Solutions to the Extended Gor'kov Equations for Anisotropic Superconductors and the Electronic Raman Response Function

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We begin with the BCS mean-field Hamiltonian $\mathcal{H}$.

$$\mathcal{H} = \mathcal{H}_0 \ + \ \mathcal{H}_{\rm BCS} \quad ,$$

$$\mathcal{H}_0 = \sum_{\boldsymbol{k},s} \varepsilon(\boldsymbol{k}) c^\dagger_{\boldsymbol{k}s} c_{\boldsymbol{k}s} \quad ,$$

$$\mathcal{H}_{\rm BCS} = \frac{1}{2} \sum_{\boldsymbol{k},s_1,s_2} \left[\Delta_{s_1 s_2}(\boldsymbol{k}) c^\dagger_{\boldsymbol{k}s_1} c^\dagger_{-\boldsymbol{k}s_2} \ - \ \Delta^*_{s_1 s_2}(-\boldsymbol{k}) c_{-\boldsymbol{k}s_1} c_{\boldsymbol{k}s_2} \right] \quad , \tag{1}$$

We introduce finite-temperature Green's functions in the Matsubara formalism as follows.

$$G_{ss'}(\boldsymbol{k},\boldsymbol{k}';\tau) = -\langle T_\tau \{ c_{\boldsymbol{k}s}(\tau) c^\dagger_{\boldsymbol{k}'s'}(0) \} \rangle \quad , \tag{2}$$

$$F_{ss'}(\boldsymbol{k},\boldsymbol{k}';\tau) = \langle T_\tau \{ c_{\boldsymbol{k}s}(\tau) c_{\boldsymbol{k}'s'}(0) \} \rangle \quad , \quad F^\dagger_{ss'}(\boldsymbol{k},\boldsymbol{k}';\tau) \ = \ \langle T_\tau \{ c^\dagger_{\boldsymbol{k}'s'}(\tau) c^\dagger_{\boldsymbol{k}s}(0) \} \rangle \quad . \tag{3}$$

Although we've written $F^\dagger_{ss'}$ here by convention, it's not the Hermitian conjugate of $F_{ss'}$. These Green's functions are c-numbers.

The Fourier transform from the time variable τ to $i\omega_n$ is defined as follows.

$$G_{ss'}(\boldsymbol{k},\boldsymbol{k}';\tau) = \frac{1}{\beta} \sum_n G_{ss'}(\boldsymbol{k},\boldsymbol{k}';i\omega_n) e^{-i\omega_n \tau} \quad ,$$

$$F^{(\dagger)}_{ss'}(\boldsymbol{k},\boldsymbol{k}';\tau) = \frac{1}{\beta} \sum_n F^{(\dagger)}_{ss'}(\boldsymbol{k},\boldsymbol{k}';i\omega_n) e^{-i\omega_n \tau} \quad . \tag{4}$$

Here, $\omega_n = (2n+1)\pi k_B T$ $(n \in \mathbb{Z})$ is the fermionic Matsubara frequency. For a homogeneous system, the momentum variables of these Green's functions become $\boldsymbol{k} = \boldsymbol{k}'$ for the G function and $\boldsymbol{k} = -\boldsymbol{k}'$ for the $F^{(\dagger)}$ function, and can be specified by a single momentum value.

$$G_{ss'}(\boldsymbol{k},i\omega_n) = -\int_0^\beta d\tau \langle T_\tau \{ c_{\boldsymbol{k}s}(\tau) c^\dagger_{\boldsymbol{k}s'}(0) \} \rangle e^{i\omega_n \tau} \quad , \tag{5}$$

$$F_{ss'}(\boldsymbol{k},i\omega_n) = \int_0^\beta d\tau \langle T_\tau \{ c_{\boldsymbol{k}s}(\tau) c_{-\boldsymbol{k}s'}(0) \} \rangle e^{i\omega_n \tau} \quad , \quad F^\dagger_{ss'}(\boldsymbol{k},i\omega_n) \ = \ \int_0^\beta d\tau \langle T_\tau \{ c^\dagger_{-\boldsymbol{k}s'}(\tau) c^\dagger_{\boldsymbol{k}s}(0) \} \rangle e^{i\omega_n \tau} \tag{6}$$

In the following, we only consider a homogeneous system.

To find the spectral representation of the Green's functions, we determine the time evolution of the operators. We use the Heisenberg equation of motion.

$$\partial_\tau c_{\boldsymbol{k}s}(\tau) = [\mathcal{H} \ , \ c_{\boldsymbol{k}s}(\tau)] \ = \ e^{\mathcal{H}\tau} \left[\mathcal{H} \ , \ c_{\boldsymbol{k}s} \right] e^{-\mathcal{H}\tau} \tag{7}$$

For this calculation, the relationship between commutators and anticommutators

$$[AB\ ,\ C] = A\{B\ ,\ C\} - \{A\ ,\ C\}B\ \ , \tag{8}$$

is also useful. For $\mathcal{H}_0$,

$$\begin{aligned}
[\mathcal{H}_0\ ,\ c_{\boldsymbol{k}s}] &= \sum_{\boldsymbol{k}',s'} \varepsilon(\boldsymbol{k}') \left[c^\dagger_{\boldsymbol{k}'s'} c_{\boldsymbol{k}'s'}\ ,\ c_{\boldsymbol{k}s} \right] \\
&= -\sum_{\boldsymbol{k}',s'} \varepsilon(\boldsymbol{k}') \left\{ c^\dagger_{\boldsymbol{k}'s'}\ ,\ c_{\boldsymbol{k}s} \right\} c_{\boldsymbol{k}'s'} \\
&= -\varepsilon(\boldsymbol{k}) c_{\boldsymbol{k}s}\ \ .
\end{aligned} \tag{9}$$

and next, for $\mathcal{H}_{\text{BCS}}$,

$$\begin{aligned}
[\mathcal{H}_{\text{BCS}}\ ,\ c_{\boldsymbol{k}s}] &= \left[\frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \left(\Delta_{s_1s_2}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_1} c^\dagger_{-\boldsymbol{k}'s_2}\ -\ \Delta^*_{s_1s_2}(-\boldsymbol{k}') c_{-\boldsymbol{k}'s_1} c_{\boldsymbol{k}'s_2} \right)\ ,\ c_{\boldsymbol{k}s} \right] \\
&= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') \left[c^\dagger_{\boldsymbol{k}'s_1} c^\dagger_{-\boldsymbol{k}'s_2}\ ,\ c_{\boldsymbol{k}s} \right] \\
&= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') \Big(c^\dagger_{\boldsymbol{k}'s_1} \left\{ c^\dagger_{-\boldsymbol{k}'s_2}\ ,\ c_{\boldsymbol{k}s} \right\} - \left\{ c^\dagger_{\boldsymbol{k}'s_1}\ ,\ c_{\boldsymbol{k}s} \right\} c^\dagger_{-\boldsymbol{k}'s_2} \Big) \\
&= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') \Big(c^\dagger_{\boldsymbol{k}'s_1} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_2}\ -\ c^\dagger_{-\boldsymbol{k}'s_2} \delta_{\boldsymbol{k},\boldsymbol{k}'} \delta_{s,s_1} \Big)\ \ .
\end{aligned} \tag{10}$$

By carefully swapping the indices, we can see that the two terms on the right-hand side are equal. Specifically, we should carefully perform the index swap on only the second term as follows:

$$\begin{aligned}
[\mathcal{H}_{\text{BCS}}\ ,\ c_{\boldsymbol{k}s}] &= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_1} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_2}\ -\ \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(-\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_2} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_1} \\
&= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_1} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_2}\ +\ \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_2s_1}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_2} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_1} \\
&= \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_1} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_2}\ +\ \frac{1}{2} \sum_{\boldsymbol{k}',s_1,s_2} \Delta_{s_1s_2}(\boldsymbol{k}') c^\dagger_{\boldsymbol{k}'s_1} \delta_{\boldsymbol{k},-\boldsymbol{k}'} \delta_{s,s_2} \\
&= \sum_{s'} \Delta_{s's}(-\boldsymbol{k}) c^\dagger_{-\boldsymbol{k}s'} \\
&= -\sum_{s'} \Delta_{ss'}(\boldsymbol{k}) c^\dagger_{-\boldsymbol{k}s'}\ \ .
\end{aligned} \tag{11}$$

From the two terms above, the time evolution of the operator is:

$$\partial_\tau c_{\boldsymbol{k}s}(\tau) = -\varepsilon(\boldsymbol{k}) c_{\boldsymbol{k}s}(\tau)\ -\ \sum_{s'} \Delta_{ss'}(\boldsymbol{k}) c^\dagger_{-\boldsymbol{k}s'}(\tau)\ \ . \tag{12}$$

Using this relation, we derive the equation of motion for the Green's functions. The time evolution of the Green's functions is:

$$\begin{aligned}
\partial_\tau G_{ss'}(\boldsymbol{k},\tau) &= \partial_\tau \Big(-\theta(\tau)\langle c_{\boldsymbol{k}s}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle \ + \ \theta(-\tau)\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}(\tau)\rangle\Big) \\
&= -[\partial_\tau\theta(\tau)]\langle c_{\boldsymbol{k}s}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle \ - \ \theta(\tau)\langle[\partial_\tau c_{\boldsymbol{k}s}(\tau)]c^\dagger_{\boldsymbol{k}s'}\rangle \\
&\quad + [\partial_\tau\theta(-\tau)]\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}(\tau)\rangle \ + \ \theta(-\tau)\langle c^\dagger_{\boldsymbol{k}s'}[\partial_\tau c_{\boldsymbol{k}s}(\tau)]\rangle \\
&= -\delta(\tau)\langle c_{\boldsymbol{k}s}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle \ + \ \theta(\tau)\varepsilon(\boldsymbol{k})\langle c_{\boldsymbol{k}s}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle \ + \ \theta(\tau)\sum_{s''}\Delta_{ss''}(\boldsymbol{k})\langle c^\dagger_{-\boldsymbol{k}s''}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle \\
&\quad - \delta(\tau)\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}(\tau)\rangle \ - \ \theta(-\tau)\varepsilon(\boldsymbol{k})\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}(\tau)\rangle \ - \ \theta(-\tau)\sum_{s''}\Delta_{ss''}(\boldsymbol{k})\langle c^\dagger_{\boldsymbol{k}s'}c^\dagger_{-\boldsymbol{k}s''}(\tau)\rangle \\
&= \varepsilon(\boldsymbol{k})G_{ss'}(\boldsymbol{k},\tau) \ + \ \sum_{s''}\Delta_{ss''}(\boldsymbol{k})F^\dagger_{s's''}(\boldsymbol{k},\tau) \qquad (13)
\end{aligned}$$

The terms proportional to the delta function cancel out due to the mechanism:

$$\delta(\tau)\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}(\tau)\rangle \ = \ \delta(0)\langle c^\dagger_{\boldsymbol{k}s'}c_{\boldsymbol{k}s}\rangle \ = \ -\delta(0)\langle c_{\boldsymbol{k}s}c^\dagger_{\boldsymbol{k}s'}\rangle \ = \ -\delta(\tau)\langle c_{\boldsymbol{k}s}(\tau)c^\dagger_{\boldsymbol{k}s'}\rangle$$

Fourier transforming both sides so that the variables become $\tau \to i\omega_n$:

$$\begin{aligned}
&\int_0^\beta d\tau\partial_\tau G_{ss'}(\boldsymbol{k},\tau)e^{i\omega_n\tau} = \int_0^\beta d\tau\varepsilon(\boldsymbol{k})G_{ss'}(\boldsymbol{k},\tau)e^{i\omega_n\tau} \ + \ \int_0^\beta d\tau\sum_{s''}\Delta_{s''s}(\boldsymbol{k})F^\dagger_{s's''}(\boldsymbol{k},\tau)e^{i\omega_n\tau} \\
\longleftrightarrow \quad & i\omega_n G_{ss'}(\boldsymbol{k},i\omega_n) = \varepsilon(\boldsymbol{k})G_{ss'}(\boldsymbol{k},i\omega_n) \ + \ \sum_{s''}\Delta_{ss''}(\boldsymbol{k})F^\dagger_{s's''}(\boldsymbol{k},i\omega_n) \qquad (14)
\end{aligned}$$

This equation is not closed for $G_{ss'}(\boldsymbol{k},i\omega_n)$, and we need to set up a similar equation of motion for $F^\dagger_{ss'}(\boldsymbol{k},i\omega_n)$ and solve them simultaneously. Taking the Hermitian conjugate of equation 12:

$$\partial_\tau c^\dagger_{\boldsymbol{k}s}(\tau) = -\varepsilon(\boldsymbol{k})c^\dagger_{\boldsymbol{k}s}(\tau) \ - \ \sum_{s'}\Delta^*_{ss'}(\boldsymbol{k})c_{-\boldsymbol{k}s'}(\tau) \quad , \qquad (15)$$

and using this relation:

$$\begin{aligned}
\partial_\tau F^\dagger_{ss'}(\boldsymbol{k},\tau) &= \partial_\tau\Big(\theta(\tau)\langle c^\dagger_{-\boldsymbol{k}s'}(\tau)c^\dagger_{\boldsymbol{k}s}\rangle \ - \ \theta(-\tau)\langle c^\dagger_{\boldsymbol{k}s}c_{-\boldsymbol{k}s'}(\tau)\rangle\Big) \\
&= [\partial_\tau\theta(\tau)]\langle c^\dagger_{-\boldsymbol{k}s'}(\tau)c^\dagger_{\boldsymbol{k}s}\rangle \ + \ \theta(\tau)\langle[\partial_\tau c^\dagger_{-\boldsymbol{k}s'}(\tau)]c^\dagger_{\boldsymbol{k}s}\rangle \\
&\quad - [\partial_\tau\theta(-\tau)]\langle c^\dagger_{\boldsymbol{k}s}c^\dagger_{-\boldsymbol{k}s'}(\tau)\rangle \ - \ \theta(-\tau)\langle c^\dagger_{\boldsymbol{k}s}[\partial_\tau c^\dagger_{-\boldsymbol{k}s'}(\tau)]\rangle \\
&= \delta(\tau)\langle c^\dagger_{-\boldsymbol{k}s'}(\tau)c^\dagger_{\boldsymbol{k}s}\rangle \ - \ \varepsilon(\boldsymbol{k})\theta(\tau)\langle c^\dagger_{-\boldsymbol{k}s'}(\tau)c^\dagger_{\boldsymbol{k}s}\rangle \ - \ \sum_{s''}\Delta^*_{s's''}(-\boldsymbol{k})\theta(\tau)\langle c_{\boldsymbol{k}s''}(\tau)c^\dagger_{\boldsymbol{k}s}\rangle \\
&\quad + \delta(\tau)\langle c^\dagger_{\boldsymbol{k}s}c^\dagger_{-\boldsymbol{k}s'}(\tau)\rangle \ + \ \theta(-\tau)\varepsilon(\boldsymbol{k})\langle c^\dagger_{\boldsymbol{k}s}c^\dagger_{-\boldsymbol{k}s'}(\tau)\rangle \ + \ \sum_{s''}\Delta^*_{s's''}(-\boldsymbol{k})\theta(-\tau)\langle c^\dagger_{\boldsymbol{k}s}c_{\boldsymbol{k}s''}(\tau)\rangle \\
&= -\varepsilon(\boldsymbol{k})F^\dagger_{ss'}(\boldsymbol{k},\tau) \ - \ \sum_{s''}\Delta^*_{s's''}(-\boldsymbol{k})G_{s''s}(\boldsymbol{k},\tau) \quad , \qquad (16)
\end{aligned}$$

$$\longleftrightarrow \quad i\omega_n F^\dagger_{ss'}(\boldsymbol{k},i\omega_n) = -\varepsilon(\boldsymbol{k})F^\dagger_{ss'}(\boldsymbol{k},i\omega_n) \ - \ \sum_{s''}\Delta^*_{s's''}(-\boldsymbol{k})G_{s''s}(\boldsymbol{k},i\omega_n) \quad . \qquad (17)$$

From the above, the simultaneous equations to solve to obtain the spectral representations of the respective Green's functions are:

$$G_{ss'}(\boldsymbol{k}, i\omega_n) = \sum_{s''} \frac{\Delta_{ss''}(\boldsymbol{k})}{i\omega_n - \varepsilon(\boldsymbol{k})} F^{\dagger}_{s's''}(\boldsymbol{k}, i\omega_n) \quad , \tag{18}$$

$$F^{\dagger}_{ss'}(\boldsymbol{k}, i\omega_n) = -\sum_{s''} \frac{\Delta^{*}_{s's''}(-\boldsymbol{k})}{i\omega_n + \varepsilon(\boldsymbol{k})} G_{s''s}(\boldsymbol{k}, i\omega_n) \quad . \tag{19}$$

The solutions to the Gor'kov equations for the case where the pair potential representation matrix is unitary can be expressed as: Let the elementary excitation energy spectrum be

$$E_{\boldsymbol{k}} = \sqrt{\varepsilon^2(\boldsymbol{k}) + |\boldsymbol{d}(\boldsymbol{k})|^2} \quad , \tag{20}$$

then,

$$\hat{G}(\boldsymbol{k}, i\omega_n) = -\frac{i\omega_n + \varepsilon(\boldsymbol{k})}{\omega_n^2 + E_{\boldsymbol{k}}^2} \hat{\sigma}_0 \quad , \tag{21}$$

$$\hat{F}(\boldsymbol{k}, i\omega_n) = \frac{i\boldsymbol{d}(\boldsymbol{k}) \cdot \hat{\boldsymbol{\sigma}} \hat{\sigma}_y}{\omega_n^2 + E_{\boldsymbol{k}}^2} = \frac{\hat{\Delta}(\boldsymbol{k})}{\omega_n^2 + E_{\boldsymbol{k}}^2} \quad . \tag{22}$$

Using these, we calculate the electronic Raman response function. Using the bosonic Matsubara frequency $\nu_n = 2m\pi k_B T$ $(m \in \mathbb{Z})$,

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\boldsymbol{q}, i\nu_m) &= -\int_0^{\beta} d\tau \langle T_\tau[\tilde{\rho}^{\dagger}_{\boldsymbol{q}}(\tau)\tilde{\rho}_{\boldsymbol{q}}]\rangle e^{i\nu_m\tau} \\
&= -\int_0^{\beta} d\tau \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2} \langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c_{\boldsymbol{k}_1,s_1}(\tau)c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}c_{\boldsymbol{k}_2,s_2}]\rangle e^{i\nu_m\tau} \\
&= -\int_0^{\beta} d\tau \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2} \Big\{ \langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c_{\boldsymbol{k}_1,s_1}(\tau)]\rangle\langle T_\tau[c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}c_{\boldsymbol{k}_2,s_2}]\rangle \\
&\qquad - \langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle \\
&\qquad + \langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle \Big\} e^{i\nu_m\tau} \quad .
\end{aligned} \tag{23}$$

The term in the first parenthesis on the right-hand side is a Green's function at the same time, so it's a constant and can be dropped. Rearranging the signs:

$$\begin{aligned}
\chi_{\tilde{\rho}\tilde{\rho}}(\boldsymbol{q}, i\nu_n) = \int_0^{\beta} d\tau \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2} \Big\{ &\langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle e^{i\nu_m\tau} \\
&- \langle T_\tau[c^{\dagger}_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c^{\dagger}_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle \Big\} e^{i\nu_m\tau} \quad .
\end{aligned} \tag{24}$$

We expand the sum over spins s_1, s_2. The first term inside the curly braces is:

$$\begin{aligned}
&\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2}\langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle \qquad (25)\\
=&\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2}\gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2}\Big\{\langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\uparrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\uparrow}(\tau)c_{\boldsymbol{k}_2,\uparrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\uparrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\uparrow}(\tau)c_{\boldsymbol{k}_2,\downarrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\downarrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\downarrow}(\tau)c_{\boldsymbol{k}_2,\uparrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\downarrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\downarrow}(\tau)c_{\boldsymbol{k}_2,\downarrow}]\rangle\Big\}\quad . \qquad (26)
\end{aligned}$$

The sum over momenta only remains when $\boldsymbol{k}_2 = -\boldsymbol{k}_1$. Performing the sum over $\boldsymbol{k}_2$ yields:

$$\begin{aligned}
&\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2}\langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle \qquad (27)\\
=&\sum_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}\Big\{\langle T_\tau[c^\dagger_{-\boldsymbol{k}+\boldsymbol{q},\uparrow}(\tau)c^\dagger_{\boldsymbol{k}-\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{-\boldsymbol{k},\uparrow}(\tau)c_{\boldsymbol{k},\uparrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{-\boldsymbol{k}+\boldsymbol{q},\uparrow}(\tau)c^\dagger_{\boldsymbol{k}-\boldsymbol{q},\downarrow}]\rangle\langle T_\tau[c_{-\boldsymbol{k},\uparrow}(\tau)c_{\boldsymbol{k},\downarrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{-\boldsymbol{k}+\boldsymbol{q},\downarrow}(\tau)c^\dagger_{\boldsymbol{k}-\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{-\boldsymbol{k},\downarrow}(\tau)c_{\boldsymbol{k},\uparrow}]\rangle\\
&+\ \langle T_\tau[c^\dagger_{-\boldsymbol{k}+\boldsymbol{q},\downarrow}(\tau)c^\dagger_{\boldsymbol{k}-\boldsymbol{q},\downarrow}]\rangle\langle T_\tau[c_{-\boldsymbol{k},\downarrow}(\tau)c_{\boldsymbol{k},\downarrow}]\rangle\Big\}\\
=&\sum_{\boldsymbol{k}}\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}\Big\{F^\dagger_{\uparrow\uparrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\uparrow\uparrow}(-\boldsymbol{k},\tau)\\
&+\ F^\dagger_{\downarrow\uparrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\uparrow\downarrow}(-\boldsymbol{k},\tau)\\
&+\ F^\dagger_{\uparrow\downarrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\downarrow\uparrow}(-\boldsymbol{k},\tau)\\
&+\ F^\dagger_{\downarrow\downarrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\downarrow\downarrow}(-\boldsymbol{k},\tau)\Big\}\quad . \qquad (28)
\end{aligned}$$

Next, we transform the second term on the right-hand side of equation (24) in the same way. After summing over spins, the sum over momentum $\boldsymbol{k}_2$ only remains for the case where $\boldsymbol{k}_2 = \boldsymbol{k}_1 + \boldsymbol{q}$.

$$\begin{aligned}
&\sum_{\boldsymbol{k}_1,\boldsymbol{k}_2,s_1,s_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2}\langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},s_1}(\tau)c_{\boldsymbol{k}_2,s_2}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,s_1}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},s_2}]\rangle \qquad (29)\\
&= \sum_{\boldsymbol{k}_1,\boldsymbol{k}_2} \gamma_{\boldsymbol{k}_1}\gamma_{\boldsymbol{k}_2}\Big\{\langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\uparrow}(\tau)c_{\boldsymbol{k}_2,\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\uparrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\uparrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\uparrow}(\tau)c_{\boldsymbol{k}_2,\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\uparrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\downarrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\downarrow}(\tau)c_{\boldsymbol{k}_2,\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\downarrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\uparrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}_1+\boldsymbol{q},\downarrow}(\tau)c_{\boldsymbol{k}_2,\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k}_1,\downarrow}(\tau)c^\dagger_{\boldsymbol{k}_2-\boldsymbol{q},\downarrow}]\rangle\Big\}\\
&= \sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}+\boldsymbol{q}}\Big\{\langle T_\tau[c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\uparrow}(\tau)c_{\boldsymbol{k}+\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k},\uparrow}(\tau)c^\dagger_{\boldsymbol{k},\uparrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k},\uparrow}(\tau)c_{\boldsymbol{k},\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k},\uparrow}(\tau)c^\dagger_{\boldsymbol{k},\downarrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\downarrow}(\tau)c_{\boldsymbol{k}+\boldsymbol{q},\uparrow}]\rangle\langle T_\tau[c_{\boldsymbol{k},\downarrow}(\tau)c^\dagger_{\boldsymbol{k},\uparrow}]\rangle\\
&\quad+\ \langle T_\tau[c^\dagger_{\boldsymbol{k}+\boldsymbol{q},\downarrow}(\tau)c_{\boldsymbol{k}+\boldsymbol{q},\downarrow}]\rangle\langle T_\tau[c_{\boldsymbol{k},\downarrow}(\tau)c^\dagger_{\boldsymbol{k},\downarrow}]\rangle\Big\}\\
&= -\sum_{\boldsymbol{k}} \gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}+\boldsymbol{q}}\Big\{G_{\uparrow\uparrow}(\boldsymbol{k}+\boldsymbol{q},\tau)G_{\uparrow\uparrow}(\boldsymbol{k},-\tau)\\
&\quad+\ G_{\downarrow\downarrow}(\boldsymbol{k}+\boldsymbol{q},\tau)G_{\downarrow\downarrow}(\boldsymbol{k},-\tau)\Big\}\ . \qquad (30)
\end{aligned}$$

Here we have used the unitary condition $G_{\uparrow\downarrow} = G_{\downarrow\uparrow} = 0$.

In summary,

$$\begin{aligned}
\chi_{\tilde\rho\tilde\rho}(\boldsymbol{q}, i\nu_m) \;&=\; -\int_0^\beta d\tau \langle T_\tau[\tilde\rho^\dagger_{\boldsymbol{q}}(\tau)\tilde\rho_{\boldsymbol{q}}]\rangle e^{i\nu_m\tau}\\
= \sum_{\boldsymbol{k}}\int_0^\beta d\tau\Big\{\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}\Big[&F^\dagger_{\uparrow\uparrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\uparrow\uparrow}(-\boldsymbol{k},\tau) \;+\; F^\dagger_{\downarrow\uparrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\uparrow\downarrow}(-\boldsymbol{k},\tau)\\
&+ F^\dagger_{\uparrow\downarrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\downarrow\uparrow}(-\boldsymbol{k},\tau) \;+\; F^\dagger_{\downarrow\downarrow}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{\downarrow\downarrow}(-\boldsymbol{k},\tau)\Big]\\
- \gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}+\boldsymbol{q}}\Big[&G_{\uparrow\uparrow}(\boldsymbol{k}+\boldsymbol{q},\tau)G_{\uparrow\uparrow}(\boldsymbol{k},-\tau) \;+\; G_{\downarrow\downarrow}(\boldsymbol{k}+\boldsymbol{q},\tau)G_{\downarrow\downarrow}(\boldsymbol{k},-\tau)\Big]\Big\}e^{i\nu_m\tau} \qquad (31)
\end{aligned}$$

Next, we perform the Fourier transform with respect to time. For the anomalous Green's functions:

$$\begin{aligned}
\int_0^\beta d\tau F^\dagger_{s's}(\boldsymbol{k}-\boldsymbol{q},\tau)F_{ss'}(-\boldsymbol{k},\tau)e^{i\nu_m\tau} &= \frac{1}{\beta^2}\int_0^\beta d\tau \sum_{n_1,n_2} F^\dagger_{s's}(\boldsymbol{k}-\boldsymbol{q},i\omega_{n_1})F_{ss'}(-\boldsymbol{k},i\omega_{n_2})e^{i(\nu_m-\omega_{n_1}-\omega_{n_2})\tau}\\
&= \frac{1}{\beta^2}\sum_{n_1,n_2} F^\dagger_{s's}(\boldsymbol{k}-\boldsymbol{q},i\omega_{n_1})F_{ss'}(-\boldsymbol{k},i\omega_{n_2})\quad \beta\delta_{\omega_{n_1},\nu_m-\omega_{n_2}}\\
&= \frac{1}{\beta}\sum_{n} F^\dagger_{s's}(\boldsymbol{k}-\boldsymbol{q},i\nu_m-i\omega_n)F_{ss'}(-\boldsymbol{k},i\omega_n) \qquad (32)
\end{aligned}$$

Similarly,

$$\int_0^\beta d\tau G_{ss}(\boldsymbol{k}+\boldsymbol{q},\tau)G_{ss}(\boldsymbol{k},-\tau)e^{i\nu_m\tau} = \frac{1}{\beta^2}\int_0^\beta d\tau \sum_{n_1,n_2} G_{ss}(\boldsymbol{k}+\boldsymbol{q},i\omega_{n_1})G_{ss}(\boldsymbol{k},i\omega_{n_2})e^{i(\nu_m-\omega_{n_1}+\omega_{n_2})\tau}$$

$$= \frac{1}{\beta^2}\sum_{n_1,n_2} G_{ss}(\boldsymbol{k}+\boldsymbol{q},i\omega_{n_1})G_{ss}(\boldsymbol{k},i\omega_{n_2}) \quad \beta\delta_{\omega_{n_1},\nu_m+\omega_{n_2}}$$

$$= \frac{1}{\beta}\sum_{n} G_{ss}(\boldsymbol{k}+\boldsymbol{q},i\nu_m+i\omega_{n})G_{ss}(\boldsymbol{k},i\omega_{n}) \tag{33}$$

From the above, the response function is as follows.

$$\chi_{\tilde{\rho}\tilde{\rho}}(\boldsymbol{q},i\nu_m) \;=\; -\int_0^\beta d\tau \langle T_\tau[\tilde{\rho}^\dagger_{\boldsymbol{q}}(\tau)\tilde{\rho}_{\boldsymbol{q}}]\rangle e^{i\nu_m\tau}$$

$$= \frac{1}{\beta}\sum_n\sum_{\boldsymbol{k}}\sum_{ss'} \tag{34}$$

$$\times\Big[\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}F^\dagger_{s's}(\boldsymbol{k}-\boldsymbol{q},i\nu_m-i\omega_n)F_{ss'}(-\boldsymbol{k},i\omega_n)-\gamma_{\boldsymbol{k}}\gamma_{\boldsymbol{k}+\boldsymbol{q}}G_{ss}(\boldsymbol{k}+\boldsymbol{q},i\nu_m+i\omega_n)G_{s's'}(\boldsymbol{k},i\omega_n)\Big] \tag{35}$$

We consider the limit $\boldsymbol{q}\to\boldsymbol{0}$. From now on, we will only consider this case, so we will abbreviate $\chi_{\tilde{\rho}\tilde{\rho}}(\boldsymbol{0},i\nu_m)=\chi_{\tilde{\rho}\tilde{\rho}}(i\nu_m)$.

$$\chi_{\tilde{\rho}\tilde{\rho}}(i\nu_m) = \frac{1}{\beta}\sum_n\sum_{\boldsymbol{k}}\sum_{ss'}\Big[\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}F^\dagger_{s's}(\boldsymbol{k},i\nu_m-i\omega_n)F_{ss'}(-\boldsymbol{k},i\omega_n)-\gamma^2_{\boldsymbol{k}}G_{ss}(\boldsymbol{k},i\nu_m+i\omega_n)G_{s's'}(\boldsymbol{k},i\omega_n)\Big]$$

$$= \frac{1}{\beta}\sum_n\sum_{\boldsymbol{k}}\left[\gamma_{\boldsymbol{k}}\gamma_{-\boldsymbol{k}}\sum_{ss'}\frac{\Delta_{s's}(\boldsymbol{k})}{(\nu_m-\omega_n)^2+E^2_{\boldsymbol{k}}}\;\frac{\Delta_{ss'}(-\boldsymbol{k})}{\omega_n^2+E^2_{-\boldsymbol{k}}}-\gamma^2_{\boldsymbol{k}}\frac{i\omega_n+i\nu_m+\varepsilon(\boldsymbol{k})}{(\nu_m+\omega_n)^2+E^2_{\boldsymbol{k}}}\;\frac{i\omega_n+\varepsilon(\boldsymbol{k})}{\omega_n^2+E^2_{\boldsymbol{k}}}\right] \tag{36}$$