Stochastic Volatility and Local Volatility

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1 Stochastic Volatility

1.1 Derivation of the Equation

Following the arguments of Wilmott(2000), let's proceed.

Assume that the stock price S_t and its variance ν_t at time t follow the following equations:

$$\begin{aligned} dS_t &= \mu_t S_t dt \ + \ \sqrt{\nu_t} S_t dZ_1 \\ d\nu_t &= \alpha(S_t, \nu_t, t)dt \ + \ \eta\beta(S_t, \nu_t, t)\sqrt{\nu_t} dZ_2 \\ \langle dZ_1 dZ_2 \rangle &= \rho dt \end{aligned}$$

Here, μ_t is a deterministic function, representing the instantaneous drift of the stock's return.

η represents the volatility of volatility.

dZ_1 and dZ_2 are Wiener processes, and ρ is the correlation between the stock's return and the change in ν_t.

The first equation was assumed by Black and Scholes (1973).

In fact, Wilmott(2000) shows in Section 8.3 that this system of equations becomes the Black-Scholes model if we take the limit $\eta \to 0$ in the second equation.

Assuming the variance ν_t follows this second equation allows for a very general discussion.

For now, the functional forms of α and β are not determined.

Furthermore, we haven't specified the process that $\sqrt{\nu_t}$ follows (e.g., assuming it's a Wiener process).

When constructing a risk-free portfolio, the source of randomness in the Black-Scholes framework was solely the stock price, so only the stock itself was needed for hedging.

In this case, Wilmott(2000)'s equations show that we must also hedge the volatility.

Consider a portfolio Π that includes an asset with value $V(S, \nu, t)$. Let the holdings of stock S be $(-\Delta)$ and the holdings of a volatility-dependent asset V_1 be $(-\Delta_1)$.

$$\Pi = V - \Delta S - \Delta_1 V_1$$

We want to find the change in the portfolio's value, $d\Pi$, over a small time interval dt. Using Itô's lemma, we get:

$$\begin{aligned} d\Pi &= dV - d(\Delta S) - d(\Delta_1 V_1) \\ &= dV - \Delta dS - \Delta_1 dV_1 \end{aligned}$$

Let's expand each term.

First, expanding dS^2, $d\nu^2$, and $dSd\nu$:

$$\begin{aligned} dS^2 &= (\sqrt{\nu} S dZ_1 + \mu S dt)^2 \\ &= \nu S^2 dZ_1^2 + 2\mu S\sqrt{\nu} S dt dZ_1 + \mu^2 S^2 dt^2 \\ &= \nu S^2 dt \end{aligned}$$

We use $dZ_1^2 = dt$, as higher-order infinitesimals like $dZ_1 dt$ and dt^2 are of order $o(dt)$ and therefore vanish. Similarly,

$$\begin{aligned} d\nu^2 &= (\alpha dt + \eta\beta\sqrt{\nu} dZ_2)^2 \\ &= \eta^2 \nu \beta^2 dt \end{aligned}$$

$$\begin{aligned} dSd\nu &= (\sqrt{\nu} S dZ_1 + \mu S dt)(\alpha dt + \eta\beta\sqrt{\nu} dZ_2) \\ &= \rho\eta\nu\beta dt \end{aligned}$$

From these, we can express dV as:

$$\begin{aligned} dV &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \nu} d\nu \\ &\quad + \frac{1}{2}\frac{\partial^2 V}{\partial S^2} dS^2 + \frac{\partial^2 V}{\partial S \partial \nu} dSd\nu + \frac{1}{2}\frac{\partial^2 V}{\partial \nu^2} d\nu^2 \\ &= \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \nu} d\nu \\ &\quad + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} dt + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} dt + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} dt \\ &= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} \right. \\ &\quad \left. + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\ &\quad + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \nu} d\nu \end{aligned}$$

Similarly, for $V_1 = V_1(S, \nu, t)$, we get:

$$\begin{aligned} dV_1 \;&=\; \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} \right. \\ &\quad \left. + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} \right) dt \\ &\quad + \frac{\partial V_1}{\partial S} dS + \frac{\partial V_1}{\partial \nu} d\nu \end{aligned}$$

Substituting these into the portfolio's stochastic differential, we have:

$$
\begin{aligned}
& dV - \Delta dS - \Delta_1 dV_1 \\
= & \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} \right. \\
& \left. \quad + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
& + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \nu} d\nu \\
& - \Delta dS \\
& - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} \right. \\
& \left. \quad + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} \right) dt \\
& - \Delta_1 \frac{\partial V_1}{\partial S} dS - \Delta_1 \frac{\partial V_1}{\partial \nu} d\nu
\end{aligned}
$$

Combining the terms:

$$
\begin{aligned}
d\Pi = & \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} \right. \\
& \left. + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
& - \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} \right. \\
& \left. \quad + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} \right) dt \\
& + \left(\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} \right) dS \\
& + \left(\frac{\partial V}{\partial \nu} - \Delta_1 \frac{\partial V_1}{\partial \nu} \right) d\nu
\end{aligned}
$$

We now hedge this portfolio to make it instantaneously risk-free. To eliminate the dS and $d\nu$ terms, we choose the following constraints:

$$
\begin{aligned}
\frac{\partial V}{\partial S} - \Delta - \Delta_1 \frac{\partial V_1}{\partial S} &= 0 \\
\frac{\partial V}{\partial \nu} - \Delta_1 \frac{\partial V_1}{\partial \nu} &= 0
\end{aligned}
$$

Solving these yields the hedge quantities we need:

$$
\begin{aligned}
\Delta &= \frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial \nu} \frac{\partial V_1}{\partial S}}{\frac{\partial V_1}{\partial \nu}} \\
\Delta_1 &= \frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}}
\end{aligned}
$$

Under these conditions, the portfolio can be expressed using the risk-free rate r:

$$
\begin{aligned}
d\Pi &= r\Pi dt \\
&= r(V - \Delta S - \Delta_1 V_1)dt \\
&= r\left\{ V - \left(\frac{\partial V}{\partial S} - \frac{\frac{\partial V}{\partial \nu}\frac{\partial V_1}{\partial S}}{\frac{\partial V_1}{\partial \nu}} \right) S - \left(\frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}} \right) V_1 \right\} dt
\end{aligned}
$$

At the same time, we have:

$$
\begin{aligned}
& d\Pi \\
=& \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
-& \Delta_1 \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} \right) dt \\
=& \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
& - \left(\frac{\frac{\partial V}{\partial \nu}}{\frac{\partial V_1}{\partial \nu}} \right) \\
& \times \left(\frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} \right) dt
\end{aligned}
$$

Since these expressions for $d\Pi$ are equal, we can rearrange the terms by moving the terms related to V to the left side and those related to V_1 to the right side:

$$
\begin{aligned}
& \frac{\begin{array}{l} \frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} \\ \qquad + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} + rS\frac{\partial V}{\partial S} - rV \end{array}}{\frac{\partial V}{\partial \nu}} \\
=& \frac{\begin{array}{l} \frac{\partial V_1}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V_1}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V_1}{\partial S \partial \nu} \\ \qquad + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V_1}{\partial \nu^2} + rS\frac{\partial V_1}{\partial S} - rV_1 \end{array}}{\frac{\partial V_1}{\partial \nu}}
\end{aligned}
$$

The left side of the equation depends only on V, and the right side only on V_1. For this equation to hold identically, both sides must equal an arbitrary function that is independent of S, ν, and t. Without loss of generality, we can set this function equal to $-(\alpha - \phi\beta\sqrt{\nu})$. Thus, we have:

$$
\begin{aligned}
& \frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} \\
& \qquad + \frac{1}{2}\eta^2 \nu \beta^2 \frac{\partial^2 V}{\partial \nu^2} + rS\frac{\partial V}{\partial S} - rV \;=\; -(\alpha - \phi\beta\sqrt{\nu})\frac{\partial V}{\partial \nu}
\end{aligned}
$$

Here, $\phi = \phi(S, \nu, t)$ is called the **market price of volatility risk**.

1.2 Market Price of Volatility Risk

Let's examine why Wilmott's discussion gives ϕ the name 'market price of volatility risk'. Consider a portfolio Π_1 composed of V, which is delta-hedged but not vega-hedged:

$$\Pi_1 = V - \frac{\partial V}{\partial S} S$$

Applying Itô's lemma just as before, we find the change in the portfolio's value, $d\Pi_1$:

$$\begin{aligned}
&d\Pi_1 \\
&= dV - Sd\left(\frac{\partial V}{\partial S}\right) - \frac{\partial V}{\partial S} dS \\
&= \Bigg\{ \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2\nu\beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
&\quad + \frac{\partial V}{\partial S} dS + \frac{\partial V}{\partial \nu} d\nu \Bigg\} \\
&\quad - \left(\Delta dS - \frac{\partial V}{\partial S} dS \right) \quad - \frac{\partial V}{\partial S} dS \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2\nu\beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
&\quad + \left(\frac{\partial V}{\partial S} - \Delta \right) dS + \frac{\partial V}{\partial \nu} d\nu
\end{aligned}$$

*1

Since this portfolio is delta-hedged, the dS term should vanish, meaning $\Delta dS = \frac{\partial V}{\partial S} dS$. This simplifies the expression to:

$$\begin{aligned}
&d\Pi_1 \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2\nu\beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
&\quad + \frac{\partial V}{\partial \nu} d\nu
\end{aligned}$$

Now, let's consider the difference in price between this portfolio, which is only delta-hedged (and therefore not vega-hedged), and a fully hedged risk-free portfolio, whose price change is $r\Pi_1 dt$:

*1 A point of confusion. The textbook seems to handle the product rule for $Sd\left(\frac{\partial V}{\partial S}\right)$ in a non-standard way. I can't figure out the exact reasoning behind the sudden appearance of the ΔdS term. To follow the text's flow for now, it's assumed that $Sd\left(\frac{\partial V}{\partial S}\right) = \Delta dS - \frac{\partial V}{\partial S} dS$.

$$
\begin{aligned}
& d\Pi_1 - r\Pi_1 dt \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} + \frac{1}{2}\eta^2\nu\beta^2 \frac{\partial^2 V}{\partial \nu^2} \right) dt \\
&\quad + \frac{\partial V}{\partial \nu} d\nu \\
&\quad - \left(rV - r\frac{\partial V}{\partial S} S \right) dt \\
&= \left(\frac{\partial V}{\partial t} + \frac{1}{2}\nu S^2 \frac{\partial^2 V}{\partial S^2} + \rho\eta\nu\beta \frac{\partial^2 V}{\partial S \partial \nu} \right. \\
&\qquad \left. + \frac{1}{2}\eta^2\nu\beta^2 \frac{\partial^2 V}{\partial \nu^2} - rV + r\frac{\partial V}{\partial S} S \right) dt \\
&\quad + \frac{\partial V}{\partial \nu} d\nu \\
&= -(\alpha - \phi\beta\sqrt{\nu}) \frac{\partial V}{\partial \nu} dt \quad + \frac{\partial V}{\partial \nu} d\nu
\end{aligned}
$$

*2

Using the second equation from the fundamental set, $d\nu = \alpha dt + \eta\beta\sqrt{\nu} dZ_2$, we can substitute it into the expression:

$$
\begin{aligned}
& d\Pi_1 - r\Pi_1 dt \\
&= -(\alpha - \phi\beta\sqrt{\nu}) \frac{\partial V}{\partial \nu} dt \; + \; \frac{\partial V}{\partial \nu} (\alpha dt + \eta\beta\sqrt{\nu} dZ_2) \\
&= \beta\sqrt{\nu} \frac{\partial V}{\partial \nu} (\phi dt + \eta dZ_2)
\end{aligned}
$$

Recalling the Capital Asset Pricing Model (CAPM):

Expected Return $=$ Risk-Free Rate $+$ Risk Premium

This leads to the following equation:

$$
d\Pi_1 \;=\; r\Pi_1 dt \;+\; \beta\sqrt{\nu} \frac{\partial V}{\partial \nu} (\phi dt + \eta dZ_2)
$$

By analogy with CAPM, $\phi(S, \nu, t)$ is called the **market price of volatility risk**.

The risk-neutral drift is defined as:

$$
\alpha' \;=\; \alpha - \beta\sqrt{\nu}\phi
$$

(Work in progress ......).

References

[1] Jim Gatheral (2006) The Volatility Surface: A Practitioner's Guide (Wiley Finance)

*2 The textbook has an incorrect sign for the $rS\frac{\partial V}{\partial S}$ term in the equation between the second and third equality signs in my notes.