American Options Without Dividends

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Abstract

This note is a memo from a discussion I had with a colleague on November 12, 2019. It's about how an American option on a non-dividend-paying stock is never worth more than its European counterpart at expiration.

1 Definition of a Convex Function

Consider a real-valued function h of x.

The function $h(x)$ is called a convex function if the following holds for any λ where $0 \leq \lambda \leq 1$ and for any x_1, x_2 where $0 < x_1 < x_2$:

$$h\Big((1-\lambda)x_1 + \lambda x_2\Big) \leq (1-\lambda)h(x_1) + \lambda h(x_2)$$

2 Jensen's Inequality

If a function $\phi(x)$ is convex, then

$$\mathbb{E}\Big(\phi(X)\Big|\mathcal{F}\Big) \geq \phi\Big(\mathbb{E}(X|\mathcal{F})\Big)$$

holds true. Here, X is a random variable and $\mathcal{F}$ is a σ-algebra generated from a subset of the sample space Ω.

Essentially, the expected value of a convex function, $\mathbb{E}(\phi(X))$, is greater than the convex function of the expected value, $\phi(\mathbb{E}(X))$.

$$\mathbb{E}(\phi(X)) \ \geq \ \phi(\mathbb{E}(X))$$

(A way to remember it: The expectation of the future is greater than the future of the expectation.)

3 Proof of Jensen's Inequality

(Will read this later.)

4 Martingales

(Checking the definitions)

A stochastic process M_t is called a **submartingale** if, for all u, t satisfying $0 \leq u \leq t \leq T$, the following holds:

$$\mathbb{E}(M_t|\mathcal{F}_u) \geq M_u$$

Conversely, it is called a **supermartingale** if

$$\mathbb{E}(M_t|\mathcal{F}_u) \leq M_u$$

In a submartingale, the expected value at a future time t, conditional on the history up to time u ($\mathcal{F}_u$), is greater than or equal to the value at the past time u. This means the process has an increasing trend.

(Even though it has 'sub' or 'inferior' in the name, the expected value of a submartingale is larger in the future.)

5 Underlying Assets Without Dividends

Let's consider a stock with a price process S_t given by:

$$\frac{dS_t}{S_t} = rdt + \sigma d\tilde{W}_t$$

The interest rate r and volatility σ are always positive. W_t is a process that becomes a Brownian motion under the risk-neutral probability measure $\tilde{\mathbb{P}}$.

6 Lemma 8.5.1[1]

Consider an American option that pays an amount of $h(S_t)$ upon exercise.

Let's assume the function $h(x)$ is convex for $x \geq 0$.

Also, assume $h(0) = 0$.

In this case, the discounted price $e^{-rt}h(S_t)$ is a submartingale.

7 Proof of Lemma 8.5.1[1]

Since $h(x)$ is a convex function, for any λ such that $0 \leq \lambda \leq 1$ and for any x_1, x_2 such that $0 < x_1 < x_2$, the following holds:

$$h\Big((1-\lambda)x_1 + \lambda x_2\Big) \leq (1-\lambda)h(x_1) + \lambda h(x_2)$$

Specifically, if we let $x_1 = 0$ and $x_2 = S_t$, we get:

$$h\Big((1-\lambda) \times 0 + \lambda S_t\Big) \leq (1-\lambda)h(0) + \lambda h(S_t)$$
$$h(\lambda S_t) \leq \lambda h(S_t)$$

Here, we've used the assumption $h(0) = 0$.

Furthermore, for all u, t such that $0 \leq u \leq t \leq T$, r is positive ($0 < r < \infty$), so:

$$\begin{array}{ccccc} 0 & \leq & r(t-u) & (< & \infty) \\ (-\infty & <) & -r(t-u) & \leq & 0 \\ 0 & \leq & e^{-r(t-u)} & \leq & 1 \end{array}$$

Since $e^{-r(t-u)}$ has the same range as λ ($0 \leq \lambda \leq 1$), we can substitute it for λ:

$$\lambda h(S_t) \geq h(\lambda S_t)$$
$$e^{-r(t-u)} h(S_t) \geq h\Big(e^{-r(t-u)} S_t\Big)$$

Taking the expected value of both sides under the measure $\tilde{\mathbb{P}}$ and conditional on $\mathcal{F}(u)$, denoted by $\tilde{\mathbb{E}}(\,\cdot\,|\mathcal{F}(u))$, we get:

$$\tilde{\mathbb{E}}\left[e^{-r(t-u)} h(S_t) \Big| \mathcal{F}(u)\right] \geq \tilde{\mathbb{E}}\left[h\Big(e^{-r(t-u)} S_t\Big) \Big| \mathcal{F}(u)\right]$$

Now, we apply Jensen's inequality to the right-hand side.

Because the expected value of a convex function $\mathbb{E}(h(X))$ is greater than the convex function of the expected value $h(\mathbb{E}(X))$, i.e., $\mathbb{E}(h(X)) \geq h(\mathbb{E}(X))$ we have:

$$\begin{aligned} \tilde{\mathbb{E}}\left[e^{-r(t-u)} h(S_t) \Big| \mathcal{F}(u)\right] &\geq \tilde{\mathbb{E}}\left[h\Big(e^{-r(t-u)} S_t\Big) \Big| \mathcal{F}(u)\right] \\ &\geq h\left(\tilde{\mathbb{E}}\left[e^{-r(t-u)} S_t \Big| \mathcal{F}(u)\right]\right) \\ &= h\left(e^{ru} \tilde{\mathbb{E}}\left[e^{-rt} S_t \Big| \mathcal{F}(u)\right]\right) \end{aligned}$$

The term inside the last expectation, $e^{-rt} S_t$, is the discounted stock price process, which is a $\tilde{\mathbb{P}}$-martingale. Therefore:

$$\begin{aligned} h\left(e^{ru} \tilde{\mathbb{E}}\left[e^{-rt} S_t \Big| \mathcal{F}(u)\right]\right) &= h\left(e^{ru} \times e^{-ru} S_u\right) \\ &= h(S_u) \end{aligned}$$

Putting the inequalities together from the beginning, we have:

$$\tilde{\mathbb{E}}\left[e^{-r(t-u)} h(S_t) \Big| \mathcal{F}(u)\right] \geq h(S_u)$$

Since a constant multiple of a convex function, $e^{-ru} h(x)$, is also convex, we can replace $h(x) \to e^{-ru} h(x)$ and follow the same logic from the start of the proof. This leads to:

$$\tilde{\mathbb{E}}\left[e^{-rt} h(S_t) \Big| \mathcal{F}(u)\right] \geq e^{-ru} h(S_u)$$

This is precisely the definition of a submartingale.

In conclusion, for an American option that pays $h(S_t)$ upon exercise, if the function $h(x)$ is convex for $x \geq 0$ and $h(0) = 0$, its discounted price process $e^{-rt} h(S_t)$ becomes a submartingale.

8 Theorem 8.5.2[1] and its Explanation

The previous discussion was general, but let's now apply it to a specific case.

When t is the expiration date T, substituting $t = T$ into the derived inequality gives us:

$$\tilde{\mathbb{E}}\left[e^{-rT}h(S_T)\middle|\mathcal{F}(u)\right] \geq e^{-ru}h(S_u)$$

The left side, $\tilde{\mathbb{E}}\left[e^{-rT}h(S_T)\middle|\mathcal{F}(u)\right]$ is the discounted price of a European option at expiration. This price is always higher than (or equal to) the discounted price of the American option on the right side, at any time u before expiration.

In other words, the price of an American option on a non-dividend-paying stock is never more than the price of a European option at expiration.

Intuitively, you might think an American option is more valuable because it gives you the freedom to exercise at any time. However, for a non-dividend-paying stock, its value is always less than or equal to the European option price at expiration.

This is all a consequence of the submartingale property. Since it's a submartingale, the expected value tends to increase.

(By the way, we haven't discussed a specific functional form for h, such as for a 'call,' 'put,' or 'forward' option yet.)

9 Corollary 8.5.3[1] and its Explanation

If we assume the convex function h is $h(S_T) = (S_T - K)^+$, we can apply this to the inequality derived in the proof of Lemma 8.5.1:

$$\tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)^+\middle|\mathcal{F}(u)\right] \geq e^{-ru}(S_u - K)^+$$

As with Theorem 8.5.2, this shows that the discounted process $e^{-rT}(S_t - K)^+$ is a submartingale, and because it has an upward trend, the expected value (conditional on a future time) is greater than or equal to the current value.

This means the price is maximized when the option is exercised at expiration $t = T$. Consequently, an American call option on a non-dividend-paying stock will not be exercised before expiration, and thus has the same value as a European call option.

The reason an American option on a non-dividend-paying stock is a submartingale is that the convexity of the function and the assumption that $h(0) = 0$ allow Jensen's inequality to hold, leading to the submartingale logic.

When we apply this line of thinking to an American put option, $h(x) = (K - x)^+$, while $h(x)$ is still a convex function, $h(0) = \text{Max}\{K\}$. This doesn't satisfy the condition $h(0) = 0$ unless $K \leq 0$, which makes the argument more complex[*1].

References

[1] Steven Shreve, "Stochastic Calculus for Finance II: Continuous-Time Models (Springer Finance)" (2004)

[*1] Although it says 'the argument becomes more complex,' I haven't yet experimented with what happens when $K > 0$.