Gaussian Integral using the Functional Renormalization Group

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Abstract

A note presenting an introductory computational example of the fRG: using the functional renormalization group (fRG) to solve a Gaussian integral, for which an exact solution is known. The purpose of this note is to demonstrate how the fRG sequentially integrates out fluctuations from high to low energies, using a Gaussian integral as an exactly solvable example.

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1 Flow of this note

1. Problem Setup: We want to calculate $Z = \int dx e^{-S(x)}$ (where $S(x) = \frac{1}{2}m^2x^2$).
2. Introduction of fRG: Introduce a regulator R_k and 'flow' the effective action Γ_k from a high energy scale Λ down to $k = 0$.
3. Wetterich Equation: This flow is described by the Wetterich Equation.
4. Calculation: Solve the equation to find the effective action at $k = 0$, $\Gamma_{k=0}$.
5. Conclusion: Calculating $Z = e^{-\Gamma_{k=0}(\phi=0)}$ from $\Gamma_{k=0}$ yields the correct answer for the Gaussian integral, $\frac{\sqrt{2\pi}}{m}$.

2 Wetterch Equation

$$\partial_k \Gamma_k = \frac{1}{2} \mathrm{STr} \left[(\Gamma_k^{(2)} + R_k)^{-1} (\partial_k R_k) \right]$$

Here, STr denotes the supertrace.

This is the flow equation that lies at the heart of the fRG.

Γ_k represents the effective average action at scale k. This is an action in which fluctuations from high energy scales above k have been integrated out, while fluctuations below k have not yet been included.

R_k is a function called the regulator, which acts as a 'cap' to suppress fluctuations with energies lower than k. Changing k from Λ down to 0 corresponds to the operation of gradually removing this cap and incorporating the fluctuations.

$\Gamma_k^{(2)} = \frac{\partial^2 \Gamma_k}{\partial \phi^2}$ is the quantity corresponding to the Hessian, the second derivative with respect to the field $\phi = \langle x \rangle$.

3 Specifying the Regulator R_k and the Flow Equation

3.1 Gaussian Integral

The following Gaussian integral can be obtained elementarily.

$$Z = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2} m^2 x^2} = \frac{\sqrt{2\pi}}{m}$$

The objective of this note is to reproduce this integral's result using the fRG framework.

The relationship between the partition function and the action is as follows:

$$Z = e^{-\Gamma_{k=0}(\phi=0)}$$

We will find the partition function by first deriving the functional form of $\Gamma_k(\phi)$ using the fRG framework, and then evaluating it at $\Gamma_{k=0}(\phi = 0)$.

3.2 Assumption for the Effective Action

Since this problem is a non-interacting theory, we assume that the part of Γ_k dependent on the field ϕ does not depend on k and retains the same form as the classical action $S(\phi) = \frac{1}{2} m^2 \phi^2$. Therefore, we set

$$\Gamma_k(\phi) = \frac{1}{2} m^2 \phi^2 + C_k$$

Here, C_k is a constant term independent of ϕ. All contributions from field fluctuations are accumulated here.

From this assumption, it follows that

$$\Gamma_k^{(2)} = \frac{\partial^2 \Gamma_k}{\partial \phi^2} = m^2 = \mathrm{const.}$$

3.3 The Regulator R_k

We choose the following for the regulator R_k:

$$\begin{aligned} R_k &= k^2 \\ \partial_k R_k &= 2k \end{aligned}$$

$$Z_k = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}m^2x^2 - \frac{1}{2}R_k x^2}$$

In the limit of large R_k, the system becomes classical.

Conversely, in the limit $R_k \to 0$, we recover the desired partition function $Z_{k=0}$.

4 Wetterich Equation

4.1 Substitution into the Wetterich Equation

Since this problem is 0-dimensional, the STr in the Wetterich Equation is unnecessary, and it becomes a simple scalar equation.

$$\partial_k \Gamma_k = \frac{1}{2}(\Gamma_k^{(2)} + R_k)^{-1}(\partial_k R_k) = \frac{1}{2}(m^2 + k^2)^{-1}(2k) = \frac{k}{m^2 + k^2}$$

This equation shows that only the ϕ-independent constant term C_k within Γ_k 'flows' with k.

4.2 Integration of the Flow

Now that $\partial_k \Gamma_k$ has been found, we can integrate it to find the action Γ_k.

$$\begin{aligned} \Gamma_k(\phi) &= \Gamma_\Lambda(\phi) - \int_k^\Lambda dk' \partial_{k'} \Gamma_{k'}(\phi) \\ &= \Gamma_\Lambda(\phi) - \frac{1}{2} \ln\left(\frac{m^2 + \Lambda^2}{m^2 + k^2}\right) \end{aligned}$$

Here, Λ is a very large value (the UV cutoff).

4.2.1 Initial Condition: $k = \Lambda$

At $k = \Lambda$, fluctuations are suppressed by the regulator $R_\Lambda = \Lambda^2$.

$\Gamma_\Lambda(\phi)$ is in the classical limit.

The partition function at this scale, Z_Λ, can be calculated as follows:

$$Z_\Lambda = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}m^2x^2 - \frac{1}{2}R_\Lambda x^2} = \int_{-\infty}^{\infty} dx e^{-\frac{1}{2}(m^2+\Lambda^2)x^2} = \sqrt{\frac{2\pi}{m^2 + \Lambda^2}}$$

Γ_Λ is determined from this Z_Λ and $S(\phi)$.

$$\Gamma_\Lambda(\phi) = S(\phi) - \ln Z_\Lambda = \frac{1}{2}m^2\phi^2 - \ln\left(\sqrt{\frac{2\pi}{m^2 + \Lambda^2}}\right)$$

$$\Gamma_\Lambda(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\ln\left(\frac{m^2 + \Lambda^2}{2\pi}\right)$$

4.2.2 Functional Form of the Action $\Gamma_k(\phi)$

We substitute this $\Gamma_\Lambda(\phi)$ into the integral result.

$$\Gamma_k(\phi) = \left[\frac{1}{2}m^2\phi^2 + \frac{1}{2}\ln\left(\frac{m^2+\Lambda^2}{2\pi}\right)\right] - \frac{1}{2}\ln\left(\frac{m^2+\Lambda^2}{m^2+k^2}\right)$$

The ln terms cancel out, leaving:

$$\Gamma_k(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\ln\left(\frac{m^2+k^2}{2\pi}\right)$$

4.2.3 Functional Form of the Desired Action at $k = 0$, $\Gamma_0(\phi)$

At $k = 0$, the regulator becomes $R_0 = 0$, and we obtain the true effective action $\Gamma_{k=0}$ of the original theory, which incorporates all fluctuations.

$$\Gamma_{k=0}(\phi) = \frac{1}{2}m^2\phi^2 + \frac{1}{2}\ln\left(\frac{m^2}{2\pi}\right)$$

4.2.4 Value of the Gaussian Integral

The desired Z is given by $Z = e^{-\Gamma_{k=0}(\phi=0)}$.

$$\Gamma_{k=0}(0) = \frac{1}{2}\ln\left(\frac{m^2}{2\pi}\right)$$

$$Z = e^{-\Gamma_{k=0}(0)} = \exp\left[-\frac{1}{2}\ln\left(\frac{m^2}{2\pi}\right)\right] = \exp\left[\ln\left(\left(\frac{m^2}{2\pi}\right)^{-1/2}\right)\right] = \left(\frac{m^2}{2\pi}\right)^{-1/2} = \sqrt{\frac{2\pi}{m^2}} = \frac{\sqrt{2\pi}}{m}$$

5 Conclusion

In this note, the advanced theoretical method of the functional renormalization group (fRG) was applied to a simple Gaussian integral (a non-interacting theory).

It has been shown how, through the fRG flow, the effect of fluctuations (in this case, the ϕ-independent constant term, i.e., the partition function itself) is rigorously and systematically incorporated.

References

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[2] J. Berges, N. Tetradis, and C. Wetterich, "Non-perturbative renormalization flow in quantum field theory and statistical mechanics", *Phys. Rep.*, **363** (4 – 6): 223 – 386 (2002), arXiv:hep-ph/0005122.