A Unified View of Monoids,
Groups, Topological Groups, and Lie Groups

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Abstract

We'll review the definitions of monoids, groups, topological groups, and Lie groups, and see how writing them in terms of category-theoretic diagrams makes them magically unified and easy to understand.

1 Naive Definitions

First, let's define monoids, groups, topological groups, and Lie groups naively.

1.1 Definition of a Monoid

A monoid is a semigroup with an identity element, satisfying associativity and having an identity.
A set M with a binary operation $\cdot : M \times M \to M$

1. Associativity: For any elements of M, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. Existence of an identity element: There exists an element e in M such that for any element a in M, $e \cdot a = a \cdot e = a$.

1.2 Definition of a Group

A group is a monoid that also has an inverse element.
A set G with a binary operation $\cdot : G \times G \to G$

1. Associativity: For any elements of G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
2. Existence of an identity element: There exists an element e in M such that for any element a in G, $e \cdot a = a \cdot e = a$.
3. Existence of an inverse element: For any element a in G, there exists an element a^{-1} that satisfies $a^{-1} \cdot a = a \cdot a^{-1} = e$.

1.3 Definition of a Topological Group

A set G is a topological group if it is a group and satisfies the axioms of a topological space, and there exist continuous maps for the product and inverse.

1. G is a group.
2. G is a topological space.
3. The maps $\mu : G \times G \to G$ and $\nu : G \to G$, defined as $\mu(x, y) = xy$ and $\nu(x) = x^{-1}$ respectively, are continuous.

1.4 Definition of a Lie Group

A set G is a Lie group if it is a differentiable manifold, and satisfies the axioms of a topological group, and the maps for the product and inverse are differentiable.

1. G is a differentiable manifold.
2. The maps $\mu : G \times G \to G$ and $\nu : G \to G$, defined as $\mu(x, y) = xy$ and $\nu(x) = x^{-1}$ respectively, are differentiable.

2 A Unified View

2.1 Monoid

Next, let's try to understand these concepts in a unified way using elementary category theory.

A one-point set $1 = \{0\}$ acts as an identity element for the product (Cartesian product), considering the bijections

$$1 \times X \xrightarrow{\lambda} X \xleftarrow{\rho} X \times 1$$

given by $\lambda < 0, x >= x$, $\rho < x, 0 >= x$.

A monoid M is a set M equipped with two functions

$$\mu : M \times M \to M \ , \quad \eta : 1 \to M$$

that make the following diagrams commute with respect to μ and η.

$$\begin{tikzcd}
M \times M \times M \arrow[r, "1\times\mu"] \arrow[d, "\mu\times 1"'] & M \times M \arrow[d, "\mu"] \\
M \times M \arrow[r, "\mu"] & M
\end{tikzcd}$$

$$\begin{tikzcd}
1 \times M \arrow[r, "\eta\times 1"] \arrow[dr, "\lambda"'] & M \times M \arrow[d, "\mu"] & M \times 1 \arrow[l, "1\times\eta"'] \arrow[dl, "\rho"] \\
& M &
\end{tikzcd}$$

2.2 Introducing a Map for the Inverse Element

A group has an inverse element. A group is a monoid that makes the following diagram commute using a map $\xi : M \to M$ where $\xi : x \mapsto x^{-1}$.

$$\begin{tikzcd}
M \arrow[r, "\delta"] \arrow[d, "\eta\circ !"] & M \times M \arrow[r, "1\times\xi"] & M \times M \arrow[d, "\mu"] \\
1 \arrow[rr, "\eta"] & & M
\end{tikzcd}$$

where η is the arrow to the identity element and ! is the unique arrow from M to the one-point set 1.

$$\begin{tikzcd}
x \arrow[r, mapsto] \arrow[d, mapsto] & <x, x> \arrow[r, mapsto] & <x, x^{-1}> \arrow[d, mapsto] \\
0 \arrow[rr, mapsto] & & xx^{-1}
\end{tikzcd}$$

Here, $\delta : M \to M \times M$ is the diagonal map given by $\delta : x \mapsto <x, x>$.

A group is what you get when M is a set and each morphism is just a function that makes the above diagrams commute.

A topological group is what you get when M is a topological space and each morphism is a continuous map that makes the diagrams commute.

A Lie group is what you get when M is a differentiable manifold and each morphism is a smooth map of manifolds that makes the diagrams commute.