

Solutions to the Extended Gor'kov Equations for Anisotropic Superconductors and the Electronic Raman Response Function

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We begin with the BCS mean-field Hamiltonian \mathcal{H} .

$$\begin{aligned}\mathcal{H} &= \mathcal{H}_0 + \mathcal{H}_{\text{BCS}} , \\ \mathcal{H}_0 &= \sum_{\mathbf{k}, s} \varepsilon(\mathbf{k}) c_{\mathbf{k}s}^\dagger c_{\mathbf{k}s} , \\ \mathcal{H}_{\text{BCS}} &= \frac{1}{2} \sum_{\mathbf{k}, s_1, s_2} \left[\Delta_{s_1 s_2}(\mathbf{k}) c_{\mathbf{k}s_1}^\dagger c_{-\mathbf{k}s_2}^\dagger - \Delta_{s_1 s_2}^*(-\mathbf{k}) c_{-\mathbf{k}s_1} c_{\mathbf{k}s_2} \right] ,\end{aligned}\tag{1}$$

We introduce finite-temperature Green's functions in the Matsubara formalism as follows.

$$G_{ss'}(\mathbf{k}, \mathbf{k}'; \tau) = -\langle T_\tau \{ c_{\mathbf{k}s}(\tau) c_{\mathbf{k}'s'}^\dagger(0) \} \rangle ,\tag{2}$$

$$F_{ss'}(\mathbf{k}, \mathbf{k}'; \tau) = \langle T_\tau \{ c_{\mathbf{k}s}(\tau) c_{\mathbf{k}'s'}(0) \} \rangle , \quad F_{ss'}^\dagger(\mathbf{k}, \mathbf{k}'; \tau) = \langle T_\tau \{ c_{\mathbf{k}'s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger(0) \} \rangle .\tag{3}$$

Although we've written $F_{ss'}^\dagger$ here by convention, it's not the Hermitian conjugate of $F_{ss'}$. These Green's functions are c-numbers.

The Fourier transform from the time variable τ to $i\omega_n$ is defined as follows.

$$\begin{aligned}G_{ss'}(\mathbf{k}, \mathbf{k}'; \tau) &= \frac{1}{\beta} \sum_n G_{ss'}(\mathbf{k}, \mathbf{k}'; i\omega_n) e^{-i\omega_n \tau} , \\ F_{ss'}^{(\dagger)}(\mathbf{k}, \mathbf{k}'; \tau) &= \frac{1}{\beta} \sum_n F_{ss'}^{(\dagger)}(\mathbf{k}, \mathbf{k}'; i\omega_n) e^{-i\omega_n \tau} .\end{aligned}\tag{4}$$

Here, $\omega_n = (2n + 1)\pi k_B T$ ($n \in \mathbb{Z}$) is the fermionic Matsubara frequency. For a homogeneous system, the momentum variables of these Green's functions become $\mathbf{k} = \mathbf{k}'$ for the G function and $\mathbf{k} = -\mathbf{k}'$ for the $F^{(\dagger)}$ function, and can be specified by a single momentum value.

$$G_{ss'}(\mathbf{k}, i\omega_n) = -\int_0^\beta d\tau \langle T_\tau \{ c_{\mathbf{k}s}(\tau) c_{\mathbf{k}s'}^\dagger(0) \} \rangle e^{i\omega_n \tau} ,\tag{5}$$

$$F_{ss'}(\mathbf{k}, i\omega_n) = \int_0^\beta d\tau \langle T_\tau \{ c_{\mathbf{k}s}(\tau) c_{-\mathbf{k}s'}(0) \} \rangle e^{i\omega_n \tau} , \quad F_{ss'}^\dagger(\mathbf{k}, i\omega_n) = \int_0^\beta d\tau \langle T_\tau \{ c_{-\mathbf{k}s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger(0) \} \rangle e^{i\omega_n \tau} .\tag{6}$$

In the following, we only consider a homogeneous system.

To find the spectral representation of the Green's functions, we determine the time evolution of the operators. We use the Heisenberg equation of motion.

$$\partial_\tau c_{\mathbf{k}s}(\tau) = [\mathcal{H}, c_{\mathbf{k}s}(\tau)] = e^{\mathcal{H}\tau} [\mathcal{H}, c_{\mathbf{k}s}] e^{-\mathcal{H}\tau} \quad (7)$$

For this calculation, the relationship between commutators and anticommutators

$$[AB, C] = A\{B, C\} - \{A, C\}B, \quad (8)$$

is also useful.

For \mathcal{H}_0 ,

$$\begin{aligned} [\mathcal{H}_0, c_{\mathbf{k}s}] &= \sum_{\mathbf{k}', s'} \varepsilon(\mathbf{k}') \left[c_{\mathbf{k}'s'}^\dagger c_{\mathbf{k}'s'} c_{\mathbf{k}s} \right] \\ &= - \sum_{\mathbf{k}', s'} \varepsilon(\mathbf{k}') \left\{ c_{\mathbf{k}'s'}^\dagger, c_{\mathbf{k}s} \right\} c_{\mathbf{k}'s'} \\ &= -\varepsilon(\mathbf{k}) c_{\mathbf{k}s}. \end{aligned} \quad (9)$$

and next, for \mathcal{H}_{BCS} ,

$$\begin{aligned} [\mathcal{H}_{\text{BCS}}, c_{\mathbf{k}s}] &= \left[\frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \left(\Delta_{s_1 s_2}(\mathbf{k}') c_{\mathbf{k}'s_1}^\dagger c_{-\mathbf{k}'s_2}^\dagger - \Delta_{s_1 s_2}^*(-\mathbf{k}') c_{-\mathbf{k}'s_1} c_{\mathbf{k}'s_2} \right), c_{\mathbf{k}s} \right] \\ &= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') \left[c_{\mathbf{k}'s_1}^\dagger c_{-\mathbf{k}'s_2}^\dagger, c_{\mathbf{k}s} \right] \\ &= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') \left(c_{\mathbf{k}'s_1}^\dagger \left\{ c_{-\mathbf{k}'s_2}^\dagger, c_{\mathbf{k}s} \right\} - \left\{ c_{\mathbf{k}'s_1}^\dagger, c_{\mathbf{k}s} \right\} c_{-\mathbf{k}'s_2}^\dagger \right) \\ &= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') \left(c_{\mathbf{k}'s_1}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_2} - c_{-\mathbf{k}'s_2}^\dagger \delta_{\mathbf{k}, \mathbf{k}'} \delta_{s, s_1} \right). \end{aligned} \quad (10)$$

By carefully swapping the indices, we can see that the two terms on the right-hand side are equal. Specifically, we should carefully perform the index swap on only the second term as follows:

$$\begin{aligned}
[\mathcal{H}_{\text{BCS}}, c_{\mathbf{k}s}] &= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') c_{\mathbf{k}' s_1}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_2} - \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(-\mathbf{k}') c_{\mathbf{k}' s_2}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_1} \\
&= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') c_{\mathbf{k}' s_1}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_2} + \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_2 s_1}(\mathbf{k}') c_{\mathbf{k}' s_2}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_1} \\
&= \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') c_{\mathbf{k}' s_1}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_2} + \frac{1}{2} \sum_{\mathbf{k}', s_1, s_2} \Delta_{s_1 s_2}(\mathbf{k}') c_{\mathbf{k}' s_1}^\dagger \delta_{\mathbf{k}, -\mathbf{k}'} \delta_{s, s_2} \\
&= \sum_{s'} \Delta_{s' s}(-\mathbf{k}) c_{-\mathbf{k} s'}^\dagger \\
&= - \sum_{s'} \Delta_{s s'}(\mathbf{k}) c_{-\mathbf{k} s'}^\dagger .
\end{aligned} \tag{11}$$

From the two terms above, the time evolution of the operator is:

$$\partial_\tau c_{\mathbf{k}s}(\tau) = -\varepsilon(\mathbf{k}) c_{\mathbf{k}s}(\tau) - \sum_{s'} \Delta_{s s'}(\mathbf{k}) c_{-\mathbf{k} s'}^\dagger(\tau) . \tag{12}$$

Using this relation, we derive the equation of motion for the Green's functions. The time evolution of the Green's functions is:

$$\begin{aligned}
\partial_\tau G_{ss'}(\mathbf{k}, \tau) &= \partial_\tau \left(-\theta(\tau) \langle c_{\mathbf{k}s}(\tau) c_{\mathbf{k} s'}^\dagger \rangle + \theta(-\tau) \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s}(\tau) \rangle \right) \\
&= -[\partial_\tau \theta(\tau)] \langle c_{\mathbf{k}s}(\tau) c_{\mathbf{k} s'}^\dagger \rangle - \theta(\tau) \langle [\partial_\tau c_{\mathbf{k}s}(\tau)] c_{\mathbf{k} s'}^\dagger \rangle \\
&\quad + [\partial_\tau \theta(-\tau)] \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s}(\tau) \rangle + \theta(-\tau) \langle c_{\mathbf{k} s'}^\dagger [\partial_\tau c_{\mathbf{k}s}(\tau)] \rangle \\
&= -\delta(\tau) \langle c_{\mathbf{k}s}(\tau) c_{\mathbf{k} s'}^\dagger \rangle + \theta(\tau) \varepsilon(\mathbf{k}) \langle c_{\mathbf{k}s}(\tau) c_{\mathbf{k} s'}^\dagger \rangle + \theta(\tau) \sum_{s''} \Delta_{s s''}(\mathbf{k}) \langle c_{-\mathbf{k} s''}^\dagger(\tau) c_{\mathbf{k} s'}^\dagger \rangle \\
&\quad - \delta(\tau) \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s}(\tau) \rangle - \theta(-\tau) \varepsilon(\mathbf{k}) \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s}(\tau) \rangle - \theta(-\tau) \sum_{s''} \Delta_{s s''}(\mathbf{k}) \langle c_{\mathbf{k} s'}^\dagger c_{-\mathbf{k} s''}^\dagger(\tau) \rangle \\
&= \varepsilon(\mathbf{k}) G_{ss'}(\mathbf{k}, \tau) + \sum_{s''} \Delta_{s s''}(\mathbf{k}) F_{s' s''}^\dagger(\mathbf{k}, \tau)
\end{aligned} \tag{13}$$

The terms proportional to the delta function cancel out due to the mechanism:

$$\delta(\tau) \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s}(\tau) \rangle = \delta(0) \langle c_{\mathbf{k} s'}^\dagger c_{\mathbf{k}s} \rangle = -\delta(0) \langle c_{\mathbf{k}s} c_{\mathbf{k} s'}^\dagger \rangle = -\delta(\tau) \langle c_{\mathbf{k}s}(\tau) c_{\mathbf{k} s'}^\dagger \rangle$$

Fourier transforming both sides so that the variables become $\tau \rightarrow i\omega_n$:

$$\begin{aligned}
\int_0^\beta d\tau \partial_\tau G_{ss'}(\mathbf{k}, \tau) e^{i\omega_n \tau} &= \int_0^\beta d\tau \varepsilon(\mathbf{k}) G_{ss'}(\mathbf{k}, \tau) e^{i\omega_n \tau} + \int_0^\beta d\tau \sum_{s''} \Delta_{s s''}(\mathbf{k}) F_{s' s''}^\dagger(\mathbf{k}, \tau) e^{i\omega_n \tau} \\
\longleftrightarrow i\omega_n G_{ss'}(\mathbf{k}, i\omega_n) &= \varepsilon(\mathbf{k}) G_{ss'}(\mathbf{k}, i\omega_n) + \sum_{s''} \Delta_{s s''}(\mathbf{k}) F_{s' s''}^\dagger(\mathbf{k}, i\omega_n)
\end{aligned} \tag{14}$$

This equation is not closed for $G_{ss'}(\mathbf{k}, i\omega_n)$, and we need to set up a similar equation of motion for $F_{ss'}^\dagger(\mathbf{k}, i\omega_n)$ and solve them simultaneously. Taking the Hermitian conjugate of equation 12:

$$\partial_\tau c_{\mathbf{k}s}^\dagger(\tau) = -\varepsilon(\mathbf{k})c_{\mathbf{k}s}^\dagger(\tau) - \sum_{s'} \Delta_{ss'}^*(\mathbf{k})c_{-\mathbf{k}s'}(\tau) , \quad (15)$$

and using this relation:

$$\begin{aligned} \partial_\tau F_{ss'}^\dagger(\mathbf{k}, \tau) &= \partial_\tau \left(\theta(\tau) \langle c_{-\mathbf{k}s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger \rangle - \theta(-\tau) \langle c_{\mathbf{k}s}^\dagger c_{-\mathbf{k}s'}(\tau) \rangle \right) \\ &= [\partial_\tau \theta(\tau)] \langle c_{-\mathbf{k}s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger \rangle + \theta(\tau) \langle [\partial_\tau c_{-\mathbf{k}s'}^\dagger(\tau)] c_{\mathbf{k}s}^\dagger \rangle \\ &\quad - [\partial_\tau \theta(-\tau)] \langle c_{\mathbf{k}s}^\dagger c_{-\mathbf{k}s'}(\tau) \rangle - \theta(-\tau) \langle c_{\mathbf{k}s}^\dagger [\partial_\tau c_{-\mathbf{k}s'}(\tau)] \rangle \\ &= \delta(\tau) \langle c_{-\mathbf{k}s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger \rangle - \varepsilon(\mathbf{k}) \theta(\tau) \langle c_{-\mathbf{k}s'}^\dagger(\tau) c_{\mathbf{k}s}^\dagger \rangle - \sum_{s''} \Delta_{s's''}^*(-\mathbf{k}) \theta(\tau) \langle c_{\mathbf{k}s''}(\tau) c_{\mathbf{k}s}^\dagger \rangle \\ &\quad + \delta(\tau) \langle c_{\mathbf{k}s}^\dagger c_{-\mathbf{k}s'}(\tau) \rangle + \theta(-\tau) \varepsilon(\mathbf{k}) \langle c_{\mathbf{k}s}^\dagger c_{-\mathbf{k}s'}(\tau) \rangle + \sum_{s''} \Delta_{s's''}^*(-\mathbf{k}) \theta(-\tau) \langle c_{\mathbf{k}s}^\dagger c_{\mathbf{k}s''}(\tau) \rangle \\ &= -\varepsilon(\mathbf{k}) F_{ss'}^\dagger(\mathbf{k}, \tau) - \sum_{s''} \Delta_{s's''}^*(-\mathbf{k}) G_{s''s}(\mathbf{k}, \tau) , \end{aligned} \quad (16)$$

$$\longleftrightarrow \quad i\omega_n F_{ss'}^\dagger(\mathbf{k}, i\omega_n) = -\varepsilon(\mathbf{k}) F_{ss'}^\dagger(\mathbf{k}, i\omega_n) - \sum_{s''} \Delta_{s's''}^*(-\mathbf{k}) G_{s''s}(\mathbf{k}, i\omega_n) . \quad (17)$$

From the above, the simultaneous equations to solve to obtain the spectral representations of the respective Green's functions are:

$$G_{ss'}(\mathbf{k}, i\omega_n) = \sum_{s''} \frac{\Delta_{ss''}(\mathbf{k})}{i\omega_n - \varepsilon(\mathbf{k})} F_{s's''}^\dagger(\mathbf{k}, i\omega_n) , \quad (18)$$

$$F_{ss'}^\dagger(\mathbf{k}, i\omega_n) = - \sum_{s''} \frac{\Delta_{s's''}^*(-\mathbf{k})}{i\omega_n + \varepsilon(\mathbf{k})} G_{s''s}(\mathbf{k}, i\omega_n) . \quad (19)$$

The solutions to the Gor'kov equations for the case where the pair potential representation matrix is unitary can be expressed as: Let the elementary excitation energy spectrum be

$$E_{\mathbf{k}} = \sqrt{\varepsilon^2(\mathbf{k}) + |\mathbf{d}(\mathbf{k})|^2} , \quad (20)$$

then,

$$\hat{G}(\mathbf{k}, i\omega_n) = -\frac{i\omega_n + \varepsilon(\mathbf{k})}{\omega_n^2 + E_{\mathbf{k}}^2} \hat{\sigma}_0 , \quad (21)$$

$$\hat{F}(\mathbf{k}, i\omega_n) = \frac{i\mathbf{d}(\mathbf{k}) \cdot \hat{\boldsymbol{\sigma}} \hat{\sigma}_y}{\omega_n^2 + E_{\mathbf{k}}^2} = \frac{\hat{\Delta}(\mathbf{k})}{\omega_n^2 + E_{\mathbf{k}}^2} . \quad (22)$$

Using these, we calculate the electronic Raman response function. Using the bosonic Matsubara frequency $\nu_n = 2m\pi k_B T$ ($m \in \mathbb{Z}$),

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\mathbf{q}, i\nu_m) &= - \int_0^\beta d\tau \langle T_\tau [\tilde{\rho}_{\mathbf{q}}^\dagger(\tau) \tilde{\rho}_{\mathbf{q}}] \rangle e^{i\nu_m \tau} \\
&= - \int_0^\beta d\tau \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger c_{\mathbf{k}_2, s_2}] \rangle e^{i\nu_m \tau} \\
&= - \int_0^\beta d\tau \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \left\{ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_1, s_1}(\tau)] \rangle \langle T_\tau [c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger c_{\mathbf{k}_2, s_2}] \rangle \right. \\
&\quad - \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2, s_2}] \rangle \\
&\quad \left. + \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2, s_2}] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \right\} e^{i\nu_m \tau} . \quad (23)
\end{aligned}$$

The term in the first parenthesis on the right-hand side is a Green's function at the same time, so it's a constant and can be dropped. Rearranging the signs:

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\mathbf{q}, i\nu_n) &= \int_0^\beta d\tau \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \left\{ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2, s_2}] \rangle e^{i\nu_m \tau} \right. \\
&\quad \left. - \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2, s_2}] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \right\} e^{i\nu_m \tau} . \quad (24)
\end{aligned}$$

We expand the sum over spins s_1, s_2 . The first term inside the curly braces is:

$$\sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2, s_2}] \rangle \quad (25)$$

$$\begin{aligned}
&= \sum_{\mathbf{k}_1, \mathbf{k}_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \left\{ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, \uparrow}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, \uparrow}(\tau) c_{\mathbf{k}_2, \uparrow}] \rangle \right. \\
&\quad + \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, \downarrow}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, \uparrow}(\tau) c_{\mathbf{k}_2, \downarrow}] \rangle \\
&\quad + \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, \uparrow}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, \downarrow}(\tau) c_{\mathbf{k}_2, \uparrow}] \rangle \\
&\quad \left. + \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, \downarrow}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, \downarrow}(\tau) c_{\mathbf{k}_2, \downarrow}] \rangle \right\} . \quad (26)
\end{aligned}$$

The sum over momenta only remains when $\mathbf{k}_2 = -\mathbf{k}_1$. Performing the sum over \mathbf{k}_2 yields:

$$\begin{aligned}
& \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2, s_2}] \rangle \\
&= \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} \left\{ \langle T_\tau [c_{-\mathbf{k}+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}-\mathbf{q}, \uparrow}^\dagger] \rangle \langle T_\tau [c_{-\mathbf{k}, \uparrow}(\tau) c_{\mathbf{k}, \uparrow}] \rangle \right. \\
&+ \langle T_\tau [c_{-\mathbf{k}+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}-\mathbf{q}, \downarrow}^\dagger] \rangle \langle T_\tau [c_{-\mathbf{k}, \uparrow}(\tau) c_{\mathbf{k}, \downarrow}] \rangle \\
&+ \langle T_\tau [c_{-\mathbf{k}+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}-\mathbf{q}, \uparrow}^\dagger] \rangle \langle T_\tau [c_{-\mathbf{k}, \downarrow}(\tau) c_{\mathbf{k}, \uparrow}] \rangle \\
&+ \left. \langle T_\tau [c_{-\mathbf{k}+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}-\mathbf{q}, \downarrow}^\dagger] \rangle \langle T_\tau [c_{-\mathbf{k}, \downarrow}(\tau) c_{\mathbf{k}, \downarrow}] \rangle \right\} \\
&= \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} \left\{ F_{\uparrow\uparrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\uparrow\uparrow}(-\mathbf{k}, \tau) \right. \\
&+ F_{\downarrow\uparrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\uparrow\downarrow}(-\mathbf{k}, \tau) \\
&+ F_{\uparrow\downarrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\downarrow\uparrow}(-\mathbf{k}, \tau) \\
&+ \left. F_{\downarrow\downarrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\downarrow\downarrow}(-\mathbf{k}, \tau) \right\} . \tag{27}
\end{aligned}$$

Next, we transform the second term on the right-hand side of equation (24) in the same way. After summing over spins, the sum over momentum \mathbf{k}_2 only remains for the case where $\mathbf{k}_2 = \mathbf{k}_1 + \mathbf{q}$.

$$\begin{aligned}
& \sum_{\mathbf{k}_1, \mathbf{k}_2, s_1, s_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, s_1}^\dagger(\tau) c_{\mathbf{k}_2, s_2}] \rangle \langle T_\tau [c_{\mathbf{k}_1, s_1}(\tau) c_{\mathbf{k}_2-\mathbf{q}, s_2}^\dagger] \rangle \\
&= \sum_{\mathbf{k}_1, \mathbf{k}_2} \gamma_{\mathbf{k}_1} \gamma_{\mathbf{k}_2} \left\{ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}_2, \uparrow}] \rangle \langle T_\tau [c_{\mathbf{k}_1, \uparrow}(\tau) c_{\mathbf{k}_2-\mathbf{q}, \uparrow}^\dagger] \rangle \right. \\
&+ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}_2, \downarrow}] \rangle \langle T_\tau [c_{\mathbf{k}_1, \uparrow}(\tau) c_{\mathbf{k}_2-\mathbf{q}, \downarrow}^\dagger] \rangle \\
&+ \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}_2, \uparrow}] \rangle \langle T_\tau [c_{\mathbf{k}_1, \downarrow}(\tau) c_{\mathbf{k}_2-\mathbf{q}, \uparrow}^\dagger] \rangle \\
&+ \left. \langle T_\tau [c_{\mathbf{k}_1+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}_2, \downarrow}] \rangle \langle T_\tau [c_{\mathbf{k}_1, \downarrow}(\tau) c_{\mathbf{k}_2-\mathbf{q}, \downarrow}^\dagger] \rangle \right\} \\
&= \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \gamma_{\mathbf{k}+\mathbf{q}} \left\{ \langle T_\tau [c_{\mathbf{k}+\mathbf{q}, \uparrow}^\dagger(\tau) c_{\mathbf{k}+\mathbf{q}, \uparrow}] \rangle \langle T_\tau [c_{\mathbf{k}, \uparrow}(\tau) c_{\mathbf{k}, \uparrow}^\dagger] \rangle \right. \\
&+ \langle T_\tau [c_{\mathbf{k}, \uparrow}^\dagger(\tau) c_{\mathbf{k}, \downarrow}] \rangle \langle T_\tau [c_{\mathbf{k}, \uparrow}(\tau) c_{\mathbf{k}, \downarrow}^\dagger] \rangle \\
&+ \langle T_\tau [c_{\mathbf{k}+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}+\mathbf{q}, \uparrow}] \rangle \langle T_\tau [c_{\mathbf{k}, \downarrow}(\tau) c_{\mathbf{k}, \uparrow}^\dagger] \rangle \\
&+ \left. \langle T_\tau [c_{\mathbf{k}+\mathbf{q}, \downarrow}^\dagger(\tau) c_{\mathbf{k}+\mathbf{q}, \downarrow}] \rangle \langle T_\tau [c_{\mathbf{k}, \downarrow}(\tau) c_{\mathbf{k}, \downarrow}^\dagger] \rangle \right\} \\
&= - \sum_{\mathbf{k}} \gamma_{\mathbf{k}} \gamma_{\mathbf{k}+\mathbf{q}} \left\{ G_{\uparrow\uparrow}(\mathbf{k} + \mathbf{q}, \tau) G_{\uparrow\uparrow}(\mathbf{k}, -\tau) \right. \\
&+ \left. G_{\downarrow\downarrow}(\mathbf{k} + \mathbf{q}, \tau) G_{\downarrow\downarrow}(\mathbf{k}, -\tau) \right\} . \tag{28}
\end{aligned}$$

Here we have used the unitary condition $G_{\uparrow\downarrow} = G_{\downarrow\uparrow} = 0$.

In summary,

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\mathbf{q}, i\nu_m) &= - \int_0^\beta d\tau \langle T_\tau [\tilde{\rho}_{\mathbf{q}}^\dagger(\tau) \tilde{\rho}_{\mathbf{q}}] \rangle e^{i\nu_m \tau} \\
&= \sum_{\mathbf{k}} \int_0^\beta d\tau \left\{ \gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} \left[F_{\uparrow\uparrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\uparrow\uparrow}(-\mathbf{k}, \tau) + F_{\downarrow\uparrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\uparrow\downarrow}(-\mathbf{k}, \tau) \right. \right. \\
&\quad \left. \left. + F_{\uparrow\downarrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\downarrow\uparrow}(-\mathbf{k}, \tau) + F_{\downarrow\downarrow}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{\downarrow\downarrow}(-\mathbf{k}, \tau) \right] \right. \\
&\quad \left. - \gamma_{\mathbf{k}} \gamma_{\mathbf{k}+\mathbf{q}} \left[G_{\uparrow\uparrow}(\mathbf{k} + \mathbf{q}, \tau) G_{\uparrow\uparrow}(\mathbf{k}, -\tau) + G_{\downarrow\downarrow}(\mathbf{k} + \mathbf{q}, \tau) G_{\downarrow\downarrow}(\mathbf{k}, -\tau) \right] \right\} e^{i\nu_m \tau} \quad (31)
\end{aligned}$$

Next, we perform the Fourier transform with respect to time. For the anomalous Green's functions:

$$\begin{aligned}
\int_0^\beta d\tau F_{s's}^\dagger(\mathbf{k} - \mathbf{q}, \tau) F_{ss'}(-\mathbf{k}, \tau) e^{i\nu_m \tau} &= \frac{1}{\beta^2} \int_0^\beta d\tau \sum_{n_1, n_2} F_{s's}^\dagger(\mathbf{k} - \mathbf{q}, i\omega_{n_1}) F_{ss'}(-\mathbf{k}, i\omega_{n_2}) e^{i(\nu_m - \omega_{n_1} - \omega_{n_2})\tau} \\
&= \frac{1}{\beta^2} \sum_{n_1, n_2} F_{s's}^\dagger(\mathbf{k} - \mathbf{q}, i\omega_{n_1}) F_{ss'}(-\mathbf{k}, i\omega_{n_2}) \beta \delta_{\omega_{n_1}, \nu_m - \omega_{n_2}} \\
&= \frac{1}{\beta} \sum_n F_{s's}^\dagger(\mathbf{k} - \mathbf{q}, i\nu_m - i\omega_n) F_{ss'}(-\mathbf{k}, i\omega_n) \quad (32)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\int_0^\beta d\tau G_{ss}(\mathbf{k} + \mathbf{q}, \tau) G_{ss}(\mathbf{k}, -\tau) e^{i\nu_m \tau} &= \frac{1}{\beta^2} \int_0^\beta d\tau \sum_{n_1, n_2} G_{ss}(\mathbf{k} + \mathbf{q}, i\omega_{n_1}) G_{ss}(\mathbf{k}, i\omega_{n_2}) e^{i(\nu_m - \omega_{n_1} + \omega_{n_2})\tau} \\
&= \frac{1}{\beta^2} \sum_{n_1, n_2} G_{ss}(\mathbf{k} + \mathbf{q}, i\omega_{n_1}) G_{ss}(\mathbf{k}, i\omega_{n_2}) \beta \delta_{\omega_{n_1}, \nu_m + \omega_{n_2}} \\
&= \frac{1}{\beta} \sum_n G_{ss}(\mathbf{k} + \mathbf{q}, i\nu_m + i\omega_n) G_{ss}(\mathbf{k}, i\omega_n) \quad (33)
\end{aligned}$$

From the above, the response function is as follows.

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\mathbf{q}, i\nu_m) &= - \int_0^\beta d\tau \langle T_\tau [\tilde{\rho}_{\mathbf{q}}^\dagger(\tau) \tilde{\rho}_{\mathbf{q}}] \rangle e^{i\nu_m \tau} \\
&= \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \sum_{ss'} \left[\gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} F_{s's}^\dagger(\mathbf{k} - \mathbf{q}, i\nu_m - i\omega_n) F_{ss'}(-\mathbf{k}, i\omega_n) \right. \\
&\quad \left. - \gamma_{\mathbf{k}} \gamma_{\mathbf{k}+\mathbf{q}} G_{ss}(\mathbf{k} + \mathbf{q}, i\nu_m + i\omega_n) G_{s's'}(\mathbf{k}, i\omega_n) \right] \quad (34)
\end{aligned}$$

We consider the limit $\mathbf{q} \rightarrow \mathbf{0}$. From now on, we will only consider this case, so we will abbreviate $\chi_{\tilde{\rho}\tilde{\rho}}(\mathbf{0}, i\nu_m) = \chi_{\tilde{\rho}\tilde{\rho}}(i\nu_m)$.

$$\begin{aligned}
\chi_{\bar{\rho}\bar{\rho}}(i\nu_m) &= \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \sum_{ss'} \left[\gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} F_{s's}^\dagger(\mathbf{k}, i\nu_m - i\omega_n) F_{ss'}(-\mathbf{k}, i\omega_n) \right. \\
&\quad \left. - \gamma_{\mathbf{k}}^2 G_{ss}(\mathbf{k}, i\nu_m + i\omega_n) G_{s's'}(\mathbf{k}, i\omega_n) \right] \\
&= \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \left[\gamma_{\mathbf{k}} \gamma_{-\mathbf{k}} \sum_{ss'} \frac{\Delta_{s's}(\mathbf{k})}{(\nu_m - \omega_n)^2 + E_{\mathbf{k}}^2} \frac{\Delta_{ss'}(-\mathbf{k})}{\omega_n^2 + E_{-\mathbf{k}}^2} \right. \\
&\quad \left. - \gamma_{\mathbf{k}}^2 \frac{i\omega_n + i\nu_m + \varepsilon(\mathbf{k})}{(\nu_m + \omega_n)^2 + E_{\mathbf{k}}^2} \frac{i\omega_n + \varepsilon(\mathbf{k})}{\omega_n^2 + E_{\mathbf{k}}^2} \right] \tag{35}
\end{aligned}$$