On Limits in Category Theory

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Abstract

Notes on limits in category theory.

1 Constructing New Sets from Sets, and New Categories from Categories

There are operations for constructing new sets from existing sets.

1. The operation to create a product set from sets X and Y: $X \times Y$
2. The operation to create a subset from a condition $\phi(x)$ on an element $x \in X$: $\{x \in X | \phi(x)\}$
3. The operation to create a disjoint union (coproduct) from sets X and Y: $X \coprod Y$
4. The operation to create a quotient set from a set X by an equivalence relation $\sim$: $X/\sim$

These correspond in category theory to:

1. Product
2. Equalizer
3. Coproduct
4. Coequalizer

Furthermore, these are understood uniformly as the following two concepts:

1. Limits (a generalization of products and equalizers)
2. Colimits (a generalization of coproducts and coequalizers)

	In Set Theory	In Category Theory	More Unified View
$X \times Y$	Product	Product	Limit
$\{x \in X \vert \phi(x)\}$	Subset	Equalizer	Limit
$X \coprod Y$	Disjoint Union	Coproduct	Colimit
$X/\sim$	Quotient Set	Coequalizer	Colimit

2 Product

2.1 Definition of Product

For objects a, b in a category C, a product of a and b is a triple $\langle u, p_0, p_1 \rangle$ satisfying the following two conditions.

$$\begin{tikzcd}
& u \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
a & & b
\end{tikzcd}$$

1. (Product object) $u \in C$ with morphisms $p_0 : u \to a$ and $p_1 : u \to b$.
2. (Universal property) For any triple $\langle v, q_0, q_1 \rangle$ satisfying the same condition, there exists a unique morphism $\exists! h : v \to u$ such that $p_0 \circ h = q_0$ and $p_1 \circ h = q_1$.

$$\begin{tikzcd}
& v \arrow[d, dashed, "\exists!\ h"] \arrow[ddl, bend right, "q_0"'] \arrow[ddr, bend left, "q_1"] & \\
& u \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
a & & b
\end{tikzcd}$$

2.2 Example of a Product

In the category of sets, **Set**, for $x, y \in \mathbf{Set}$, if we define $p_0 : x \times y \to x$ by $\langle a, b \rangle \mapsto a$ and $p_1 : x \times y \to y$ by $\langle a, b \rangle \mapsto b$, then $\langle x \times y, p_0, p_1 \rangle$ is the product of x and y.

2.2.1 Verification

■Checking Commutativity Let $\langle v, q_0, q_1 \rangle$ be taken as follows:

$$\begin{tikzcd}
& v \arrow[d, dashed, "h"] \arrow[ddl, bend right, "q_0"'] \arrow[ddr, bend left, "q_1"] & \\
& x \times y \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
x & & y
\end{tikzcd}$$

That is, for $a \in v$, we take $h(a) = \langle q_0(a), q_1(a) \rangle \in x \times y$.

Then for the morphism $h : v \to x \times y$, we have: $p_0 \circ h(a) = p_0(\langle q_0(a), q_1(a) \rangle) = q_0(a)$ which implies

$$p_0 \circ h = q_0$$

Similarly,

$$p_1 \circ h = q_1$$

Thus, the diagram commutes.

■Checking the Universal Property Next, suppose there is another $h' : v \to x \times y$ such that:

$$\begin{tikzcd}
& v \arrow[d, dashed, "h'"] \arrow[ddl, bend right, "q_0"'] \arrow[ddr, bend left, "q_1"] & \\
& x \times y \arrow[dl, "p_0"'] \arrow[dr, "p_1"] & \\
x & & y
\end{tikzcd}$$

If we can show that $h' = h$, then the uniqueness holds, and the universal property is satisfied.
Let $h'(a) = \langle q'_0(a), q'_1(a) \rangle \in x \times y$. Then,
$q_0(a) = p_0 \circ h'(a) = p_0(\langle q'_0(a), q'_1(a) \rangle) = q'_0(a)$ which implies

$$q'_0 = q_0$$

Similarly,

$$q'_1 = q_1$$

Therefore, $h'(a) = \langle q'_0(a), q'_1(a) \rangle = \langle q_0(a), q_1(a) \rangle = h(a)$. This holds for any a, so

$$h' = h$$

Thus, h is unique, and the universal property of the product is satisfied.

3 Equalizer

3.1 Definition of Equalizer

$$\begin{tikzcd}
u \arrow[r, "p"] & a \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & b \\
v \arrow[u, dashed, "\exists!\ h"] \arrow[ur, "q"'] & &
\end{tikzcd}$$

For morphisms $f, g : a \to b$ in a category C, an equalizer of f and g is a pair $\langle u, p \rangle$ satisfying:

1. (Equalizer object) $u \in C$ with a morphism $p : u \to a$ such that $f \circ p = g \circ p$.
2. (Universal property) For any pair $\langle v, q \rangle$ satisfying the same condition, there exists a unique morphism $\exists! h : v \to u$ such that $p \circ h = q$.

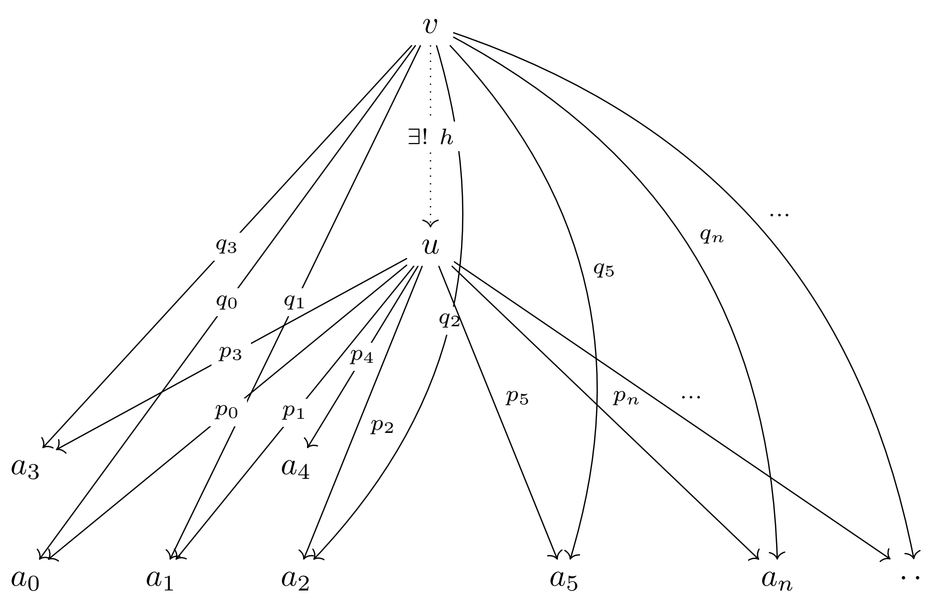
3.2 Example of an Equalizer

In the category of sets, **Set**, where $f, g : x \to y$ are functions, the equalizer is something like $u = \{a \in x | f(a) = g(a)\}$ and $p : u \hookrightarrow x$.

4 Limit

4.1 The "Limit-ness" of a Limit

The definitions of a product and an equalizer are such that, given some objects and morphisms, there is a certain object and morphism, say u and p, and if there is another object and morphism, say v and q, that satisfy the same conditions, there exists a unique morphism between them.



A limit is an object and collection of morphisms, $u, p_0, p_1, p_2, \cdots$, corresponding to a large number of objects. If there is another object and collection of morphisms, $v, q_0, q_1, q_2, , \cdots$, that satisfy the same conditions, there exists a unique morphism h between them.

It is known that when there are infinitely many objects, a limit can be expressed as a combination of (infinitely many) products and (infinitely many) equalizers.

4.2 Example of a Limit

Let's consider an example in **Set**.

Let p be a prime number and $X_n = \mathbf{Z}/p^n\mathbf{Z} = \{0, 1, \cdots p^n - 1\}$.

Consider the following sequence:

$$X_0 \xleftarrow{f_0} X_1 \xleftarrow{f_1} X_2 \cdots \xleftarrow{f_n} X_{n+1} \cdots$$

The sets are:

$X_0 = \{0\}$

$X_1 = \{0, 1, \cdots p - 1\}$

$X_2 = \{0, 1, \cdots p^2 - 1\}$

The morphisms are:

$$\begin{array}{cccc} f_n : & X_{n+1} & \to & X_n \\ & \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ & x & \to & x \bmod p^n \end{array}$$

In this case, the limit is denoted as $\mathbf{Z}_p$, and

$$\mathbf{Z}_p = \left\{ x \in \prod_{n=0}^{\infty} X_n \,\middle|\, \forall n,\ f_n(x_{n+1}) = x_n \right\}$$

For example, in this case, the limit is written as an infinite product combined with a single equalizer (the operation of taking a subset defined by an equation).

5 Coproduct

This is the dual of a product, with the direction of the morphisms reversed.

$$
\begin{tikzcd}
& v & \\
& u \arrow[u, dashed, "\exists!\ h"] & \\
a \arrow[ur, "p_0"'] \arrow[uur, bend left, "q_0"] & & b \arrow[ul, "p_1"] \arrow[uul, bend right, "q_1"']
\end{tikzcd}
$$

6 Coequalizer

This is the dual of an equalizer, with the direction of the morphisms reversed. The positions of objects a and b are also interchanged.

$$
\begin{tikzcd}
u \arrow[d, dashed, "\exists!\ h"] & b \arrow[l, "p"] \arrow[dl, "q"] & a \arrow[l, shift right, "f"'] \arrow[l, shift left, "g"] \\
v & &
\end{tikzcd}
$$