Foreign Exchange and Numeraire

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Abstract

These are my self-study notes for Chapter 3 of 'Financial Calculus: An Introduction to Derivative Pricing' by Martin Baxter and Andrew Rennie. Written on June 3, 2020.

The fundamental assets in the foreign exchange market are currencies.

Holding currency, just like holding a stock, involves risk.

For example, the exchange rate for one Japanese yen to US dollars, like a stock price, fluctuates from moment to moment.

This inherent risk gives rise to the demand for derivatives.

1 The Black-Scholes Currency Model

Let's denote the US dollar bond as B_t, the Japanese yen bond as D_t, and the exchange rate as C_t (where 1 JPY $= C_t$ USD).

The Black-Scholes currency model then gives us the following equations:

$$
\begin{aligned}
B_t &= e^{rt} \\
D_t &= e^{ut} \\
C_t &= C_0 \exp(\sigma W_t + \mu t)
\end{aligned}
$$

Here, W_t is a $\mathbb{P}$-Brownian motion, and r, u, σ, μ are constants.

1.1 For the US Dollar-Based Investor

A US dollar-based investor can trade two types of assets: the US dollar bond B_t and the yen bond converted to dollars, $C_t D_t$. Just as in the standard Black-Scholes model for stocks and bonds, a replicating portfolio can be constructed.

While C_t represents the dollar price of one yen, the dollar-based investor cannot trade the yen in its raw cash form. If this were possible, it would create an arbitrage opportunity against holding yen bonds. Cash has a zero interest rate, while the yen bond yields an interest rate of u. As a result, market participants could make infinite profits by going long on an arbitrary amount of yen bonds and shorting the cash itself.

The asset $C_t D_t$ is tradable in dollars. It represents the dollar-denominated price of the yen bond, D_t.

With these two stochastic processes, B_t and $C_t D_t$, we can construct a replicating portfolio.

1.1.1 Constructing the Replicating Portfolio

Using the tradable assets B_t and $C_t D_t$, we can construct a replicating portfolio for a contract X and determine its price using the no-arbitrage principle. This is done in three main steps:

1. Find a measure $\mathbb{Q}^\$$ under which the process for the yen bond discounted by the dollar bond, $Z_t = B_t^{-1} C_t D_t$, becomes a martingale.
2. Transform the contract X into a process $E_t = \mathbb{E}_{\mathbb{Q}^\$}(B_T^{-1} X | \mathcal{F}_t)$.
3. Find a predictable process ϕ_t such that $dE_t = \phi_t dZ_t$.

The process for the yen bond discounted by the dollar bond, Z_t, is:

$$\begin{aligned} Z_t &= B_t^{-1} C_t D_t \\ &= e^{-rt} e^{ut} C_0 \exp(\sigma W_t + \mu t) \\ &= C_0 \exp\left[\sigma W_t + (\mu + u - r)t\right] \end{aligned}$$

Its stochastic differential is:

$$\begin{aligned} dZ_t &= \left(\frac{\partial Z_t}{\partial t}\right) dt + \left(\frac{\partial Z_t}{\partial x}\right) dW_t + \frac{1}{2!}\left(\frac{\partial^2 Z_t}{\partial x^2}\right)(dW_t)^2 \\ \frac{dZ_t}{Z_t} &= \left(\mu + u - r + \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \end{aligned}$$

Now, we apply Girsanov's theorem.

$$\gamma \;=\; \frac{\mu + u - r + \frac{1}{2}\sigma^2}{\sigma}$$

By using this γ, the stochastic differential for the Brownian motion $W_t^\$$ under the measure $\mathbb{Q}^\$$ that makes Z_t a martingale should be:

$$dW_t^\$ = dW_t + \gamma dt$$

Furthermore, according to the Radon-Nikodym theorem, such a measure $\mathbb{Q}^\$$ is defined by:

$$\begin{aligned} \frac{d\mathbb{Q}^\$}{d\mathbb{P}} &= \exp\left(-\int_0^T \gamma dW_t - \frac{1}{2}\int_0^T \gamma^2 dt\right) \\ &= \exp\left(-\gamma W_T - \frac{1}{2}\gamma^2 T\right) \end{aligned}$$

Under this measure $\mathbb{Q}^\$$, we have:

$$\begin{aligned} \frac{dZ_t}{Z_t} &= \sigma dW_t^\$ \\ Z_t &= Z_0 \exp\left(\int_0^t \sigma dW_s^\$ - \frac{1}{2}\int_0^t \sigma^2 ds\right) \\ &= C_0 \exp\left(\sigma W_t^\$ - \frac{1}{2}\sigma^2 t\right) \\ C_t &= B_t Z_t D_t^{-1} \\ &= e^{rt} C_0 \exp\left(\sigma W_t^\$ - \frac{1}{2}\sigma^2 t\right) e^{-ut} \\ &= C_0 \exp\left[\sigma W_t^\$ + \left(r - u - \frac{1}{2}\sigma^2\right) t\right] \end{aligned}$$

This holds.

We define the conditional expectation under the measure $\mathbb{Q}^{\$}$ and the filtration $\mathcal{F}_t$ as:

$$E_t = \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1}X|\mathcal{F}_t)$$

For $s(<t)$, we get:

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}^{\$}}(E_t|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{Q}^{\$}}\left(\mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1}X|\mathcal{F}_t)\Big|\mathcal{F}_s\right) \\ &= \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1}X|\mathcal{F}_s) \\ &= E_s\end{aligned}$$

So, E_t is a $\mathbb{Q}^{\$}$-martingale. By the martingale representation theorem, a predictable process exists such that:

$$dE_t = \phi_t dZ_t$$

We want to find the holdings of dollar-denominated currency, $S_t = C_t D_t$, and the holdings of the dollar bond, B_t, needed to construct the replicating portfolio at time t. Let these be ϕ_t and ψ_t, respectively.

The value of the replicating portfolio, V_t, is:

$$V_t = \phi_t S_t + \psi_t B_t$$

At maturity, the portfolio is identical to the contract, so:

$$X = \phi_T S_T + \psi_T B_T$$

The $\mathbb{Q}^{\$}$-martingale E_t we constructed earlier becomes at $t = T$:

$$\begin{aligned}E_T &= \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1}X|\mathcal{F}_T) \\ &= B_T^{-1}X\end{aligned}$$

Which means:

$$B_T E_T = X = \phi_T S_T + \psi_T B_T$$

If this equality, $B_t E_t = V_t = \phi_t S_t + \psi_t B_t$, holds for all t, not just at T, then the dollar bond holdings ψ_t of the replicating portfolio can be found by rearranging the equation:

$$\psi_t = E_t - \phi_t Z_t$$

Let's check if this assumption is correct.

Taking the stochastic differential of $V_t = B_t E_t$, and using the facts that $dE_t = \phi_t dZ_t$ and $E_t = \phi_t Z_t + \psi_t$, we get:

$$\begin{aligned}dV_t &= B_t dE_t + E_t dB_t \\ &= B_t(\phi_t dZ_t) + (\phi_t Z_t + \psi_t)dB_t \\ &= \phi_t(B_t dZ_t + Z_t dB_t) + \psi_t dB_t \\ &= \phi_t dS_t + \psi_t dB_t\end{aligned}$$

The portfolio is indeed self-financing. This confirms that we can construct a replicating portfolio by assuming dollar bond holdings are $\psi_t = E_t - \phi_t Z_t$.

The value of the portfolio that replicates contract X is V_t. We've found that it can be expressed using the measure $\mathbb{Q}^{\$}$ that makes the dollar-denominated currency price Z_t a martingale, as follows:

$$V_t = B_t E_t = B_t \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1}X|\mathcal{F}_t)$$

1.2 The Forward Contract

Let's consider a contract to buy one yen for k dollars at a future time $T(> t)$.

The payoff at time T is:

$$\begin{aligned} X &= C_T - k \\ &= C_0 \exp(\sigma W_T + \mu T) - k \\ &= C_0 \exp\left[\sigma W_T^{\$} + \left(r - u - \frac{1}{2}\sigma^2\right) T\right] - k \end{aligned}$$

The value at any time t is:

$$\begin{aligned} V_t &= B_t \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1} X | \mathcal{F}_t) \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^{\$}}(C_T - k | \mathcal{F}_t) \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^{\$}}\left(C_0 \exp\left[\sigma W_T^{\$} + \left(r - u - \frac{1}{2}\sigma^2\right) T\right] - k \middle| \mathcal{F}_t\right) \end{aligned}$$

The value of the contract at time zero (the present) should be zero under the no-arbitrage condition.

$$\begin{aligned} 0 &= V_0 \\ &= e^{-rT} \mathbb{E}_{\mathbb{Q}^{\$}} C_0 \exp\left[\sigma W_T^{\$} + \left(r - u - \frac{1}{2}\sigma^2\right) T\right] - e^{-rT} k \end{aligned}$$

Therefore, the no-arbitrage delivery price F (the value of k for which $V_0 = 0$ at $t = 0$) is:

$$\begin{aligned} F &= \mathbb{E}_{\mathbb{Q}^{\$}} C_0 \exp\left[\sigma W_T^{\$} + \left(r - u - \frac{1}{2}\sigma^2\right) T\right] \\ &= C_0 \exp\left[\left(r - u - \frac{1}{2}\sigma^2\right) T\right] \times \mathbb{E}_{\mathbb{Q}^{\$}} \exp \sigma W_T^{\$} \\ &= C_0 \exp\left[\left(r - u - \frac{1}{2}\sigma^2\right) T\right] \times \exp\left(\frac{1}{2}\sigma^2 T\right) \\ &= C_0 e^{(r-u)T} \end{aligned}$$

This value is equal to the yen-dollar exchange rate discounted by the interest rate difference between the two currencies.

Using F, we can also find the value of the forward contract at time t, V_t.

$$\begin{aligned} V_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^{\$}}(C_T - F | \mathcal{F}_t) \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}^{\$}}(C_T | \mathcal{F}_t) - e^{-r(T-t)} F \\ &= B_t \mathbb{E}_{\mathbb{Q}^{\$}}(B_T^{-1} C_T | \mathcal{F}_t) - e^{-r(T-t)} C_0 e^{(r-u)T} \\ &= C_t - e^{-uT} e^{rt} C_0 \\ &= e^{-uT}\left(e^{uT} C_t - e^{rt} C_0\right) \end{aligned}$$

The discounted value of the portfolio is:

$$\begin{aligned} E_t &= B_t^{-1} V_t \\ &= e^{-rt} e^{-uT}\left(e^{uT} C_t - e^{rt} C_0\right) \\ &= e^{-rt} C_t - e^{-uT} C_0 \\ &= e^{uT} Z_t - e^{-uT} C_0 \end{aligned}$$

[*1] The stochastic differential is $dE_t = e^{-uT} dZ_t$. The holdings of stock ϕ_t and bonds ψ_t needed to construct the replicating portfolio are constant with respect to t:

$$\phi_t = e^{-uT} = D_T^{-1}$$

$$\begin{aligned}
\psi_t &= E_t - \phi_t Z_t \\
&= (e^{uT} Z_t - e^{-uT} C_0) - e^{-uT} Z_t \\
&= -e^{-uT} C_0 \\
&= -D_T^{-1} C_0
\end{aligned}$$

[*2]

1.3 For the Japanese Yen-Based Investor

Unlike the US dollar-based investor, the Japanese yen-based investor is interested in the yen-denominated prices of tradable assets.

First, the yen bond $D_t = e^{ut}$ is tradable.

Furthermore, the dollar bond denominated in yen, $C_t^{-1} B_t$, is also tradable.

If we consider the exchange rate of one dollar to yen, C_t^{-1}, we have:

$$C_t^{-1} = C_0^{-1} \exp(-\sigma W_t - \mu t)$$

With these two assets, the yen bond D_t and the yen-denominated dollar bond $C_t^{-1} B_t$, we can replicate a risk-free portfolio.

The price of the dollar bond discounted by the yen bond is:

$$\begin{aligned}
Y_t &= D_t^{-1} C_t^{-1} B_t \\
&= e^{-ut} C_0^{-1} \exp(-\sigma W_t - \mu t) e^{rt} \\
&= C_0^{-1} \exp(-\sigma W_t - (\mu + u - r)t)
\end{aligned}$$

The stochastic differential is:

$$\begin{aligned}
dY_t &= \frac{\partial Y_t}{\partial t} dt + \frac{\partial Y_t}{\partial x} dW_t + \frac{1}{2!} \frac{\partial^2 Y_t}{\partial x^2} (dW_t)^2 \\
&= -(\mu + u - r) Y_t dt - \sigma Y_t dW_t + \frac{1}{2} \sigma^2 Y_t dt
\end{aligned}$$

$$\frac{dY_t}{Y_t} = -\sigma dW_t - \left(\mu + u - r + \frac{1}{2}\sigma^2 \right) dt$$

Therefore, for $W_t^{\tilde{\mathbb{Q}}}$ to be a $\tilde{\mathbb{Q}}$-Brownian motion, we need to introduce a new measure $\tilde{\mathbb{Q}}$ such that the discounted price Y_t becomes a martingale.

$$dW_t^{\tilde{\mathbb{Q}}} = dW_t + \frac{\mu + u - r + \frac{1}{2}\sigma^2}{\sigma} dt$$

$$W_t^{\tilde{\mathbb{Q}}} = W_t + \frac{\mu + u - r + \frac{1}{2}\sigma^2}{\sigma} t$$

[*1] The final equation isn't right. Working backward, we see $e^{uT} Z_t = e^{uT}(B_t^{-1} C_t D_t) = e^{uT} e^{-rt} C_t e^{ut}$. Some calculation must be incorrect.

[*2] The value for ψ_t doesn't match the textbook. The exponent sign for the coefficient of Z_t is different.

Option Pricing in the Yen World

A yen-denominated payoff X at time T has a value at time t of:

$$U_t \;=\; D_t \mathbb{E}_{\tilde{\mathbb{Q}}}(D_T^{-1} X | \mathcal{F}_t)$$

Here, $\tilde{\mathbb{Q}}$ is the martingale measure for Y_t, the asset value discounted by the yen bond.

1.4 Changing the Numeraire

A concern arises: will the US dollar-based investor and the Japanese yen-based investor value the same security differently?

In the dollar world, the value of a payoff X at time t is:

$$V_t \;=\; B_t \mathbb{E}_{\mathbb{Q}^\$}(B_T^{-1} X | \mathcal{F}_t)$$

The unit is dollars.

In the yen world, the same contract is a payment of $C_T^{-1} X$ yen, not X dollars. Therefore, its value at time t is:

$$U_t \;=\; D_t \mathbb{E}_{\tilde{\mathbb{Q}}}(D_T^{-1}(C_T^{-1} X) | \mathcal{F}_t)$$

The unit is yen.

Do these two values actually coincide?

Is the dollar-equivalent value of the price determined in the yen world, $C_t U_t$, equal to the original V_t?

The $\mathbb{Q}^\$$-Brownian motion $W_t^{\mathbb{Q}^\$}$ and the $\tilde{\mathbb{Q}}$-Brownian motion $W_t^{\tilde{\mathbb{Q}}}$ are expressed using the $\mathbb{P}$-Brownian motion W_t as follows:

$$\begin{aligned} W_t^{\mathbb{Q}^\$} &= W_t + \frac{\mu + u - r - \frac{1}{2}\sigma^2}{\sigma} t \\ W_t^{\tilde{\mathbb{Q}}} &= W_t + \frac{\mu + u - r + \frac{1}{2}\sigma^2}{\sigma} t \end{aligned}$$

This means:

$$\begin{aligned} W_t^{\tilde{\mathbb{Q}}} &= W_t^{\mathbb{Q}^\$} - \sigma t \\ dW_t^{\tilde{\mathbb{Q}}} &= dW_t^{\mathbb{Q}^\$} - \sigma dt \end{aligned}$$

Therefore, by the reverse of Girsanov's theorem, the Radon-Nikodym derivative must be:

$$\begin{aligned} \frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}^\$} &= \exp\left(- \int_0^T (-\sigma) dW_t^{\mathbb{Q}^\$} - \frac{1}{2}\int_0^T (-\sigma)^2 dt \right) \\ &= \exp\left(\sigma W_T^{\mathbb{Q}^\$} - \frac{1}{2}\sigma^2 T \right) \end{aligned}$$

Taking the conditional expectation of this Radon-Nikodym derivative under the measure $\mathbb{Q}^\$$ and the filtration $\mathcal{F}_t$, we get:

$$
\begin{aligned}
\xi_t &= \mathbb{E}_{Q^{\$}}\left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}^{\$}}\middle|\mathcal{F}_t\right)\\
&= \mathbb{E}_{Q^{\$}}\left[\exp\left(\sigma W_T^{\mathbb{Q}^{\$}} - \frac{1}{2}\sigma^2 T\right)\middle|\mathcal{F}_t\right]\\
&= \exp\left(-\frac{1}{2}\sigma^2 T\right)\\
&\times \mathbb{E}_{Q^{\$}}\left[\exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left\{\sigma\left(W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}\right)\right\}\middle|\mathcal{F}_t\right]
\end{aligned}
$$

The factor inside the expectation is:

$$
\begin{aligned}
&\exp\left\{\sigma\left(W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}\right)\right\}\\
&= \exp\left\{\sigma\sqrt{T-t}\frac{W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}}{\sqrt{T-t}}\right\}
\end{aligned}
$$

This factor,

$$
\frac{W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}}{\sqrt{T-t}}
$$

is a standard normal random variable following an $N(0,1)$ distribution under the measure $\mathbb{Q}^{\$}$. Let's call this variable Z.

The expectation can then be broken down into a product of a $\mathcal{F}_t$-measurable factor,

$$
\exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)
$$

and an $\mathcal{F}_t$-independent random variable,

$$
\exp\left(Z\sigma\sqrt{T-t}\right)
$$

Thus:

$$
\begin{aligned}
&\mathbb{E}_{Q^{\$}}\left[\exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left(\sigma\sqrt{T-t}\frac{W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}}{\sqrt{T-t}}\right)\middle|\mathcal{F}_t\right]\\
&= \mathbb{E}\left[\exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left(Z\sigma\sqrt{T-t}\right)\right]\\
&= \exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp\left(z\sigma\sqrt{T-t}\right)e^{-\frac{1}{2}z^2}dz\\
&= \exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left(\frac{1}{2}\left(\sigma\sqrt{T-t}\right)^2\right)
\end{aligned}
$$

From the above, we can derive:

$$
\begin{aligned}
\xi_t &= \mathbb{E}_{Q^{\$}}\left(\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}^{\$}}\middle|\mathcal{F}_t\right)\\
&= \exp\left(-\frac{1}{2}\sigma^2 T\right)\\
&\times \mathbb{E}_{Q^{\$}}\left[\exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left\{\sigma\left(W_T^{\mathbb{Q}^{\$}} - W_t^{\mathbb{Q}^{\$}}\right)\right\}\middle|\mathcal{F}_t\right]\\
&= \exp\left(-\frac{1}{2}\sigma^2 T\right)\\
&\times \exp\left(\sigma W_t^{\mathbb{Q}^{\$}}\right)\exp\left(\frac{1}{2}\left(\sigma\sqrt{T-t}\right)^2\right)\\
&= \exp\left(\sigma W_t^{\mathbb{Q}^{\$}} - \frac{1}{2}\sigma^2 t\right)
\end{aligned}
$$

We have already shown that the price of the yen bond discounted by the dollar bond is:

$$\begin{aligned} Z_t &= B_t^{-1} C_t D_t \\ &= C_0 \exp\left(\sigma W_t^{\mathbb{Q}^\$} - \frac{1}{2}\sigma^2 t\right) \\ &= C_0 \xi_t \end{aligned}$$

Therefore, the price of the yen bond discounted by the dollar bond, Z_t, is proportional to the Radon-Nikodym process ξ_t.

Using this process, the dollar-converted value of the price set in the yen world, $C_t U_t$ (where U_t is the price for the payoff $C_T^{-1} X$ described earlier), is:

$$\begin{aligned} C_t U_t &= C_t D_t \mathbb{E}_{\tilde{\mathbb{Q}}}(D_T^{-1}(C_T^{-1} X)|\mathcal{F}_t) \\ &= C_t D_t \xi_t^{-1} \mathbb{E}_{\mathbb{Q}^\$}(\xi_T^{-1} D_T^{-1} \quad C_T^{-1} X|\mathcal{F}_t) \\ &= B_t \mathbb{E}_{\mathbb{Q}^\$}(B_T^{-1} X|\mathcal{F}_t) \\ &= V_t \end{aligned}$$

This shows that a dollar-denominated payoff X at time T has the same value at any time $t(< T)$, regardless of whether it is priced from the dollar world or the yen world.

References

[1] Financial Calculus - An Introduction to Derivative Pricing - Martin Baxter, Andrew Rennie