Market Price of Risk

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Abstract

These are notes from a self-study session on Chapter 3 of 'Financial Calculus - An Introduction to Derivative Pricing' by Martin Baxter and Andrew Rennie. Written on June 3, 2020. The note explores what it means for an asset to be tradable.

Contents

Stochastic processes can be either tradable or non-tradable.

For example, a foreign exchange rate itself is not directly tradable. For a given exchange rate between another country's currency and one's own, what is actually traded are assets denominated in a currency, converted at that rate. For instance, the price of an asset denominated in a foreign currency is multiplied by the rate to convert it to the price of an asset in the domestic currency. It is the converted asset that is traded, not the exchange rate itself.

Let's consider how the distinction between tradable and non-tradable is determined.

Ultimately, however, this distinction can also be a matter of human sentiment: whether market participants want to trade something, or more precisely, whether there is a counterparty willing to engage in the trade.

First, it is necessary to consider what 'tradable' truly means.

If we determine that an asset represented by a specific stochastic process S_t is tradable, and we can skillfully choose an appropriate stochastic process B_t for discounting it, we can then proceed to analyze the market constructed from S_t and B_t.

1 Martingales are Tradable

Suppose there exists a measure $\mathbb{Q}$ that makes a tradable asset $Z_t = B_t^{-1} S_t$ a martingale. Let's consider a case where another stochastic process V_t, adapted to the same filtration $\mathcal{F}_t$, when discounted, gives a price $E_t = B_t^{-1} V_t$ that is also a $\mathbb{Q}$-martingale.

If the volatility of Z_t is non-zero, the Martingale Representation Theorem guarantees the existence of an $\mathcal{F}_t$-predictable stochastic process ϕ_t such that

$$dE_t \;=\; \phi_t dZ_t$$

Following the same logic as in previous examples, we construct a portfolio consisting of ϕ_t and ψ_t based on the following strategy:

· Hold ϕ_t units of S_t.

· Hold ψ_t units of B_t.

Here, ψ_t is chosen to satisfy

$$\phi_t S_t + \psi_t B_t \;=\; B_t E_t \;=\; V_t$$

This means $\psi_t = E_t - \phi_t Z_t$. The value of ψ_t is chosen so that the price of the portfolio constructed with this strategy is always equal to V_t.

This strategy is self-financing. In other words, the change in the portfolio's price is caused solely by the changes in the asset prices.

From the above, we have successfully replicated V_t using S_t and B_t.

Such a V_t is called a tradable asset.

For a discounted stochastic process to be a $\mathbb{Q}$-martingale means that the process can be replicated without any cost using tradable assets (in this case, S_t and B_t).

Conversely, this implies that such a $\mathbb{Q}$-martingale is itself a tradable asset.

2 If it is not a Martingale, it is Non-Tradable

Let's assume that $E_t = B_t^{-1} V_t$ is not a $\mathbb{Q}$-martingale.

In this case, for some times T and s, the following holds with positive probability:

$$\mathbb{E}_{\mathbb{Q}}(B_T^{-1} V_T | \mathcal{F}_s) \;\neq\; B_s^{-1} V_s$$

Let U_t be the cost of replicating the contract $X = V_T$. That is,

$$U_t = B_t^{-1}\mathbb{E}_{\mathbb{Q}}(B_T^{-1}V_T|\mathcal{F}_t)$$

At $t = T$, we have $U_T = V_T$, but for t (where $0 < t < T$), $U_t \neq V_t$ with positive probability. Let's examine this case.

First, when $U_t > V_t$, one can gain an infinite profit by buying V_t and selling U_t.
Next, when $U_t < V_t$, one can gain an infinite profit by buying U_t and selling V_t.

From this, assuming that $E_t = B_t^{-1}V_t$ is not a $\mathbb{Q}$-martingale leads to the creation of an arbitrage opportunity through trading V_t. Such opportunities vanish in an extremely short time. (If they were to persist, the parties involved would generate either infinite wealth or infinite loss.)

In other words, V_t is non-tradable (except for the fleeting moments when an arbitrage opportunity exists).

In a market consisting of tradable assets S_t and B_t, we can determine whether another process is tradable based on the reasoning above.

In summary, if we let $\mathbb{Q}$ be the martingale measure for $B_t^{-1}S_t$, a process is tradable if it is a $\mathbb{Q}$-martingale, and non-tradable if it is not.

3 Exercise 4.1

Under measure $\mathbb{Q}$, assume the stock price S_t and bond price B_t are given by

$$\begin{aligned}
S_t &= \exp\left(\sigma\tilde{W}_t + \left(r - \frac{1}{2}\sigma^2\right)t\right) \\
B_t &= e^{rt}
\end{aligned}$$

3.1 (1) $X_t = S_t^2$

Discounting the process $X_t = S_t^2$, we get:

$$\begin{aligned}
Z_t &= B_t^{-1}X_t \\
&= \exp\left(2\sigma\tilde{W}_t + \left(2r - \sigma^2\right)t - rt\right) \\
&= \exp\left(2\sigma\tilde{W}_t + \left(r - \sigma^2\right)t\right)
\end{aligned}$$

Its stochastic differential equation (SDE) is:

$$\begin{aligned}
\frac{dZ_t}{Z_t} &= \exp\left(2\sigma d\tilde{W}_t + \left(r - \sigma^2 + \frac{(2\sigma)^2}{2}\right)dt\right) \\
&= \exp\left(2\sigma d\tilde{W}_t + \left(r + \sigma^2\right)dt\right)
\end{aligned}$$

When the drift $r + \sigma^2 = 0$, this process is a martingale and thus tradable. However, if the drift $r + \sigma^2 \neq 0$, it is not a martingale and is non-tradable.

3.2 (2) $X_t = S_t^{-2r/\sigma^2}$

Discounting the process $X_t = S_t^{-2r/\sigma^2}$, we get:

$$
\begin{aligned}
Z_t &= B_t^{-1} X_t \\
&= \exp\left(-\frac{2r}{\sigma^2}\sigma\tilde{W}_t - \frac{2r}{\sigma^2}\left(r - \frac{1}{2}\sigma^2\right)t - rt\right) \\
&= \exp\left(-\frac{2r}{\sigma}\tilde{W}_t - \frac{2r^2}{\sigma^2}t\right)
\end{aligned}
$$

The SDE for this process is:

$$
\begin{aligned}
\frac{dZ_t}{Z_t} &= \exp\left(-\frac{2r}{\sigma}d\tilde{W}_t + \left(-\frac{2r^2}{\sigma^2} + \frac{1}{2}\left(\frac{2r}{\sigma}\right)^2\right)dt\right) \\
&= \exp\left(-\frac{2r}{\sigma}d\tilde{W}_t\right)
\end{aligned}
$$

The drift is zero (regardless of the conditions on r or σ), so it is a martingale. Therefore, it is tradable.

4 Tradable Assets and the Market Price of Risk

In the most basic Black-Scholes model presented so far, the stock price S_t is

$$S_t = S_0 \exp(\sigma W_t + \mu t)$$

and its SDE is

$$dS_t = S_t \exp\left(\sigma dW_t + \left(\mu + \frac{1}{2}\sigma^2\right)dt\right)$$

However, for simplicity in this section, we will define it as

$$dS_t = S_t \exp(\sigma dW_t + \mu dt)$$

This means we start by defining the stock price as

$$S_t = S_0 \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right)$$

This definition simplifies discussions involving SDEs.

Consider a market with two different risky assets, S_t^1 and S_t^2.

Assume these risky assets are adapted to the same Brownian motion W_t.

$$
\begin{aligned}
dS_t^1 &= S_t^1 \exp(\sigma_1 dW_t + \mu_1 dt) \\
dS_t^2 &= S_t^2 \exp(\sigma_2 dW_t + \mu_2 dt)
\end{aligned}
$$

Since we assume these risky assets are tradable, a common martingale measure $\mathbb{Q}$ must exist for their respective discounted processes. If the numéraire is $B_t = \exp(rt)$, then using $i = 1, 2$ for abbreviation, the discounted processes are:

$$
\begin{aligned}
S_t^i &= S_0^i \exp\left(\sigma_i dW_t + \left(\mu_i - \frac{1}{2}\sigma_i^2\right)dt\right) \\
B_t^{-1} S_t^i &= S_0^i \exp\left(\sigma_i dW_t + \left(\mu_i - r - \frac{1}{2}\sigma_i^2\right)dt\right) \\
d(B_t^{-1} S_t^i) &= S_0^i \exp(\sigma_i dW_t + (\mu_i - r)\,dt)
\end{aligned}
$$

Therefore, under the measure that makes the discounted process $B_t^{-1}S_t^i$ a martingale,

$$\tilde{W}_t = W_t + \frac{\mu_i - r}{\sigma_i}t$$

must hold for both $i = 1$ and $i = 2$.

This implies that

$$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

Let's denote this value by γ.

$$\gamma \;=\; \frac{\mu_i - r}{\sigma_i}$$

This value represents the excess asset return μ over the cash bond's growth rate r, per unit of risk (σ). Hence, $\gamma = \dfrac{\mu - r}{\sigma}$ is called the **market price of risk**.

The market price of risk is the very factor that appears in the drift transformation of Brownian motion under Girsanov's theorem.

Using the term 'market price of risk', we can state that: 'All tradable assets in the same market have the same market price of risk.'

The above was for the case where μ and σ are constants. In the more general case where μ_t and σ_t are predictable processes, the market price of risk $\gamma_t = \dfrac{\mu_t - r}{\sigma_t}$ becomes a stochastic process, but it is the same stochastic process for all tradable assets in the market.

5 Risk-Neutral Probability Measure

The martingale measure $\mathbb{Q}$ for the discounted process is also called the risk-neutral probability measure.

The reason is as follows:

$$\begin{aligned} d(B_t^{-1}S_t^i) &= B_t^{-1}S_t^i \exp\left(\sigma_i dW_t + (\mu_i - r)\, dt\right) \\ &= B_t^{-1}S_t^i \exp\left(\sigma_i d\tilde{W}_t\right) \end{aligned}$$

Therefore,

$$\begin{aligned} B_t^{-1}S_t^i &= B_0^{-1}S_0^i \exp\left(\int_0^t \sigma_i d\tilde{W}_s - \frac{1}{2}\int_0^t \sigma_i^2 ds\right) \\ &= B_0^{-1}S_0^i \exp\left(\sigma_i \tilde{W}_t - \frac{1}{2}\sigma_i^2 t\right) \\ S_t^i &= S_0^i \exp\left(\sigma_i \tilde{W}_t + \left(r - \frac{1}{2}\sigma_i^2\right) t\right) \\ dS_t^i &= S_t^i \exp\left(\sigma_i \tilde{W}_t + \left(r - \frac{1}{2}\sigma_i^2 + \frac{1}{2}\sigma_i^2\right) t\right) \\ &= S_t^i \exp\left(\sigma_i \tilde{W}_t + rt\right) \end{aligned}$$

Thus, under $\mathbb{Q}$, the drift of the SDE for the price S_t^i of a tradable risky asset becomes the growth rate of the cash bond.

It is particularly important to note that under $\mathbb{Q}$, the drift r in the SDE does not depend on the risk (σ_i).

This means that, under $\mathbb{Q}$, all tradable assets have the same growth rate as the cash bond, regardless of their risk (σ_i).

(This example uses $i = 1, 2$, but it can be shown to hold for any $i \in \mathbb{N}$ by the exact same argument.)

Since $\mathbb{Q}$ is a probability measure independent of risk (σ_i), it is called the risk-neutral probability measure.

Let's revisit the above discussion from the perspective of the market price of risk to provide an alternative explanation.

In general, the SDE for S_t under $\mathbb{Q}$ can be written in the form

$$dS_t = S_t \exp\left(\sigma \tilde{W}_t + \tilde{\mu} t\right)$$

where we provisionally set the drift as $\tilde{\mu}$.

Recalling Girsanov's theorem, under a $\mathbb{Q}$-martingale measure, the market price of risk for a tradable asset must be zero.

That is,

$$\frac{\tilde{\mu} - r}{\sigma} \;=\; 0$$

This implies

$$\tilde{\mu} \;=\; r$$

Therefore, for any tradable asset, the SDE for S_t under $\mathbb{Q}$ is generally

$$dS_t = S_t \exp\left(\sigma \tilde{W}_t + rt\right)$$

(simply substituting $\tilde{\mu} \leftarrow r$ into the previous equation). This shows that under $\mathbb{Q}$, the expression is risk-neutral, independent of σ.

6 On Non-Tradable Processes

Let's consider a non-tradable stochastic process X_t. Assume its SDE is

$$dX_t \;=\; \sigma_t dW_t + \mu_t dt$$

where W_t is a $\mathbb{P}$-Brownian motion, and σ_t, μ_t are predictable processes.

(Note that the right-hand side is dX_t this time, not dX_t/X_t as in the previous section. So, this is not an exponential process as before.) While X_t itself is non-tradable, let's assume that a transformation of it, $Y_t = f(X_t, t)$, is tradable.

Its SDE, using Itô's lemma, is:

$$\begin{aligned}
dY_t &= \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial x} dx + \frac{1}{2}\frac{\partial^2 Y_t}{\partial x^2} dx^2 \\
&= \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial x} dX_t + \frac{1}{2}\frac{\partial^2 Y_t}{\partial x^2} dX_t^2 \\
&= \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial x} (\sigma_t dW_t + \mu_t dt) + \frac{1}{2}\frac{\partial^2 Y_t}{\partial x^2} \sigma_t^2 dt \\
&= \left(\frac{\partial Y_t}{\partial t} + \mu_t \frac{\partial Y_t}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 Y_t}{\partial x^2}\right) dt + \sigma_t \frac{\partial Y_t}{\partial x} dW_t \\
\frac{dY_t}{Y_t} &= \frac{1}{Y_t}\left(\frac{\partial Y_t}{\partial t} + \mu_t \frac{\partial Y_t}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 Y_t}{\partial x^2}\right) dt + \frac{\sigma_t}{Y_t}\frac{\partial Y_t}{\partial x} dW_t
\end{aligned}$$

The last equality was obtained by dividing both sides by Y_t to align with the market price of risk expression from the previous section. (Since the SDE for X_t at the beginning of this discussion was $dX_t = \sigma_t dW_t + \mu_t dt$ and not $dX_t/X_t = \sigma_t dW_t + \mu_t dt$, we needed to create dY_t/Y_t on the left to match the formulation of the previous section.)

Since Y_t is tradable, we can consider its market price of risk.

The drift is

$$\tilde{\mu}_t \;=\; \frac{1}{Y_t}\left(\frac{\partial Y_t}{\partial t} + \mu_t \frac{\partial Y_t}{\partial x} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 Y_t}{\partial x^2}\right)$$

and the risk is

$$\tilde{\sigma}_t \;=\; \frac{\sigma_t}{Y_t}\frac{\partial Y_t}{\partial x}$$

Setting these allows us to relate this to the market price of risk from the previous section. If the cash bond interest rate follows a stochastic process r_t, then

$$\begin{aligned}\gamma_t &= \frac{\tilde{\mu}_t - r_t}{\tilde{\sigma}_t} \\ &= \frac{\dfrac{\partial Y_t}{\partial t} + \mu_t \dfrac{\partial Y_t}{\partial x} + \dfrac{1}{2}\sigma_t^2 \dfrac{\partial^2 Y_t}{\partial x^2} - r_t Y_t}{\sigma_t \dfrac{\partial Y_t}{\partial x}}\end{aligned}$$

Note that this γ_t represents the change of measure from $\mathbb{P}$ to $\mathbb{Q}$.

If $\tilde{W}_t$ is the $\mathbb{Q}$-Brownian motion, then from Girsanov's theorem,

$$d\tilde{W}_t \;=\; dW_t + \gamma_t dt$$

This allows us to find the SDE for X_t under $\mathbb{Q}$.

$$\begin{aligned}& dX_t \\ &= \sigma_t dW_t + \mu_t dt \\ &= \sigma_t (d\tilde{W}_t - \gamma_t dt) + \mu_t dt \\ &= \sigma_t d\tilde{W}_t - \sigma_t \frac{\dfrac{\partial Y_t}{\partial t} + \mu_t \dfrac{\partial Y_t}{\partial x} + \dfrac{1}{2}\sigma_t^2 \dfrac{\partial^2 Y_t}{\partial x^2} - r_t Y_t}{\sigma_t \dfrac{\partial Y_t}{\partial x}} dt + \mu_t dt \\ &= \sigma_t d\tilde{W}_t - \frac{\dfrac{\partial Y_t}{\partial t} + \dfrac{1}{2}\sigma_t^2 \dfrac{\partial^2 Y_t}{\partial x^2} - r_t Y_t}{\dfrac{\partial Y_t}{\partial x}} dt \\ &= \sigma_t d\tilde{W}_t + \frac{r_t Y_t - \dfrac{1}{2}\sigma_t^2 \dfrac{\partial^2 Y_t}{\partial x^2} - \dfrac{\partial Y_t}{\partial t}}{\dfrac{\partial Y_t}{\partial x}} dt\end{aligned}$$

where $Y_t = f(X_t, t)$.

7 Some Examples

Let's use a few examples to find the SDE that a non-tradable process follows under the risk-neutral measure $\mathbb{Q}$, given its SDE under the measure $\mathbb{P}$.

7.1 Black-Scholes Model

Suppose X_t is the logarithm of a tradable asset Y_t. That is, $X_t = \log Y_t$ from which

$$Y_t \;=\; \exp X_t$$

Further, assume that $\sigma_t = \sigma = \text{const.}$ and $\mu_t = \mu = \text{const.}$ Then, using the $\mathbb{P}$-Brownian motion W_t, we can write

$$\begin{aligned} X_t &= \sigma W_t + \mu t \\ \iff \quad dX_t &= \sigma dW_t + \left(\mu + \frac{1}{2}\sigma^2\right) dt \end{aligned}$$

The numéraire is the risk-free asset $B_t = e^{rt}$ with interest rate r.

Under this setup, we want to find the SDE that X_t satisfies under measure $\mathbb{Q}$.

In this case, the market price of risk is

$$\begin{aligned} \gamma &= \frac{\tilde{\mu} - r}{\tilde{\sigma}} \\ &= \frac{\left(\mu + \frac{1}{2}\sigma^2\right) - r}{\sigma} \end{aligned}$$

Under $\mathbb{Q}$, this must be zero, so $\mu + \frac{1}{2}\sigma^2 - r = 0$.

This means

$$\mu \;=\; r - \frac{1}{2}\sigma^2$$

Therefore, the desired answer, using the $\mathbb{Q}$-Brownian motion $\tilde{W}_t$, is

$$dX_t \;=\; \sigma d\tilde{W}_t + \left(r - \frac{1}{2}\sigma^2\right) dt$$

This is the solution, but for practice, let's use the formula we derived to provide an alternative solution.

$$\begin{aligned} dX_t &= \sigma_t d\tilde{W}_t + \frac{r_t Y_t - \frac{1}{2}\sigma_t^2 \frac{\partial^2 Y_t}{\partial x^2} - \frac{\partial Y_t}{\partial t}}{\frac{\partial Y_t}{\partial x}} dt \\ &= \sigma_t d\tilde{W}_t + \frac{r_t \exp X_t - \frac{1}{2}\sigma_t^2 \frac{\partial^2 \exp X_t}{\partial x^2} - \frac{\partial \exp X_t}{\partial t}}{\frac{\partial \exp X_t}{\partial x}} dt \\ &= \sigma_t d\tilde{W}_t + \frac{r_t \exp X_t - \frac{1}{2}\sigma_t^2 \exp X_t - 0}{\exp X_t} dt \\ &= \sigma d\tilde{W}_t + \left(r - \frac{1}{2}\sigma^2\right) dt \end{aligned}$$

As expected, the results match.

7.2 Black-Scholes with Continuous Dividends

Suppose a stock price S_t and a bond price B_t follow the Black-Scholes model

$$S_t = \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right) t\right)$$
$$B_t = \exp(rt)$$

Furthermore, assume that holding the stock pays a dividend of $\delta S_t dt$ over the time interval from t to $t + dt$.

Under measure $\mathbb{P}$, it follows the SDE below, using $\mathbb{P}$-Brownian motion W_t.

$$\frac{dS_t}{S_t} = \exp\left(\sigma dW_t + \mu dt\right)$$

Because of the dividends, S_t is not a tradable asset in this case.

If one were to buy the stock for S_0 at $t = 0$ and hold it until time t, its value would not be just S_t, but S_t plus the accumulated dividends.

If we assume two strategies exist - one where dividends are held as cash bonds, and another where dividends are continuously reinvested in the stock - an arbitrage opportunity would arise.

Therefore, we must consider the strategy of continuously reinvesting dividends into the stock.

Let's consider this case of continuous dividend reinvestment. If one holds ϕ_t shares of the stock at time t, the number of shares increases by $d\phi_t$ in time dt as follows:

$$d\phi_t = \delta dt$$

Solving this gives

$$\phi_t \;=\; \exp(\delta t)$$

Thus, the number of shares of stock S_t held at time t is $\phi_t = \exp(\delta t)$, and its price is $Y_t = \phi_t S_t$.

The SDE for Y_t is

$$Y_t = Y_0 \exp\left(\sigma W_t + (\mu + \delta)t\right)$$

so

$$\frac{dY_t}{Y_t} = \exp\left(\sigma dW_t + (\mu + \delta)dt\right)$$

Now, we consider the probability measure $\mathbb{Q}$ that makes the discounted asset process $B_t^{-1}S_t$ a martingale, i.e., the risk-neutral probability measure $\mathbb{Q}$.

By Girsanov's theorem, using $\mathbb{Q}$-Brownian motion $\tilde{W}_t$,

$$d\tilde{W}_t = dW_t + \gamma dt$$

where the market price of risk γ is

$$\gamma = \frac{(\mu + \delta) - r}{\sigma}$$

For the asset to be tradable (i.e., for the discounted asset process $B_t^{-1}S_t$ to be a martingale), we need $\gamma = 0$, which means

$$\mu = r - \delta$$

From this, the SDE that the non-tradable process S_t must satisfy under the risk-neutral measure $\mathbb{Q}$ is

$$\begin{aligned}\frac{dS_t}{S_t} &= \exp\left(\sigma dW_t + \mu dt\right) \\ &= \exp\left(\sigma(d\tilde{W}_t - \gamma dt) + (r-\delta)dt\right) \\ &= \exp\left(\sigma(d\tilde{W}_t - 0dt) + (r-\delta)dt\right) \\ &= \exp\left(\sigma d\tilde{W}_t + (r-\delta)dt\right)\end{aligned}$$

(Note that the equality from the second to the third line uses the fact that $\gamma = 0$.)

7.3 Foreign Exchange

Let C_t be the yen-dollar exchange rate, i.e., 1 dollar $= C_t$ yen. Let the dollar interest rate be r, and the yen interest rate be u.

To summarize, our model consists of: B_t as the dollar cash bond, D_t as the yen cash bond,

$$\begin{aligned}C_t &= \exp\left(\sigma W_t + \left(\mu - \frac{1}{2}\sigma^2\right)t\right) \\ B_t &= \exp(rt) \\ D_t &= \exp(ut)\end{aligned}$$

The units are C_t (yen/dollar), B_t (dollars), and D_t (yen), respectively.

In this case, B_t and D_t are tradable, but C_t, being cash itself, is not tradable on its own.

If we were to assume cash is tradable, an arbitrage would arise between two strategies:

· A strategy of simply holding cash from $t = 0$ to $t = T$.

· A strategy of holding a cash bond from $t = 0$ to $t = T$.

(Naturally, the strategy of holding the cash bond yields a higher profit with probability 1 than simply holding cash.)

First, let's examine the SDE of the non-tradable process C_t. Under measure $\mathbb{P}$, it follows this SDE with $\mathbb{P}$-Brownian motion W_t:

$$\frac{dC_t}{C_t} = \exp\left(\sigma dW_t + \mu dt\right)$$

We want to find the SDE that this process C_t follows under the risk-neutral measure $\mathbb{Q}$, using a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$.

Assets that are tradable in dollars are the dollar cash bond B_t and the yen cash bond D_t converted by the yen-dollar exchange rate C_t, which is the process $Y_t = C_t^{-1} D_t$.

$$Y_t = \exp\left(-\sigma W_t + \left(-\mu + \frac{1}{2}\sigma^2 + u\right)t\right)$$

The risk-neutral measure $\mathbb{Q}$ is the measure under which the discounted asset process $Z_t = B_t^{-1} Y_t$ is a martingale.

$$Z_t = \exp\left(-\sigma W_t + \left(-\mu + \frac{1}{2}\sigma^2 + u - r\right)t\right)$$

The SDE that Z_t satisfies is

$$\begin{aligned}\frac{dZ_t}{Z_t} &= \exp\left(-\sigma dW_t + \left(-\mu + \frac{1}{2}\sigma^2 + u - r + \frac{1}{2}(-\sigma)^2\right) dt\right) \\ &= \exp\left(-\sigma dW_t + \left(-\mu + \sigma^2 + u - r\right) dt\right)\end{aligned}$$

From Girsanov's theorem, since $\mathbb{P}$ and $\mathbb{Q}$ are equivalent measures, there exists a predictable process γ such that the following holds between the $\mathbb{P}$-Brownian motion W_t and the $\mathbb{Q}$-Brownian motion $\tilde{W}_t$:

$$d\tilde{W}_t = dW_t + \gamma dt$$

And this γ makes Z_t a $\mathbb{Q}$-martingale. So, considering

$$-\sigma d\tilde{W}_t = -\sigma dW_t + \left(-\mu + \sigma^2 + u - r\right) dt$$

we get

$$\gamma = \frac{-\mu + \sigma^2 + u - r}{-\sigma}$$

For Z_t to be tradable, we need $\gamma = 0$, which means $\mu = \sigma^2 + u - r$.

From this, the non-tradable process C_t can be expressed using the $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ as follows:

$$\begin{aligned}\frac{dC_t}{C_t} &= \exp\left(\sigma dW_t + \mu dt\right) \\ &= \exp\left(\sigma(d\tilde{W}_t - \gamma dt) + (\sigma^2 + u - r)dt\right) \\ &= \exp\left(\sigma(d\tilde{W}_t - 0dt) + (\sigma^2 + u - r)dt\right) \\ &= \exp\left(\sigma d\tilde{W}_t + (\sigma^2 + u - r)dt\right)\end{aligned}$$

References

[1] Financial Calculus - An Introduction to Derivative Pricing - Martin Baxter, Andrew Rennie