Adjunction

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Contents

1 Definition

1.1 Definition of Adjunction

Given categories C and D, we have functors $F : C \to D$ and $G : D \to C$. We say that 'F and G are adjoint', denoted $F \dashv G : C \to D$,

Adjunction

if for every object c in C and every object d in D, there exists a natural bijection

$$\phi_{cd} : \mathrm{Hom}_D(Fc, d) \to \mathrm{Hom}_C(c, Gd)$$

1.2 Meaning of 'Natural'

The meaning of 'natural' here is that it is:

1. natural in c when d is fixed, and
2. natural in d when c is fixed.

In other words, there are two conditions.

First, 'natural in c' means:

Natural in c

$$\theta_c : \mathrm{Hom}_D(F-, d) \to \mathrm{Hom}_C(-, Gd)$$

that there exists a natural transformation θ_c.

Since this is a natural transformation, both its domain and codomain must be functors.

Recall that Hom is a functor.

In general, for an object $a \in C$ in a category C, the functor $\mathrm{Hom}_C(-, a)$ is a functor from C^{op} to **Set**.

Similarly, $\mathrm{Hom}_C(-, Gd)$, for the object $Gd \in C$, is a functor $C^{op} \to \mathbf{Set}$.

Let's consider the meaning of the other functor, $\mathrm{Hom}_D(F-, d)$.

First, without F, $\mathrm{Hom}_D(-, d)$ is a functor $D^{op} \to \mathbf{Set}$.

Furthermore, we have the functor $F : C \to D$,

$$C^{op} \xrightarrow{F} D^{op} \xrightarrow{\mathrm{Hom}_D(-,d)} \mathbf{Set}$$

The composition of these two functors is $\mathrm{Hom}_D(F-, d)$.

The existence of a natural transformation θ_c between these functors is the meaning of 'natural in c'.

'Natural in d' is similar,

$$\theta_d : \mathrm{Hom}_D(Fc, -) \to \mathrm{Hom}_C(c, G-)$$

meaning that there exists a natural transformation θ_d.

1.3 Left and Right

When $F \dashv G : C \to D$, that is,

for every object c in C and every object d in D, there exists a natural bijection

$$\phi_{cd} : \mathrm{Hom}_D(Fc, d) \to \mathrm{Hom}_C(c, Gd)$$

F is called the **left adjoint functor** of G, and G is called the **right adjoint functor** of F.

The functor applied to the object on the left (in $\mathrm{Hom}_D(Fc, d)$) is the left adjoint, and the functor applied to the object on the right (in $\mathrm{Hom}_C(c, Gd)$) is the right adjoint. Apparently, the left/right convention is sometimes reversed depending on the literature.

This is tough for those with left-right confusion, but when writing $F \dashv G : C \to D$, we say 'F is left adjoint' and, meaning the same thing, 'F has a right adjoint'.

2 Examples

2.1 Adjunction between the Category of Vector Spaces and the Category of Sets

Let V be an object from the category of vector spaces, **Vect**.

V is equipped with axioms, such as scalar multiplication and addition preserving linearity. We can take a functor U that 'forgets' these conditions (a **forgetful functor**).

$$\begin{array}{rccc} U: & \mathbf{Vect} & \to & \mathbf{Set} \\ & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ & V & \mapsto & U(V) \end{array}$$

A morphism in **Vect** is a linear map $f : V \to W$. Forgetting that it preserves the linear structure and regarding it as merely a map, we get $U(f) : U(V) \to U(W)$, which is a morphism in **Set**.

Consider an adjoint for this U. That is, we seek a functor F such that for any set X, $F(X)$ becomes a vector space.

This is obtained by taking the vector space with X as its basis.

In this case, F is a functor $F : \mathbf{Set} \to \mathbf{Vect}$.

Moreover, F is adjoint to U. This can be shown by explicitly constructing the natural bijection

$$\phi_{cd} : \mathrm{Hom}_{\mathbf{Vect}}(Fc, d) \to \mathrm{Hom}_{\mathbf{Set}}(c, Ud)$$

2.2 Adjunction between the Category of Abelian Groups and the Category of Sets

There is an adjunction between the forgetful functor U from the category of abelian groups **Ab** to the category of sets **Set** (forgetting the structure), and the functor F, where $F(X)$ forms the **free abelian group** generated by the set X.

$$\begin{tikzcd} \mathbf{Ab} \arrow[rr, bend right, "U"'] \arrow[rr, phantom, "\perp"] & & \mathbf{Set} \arrow[ll, bend right, "F"'] \end{tikzcd}$$

2.3 Adjunction between the Category of Topological Spaces and the Category of Sets

Between the forgetful functor U from the category of topological spaces **Top** to the category of sets **Set** (forgetting the structure), and the functor F, where $F(X)$ equips the set X with the **discrete topology**, there is an adjunction.

The functor G that equips X with the **indiscrete topology** is also an adjoint of U; it is the right adjoint of U.

$$F \dashv U \dashv G$$

$$\begin{tikzcd} \mathbf{Top} \arrow[r, "U"'] & \mathbf{Set} \arrow[l, bend right=50, "F"'] \arrow[l, bend left=50, "G"] \end{tikzcd}$$