 \exp\left(\sigma_1 W_1(t) + \mu t\right) \ , & B_t & = & \exp(rt) \\ C_t & = & C_0 \exp\left(\rho\sigma_2 W_1(t) + \bar{\rho}\sigma_2 W_2(t) + \nu t\right) \ , & D_t & = & \exp(ut) \end{array} \tag{1}$$

Here, μ, ν, r, u are constants, with r and u being positive. ρ is a constant between 0 and 1, inclusive, and we use the shorthand $\bar{\rho} = \sqrt{1-\rho^2}$. S_t is the stock price in pounds at time t, C_t is the exchange rate (1 pound $= C_t$ dollars), B_t is the dollar cash bond, and D_t is the pound cash bond.

3 Interpretation of the Model

By calculating the variance-covariance matrix and correlation matrix of the vector-valued random variable $(\log S_t, \log C_t)$, we find that:

· The volatility of $\log S_t$ is σ_1

· The volatility of $\log C_t$ is σ_2

· The correlation between $\log S_t$ and $\log C_t$ is ρ

4 SDE of Tradable Assets

In this model, there are three assets that can be traded in dollars.

1. The pound cash bond, denominated in dollars: $C_t D_t$
2. The stock price, denominated in dollars: $C_t S_t$
3. The dollar cash bond: B_t

Taking B_t as the numeraire, we consider the two discounted processes $Y_t = B_t^{-1} C_t D_t$ and $Z_t = B_t^{-1} C_t S_t$. The stochastic differential equations (SDEs) that these processes satisfy are:

$$\frac{dY_t}{Y_t} = \rho\sigma_2 dW_1(t) + \bar{\rho}\sigma_2 dW_2(t) \ + \ \left(\nu + u + \frac{1}{2}\sigma_2^2 - r\right) dt \tag{2}$$

$$\frac{dZ_t}{Z_t} = (\sigma_1 + \rho\sigma_2) dW_1(t) + \bar{\rho}\sigma_2 dW_2(t) \ + \ \left(\mu + \nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r\right) dt \tag{3}$$

5 SDE of Tradable Assets (Matrix Representation)

The stochastic differential equations can be expressed in matrix form as:

$$\begin{pmatrix} dY_t/Y_t \\ dZ_t/Z_t \end{pmatrix} = \begin{pmatrix} \rho\sigma_2 & \bar{\rho}\sigma_2 & \nu + u + \frac{1}{2}\sigma_2^2 - r \\ \sigma_1 + \rho\sigma_2 & \bar{\rho}\sigma_2 & \mu + \nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \\ dt \end{pmatrix} \tag{4}$$

$$= \begin{pmatrix} \rho\sigma_2 & \bar{\rho}\sigma_2 \\ \sigma_1 + \rho\sigma_2 & \bar{\rho}\sigma_2 \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} \ + \ \begin{pmatrix} \nu + u + \frac{1}{2}\sigma_2^2 - r \\ \mu + \nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r \end{pmatrix} dt \tag{5}$$

We have separated the terms involving the differentials of Brownian motion from the time differential term.

The coefficient matrix of the Brownian motion differential terms is called the volatility matrix, denoted by $\mathbf{\Sigma}$.

$$\mathbf{\Sigma} = \begin{pmatrix} \rho\sigma_2 & \bar{\rho}\sigma_2 \\ \sigma_1 + \rho\sigma_2 & \bar{\rho}\sigma_2 \end{pmatrix} \tag{6}$$

The drift vector $\boldsymbol{\mu}$ is defined as follows.

$$\boldsymbol{\mu} = \begin{pmatrix} \nu + u + \frac{1}{2}\sigma_2^2 \\ \mu + \nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 \end{pmatrix} \tag{7}$$

Using the volatility matrix $\mathbf{\Sigma}$ and the drift vector $\boldsymbol{\mu}$, the SDE can be written more clearly as:

$$\begin{pmatrix} dY_t/Y_t \\ dZ_t/Z_t \end{pmatrix} = \mathbf{\Sigma} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + (\boldsymbol{\mu} - r\mathbf{1})dt \tag{8}$$

6 Condition for No Drift

To eliminate the drift term, we need to be able to write the SDE using a Brownian motion $(\tilde{W}_1(t), \tilde{W}_2(t))$ under some measure $\mathbb{Q}$, such that:

$$\begin{pmatrix} dY_t/Y_t \\ dZ_t/Z_t \end{pmatrix} = \mathbf{\Sigma} \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \end{pmatrix} \tag{9}$$

Comparing the right-hand sides gives us

$$\mathbf{\Sigma} \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \end{pmatrix} = \mathbf{\Sigma} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + (\boldsymbol{\mu} - r\mathbf{1})dt \tag{10}$$

If the volatility matrix $\mathbf{\Sigma}$ is invertible, then

$$\begin{aligned} \begin{pmatrix} d\tilde{W}_1(t) \\ d\tilde{W}_2(t) \end{pmatrix} &= \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1})dt && (11) \\ &= \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix} + \boldsymbol{\gamma} dt && (12) \end{aligned}$$

where $\boldsymbol{\gamma}$ is the market price of risk corresponding to $(W_1(t), W_2(t))$, $\boldsymbol{\gamma}^T = (\gamma_1(t), \gamma_2(t))$.

$$\boldsymbol{\gamma} = \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) \tag{13}$$

To describe the SDE using the $\mathbb{Q}$-Brownian motion $(\tilde{W}_1(t), \tilde{W}_2(t))$, we just need to calculate the market price of risk.

$$\begin{aligned} \boldsymbol{\gamma} &= \mathbf{\Sigma}^{-1}(\boldsymbol{\mu} - r\mathbf{1}) && (14) \\ &= \frac{1}{\bar{\rho}\sigma_1\sigma_2} \begin{pmatrix} -\bar{\rho}\sigma_2 & \bar{\rho}\sigma_2 \\ \sigma_1 + \rho\sigma_2 & -\rho\sigma_2 \end{pmatrix} \begin{pmatrix} \nu + u + \frac{1}{2}\sigma_2^2 - r \\ \mu + \nu + \frac{1}{2}\sigma_1^2 + \rho\sigma_1\sigma_2 + \frac{1}{2}\sigma_2^2 - r \end{pmatrix} && (15) \end{aligned}$$

From this, the components of the market price of risk are respectively

$$\gamma_1 = \frac{u - \mu - \frac{1}{2}\sigma_1^2 - \rho\sigma_1\sigma_2}{\sigma_1} \tag{16}$$

$$\gamma_2 = \frac{\nu + u + \frac{1}{2}\sigma_2^2 - r - \rho\sigma_2\gamma_1}{\bar{\rho}\sigma_2} \tag{17}$$

Using the Condition that $\boldsymbol{\gamma}$ Satisfies When Tradable

The stochastic differential equations can be rewritten using the $\mathbb{Q}$-Brownian motion as follows.

$$\begin{pmatrix} dY_t/Y_t \\ dZ_t/Z_t \end{pmatrix} = \begin{pmatrix} \rho\sigma_2 d\tilde{W}_1(t) \ + \ \bar{\rho}\sigma_2 d\tilde{W}_2(t) \\ (\sigma_1 + \rho\sigma_2) d\tilde{W}_1(t) \ + \ \bar{\rho}\sigma_2 d\tilde{W}_2(t) \end{pmatrix} \tag{18}$$

This is a system of SDEs. Attempting to solve it directly in this form seems quite cumbersome, as it would require diagonalizing the volatility matrix and so on.

However, using the concept of the market price of risk from the previous section, we only need to find the values of $\boldsymbol{\mu} = (\mu, \nu)$ that make $\boldsymbol{\gamma} = \mathbf{0}$ and substitute them back into the processes for the original assets.

From $\gamma_1 = 0$, we have

$$\mu \ = \ u - \frac{1}{2}\sigma_1^2 - \rho\sigma_1\sigma_2 \tag{19}$$

From $\gamma_2 = 0$, we have

$$\nu \ = \ r - u - \frac{1}{2}\sigma_2^2 \tag{20}$$

The original asset processes were expressed using the $\mathbb{P}$-Brownian motion as

$$S_t = S_0 \exp\left(\sigma_1 W_1(t) + \mu t\right) \tag{21}$$

$$C_t = C_0 \exp\left(\rho\sigma_2 W_1(t) + \bar{\rho}\sigma_2 W_2(t) + \nu t\right) \tag{22}$$

so, describing them with the $\mathbb{Q}$-Brownian motion and substituting the values of $\boldsymbol{\mu}$ we just found gives:

$$S_t = S_0 \exp\left(\sigma_1 \tilde{W}_1(t) + \left(u - \frac{1}{2}\sigma_1^2 - \rho\sigma_1\sigma_2\right) t\right) \tag{23}$$

$$C_t = C_0 \exp\left(\rho\sigma_2 \tilde{W}_1(t) + \bar{\rho}\sigma_2 \tilde{W}_2(t) + \left(r - u - \frac{1}{2}\sigma_2^2\right) t\right) \tag{24}$$

For C_t, by defining a $\mathbb{Q}$-Brownian motion

$$\tilde{W}_3(t) \ = \ \rho\tilde{W}_1(t) + \bar{\rho}\tilde{W}_2(t)$$

we can write

$$C_t = e^{(r-u)t} C_0 \exp\left(\sigma_2 \tilde{W}_3(t) - \frac{1}{2}\sigma_2^2 t\right) \tag{25}$$

Therefore, if we choose the numeraire $e^{(r-u)t} = B_t D_t^{-1}$, the process becomes a $\mathbb{Q}$-martingale, and is thus tradable.

S_t, on the other hand, is given by

$$S_t = e^{ut} S_0 e^{-\rho\sigma_1\sigma_2} \exp\left(\sigma_1 \tilde{W}_1(t) - \frac{1}{2}\sigma_1^2 t\right) \tag{26}$$

Because it contains a factor of $e^{-\rho\sigma_1\sigma_2}$, it does not become tradable even if we choose the pound cash bond $e^{ut} = D_t$ as the numeraire.

7 Quanto Forward

We have successfully expressed the tradable assets using a Brownian motion under a measure $\mathbb{Q}$ that renders them martingales. Our next step is to determine the price of a quanto contract by calculating the expected value under this $\mathbb{Q}$ measure.

Regarding the expression for S_t written with the $\mathbb{Q}$-Brownian motion, and using the forward price at maturity T, $F = D_T S_0 = e^{uT} S_0$, we can write:

$$S_t = Fe^{-\rho\sigma_1\sigma_2} \exp\left(\sigma_1 \tilde{W}_1(t) - \frac{1}{2}\sigma_1^2 t\right) \tag{27}$$

The present value of a forward contract with a delivery price of K (dollars) can be calculated as follows.

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - K) && (28)\\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}\left[Fe^{-\rho\sigma_1\sigma_2}\exp\left(\sigma_1\tilde{W}_1(T) - \frac{1}{2}\sigma_1^2 T\right) - K\right] && (29)\\ &= Fe^{-\rho\sigma_1\sigma_2}e^{-rT}\exp\left(-\frac{1}{2}\sigma_1^2 T\right)\mathbb{E}_{\mathbb{Q}}e^{\sigma_1\tilde{W}_1(T)} - Ke^{-rT} && (30) \end{aligned}$$

Here, the expected value $\mathbb{E}_{\mathbb{Q}}e^{\sigma_1\tilde{W}_1(T)}$ is calculated by noting that $\dfrac{\tilde{W}_1(T)}{\sqrt{T}}$ follows a standard normal distribution (a random variable Z) under the measure $\mathbb{Q}$.

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}e^{\sigma_1\tilde{W}_1(T)} &= \mathbb{E}_{\mathbb{Q}}\exp\left(\sigma_1\sqrt{T}\frac{\tilde{W}_1(T)}{\sqrt{T}}\right) && (31)\\ &= \mathbb{E}\exp\left(\sigma_1\sqrt{T}Z\right) && (32)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(\sigma_1\sqrt{T}z\right)\ e^{-\frac{1}{2}z^2}dz && (33)\\ &= \exp\left(\frac{1}{2}(\sigma_1\sqrt{T})^2\right) && (34) \end{aligned}$$

In summary,

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - K) && (35)\\ &= Fe^{-rT}e^{-\rho\sigma_1\sigma_2}\exp\left(-\frac{1}{2}\sigma_1^2 T\right)\exp\left(\frac{1}{2}\sigma_1^2 T\right) - Ke^{-rT} && (36)\\ &= Fe^{-rT}e^{-\rho\sigma_1\sigma_2} - Ke^{-rT} && (37) \end{aligned}$$

The delivery price K is set such that the present value is zero, ensuring that neither party to the trade makes a profit or loss at inception.

Thus, if we write the value of K when $V_0 = 0$ as F_Q, we have:

$$\begin{aligned} 0 = V_0 &= Fe^{-rT}e^{-\rho\sigma_1\sigma_2} - F_Q e^{-rT} && (38)\\ \iff F_Q &= Fe^{-\rho\sigma_1\sigma_2} && (39) \end{aligned}$$

Since the values of σ_1 and σ_2 are both positive, the price of the quanto forward is higher than the standard forward price F by a factor of $e^{-\rho\sigma_1\sigma_2}$ only when the stock price and the exchange rate are negatively correlated ($\rho < 0$).

8 Quanto Digital Option

A quanto digital option is a contract that, for example, pays 1 dollar (not 1 pound) if the stock price S_T (in pounds) at maturity T exceeds a predetermined price of K pounds.

The payoff is

$$X = 1_{\{S_T > K\}}$$

and its present value V_0 is

$$\begin{aligned} V_0 &= B_T^{-1}\mathbb{E}_{\mathbb{Q}}X && (40)\\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}1_{\{S_T>K\}} && (41)\\ &= e^{-rT}\mathbb{Q}\{S_T > K\} && (42)\end{aligned}$$

Here, $\mathbb{Q}\{B\}$ is the probability of condition B being met under the measure $\mathbb{Q}$.

S_T could be written using the $\mathbb{Q}$-Brownian motion as follows.

$$\begin{aligned} S_T &= F\exp\left(\sigma_1\tilde{W}_1(T) - \frac{1}{2}\sigma_1^2 T - \rho\sigma_1\sigma_2\right) && (43)\\ &= F\exp\left(\sigma_1\sqrt{T}\frac{\tilde{W}_1(T)}{\sqrt{T}} - \frac{1}{2}\sigma_1^2 T - \rho\sigma_1\sigma_2\right) && (44)\\ &= F\exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T - \rho\sigma_1\sigma_2\right) && (45)\end{aligned}$$

where F is the forward price at maturity T, $F = e^{uT}S_0$. Also, Z is a standard normal random variable under $\mathbb{Q}$, following $N(0,1)$.

Furthermore, for simplicity of notation, if we use

$$F_Q = Fe^{-\rho\sigma_1\sigma_2}$$

then,

$$S_T = F_Q\exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\right)$$

Rearranging the condition $S_T > K$, we get:

$$\begin{aligned} S_T &> K && (46)\\ F_Q\exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\right) &> K && (47)\\ Z &> \frac{\frac{1}{2}\sigma_1^2 T - \log\frac{F_Q}{K}}{\sigma_1\sqrt{T}} = z_0 && (48)\end{aligned}$$

Since the right-hand side is complex, we will temporarily set it as z_0. Continuing the calculation,

$$\begin{aligned} V_0 &= e^{-rT}\mathbb{Q}\{S_T > K\} && (49)\\ &= e^{-rT}\mathbb{Q}\{Z > z_0\} && (50)\\ &= e^{-rT}\frac{1}{\sqrt{2\pi}}\int_{z_0}^{\infty} e^{-\frac{1}{2}z^2}dz && (51)\\ &= e^{-rT}\Phi\left(\frac{\log\frac{F_Q}{K} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right) && (52)\end{aligned}$$

9 Quanto Call Option

A quanto call option is a contract that, for example, pays $S_T - k$ dollars (not $S_T - k$ pounds) if the stock price S_T (in pounds) at maturity T exceeds a predetermined strike price of k pounds.

The payoff is

$$X = (S_T - k)^+ \tag{53}$$

and its present value V_0 is

$$\begin{aligned} V_0 &= B_T^{-1}\mathbb{E}_{\mathbb{Q}}X \quad (54) \\ &= e^{-rT}\mathbb{E}_{\mathbb{Q}}(S_T - k)^+ \quad (55) \end{aligned}$$

As in the previous section, using $F_Q = Fe^{-\rho\sigma_1\sigma_2}$ and a standard normal random variable Z,

$$S_T = F_Q \exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\right) \tag{56}$$

S_T can be written as above, therefore:

$$V_0 = e^{-rT}\mathbb{E}\left(F_Q \exp\left(\sigma_1\sqrt{T}Z - \frac{1}{2}\sigma_1^2 T\right) - k\right)^+ \tag{57}$$

Using the 'calculation formula for the price of a call option in the log-normal case', which was already discussed in Section 4.1, we get:

$$V_0 = e^{-rT}\left\{F_Q\Phi\left(\frac{\log\frac{F_Q}{k} + \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right) - k\Phi\left(\frac{\log\frac{F_Q}{k} - \frac{1}{2}\sigma_1^2 T}{\sigma_1\sqrt{T}}\right)\right\} \tag{58}$$

10 Exercise 4.3

Consider the case where the underlying asset is S_t yen, the exchange rate C_t is in units of dollars per yen, and the correlation coefficient is ρ. Determine the formula for the quanto forward price in this scenario. In the previous model, the payoff currency was in the denominator of the exchange rate convention, such as 'pounds per dollar'.

In the current case, the payoff currency is in the numerator, which means the correlation coefficient effectively becomes $\rho \to -\rho$.

In this case, the formula becomes: $F_Q = Fe^{\rho\sigma_1\sigma_2}$.

The pricing formulas for the digital and call options are also modified in accordance with this change to the forward price formula.

References

[1] Financial Calculus - An Introduction to Derivative Pricing - Martin Baxter, Andrew Rennie