Cauchy's Integral Theorem

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Theorem Cauchy's Integral Theorem.

Let $\Omega \subset \mathbb{C}$ be a simply connected domain, and let f be analytic on Ω. Then for any piecewise smooth, closed curve $C \subset \Omega$:

$$\oint_C f(z)dz = 0$$

Proof. Let u and v be real part of f and imaginary part of f, respectively. Additionally, let x and y be the real part and imaginary part of $z \in \mathbb{C}$. This means

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

Hence,

$$\begin{aligned} f(z)\Delta z &= (u(x,y) + iv(x,y))\,(\Delta x + i\Delta y) \\ &= u(x,y)\Delta x - v(x,y)\Delta y + i\,(v(x,y)\Delta x + u(x,y)\Delta y) \end{aligned}$$

Therefore,

$$\begin{aligned} \oint_C f(z)dz &= \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx - u\,dy) \\ &= \int_\Omega \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dxdy + i\int_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dxdy \quad \text{(Green's Theorem)} \\ &= 0 \quad \text{(Cauchy–Riemann Equations)} \end{aligned}$$

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