Laplace Transform of the Convolution Integral

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When we consider the Laplace transform, the convolution of two functions f and g is defined as

$$f * g = \int_0^t f(\tau)g(t-\tau)\,d\tau$$

Note that this definition differs from that used in the Fourier transform.

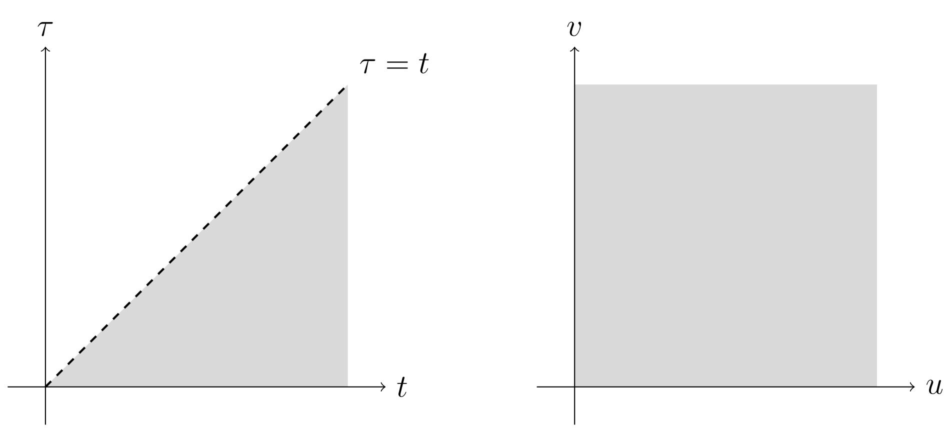
The Laplace transform of the convolution is given by

$$\begin{aligned}\mathcal{L}\left[f * g\right] &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)\,d\tau\, e^{-st}\,dt \\ &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)e^{-st}\,d\tau\,dt\end{aligned}$$

We now perform the change of variables $\tau = u$, $t - \tau = v$, which is equivalent to

$$\begin{aligned}\tau &= u \\ t &= u + v\end{aligned}$$

The integration region transforms as follows:



$0 \le \tau \le t$ and $0 \le t < \infty$ $\qquad$ $0 \le u < \infty$ and $0 \le v < \infty$

The determinant of the Jacobian matrix is 1. Hence,

$$
\begin{aligned}
\mathcal{L}\left[f * g\right] &= \int_0^\infty \int_0^t f(\tau)g(t-\tau)e^{-st}\,d\tau\,dt \\
&= \int_0^\infty \int_0^\infty f(u)g(v)e^{-s(u+v)}\,du\,dv \\
&= \int_0^\infty f(u)e^{-su}\,du \int_0^\infty g(v)e^{-sv}\,dv \\
&= \mathcal{L}\left[f\right]\mathcal{L}\left[g\right]
\end{aligned}
$$