The Values of the Riemann Zeta Function at Negative Integers

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Abstract

A memo on the topic of the divergent infinite series $1 + 2 + 3 + \cdots$ which is known to be represented as the finite value $-1/12$ via analytic continuation. Still in progress.

1 Problem

For a complex number s, the Riemann zeta function (hereafter, simply the zeta function) defined for $\mathrm{Re}(s) > 1$ is given by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{1}$$

It is clear that the function converges for $\mathrm{Re}(s) > 1$. At $s = 1$, it becomes the harmonic series and diverges. However, it can be analytically continued to $\mathrm{Re}(s) < 0$ by using the following functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s) \tag{2}$$

1.1 At $s = -1$

What about $\zeta(s)$ when s takes a negative value? For example, when $s = -1$,

$$\begin{aligned} \zeta(-1) &= 1 + \frac{1}{2^{-1}} + \frac{1}{3^{-1}} + \cdots + \frac{1}{n^{-1}} + \cdots \\ &= 1 + 2 + 3 + \cdots + n + \cdots \end{aligned} \tag{3}$$

This can be expressed as an infinite sum that, of course, seems to diverge. On the other hand, using the functional equation (2) used for analytic continuation,

$$\begin{aligned} \zeta(-1) &= 2^{-1}\pi^{-2} \sin\left(\frac{-\pi}{2}\right) \Gamma(2)\,\zeta(2) \\ &= \frac{1}{2\pi^2} \times (-1) \times (1!) \times \frac{\pi^2}{6} \\ &= -\frac{1}{12} \end{aligned} \tag{4}$$

The value remains finite. It seems as if these equations suggest a contradiction.

2 Bernoulli number

As a preparation, let's briefly summarize Bernoulli numbers B_n. B_n are defined as the coefficients of the series expansion of the following function:

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} \tag{5}$$

The general term is also known and is given by

$$B_n = \sum_{k=0}^{n} (-1)^k k^n \sum_{m=k}^{n} \frac{{}_m\mathrm{C}_r}{m+1} \tag{6}$$

Here, the binomial coefficient is denoted as

$${}_n\mathrm{C}_k = \frac{n!}{(n-k)!k!} \tag{7}$$

However, since it is a double series, using this formula directly can make the calculation heavy. Therefore, to actually find B_n, the following recurrence relation is used:

$$\begin{cases} B_0 = 1 \\ B_0 = -\dfrac{1}{n+1} \displaystyle\sum_{k=0}^{n-1} {}_{n+1}\mathrm{C}_k B_k \end{cases} \tag{8}$$

All B_n are rational numbers. The first few terms are: $B_0 = 1$, $B_1 = -\dfrac{1}{2}$, and $B_2 = \dfrac{1}{6}$. However, as n gets larger, the numerator and denominator of B_n become large, making them unsuitable for floating-point arithmetic. For example, $B_{24} = -\dfrac{236364091}{2730}$, $B_{28} = -\dfrac{1869628555}{58}$.

It is also known that $B_n = 0$ for odd $n \geq 3$ (all odd n except for $n = 1$). This can be proven as follows:

$$\begin{aligned} \frac{x}{e^x - 1} - B_1 \frac{x^1}{1!} &= \frac{x}{e^x - 1} + \frac{1}{2}x \\ &= \frac{2x + x(e^x - 1)}{2(e^x - 1)} \\ &= \frac{x}{2} \frac{(e^x + 1) \times e^{-x/2}}{(e^x - 1) \times e^{-x/2}} \\ &= \frac{x}{2} \coth \frac{x}{2} \end{aligned} \tag{9}$$

Since the coth function is an odd function, the expression above becomes an even function. Therefore, when we expand the above expression as a series, only terms of even powers remain. That is, $B_n = 0$ for odd $n \geq 3$ (all odd n except for $n = 1$).

2.1 Series expansion of coth

Solving in reverse,

$$\coth x = \frac{2}{2x} \left(-B_1 \frac{(2x)^1}{1!} + \frac{2x}{e^{2x} - 1} \right)$$

$$
\begin{aligned}
&= \frac{1}{x}\left(-B_1(2x) + \sum_{n=0}^{\infty} B_n \frac{(2x)^n}{n!}\right) \\
&= \frac{1}{x}\left(-B_1(2x) + B_0 + (2x)B_1 + \frac{1}{2!}(2x)^2 B_2 + \frac{1}{3!}(2x)^3 B_3 + \cdots + \frac{1}{n!}(2x)^n B_n + \cdots\right) \\
&= \frac{1}{x}\left(B_0 + \frac{1}{2!}(2x)^2 B_2 + \frac{1}{3!}(2x)^3 B_3 + \cdots + \frac{1}{n!}(2x)^n B_n + \cdots\right) \qquad (10)
\end{aligned}
$$

$B_0 = 1$. Furthermore, since $B_n = 0$ for odd $n \geq 3$,

$$
\begin{aligned}
\text{eqn.}(10) &= \frac{1}{x}\left(1 + \frac{1}{2!}(2x)^2 B_2 + \frac{1}{4!}(2x)^4 B_4 + \cdots + \frac{1}{(2n)!}(2x)^{2n} B_{2n} + \cdots\right) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{2^{2n} B_{2n}}{(2n)!} x^{2n-1} \qquad (11)
\end{aligned}
$$

2.2 Series expansion of cot

$$
\coth x = \frac{\cosh x}{\sinh x} = \frac{\cos(ix)}{-i\sin(ix)} = i\cot(ix) \qquad (12)
$$

Therefore, substituting $ix \to y$,

$$
\begin{aligned}
\cot y &= \frac{1}{i}\coth\frac{y}{i} = -i\coth(-iy) \\
&= i\coth(iy) \\
&= i\left\{\frac{1}{iy} + \left(\frac{1}{2!}2^2(iy)^1 B_2 + \frac{1}{4!}2^4(iy)^3 B_4 + \cdots + \frac{1}{(2n)!}2^{2n}(iy)^{2n-1} B_{2n} + \cdots\right)\right\} \\
&= \frac{1}{y} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} y^{2n-1} \qquad (13)
\end{aligned}
$$

2.3 Series expansion of tan

Taking the reciprocal of both sides of the double-angle formula

$$
\tan(2x) = \frac{2\tan x}{1 - \tan^2 x} \qquad (14)
$$

we get

$$
\begin{aligned}
\cot(2x) &= \frac{1 - \tan^2 x}{2\tan x} \\
&= \frac{1}{2\tan x} - \frac{\tan x}{2} \qquad (15)
\end{aligned}
$$

That is,

$$
\begin{aligned}
\tan x &= \cot x - 2\cot(2x) \\
&= \frac{1}{x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} x^{2n-1} - 2\left(\frac{1}{2x} + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}(2x)^{2n-1}\right) \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!}\left\{x^{2n-1} - 2(2x)^{2n-1}\right\} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n}(1 - 2^{2n}) B_{2n}}{(2n)!} x^{2n-1} \qquad (16)
\end{aligned}
$$

Thus, it is clear that B_n are basically involved in the series expansion coefficients of trigonometric functions. As for the sin and cos functions,

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \tag{17}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \tag{18}$$

The exponential functions are only in the numerator, not in the denominator, so B_n does not appear.

3 Some special cases

The value of $\zeta(s)$ for a general s is calculated numerically. However, for special values of s, the analytic values are known to be as follows.

For any positive even integer $2m$,

$$\zeta(2m) = \frac{(-1)^{2m} B_{2m} (2\pi)^{2m}}{(2m)!2} \tag{19}$$

For any negative integer $-m$,

$$\zeta(-m) = -\frac{B_{m+1}}{m+1} \tag{20}$$

There is no known elementary expression for the value of $\zeta(s)$ when s is an odd number. However, a series representation was given by Ramanujan. For any odd number $2m-1$,

$$\zeta(2m-1) = -2^{2m}\pi^{2m-1} \sum_{k=0}^{m} (-1)^{k+1} \frac{B_{2k}}{(2k)!2} \frac{B_{2m-2k}}{(2m-2k)!} - 2\sum_{k=1}^{\infty} \frac{k^{-2m+1}}{e^{2\pi k} - 1} \tag{21}$$

4 Analytic continuation

In the Gamma function

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1} e^{-t} \tag{22}$$

by changing the variable to $t = nx$, we get

$$\Gamma(s) = \int_0^\infty ndx\, (nx)^{s-1} e^{-nx} \tag{23}$$

Furthermore, by dividing both sides by n^s and taking the sum over all natural numbers n,

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\Gamma(s)}{n^s} &= \sum_{n=1}^{\infty} \int_0^\infty dx\, x^{s-1} e^{-nx} \\ \Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} &= \int_0^\infty dx\, x^{s-1} \sum_{n=1}^{\infty} e^{-nx} \end{aligned} \tag{24}$$

Thus, the expression

$$\Gamma(s)\zeta(s) = \int_0^\infty dx\, \frac{x^{s-1}}{e^x - 1} \tag{25}$$

is obtained.

By the way, using the following identity for alternating series

$$1 - \frac{1}{2^s} + \frac{1}{3^s} - \cdots + \frac{(-1)^{n+1}}{n^s} + \cdots = \left\{1 + \frac{1}{2^s} + \frac{1}{3^s} + \cdots + \frac{1}{n^s} + \cdots\right\} - 2\left\{\frac{1}{2^s} + \frac{1}{4^s} + \cdots + \frac{1}{(2n)^s} + \cdots\right\}$$

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} &= \zeta(s) - 2\sum_{n=1}^{\infty} \frac{1}{(2n)^s} \\ &= \zeta(s) - \frac{2}{2^s}\zeta(s) \\ &= \zeta(s)\left(1 - \frac{1}{2^{s-1}}\right) \end{aligned} \tag{26}$$

and equation (23), if we calculate the right and left sides independently,

$$\begin{aligned} \sum_{n=1}^{\infty} \Gamma(s)\frac{(-1)^{n+1}}{n^s} &= \sum_{n=1}^{\infty} \left\{\int_0^{\infty} ndx\,(nx)^{s-1}e^{-nx}\right\}\frac{(-1)^{n+1}}{n^s} \\ \Gamma(s)\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} &= \int_0^{\infty} dx\, x^{s-1}\sum_{n=1}^{\infty}(-1)^{n+1}e^{-nx} \\ \Gamma(s)\zeta(s)\left(1 - \frac{1}{2^{s-1}}\right) &= \int_0^{\infty} dx\, x^{s-1}\frac{1}{e^x + 1} \end{aligned} \tag{27}$$

is newly obtained.

From equation (23),

$$\frac{1}{n^s} = \frac{1}{\Gamma(s)}\int_0^{\infty} dt\, t^{s-1}e^{-nt} \tag{28}$$

Using this fact,

$$\begin{aligned} \pi^{-s/2}\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{(\pi n)^{s/2}} \\ &= \sum_{n=1}^{\infty} \frac{1}{\Gamma(s/2)}\int_0^{\infty} dt\, t^{s/2\,-1}e^{-(\pi n^2)t} \\ &= \frac{1}{\Gamma(s/2)}\int_0^{\infty} dt\, t^{s/2\,-1}\sum_{n=1}^{\infty} e^{-\pi n^2 t} \end{aligned} \tag{29}$$

Here, as a special case of the Jacobi theta function, we define $\psi(x)$ as

$$\psi(x) = \sum_{n=1}^{\infty} e^{-\pi n^2 x} \tag{30}$$

From the Poisson summation formula, $\psi(x)$ satisfies the following.

$$2\psi(1/x) + 1 = \sqrt{x}\Big(2\psi(x) + 1\Big) \tag{31}$$

(The following is still being written...)