Martingale Representation Theorem

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Abstract

These are my self-study notes for Chapter 3 of **Financial Calculus: An Introduction to Derivative Pricing** by Martin Baxter and Andrew Rennie, written on May 20, 2020.

1 Current Status and Review of Problems

In Section 2.3, we saw what a martingale process looks like in the context of discrete processes.

(Review) Definition (vii) from Section 2.3, 'A Diagrammatic Definition'

A process S is said to be a martingale with respect to the probability measure $\mathbb{P}$ and the filtration $\mathcal{F}_i$ if, for all $i \leq j$,

$$\mathbf{E}_{\mathbb{P}}(S_j|\mathcal{F}_i) = S_i$$

In continuous-time processes, the same holds true. Let's see how.

2 Martingale Conditions for Continuous-Time Processes

A stochastic process M_t is a martingale with respect to a measure $\mathbb{P}$ if it satisfies the following conditions.

Martingale Conditions

1. For all t, $\mathbf{E}_{\mathbb{P}}(|M_t|) < \infty$.
2. For $s(< t)$, $\mathbf{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = M_s$.

The second condition is the particularly crucial one for a martingale.

Just as with discrete processes, this expresses that the expected value in the future is equal to the present value.

To get a better feel for this, the textbook provides three examples.

2.1 Example 1: The Constant Process

A process where $S_t = c$ (a constant) at all times t is a martingale under any measure.

For any future times s, t (where $s(< t)$), we have $S_t = S_s = c$, so

$$\begin{aligned} c &= \mathbf{E}_{\mathbb{P}}(S_t|\mathcal{F}_s) \\ &= \mathbf{E}_{\mathbb{P}}(S_s|\mathcal{F}_s) \\ &= S_s \end{aligned}$$

This holds true for any measure $\mathbb{P}$.

2.2 Example 2: A $\mathbb{P}$-Brownian Motion under Measure $\mathbb{P}$

Let's confirm that a $\mathbb{P}$-Brownian motion is a $\mathbb{P}$-martingale.

For times s, t $(s < t)$, with W_t as the $\mathbb{P}$-Brownian motion, we have:

$$\begin{aligned} \mathbf{E}_{\mathbb{P}}(W_t|\mathcal{F}_s) &= \mathbf{E}_{\mathbb{P}}(W_t + W_s - W_s|\mathcal{F}_s) \\ &= \mathbf{E}_{\mathbb{P}}(W_s|\mathcal{F}_s) + \mathbf{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) \\ &= W_s + 0 \end{aligned}$$

Here, $\mathbf{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) = 0$ because of the property of Brownian motion that $W_t - W_s$ is independent of $\mathcal{F}_s$ and has a $N(0, t-s)$ distribution under $\mathbb{P}$.

Thus, W_t is a martingale.

2.3 Example 3: The Tower Law: The Process of Conditional Expectation under Measure $\mathbb{P}$

For a contract X with a payoff fixed at maturity T, let's confirm that the process $N_t = \mathbf{E}_{\mathbb{P}}(X|\mathcal{F}_t)$ is a $\mathbb{P}$-martingale.

To show this, we use the tower law, which holds for times s, t $(s \leq t)$:

$$\mathbf{E}_{\mathbb{P}}\Big(\mathbf{E}_{\mathbb{P}}\left(X|\mathcal{F}_t\right)\Big|\mathcal{F}_s\Big) \;=\; \mathbf{E}_{\mathbb{P}}(X|\mathcal{F}_s)$$

This is the same result we saw for discrete processes: the expectation of an expectation, first conditional on the history up to time t and then on the history up to time s, is equal to the expectation conditional on the history up to time s from the start.

Using this, we can show that:

$$\begin{aligned} \mathbf{E}_{\mathbb{P}}(N_t|\mathcal{F}_s) &= \mathbf{E}_{\mathbb{P}}\Big(\mathbf{E}_{\mathbb{P}}\left(X|\mathcal{F}_t\right)\Big|\mathcal{F}_s\Big) \\ &= \mathbf{E}_{\mathbb{P}}(X|\mathcal{F}_s) \\ &= N_s \end{aligned}$$

This proves that N_t is a martingale.

3 Exercise 3.10

3.1 Problem

Let W_t be a $\mathbb{P}$-Brownian motion. Show that the stochastic process $X_t = W_t + \gamma t$ is a $\mathbb{P}$-martingale only when $\gamma = 0$.

3.2 Solution

Taking the expectation under measure $\mathbb{P}$ conditional on the history $\mathcal{F}_s$ at time $s(< t)$:

$$\begin{aligned}\mathbf{E}_{\mathbb{P}}(X_t|\mathcal{F}_s) &= \mathbf{E}_{\mathbb{P}}(W_t + \gamma t|\mathcal{F}_s)\\ &= W_s + \gamma t\\ &= X_s + \gamma(t-s)\end{aligned}$$

Therefore, X_t is a $\mathbb{P}$-martingale only if $\gamma = 0$.

3.3 Digging a Bit Deeper

What if we introduce a positive constant volatility $\sigma(> 0)$ to the process, like $X_t = \sigma W_t + \gamma t$? In this case,

$$\begin{aligned}\mathbf{E}_{\mathbb{P}}(X_t|\mathcal{F}_s) &= \mathbf{E}_{\mathbb{P}}(\sigma W_t + \gamma t|\mathcal{F}_s)\\ &= \sigma W_s + \gamma t\\ &= X_s + \gamma(t-s)\end{aligned}$$

Similarly, it's not a $\mathbb{P}$-martingale unless the drift $\gamma = 0$.

Right after this, we'll see that "being a martingale is equivalent to having no drift term." The textbook will then discuss the necessary conditions when the drift is not constant but time-dependent.

3.4 (Addendum) Martingale of the Square of a Brownian Motion

Let W_t be a $\mathbb{P}$-Brownian motion. Taking the expectation of W_t^2 under measure $\mathbb{P}$ conditional on the history $\mathcal{F}_s$ at time $s(< t)$, we get:

$$\begin{aligned}\mathbf{E}_{\mathbb{P}}(W_t^2|\mathcal{F}_s) &= \mathbf{E}_{\mathbb{P}}[\{W_s + (W_t - W_s)\}^2|\mathcal{F}_s]\\ &= W_s^2 \ + \ 2W_s\mathbf{E}_{\mathbb{P}}(W_t - W_s|\mathcal{F}_s) \ + \ \mathbf{E}_{\mathbb{P}}[(W_t - W_s)^2|\mathcal{F}_s]\\ &= W_s^2 \ + \ 0 \ + \ (t-s)\end{aligned}$$

So, by subtracting t from both sides,

$$\mathbf{E}_{\mathbb{P}}(W_t^2 - t|\mathcal{F}_s) = W_s^2 - s$$

This shows that $W_t^2 - t$ is a martingale. (This uses the fact that $W_t - W_s$ is independent of $\mathcal{F}_s$ and has variance $(t-s)$.)

3.5 (Addendum) Exponential Martingale with Constant Volatility

Let W_t be a $\mathbb{P}$-Brownian motion. Taking the expectation of $\exp(\sigma W_t)$ under measure $\mathbb{P}$ conditional on the history $\mathcal{F}_s$ at time $s(< t)$:

$$\mathbf{E}_{\mathbb{P}}[e^{\sigma W_t}|\mathcal{F}_s] = e^{\sigma W_s}\mathbf{E}_{\mathbb{P}}[e^{\sigma(W_t - W_s)}|\mathcal{F}_s]$$

Recalling the moment generating function from the previous section (since $W_t - W_s$ is independent of $\mathcal{F}_s$ and has variance $(t-s)$), we find that:

$$\begin{aligned}\mathbf{E}_{\mathbb{P}}[e^{\sigma W_t}|\mathcal{F}_s] &= e^{\sigma W_s}\exp\left(0\times\sigma + \frac{1}{2}\sigma^2(t-s)\right)\\ &= \exp\left(\frac{1}{2}\sigma^2 t\right)\exp\left(\sigma W_s - \frac{1}{2}\sigma^2 s\right)\end{aligned}$$

Multiplying both sides by $\exp\left(-\frac{1}{2}\sigma^2 t\right)$ gives us:

$$\mathbf{E}_{\mathbb{P}}[e^{(\sigma W_t - \frac{1}{2}\sigma^2 t)}|\mathcal{F}_s] = e^{(\sigma W_s - \frac{1}{2}\sigma^2 s)}$$

Therefore, $e^{(\sigma W_t - \frac{1}{2}\sigma^2 t)}$ is also a martingale.

4 Martingale Representation Theorem

(Review) Binomial Martingale Representation Theorem

Let the binomial process M_i at time i be a $\mathbb{Q}$-martingale. If another process N_i is also a $\mathbb{Q}$-martingale, there exists a predictable process ϕ_i such that N_i can be expressed as:

$$N_i \;=\; N_0 + \sum_{k=1}^{i}\phi_k \Delta M_k$$

(A predictable process ϕ_i exists that allows N_i to be represented in this way.)

Martingale Representation Theorem

Let the process M_t be a $\mathbb{Q}$-martingale, and assume its volatility σ_t is never zero. If another process N_t is also a $\mathbb{Q}$-martingale, there exists a predictable process ϕ_t that always satisfies:

$$\int_0^T \phi_t^2\sigma_t^2 dt < \infty$$

and N_t can be expressed as:

$$N_t \;=\; N_0 + \int_0^t \phi_s dM_s$$

This is the same theorem as the binomial process version, with the summation replaced by an integral.

Let's consider a point made in the textbook's explanation: "If a measure $\mathbb{Q}$ exists such that M_t is a $\mathbb{Q}$-martingale, then any other $\mathbb{Q}$-martingale can be expressed using M_t as shown above. The process ϕ_t is simply the ratio of their respective volatilities."

Expressed in differential form, the martingale representation theorem is $dN_t = \phi_t dM_t$.

Since M_t is a $\mathbb{Q}$-martingale, we can write $dM_t = \sigma_t^{(M)} d\tilde{W}_t$ using a $\mathbb{Q}$-Brownian motion $\tilde{W}_t$ (assuming M's volatility $\sigma_t^{(M)} > 0$). Substituting this in, we get $dN_t = \phi_t\sigma_t^{(M)} d\tilde{W}_t$.

On the other hand, N_t is also a $\mathbb{Q}$-martingale. So, if we denote N's volatility as $\sigma_t^{(N)}$, we can express dN_t as $dN_t = \sigma_t^{(N)} d\tilde{W}_t$.

Thus, expressing dN_t using the volatilities of M and N, $\sigma_t^{(M)}$ and $\sigma_t^{(N)}$, we have: $\phi_t \sigma_t^{(M)} d\tilde{W}_t = \sigma_t^{(N)} d\tilde{W}_t$ This can be rearranged to:

$$\phi_t \;=\; \frac{\sigma_t^{(N)}}{\sigma_t^{(M)}}$$

This confirms the textbook's statement: "the process ϕ_t is simply the ratio of their respective volatilities."

5 No-Drift Condition

As we saw briefly in Exercise 3.10, the conditions for a process to be a martingale if it has no drift are stated here in more detail.

A stochastic process X_t that satisfies $dX_t = \sigma_t dW_t + \mu_t dt$ is a martingale if and only if $\mu_t = 0$, provided it meets the condition:

$$\mathbf{E}\left[\left(\int_0^t \sigma_s^2 ds\right)^{\frac{1}{2}}\right] \;<\; \infty$$

Processes that do not satisfy the above condition are called local martingales.

6 Exponential Martingales

For a geometric Brownian motion with no drift, $dX_t = \sigma_t X_t dW_t$, the condition for X_t to be a martingale, when applied directly from the above, would be:

$$\mathbf{E}\left[\left(\int_0^t \sigma_s^2 X_s^2 ds\right)^{\frac{1}{2}}\right] \;<\; \infty$$

However, the textbook states that the condition can actually be simplified to the more concise:

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^t \sigma_s^2 ds\right)\right] \;<\; \infty$$

is sufficient.

In this case, the solution can be written explicitly as:

$$X_t \;=\; X_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right)$$

Let's verify this.

Let $X_t = X_0 e^{Y_t}$. We have $Y_t = \displaystyle\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds$, so its differential is:

$$dY_t \;=\; \sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt$$

The square of dY_t is:

$$\begin{aligned} dY_t^2 &= \left(\sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt\right)^2 \\ &= \sigma_t^2 dt \end{aligned}$$

Using Itô's formula with the function $f(y) = X_0 e^y$, where $f'(y) = f''(y) = X_0 e^y$, we find:

$$\begin{aligned}
dX_t &= df(y) \\
&= f'(y)dY_t \;+\; \frac{1}{2}f''(y)dY_t^2 \\
&= X_0 e^{Y_t}\left(\sigma_t dW_t - \frac{1}{2}\sigma_t^2 dt\right) \;+\; \frac{1}{2}X_0 e^{Y_t}\sigma_t^2 dt \\
&= X_0 e^{Y_t}\sigma_t dW_t \\
&= \sigma_t X_t dW_t
\end{aligned}$$

This confirms that the solution to $dX_t = \sigma_t X_t dW_t$ is indeed:

$$X_t \;=\; X_0 \exp\left(\int_0^t \sigma_s dW_s - \frac{1}{2}\int_0^t \sigma_s^2 ds\right)$$

7 Exercise 3.11

7.1 Problem

If σ is a function bounded in both time and path, show that $dX_t = \sigma_t X_t dW_t$ is a $\mathbb{P}$-martingale.

7.2 Solution

First, since $dX_t = \sigma_t X_t dW_t$ has no drift term, it will be a martingale if it satisfies the condition for an exponential martingale:

$$\mathbf{E}\left[\exp\left(\frac{1}{2}\int_0^t \sigma_s^2 ds\right)\right] \;<\; \infty$$

Let's express σ_t using time t and path ω as $\sigma_t = \sigma(t,\omega)$. Since it's a function bounded in both time and path, there exists a constant K such that for any (t,ω), we have $|\sigma(t,\omega)| < K$. Squaring this, we get $\sigma^2(t,\omega) < K^2$. Therefore,

$$\mathbf{E}_{\mathbb{P}}\left[\exp\left(\frac{1}{2}\int_0^t \sigma_s^2 ds\right) \;\middle|\; \omega\right] < \exp\left(\frac{1}{2}\int_0^t K^2 ds\right) \;=\; \text{const.}$$

Since this is bounded by a finite value, the condition given in the problem is sufficient to prove that X_t is a martingale.

This concludes Section 3.5.

References

[1] Financial Calculus - An Introduction to Derivative Pricing - Martin Baxter, Andrew Rennie