Cauchy's Integral Formula

Masaru Sawata

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Theorem Cauchy's Integral Formula.

If a complex-valued function $f : \mathbb{C} \to \mathbb{C}$ is analytic at any point in a domain Ω closed by the contour C,

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz = f(z_0)$$

Proof. Since f is an analytic function in a domain Ω, $\dfrac{f(z)}{z - z_0}$ is analytic except $z = z_0$. Applying the Cauchy's Integral Theorem to the integral along the contour shown in Figure 1,

$$\oint_C \frac{f(z)}{z - z_0}\, dz + \oint_{-C'} \frac{f(z)}{z - z_0}\, dz = 0$$

where $-C'$ is the curve C' traversed in the oposite direction. Thus,

$$\oint_C \frac{f(z)}{z - z_0}\, dz - \oint_{C'} \frac{f(z)}{z - z_0}\, dz = 0$$

$$\oint_C \frac{f(z)}{z - z_0}\, dz = \oint_{C'} \frac{f(z)}{z - z_0}\, dz$$

Therefore, we can use the smaller contour to evaluate the integral. Let C' be the circle whose radius is r centered at z_0. As mentioned earlier, the integral does not change by r as long as C' is in the C. Then,

$$\begin{aligned}
\oint_C \frac{f(z)}{z - z_0}\, dz &= \oint_{C'} \frac{f(z)}{z - z_0}\, dz \\
&= \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{z_0 + re^{i\theta} - z_0} ire^{i\theta}\, d\theta \\
&= i \int_0^{2\pi} f(z_0 + re^{i\theta})\, d\theta
\end{aligned}$$

Since f is an analytic function, it is continuous at $z = z_0$. Hence, for any positive real value ε, there exists δ such that

$$\left| f(z_0 + re^{i\theta}) - f(z_0) \right| < \varepsilon \quad (0 < r < \delta)$$

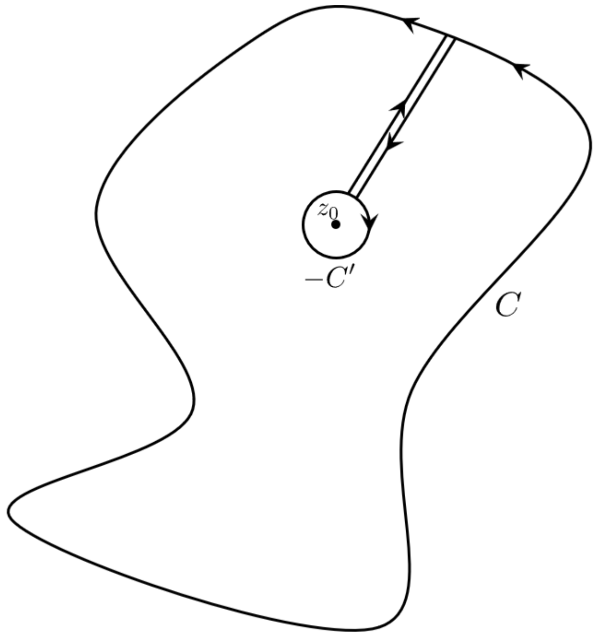

Figure 1: Contour for integration

Thus,

$$\begin{aligned}
\left|\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta - \int_0^{2\pi} f(z_0)\,d\theta\right| &= \left|\int_0^{2\pi} \big(f(z_0+re^{i\theta}) - f(z_0)\big)\ d\theta\right| \\
&\le \int_0^{2\pi} \left|f(z_0+re^{i\theta}) - f(z_0)\right|\ d\theta \\
&< \int_0^{2\pi} \varepsilon\,d\theta \\
&= 2\pi\varepsilon
\end{aligned}$$

The above indicates that there is $r(>0)$ which makes the absolute value of the difference between $\displaystyle\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$ and $\displaystyle\int_0^{2\pi} f(z_0)\,d\theta$ smaller than any positive real number.

Additionally, due to the fact that $\displaystyle\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$ does not change by r (Cauchy's integral theorem), both integral must be the same. Hence,

$$\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta = \int_0^{2\pi} f(z_0)\,d\theta = 2\pi f(z_0)$$

Therefore,

$$\begin{aligned}
\oint_C \frac{f(z)}{z-z_0}\,dz &= i\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta \\
&= 2\pi i f(z_0)
\end{aligned}$$

Thus,

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \, dz = f(z_0)$$

$\square$