# Yoneda's Lemma, Analogies and Proof

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Abstract

On Yoneda's Lemma. Analogies and proof for a deeper understanding.

## 1 Statement of Yoneda's Lemma

Let $C$ be a category and $a \in C$ be an object.

We define $y(a)$ as $y(a) = \mathrm{Hom}_C(-, a)$ *1 :

$$\begin{array}{cccc} y(a): & C^{op} & \to & \mathbf{Set} \\ & \cup & & \cup \\ & x & \mapsto & \mathrm{Hom}_C(-, a) \end{array}$$

For a functor $F : C^{op} \to \mathbf{Set}$,

$$\mathrm{Hom}_C(y(a), F) \cong Fa$$

holds.

The left-hand side, $\mathrm{Hom}_C(y(a), F)$, is the "set" of all natural transformations from $y(a)$ to $F$.

The right-hand side, $Fa$, is a "set" because the functor $F$ maps it to a set.

The fact that two sets are isomorphic means that there is a bijection between the left-hand side and the right-hand side.

### 1.1 Simple Analogy: The Example of Linear Algebra

Let $V$ be a real vector space.

It is fine to take $V = \mathbb{R}^3$.

In this case, the set of all homomorphisms (i.e., linear maps here) from real numbers to the real vector space is isomorphic to the real vector space.

$$\mathrm{Hom}(\mathbb{R}, V) \cong V$$

This isomorphism is a basic one in linear algebra, but it serves as a good analogy for Yoneda's Lemma.

In other words, it is the case where we take the category $C$ to be the category of real vector spaces and $a$ to be the real vector space $\mathbb{R}$.

*1 In other words, we define $y(a) : C^{op} \to \mathbf{Set}$ such that $y(a)(x) = \mathrm{Hom}_C(x, a)$ and for any morphism $f : x \to x'$, we define $y(a)(f) = \mathrm{Hom}_C(f, a)$.

Here, $\mathrm{Hom}_C(f, a) : \mathrm{Hom}_C(x', a) \to \mathrm{Hom}_C(x, a)$ is defined by mapping $\phi \in \mathrm{Hom}_C(x', a)$ to $\phi \circ f$.

Looking more closely,

$$\begin{array}{ccc} \mathrm{Hom}(\mathbb{R}, V) & \cong & V \\ \cup\!\!\!\!\!\!\cup & & \cup\!\!\!\!\!\!\cup \\ f & \mapsto & f(1) \end{array}$$

Using the identity element 1 of $\mathbb{R}$, we map $f$ to $f(1)$ (i.e., we substitute 1).

Since $f$ is a linear map, for any real number $\lambda$, $f(\lambda) = f(\lambda \cdot 1) = \lambda f(1)$ holds.

This means that the value of $f(\lambda)$ for any $\lambda$ is uniquely determined by the constant $\lambda$ and the value of $f(1)$.

In short, the values of $f$ are uniquely determined once we fix a representative value, $f(1)$.

Due to this, there is a one-to-one correspondence, a bijection, between the set of all linear maps $f$ and the set of all possible values of $f(1)$ (i.e., the elements of the vector space $V$).

## 1.2 Simple Analogy: The Example of Set Theory

Yoneda's Lemma can also be seen as an analogy that reinterprets elements in set theory as maps.

In the usual way of thinking, to identify an element $a$ of a set $A$, we simply pick that element from $A$.

However, from the perspective of Yoneda's Lemma, to identify an element $a$ of a set $A$, it is essential to consider maps from other sets to $A$.

As a concrete example, we consider the category of sets, **Set**, and a set $A$ and a set with only a single element, $\{*\}$.

$$\begin{array}{ccc} \mathrm{Hom}(\{*\}, A) & \cong & A \\ \cup\!\!\!\!\!\!\cup & & \cup\!\!\!\!\!\!\cup \\ f & \mapsto & f(*) \end{array}$$

This shows that a map $f : \{*\} \to A$ is completely determined by which element of $A$ the single element $*$ is mapped to.

In other words, the map $f$ corresponds one-to-one with the element $f(*)$ of $A$.

1. **Identification of an element**: An element $a$ of a set $A$ corresponds one-to-one with a map $f : \{*\} \to A$ such that $f(*) = a$.
2. **Identification of a map**: The set of all maps from any set $X$ to $A$, denoted as $\mathrm{Hom}(\{*\}, A)$, corresponds one-to-one with $A$.

This analogy shows the essential connection between "the object itself" and "the maps to that object," which is the essence of Yoneda's Lemma.

## 1.3 Simple Analogy: The Example of Group Theory

Yoneda's Lemma can be viewed as an analogy that reinterprets "elements of a group" in terms of "homomorphisms."

In the usual way of thinking, to identify an element $g$ of a group $G$, we simply pick that element from $G$.

However, from the perspective of Yoneda's Lemma, to identify an element $g$ of a group $G$, it is essential to consider a homomorphism from another group to $G$.

As a concrete example, we consider a homomorphism $\phi : \mathbb{Z} \to G$ from the infinite cyclic group $\mathbb{Z}$ to $G$.

In the category of groups, **Grp**, we consider an object $G$. The important thing to consider for the analogy of Yoneda's Lemma is the infinite cyclic group $\mathbb{Z}$.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G) & \cong & G \\ \Cup & & \Cup \\ \phi & \mapsto & \phi(1) \end{array}$$

This shows that a group homomorphism $\phi : \mathbb{Z} \to G$ is completely determined by the value of $\phi(1)$.

Since any element $n$ of $\mathbb{Z}$ can be expressed as the sum of $n$ ones, the property of group homomorphisms means that $\phi(n) = n\phi(1)$. Thus, if $\phi(1)$ is determined, the entire map is determined. This creates a one-to-one correspondence between the homomorphism $\phi$ and the element $\phi(1)$ of the group $G$.

Why does this serve as an analogy?

1. **Identification of an element**: An element $g$ of a group $G$ corresponds one-to-one with a homomorphism $\phi : \mathbb{Z} \to G$ such that $\phi(1) = g$.
2. **Identification of a map**: The set of all homomorphisms from any group $H$ to $G$, denoted as $\mathrm{Hom}_{\mathbf{Grp}}(\mathbb{Z}, G)$, corresponds one-to-one with $G$. This is because any homomorphism $\phi$ is completely determined by the value of $\phi(1)$.

This analogy shows the essential connection between "the object itself" and "the maps to that object" within the framework of group theory, which is the essence of Yoneda's Lemma.

# 2 Proof of Yoneda's Lemma

Let $\sigma$ be the morphism that maps the set $\mathrm{Hom}_C(y(a), F)$ to the set $Fa$:

$$\begin{array}{cccc} \sigma : & \mathrm{Hom}_C(y(a), F) & \to & Fa \\ & \Cup & & \Cup \\ & \phi & \mapsto & \sigma(\phi) = \phi_a(\mathrm{id}_a) \end{array}$$

Here, $\phi : y(a) \to F$ is a natural transformation.

For an object $x \in C$, the $x$-component is a homomorphism $\phi_x : y(a)(x) \to Fx$. The same is true for $y(a)(x) = \mathrm{Hom}_C(x, a)$, so $\phi_x : \mathrm{Hom}_C(x, a) \to Fx$.

In particular,

$$\begin{array}{cccc} \phi_a : & \mathrm{Hom}_C(a, a) & \to & Fa \\ & \Cup & & \Cup \\ & \mathrm{id}_a & \mapsto & \phi_a(\mathrm{id}_a) \end{array}$$

If we can show that this defined $\sigma$ is a bijection, Yoneda's Lemma will be proven.

## 2.1 Injectivity

Here, for any object $x$ in $C$, we take a morphism $p : x \to a$ from $\mathrm{Hom}_C(x, a)$.

In this case, we can consider $\phi_x(p)$.

In the commutative diagram,

$$y(a) \xrightarrow{\phi} F$$

$$\begin{tikzcd}
x \arrow[d, "p"'] & y(a)(x) \arrow[r, "\phi_x"] \arrow[d, "y(a)(p)"'] & Fx \arrow[d, "F(p)"] \\
a & y(a)(a) \arrow[r, "\phi_a"] & Fa
\end{tikzcd}$$

This diagram shows that $F(p) \circ \phi_a = \phi_x \circ y(a)(p)$.

Here, $y(a)(p)$ is a map defined by the composition of morphisms, and specifically, $y(a)(p) : y(a)(a) \to y(a)(x)$ is the map that sends any $\varphi \in y(a)(a) = \mathrm{Hom}_C(a, a)$ to $\varphi \circ p$.

Applying the element $\mathrm{id}_a$ of $y(a)(a)$ to the above commutative diagram, we get $F(p)(\phi_a(\mathrm{id}_a)) = \phi_x(y(a)(p)(\mathrm{id}_a))$

Since $y(a)(p)(\mathrm{id}_a) = \mathrm{id}_a \circ p = p$, $F(p)(\phi_a(\mathrm{id}_a)) = \phi_x(p)$

Writing it out in a more organized way,

$$\phi_x(p) = \phi_x(\mathrm{id}_a \circ p) = \phi_x(\mathrm{id}_a) * p$$

Here, $\phi_x(\mathrm{id}_a) \in Fa$. This $F$ is a functor and a $C$-set. $- * p$ means the map $F(p)(-)$ induced by the functor $F$.

Since we defined $\phi_x(\mathrm{id}_a) = \sigma(\phi)$,

$$\phi_x(p) = F(p)(\sigma(\phi))$$

This proves injectivity

$$\sigma(\phi) = \sigma(\phi') \quad \Rightarrow \quad \phi = \phi'$$

because if $\sigma(\phi)$ is determined, $\phi_x(p)$ is also determined.

Thus, $\sigma$ is injective.

## 2.2 Surjectivity

To show that $\sigma$ is surjective, we must show that for any element $u$ of the set $Fa$, there exists a natural transformation $\phi : y(a) \to F$ such that $\sigma(\phi) = u$.

Let $u$ be any element of $Fa$. Using this $u$, we define a map $\phi_x : y(a)(x) \to Fx$ for each object $x \in C$ as follows.

For any morphism $p : x \to a$,

$$\phi_x(p) = F(p)(u)$$

Here, $F(p)$ is the map $F(p) : Fa \to Fx$ obtained by the action of the functor $F$ on the morphism $p$.

Next, we confirm that this defined family of maps $\phi = \{\phi_x\}_{x \in C}$ is indeed a natural transformation from $y(a)$ to $F$. By the definition of a natural transformation, the following commutative diagram must hold for any morphism $q : x' \to x$.

$$
\begin{tikzcd}
y(a) \arrow[r, "\phi"] & F
\end{tikzcd}
$$

$$
\begin{tikzcd}
x' \arrow[d, "q"'] & y(a)(x') \arrow[r, "\phi_{x'}"] \arrow[d, "y(a)(q)"'] & Fx' \arrow[d, "F(q)"] \\
x & y(a)(x) \arrow[r, "\phi_x"] & Fx
\end{tikzcd}
$$

We check that this diagram commutes, i.e., $F(q) \circ \phi_{x'} = \phi_x \circ y(a)(q)$.

Left-hand side: $F(q) \circ \phi_{x'} = F(q)(\phi_{x'}(p'))$ Here, $p'$ is any element of $y(a)(x')$ (i.e., a morphism $p' : x' \to a$). By definition, $\phi_{x'}(p') = F(p')(u)$, so

$$F(q)(F(p')(u)) = F(q \circ p')(u) \quad \text{(property of functor } F\text{)}$$

Right-hand side: $\phi_x \circ y(a)(q)$ $y(a)(q)$ is the map that sends the morphism $p'$ to $p' \circ q$. Thus,

$$\phi_x(y(a)(q)(p')) = \phi_x(p' \circ q)$$

By definition, $\phi_x(p' \circ q) = F(p' \circ q)(u)$.

Since both sides are equal, we have confirmed that $\phi$ is a natural transformation.

Finally, we show that this natural transformation $\phi$ satisfies $\sigma(\phi) = u$. By the definition of $\sigma(\phi)$,

$$\sigma(\phi) = \phi_a(\mathrm{id}_a)$$

According to the definition of $\phi_a$, substituting $\mathrm{id}_a$, we get

$$\phi_a(\mathrm{id}_a) = F(\mathrm{id}_a)(u) = \mathrm{id}_{Fa}(u) = u$$

Thus, $\sigma$ is surjective.

Therefore, $\sigma$ is a bijection, and Yoneda's Lemma

$$\mathrm{Hom}_C(y(a), F) \cong Fa$$

is proven.