Construction Strategy of a Hedging Portfolio Using a Replicating Portfolio

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Abstract

This note outlines a strategy for constructing a hedging portfolio using a replicating portfolio.

1 Overview of Replicating Portfolio Construction

The replicating strategy to be described proceeds in the following three stages:

1. Find a martingale measure $\mathbb{Q}$ that makes the discounted stock price process Z_t a martingale.
2. Transform the contract into a process. $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1}X|\mathcal{F}_t)$
3. Find a predictable process ϕ_t such that $dE_t = \phi_t dZ_t$.

I will summarize these steps roughly.

If a proof of a theorem is needed, please refer to an appropriate textbook or reference accordingly.

This is a memo to check the overall picture, so let's set aside the details for now.

First, I will try these three steps in a simple case where the risk-free interest rate is zero ($r = 0$).

Next, I will construct a replicating portfolio in the case where the risk-free interest rate is not zero (by modifying the results derived for $r = 0$).

2 Replicating Portfolio Construction under the Black-Scholes Model

Pricing a call option according to the Black-Scholes model is a good example of creating a replicating portfolio.

The Black-Scholes model is a model for the price process of a continuously tradable underlying asset (here, assumed to be a stock price) S_t and a bond price process B_t, and is expressed using constants r, σ, μ as follows:

- $S_t = S_0 \exp(\sigma W_t + \mu t)$
- $B_t = \exp(rt)$

Here, S_0 is the stock price at the present time ($t = 0$) (with the bond as the numeraire) and is a positive value.

t represents time as a real number, and T denotes the contract's expiration time.

r is a real number representing the risk-free interest rate, and σ is a non-zero value representing the volatility of the stock (if it were zero, S_t would not be a stochastic process).

μ is the drift (= expected return of a person on the $\mathbb{P}$ measure) and is a real number.
W_t is a $\mathbb{P}$ - Brownian motion. That is,

1. W_t is continuous and satisfies $W_0 = 0$.
2. Under the measure $\mathbb{P}$, the distribution of W_t follows a normal distribution $N(0, t)$.
3. The increment $W_{s+t} - W_s$ follows a normal distribution $N(0, t)$ under the probability measure $\mathbb{P}$ and is independent of the history $\mathcal{F}_s$ up to time s.

In this Black-Scholes model, the market is assumed to consist of a risk-free cash bond with a fixed interest rate and a stock (which has risk) that follows a geometric Brownian motion.

A model for stock fluctuations that is too complex makes it impossible to find a replicating strategy, while a model that is too simple lacks the validity to represent reality.

The model that represents stock prices as a Brownian motion satisfies the minimum properties necessary to represent the real world.

2.1 $r = 0$: When the risk-free interest rate is zero

First, let's try to construct a replicating portfolio in the case where the risk-free interest rate $r = 0$. In this case, the construction of the replicating strategy is simplified into the following three steps:

1. Find a measure $\mathbb{Q}$ that makes S_t a martingale.
2. Transform the contract into a process. $E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$
3. Find a predictable process ϕ_t such that $dE_t = \phi_t dS_t$.

At each of these stages, the following theorems are used:

- Girsanov's Theorem and the Radon-Nikodym Theorem (Stage 1, Stage 2)
- The Martingale Representation Theorem (Stage 3)

2.2 Step 1: Find a measure $\mathbb{Q}$ that makes S_t a martingale.

We find the stochastic differential equation that S_t satisfies. This is the task of applying Girsanov's theorem to investigate whether S_t is a martingale under a given measure $\mathbb{Q}$.

Based on the assumption of the Black-Scholes model, the stock price is a geometric Brownian motion

$$S_t = \exp(\sigma W_t + \mu t)$$

Therefore, the logarithm of the stock price

$$Y_t = \log S_t = \sigma W_t + \mu t$$

is simply a Brownian motion with a drift, so the stochastic differential equation can be found by taking the stochastic differential (applying Itô's lemma).

According to Itô's lemma, the stochastic differential $df(X_t)$ of a function $f(X_t)$ of a stochastic process X_t satisfying $dX_t = \sigma_t dW_t + \mu_t dt$ is given as follows:

$$df(X_t) = \sigma_t f'(X_t) dW_t + \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t) \right) dt$$

In this case, $S_t = f(X_t) = \exp X_t = \exp(\sigma W_t + \mu t)$, so

$$\begin{aligned} df(X_t) &= \sigma_t f'(X_t) dW_t + \left(\mu_t f'(X_t) + \frac{1}{2}\sigma_t^2 f''(X_t)\right) dt \\ dS_t &= \sigma \exp X_t dW_t + \left(\mu \exp X_t + \frac{1}{2}\sigma^2 \exp X_t\right) dt \\ &= \sigma S_t dW_t + \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt \end{aligned}$$

Dividing both sides by S_t, we get

$$\frac{dS_t}{S_t} = \sigma dW_t + \left(\mu + \frac{1}{2}\sigma^2\right) dt$$

Now, we eliminate the drift of S_t by using the inverse of Girsanov's theorem.

That is, the inverse of Girsanov's theorem guarantees the existence of a measure $\mathbb{Q}$ such that when W_t is a $\mathbb{P}$-Brownian motion, $\tilde{W}_t = W_t + \displaystyle\int_0^t \gamma_s ds$ becomes a $\mathbb{Q}$-Brownian motion, and the measure $\mathbb{Q}$ is defined such that its Radon-Nikodym derivative with respect to the measure $\mathbb{P}$ is $\dfrac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma_t dW_t - \dfrac{1}{2}\int_0^T \gamma_t^2 dt\right)$.

However, the condition for applying Girsanov's theorem, $\mathbb{E}_{\mathbb{P}} \exp\left(\dfrac{1}{2}\int_0^T \gamma_t^2 dt\right) < \infty$, must be satisfied. (This is Novikov's condition. It is a technical requirement, so one often does not need to worry too much about it in actual hedging.)

The $\tilde{W}_t$ obtained by this transformation is a $\mathbb{Q}$-Brownian motion, so S_t satisfies the following stochastic differential equation:

$$\frac{dS_t}{S_t} = \sigma d\tilde{W}_t$$

Therefore, the $\mathbb{P}$-Brownian motion and the $\mathbb{Q}$-Brownian motion, $d\tilde{W}_t$ and dW_t, must have the following relationship:

$$\begin{aligned} \sigma S_t d\tilde{W}_t &= \sigma S_t dW_t + \left(\mu + \frac{1}{2}\sigma^2\right) S_t dt \\ d\tilde{W}_t &= dW_t + \frac{\mu + \frac{1}{2}\sigma^2}{\sigma} dt \end{aligned}$$

Thus, from the inverse of Girsanov's theorem, when we set

$$\gamma = \frac{\mu + \frac{1}{2}\sigma^2}{\sigma}$$

a measure $\mathbb{Q}$ exists such that

$$\tilde{W}_t = W_t + \int_0^t \gamma ds = W_t + \gamma t$$

becomes a $\mathbb{Q}$-Brownian motion, and is defined as follows:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \gamma dW_t - \frac{1}{2}\int_0^T \gamma^2 dt\right) = \exp\left(-\gamma W_T - \frac{1}{2}\gamma^2 T\right)$$

2.3 Step 2: Transform the contract into a process. $E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$

Next, we transform a certain contract X into a $\mathbb{Q}$-martingale E_t under the measure $\mathbb{Q}$ defined in the first stage.

The conditional expectation of the contract X under the filtration $\mathcal{F}_t$ and measure $\mathbb{Q}$ is

$$E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$$

For this to be a $\mathbb{Q}$-martingale, it must satisfy $\mathbb{E}_{\mathbb{Q}}(E_t|\mathcal{F}_s) = E_s$ for time $s < t$. In fact, for $s < t$,

$$\mathbb{E}_{\mathbb{Q}}(E_t|\mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}\Big(\mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)\ |\mathcal{F}_s\Big) = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_s) = E_s$$

is true (due to the tower property), so by defining

$$E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t)$$

the contract X becomes a $\mathbb{Q}$-martingale under the measure $\mathbb{Q}$.

2.4 Step 3: Find a predictable process ϕ_t such that $dE_t = \phi_t dS_t$.

According to the Martingale Representation Theorem, there exists a predictable process ϕ_t that satisfies

$$dE_t = \phi_t dS_t$$

In integral form, this is

$$E_t = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_t) = \mathbb{E}_{\mathbb{Q}}X + \int_0^t \phi_s dS_s$$

Now, we are ready to find the specific holdings of stock ϕ_t and bond ψ_t required for the construction of the replicating portfolio.

In conclusion, the following strategies 1 and 2 apply:

- Strategy 1. Hold ϕ_t units of stock at time t.
- Strategy 2. Hold $\psi_t = E_t - \phi_t S_t$ units of bond at time t.

To understand this, let's check if a replicating portfolio that satisfies strategies 1 and 2 is self-financing. The value of the replicating portfolio at time t, V_t, is (noting that $B_t = 1$ since we are considering $r = 0$):

$$V_t = \phi_t S_t + \psi_t B_t = \phi_t S_t + (E_t - \phi_t S_t) \times 1 = E_t$$

It is trivially true that $dV_t = dE_t$. Since $dB_t = 0$, by using the Martingale Representation Theorem, we get

$$dV_t = \phi_t dS_t = \phi_t dS_t + 0 = \phi_t dS_t + \psi_t dB_t$$

Indeed, it is self-financing.

The value of the replicating portfolio at the end time is

$$V_T = E_T = \mathbb{E}_{\mathbb{Q}}(X|\mathcal{F}_T) = X$$

Therefore, this strategy V_t is the no-arbitrage price of X at all times t.

At the start time,

$$V_0 = E_0 = \mathbb{E}_{\mathbb{Q}}X$$

we can also see that the price of contract X is the expected value under the measure $\mathbb{Q}$ that makes the stock price process S_t a martingale.

2.5 Why must the prices be the same?

If the price of the replicating portfolio (V_t) and the theoretical price of the derivative (E_t) are not the same, an "**arbitrage opportunity**" exists in the market.

This is a magical trading opportunity to make a profit without taking any risk.

2.6 Case 1: Replicating portfolio is undervalued ($V_t < E_t$)

If the derivative's price is higher than the cost of reproducing that derivative (the replicating portfolio), market participants will act as follows:

1. **Sell (short) the expensive derivative.** This brings money into their hands.
2. Alternatively, **buy (long) the cheap replicating portfolio.** They use the money from the sale for this.

When this trade is set up, the replicating portfolio will have the same value as the derivative at expiration, so they offset each other's risk. However, the initial price difference remains as a guaranteed profit in their hands.

2.7 Case 2: Replicating portfolio is overvalued ($V_t > E_t$)

Conversely, if the cost of reproducing the derivative is higher than the derivative's price, the same logic applies:

1. **Buy (long) the cheap derivative.**
2. **Sell (short) the expensive replicating portfolio.**

In this case as well, the risks offset each other at expiration, so the profit gained initially is locked in.

2.8 The market's self-correcting function

Such risk-free profit opportunities are instantly found by arbitrageurs.

They will continue to trade until the opportunity disappears, causing the derivative's price (E_t) and the replicating portfolio's price (V_t) to quickly converge to be **equal**.

This mechanism is the reason why the replicating portfolio price is determined as a single no-arbitrage price.

3 Introducing the Risk-Free Interest Rate

I have so far explained that when the interest rate r is zero, any contract X satisfies the no-arbitrage condition and its no-arbitrage price is obtained by taking the expected value under a measure $\mathbb{Q}$ that makes S_t a martingale.

If the interest rate r is not zero, the method so far does not work well, and an improvement is needed.

For example, the strike price for which the value of a forward contract with strike K (a contract to exchange funds of $S_T - K$ at $t = T$) becomes zero is $K = S_0 e^{rT}$ (for any other K, an arbitrage would occur).

$$\mathbb{E}_{\mathbb{Q}}(S_T - K) = \mathbb{E}_{\mathbb{Q}}(S_T - S_0 e^{rT}) = S_0 - S_0 e^{rT} \neq 0$$

However, this problem can be easily solved.

It would be sufficient to have a new measure $\tilde{\mathbb{Q}}$ different from the measure $\mathbb{Q}$ considered so far, such that it satisfies $\mathbb{E}_{\tilde{\mathbb{Q}}} S_t = S_0 e^{rt}$.

This is a new measure $\tilde{\mathbb{Q}}$ (different from the measure we have been considering) that makes the discounted stock price process $Z_t = B_t^{-1} S_t$, where the discounted process is $B_t^{-1} = e^{-rt}$, a martingale.

In economic terms, this is equivalent to thinking about a world where the growth of money's value has stopped (a world where the numeraire is B_t).

In the following, I will attempt to construct a replicating portfolio in this world.

3.1 Step 1: Find a measure $\mathbb{Q}$ that makes the discounted stock price process Z_t a martingale.

Using Itô's lemma,

$$dB_t = \left(\frac{\partial B_t}{\partial t} + \frac{1}{2} \frac{\partial^2 B_t}{\partial x^2} \right) dt + \frac{\partial B_t}{\partial x} dW_t = r B_t dt$$
$$dS_t = \left(\frac{\partial S_t}{\partial t} + \frac{1}{2} \frac{\partial^2 S_t}{\partial x^2} \right) dt + \frac{\partial S_t}{\partial x} dW_t = \left(\mu + \frac{1}{2}\sigma^2 \right) S_t dt + \sigma S_t dW_t$$

so

$$\frac{d\left(\dfrac{S_t}{B_t}\right)}{\dfrac{S_t}{B_t}} = \frac{dS_t}{S_t} - \frac{dB_t}{B_t} - \frac{dS_t}{S_t}\frac{dB_t}{B_t} + \left(\frac{dB_t}{B_t}\right)^2 = \sigma dW_t + \left(\mu - r + \frac{1}{2}\sigma^2\right) dt$$

We now use Girsanov's theorem. Setting $\theta = \dfrac{\mu - r + \frac{1}{2}\sigma^2}{\sigma}$, we define a new measure $\mathbb{Q}$ with the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\int_0^T \theta dW_t - \frac{1}{2}\int_0^T \theta^2 dt \right) = \exp\left(-\theta W_T - \frac{1}{2}\theta^2 T \right)$$

Under this measure, Z_t becomes a martingale.

At this point, the stochastic differential equation to be solved is

$$\frac{dZ_t}{Z_t} = \sigma d\tilde{W}_t$$

where we have set the desired process as $d\tilde{W}_t = dW_t + \theta dt$.

3.2 Step 2: Transform the contract into a process. $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$

For $s < t$,

$$E_s = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_s) = e^{-rT} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_s)$$

and

$$\mathbb{E}_{\mathbb{Q}}(E_t | \mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}\left(e^{-rT} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_t) \middle| \mathcal{F}_s \right) = e^{-rT} \mathbb{E}_{\mathbb{Q}}(X | \mathcal{F}_s) = E_s$$

Thus, $E_t = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_t)$ becomes a martingale under the new measure $\mathbb{Q}$.

3.3 Step 3: Find a predictable process ϕ_t such that $dE_t = \phi_t dZ_t$.

We can proceed in the same way as when we were thinking in a world where the numeraire was not the bond.

The stock holding amount for replication, ϕ_t, and the bond holding amount, ψ_t, give the price of the replicating portfolio at time t as

$$V_t = \phi_t S_t + \psi_t B_t$$

Since the price of the contract and the price of the replicating portfolio are the same at time T,

$$X = V_T = \phi_T S_T + \psi_T B_T$$

On the other hand,

$$E_T = \mathbb{E}_{\mathbb{Q}}(B_T^{-1} X | \mathcal{F}_T) = B_T^{-1} X$$

Therefore,

$$V_t = B_t E_t$$

Similar to the $r = 0$ case, we consider a replicating portfolio that satisfies strategies 1 and 2:

- Strategy 1. Hold ϕ_t units of stock at time t.
- Strategy 2. Hold $\psi_t = E_t - \phi_t S_t$ units of bond at time t.

$$\begin{aligned}
dV_t &= dB_t E_t + B_t dE_t \\
&= dB_t B_t^{-1}(\phi_t S_t + \psi_t B_t) + B_t(\phi_t dZ_t + \psi_t) \\
&= \phi_t(Z_t dB_t + B_t dZ_t) + \psi_t dB_t \\
&= \phi_t dS_t + \psi_t dB_t
\end{aligned}$$

From the above, similar to the $r = 0$ case, under this strategy, (ϕ_t, ψ_t) becomes self-financing.

4 Concrete Example

Finally, as a practice problem to summarize what has been discussed, I will apply the framework described so far to a simple contract.

4.1 Pricing a European Call Option

The contract X for a European call option with maturity T and strike price K is

$$X = \mathrm{Max}(S_T - K, 0)$$

Therefore, the present value of the replicating portfolio under the no-arbitrage condition can be written as

$$V_0 = e^{-rT} \mathbb{E}_{\mathbb{Q}} \mathrm{Max}(S_T - K, 0)$$

Here, the measure $\mathbb{Q}$ is the martingale measure for the process $B_t^{-1} S_t$.

Representing the underlying asset S_t using the $\mathbb{Q}$-Brownian motion $\tilde{W}_t$, we have

$$\begin{aligned}
d(\log S_t) &= \sigma d\tilde{W}_t + \left(r - \frac{1}{2}\sigma^2\right) dt \\
S_t &= S_0 \exp\left[\sigma \tilde{W}_t + \left(r - \frac{1}{2}\sigma^2\right) t\right]
\end{aligned}$$

Therefore, if we write a normal distribution with mean m and variance v as $N(m, v)$, we can see that the marginal distribution of S_T is the product of S_0 and the exponential of $N\left(\left(r - \frac{1}{2}\sigma^2\right)T, \sigma^2 T\right)$.

If we let Z be a random variable that follows $N\left(-\frac{1}{2}\sigma^2 T, \sigma^2 T\right)$, then $S_T = S_0 \exp(Z + rT)$, so we can express the (usual, not measure-aware) expected value as

$$\begin{aligned}
V_0 &= e^{-rT}\mathbb{E}\mathrm{Max}\Big(S_0 \exp(Z + rT) - K, 0\Big) \\
&= \frac{1}{\sqrt{2\pi\sigma}}\frac{1}{\sqrt{T}}\int_{\log(K/S_0)-rT}^{\infty} \left(S_0 e^x - Ke^{-rT}\right) \exp\left[-\frac{1}{2\sigma^2 T}\left(x + \frac{1}{2}\sigma^2 T\right)\right] dx \\
&= S_0\Phi(d_+) - Ke^{-rT}\Phi(d_-)
\end{aligned}$$

and solve it analytically. Here, $\Phi(x)$ is the probability that $N(0, 1)$ is less than or equal to x, and $d_\pm$ are

$$\begin{aligned}
d_\pm &= \frac{\log\left(\frac{S_0}{K}\right) + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} \\
\Phi(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} \exp\left(-\frac{y^2}{2}\right) dy
\end{aligned}$$

The replicating portfolio after a time $t (< T)$ has elapsed from the present $(t = 0)$ has a value, from the no-arbitrage condition, of

$$V_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left(\mathrm{Max}(S_T - K, 0)\middle|\mathcal{F}_t\right)$$

which is the present price of the contract if the time to maturity is changed from T to $T - t$, so the no-arbitrage price can be obtained by simply replacing T with $T - t$ in the equation for V_0.

The holdings of the underlying asset ϕ_t and the numeraire ψ_t required for hedging are

$$\begin{aligned}
\phi_t &= \frac{\partial V_t}{\partial S_0} = \Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}\right) \\
B_t\psi_t &= -Ke^{r(T-t)}\Phi\left(\frac{\log\left(\frac{S_0}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}\right)
\end{aligned}$$

By trading in this way, the portfolio becomes self-financing and replication becomes possible.

The above is the pricing of a European option based on replicating portfolio theory. There are various other methods for pricing European options besides this approach, such as:

- Solving the Black-Scholes equation.
- Solving the partial differential equation converted from it by Feynman-Kac.
- Taking the continuous limit of the price process of a discrete binomial tree.

Having multiple alternative solutions that lead to the same phenomenon and conclusion will broaden your understanding. I hope this note serves as an aid to that end.

4.2 References

- Financial Calculus - An Introduction to Derivative Pricing - Martin Baxter, Andrew Rennie