Inverse Fourier Transform Calculation Example Using Cauchy's Integral Formula

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Although there are several conventions for the Fourier Transform, the following definition is used here.

$$\mathcal{F}[f](\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$$

The corresponding inverse Fourier Transform is given by

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\, d\xi$$

As an example, we consider the following function:

$$\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + \xi^2}$$

The Inverse Fourier Transform is

$$
\begin{aligned}
\mathcal{F}^{-1}\left[\hat{f}\right](x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\xi)e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2a}{a^2 + \xi^2} e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left[\frac{1}{\xi - ia} - \frac{1}{\xi + ia}\right] e^{i\xi x}\, d\xi \\
&= \frac{1}{2\pi i} \left[\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - ia}\, d\xi - \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi + ia}\, d\xi\right]
\end{aligned}
$$

We can evaluate the integral using Cauchy's integral formula, which is

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\, dz$$

where f is an analytic function within the domain closed by C. In addition, z_0 is in the domain closed by C. ($e^{i\xi x}$ is analytic at any point on the complex plane.)

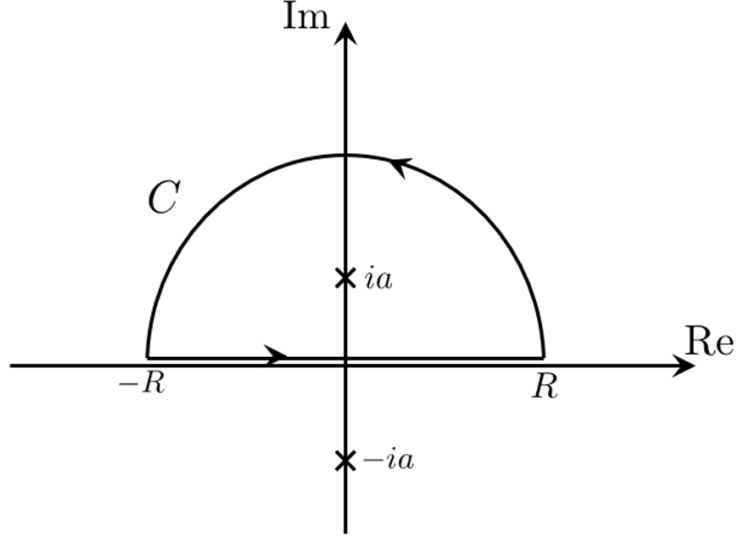If $x > 0$, we consider the semi-disk shown in Figure 1.

Figure 1: Contour C for integration

Since there are no singularities for $\dfrac{e^{izx}}{z + ia}$ inside the contour, we can apply Cauchy's Integral Theorem:

$$\frac{1}{2\pi i} \oint_C \frac{e^{izx}}{z + ia}\, dz = 0$$

Since e^{izx} does not have singularity inside the contour, we can apply Cauchy's integral formula:

$$\frac{1}{2\pi i} \oint_C \frac{e^{izx}}{z - ia}\, dz = e^{-ax}$$

Additionally, the above contour integrals can be written as

$$\frac{1}{2\pi i} \oint_C \frac{e^{izx}}{z + ia}\, dz = \frac{1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi + ia}\, d\xi + \frac{1}{2\pi i} \int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} + ia}\, iRe^{i\theta}\, d\theta = 0$$

$$\frac{1}{2\pi i} \oint_C \frac{e^{izx}}{z - ia}\, dz = \frac{1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi - ia}\, d\xi + \frac{1}{2\pi i} \int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia}\, iRe^{i\theta}\, d\theta = e^{-ax}$$

Here, we evaluate the following integral

$$\int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia}\, iRe^{i\theta}\, d\theta$$

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$$\left| \int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \, d\theta \right| \le \int_0^\pi \left| \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \right| d\theta$$

$$= \int_0^\pi \frac{\left| e^{iRe^{i\theta}x} \right|}{\left| Re^{i\theta} \pm ia \right|} R \, d\theta$$

$$= \int_0^\pi \frac{\left| e^{iR(\cos\theta + i\sin\theta)x} \right|}{\left| Re^{i\theta} \pm ia \right|} R \, d\theta$$

$$= \int_0^\pi \frac{e^{-R\sin\theta x}}{\left| Re^{i\theta} \pm ia \right|} R \, d\theta$$

$$\le \int_0^\pi \frac{e^{-R\sin\theta x}}{|R - a|} R \, d\theta \quad (\text{Using } ||z_1| - |z_2|| \le |z_1 - z_2|)$$

$$= \int_0^\pi \frac{e^{-R\sin\theta x}}{R - a} R \, d\theta \quad (R > a \text{ since } ia \text{ is in the semi-disk.})$$

$$= \frac{R}{R - a} \int_0^\pi e^{-R\sin\theta x} \, d\theta$$

$$= \frac{R}{R - a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R\sin\left(\theta + \frac{\pi}{2}\right)x} \, d\theta$$

$$= \frac{R}{R - a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-R\cos\theta x} \, d\theta$$

$$= \frac{2R}{R - a} \int_0^{\frac{\pi}{2}} e^{-R\cos\theta x} \, d\theta \quad (\cos\theta \text{ is an even function})$$

For the interval of $0 \le \theta \le \dfrac{\pi}{2}$, we can graphically confirm that $\cos\theta \ge -\dfrac{2}{\pi}\theta + 1$. Thus,

$$\left| \int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \, d\theta \right| \le \frac{2R}{R - a} \int_0^{\frac{\pi}{2}} e^{-R\left(-\frac{2}{\pi}\theta + 1\right)x} \, d\theta$$

$$= \frac{2Re^{-Rx}}{R - a} \int_0^{\frac{\pi}{2}} e^{R\frac{2}{\pi}\theta x} \, d\theta$$

$$= \frac{2Re^{-Rx}}{R - a} \left[\frac{\pi}{2Rx} e^{R\frac{2}{\pi}\theta x} \right]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi e^{-Rx}}{(R - a)x} \left(e^{Rx} - 1 \right)$$

$$= \frac{\pi \left(1 - e^{-Rx} \right)}{(R - a)x} \to 0 \quad (R \to \infty)$$

Therefore,

$$\lim_{R \to \infty} \left[\frac{1}{2\pi i} \int_{-R}^R \frac{e^{i\xi x}}{\xi + ia} \, d\xi + \frac{1}{2\pi i} \int_0^\pi \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} + ia} iRe^{i\theta} \, d\theta \right] = \frac{1}{2\pi i} \int_{-\infty}^\infty \frac{e^{i\xi x}}{\xi + ia} \, d\xi = 0$$

$$\lim_{R \to \infty} \left[\frac{1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi - ia} \, d\xi + \frac{1}{2\pi i} \int_{0}^{\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia} iRe^{i\theta} \, d\theta \right] = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - ia} \, d\xi = e^{-ax}$$

Hence, the Inverse Fourier Transform for $x > 0$ is

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \frac{1}{2\pi i} \left[\int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - ia} \, d\xi - \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi + ia} \, d\xi \right]$$
$$= e^{-ax}$$

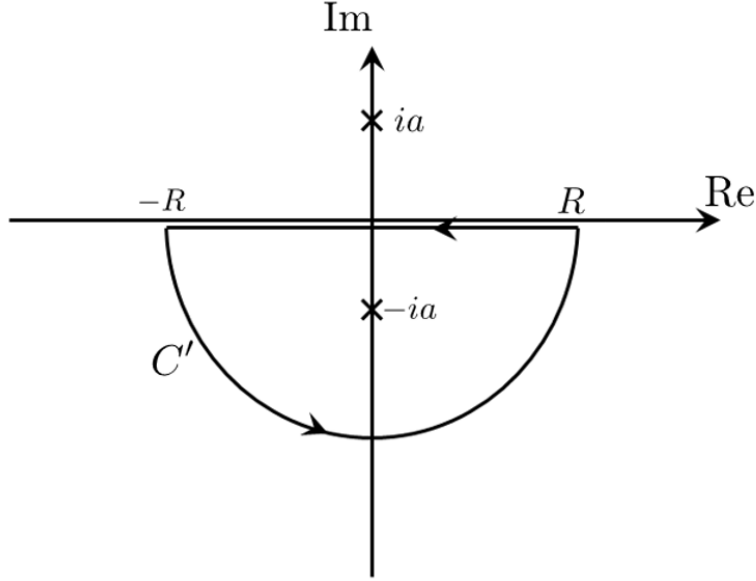If $x < 0$, we consider the semi-disk shown in Figure 2.

Figure 2: Contour C' for integration

We repeat the same calculation using the different contour. Since there are no singularities for $\dfrac{e^{izx}}{z - ia}$ inside the contour, we can apply Cauchy's Integral Theorem:

$$\frac{1}{2\pi i} \oint_{C'} \frac{e^{izx}}{z - ia} \, dz = 0$$

Since e^{izx} does not have singularity inside the contour, we can apply Cauchy's integral formula:

$$\frac{1}{2\pi i} \oint_{C'} \frac{e^{izx}}{z + ia} \, dz = e^{ax}$$

Additionally, the above contour integrals can be written as

$$\frac{1}{2\pi i} \oint_{C'} \frac{e^{izx}}{z - ia} \, dz = \frac{1}{2\pi i} \int_{R}^{-R} \frac{e^{i\xi x}}{\xi - ia} \, d\xi + \frac{1}{2\pi i} \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia} iRe^{i\theta} \, d\theta = 0$$

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$$\frac{1}{2\pi i} \oint_{C'} \frac{e^{izx}}{z+ia} \, dz = \frac{1}{2\pi i} \int_{R}^{-R} \frac{e^{i\xi x}}{\xi + ia} \, d\xi + \frac{1}{2\pi i} \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} + ia} iRe^{i\theta} \, d\theta = e^{ax}$$

Here, we evaluate the following integral

$$\int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \, d\theta$$

$$\left| \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \, d\theta \right| \leq \frac{R}{R-a} \int_{\pi}^{2\pi} e^{-R\sin\theta x} \, d\theta \quad \text{(The same calculation as } x > 0)$$

$$= \frac{R}{R-a} \int_{0}^{\pi} e^{R\sin\theta x} \, d\theta$$

$$= \frac{R}{R-a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{R\sin\left(\theta+\frac{\pi}{2}\right)x} \, d\theta$$

$$= \frac{R}{R-a} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{R\cos\theta x} \, d\theta$$

$$= \frac{2R}{R-a} \int_{0}^{\frac{\pi}{2}} e^{R\cos\theta x} \, d\theta \quad (\cos\theta \text{ is an even function})$$

Since $\cos\theta \geq -\dfrac{2}{\pi}\theta + 1 \; \left(0 \leq \theta \leq \dfrac{\pi}{2}\right)$ and $x < 0$,

$$\left| \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} \pm ia} iRe^{i\theta} \, d\theta \right| \leq \frac{2R}{R-a} \int_{0}^{\frac{\pi}{2}} e^{R\left(-\frac{2}{\pi}\theta+1\right)x} \, d\theta$$

$$= \frac{2Re^{Rx}}{R-a} \int_{0}^{\frac{\pi}{2}} e^{-R\frac{2}{\pi}\theta x} \, d\theta$$

$$= \frac{2Re^{Rx}}{R-a} \left[\frac{\pi}{2Rx} e^{-R\frac{2}{\pi}\theta x} \right]_{0}^{\frac{\pi}{2}}$$

$$= \frac{\pi e^{Rx}}{(R-a)x} \left(e^{-Rx} - 1 \right)$$

$$= \frac{\pi \left(1 - e^{Rx}\right)}{(R-a)x} \to 0 \quad (R \to \infty)$$

Therefore,

$$\lim_{R\to\infty} \left[\frac{-1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi - ia} \, d\xi + \frac{1}{2\pi i} \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} - ia} iRe^{i\theta} \, d\theta \right] = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi - ia} \, d\xi = 0$$

$$\lim_{R\to\infty} \left[\frac{-1}{2\pi i} \int_{-R}^{R} \frac{e^{i\xi x}}{\xi + ia} \, d\xi + \frac{1}{2\pi i} \int_{\pi}^{2\pi} \frac{e^{iRe^{i\theta}x}}{Re^{i\theta} + ia} iRe^{i\theta} \, d\theta \right] = \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi x}}{\xi + ia} \, d\xi = e^{ax}$$

Hence, the Inverse Fourier Transform for $x < 0$ is

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \frac{1}{2\pi i}\left[\int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi - ia}\,d\xi - \int_{-\infty}^{\infty}\frac{e^{i\xi x}}{\xi + ia}\,d\xi\right]$$
$$= e^{ax}$$

If $x = 0$, The Inverse Fourier Transform is

$$\mathcal{F}^{-1}\left[\hat{f}\right](0) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{2a}{a^2 + \xi^2}\,d\xi$$
$$= \frac{a}{\pi}\int_{-\infty}^{\infty}\frac{1}{a^2 + \xi^2}\,d\xi$$
$$= \frac{a}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{a^2 + a^2\tan^2\phi}\frac{a}{\cos^2\phi}\,d\phi \quad (\xi = a\tan\phi)$$
$$= \frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}d\phi$$
$$= 1$$

Therefore,

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = \begin{cases} e^{-ax} & \text{if } x > 0 \\ 1 & \text{if } x = 0 \\ e^{ax} & \text{if } x < 0 \end{cases}$$

This is equivalent to

$$\mathcal{F}^{-1}\left[\hat{f}\right](x) = e^{-a|x|}$$