Anisotropic BCS-Nambu Green's Function in Spin Space

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Abstract

In this note, an anisotropic BCS mean-field Hamiltonian in the Nambu $\otimes$ spin space, capable of treating spin-singlet and triplet states, is introduced. Its eigenvalues and transformation matrix are then derived by diagonalizing it using a Bogoliubov transformation. Furthermore, the Green's function (normal and anomalous components) in the Nambu $\otimes$ spin space is derived using the equation of motion method, and its explicit expression is provided.

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1 Mean-Field Hamiltonian in (Nambu$\otimes$Spin) Space

1.1 The Spin-Singlet (Conventional BCS) Case

We begin by setting up the Hamiltonian.

$$H = H_0 + H_{\mathrm{MF}} \tag{1}$$

Here,

$$H_0 = \sum_{\boldsymbol{k},\sigma} \xi_{\boldsymbol{k}} c^\dagger_{\boldsymbol{k}\sigma} c_{\boldsymbol{k}\sigma} \quad = \quad \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & , & c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\boldsymbol{k}} & 0 \\ 0 & -\xi_{\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix}, \tag{2}$$

$$\begin{aligned} H_{\mathrm{MF}} &= \Delta^* \sum_{\boldsymbol{k}} \left(c_{-\boldsymbol{k}\downarrow} c_{\boldsymbol{k}\uparrow} - c_{-\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\downarrow} \right) + \Delta \sum_{\boldsymbol{k}} \left(c^\dagger_{\boldsymbol{k}\uparrow} c^\dagger_{-\boldsymbol{k}\downarrow} - c^\dagger_{-\boldsymbol{k}\uparrow} c^\dagger_{\boldsymbol{k}\downarrow} \right) && (3) \\ &= \Delta \sum_{\boldsymbol{k},\sigma} c^\dagger_{\boldsymbol{k}\sigma} c^\dagger_{-\boldsymbol{k}\bar{\sigma}} \quad + \mathrm{H.c.} && (4) \\ &= \sum_{\boldsymbol{k}} \begin{pmatrix} c^\dagger_{\boldsymbol{k}\uparrow} & , & c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \begin{pmatrix} 0 & \Delta \\ \Delta^* & 0 \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix}, && (5) \end{aligned}$$

Then,

$$\boldsymbol{c_k}^\dagger = \left(c^\dagger_{\boldsymbol{k}\uparrow} \quad , \quad c_{-\boldsymbol{k}\downarrow} \right) \;, \qquad \boldsymbol{c_k} \quad = \quad \left(\begin{array}{c} c_{\boldsymbol{k}\uparrow} \\ \\ c^\dagger_{-\boldsymbol{k}\downarrow} \end{array} \right) , \tag{6}$$

are called two-component (Nambu) spinors. Also, the anomalous expectation value Δ is defined as follows.

$$\Delta = \sum_{\boldsymbol{k}} \langle c_{\boldsymbol{k}\uparrow} c_{-\boldsymbol{k}\downarrow} \rangle \tag{7}$$

This Hamiltonian H can now be expressed using the spinors $\boldsymbol{c_k}^{(\dagger)}$ and the 2×2 matrix $\hat{H}$.

$$H = \sum_{\boldsymbol{k}} \boldsymbol{c_k}^\dagger \left(\begin{array}{cc} \xi_{\boldsymbol{k}} & \Delta \\ \\ \Delta^* & -\xi_{\boldsymbol{k}} \end{array} \right) \boldsymbol{c_k} \quad = \quad \sum_{\boldsymbol{k}} \boldsymbol{c_k}^\dagger \; \hat{H} \; \boldsymbol{c_k}. \tag{8}$$

$\hat{H}$ can be easily diagonalized using an arbitrary real parameter λ.

$$\det(\hat{H} - \lambda \, \hat{1}_{2\times 2}) = 0 \tag{9}$$

$$\longrightarrow \qquad \lambda = \pm\sqrt{\xi^2_{\boldsymbol{k}} + |\Delta_{\boldsymbol{k}}|^2} \;=\; \pm E_{\boldsymbol{k}}. \tag{10}$$

This defines the eigenvalue $E_{\boldsymbol{k}}$.

To obtain the diagonalized basis $\boldsymbol{a}$, we use the Bogoliubov transformation matrix $\hat{U}$.

$$\boldsymbol{c_k} = \hat{U} \boldsymbol{a_k} \;=\; \left(\begin{array}{cc} u_{\boldsymbol{k}} & -v_{\boldsymbol{k}} e^{i\varphi} \\ \\ v_{\boldsymbol{k}} e^{-i\varphi} & u_{\boldsymbol{k}} \end{array} \right) \left(\begin{array}{c} a_{\boldsymbol{k}\uparrow} \\ \\ a^\dagger_{-\boldsymbol{k}\downarrow} \end{array} \right) \tag{11}$$

$$H = \sum_{\boldsymbol{k}} \boldsymbol{c_k}^\dagger \; \hat{H} \; \boldsymbol{c_k} \;=\; \sum_{\boldsymbol{k}} \boldsymbol{a_k}^\dagger \; \hat{U}^\dagger \; \hat{H} \; \hat{U} \; \boldsymbol{a_k} \;=\; \sum_{\boldsymbol{k}} \boldsymbol{a_k}^\dagger \left(\begin{array}{cc} E_{\boldsymbol{k}} & 0 \\ \\ 0 & -E_{\boldsymbol{k}} \end{array} \right) \boldsymbol{a_k}. \tag{12}$$

We only need to consider the unitary case (where $\hat{U}^\dagger = \hat{U}^{-1}$, allowing us to set $\varphi = 2\pi n$ for an integer n).

$$\hat{1}_{2\times 2} = \hat{U}^\dagger \hat{U} \;=\; \left(\begin{array}{cc} u_{\boldsymbol{k}} & v_{\boldsymbol{k}} e^{i\varphi} \\ \\ -v_{\boldsymbol{k}} e^{-i\varphi} & u_{\boldsymbol{k}} \end{array} \right) \left(\begin{array}{cc} u_{\boldsymbol{k}} & -v_{\boldsymbol{k}} e^{i\varphi} \\ \\ v_{\boldsymbol{k}} e^{-i\varphi} & u_{\boldsymbol{k}} \end{array} \right) \;=\; \left(\begin{array}{cc} u^2_{\boldsymbol{k}} + v^2_{\boldsymbol{k}} & 0 \\ \\ 0 & u^2_{\boldsymbol{k}} + v^2_{\boldsymbol{k}} \end{array} \right). \tag{13}$$

The condition $u^2_{\boldsymbol{k}} + v^2_{\boldsymbol{k}} = 1$ emerges. The simultaneous equations for the components of the matrix $\hat{U}$ can also be solved.

$$\hat{H}\ \hat{U} = \hat{U}\begin{pmatrix} E_{\boldsymbol{k}} & 0 \\ 0 & -E_{\boldsymbol{k}} \end{pmatrix} \qquad \longleftarrow \qquad \hat{U}^{\dagger}\ \hat{H}\ \hat{U} \ = \ \begin{pmatrix} E_{\boldsymbol{k}} & 0 \\ 0 & -E_{\boldsymbol{k}} \end{pmatrix} \tag{14}$$

$$\begin{pmatrix} u_{\boldsymbol{k}}\xi_{\boldsymbol{k}} + v_{\boldsymbol{k}}\Delta & u_{\boldsymbol{k}}\Delta - v_{\boldsymbol{k}}\xi_{\boldsymbol{k}} \\ \Delta^* u_{\boldsymbol{k}} - v_{\boldsymbol{k}}\xi_{\boldsymbol{k}} & -v_{\boldsymbol{k}}\Delta^* - u_{\boldsymbol{k}}\xi_{\boldsymbol{k}} \end{pmatrix} = E_{\boldsymbol{k}}\begin{pmatrix} u_{\boldsymbol{k}} & v_{\boldsymbol{k}} \\ v_{\boldsymbol{k}} & -u_{\boldsymbol{k}} \end{pmatrix} \tag{15}$$

$$\longrightarrow \qquad u_{\boldsymbol{k}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right), \qquad v_{\boldsymbol{k}}^2 \ = \ \frac{1}{2}\left(1 - \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right) \tag{16}$$

Now, for the spin-singlet case, the eigenvalue $E_{\boldsymbol{k}}$ and the Bogoliubov transformation matrix components $u_{\boldsymbol{k}}$, $v_{\boldsymbol{k}}$ have been expressed in terms of the known values $\xi_{\boldsymbol{k}}$ and Δ.

1.2 The Generalized Spin Case

Next, to enable the calculation of physical quantities not just for the spin-singlet case but also for any spin-triplet case, the two-component Nambu spinor $\boldsymbol{c}_{\boldsymbol{k}}$ is generalized to a four-component one.

$$H \ = \ \frac{1}{2}\sum_{\boldsymbol{k}} \boldsymbol{c}_{\boldsymbol{k}}{}^{\dagger}\hat{H}\boldsymbol{c}_{\boldsymbol{k}} = \frac{1}{2}\sum_{\boldsymbol{k}}\left(c_{\boldsymbol{k}\uparrow}^{\dagger}\ ,\ c_{\boldsymbol{k}\downarrow}^{\dagger}\ ,\ c_{-\boldsymbol{k}\uparrow}\ ,\ c_{-\boldsymbol{k}\downarrow}\right)\begin{pmatrix} \xi_{\boldsymbol{k}}\hat{1}_{2\times 2} & \hat{\Delta}_{\boldsymbol{k}} \\ \hat{\Delta}_{\boldsymbol{k}}^* & -\xi_{\boldsymbol{k}}\hat{1}_{2\times 2} \end{pmatrix}\begin{pmatrix} c_{\boldsymbol{k}\uparrow} \\ c_{\boldsymbol{k}\downarrow} \\ c_{-\boldsymbol{k}\uparrow}^{\dagger} \\ c_{-\boldsymbol{k}\downarrow}^{\dagger} \end{pmatrix} \tag{17}$$

The definition of the matrix $\hat{\Delta}_{\boldsymbol{k}}$, composed of anomalous expectation values, is:

$$(\hat{\Delta}_{\boldsymbol{k}})_{\sigma\sigma'} = -V\sum_{\boldsymbol{k'}} \boldsymbol{k}\cdot\boldsymbol{k'}\langle c_{\boldsymbol{k'}\sigma}c_{-\boldsymbol{k'}\sigma'}\rangle. \tag{18}$$

The components of the vector $\boldsymbol{d}$, which is perpendicular to the total angular momentum of the Cooper pair, can be chosen as the basis for the matrix $\hat{\Delta}_{\boldsymbol{k}}$.

$$\hat{\Delta}_{\boldsymbol{k}} = \begin{pmatrix} -d_{\boldsymbol{k}}^x + id_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + id_{\boldsymbol{k}}^y \end{pmatrix}. \tag{19}$$

When the matrix $\hat{\Delta}_{\boldsymbol{k}}$ is unitary,

$$\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}^{\dagger} = \begin{pmatrix} {d_{\boldsymbol{k}}^x}^2 + {d_{\boldsymbol{k}}^y}^2 + {d_{\boldsymbol{k}}^z}^2 & 0 \\ 0 & {d_{\boldsymbol{k}}^x}^2 + {d_{\boldsymbol{k}}^y}^2 + {d_{\boldsymbol{k}}^z}^2 \end{pmatrix} \ \propto \ \hat{1}_{2\times 2} \tag{20}$$

Namely,

$${d_{\boldsymbol{k}}^x}^2 + {d_{\boldsymbol{k}}^y}^2 + {d_{\boldsymbol{k}}^z}^2 \ = \ \frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}^{\dagger}\big], \qquad \hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}^{\dagger} \ = \ \frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}^{\dagger}\big] \ \ \hat{1}_{2\times 2}. \tag{21}$$

Then, the 4×4 Hamiltonian can be written specifically as follows.

$$\hat{H} = \begin{pmatrix} \xi_{\boldsymbol{k}} & 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ 0 & \xi_{\boldsymbol{k}} & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ -d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z & -\xi_{\boldsymbol{k}} & 0 \\ d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} \end{pmatrix}. \tag{22}$$

The eigenvalue equation $\det(\hat{H} - \lambda \hat{1}_{4\times 4})$ can be solved through several tedious processes.

$$\begin{aligned} 0 &= \begin{vmatrix} \xi_{\boldsymbol{k}} - \lambda & 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ 0 & \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ -d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z & -\xi_{\boldsymbol{k}} - \lambda & 0 \\ d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} - \lambda \end{vmatrix} \\ &= (\xi_{\boldsymbol{k}} - \lambda) \begin{vmatrix} \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ d_{\boldsymbol{k}}^z & -\xi_{\boldsymbol{k}} - \lambda & 0 \\ d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} - \lambda \end{vmatrix} \\ &\quad + (-d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y) \begin{vmatrix} 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} - \lambda \end{vmatrix} \\ &\quad - d_{\boldsymbol{k}}^z \begin{vmatrix} 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ d_{\boldsymbol{k}}^z & -\xi_{\boldsymbol{k}} - \lambda & 0 \end{vmatrix}. \end{aligned} \tag{23}$$

Each term on the right-hand side can be expanded as follows.

$$(\xi_{\boldsymbol{k}} - \lambda) \begin{vmatrix} \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ d_{\boldsymbol{k}}^z & -\xi_{\boldsymbol{k}} - \lambda & 0 \\ d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} - \lambda \end{vmatrix} = ({\xi_{\boldsymbol{k}}}^2 - \lambda^2)({\xi_{\boldsymbol{k}}}^2 - \lambda^2 + {d_{\boldsymbol{k}}^x}^2 + {d_{\boldsymbol{k}}^y}^2 + {d_{\boldsymbol{k}}^z}^2) \tag{24}$$

$$(-d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y) \begin{vmatrix} 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y \\ d_{\boldsymbol{k}}^x - i d_{\boldsymbol{k}}^y & 0 & -\xi_{\boldsymbol{k}} - \lambda \end{vmatrix} = ({d_{\boldsymbol{k}}^x}^2 + {dy_{\boldsymbol{k}}}^2)({\xi_{\boldsymbol{k}}}^2 - \lambda^2 + {d_{\boldsymbol{k}}^x}^2 + {dy_{\boldsymbol{k}}}^2 + {d_{\boldsymbol{k}}^z}^2) \tag{25}$$

$$-d_{\boldsymbol{k}}^z \begin{vmatrix} 0 & -d_{\boldsymbol{k}}^x + i d_{\boldsymbol{k}}^y & d_{\boldsymbol{k}}^z \\ \xi_{\boldsymbol{k}} - \lambda & d_{\boldsymbol{k}}^z & d_{\boldsymbol{k}}^x + iy \\ z & -\xi_{\boldsymbol{k}} - \lambda & 0 \end{vmatrix} = {d_{\boldsymbol{k}}^z}^2({\xi_{\boldsymbol{k}}}^2 - \lambda^2 + {d_{\boldsymbol{k}}^x}^2 + {dy_{\boldsymbol{k}}}^2 + {d_{\boldsymbol{k}}^z}^2) \tag{26}$$

Collecting all terms yields the following.

$$({\xi_{\boldsymbol{k}}}^2 - \lambda^2 + {d_{\boldsymbol{k}}^x}^2 + {dy_{\boldsymbol{k}}}^2 + {d_{\boldsymbol{k}}^z}^2)^2 = 0 \tag{27}$$

$$\rightarrow \quad \lambda = \pm\sqrt{\xi_{\boldsymbol{k}}^2 + {d_{\boldsymbol{k}}^x}^2 + {d_{\boldsymbol{k}}^y}^2 + {d_{\boldsymbol{k}}^z}^2} = \pm\sqrt{\xi_{\boldsymbol{k}}^2 + \frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}_{\boldsymbol{k}}^\dagger\big]} = \pm E_{\boldsymbol{k}} \tag{28}$$

Finally, the eigenvalue $E_{\boldsymbol{k}}$ for the 4×4 matrix $\hat{H}$ has been obtained.

To know how to represent the diagonalized basis $\boldsymbol{a}$ (which is a four-component vector, not a two-component one), the 4×4 Bogoliubov transformation matrix $\hat{U}$ is defined using 2×2 block matrices $\hat{u}_{\boldsymbol{k}}$ and $\hat{v}_{\boldsymbol{k}}$.

$$\boldsymbol{c}_{\boldsymbol{k}} = \begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{k}\uparrow} \\ a_{\boldsymbol{k}\downarrow} \\ a^\dagger_{-\boldsymbol{k}\uparrow} \\ a^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} = \hat{U} \boldsymbol{a}_{\boldsymbol{k}} \tag{29}$$

$$H = \frac{1}{2} \sum_{\boldsymbol{k}} \boldsymbol{c}_{\boldsymbol{k}}^\dagger \ \hat{H} \ \boldsymbol{c}_{\boldsymbol{k}} = \frac{1}{2} \sum_{\boldsymbol{k}} \boldsymbol{a}_{\boldsymbol{k}}^\dagger \ \hat{U}^\dagger \ \hat{H} \ \hat{U} \ \boldsymbol{a}_{\boldsymbol{k}} = \frac{1}{2} \sum_{\boldsymbol{k}} \boldsymbol{a}_{\boldsymbol{k}}^\dagger \begin{pmatrix} E_{\boldsymbol{k}} & 0 & 0 & 0 \\ 0 & E_{\boldsymbol{k}} & 0 & 0 \\ 0 & 0 & -E_{\boldsymbol{k}} & 0 \\ 0 & 0 & 0 & -E_{\boldsymbol{k}} \end{pmatrix} \boldsymbol{a}_{\boldsymbol{k}}. \tag{30}$$

At this point, the 2×2 block matrices $\hat{u}_{\boldsymbol{k}}$ and $\hat{v}_{\boldsymbol{k}}$ are unknown.

$$\begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix}^\dagger \begin{pmatrix} \xi_{\boldsymbol{k}} \hat{1}_{2\times2} & \hat{\Delta}_{\boldsymbol{k}} \\ \hat{\Delta}^*_{\boldsymbol{k}} & -\xi_{\boldsymbol{k}} \hat{1}_{2\times2} \end{pmatrix} \begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} E_{\boldsymbol{k}} \hat{1}_{2\times2} & 0 \\ 0 & -E_{\boldsymbol{k}} \hat{1}_{2\times2} \end{pmatrix} \tag{31}$$

$$\begin{pmatrix} \xi_{\boldsymbol{k}} \hat{1}_{2\times2} & \hat{\Delta}_{\boldsymbol{k}} \\ \hat{\Delta}^*_{\boldsymbol{k}} & -\xi_{\boldsymbol{k}} \hat{1}_{2\times2} \end{pmatrix} \begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} = \begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} E_{\boldsymbol{k}} \hat{1}_{2\times2} & 0 \\ 0 & -E_{\boldsymbol{k}} \hat{1}_{2\times2} \end{pmatrix} \tag{32}$$

This procedure is permissible because $\hat{U}$ is unitary ($\hat{U}^\dagger = \hat{U}^{-1}$). Immediately, several constraints on the matrices $\hat{u}_{\boldsymbol{k}}$ and $\hat{v}_{\boldsymbol{k}}$ are obtained.

$$\begin{cases} \hat{u}_{\boldsymbol{k}} = \dfrac{\hat{\Delta}_{\boldsymbol{k}}}{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}} \hat{v}^*_{-\boldsymbol{k}} & \cdots \ (1,1,\mathrm{A}) \\[2ex] \hat{v}_{\boldsymbol{k}} = \dfrac{\hat{\Delta}_{\boldsymbol{k}}}{E_{\boldsymbol{k}} + \xi_{\boldsymbol{k}}} \hat{u}_{-\boldsymbol{k}} & \cdots \ (1,2,\mathrm{A}) \\[2ex] \hat{v}^*_{-\boldsymbol{k}} = \dfrac{\hat{\Delta}^*_{\boldsymbol{k}}}{E_{\boldsymbol{k}} + \xi_{\boldsymbol{k}}} \hat{u}_{\boldsymbol{k}} & \cdots \ (2,1,\mathrm{A}) \\[2ex] \hat{u}_{-\boldsymbol{k}} = \dfrac{\hat{\Delta}^*_{\boldsymbol{k}}}{E_{\boldsymbol{k}} - \xi_{\boldsymbol{k}}} \hat{v}_{\boldsymbol{k}} & \cdots \ (2,2,\mathrm{A}) \end{cases} \tag{33}$$

These relations indicate that if $\hat{u}_{\boldsymbol{k}}$ (or $\hat{v}_{\boldsymbol{k}}$) is proportional to the identity matrix $\hat{1}_{2\times2}$, then $\hat{v}_{\boldsymbol{k}}$ (or $\hat{u}_{\boldsymbol{k}}$) is proportional to $\hat{\Delta}_{\boldsymbol{k}}$. Let us here choose the condition $\hat{u}_{\boldsymbol{k}} \propto \hat{1}_{2\times2}$.

$$\hat{u}_{\boldsymbol{k}} = \frac{\hat{1}_{2\times2}}{f(\boldsymbol{k})}, \qquad \hat{u}_{\boldsymbol{k}}^{-1} = f(\boldsymbol{k}) \hat{1}_{2\times2} \tag{34}$$

At this time, $f(\boldsymbol{k})$ is an unknown function. The problem comes down to how to obtain an expression for the function $f(\boldsymbol{k})$.

$$
\begin{cases}
\hat{v}_{\boldsymbol{k}} &= \dfrac{\hat{\Delta}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}}\hat{u}_{-\boldsymbol{k}} &= \dfrac{\hat{\Delta}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}}\cdot\dfrac{1}{f(-\boldsymbol{k})} \\
\hat{u}_{-\boldsymbol{k}} &= \dfrac{\hat{\Delta}^{\dagger}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}-\xi_{\boldsymbol{k}}}\hat{v}_{\boldsymbol{k}} &= \dfrac{\frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\big]}{E_{\boldsymbol{k}}^2-\xi_{\boldsymbol{k}}^2}\cdot\dfrac{1}{f(-\boldsymbol{k})}\hat{1}_{2\times2} &= \dfrac{\hat{1}_{2\times2}}{f(-\boldsymbol{k})} \qquad \cdots\ (\text{Trivial}) \\
\hat{v}^{*}_{-\boldsymbol{k}} &= \dfrac{\hat{\Delta}^{\dagger}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}}\hat{u}_{\boldsymbol{k}} &= \dfrac{\hat{\Delta}^{\dagger}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}}\cdot\dfrac{1}{f(\boldsymbol{k})} \\
\hat{u}_{\boldsymbol{k}} &= \dfrac{\hat{\Delta}_{\boldsymbol{k}}}{E_{\boldsymbol{k}}-\xi_{\boldsymbol{k}}}\hat{v}^{*}_{-\boldsymbol{k}} &= \dfrac{\frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\big]}{E_{\boldsymbol{k}}^2-\xi_{\boldsymbol{k}}^2}\cdot\dfrac{1}{f(\boldsymbol{k})}\hat{1}_{2\times2} &= \dfrac{\hat{1}_{2\times2}}{f(\boldsymbol{k})} \qquad \cdots\ (\text{Trivial})
\end{cases}
\tag{35}
$$

In addition to these constraints, the unitary condition for $\hat{U}$ can be described as follows.

$$
\begin{aligned}
1 &= \left|\det\begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^{*}_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix}\right| = \left|\det\Big(\hat{u}_{\boldsymbol{k}}\Big)\ \det\Big[\hat{u}_{-\boldsymbol{k}} - \hat{v}^{*}_{-\boldsymbol{k}}\hat{u}^{-1}_{\boldsymbol{k}}(-\hat{v}_{\boldsymbol{k}})\Big]\right| \\
\longleftrightarrow \qquad & \left|\det\Big(\hat{u}_{\boldsymbol{k}}\Big)\det\Big(\hat{u}_{-\boldsymbol{k}}\Big) + \det\Big(\hat{v}_{\boldsymbol{k}}\Big)\det\Big(\hat{v}^{*}_{-\boldsymbol{k}}\Big)\right| = 1,
\end{aligned}
\tag{36}
$$

This leads to the following relation.

$$
\begin{aligned}
1 &= \left|\det\Big(\hat{u}_{\boldsymbol{k}}\Big)\det\Big(\hat{u}_{-\boldsymbol{k}}\Big) + \det\Big(\hat{v}_{\boldsymbol{k}}\Big)\det\Big(\hat{v}^{*}_{-\boldsymbol{k}}\Big)\right| \\
\longleftrightarrow \qquad & \Big|f(\boldsymbol{k})f(-\boldsymbol{k})\Big| = 1 + \frac{\frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\big]}{\Big(E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}\Big)^2}
\end{aligned}
\tag{37}
$$

Therefore, when $\hat{u}_{\boldsymbol{k}} = \hat{u}_{-\boldsymbol{k}}$, we obtain:

$$
\begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^{*}_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} = \frac{1}{\sqrt{\Big(E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}\Big)^2 + \frac{1}{2}\mathrm{Tr}\big[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^{\dagger}_{\boldsymbol{k}}\big]}}\begin{pmatrix} (E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}})\hat{1}_{2\times2} & -\hat{\Delta}_{\boldsymbol{k}} \\ \hat{\Delta}^{\dagger}_{\boldsymbol{k}} & (E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}})\hat{1}_{2\times2} \end{pmatrix}
\tag{38}
$$

This is the goal of this subsection.

Can this result be reduced from the triplet case to the singlet case? Let's consider the (2,1) component of the order parameter, $(\hat{\Delta}_{\boldsymbol{k}})_{\uparrow\downarrow} = d_z = \Delta$.

$$
\begin{aligned}
{(\hat{v}_{\boldsymbol{k}})_{\uparrow\downarrow}}^2 &= \frac{\Delta^2}{\Big(E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}\Big)^2 + E_{\boldsymbol{k}}^2 - \xi_{\boldsymbol{k}}^2} \\
&= \frac{1}{2}\left(1 - \frac{\xi_{\boldsymbol{k}}}{E_{\boldsymbol{k}}}\right)
\end{aligned}
\tag{39}
$$

This value matches $v_{\boldsymbol{k}}$ for the singlet case that appeared in the previous subsection.

2 Green's Function in (Nambu⊗Spin) Space

In this section, the Green's function for the anisotropic (spin-dependent) BCS model is derived. We begin with the equation of motion.

2.1 The Spin-Singlet (Conventional BCS) Case

The Green's function (2×2 matrix) in Nambu space is now defined as follows.

$$
\begin{aligned}
i\hat{G}(k) = \int dx \left\langle \hat{\mathrm{T}} \left[\boldsymbol{c}_{\boldsymbol{k}}(x) \boldsymbol{c}_{\boldsymbol{k}}^{\dagger} \right] \right\rangle e^{ik\cdot x} \quad &= \quad \int dx \left\langle \hat{\mathrm{T}} \left[\begin{pmatrix} c_{\boldsymbol{k}\uparrow}(x) \\ c^{\dagger}_{-\boldsymbol{k}\downarrow}(x) \end{pmatrix} \begin{pmatrix} c^{\dagger}_{\boldsymbol{k}\uparrow} & , & c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \right] \right\rangle e^{ik\cdot x} \\
= \int dx \left\langle \hat{\mathrm{T}} \left[\begin{pmatrix} c_{\boldsymbol{k}\uparrow}(x) c^{\dagger}_{\boldsymbol{k}\uparrow} & c_{\boldsymbol{k}\uparrow}(x) c_{-\boldsymbol{k}\downarrow} \\ c^{\dagger}_{-\boldsymbol{k}\downarrow}(x) c^{\dagger}_{\boldsymbol{k}\uparrow} & c^{\dagger}_{-\boldsymbol{k}\downarrow}(x) c_{-\boldsymbol{k}\downarrow} \end{pmatrix} \right] \right\rangle e^{ik\cdot x} \quad &= \quad i \int dx \begin{pmatrix} G(x) & F(x) \\ \bar{F}(x) & \bar{G}(x) \end{pmatrix} e^{ik\cdot x}
\end{aligned} \tag{40}
$$

Here $\hat{\mathrm{T}}\left[\cdots\right]$ is the time-ordering operator. $x^{\mu} = (t, \boldsymbol{r})$ and $k^{\mu} = (\omega, \boldsymbol{k})$ (i.e., $k\cdot x = g_{\mu\nu}k^{\mu}x^{\nu} = \omega t - \boldsymbol{k}\cdot\boldsymbol{r}$) are the four-momentum vectors written in abbreviated notation. The retarded parts of these functions are also defined as follows.

$$
i\hat{G}^{R}(k) = \int dx \left\langle \begin{pmatrix} \left\{ c_{\boldsymbol{k}\uparrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow} \right\} & \left\{ c_{\boldsymbol{k}\uparrow}(x), c_{-\boldsymbol{k}\downarrow} \right\} \\ \left\{ c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow} \right\} & \left\{ c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c_{-\boldsymbol{k}\downarrow} \right\} \end{pmatrix} \right\rangle e^{ik\cdot x} \quad = \quad i \int dx \begin{pmatrix} G^{R}(x) & F^{R}(x) \\ \bar{F}^{R}(x) & \bar{G}^{R}(x) \end{pmatrix} e^{ik\cdot x} \tag{41}
$$

The equation of motion for the diagonalized basis (quasiparticle) $a^{(\dagger)}_{\boldsymbol{k}\sigma}$ (which can be regarded as a vector by the spin index $\sigma = \uparrow, \downarrow$) is obtained below:

$$
\begin{aligned}
i\frac{da_{\boldsymbol{k}\uparrow}(t)}{dt} &= \left[a_{\boldsymbol{k}\uparrow}, H \right] \\
&= \sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow}, \boldsymbol{c}_{\boldsymbol{q}}^{\dagger} \hat{H} \boldsymbol{c}_{\boldsymbol{q}} \right] \quad = \quad \sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \quad , \quad \begin{pmatrix} c^{\dagger}_{\boldsymbol{q}\uparrow} & , & c_{-\boldsymbol{q}\downarrow} \end{pmatrix} \begin{pmatrix} \xi_{\boldsymbol{q}} & \Delta \\ \Delta^{*} & -\xi_{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{q}\uparrow} \\ c^{\dagger}_{-\boldsymbol{q}\downarrow} \end{pmatrix} \right] \\
&= \sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow}, \boldsymbol{a}_{\boldsymbol{q}}^{\dagger} \hat{U}^{\dagger} \hat{H} \hat{U} \boldsymbol{a}_{\boldsymbol{q}} \right] \quad = \quad \sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \quad , \quad \begin{pmatrix} a^{\dagger}_{\boldsymbol{q}\uparrow} & , & a_{-\boldsymbol{q}\downarrow} \end{pmatrix} \begin{pmatrix} E_{\boldsymbol{q}} & 0 \\ 0 & -E_{\boldsymbol{q}} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{q}\uparrow} \\ a^{\dagger}_{-\boldsymbol{q}\downarrow} \end{pmatrix} \right] \\
&= \sum_{\boldsymbol{q}} E_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \quad , \quad a^{\dagger}_{\boldsymbol{q}\uparrow} a_{\boldsymbol{q}\uparrow} - a_{-\boldsymbol{q}\downarrow} a^{\dagger}_{-\boldsymbol{q}\downarrow} \right] \\
&= \sum_{\boldsymbol{q}} E_{\boldsymbol{q}} \left(\left\{ a_{\boldsymbol{k}\uparrow} \ , \ a^{\dagger}_{\boldsymbol{q}\uparrow} \right\} a_{\boldsymbol{q}\uparrow} - a^{\dagger}_{\boldsymbol{q}\uparrow} \left\{ a_{\boldsymbol{k}\uparrow} \ , \ a_{\boldsymbol{q}\uparrow} \right\} - \left\{ a_{\boldsymbol{k}\uparrow} \ , \ a_{-\boldsymbol{q}\downarrow} \right\} a^{\dagger}_{-\boldsymbol{q}\downarrow} + a_{-\boldsymbol{q}\downarrow} \left\{ a_{\boldsymbol{k}\uparrow} \ , \ a^{\dagger}_{-\boldsymbol{q}\downarrow} \right\} \right) \\
&= E_{\boldsymbol{k}} a_{\boldsymbol{k}\uparrow}
\end{aligned} \tag{42}
$$

Integrating this gives,

$$
a_{\boldsymbol{k}\uparrow}(t) = e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow}(0) \tag{43}
$$

In the same way, the equation for $a^{\dagger}_{-\boldsymbol{k}\downarrow}$ is obtained.

$$
i\frac{da^{\dagger}_{-\boldsymbol{k}\downarrow}}{dt} = \left[a^{\dagger}_{-\boldsymbol{k}\downarrow}, H \right] \quad = \quad -E_{\boldsymbol{k}} a^{\dagger}_{-\boldsymbol{k}\downarrow} \tag{44}
$$

And,

$$a^\dagger_{-\boldsymbol{k}\downarrow}(t) = e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow}(0) \tag{45}$$

The relationship between the spinors $\boldsymbol{c}$ and $\boldsymbol{a}$ is given by.

$$\begin{aligned} c_{\boldsymbol{k}\uparrow}(t) &= u_{\boldsymbol{k}} a_{\boldsymbol{k}\uparrow}(t) - v_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}\downarrow}(t) \\ &= u_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} \end{aligned} \tag{46}$$

$$\begin{aligned} c^\dagger_{-\boldsymbol{k}\downarrow}(t) &= u_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}\downarrow}(t) + v_{\boldsymbol{k}} a_{\boldsymbol{k}\uparrow}(t) \\ &= u_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} \end{aligned} \tag{47}$$

The (1,1) component of the retarded Green's function becomes clear as follows.

$$\begin{aligned}
iG^R_{\boldsymbol{k}}(t) \quad &= \quad \theta(t) \left\langle \left\{ c_{\boldsymbol{k}\uparrow}(t), c^\dagger_{\boldsymbol{k}\uparrow} \right\} \right\rangle \quad = \quad \theta(t) \Big[\left\langle c_{\boldsymbol{k}\uparrow}(t) c^\dagger_{\boldsymbol{k}\uparrow} \right\rangle + \left\langle c^\dagger_{\boldsymbol{k}\uparrow} c_{\boldsymbol{k}\uparrow}(t) \right\rangle \Big] \\
&= \theta(t) \left\langle \Big(u_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} \Big) \Big(u_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} a_{-\boldsymbol{k}\downarrow} \Big) \right\rangle \\
&\quad + \theta(t) \left\langle \Big(u_{\boldsymbol{k}} a^\dagger_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} a_{-\boldsymbol{k}\downarrow} \Big) \Big(u_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} \Big) \right\rangle \\
&= \theta(t) \left\langle u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} a^\dagger_{\boldsymbol{k}\uparrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow} a_{-\boldsymbol{k}\downarrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} a^\dagger_{\boldsymbol{k}\uparrow} + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow} a_{-\boldsymbol{k}\downarrow} \right\rangle \\
&\quad + \theta(t) \left\langle u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a^\dagger_{\boldsymbol{k}\uparrow} a_{\boldsymbol{k}\uparrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a^\dagger_{\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\downarrow} - u_{\boldsymbol{k}} v_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} a_{-\boldsymbol{k}\downarrow} a_{\boldsymbol{k}\uparrow} + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} a_{-\boldsymbol{k}\downarrow} a^\dagger_{-\boldsymbol{k}\downarrow} \right\rangle \\
&= \theta(t) \Big(u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} \left\langle a_{\boldsymbol{k}\uparrow} a^\dagger_{\boldsymbol{k}\uparrow} \right\rangle + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{-\boldsymbol{k}\downarrow} a_{-\boldsymbol{k}\downarrow} \right\rangle + u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{\boldsymbol{k}\uparrow} a_{\boldsymbol{k}\uparrow} \right\rangle + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \left\langle a_{-\boldsymbol{k}\downarrow} a^\dagger_{-\boldsymbol{k}\downarrow} \right\rangle \Big) \\
&= \theta(t) \\
&\quad \times \left[u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} \frac{1}{2} \Big(1 + \tanh \frac{\beta E_{\boldsymbol{k}}}{2} \Big) + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \frac{1}{2} \Big(1 - \tanh \frac{\beta E_{\boldsymbol{k}}}{2} \Big) \right. \\
&\quad \left. + u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} \frac{1}{2} \Big(1 - \tanh \frac{\beta E_{\boldsymbol{k}}}{2} \Big) + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \frac{1}{2} \Big(1 + \tanh \frac{\beta E_{\boldsymbol{k}}}{2} \Big) \right] \\
&= \theta(t) \left(u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \right)
\end{aligned} \tag{48}$$

Or, in Fourier space,

$$\begin{aligned}
G^R(k) \quad &= \quad \int dt \; G^R_{\boldsymbol{k}}(t) e^{i\omega t} \\
&= -i \lim_{\eta \to +0} \int_0^\infty dt \;\; \left(u^2_{\boldsymbol{k}} e^{-iE_{\boldsymbol{k}}t} + v^2_{\boldsymbol{k}} e^{iE_{\boldsymbol{k}}t} \right) e^{i\omega t - \eta t} \\
&= \lim_{\eta \to +0} \left(\frac{u^2_{\boldsymbol{k}}}{\omega - E_{\boldsymbol{k}} + i\eta} + \frac{v^2_{\boldsymbol{k}}}{\omega + E_{\boldsymbol{k}} + i\eta} \right).
\end{aligned} \tag{49}$$

The imaginary part is related to the density of states.

$$
\begin{aligned}
& -\frac{1}{\pi}\sum_{\boldsymbol{k}}\mathrm{Im}G^R(k) \\
&= -\frac{1}{\pi}\lim_{\eta\to+0}\sum_{\boldsymbol{k}}\left(\mathrm{Im}\frac{u_{\boldsymbol{k}}^2}{\omega-E_{\boldsymbol{k}}+i\eta}+\mathrm{Im}\frac{v_{\boldsymbol{k}}^2}{\omega+E_{\boldsymbol{k}}+i\eta}\right) \\
&= \sum_{\boldsymbol{k}}\left[u_{\boldsymbol{k}}^2\delta(\omega-E_{\boldsymbol{k}})+v_{\boldsymbol{k}}^2\delta(\omega+E_{\boldsymbol{k}})\right]
\end{aligned} \tag{50}
$$

The $(1,2)$ component of the retarded part, $F^R(k)$, is given similarly.

$$
\begin{aligned}
& iF_{\boldsymbol{k}}^R(t) \\
&= \theta(t)\left\langle\{c_{\boldsymbol{k}\uparrow}(t),c_{-\boldsymbol{k}\downarrow}\}\right\rangle \\
&= \theta(t)\Big[\left\langle c_{\boldsymbol{k}\uparrow}(t)c_{-\boldsymbol{k}\downarrow}\right\rangle+\left\langle c_{-\boldsymbol{k}\downarrow}c_{\boldsymbol{k}\uparrow}(t)\right\rangle\Big] \\
&= \theta(t)\left\langle\left(u_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}-v_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a_{-\boldsymbol{k}\downarrow}^\dagger\right)\left(u_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow}+v_{\boldsymbol{k}}a_{\boldsymbol{k}\uparrow}^\dagger\right)\right\rangle \\
&\quad +\theta(t)\left\langle\left(u_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow}+v_{\boldsymbol{k}}a_{\boldsymbol{k}\uparrow}^\dagger\right)\left(u_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}-v_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a_{-\boldsymbol{k}\downarrow}^\dagger\right)\right\rangle \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}^\dagger\right\rangle-e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}^\dagger a_{-\boldsymbol{k}\downarrow}\right\rangle+e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}^\dagger a_{\boldsymbol{k}\uparrow}\right\rangle-e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}^\dagger\right\rangle\right) \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left[e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\left(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\right)-e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\left(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\right)\right. \\
&\quad \left.+e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\left(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\right)-e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\left(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\right)\right] \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(e^{-iE_{\boldsymbol{k}}t}-e^{iE_{\boldsymbol{k}}t}\right),
\end{aligned} \tag{51}
$$

$$
\begin{aligned}
F^R(k) &= -iu_{\boldsymbol{k}}v_{\boldsymbol{k}}\lim_{\eta\to+0}\int_0^\infty dt\ \left(e^{iE_{\boldsymbol{k}}t}-e^{-iE_{\boldsymbol{k}}t}\right)e^{i\omega t-\eta t} \\
&= -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\lim_{\eta\to+0}\left(\frac{1}{\omega-E_{\boldsymbol{k}}+i\eta}-\frac{1}{\omega+E_{\boldsymbol{k}}+i\eta}\right),
\end{aligned} \tag{52}
$$

The density of states corresponding to the anomalous Green's function is

$$
\begin{aligned}
& -\frac{1}{\pi}\sum_{\boldsymbol{k}}\mathrm{Im}F^R(k) \\
&= \frac{1}{\pi}\lim_{\eta\to+0}\sum_{\boldsymbol{k}}u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(\mathrm{Im}\frac{1}{\omega-E_{\boldsymbol{k}}+i\eta}-\mathrm{Im}\frac{1}{\omega+E_{\boldsymbol{k}}+i\eta}\right) \\
&= \sum_{\boldsymbol{k}}u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left[\delta(\omega+E_{\boldsymbol{k}})-\delta(\omega-E_{\boldsymbol{k}})\right].
\end{aligned} \tag{53}
$$

The $(2,1)$ component of the retarded part, $\bar{F}^R(k)$, is equal to the $(1,2)$ component $F^R(k)$.

$$
\begin{aligned}
i\bar{F}^R_{\boldsymbol{k}}(t) \quad &= \quad \theta(t)\left\langle\left\{c^\dagger_{-\boldsymbol{k}\downarrow}(t), c^\dagger_{\boldsymbol{k}\uparrow}\right\}\right\rangle \quad = \quad \theta(t)\Big[\left\langle c^\dagger_{-\boldsymbol{k}\downarrow}(t)c^\dagger_{\boldsymbol{k}\uparrow}\right\rangle + \left\langle c^\dagger_{\boldsymbol{k}\uparrow}c^\dagger_{-\boldsymbol{k}\downarrow}(t)\right\rangle\Big] \\
&= \theta(t)\left\langle\Big(u_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a^\dagger_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}\Big)\Big(u_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow}\Big)\right\rangle \\
&\quad +\theta(t)\left\langle\Big(u_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow} - v_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow}\Big)\Big(u_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a^\dagger_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}\Big)\right\rangle \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(-e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}\right\rangle + e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a^\dagger_{\boldsymbol{k}\uparrow}\right\rangle + e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}\right\rangle - e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}\right\rangle\right) \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\bigg[-e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) \\
&\quad + e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) - e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big)\bigg] \\
&= \theta(t)u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(-e^{iE_{\boldsymbol{k}}t} + e^{-iE_{\boldsymbol{k}}t}\right) \\
&= iF^R_{\boldsymbol{k}}(t)
\end{aligned}
\tag{54}
$$

The $(2,2)$ component of the retarded part, $\bar{G}^R(k)$, is also equal to $G^R(k)$.

$$
\begin{aligned}
i\bar{G}^R_{\boldsymbol{k}}(t) \quad &= \quad \theta(t)\left\langle\left\{c^\dagger_{-\boldsymbol{k}\downarrow}(t), c_{-\boldsymbol{k}\downarrow}\right\}\right\rangle \quad = \quad \theta(t)\Big[\left\langle c^\dagger_{-\boldsymbol{k}\downarrow}(t)c_{-\boldsymbol{k}\downarrow}\right\rangle + \left\langle c_{-\boldsymbol{k}\downarrow}c^\dagger_{-\boldsymbol{k}\downarrow}(t)\right\rangle\Big] \\
&= \theta(t)\left\langle\Big(u_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a^\dagger_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}\Big)\Big(u_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow}\Big)\right\rangle \\
&\quad +\theta(t)\left\langle\Big(u_{\boldsymbol{k}}a_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow}\Big)\Big(u_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}a^\dagger_{-\boldsymbol{k}\downarrow} + v_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}a_{\boldsymbol{k}\uparrow}\Big)\right\rangle \\
&= \theta(t)\left(u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}\right\rangle + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a^\dagger_{\boldsymbol{k}\uparrow}\right\rangle\right. \\
&\quad \left. + u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}\right\rangle + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}\right\rangle\right) \\
&= \theta(t)\bigg[u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) \\
&\quad + u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big)\bigg] \\
&= \theta(t)\left(u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t} + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\right) \\
&= iG^R_{\boldsymbol{k}}(t)
\end{aligned}
\tag{55}
$$

Finally, we arrive at the expression for $\hat{G}^R$ in terms of known values.

$$
i\begin{pmatrix} G^R_{\boldsymbol{k}}(t) & F^R_{\boldsymbol{k}}(t) \\ \bar{F}^R_{\boldsymbol{k}}(t) & \bar{G}^R_{\boldsymbol{k}}(t) \end{pmatrix} = \theta(t)\begin{pmatrix} u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t} + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t} & -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(e^{iE_{\boldsymbol{k}}t} - e^{-iE_{\boldsymbol{k}}t}\right) \\ -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(e^{iE_{\boldsymbol{k}}t} - e^{-iE_{\boldsymbol{k}}t}\right) & u^2_{\boldsymbol{k}}e^{iE_{\boldsymbol{k}}t} + v^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t} \end{pmatrix} \tag{56}
$$

$$
\begin{pmatrix} G^R(k) & F^R(k) \\ \bar{F}^R(k) & \bar{G}^R(k) \end{pmatrix} = \lim_{\eta\to+0}\begin{pmatrix} \dfrac{u^2_{\boldsymbol{k}}}{\omega - E_{\boldsymbol{k}} + i\eta} + \dfrac{v^2_{\boldsymbol{k}}}{\omega + E_{\boldsymbol{k}} + i\eta} & -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(\dfrac{1}{\omega - E_{\boldsymbol{k}} + i\eta} - \dfrac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\right) \\ -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\left(\dfrac{1}{\omega - E_{\boldsymbol{k}} + i\eta} - \dfrac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\right) & \dfrac{u^2_{\boldsymbol{k}}}{\omega - E_{\boldsymbol{k}} + i\eta} + \dfrac{v^2_{\boldsymbol{k}}}{\omega + E_{\boldsymbol{k}} + i\eta} \end{pmatrix} \tag{57}
$$

2.2 The Generalized Spin Case

The Green's function (4×4 matrix) in Nambu space is defined as follows.

$$
\begin{aligned}
i\hat{G}(k) &= \int dx \left\langle \hat{\mathrm{T}}\left[\boldsymbol{c}_{\boldsymbol{k}}(x)\boldsymbol{c}_{\boldsymbol{k}}^{\dagger}\right]\right\rangle e^{ik\cdot x} \\
&= \int dx \left\langle \hat{\mathrm{T}}\left[\begin{pmatrix} c_{\boldsymbol{k}\uparrow}(x) \\ c_{\boldsymbol{k}\downarrow}(x) \\ c^{\dagger}_{-\boldsymbol{k}\uparrow}(x) \\ c^{\dagger}_{-\boldsymbol{k}\downarrow}(x) \end{pmatrix}\left(c^{\dagger}_{\boldsymbol{k}\uparrow}\ ,\ c^{\dagger}_{\boldsymbol{k}\downarrow}\ ,\ c_{-\boldsymbol{k}\uparrow}\ ,\ c_{-\boldsymbol{k}\downarrow}\right)\right]\right\rangle e^{ik\cdot x}
\end{aligned} \tag{58}
$$

Here $\hat{\mathrm{T}}\left[\cdots\right]$ is the time-ordering operator. $x^{\mu} = (t, \boldsymbol{r})$ and $k^{\mu} = (\omega, \boldsymbol{k})$ (i.e., $k\cdot x = g_{\mu\nu}k^{\mu}x^{\nu} = \omega t - \boldsymbol{k}\cdot\boldsymbol{r}$) are the four-momentum vectors written in abbreviated notation. The retarded parts of these functions are also defined as follows.

$$
\begin{aligned}
&i\hat{G}^R(k) \\
&= \int dx \left\langle \begin{pmatrix} \{c_{\boldsymbol{k}\uparrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow}\} & \{c_{\boldsymbol{k}\uparrow}(x), c^{\dagger}_{\boldsymbol{k}\downarrow}\} & \{c_{\boldsymbol{k}\uparrow}(x), c_{-\boldsymbol{k}\uparrow}\} & \{c_{\boldsymbol{k}\uparrow}(x), c_{-\boldsymbol{k}\downarrow}\} \\ \{c_{\boldsymbol{k}\downarrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow}\} & \{c_{\boldsymbol{k}\downarrow}(x), c^{\dagger}_{\boldsymbol{k}\downarrow}\} & \{c_{\boldsymbol{k}\downarrow}(x), c_{-\boldsymbol{k}\uparrow}\} & \{c_{\boldsymbol{k}\downarrow}(x), c_{-\boldsymbol{k}\downarrow}\} \\ \{c^{\dagger}_{-\boldsymbol{k}\uparrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\uparrow}(x), c^{\dagger}_{\boldsymbol{k}\downarrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\uparrow}(x), c_{-\boldsymbol{k}\uparrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\uparrow}(x), c_{-\boldsymbol{k}\downarrow}\} \\ \{c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c^{\dagger}_{\boldsymbol{k}\uparrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c^{\dagger}_{\boldsymbol{k}\downarrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c_{-\boldsymbol{k}\uparrow}\} & \{c^{\dagger}_{-\boldsymbol{k}\downarrow}(x), c_{-\boldsymbol{k}\downarrow}\} \end{pmatrix}\right\rangle e^{ik\cdot x}
\end{aligned} \tag{59}
$$

The equation of motion for the diagonalized basis (quasiparticle) $a^{(\dagger)}_{\boldsymbol{k}\sigma}$ is obtained below:

$$
\begin{aligned}
i\frac{da_{\boldsymbol{k}\uparrow}(t)}{dt} &= [a_{\boldsymbol{k}\uparrow}, H] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow}, \boldsymbol{c_q}^\dagger \hat{H} \boldsymbol{c_q}\right] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \ , \ (c^\dagger_{\boldsymbol{q}\uparrow} \ , \ c^\dagger_{\boldsymbol{q}\downarrow} \ , \ c_{-\boldsymbol{q}\uparrow} \ , \ c_{-\boldsymbol{q}\downarrow}) \begin{pmatrix} \xi_{\boldsymbol{q}} \hat{1}_{2\times 2} & \hat{\Delta}_{\boldsymbol{q}} \\ \hat{\Delta}^*_{\boldsymbol{q}} & -\xi_{\boldsymbol{q}} \hat{1}_{2\times 2} \end{pmatrix} \begin{pmatrix} c_{\boldsymbol{q}\uparrow} \\ c_{\boldsymbol{q}\downarrow} \\ c^\dagger_{-\boldsymbol{q}\uparrow} \\ c^\dagger_{-\boldsymbol{q}\downarrow} \end{pmatrix} \right] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow}, \boldsymbol{a_q}^\dagger \hat{U}^\dagger \hat{H} \hat{U} \boldsymbol{a_q}\right] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \ , \ (a^\dagger_{\boldsymbol{q}\uparrow} \ , \ a^\dagger_{\boldsymbol{q}\downarrow} \ , \ a_{-\boldsymbol{q}\uparrow} \ , \ a_{-\boldsymbol{q}\downarrow}) \begin{pmatrix} E_{\boldsymbol{q}} \hat{1}_{2\times 2} & \\ & -E_{\boldsymbol{q}} \hat{1}_{2\times 2} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{q}\uparrow} \\ a_{\boldsymbol{q}\downarrow} \\ a^\dagger_{-\boldsymbol{q}\uparrow} \\ a^\dagger_{-\boldsymbol{q}\downarrow} \end{pmatrix} \right] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} E_{\boldsymbol{q}} \left[a_{\boldsymbol{k}\uparrow} \ , \ a^\dagger_{\boldsymbol{q}\uparrow} a_{\boldsymbol{q}\uparrow} + a^\dagger_{\boldsymbol{q}\downarrow} a_{\boldsymbol{q}\downarrow} - a_{-\boldsymbol{q}\uparrow} a^\dagger_{-\boldsymbol{q}\uparrow} - a_{-\boldsymbol{q}\downarrow} a^\dagger_{-\boldsymbol{q}\downarrow}\right] \\
&= \frac{1}{2}\sum_{\boldsymbol{q}} E_{\boldsymbol{q}} \Big(\left\{a_{\boldsymbol{k}\uparrow} \ , \ a^\dagger_{\boldsymbol{q}\uparrow}\right\} a_{\boldsymbol{q}\uparrow} - a^\dagger_{\boldsymbol{q}\uparrow} \{a_{\boldsymbol{k}\uparrow} \ , \ a_{\boldsymbol{q}\uparrow}\} + \left\{a_{\boldsymbol{k}\uparrow} \ , \ a^\dagger_{\boldsymbol{q}\downarrow}\right\} a_{\boldsymbol{q}\downarrow} - a^\dagger_{\boldsymbol{q}\downarrow} \{a_{\boldsymbol{k}\uparrow} \ , \ a_{\boldsymbol{q}\downarrow}\} \\
&\qquad - \{a_{\boldsymbol{k}\uparrow} \ , \ a_{-\boldsymbol{q}\uparrow}\} a^\dagger_{-\boldsymbol{q}\uparrow} + a_{-\boldsymbol{q}\uparrow} \left\{a_{\boldsymbol{k}\uparrow} \ , \ a^\dagger_{-\boldsymbol{q}\uparrow}\right\} - \{a_{\boldsymbol{k}\uparrow} \ , \ a_{-\boldsymbol{q}\downarrow}\} a^\dagger_{-\boldsymbol{q}\downarrow} + a_{-\boldsymbol{q}\downarrow} \left\{a_{\boldsymbol{k}\uparrow} \ , \ a^\dagger_{-\boldsymbol{q}\downarrow}\right\} \Big) \\
&= E_{\boldsymbol{k}} a_{\boldsymbol{k}\uparrow}
\end{aligned} \tag{60}
$$

Integrating this gives,

$$
a_{\boldsymbol{k}\uparrow}(t) = e^{-iE_{\boldsymbol{k}}t} a_{\boldsymbol{k}\uparrow}(0) \tag{61}
$$

In the same way, the equation for $a^\dagger_{-\boldsymbol{k}\downarrow}$ is obtained.

$$
i\frac{da^\dagger_{-\boldsymbol{k}\downarrow}(t)}{dt} = \left[a^\dagger_{-\boldsymbol{k}\downarrow}, H\right] \ = \ -E_{\boldsymbol{k}} a^\dagger_{-\boldsymbol{k}\downarrow} \tag{62}
$$

And,

$$
a^\dagger_{-\boldsymbol{k}\downarrow}(t) = e^{iE_{\boldsymbol{k}}t} a^\dagger_{-\boldsymbol{k}\downarrow}(0). \tag{63}
$$

The relationship between the spinors $\boldsymbol{c}$ and $\boldsymbol{a}$ is given by.

$$
\boldsymbol{c_k} = \begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} \begin{pmatrix} a_{\boldsymbol{k}\uparrow} \\ a_{\boldsymbol{k}\downarrow} \\ a^\dagger_{-\boldsymbol{k}\uparrow} \\ a^\dagger_{-\boldsymbol{k}\downarrow} \end{pmatrix} \ = \ \hat{U} \boldsymbol{a_k}, \tag{64}
$$

Here,

$$\begin{pmatrix} \hat{u}_{\boldsymbol{k}} & -\hat{v}_{\boldsymbol{k}} \\ \hat{v}^*_{-\boldsymbol{k}} & \hat{u}_{-\boldsymbol{k}} \end{pmatrix} = \frac{1}{\sqrt{\left(E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}}\right)^2 + \frac{1}{2}\mathrm{Tr}\left[\hat{\Delta}_{\boldsymbol{k}}\hat{\Delta}^\dagger_{\boldsymbol{k}}\right]}} \begin{pmatrix} (E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}})\hat{1}_{2\times 2} & -\hat{\Delta}_{\boldsymbol{k}} \\ \hat{\Delta}^\dagger_{\boldsymbol{k}} & (E_{\boldsymbol{k}}+\xi_{\boldsymbol{k}})\hat{1}_{2\times 2} \end{pmatrix}. \tag{65}$$

Therefore, we can substitute as follows.

$$(\hat{u}_{\boldsymbol{k}})_{\sigma\sigma'} = (\hat{u}_{-\boldsymbol{k}})_{\sigma\sigma'} = u_{\boldsymbol{k}}\delta_{\sigma\sigma'}, \tag{66}$$

$$(\hat{v}_{\boldsymbol{k}})_{\sigma\sigma'} = v_{\boldsymbol{k}\sigma\sigma'} \quad , \quad (\hat{v}^*_{\boldsymbol{k}})_{\sigma\sigma'} = v^*_{\boldsymbol{k}\sigma\sigma'}. \tag{67}$$

This defines the scalar quantities $u_{\boldsymbol{k}}$, $v_{\boldsymbol{k}\sigma\sigma'}$ and $v^*_{\boldsymbol{k}\sigma\sigma'}$. Using this relationship, an expression for the Green's function can be obtained.

$$\begin{aligned} c_{\boldsymbol{k}\sigma}(t) &= (\hat{u}_{\boldsymbol{k}})_{\sigma\uparrow}a_{\boldsymbol{k}\uparrow}(t) + (\hat{u}_{\boldsymbol{k}})_{\sigma\downarrow}a_{\boldsymbol{k}\downarrow}(t) - (\hat{v}_{\boldsymbol{k}})_{\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}(t) - (\hat{v}_{\boldsymbol{k}})_{\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}(t) \\ &= u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}, \end{aligned} \tag{68}$$

$$\begin{aligned} c^\dagger_{-\boldsymbol{k}\sigma}(t) &= (\hat{v}^*_{\boldsymbol{k}})_{\sigma\uparrow}a_{\boldsymbol{k}\uparrow}(t) + (\hat{v}^*_{\boldsymbol{k}})_{\sigma\downarrow}a_{\boldsymbol{k}\downarrow}(t) + (\hat{u}_{\boldsymbol{k}})_{\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}(t) + (\hat{u}_{\boldsymbol{k}})_{\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}(t) \\ &= v^*_{\boldsymbol{k}\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + v^*_{\boldsymbol{k}\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}. \end{aligned} \tag{69}$$

$$\begin{aligned} iG^R_{\boldsymbol{k}\uparrow\uparrow}(t) &= \theta(t)\left\langle\left\{c_{\boldsymbol{k}\uparrow}(t), c^\dagger_{\boldsymbol{k}\uparrow}\right\}\right\rangle = \theta(t)\left\langle c_{\boldsymbol{k}\uparrow}(t)c^\dagger_{\boldsymbol{k}\uparrow} + c^\dagger_{\boldsymbol{k}\uparrow}c_{\boldsymbol{k}\uparrow}(t)\right\rangle \\
&= \theta(t)\Big\langle\Big(u_{\boldsymbol{k}}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\uparrow\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\uparrow\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\Big)\Big(u_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\uparrow\uparrow}a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\uparrow\downarrow}a_{-\boldsymbol{k}\downarrow}\Big) \\
&\quad + \Big(u_{\boldsymbol{k}}a^\dagger_{\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\uparrow\uparrow}a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\uparrow\downarrow}a_{-\boldsymbol{k}\downarrow}\Big)\Big(u_{\boldsymbol{k}}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\uparrow\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\uparrow\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\Big)\Big\rangle \\
&= \theta(t)\Big(u^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a^\dagger_{\boldsymbol{k}\uparrow}\right\rangle + |v_{\boldsymbol{k}\uparrow\uparrow}|^2e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\uparrow}a_{-\boldsymbol{k}\uparrow}\right\rangle + |v_{\boldsymbol{k}\uparrow\downarrow}|^2e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}\right\rangle \\
&\quad + u^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}\right\rangle + |v_{\boldsymbol{k}\uparrow\uparrow}|^2e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}\right\rangle + |v_{\boldsymbol{k}\uparrow\downarrow}|^2e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}\right\rangle\Big) \\
&= \theta(t)\Big[u^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + |v_{\boldsymbol{k}\uparrow\uparrow}|^2e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + |v_{\boldsymbol{k}\uparrow\downarrow}|^2e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) \\
&\quad + u^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1-\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + |v_{\boldsymbol{k}\uparrow\uparrow}|^2e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big) + |v_{\boldsymbol{k}\uparrow\downarrow}|^2e^{iE_{\boldsymbol{k}}t}\frac{1}{2}\Big(1+\tanh\frac{\beta E_{\boldsymbol{k}}}{2}\Big)\Big] \\
&= \theta(t)\Big[u^2_{\boldsymbol{k}}e^{-iE_{\boldsymbol{k}}t} + \Big(|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2\Big)e^{iE_{\boldsymbol{k}}t}\Big] \end{aligned} \tag{70}$$

Or, in Fourier space,

$$
\begin{aligned}
G^R_{\uparrow\uparrow}(k) &= \int dt \; G^R_{\boldsymbol{k}}(t) e^{i\omega t} \\
&= -i \lim_{\eta\to+0} \int_0^\infty dt \; \left[u_{\boldsymbol{k}}^2 e^{-iE_{\boldsymbol{k}}t} + \Big(|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2 \Big) e^{iE_{\boldsymbol{k}}t} \right] e^{i\omega t - \eta t} \\
&= \lim_{\eta\to+0} \left(\frac{u_{\boldsymbol{k}}^2}{\omega - E_{\boldsymbol{k}} + i\eta} + \frac{|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2}{\omega + E_{\boldsymbol{k}} + i\eta} \right). \qquad (71)
\end{aligned}
$$

The imaginary part is related to the density of states.

$$
\begin{aligned}
&-\frac{1}{\pi} \sum_{\boldsymbol{k}} \mathrm{Im} G^R_{\uparrow\uparrow}(k) \\
&= -\frac{1}{\pi} \lim_{\eta\to+0} \sum_{\boldsymbol{k}} \left(\mathrm{Im} \frac{u_{\boldsymbol{k}}^2}{\omega - E_{\boldsymbol{k}} + i\eta} + \mathrm{Im} \frac{|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2}{\omega + E_{\boldsymbol{k}} + i\eta} \right) \\
&= \sum_{\boldsymbol{k}} \left[u_{\boldsymbol{k}}^2 \delta(\omega - E_{\boldsymbol{k}}) + \left(|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2 \right) \delta(\omega + E_{\boldsymbol{k}}) \right]. \qquad (72)
\end{aligned}
$$

$$
\left(G^R_{\boldsymbol{k}\sigma\sigma'} \right) = \begin{pmatrix} G^R_{\boldsymbol{k}\uparrow\uparrow} & G^R_{\boldsymbol{k}\uparrow\downarrow} \\ G^R_{\boldsymbol{k}\downarrow\uparrow} & G^R_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix}, \qquad (73)
$$

can be written down as follows.

$$
\begin{aligned}
iG^R_{\boldsymbol{k}\sigma\sigma'}(t) \;&=\; \theta(t) \left\langle \left\{ c_{\boldsymbol{k}\sigma}(t), c^\dagger_{\boldsymbol{k}\sigma'} \right\} \right\rangle \;=\; \theta(t) \left\langle c_{\boldsymbol{k}\sigma}(t) c^\dagger_{\boldsymbol{k}\sigma'} + c^\dagger_{\boldsymbol{k}\sigma'} c_{\boldsymbol{k}\sigma}(t) \right\rangle \\
&= \theta(t) \Big\langle \Big(u_{\boldsymbol{k}} \delta_{\sigma\uparrow} a_{\boldsymbol{k}\uparrow} e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}} \delta_{\sigma\downarrow} a_{\boldsymbol{k}\downarrow} e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\uparrow} a^\dagger_{-\boldsymbol{k}\uparrow} e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\downarrow} a^\dagger_{-\boldsymbol{k}\downarrow} e^{iE_{\boldsymbol{k}}t} \Big) \\
&\quad \times \Big(u_{\boldsymbol{k}} \delta_{\sigma'\uparrow} a^\dagger_{\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}} \delta_{\sigma'\downarrow} a^\dagger_{\boldsymbol{k}\downarrow} - v^*_{\boldsymbol{k}\sigma'\uparrow} a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\sigma'\downarrow} a_{-\boldsymbol{k}\downarrow} \Big) \\
&\qquad + \Big(u_{\boldsymbol{k}} \delta_{\sigma'\uparrow} a^\dagger_{\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}} \delta_{\sigma'\downarrow} a^\dagger_{\boldsymbol{k}\downarrow} - v^*_{\boldsymbol{k}\sigma'\uparrow} a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\sigma'\downarrow} a_{-\boldsymbol{k}\downarrow} \Big) \\
&\quad \times \Big(u_{\boldsymbol{k}} \delta_{\sigma\uparrow} a_{\boldsymbol{k}\uparrow} e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}} \delta_{\sigma\downarrow} a_{\boldsymbol{k}\downarrow} e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\uparrow} a^\dagger_{-\boldsymbol{k}\uparrow} e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\downarrow} a^\dagger_{-\boldsymbol{k}\downarrow} e^{iE_{\boldsymbol{k}}t} \Big) \Big\rangle \\
&= \theta(t) \Big(u_{\boldsymbol{k}}^2 \delta_{\sigma\uparrow} \delta_{\sigma'\uparrow} e^{-iE_{\boldsymbol{k}}t} \left\langle a_{\boldsymbol{k}\uparrow} a^\dagger_{\boldsymbol{k}\uparrow} \right\rangle + u_{\boldsymbol{k}}^2 \delta_{\sigma\downarrow} \delta_{\sigma'\downarrow} e^{-iE_{\boldsymbol{k}}t} \left\langle a_{\boldsymbol{k}\downarrow} a^\dagger_{\boldsymbol{k}\downarrow} \right\rangle \\
&\qquad + v_{\boldsymbol{k}\sigma\uparrow} v^*_{\boldsymbol{k}\sigma'\uparrow} e^{iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{-\boldsymbol{k}\uparrow} a_{-\boldsymbol{k}\uparrow} \right\rangle + v_{\boldsymbol{k}\sigma\downarrow} v^*_{\boldsymbol{k}\sigma'\downarrow} e^{iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{-\boldsymbol{k}\downarrow} a_{-\boldsymbol{k}\downarrow} \right\rangle \\
&\qquad\quad + u_{\boldsymbol{k}}^2 \delta_{\sigma\uparrow} \delta_{\sigma'\uparrow} e^{-iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{\boldsymbol{k}\uparrow} a_{\boldsymbol{k}\uparrow} \right\rangle + u_{\boldsymbol{k}}^2 \delta_{\sigma\downarrow} \delta_{\sigma'\downarrow} e^{-iE_{\boldsymbol{k}}t} \left\langle a^\dagger_{\boldsymbol{k}\downarrow} a_{\boldsymbol{k}\downarrow} \right\rangle \\
&\qquad + v_{\boldsymbol{k}\sigma\uparrow} v^*_{\boldsymbol{k}\sigma'\uparrow} e^{iE_{\boldsymbol{k}}t} \left\langle a_{-\boldsymbol{k}\uparrow} a^\dagger_{-\boldsymbol{k}\uparrow} \right\rangle + v_{\boldsymbol{k}\sigma\downarrow} v^*_{\boldsymbol{k}\sigma'\downarrow} e^{iE_{\boldsymbol{k}}t} \left\langle a_{-\boldsymbol{k}\downarrow} a^\dagger_{-\boldsymbol{k}\downarrow} \right\rangle \Big) \\
&= \theta(t) \left[u_{\boldsymbol{k}}^2 \Big(\delta_{\sigma\uparrow} \delta_{\sigma'\uparrow} + \delta_{\sigma\downarrow} \delta_{\sigma'\downarrow} \Big) e^{-iE_{\boldsymbol{k}}t} + \Big(v_{\boldsymbol{k}\sigma\uparrow} v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow} v^*_{\boldsymbol{k}\sigma'\downarrow} \Big) e^{iE_{\boldsymbol{k}}t} \right] \\
&= \theta(t) \left[u_{\boldsymbol{k}}^2 \delta_{\sigma\sigma'} e^{-iE_{\boldsymbol{k}}t} + \Big(v_{\boldsymbol{k}\sigma\uparrow} v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow} v^*_{\boldsymbol{k}\sigma'\downarrow} \Big) e^{iE_{\boldsymbol{k}}t} \right], \qquad (74)
\end{aligned}
$$

The Fourier transform gives:

$$G^R_{\sigma\sigma'}(k) \;=\; \int dt\ G^R_{\boldsymbol{k}\sigma\sigma'}(t)e^{i\omega t}$$

$$= -i \lim_{\eta\to+0}\int_0^\infty dt\ \left[u_{\boldsymbol{k}}^2\delta_{\sigma\sigma'}e^{-iE_{\boldsymbol{k}}t} + \left(v_{\boldsymbol{k}\sigma\uparrow}v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}v^*_{\boldsymbol{k}\sigma'\downarrow}\right)e^{iE_{\boldsymbol{k}}t}\right]e^{i\omega t-\eta t}$$

$$= \lim_{\eta\to+0}\left[\frac{u_{\boldsymbol{k}}^2\delta_{\sigma\sigma'}}{\omega - E_{\boldsymbol{k}} + i\eta} + \frac{v_{\boldsymbol{k}\sigma\uparrow}v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}v^*_{\boldsymbol{k}\sigma'\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta}\right], \tag{75}$$

$$-\frac{1}{\pi}\sum_{\boldsymbol{k}}\mathrm{Im}G^R_{\sigma\sigma'}(k)$$

$$= -\frac{1}{\pi}\lim_{\eta\to+0}\sum_{\boldsymbol{k}}\left[\mathrm{Im}\frac{u_{\boldsymbol{k}}^2\delta_{\sigma\sigma'}}{\omega - E_{\boldsymbol{k}} + i\eta} + \mathrm{Im}\frac{v_{\boldsymbol{k}\sigma\uparrow}v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}v^*_{\boldsymbol{k}\sigma'\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta}\right]$$

$$= \sum_{\boldsymbol{k}}\left[u_{\boldsymbol{k}}^2\delta_{\sigma\sigma'}\delta(\omega - E_{\boldsymbol{k}}) + \left(v_{\boldsymbol{k}\sigma\uparrow}v^*_{\boldsymbol{k}\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}v^*_{\boldsymbol{k}\sigma'\downarrow}\right)\delta(\omega + E_{\boldsymbol{k}})\right]. \tag{76}$$

From the above, we obtain:

$$\begin{pmatrix} G^R_{\uparrow\uparrow}(k) & G^R_{\uparrow\downarrow}(k) \\ G^R_{\downarrow\uparrow}(k) & G^R_{\downarrow\downarrow}(k) \end{pmatrix}$$

$$= \lim_{\eta\to+0}\begin{pmatrix} \dfrac{u_{\boldsymbol{k}}^2}{\omega - E_{\boldsymbol{k}} + i\eta} + \dfrac{|v_{\boldsymbol{k}\uparrow\uparrow}|^2 + |v_{\boldsymbol{k}\uparrow\downarrow}|^2}{\omega + E_{\boldsymbol{k}} + i\eta} & \dfrac{v_{\boldsymbol{k}\uparrow\uparrow}v^*_{\boldsymbol{k}\downarrow\uparrow} + v_{\boldsymbol{k}\uparrow\downarrow}v^*_{\boldsymbol{k}\downarrow\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta} \\ \dfrac{v_{\boldsymbol{k}\downarrow\uparrow}v^*_{\boldsymbol{k}\uparrow\uparrow} + v_{\boldsymbol{k}\downarrow\downarrow}v^*_{\boldsymbol{k}\uparrow\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta} & \dfrac{u_{\boldsymbol{k}}^2}{\omega - E_{\boldsymbol{k}} + i\eta} + \dfrac{|v_{\boldsymbol{k}\downarrow\uparrow}|^2 + |v_{\boldsymbol{k}\downarrow\downarrow}|^2}{\omega + E_{\boldsymbol{k}} + i\eta} \end{pmatrix}. \tag{77}$$

This result can be reduced to the singlet case by setting $(v_{\boldsymbol{k}\sigma\sigma'}) = v_{\boldsymbol{k}}\hat{\sigma}^x$ and $(v^*_{\boldsymbol{k}\sigma\sigma'}) = v^*_{\boldsymbol{k}}\hat{\sigma}^x$.

$$\begin{pmatrix} G^R_{\uparrow\uparrow}(k) & G^R_{\uparrow\downarrow}(k) \\ G^R_{\downarrow\uparrow}(k) & G^R_{\downarrow\downarrow}(k) \end{pmatrix} = \lim_{\eta\to+0}\left(\frac{u_{\boldsymbol{k}}^2}{\omega - E_{\boldsymbol{k}} + i\eta} + \frac{|v_{\boldsymbol{k}}|^2}{\omega + E_{\boldsymbol{k}} + i\eta}\right)\hat{1}_{2\times2}. \tag{78}$$

The retarded anomalous Green's function in spin space is:

$$\left(F^R_{\boldsymbol{k}\sigma\sigma'}\right) = \begin{pmatrix} F^R_{\boldsymbol{k}\uparrow\uparrow} & F^R_{\boldsymbol{k}\uparrow\downarrow} \\ F^R_{\boldsymbol{k}\downarrow\uparrow} & F^R_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix}, \tag{79}$$

and,

$$
\begin{aligned}
iF^R_{\boldsymbol{k}\sigma\sigma'}(t) \quad &= \quad \theta(t)\left\langle\{c_{\boldsymbol{k}\sigma}(t), c_{-\boldsymbol{k}\sigma'}\}\right\rangle \quad = \quad \theta(t)\left\langle c_{\boldsymbol{k}\sigma}(t)c_{-\boldsymbol{k}\sigma'} + c_{-\boldsymbol{k}\sigma'}c_{\boldsymbol{k}\sigma}(t)\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \theta(t)\Big\langle\Big(u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\Big) \\
&\quad \times \Big(v_{\boldsymbol{k}\sigma'\uparrow}a^\dagger_{\boldsymbol{k}\uparrow} + v_{\boldsymbol{k}\sigma'\downarrow}a^\dagger_{\boldsymbol{k}\downarrow} + u_{\boldsymbol{k}}\delta_{\sigma'\uparrow}a_{-\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}}\delta_{\sigma'\downarrow}a_{-\boldsymbol{k}\downarrow}\Big) \\
&\qquad + \Big(v_{\boldsymbol{k}\sigma'\uparrow}a^\dagger_{\boldsymbol{k}\uparrow} + v_{\boldsymbol{k}\sigma'\downarrow}a^\dagger_{\boldsymbol{k}\downarrow} + u_{\boldsymbol{k}}\delta_{\sigma'\uparrow}a_{-\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}}\delta_{\sigma'\downarrow}a_{-\boldsymbol{k}\downarrow}\Big) \\
&\quad \times \Big(u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} - v_{\boldsymbol{k}\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\Big)\Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \theta(t)\Big(u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a^\dagger_{\boldsymbol{k}\uparrow}\right\rangle + u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\downarrow}a^\dagger_{\boldsymbol{k}\downarrow}\right\rangle \\
&\qquad - u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow}e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\uparrow}a_{-\boldsymbol{k}\uparrow}\right\rangle - u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}\right\rangle \\
&\qquad + u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}\right\rangle + u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\downarrow}a_{\boldsymbol{k}\downarrow}\right\rangle \\
&\qquad - u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow}e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}\right\rangle - u_{\boldsymbol{k}}v_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}\right\rangle\Big)
\end{aligned}
$$

$$
= u_{\boldsymbol{k}}\theta(t)\left[\Big(v_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}\Big)e^{-iE_{\boldsymbol{k}}t} - \Big(v_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}\Big)e^{iE_{\boldsymbol{k}}t}\right], \tag{80}
$$

$$
F^R_{\sigma\sigma'}(k) \;=\; \int dt\; F^R_{\boldsymbol{k}\sigma\sigma'}(t)e^{i\omega t}
$$

$$
\begin{aligned}
&= -iu_{\boldsymbol{k}}\lim_{\eta\to+0}\int_0^\infty dt\left[\Big(v_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}\Big)e^{-iE_{\boldsymbol{k}}t} - \Big(v_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}\Big)e^{iE_{\boldsymbol{k}}t}\right]e^{i\omega t-\eta t} \\
&= -u_{\boldsymbol{k}}\lim_{\eta\to+0}\left[\frac{v_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}}{\omega - E_{\boldsymbol{k}} + i\eta} - \frac{v_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta}\right].
\end{aligned} \tag{81}
$$

We obtain:

$$
\begin{aligned}
&\begin{pmatrix} F^R_{\uparrow\uparrow}(k) & F^R_{\uparrow\downarrow}(k) \\ F^R_{\downarrow\uparrow}(k) & F^R_{\downarrow\downarrow}(k) \end{pmatrix} \\
&= -u_{\boldsymbol{k}}\lim_{\eta\to+0}\left[\frac{1}{\omega - E_{\boldsymbol{k}} + i\eta}\begin{pmatrix} v_{\boldsymbol{k}\uparrow\uparrow} & v_{\boldsymbol{k}\downarrow\uparrow} \\ v_{\boldsymbol{k}\uparrow\downarrow} & v_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix} - \frac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\begin{pmatrix} v_{\boldsymbol{k}\uparrow\uparrow} & v_{\boldsymbol{k}\uparrow\downarrow} \\ v_{\boldsymbol{k}\downarrow\uparrow} & v_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix}\right].
\end{aligned} \tag{82}
$$

This result can be reduced to the singlet case by setting $(v_{\boldsymbol{k}\sigma\sigma'}) = v_{\boldsymbol{k}}\hat{\sigma}^x$ and $(v^*_{\boldsymbol{k}\sigma\sigma'}) = v^*_{\boldsymbol{k}}\hat{\sigma}^x$.

$$
\begin{aligned}
&\begin{pmatrix} F^R_{\uparrow\uparrow}(k) & F^R_{\uparrow\downarrow}(k) \\ F^R_{\downarrow\uparrow}(k) & F^R_{\downarrow\downarrow}(k) \end{pmatrix} \\
&= -u_{\boldsymbol{k}}v_{\boldsymbol{k}}\lim_{\eta\to+0}\left(\frac{1}{\omega - E_{\boldsymbol{k}} + i\eta} - \frac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\right)\hat{\sigma}^x.
\end{aligned} \tag{83}
$$

The retarded anti-anomalous Green's function in spin space is:

$$
\left(\bar{F}^R_{\boldsymbol{k}\sigma\sigma'}\right) = \begin{pmatrix} \bar{F}^R_{\boldsymbol{k}\uparrow\uparrow} & \bar{F}^R_{\boldsymbol{k}\uparrow\downarrow} \\ \bar{F}^R_{\boldsymbol{k}\downarrow\uparrow} & \bar{F}^R_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix}, \tag{84}
$$

obtained as follows.

$$
\begin{aligned}
i\bar{F}^R_{\boldsymbol{k}\sigma\sigma'}(t) \quad &= \quad \theta(t)\left\langle\left\{c^\dagger_{-\boldsymbol{k}\sigma}(t), c^\dagger_{\boldsymbol{k}\sigma'}\right\}\right\rangle \quad = \quad \theta(t)\left\langle c^\dagger_{-\boldsymbol{k}\sigma}(t)c^\dagger_{\boldsymbol{k}\sigma'} + c^\dagger_{\boldsymbol{k}\sigma'}c^\dagger_{-\boldsymbol{k}\sigma}(t)\right\rangle \\
&= \theta(t)\left\langle\left(v^*_{\boldsymbol{k}\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + v^*_{\boldsymbol{k}\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\right)\right. \\
&\quad \times\left(u_{\boldsymbol{k}}\delta_{\sigma'\uparrow}a^\dagger_{\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}}\delta_{\sigma'\downarrow}a^\dagger_{\boldsymbol{k}\downarrow} - v^*_{\boldsymbol{k}\sigma'\uparrow}a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\sigma'\downarrow}a_{-\boldsymbol{k}\downarrow}\right) \\
&\quad + \left(u_{\boldsymbol{k}}\delta_{\sigma'\uparrow}a^\dagger_{\boldsymbol{k}\uparrow} + u_{\boldsymbol{k}}\delta_{\sigma'\downarrow}a^\dagger_{\boldsymbol{k}\downarrow} - v^*_{\boldsymbol{k}\sigma'\uparrow}a_{-\boldsymbol{k}\uparrow} - v^*_{\boldsymbol{k}\sigma'\downarrow}a_{-\boldsymbol{k}\downarrow}\right) \\
&\quad \left.\times\left(v^*_{\boldsymbol{k}\sigma\uparrow}a_{\boldsymbol{k}\uparrow}e^{-iE_{\boldsymbol{k}}t} + v^*_{\boldsymbol{k}\sigma\downarrow}a_{\boldsymbol{k}\downarrow}e^{-iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}e^{iE_{\boldsymbol{k}}t} + u_{\boldsymbol{k}}\delta_{\sigma\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}e^{iE_{\boldsymbol{k}}t}\right)\right\rangle \\
&= \theta(t)\left(v^*_{\boldsymbol{k}\sigma\uparrow}u_{\boldsymbol{k}}\delta_{\sigma'\uparrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\uparrow}a^\dagger_{\boldsymbol{k}\uparrow}\right\rangle + v^*_{\boldsymbol{k}\sigma\downarrow}u_{\boldsymbol{k}}\delta_{\sigma'\downarrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a_{\boldsymbol{k}\downarrow}a^\dagger_{\boldsymbol{k}\downarrow}\right\rangle\right. \\
&\quad -u_{\boldsymbol{k}}\delta_{\sigma\uparrow}v^*_{\boldsymbol{k}\sigma'\uparrow}e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\uparrow}a_{-\boldsymbol{k}\uparrow}\right\rangle - u_{\boldsymbol{k}}\delta_{\sigma\downarrow}v^*_{\boldsymbol{k}\sigma'\downarrow}e^{iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{-\boldsymbol{k}\downarrow}a_{-\boldsymbol{k}\downarrow}\right\rangle \\
&\quad + u_{\boldsymbol{k}}v^*_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\uparrow}a_{\boldsymbol{k}\uparrow}\right\rangle + u_{\boldsymbol{k}}v^*_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}e^{-iE_{\boldsymbol{k}}t}\left\langle a^\dagger_{\boldsymbol{k}\downarrow}a_{\boldsymbol{k}\downarrow}\right\rangle \\
&\quad \left. - v^*_{\boldsymbol{k}\sigma'\uparrow}u_{\boldsymbol{k}}\delta_{\sigma\uparrow}e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\uparrow}a^\dagger_{-\boldsymbol{k}\uparrow}\right\rangle - v^*_{\boldsymbol{k}\sigma'\downarrow}u_{\boldsymbol{k}}\delta_{\sigma\downarrow}e^{iE_{\boldsymbol{k}}t}\left\langle a_{-\boldsymbol{k}\downarrow}a^\dagger_{-\boldsymbol{k}\downarrow}\right\rangle\right) \\
&= u_{\boldsymbol{k}}\theta(t)\left[\left(v^*_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v^*_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}\right)e^{-iE_{\boldsymbol{k}}t} - \left(v^*_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v^*_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}\right)e^{iE_{\boldsymbol{k}}t}\right], \qquad (85)
\end{aligned}
$$

$$
\begin{aligned}
&\bar{F}^R_{\sigma\sigma'}(k) \\
&= \int dt\ \bar{F}^R_{\boldsymbol{k}\sigma\sigma'}(t)e^{i\omega t} \\
&= -iu_{\boldsymbol{k}}\lim_{\eta\to+0}\int_0^\infty dt\left[\left(v^*_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v^*_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}\right)e^{-iE_{\boldsymbol{k}}t} - \left(v^*_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v^*_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}\right)e^{iE_{\boldsymbol{k}}t}\right]e^{i\omega t-\eta t} \\
&= -u_{\boldsymbol{k}}\lim_{\eta\to+0}\left[\frac{v^*_{\boldsymbol{k}\sigma\uparrow}\delta_{\sigma'\uparrow} + v^*_{\boldsymbol{k}\sigma\downarrow}\delta_{\sigma'\downarrow}}{\omega - E_{\boldsymbol{k}} + i\eta} - \frac{v^*_{\boldsymbol{k}\sigma'\uparrow}\delta_{\sigma\uparrow} + v^*_{\boldsymbol{k}\sigma'\downarrow}\delta_{\sigma\downarrow}}{\omega + E_{\boldsymbol{k}} + i\eta}\right]. \qquad (86)
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
&\begin{pmatrix} \bar{F}^R_{\uparrow\uparrow}(k) & \bar{F}^R_{\uparrow\downarrow}(k) \\ \bar{F}^R_{\downarrow\uparrow}(k) & \bar{F}^R_{\downarrow\downarrow}(k) \end{pmatrix} \\
&= -u_{\boldsymbol{k}}\lim_{\eta\to+0}\left[\frac{1}{\omega - E_{\boldsymbol{k}} + i\eta}\begin{pmatrix} v^*_{\boldsymbol{k}\uparrow\uparrow} & v^*_{\boldsymbol{k}\uparrow\downarrow} \\ v^*_{\boldsymbol{k}\downarrow\uparrow} & v^*_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix} - \frac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\begin{pmatrix} v^*_{\boldsymbol{k}\uparrow\uparrow} & v^*_{\boldsymbol{k}\downarrow\uparrow} \\ v^*_{\boldsymbol{k}\uparrow\downarrow} & v^*_{\boldsymbol{k}\downarrow\downarrow} \end{pmatrix}\right]. \qquad (87)
\end{aligned}
$$

This result can be reduced to the singlet case by setting $(v_{\boldsymbol{k}\sigma\sigma'}) = v_{\boldsymbol{k}}\hat{\sigma}^x$ and $(v^*_{\boldsymbol{k}\sigma\sigma'}) = v^*_{\boldsymbol{k}}\hat{\sigma}^x$.

$$
\begin{pmatrix} \bar{F}^R_{\uparrow\uparrow}(k) & \bar{F}^R_{\uparrow\downarrow}(k) \\ \bar{F}^R_{\downarrow\uparrow}(k) & \bar{F}^R_{\downarrow\downarrow}(k) \end{pmatrix} = -u_{\boldsymbol{k}}v^*_{\boldsymbol{k}}\lim_{\eta\to+0}\left(\frac{1}{\omega - E_{\boldsymbol{k}} + i\eta} - \frac{1}{\omega + E_{\boldsymbol{k}} + i\eta}\right)\hat{\sigma}^x. \qquad (88)
$$