

Massless Real Scalar Field in 5D Minkowski Space Gaining Mass Upon Compactification of One Spatial Dimension

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Abstract

These are notes from a chat I had with a friend.

I'll make a note of how a massless real scalar field acquires mass when one spatial dimension is compactified in 5D Minkowski spacetime.

1 Set Up

Let $\mu, \nu = 0, 1, 2, 3$ and $M, N = 0, 1, 2, 3, 4$. We define the metric $\eta_{\mu\nu}$ for 4D Minkowski space M_4 and the metric η_{MN} for 5D Minkowski space with the following signatures:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}, \quad \eta_{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$$

We compactify the 5th component (the 4th spatial dimension) into a circle with radius a . That is, we parameterize the 5th component, which has become a circle, with an angle θ . Then the coordinates become:

$$(x^0, x^1, x^2, x^3, x^4) \rightarrow (x^0, x^1, x^2, x^3, \theta)$$

The infinitesimal element of the 5th component is $dx^4 = ad\theta$, so the line element ds of this space $M_4 \times S^1$ is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + a^2 d\theta^2$$

The metric g_{MN} of this space is

$$g_{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & a^2 \end{pmatrix}, \quad g^{MN} = \begin{pmatrix} -1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \frac{1}{a^2} \end{pmatrix}$$

2 How does a Massless Scalar Field ϕ Change When Compactified from $M_5 \rightarrow M_4 \times S^1$?

For simplicity, we'll write the spacetime coordinates as x in 4D and X in 5D.

The action for a massless real scalar field $\phi(x)$ on M_4 is as follows:

$$S = \int d^4x \left(-\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) \right)$$

For the Lagrangian to be invariant in the general case of $g_{\mu\nu}$, it is necessary to write it as:

$$S = \int d^4x \sqrt{-g} \left(-\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) \right) \quad \text{where} \quad g = \det(g_{\mu\nu})$$

We will now consider the action by extending this to the 5D case:

$$S = \int d^5X \sqrt{-g} \left(-\frac{1}{2} g^{MN} \partial_M \phi(X) \partial_N \phi(X) \right) \quad \text{where} \quad g = \det(g_{MN})$$

The 5th component of the scalar field is compactified, and since its period is 2π , we can perform a Fourier series expansion.

$$\begin{aligned} \phi(X) &= \phi(x, \theta) \\ &= \frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \end{aligned}$$

Substituting this into the action:

$$S = -\frac{1}{2} \int d^5X \sqrt{-\det(g_{MN})} g^{MN} \partial_M \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) \partial_N \left(\frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \right)$$

Here,

$$\det(g_{MN}) = \begin{vmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \\ & & & & a^2 \end{vmatrix} = -a^2$$

Furthermore, when we separate the calculation into 4D Minkowski space and S^1 components, we get:

$$\begin{aligned} &= -\frac{1}{2} a \int d^4x d\theta \left\{ g_{\mu\nu} \partial_\mu \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) \partial_\nu \phi(x) \left(\frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \right) \right. \\ &\quad \left. + g^{\theta\theta} \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \right) \right\} \end{aligned}$$

2.1 Calculation of the First Term of the Action

The first term is:

$$-\frac{1}{2} a \int d^4x d\theta \sum_{n,m} \frac{1}{2\pi} e^{i(n+m)\theta} \eta^{\mu\nu} \partial_\mu \phi_m(x) \partial_\nu \phi_n(x)$$

Now, let's evaluate the θ integral.

$$\int d\theta e^{i(n+m)\theta} = 2\pi \delta_{n,-m}$$

So we can also perform the summation over m to get:

$$-\frac{1}{2} a \sum_n \int d^4x \eta^{\mu\nu} \partial_\mu \phi_n(x) \partial_\nu \phi_{-n}(x)$$

Since ϕ is a real scalar field,

$$\phi(x) = \phi^*(x)$$

Therefore,

$$\sum_n \frac{1}{\sqrt{2\pi}} \phi_n(x) e^{in\theta} = \sum_n \frac{1}{\sqrt{2\pi}} \phi_n^*(x) e^{-in\theta}$$

$$= \sum_n \frac{1}{\sqrt{2\pi}} \phi_{-n}^*(x) e^{in\theta}$$

In the last equality, we substituted $n \rightarrow -n$.

The set $\{e^{in\theta}\}_{n=0,\pm 1,\pm 2,\dots}$ is an orthonormal basis, so:

$$\phi_n(x) = \phi_{-n}^*(x)$$

In particular, when $n = 0$:

$$\phi_0(x) = \phi_0^*(x)$$

so $\phi_0(x) \in \mathbb{R}$

Thus, the first term of the action is:

$$\begin{aligned} & -\frac{1}{2}a \sum_{n \in \mathbb{Z}} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi_n^*(x) \\ &= -a \sum_{n=1}^{\infty} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi_n^*(x) - \frac{1}{2}a \int d^4x \partial_\mu \phi_0(x) \partial^\nu \phi_0^*(x) \end{aligned}$$

2.2 Calculation of the Second Term of the Action

Let's proceed with the calculation of the second term of the action.

Here, $g^{\theta\theta} = \frac{1}{a^2}$, and the θ derivative is:

$$\partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) = \frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) (im) e^{im\theta}$$

So,

$$\begin{aligned} & -\frac{1}{2}a \int d^4x d\theta g^{\theta\theta} \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_m \phi_m(x) e^{im\theta} \right) \partial_\theta \left(\frac{1}{\sqrt{2\pi}} \sum_n \phi_n(x) e^{in\theta} \right) \\ &= -\frac{1}{2}a \int d^4x \frac{1}{a^2} \frac{1}{2\pi} \sum_{m,n} \left\{ \int d\theta e^{i(n+m)\theta} \right\} (-nm) \phi_n(x) \phi_m(x) \\ &= -\frac{1}{2}a \int d^4x \frac{1}{a^2} \frac{1}{2\pi} \sum_{m,n} \{2\pi \delta_{n,-m}\} (-nm) \phi_n(x) \phi_m(x) \end{aligned}$$

Performing the summation over m , we get:

$$\begin{aligned} &= -\frac{1}{2}a \int d^4x \frac{1}{a^2} \sum_n n^2 \phi_n(x) \phi_n^*(x) \\ &= -\frac{1}{2}a \sum_{n=1}^{\infty} \int d^4x \frac{n^2}{a^2} \phi_n(x) \phi_n^*(x) \end{aligned}$$

From the above, the action for a massless real scalar field on $M_4 \times S^1$ is:

$$\begin{aligned} S &= -a \sum_{n=1}^{\infty} \int d^4x \partial_\mu \phi_m(x) \partial^\nu \phi_n^*(x) - \frac{1}{2}a \int d^4x \partial_\mu \phi_0(x) \partial^\nu \phi_0^*(x) - \frac{1}{2}a \sum_{n=1}^{\infty} \int d^4x \frac{n^2}{a^2} \phi_n(x) \phi_n^*(x) \\ &= a \int d^4x \left\{ -\frac{1}{2} \partial_\mu \phi_0(x) \partial^\nu \phi_0^*(x) - \sum_{n=1}^{\infty} \left[\partial_\mu \phi_m(x) \partial^\nu \phi_n^*(x) + \frac{n^2}{a^2} \phi_n(x) \phi_n^*(x) \right] \right\} \end{aligned}$$

3 Interpretation

Let's write the integrand of the 4D coordinate x in the action as L_4 :

$$L_4 = -\frac{1}{2}\partial_\mu\phi_0(x)\partial^\nu\phi_0^*(x) - \sum_{n=1}^{\infty} \left[\partial_\mu\phi_n(x)\partial^\nu\phi_n^*(x) + \frac{n^2}{a^2}\phi_n(x)\phi_n^*(x) \right]$$

$$= L_{\text{free}} + L_{\text{mass}}$$

The first term,

$$L_{\text{free}} = -\frac{1}{2}\partial_\mu\phi_0(x)\partial^\nu\phi_0^*(x)$$

is a massless real scalar field on 4D Minkowski space, and the second term,

$$L_{\text{mass}} = -\sum_{n=1}^{\infty} \left[\partial_\mu\phi_n(x)\partial^\nu\phi_n^*(x) + \frac{n^2}{a^2}\phi_n(x)\phi_n^*(x) \right]$$

is the sum of complex scalar fields $\phi_n(x)$ ($n = 1, 2, 3, \dots$) with an effective mass of $M_n = \left| \frac{n}{a} \right|$.

In the end, a massless real scalar field on the compactified space $M_4 \times S^1$ can be seen, from a perspective that ignores the S^1 , as a superposition on M_4 of:

$$\begin{cases} \phi_0(x) & \cdots & \text{a massless real scalar field} \\ \phi_n(x) & \cdots & \text{a complex scalar field with mass } \left| \frac{n}{a} \right| \end{cases}$$

In conclusion, it has been shown that when one spatial dimension of a 5D spacetime is compactified, a massless real scalar field from the original theory gives rise to complex scalar fields that have mass.

If the radius a of the compactified dimension is very small, the mass $M_n = \left| \frac{n}{a} \right|$ becomes very large, so the complex scalar fields $\phi_n(x)$ do not affect physical observations, and only the massless real scalar field $\phi_0(x)$ is observed.

It is known that 5D general relativity, when viewed as a theory on 4D spacetime with a compactified S^1 like this, gives rise to 4D general relativity and 4D electromagnetism.

It was thought that by extending this to higher dimensions, it might be possible to unify

$$\left\{ \begin{array}{l} \cdot \text{ gravity} \\ \cdot \text{ electromagnetism} \\ \cdot \text{ the weak force} \\ \cdot \text{ the strong force} \end{array} \right.$$

This idea led to attempts at supergravity theory, but the theory was found to contradict observations. Currently, string theory is considered the most promising candidate for a unified theory of the four forces.

It is known that supergravity theory is the low-energy limit of superstring theory.