

Gaussian Integral

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Sometimes, we need to evaluate the following integral:

$$\int_{-\infty}^{\infty} e^{-x^2} dx$$

Surprisingly, this is equal to $\sqrt{\pi}$.

There are several ways to evaluate this integral, but an elegant and simple method is shown here.

Let I be the value of the integral:

$$I = \int_{-\infty}^{\infty} e^{-x^2} dx$$

We consider I^2 :

$$\begin{aligned} I^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 \\ &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-x^2} dx \\ &= \int_{-\infty}^{\infty} e^{-x^2} dx \int_{-\infty}^{\infty} e^{-y^2} dy \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \end{aligned}$$

Then, Next, we change variables from Cartesian coordinates (x, y) to polar coordinates (r, θ) :

$$\begin{aligned} I^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy \\ &= \int_0^{2\pi} \int_0^{\infty} e^{-r^2} r dr d\theta \\ &= 2\pi \int_0^{\infty} r e^{-r^2} dr \\ &= 2\pi \left[-\frac{1}{2} e^{-r^2} \right]_0^{\infty} \\ &= \pi \end{aligned}$$

Since the integrand is always positive ($e^{-x^2} > 0$), we conclude that $I > 0$. Therefore,

$$\int_{-\infty}^{\infty} e^{-x^2} dx = I = \sqrt{\pi}$$