Adjunctions according to Steve Awodey

Masaru Okada

October 19, 2025

Abstract

This paper summarizes adjunctions following Chapter 9 of the 2nd Edition of 'Category Theory' by Steve Awodey[1].

Contents

1 Preliminary Definitions

1.1 Constructing Words with the Kleene Closure

As an example of the method for 'constructing a free monoid from an arbitrary set', let us consider a set of alphabetic characters $A = \{a, b, c, ..., y, z\}$.

A finite string of these characters (regardless of whether the string is meaningful) is called a 'word' over A. For example,

$$word, thisword, categoriesarefun, asdfasdaf, ...$$

The empty string will be represented by a hyphen '-'.

The Kleene Closure is then the operator $(\,\cdot\,)^{\text{Kleene}}$ defined by

$$A^{\text{Kleene}} = \{-, word, thisword, categoriesarefun, asdfasdaf, ...\}$$

We now introduce a string concatenation operation $++$ for the elements, or words, in the set A^{Kleene}.

This defines $++ : A^{\text{Kleene}} \times A^{\text{Kleene}} \to A^{\text{Kleene}}$ such that

$$\begin{aligned} word\ ++ - &= word \\ this\ ++\ word &= thisword \\ categories\ ++\ are\ ++\ fun &= categoriesarefun \end{aligned}$$

The empty string $-$ serves as the identity element.

Under this operation, $(A^{\text{Kleene}}, ++)$ becomes a monoid.

Furthermore, A^{Kleene} satisfies the following conditions, making it a free monoid:

1. no junk (All words can be expressed as a product of elements from A.)
2. no noise (For every word, the method of expressing it as a combination of elements from A is unique (aside from the monoid axioms). For example, if $a \neq b$, then $ab \neq ba$.)

1.2 Universal Property of Free Monoids

The two conditions for a monoid to be 'free', no junk and no noise, can be expressed very neatly using a categorical definition.

First, any monoids M, N have underlying sets $U(M), U(N)$.

And any homomorphism $f : N \to M$ has an underlying map $U(f) : U(N) \to U(M)$.

This U is a functor, known as a 'forgetful functor'.

The free monoid $M(A)$ constructed from a set A has the following universal property.

Universal Property of the Free Monoid $M(A)$

There is a map $i : A \to U(M(A))$ such that for any monoid N and any map $f : A \to U(N)$, there exists a **unique** monoid homomorphism $g : M(A) \to N$ satisfying $U(g) \circ i = f$.

This can be summarized neatly in categorical terms.

Diagram of the Universal Property of $M(A)$

Diagram in **Mon**:

$$M(A) \xrightarrow{\exists !\, g} N$$

Diagram in **Set**:

$$\begin{tikzcd} U(M(A)) \arrow[r, "U(g)"] & U(N) \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

1.3 A Simple Example of a Free-Forgetful Adjunction

Any monoid M has an underlying set $U(M)$.

Also, as constructed in the previous section, every set X has a free monoid $F(X)$.
Let us consider the map ϕ that sends g to $U(g) \circ i$.

$$\begin{array}{cccc} \phi : & \mathrm{Hom}_{\mathbf{Mon}}(F(X), M) & \to & \mathrm{Hom}_{\mathbf{Set}}(X, U(M)) \\ & \cup & & \cup \\ & g & \mapsto & U(g) \circ i \end{array}$$

From the universal property of the free monoid, this map is an isomorphism.

$$\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(M))$$

A mnemonic for this is: 'Free is left adjoint to Forgetful'.

1.4 A Simple Definition of Adjunction

By generalizing this flow to categories $\mathbf{C}$ and $\mathbf{D}$, we can define an adjunction.

Adjunction between Categories $\mathbf{C}$ and $\mathbf{D}$

An adjunction between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors F, G

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : G$$

and a natural transformation $\eta : 1_{\mathbf{C}} \to G \circ F$.
They have the following property.
For any $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $f : C \to G(D)$, there exists a **unique** $g : F(C) \to D$ such that $f = G(g) \circ \eta_C$ holds, as shown below.

$$F(C) \xrightarrow{!g} D$$

$$\begin{tikzcd} G(F(C)) \arrow[r, "G(g)"] & G(D) \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

In this case, F is called the **left adjoint** to G, and G is the **right adjoint** to F, written as $F \dashv G$.
η is called the **unit** of the adjunction.

2 Example: The Diagonal Functor

2.1 The Right Adjoint to the Diagonal Functor is the Product Functor

As an example, let us consider the diagonal functor $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.
On objects and morphisms, it is defined as follows:

$$\begin{array}{rcll} \Delta(C) & = & (C, C) & \text{for } C \in \mathrm{Obj}(\mathbf{C}) \\ \Delta(f : C \to C') & = & (f, f) : (C, C) \to (C', C') & \text{for } f \in \mathrm{Mor}(\mathbf{C}) \end{array}$$

We seek the right adjoint R to this diagonal functor.

Since it must go in the opposite direction of $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$, R will be a functor $R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Let us denote its action on objects as

$$R : \mathbf{C} \times \mathbf{C} \ni (X, Y) \mapsto R(X, Y) \in \mathbf{C}$$

Recall the construction of an adjunction.

Recalling the correspondence from the free-forgetful adjunction

$$\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(M))$$

and substituting the respective components, we get:

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X, Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, R(X, Y))$$

The left-hand side (LHS) can be expanded as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X, Y)) &\cong \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((C, C), (X, Y)) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(C, X) \times \mathrm{Hom}_{\mathbf{C}}(C, Y) \\ &\cong \mathrm{Hom}_{\mathbf{C}}(C, X \times Y) \end{aligned}$$

The first isomorphism $\cong$ uses the definition of $\Delta(C)$.

The second $\cong$ uses the definition of morphisms in the product category $\mathbf{C} \times \mathbf{C}$.

The third $\cong$ uses the universal property of the product $X \times Y$ in the category $\mathbf{C}$, which is $\mathrm{Hom}_{\mathbf{C}}(C, X \times Y) \cong \mathrm{Hom}_{\mathbf{C}}(C, X) \times \mathrm{Hom}_{\mathbf{C}}(C, Y)$.

By comparing the LHS and RHS when substituted back into the adjunction definition, we have:

$$\mathrm{Hom}_{\mathbf{C}}(C, R(X, Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, X \times Y)$$

Here, we wish to apply the Yoneda Corollary:

$$\mathrm{Hom}_{\mathbf{C}}(C, F) \cong \mathrm{Hom}_{\mathbf{C}}(C, G) \ \Rightarrow F \cong G$$

To use this corollary, the isomorphism must be natural in C. In our case, by the definition of adjunction, there is a natural isomorphism between

$$\mathrm{Hom}(-, R(X, Y)) \cong \mathrm{Hom}(-, X \times Y)$$

From the above, we can conclude that

$$R(X, Y) \cong X \times Y$$

It has been shown that the right adjoint to the diagonal functor Δ is the product functor $\times$, i.e., $\Delta \dashv \times$.

2.2 The Unit of the Adjunction

Let us examine the unit of this adjunction. By the definition of the adjunction $\Delta \dashv \times$ (i.e., $L = \Delta, R = \times$), the unit η is a natural transformation $\eta : 1_{\mathbf{C}} \to R \circ L = \times \circ \Delta$.

Its component η_C, for each object $C \in \mathbf{C}$, is a morphism to $(\times \circ \Delta)(C) = \times(\Delta(C)) = \times(C, C) = C \times C$. That is, it has the form $\eta_C : C \to C \times C$.

This η_C is defined as the morphism on the RHS that corresponds to the identity morphism $1_{\Delta(C)} : \Delta(C) \to \Delta(C)$ on the LHS, by specifically choosing $(X, Y) = \Delta(C) = (C, C)$ in the adjoint isomorphism

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X, Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, \times(X, Y))$$

Here, by the definition of the product category, $1_{\Delta(C)}$ is the pair of morphisms $(1_C, 1_C)$.

$$1_{\Delta(C)} = (1_C, 1_C) : (C, C) \to (C, C)$$

On the other hand, by the universal property of the product $C \times C$

$$\mathrm{Hom}_{\mathbf{C}}(C, C \times C) \cong \mathrm{Hom}_{\mathbf{C}}(C, C) \times \mathrm{Hom}_{\mathbf{C}}(C, C)$$

the morphism in $\mathrm{Hom}_{\mathbf{C}}(C, C \times C)$ corresponding to the pair of morphisms $(1_C, 1_C)$ is the **unique** morphism $f : C \to C \times C$ that satisfies

$$p_1 \circ f = 1_C \quad \text{and} \quad p_2 \circ f = 1_C$$

This is none other than the definition of the so-called **diagonal morphism** δ_C. Therefore, the unit of the adjunction is the diagonal morphism $\eta_C = \delta_C$.

Let us consider the universal property of the unit η.

In this context, the universal property of η is expressed as follows.

Any morphism $f : C \to X \times Y \quad (\in \mathbf{C})$ can be factored through η_C and the **unique** morphism $g : \Delta(C) \to (X, Y) \quad (\in \mathbf{C} \times \mathbf{C})$ that corresponds to f via the adjunction.

If we write the pair of morphisms $g_1 : C \to X$ and $g_2 : C \to Y$ as $g = (g_1, g_2)$, then the action of the functor $R = \times$ on this morphism is

$$R(g) = g_1 \times g_2 : C \times C \to X \times Y$$

At this time, from the definition of the adjunction

$$f = R(g) \circ \eta_C$$

it follows that

$$f = (g_1 \times g_2) \circ \delta_C$$

This relationship can be expressed by the following commutative diagram.

$$\begin{tikzcd}
(C, C) \arrow[r, "\exists!\,(g_1, g_2)"] & (X, Y)
\end{tikzcd}$$

$$\begin{tikzcd}
C \times C \arrow[r, "R(g) = g_1 \times g_2"] & X \times Y \\
C \arrow[u, "\eta_C = \delta_C"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Here, $f : C \to X \times Y$ and $g = (g_1, g_2) : (C, C) \to (X, Y)$ correspond one-to-one via the adjunction.

References

[1] Category Theory 2nd Edition - Steve Awodey