Tensor Algebra

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October 18, 2025

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1 The Dual Linear Space Formed by Linear Maps

1.1 Dual Space of a Vector Space

Let V be a finite-dimensional vector space.

A linear map from V to $\mathbb{R}$ is called a **linear functional** on V.

For linear functionals $\varphi, \psi : V \to \mathbb{R}$, addition and scalar multiplication can be defined. That is, for a real number α and $\mathbf{x} \in V$:

$$\begin{array}{rcl}(\varphi+\psi)(\mathbf{x}) & = & \varphi(\mathbf{x})+\psi(\mathbf{x}) \\ (\alpha\varphi)(\mathbf{x}) & = & \alpha\varphi(\mathbf{x})\end{array}$$

The resulting sum and scalar multiple are also linear. For $\mathbf{x}, \mathbf{y} \in V$,

For example, for addition:

$$\begin{array}{rcl}(\varphi+\psi)(\mathbf{x}+\mathbf{y}) & = & \varphi(\mathbf{x}+\mathbf{y})+\psi(\mathbf{x}+\mathbf{y}) \\ & = & \varphi(\mathbf{x}+\mathbf{y})+\psi(\mathbf{x}+\mathbf{y}) \\ & = & \varphi(\mathbf{x})+\varphi(\mathbf{y})+\psi(\mathbf{x})+\psi(\mathbf{y}) \\ & = & \varphi(\mathbf{x})+\psi(\mathbf{x})+\psi(\mathbf{y})+\varphi(\mathbf{y}) \\ & = & (\varphi+\psi)(\mathbf{x})+(\varphi+\psi)(\mathbf{y})\end{array}$$

Similarly, for scalar multiplication:

$$\begin{aligned}(\alpha\varphi)(\mathbf{x}+\mathbf{y}) &= \alpha(\varphi(\mathbf{x}+\mathbf{y})) \\ &= \alpha\varphi(\mathbf{x})+\alpha\varphi(\mathbf{y})\end{aligned}$$

The set of all linear functionals on V also forms a vector space.

That this satisfies the vector space axioms can be confirmed, for example, by:

$$\begin{aligned}\alpha(\varphi+\psi)(\mathbf{x}) &= (\alpha\varphi+\alpha\psi)(\mathbf{x}) \\ (\alpha+\beta)\varphi(\mathbf{x}) &= (\alpha\varphi+\beta\varphi)(\mathbf{x})\end{aligned}$$

The vector space formed by all linear functionals on V is called the **dual linear space** (or dual space) and is denoted by V^*.

1.2 The Dual Space of the Dual Space

An element φ of V^* provides the mapping $\varphi : x \to \varphi(x)$.

From this perspective, x is the variable, and φ looks like a linear function.

Let's consider its dual space. In this case, x is fixed, and φ can be seen as the variable, moving freely over V^*. In notation, we view this as follows:

$$x : \varphi \to \varphi(x)$$

To clarify which is the variable, we will write $\tilde{x}$ for the perspective where x is fixed and φ varies. That is,

$$\tilde{x}(\varphi) = \varphi(x)$$

In this case, the equalities that held in the linear space, for example:

$$\begin{aligned}(\varphi+\psi)(\mathbf{x}+\mathbf{y}) &= (\varphi+\psi)(\mathbf{x})+(\varphi+\psi)(\mathbf{y}) \\ (\alpha\varphi)(\mathbf{x}+\mathbf{y}) &= \alpha\varphi(\mathbf{x})+\alpha\varphi(\mathbf{y})\end{aligned}$$

These equalities can be seen, respectively, from a different perspective as:

$$\begin{aligned}(\tilde{x}+\tilde{y})(\varphi+\psi) &= \tilde{x}(\varphi+\psi)+\tilde{y}(\varphi+\psi) \\ (\tilde{x}+\tilde{y})(\alpha\varphi) &= \alpha\tilde{x}(\varphi)+\alpha\tilde{y}(\varphi)\end{aligned}$$

This $\tilde{x}$ is an element of the dual space of V^*, so we write $\tilde{x} \in V^{**}$.

1.3 Basis of a Vector Space and Dual Basis

1.3.1 Basis of a Vector Space

V is a finite-dimensional vector space, and let its dimension be n. If we denote its basis by $\{e_1, e_2, e_3, \cdots, e_n\}$, then $\mathbf{x} \in V$ can be expressed as:

$$\begin{aligned}V \ni \quad \mathbf{x} &= x^1e_1+x^2e_2+x^3e_3+\cdots+x^ne_n \\ &= \sum_{k=1}^{n} x^k e_k\end{aligned}$$

1.3.2 Dual Basis

Considering V^*, its elements were linear functionals φ. We want to consider a basis for V^*, so to make the notation look like a basis, let's represent the linear functionals as e^k (instead of φ, etc.). V^* is spanned by the linear functionals $\{e^1, e^2, e^3, \cdots, e^n\}$. Letting $\varphi(e_1) = a_1, \varphi(e_2) = a_2, \cdots, \varphi(e_n) = a_n$, in this notation, $\varphi \in V^*$ is expressed as:

$$\begin{aligned} V^* \ni \quad \varphi &= a_1 e^1 + a_2 e^2 + a_3 e^3 + \cdots + a_n e^n \\ &= \sum_{k=1}^{n} a_k e^k \end{aligned}$$

With respect to the basis $\{e_1, e_2, e_3, \cdots, e_n\}$ of V, the basis of V^*, namely the set of linear functionals $\{e^1, e^2, e^3, \cdots, e^n\}$, is called the **dual basis**.

1.3.3 Dual Basis of the Dual Space V^{**}

Using the definition of the dual basis:

$$\begin{aligned} V^* \ni \quad \varphi &= a_1 e^1 + a_2 e^2 + a_3 e^3 + \cdots + a_n e^n \\ &= \sum_{k=1}^{n} a_k e^k \end{aligned}$$

Using this, the correspondence with the basis of the vector space is as follows.

$$V^* \ni \quad x^i = e^i(x^1 e_1 + x^2 e_2 + x^3 e_3 + \cdots + x^n e_n)$$

Here, if we set the coefficient of the j-th component to 1 and all other coefficients to 0,

$$e^i(0e_1 + 0e_2 + \cdots + 1e_j + \cdots + 0e_n)$$

From this, the following relation between the bases is obtained.

$$e^i(e_j) = \begin{cases} 1, & (i = j) \\ 0, & (i \neq j) \end{cases}$$

■One-to-one Correspondence between V and V^{**}

For $x, y \in V$ and $\tilde{x}, \tilde{y} \in V^{**}$, $x \neq y \Rightarrow \tilde{x} \neq \tilde{y}$ holds.

This is because,

$$x = \sum_{k=1}^{n} x^i e_j, \quad y = \sum_{k=1}^{n} y^i e_j$$

if we express them as above, then $x \neq y$ implies that $x^j \neq y^j$ holds for some j.

Taking $\{e^1, e^2, e^3, \cdots, e^n\}$ as the dual basis,

$$\tilde{x}(e^j) \;=\; e^j(x) \;=\; x^j, \quad \tilde{y}(e^j) \;=\; e^j(y) \;=\; y^j$$

Therefore, $\tilde{x}$ and $\tilde{y}$ have different values at e^j. Thus, we can say that $x \neq y \Rightarrow \tilde{x} \neq \tilde{y}$.

■Dual Basis of V^{**}

The relation between the dual basis of V^* and the basis of V

$$e^i(e_j) = \begin{cases} 1, & (i = j) \\ 0, & (i \neq j) \end{cases}$$

Expressing this using elements of V^{**}, we get:

$$\tilde{e}_i(e^j) = \begin{cases} 1, & (i = j) \\ 0, & (i \neq j) \end{cases}$$

However, this shows that the dual basis of V^{**}, $\{\tilde{e}_1, \tilde{e}_2, \tilde{e}_3, \cdots, \tilde{e}_n\}$, is the dual basis of the dual basis of V^*, $\{e^1, e^2, e^3, \cdots, e^n\}$.

Summarizing the above in different terms,

One-to-one Correspondence between V and V^{**}

There exists a one-to-one map $\Phi : V \to V^{**}$, which is expressed as:

$$\Phi : V \ni x = \sum_{k=1}^{n} x^i e_i \mapsto \tilde{x} = \sum_{k=1}^{n} x^i \tilde{e}_i \in V^{**}$$

2 Bilinear Space

Up to this point, the dual of a vector space has been a vector space, and its dual is also a vector space (formed by a set of linear maps), so the discussion has been closed within vector spaces. From here, we will consider extending vector spaces.

Consider the Cartesian product set $V^* \times V^*$.

$$V^* \times V^* = \{\ (\tilde{x}, \tilde{y}) \mid \tilde{x} \in V^*, \tilde{y} \in V^*\}$$

We define the bilinearity of a two-variable function $\varphi(\tilde{x}, \tilde{y})$ defined on this $V^* \times V^*$ as follows.

Bilinear Function

When a two-variable function $\varphi(\tilde{x}, \tilde{y})$ defined on $V^* \times V^*$ satisfies the following properties (bilinearity), it is called a **bilinear function** (or bilinear form) on V^*. $(\alpha, \beta \in \mathbb{R})$

1. $\varphi(\alpha\tilde{x} + \beta\tilde{x}', \tilde{y}) = \alpha\varphi(\tilde{x}, \tilde{y}) + \beta\varphi(\tilde{x}', \tilde{y})$
2. $\varphi(\tilde{x}, \alpha\tilde{y} + \beta\tilde{y}') = \alpha\varphi(\tilde{x}, \tilde{y}) + \beta\varphi(\tilde{x}, \tilde{y}')$

A bilinear function is a function that is linear in each variable.

2.1 The Vector Space Formed by All Bilinear Functions

■Addition and Scalar Multiplication of Bilinear Functions

For bilinear functions φ, ψ on V^*, $\varphi + \psi$ and $\alpha\varphi$ are also bilinear functions $(\alpha \in \mathbb{R})$.
In other words, the set of all bilinear functions on V^* forms a vector space.

We will denote the vector space formed by all bilinear functions on V^* as $L_2(V^*)$. The subscript 2 signifies that it is a function of two variables.

Using this notation, the vector space V^{**} formed by all linear functions on V^* can be expressed as $L_1(V^*)$. And this vector space $L_1(V^*)$ could be identified with V.

$$V = L_1(V^*)$$

2.2 Tensor Product

We introduce the tensor product as an operation that connects $L_1(V^*)$ and $L_2(V^*)$.

Tensor Product

Let $V \otimes V = L_2(V^*)$. The vector space $V \otimes V$ is called the **tensor product of V of order 2**.

In this notation,

$$V = L_1(V^*)$$

$$V \otimes V = L_2(V^*)$$

and this is useful for defining higher-order tensor products.

For $x, y \in V$ and $\tilde{x}, \tilde{y} \in V^*$, if we define

$$x \otimes y(\tilde{x}, \tilde{y}) = x(\tilde{x})y(\tilde{y})$$

this becomes a map from $V^* \times V^*$ to $\mathbb{R}$.

$$x \otimes y : V^* \times V^* \to \mathbb{R}$$

This map is an element of the tensor product.

$$x \otimes y \in V \otimes V$$

3 Multilinear Functions and Tensor Spaces

By extending the tensor product notation, we can define multilinear functions and tensor spaces.

k-linear Function

A function $\varphi(\tilde{x}_1, \tilde{x}_1, \cdots, \tilde{x}_k)$ defined on the k-fold Cartesian product $V^* \times \cdots \times V^*$ of V^* that satisfies the following property is called a **k-linear function** on V^*.

$$\varphi(\tilde{x}_1, \tilde{x}_1, \cdots, \alpha\tilde{x}_i + \beta\tilde{y}_i, \cdots, \tilde{x}_k) = \alpha\varphi(\tilde{x}_1, \tilde{x}_1, \cdots, \tilde{x}_i, \cdots, \tilde{x}_k) + \beta\varphi(\tilde{x}_1, \tilde{x}_1, \cdots, \tilde{y}_i, \cdots, \tilde{x}_k)$$

The set of all k-linear functions on V^* also forms a vector space, which we denote by $L_k(V^*)$.

To make it a vector space, for $\varphi, \psi \in L_k(V^*)$ and $\alpha \in \mathbb{R}$, we define addition and scalar multiplication respectively as follows.

Addition and Scalar Multiplication of k-linear Functions

$$(\varphi + \psi)(\tilde{x}_1, \cdots, \tilde{x}_k) = \varphi(\tilde{x}_1, \cdots, \tilde{x}_k) + \psi(\tilde{x}_1, \cdots, \tilde{x}_k)$$

$$(\alpha\varphi)(\tilde{x}_1, \cdots, \tilde{x}_k) = \alpha\varphi(\tilde{x}_1, \cdots, \tilde{x}_k)$$

By varying the natural number k, we obtain a sequence of vector spaces formed by k-linear functions.

$$L_1(V^*)\ ,\quad L_2(V^*)\ ,\quad L_3(V^*)\ ,\quad \cdots\ ,\quad L_k(V^*)\ ,\quad \cdots$$

Just as we denoted $L_2(V^*)$ as $V \otimes V$, we call the vector space formed by the general k-linear space the **k-th order tensor space**, and define it as follows.

k-th Order Tensor Space

$$L_k(V^*) = V \otimes V \otimes \cdots \otimes V = \otimes^k V$$

Although this is just a change in notation, the sequence of vector spaces formed by k-linear functions we just saw is now expressed as follows.

$$\otimes^1 V (= V)\ ,\quad \otimes^2 V\ ,\quad \otimes^3 V\ ,\quad \cdots\ ,\quad \otimes^k V\ ,\quad \cdots$$

Similar to the case of bilinear spaces, for k elements $x_1, x_2, \cdots, x_k$ of V, their tensor product can be defined as the following map.

$$\otimes^k V \ni x_1 \otimes x_2 \otimes \cdots \otimes x_k : V \times V \times \cdots \times V \to \mathbb{R}$$

$$x_1 \otimes \cdots \otimes x_k(\tilde{x}_1, \cdots, \tilde{x}_k) = \tilde{x}_1(x_1) \cdots \tilde{x}_k(x_k)$$

As a note on terminology, one might imagine the 'tensor product' as a product operator, but it should be noted that it is, in fact, 'a map to the real numbers'.

4 Product of Polynomials

Before introducing tensor algebra, let's consider the more familiar and intuitive algebra of polynomials.

First, consider the set of all k-th degree monomials $\mathbf{P}^k = \{ax^k \mid a \in \mathbb{R}\}$.

This $\mathbf{P}^k$ is a one-dimensional vector space isomorphic to $\mathbb{R}$.

$$\begin{array}{rcl}
\mathbf{P}^0 & \ni & a_0 \\
\mathbf{P}^1 & \ni & a_1 x \\
\mathbf{P}^2 & \ni & a_2 x^2 \\
\mathbf{P}^3 & \ni & a_3 x^3 \\
 & \vdots & \\
\mathbf{P}^k & \ni & a_k x^k \\
 & \vdots &
\end{array}$$

These are distinct vector spaces. Their sum, called the **direct sum**, is expressed using $\oplus$:

$$\begin{array}{rcl}
\mathbf{P}^0 & \ni & a_0 \\
\mathbf{P}^0 \oplus \mathbf{P}^1 & \ni & a_0 + a_1 x \\
\mathbf{P}^0 \oplus \mathbf{P}^1 \oplus \mathbf{P}^2 & \ni & a_0 + a_1 x + a_2 x^2 \\
\mathbf{P}^0 \oplus \mathbf{P}^1 \oplus \mathbf{P}^2 \oplus \mathbf{P}^3 & \ni & a_0 + a_1 x + a_2 x^2 + a_3 x^3 \\
 & \vdots & \\
\mathbf{P}^0 \oplus \mathbf{P}^1 \oplus \mathbf{P}^2 \oplus \cdots \oplus \mathbf{P}^k & \ni & a_0 + a_1 x + a_2 x^2 + \cdots + a_k x^k \\
 & \vdots &
\end{array}$$

The product of monomials can be expressed as:

$$\mathbf{P}^k \times \mathbf{P}^l = \mathbf{P}^{k+l} \ni a_k a_l x^{k+l}$$

The set of all polynomials $\mathbf{P}$ is

$$\mathbf{P} = \mathbf{P}^0 \oplus \mathbf{P}^1 \oplus \mathbf{P}^2 \oplus \cdots \oplus \mathbf{P}^k \oplus \cdots$$

We know from experience that the product and sum of polynomials can be defined within this $\mathbf{P}$ constructed this way.

Let's introduce a convenient notation for the direct sum.

$$\mathbf{P} = \mathbf{P}^0 \oplus \mathbf{P}^1 \oplus \mathbf{P}^2 \oplus \cdots \oplus \mathbf{P}^k \oplus \cdots = \bigoplus_{k=0}^{\infty} \mathbf{P}^k$$

5 Tensor Algebra

(Work in Progress ....)