Dirty limit. Quasiclassical Green function θ-parameterization

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The Usadel equation in zero magnetic field is

$$iD\vec{\nabla}(\check{g}\vec{\nabla}\check{g}) + \check{H}_0\check{g} - \check{g}\check{H}_0 = 0, \tag{1}$$

where the quasiclassical Nambu-Green function $\check{g}$ and the non-perturbative Hamiltonian $\check{H}_0$ are respectively

$$\check{g} \;=\; \begin{pmatrix} g & f \\ -f^{\dagger} & -g \end{pmatrix}, \qquad \check{H}_0 \;=\; \begin{pmatrix} -i\omega_n & -\Delta \\ \Delta^* & i\omega_n \end{pmatrix}. \tag{2}$$

g (f) is the (anomalous) Green function. Δ is a constant superconducting gap.

Especially, in homogeneous state, Green functions can be written as

$$g \;=\; -\frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}}, \qquad f \;=\; \frac{\Delta}{i\sqrt{\omega_n^2 + |\Delta|^2}}. \tag{3}$$

Matsubara frequency ω_n is able to be extended (retarded) analytical continuation: $i\omega_n \to E + i\eta$

where superconducting excitation energy $E = \sqrt{\varepsilon^2 + |\Delta|^2}$ is real and η is infinitesimal positive number.

$\check{g}$ satisfy the condition of 2-dimensional rotation matrix that $\mathrm{Tr}\check{g} = 0$ and $\det\check{g} = 1$.

Therefore $\check{g}$ can be parameterized by $\theta(x)$:

$$\check{g}(x) \;=\; \begin{pmatrix} \cos[\theta(x)] & \sin[\theta(x)]e^{i\chi} \\ \sin[\theta(x)]e^{-i\chi} & -\cos[\theta(x)] \end{pmatrix}. \tag{4}$$

In zero-field, χ is constant. It can be put $\chi = 0$.

The (2,1) element of the Usadel equation in the nomal metal has the form

$$-iD\frac{\partial}{\partial x}\Big\{\cos[\theta(x)]\frac{\partial}{\partial x}\big(-\sin[\theta(x)]\big) + \sin[\theta(x)]\frac{\partial}{\partial x}\cos[\theta(x)]\Big\} = 2i\omega_n\sin[\theta(x)]. \tag{5}$$

Further, there are useful relations

$$\frac{\partial}{\partial x}\sin[\theta(x)] \;=\; \cos[\theta(x)]\frac{\partial\theta(x)}{\partial x}, \qquad \frac{\partial}{\partial x}\cos[\theta(x)] \;=\; -\sin[\theta(x)]\frac{\partial\theta(x)}{\partial x}. \tag{6}$$

After analytical continuation, the obtained form becomes much easier

$$D\frac{\partial^2\theta(x)}{\partial x^2} + 2iE\sin[\theta(x)] = 0. \tag{7}$$