# Adjunctions according to Steve Awodey

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#### Abstract

This paper summarizes adjunctions following Chapter 9 of the 2nd Edition of 'Category Theory' by Steve Awodey[1].

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# 1 Preliminary Definitions

#### 1.1 Constructing Words with the Kleene Closure

As an example of the method for 'constructing a free monoid from an arbitrary set', let us consider a set of alphabetic characters  $A = \{a, b, c, ..., y, z\}$ .

A finite string of these characters (regardless of whether the string is meaningful) is called a 'word' over A. For example,

 $word, this word, categories are fun, as dfas daf, \dots$ 

The empty string will be represented by a hyphen '-'.

The Kleene Closure is then the operator  $(\cdot)^{\text{Kleene}}$  defined by

 $A^{\rm Kleene} = \{-, word, this word, categories are fun, as dfas daf, \ldots\}$ 

We now introduce a string concatenation operation ++ for the elements, or words, in the set  $A^{\text{Kleene}}$ .

This defines  $++: A^{\text{Kleene}} \times A^{\text{Kleene}} \to A^{\text{Kleene}}$  such that

$$word ++- = word$$
 $this ++ word = thisword$ 
 $categories ++ are ++ fun = categories are fun$ 

The empty string - serves as the identity element.

Under this operation,  $(A^{\text{Kleene}}, ++)$  becomes a monoid.

Furthermore,  $A^{\text{Kleene}}$  satisfies the following conditions, making it a free monoid:

- 1. no junk (All words can be expressed as a product of elements from A.)
- 2. no noise (For every word, the method of expressing it as a combination of elements from A is unique (aside from the monoid axioms). For example, if  $a \neq b$ , then  $ab \neq ba$ .)

### 1.2 Universal Property of Free Monoids

The two conditions for a monoid to be 'free', no junk and no noise, can be expressed very neatly using a categorical definition.

First, any monoids M, N have underlying sets U(M), U(N).

And any homomorphism  $f: N \to M$  has an underlying map  $U(f): U(N) \to U(M)$ .

This U is a functor, known as a 'forgetful functor'.

The free monoid M(A) constructed from a set A has the following universal property.

Universal Property of the Free Monoid M(A) —

There is a map  $i: A \to U(M(A))$  such that for any monoid N and any map  $f: A \to U(N)$ , there exists a **unique** monoid homomorphism  $g: M(A) \to N$  satisfying  $U(g) \circ i = f$ .

This can be summarized neatly in categorical terms.

· Diagram of the Universal Property of M(A) —

Diagram in **Mon**:

$$M(A) \xrightarrow{\exists ! g} N$$

Diagram in **Set**:

$$U(M(A)) \xrightarrow{U(g)} U(N)$$

### 1.3 A Simple Example of a Free-Forgetful Adjunction

Any monoid M has an underlying set U(M).

Also, as constructed in the previous section, every set X has a free monoid F(X). Let us consider the map  $\phi$  that sends g to  $U(g) \circ i$ .

From the universal property of the free monoid, this map is an isomorphism.

$$\operatorname{Hom}_{\mathbf{Mon}}(F(X), M) \cong \operatorname{Hom}_{\mathbf{Set}}(X, U(M))$$

A mnemonic for this is: 'Free is left adjoint to Forgetful'.

### 1.4 A Simple Definition of Adjunction

By generalizing this flow to categories C and D, we can define an adjunction.

· Adjunction between Categories  $\bf C$  and  $\bf D$  —

An adjunction between categories C and D consists of functors F, G

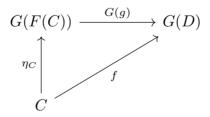
$$F: \mathbf{C} \rightleftharpoons \mathbf{D}: G$$

and a natural transformation  $\eta: 1_{\mathbf{C}} \to G \circ F$ .

They have the following property.

For any  $C \in \mathbf{C}$ ,  $D \in \mathbf{D}$  and  $f : C \to G(D)$ , there exists a **unique**  $g : F(C) \to D$  such that  $f = G(g) \circ \eta_C$  holds, as shown below.

$$F(C) \xrightarrow{!g} D$$



In this case, F is called the **left adjoint** to G, and G is the **right adjoint** to F, written as  $F \dashv G$ .  $\eta$  is called the **unit** of the adjunction.

# 2 Example: The Diagonal Functor

#### 2.1 The Right Adjoint to the Diagonal Functor is the Product Functor

As an example, let us consider the diagonal functor  $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ .

On objects and morphisms, it is defined as follows:

$$\begin{array}{ccc} \Delta(C) & = & (C,C) & \text{for } C \in \mathrm{Obj}(\mathbf{C}) \\ \Delta(f:C \to C') & = & (f,f):(C,C) \to (C',C') & \text{for } f \in \mathrm{Mor}(\mathbf{C}) \end{array}$$

We seek the right adjoint R to this diagonal functor.

Since it must go in the opposite direction of  $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$ , R will be a functor  $R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$ . Let us denote its action on objects as

$$R: \mathbf{C} \times \mathbf{C} \ni (X, Y) \mapsto R(X, Y) \in \mathbf{C}$$

Recall the construction of an adjunction.

Recalling the correspondence from the free-forgetful adjunction

$$\operatorname{Hom}_{\mathbf{Mon}}(F(X), M) \cong \operatorname{Hom}_{\mathbf{Set}}(X, U(M))$$

and substituting the respective components, we get:

$$\operatorname{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C),(X,Y)) \cong \operatorname{Hom}_{\mathbf{C}}(C,R(X,Y))$$

The left-hand side (LHS) can be expanded as follows:

$$\operatorname{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C),(X,Y)) \cong \operatorname{Hom}_{\mathbf{C}\times\mathbf{C}}((C,C),(X,Y)) 
\cong \operatorname{Hom}_{\mathbf{C}}(C,X) \times \operatorname{Hom}_{\mathbf{C}}(C,Y) 
\cong \operatorname{Hom}_{\mathbf{C}}(C,X \times Y)$$

The first isomorphism  $\cong$  uses the definition of  $\Delta(C)$ .

The second  $\cong$  uses the definition of morphisms in the product category  $\mathbf{C} \times \mathbf{C}$ .

The third  $\cong$  uses the universal property of the product  $X \times Y$  in the category  $\mathbb{C}$ , which is  $\operatorname{Hom}_{\mathbf{C}}(C, X \times Y) \cong \operatorname{Hom}_{\mathbf{C}}(C, X) \times \operatorname{Hom}_{\mathbf{C}}(C, Y)$ .

By comparing the LHS and RHS when substituted back into the adjunction definition, we have:

$$\operatorname{Hom}_{\mathbf{C}}(C, R(X, Y)) \cong \operatorname{Hom}_{\mathbf{C}}(C, X \times Y)$$

Here, we wish to apply the Yoneda Corollary:

$$\operatorname{Hom}_{\mathbf{C}}(C, F) \cong \operatorname{Hom}_{\mathbf{C}}(C, G) \Rightarrow F \cong G$$

To use this corollary, the isomorphism must be natural in C. In our case, by the definition of adjunction, there is a natural isomorphism between

$$\operatorname{Hom}(-, R(X, Y)) \cong \operatorname{Hom}(-, X \times Y)$$

From the above, we can conclude that

$$R(X,Y) \cong X \times Y$$

It has been shown that the right adjoint to the diagonal functor  $\Delta$  is the product functor  $\times$ , i.e.,  $\Delta \dashv \times$ .

### 2.2 The Unit of the Adjunction

Let us examine the unit of this adjunction. By the definition of the adjunction  $\Delta \dashv \times$  (i.e.,  $L = \Delta, R = \times$ ), the unit  $\eta$  is a natural transformation  $\eta: 1_{\mathbf{C}} \to R \circ L = \times \circ \Delta$ .

Its component  $\eta_C$ , for each object  $C \in \mathbf{C}$ , is a morphism to  $(\times \circ \Delta)(C) = \times (\Delta(C)) = \times (C, C) = C \times C$ . That is, it has the form  $\eta_C : C \to C \times C$ .

This  $\eta_C$  is defined as the morphism on the RHS that corresponds to the identity morphism  $1_{\Delta(C)}$ :  $\Delta(C) \to \Delta(C)$  on the LHS, by specifically choosing  $(X,Y) = \Delta(C) = (C,C)$  in the adjoint isomorphism

$$\operatorname{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C),(X,Y)) \cong \operatorname{Hom}_{\mathbf{C}}(C,\times(X,Y))$$

Here, by the definition of the product category,  $1_{\Delta(C)}$  is the pair of morphisms  $(1_C, 1_C)$ .

$$1_{\Delta(C)} = (1_C, 1_C) : (C, C) \to (C, C)$$

On the other hand, by the universal property of the product  $C \times C$ 

$$\operatorname{Hom}_{\mathbf{C}}(C, C \times C) \cong \operatorname{Hom}_{\mathbf{C}}(C, C) \times \operatorname{Hom}_{\mathbf{C}}(C, C)$$

the morphism in  $\operatorname{Hom}_{\mathbf{C}}(C, C \times C)$  corresponding to the pair of morphisms  $(1_C, 1_C)$  is the **unique** morphism  $f: C \to C \times C$  that satisfies

$$p_1 \circ f = 1_C$$
 and  $p_2 \circ f = 1_C$ 

This is none other than the definition of the so-called **diagonal morphism**  $\delta_C$ . Therefore, the unit of the adjunction is the diagonal morphism  $\eta_C = \delta_C$ .

Let us consider the universal property of the unit  $\eta$ .

In this context, the universal property of  $\eta$  is expressed as follows.

Any morphism  $f: C \to X \times Y \ (\in \mathbf{C})$  can be factored through  $\eta_C$  and the **unique** morphism  $g: \Delta(C) \to (X,Y) \ (\in \mathbf{C} \times \mathbf{C})$  that corresponds to f via the adjunction.

If we write the pair of morphisms  $g_1: C \to X$  and  $g_2: C \to Y$  as  $g = (g_1, g_2)$ , then the action of the functor  $R = \times$  on this morphism is

$$R(g) = g_1 \times g_2 : C \times C \to X \times Y$$

At this time, from the definition of the adjunction

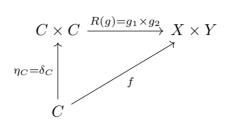
$$f = R(q) \circ \eta_C$$

it follows that

$$f = (g_1 \times g_2) \circ \delta_C$$

This relationship can be expressed by the following commutative diagram.

$$(C,C) \xrightarrow{\exists ! (g_1,g_2)} (X,Y)$$



Here,  $f: C \to X \times Y$  and  $g = (g_1, g_2): (C, C) \to (X, Y)$  correspond one-to-one via the adjunction.

# References

[1] Category Theory 2nd Edition - Steve Awodey