An Introduction to Adjunctions according to Steve Awodey

Reading from `Steve Awodey`, Chapter 9.1

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Abstract

This note summarizes the introduction to the definition of adjunctions, following Chapter 9, Section 1 of Steve Awodey's 2nd Edition [1].

Contents

1 Preparatory Definitions

1.1 Kleene Closure constructing words

Here is an example of one method for 'constructing a free monoid from an arbitrary set'.

Consider the set of alphabetic characters $A = \{a, b, c, ..., y, z\}$.

A finite string of characters (regardless of whether the string is meaningful) is called a **word** on A. For example,

$$word, thisword, categoriesarefun, asdfasdaf, ...$$

The empty string will be denoted by a hyphen '-'.

The **Kleene Closure** is then the operator $(\,\cdot\,)^{\text{Kleene}}$ defined by,

$$A^{\text{Kleene}} = \{-, word, thisword, categoriesarefun, asdfasdaf, ...\}$$

We introduce a string concatenation operation ++ for the elements (words) in the set A^{Kleene}. This defines $++ : A^{\text{Kleene}} \times A^{\text{Kleene}} \to A^{\text{Kleene}}$ such that:

$$\begin{array}{rcl} word\ ++ - & = & word \\ this\ ++\ word\& = & thisword & \\ categories\ ++\ are\ ++\ fun\& = & categoriesarefun & \end{array}$$

The empty string $-$ serves as the identity element.

With this operation, $(A^{\text{Kleene}}, ++)$ becomes a monoid.

Furthermore, A^{Kleene} satisfies the following conditions, making it a **free monoid**:

1. **no junk** (All words can be expressed as a product of elements of A.)
2. **no noise** (For every word, the way it is written as a concatenation of elements from A is unique (apart from the monoid axioms). For example, if $a \neq b$, then $ab \neq ba$.)

1.2 Universal Property of the Free Monoid

The two conditions (no junk, no noise) that make a monoid free can be expressed very neatly using a categorical definition.

First, any monoids M, N have **underlying sets** $U(M), U(N)$.

And any homomorphism $f : N \to M$ has an **underlying map** $U(f) : U(N) \to U(M)$.

This U is a functor, known as the **forgetful functor**.

The free monoid $M(A)$ constructed from a set A has the following **universal property**.

Universal Property of the Free Monoid $M(A)$

There is a map $i : A \to U(M(A))$, such that for any monoid N and any map $f : A \to U(N)$, there exists a **unique** monoid homomorphism $g : M(A) \to N$ satisfying $U(g) \circ i = f$.

This can be summarized neatly in categories.

Diagram for the Universal Property of $M(A)$

Diagram in **Mon**:

$$M(A) \xrightarrow{\exists!\, g} N$$

Diagram in **Set**:

$$\begin{tikzcd} U(M(A)) \arrow[r, "U(g)"] & U(N) \\ A \arrow[u, "i"] \arrow[ur, "f"'] & \end{tikzcd}$$

1.3 A Simple Example of a Free-Forgetful Adjunction

Any monoid M has an underlying set $U(M)$.
Also, as constructed in the previous section, every set X has a **free monoid** $F(X)$.
Consider the map ϕ that sends g to $U(g) \circ i = f$.

$$\begin{array}{rccc} \phi: & \mathrm{Hom}_{\mathbf{Mon}}(F(X), M) & \to & \mathrm{Hom}_{\mathbf{Set}}(X, U(M)) \\ & \Cup & & \Cup \\ & g & \mapsto & U(g) \circ i \end{array}$$

From the universal property of the free monoid, this map is an isomorphism.

$$\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(M))$$

A mnemonic for this is: '**Free is left adjoint to Forgetful**'.

1.4 A Simple Definition of Adjunction

We define an adjunction by generalizing this flow to categories $\mathbf{C}$ and $\mathbf{D}$.

Adjunction between Categories $\mathbf{C}$ and $\mathbf{D}$

An adjunction between categories $\mathbf{C}$ and $\mathbf{D}$ consists of functors F, G

$$F : \mathbf{C} \rightleftharpoons \mathbf{D} : G$$

and a natural transformation $\eta : 1_{\mathbf{C}} \to G \circ F$.
They have the following property.
For any $C \in \mathbf{C}$, $D \in \mathbf{D}$ and $f : C \to G(D)$, there exists a **unique** g such that $f = G(g) \circ \eta_C$ holds as follows.

$$\begin{tikzcd} F(C) \arrow[r, "!g"] & D \end{tikzcd}$$

$$\begin{tikzcd} G(F(C)) \arrow[r, "G(g)"] & G(D) \\ C \arrow[u, "\eta_C"] \arrow[ur, "f"'] & \end{tikzcd}$$

In this case, F is called the **left adjoint** to G, and G is the **right adjoint** to F, written $F \dashv G$.
η is called the **unit** of the adjunction.

2 Example: The Diagonal Functor

2.1 The Right Adjoint to the Diagonal Functor is the Product Functor

As an example, consider the **diagonal functor** $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$.

Objects and morphisms are mapped respectively:

$$\begin{array}{rcll} \Delta(C) & = & (C,C) & \text{for } C \in \mathrm{Obj}(\mathbf{C}) \\ \Delta(f : C \to C') & = & (f,f) : (C,C) \to (C',C') & \text{for } f \in \mathrm{Mor}(\mathbf{C}) \end{array}$$

Let's consider the right adjoint R to the diagonal functor.

Since it goes in the opposite direction of $\Delta : \mathbf{C} \to \mathbf{C} \times \mathbf{C}$, it will be a functor $R : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$. Let's denote its action on objects as

$$R : \mathbf{C} \times \mathbf{C} \ni (X,Y) \mapsto R(X,Y) \in \mathbf{C}$$

Recall the construction of the adjunction.

Recalling the free-forgetful adjunction

$$\mathrm{Hom}_{\mathbf{Mon}}(F(X), M) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(M))$$

and substituting into this correspondence, we get:

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, R(X,Y))$$

The left-hand side (LHS) of this is:

$$\begin{array}{rcl} \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X,Y)) & \cong & \mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((C,C), (X,Y)) \\ & \cong & \mathrm{Hom}_{\mathbf{C}}(C,X) \times \mathrm{Hom}_{\mathbf{C}}(C,Y) \\ & \cong & \mathrm{Hom}_{\mathbf{C}}(C, X \times Y) \end{array}$$

The first isomorphism uses the definition of $\Delta(C)$.

The second isomorphism uses the definition of morphisms in the product category $\mathbf{C} \times \mathbf{C}$.

The third isomorphism uses the universal property of the product $X \times Y$ in $\mathbf{C}$: $\mathrm{Hom}_{\mathbf{C}}(C, X \times Y) \cong \mathrm{Hom}_{\mathbf{C}}(C,X) \times \mathrm{Hom}_{\mathbf{C}}(C,Y)$.

Comparing the LHS and RHS when substituted into the adjunction definition:

$$\mathrm{Hom}_{\mathbf{C}}(C, R(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, X \times Y)$$

We want to apply a corollary of the Yoneda Lemma here:

$$\mathrm{Hom}_{\mathbf{C}}(C, F) \cong \mathrm{Hom}_{\mathbf{C}}(C, G) \ \Rightarrow F \cong G$$

To use this corollary of the Yoneda Lemma, the isomorphism must be natural in C. In this case, by the definition of adjunction, there is a natural isomorphism between

$$\mathrm{Hom}(-, R(X,Y)) \cong \mathrm{Hom}(-, X \times Y)$$

From the above, we can conclude:

$$R(X,Y) \cong X \times Y$$

It has been shown that the right adjoint to the diagonal functor Δ is the product functor $\times$, i.e., $\Delta \dashv \times$.

2.2 The Unit of the Adjunction

Let's consider the unit of the adjunction. By the definition of the adjunction $\Delta \dashv \times$ (i.e., $L = \Delta, R = \times$), the **unit** η is a natural transformation $\eta : 1_{\mathbf{C}} \to R \circ L = \times \circ \Delta$.

Its component η_C, for each object C in $\mathbf{C}$, is a morphism to $(\times \circ \Delta)(C) = \times(\Delta(C)) = \times(C, C) = C \times C$, thus having the form $\eta_C : C \to C \times C$.

This η_C is defined as the morphism on the RHS corresponding to the identity morphism $1_{\Delta(C)} : \Delta(C) \to \Delta(C)$ on the LHS of the adjunction isomorphism

$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}(\Delta(C), (X, Y)) \cong \mathrm{Hom}_{\mathbf{C}}(C, \times(X, Y))$$

by specifically choosing $(X, Y) = \Delta(C) = (C, C)$.

Here, $1_{\Delta(C)}$ is, by the definition of the product category, the pair of morphisms $(1_C, 1_C)$.

$$1_{\Delta(C)} = (1_C, 1_C) : (C, C) \to (C, C)$$

On the other hand, by the universal property of the product $C \times C$

$$\mathrm{Hom}_{\mathbf{C}}(C, C \times C) \cong \mathrm{Hom}_{\mathbf{C}}(C, C) \times \mathrm{Hom}_{\mathbf{C}}(C, C)$$

the morphism in $\mathrm{Hom}_{\mathbf{C}}(C, C \times C)$ corresponding to the pair of morphisms $(1_C, 1_C)$ is the unique morphism $f : C \to C \times C$ satisfying

$$p_1 \circ f = 1_C \quad \text{and} \quad p_2 \circ f = 1_C$$

This is none other than the definition of the so-called **diagonal morphism** δ_C. Therefore, the unit of the adjunction is the diagonal morphism $\eta_C = \delta_C$.

Let's consider the universal property of the unit η.

The universal property of the unit η is expressed in this context as follows.

Any morphism $f : C \to X \times Y \quad (\in \mathbf{C})$ can be factored through η_C and the unique morphism $g : \Delta(C) \to (X, Y) \quad (\in \mathbf{C} \times \mathbf{C})$ that corresponds to f via the adjunction.

If we write the pair of morphisms $g_1 : C \to X$ and $g_2 : C \to Y$ as $g = (g_1, g_2)$, the action of the functor $R = \times$ on morphisms is

$$R(g) = g_1 \times g_2 : C \times C \to X \times Y$$

In this case, from the definition of the adjunction

$$f = R(g) \circ \eta_C$$

we have

$$f = (g_1 \times g_2) \circ \delta_C$$

Expressing this relationship as a commutative diagram gives the following.

$$(C, C) \xrightarrow{\exists!\ (g_1, g_2)} (X, Y)$$

$$\begin{tikzcd}
C \times C \arrow[r, "R(g) = g_1 \times g_2"] & X \times Y \\
C \arrow[u, "\eta_C = \delta_C"] \arrow[ur, "f"'] &
\end{tikzcd}$$

Here, $f : C \to X \times Y$ and $g = (g_1, g_2) : (C, C) \to (X, Y)$ correspond one-to-one via the adjunction.

References

[1] Category Theory 2nd Edition - Steve Awodey