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# * UNIT-2 *

## * RECURRENCE RELATIONS *

- Some of the counting problems cannot be solved by using sum rule, product rule and permutations and combinations. They can be solved by finding the relation between the terms of sequence known as recurrence relation
- Definition:
- An equation that express $a_n$ in terms of one or more of the previous terms of the sequence known as namely $a_0, a_1, a_2, ---- a_{n-1}$ for all integers $n$ with $n > n_0$ where $n_0$ is a non negative integer is called a Recurrence relation for the sequence $a_n$ or a difference equation.

- Find the first 5 terms of the sequence denoted by the Recurrence relation (i) $a_n = a_{n-1}^2$, $a_0 = 2$ (ii) $a_n = na_{n-1} + n^2 a_{n-2}$, $a_0 = 1$, $a_1 = 1$

(i) given $a_n = a_{n-1}^2 \rightarrow (1)$

$a_0 = 2$

Take $n=1$ in eq (1)

$a_1 = a_0^2 = 2^2 = 4$

Take $n=2$ in eq ①

$a_2 = a_1^2 = 4^2 = 16$

Take $n=3$ in eq ①

$a_3 = a_2^2 = (16)^2 = 256$

Take $n=4$ in eq ①

$a_4 = a_3^2 = (256)^2 = 65536$

(ii) given $a_n = n a_{n-1} + n^2 a_{n-2} \rightarrow$ ①

Take $n=2$ in eq ①

$a_2 = 2a_1 + 4a_0$

$= 2(1) + 4(1) = 6$

Take $n=3$

$a_3 = 3a_2 + 9a_1$

$= 3(6) + 9(1) = 27$

Take $n=4$

$a_4 = 4a_3 + 16a_2$

$= 4(27) + 16(6)$

$= 204$

- Determine whether the sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = a_{n-1} + 2a_{n-2} + 2n - 9$ if (i) $a_n = -n+2$

(iii) $a_n = 3(-1)^n + 2^n - n + 2$

(i) $a_n = -n + 2$

$a_{n-1} = -(n-1) + 2$

$a_{n-2} = -(n-2) + 2$

consider RHS

$a_{n-1} + 2a_{n-2} + 2n - 9$

$= -(n-1) + 2 + 2[-(n-2) + 2] + 2n - 9$

$= -n + 1 + 2 + 2(-n + 2 + 2) + 2n - 9$

$= -n + 3 - 2n + 8 + 2n - 9$

$= -n + 2$

$= a_n = \text{LHS}$

(ii) $a_n = 3(-1)^n + 2^n - n + 2$

$a_{n-1} = 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2$

$a_{n-2} = 3(-1)^{n-2} + 2^{n-2} - (n-2) + 2$

consider RHS

$a_{n-1} + 2a_{n-2} + 2n - 9$

$= 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2[3(-1)^{n-2} + 2^{n-2}$

$- (n-2) + 2] + 2n - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 1 + 2 + 6(-1)^{n-2} + 2^{n-2+1}$

$- 2n + 4 + 4 + 2n - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + 6[(-1)^{n-1} \cdot (-1)^{-1}]$

$+ 2^{n-1} - \cancel{2n} + 8 + \cancel{2n} - 9$

$= 3(-1)^{n-1} + 2^{n-1} - n + 3 + (-6(-1)^{n-1}) + 2^{n-1}$

$= -3(-1)^{n-1} + 2^{1}(2^{n-1}) - n + 2$

$= -3(-1)^{n-1} + 2^{n} - n + 2$

$= -3(-1)^{n}(-1)^{-1} + 2^{n} - n + 2$

$= 3(-1)^{n} + 2^{n} - n + 2$

$= a_n = \text{LHS}$

- Let $a_n = 2^n + 5(3^n)$, for $n = 0, 1, 2, \ldots$

(i) find $a_0, a_1, a_2, a_3, a_4$

(ii) Show that $a_2 = 5a_1 - 6a_0$,

$a_3 = 5a_2 - 6a_1$,

$a_4 = 5a_3 - 6a_2$

(iii) Show that $a_n = 5a_{n-1} - 6a_{n-2}$ for all integers $n$ with $n \geq 2$

given $a_n = 2^n + 5(3^n) \rightarrow (1)$

(i) Take $n=0$ in (1)

$a_0 = 2^0 + 3^0(5)$

$= 6$

Take $n=1$ in (1)

$a_1 = 2^1 + 5(3^1)$

$= 17$

Take $n = 2$

$a_2 = 2^2 + 5(3^2)$

$= 49$

Take $n = 3$

$a_3 = 2^3 + 5(3^3)$

$= 143$

Take $n = 4$

$a_4 = 2^4 + 5(3^4)$

$= 421$

(ii) $a_2 = 5a_1 - 6a_0 \rightarrow (2)$

$a_3 = 5a_2 - 6a_1 \rightarrow (3)$

$a_4 = 5a_3 - 6a_2 \rightarrow (4)$

RHS (2) $5a_1 - 6a_0 = 5(17) - 6(6) = 49 = a_2$

$\therefore a_2 = 5a_1 - 6a_0$

RHS (3) $5a_2 - 6a_1 = 5(49) - 6(17) = 143 = a_3$

$\therefore a_3 = 5a_2 - 6a_1$

RHS (4) $5a_3 - 6a_2 = 5(143) - 6(49) = 421 = a_4$

$\therefore a_4 = 5a_3 - 6a_2$

(iii) $a_n = 5a_{n-1} - 6a_{n-2}$

$\because a_n = 2^n + 5(3^n)$

$a_{n-1} = 2^{n-1} + 5(3^{n-1})$

$$a_{n-2} = 2^{n-2} + 5(3^{n-2})$$

RHS

$$= 5a_{n-1} - 6a_{n-2}$$

$$= 5[2^{n-1} + 5(3^{n-1})] - 6[2^{n-2} + 5(3^{n-2})]$$

$$= 5(2^{n-1}) + 25(3^{n-1}) - 6(2^{n-1})2^{-1} - 30(3^{n-1})3^{-1}$$

$$= 2^{n-1}[5-3] + 3^{n-1}[25-10]$$

$$= 2^1(2^{n-1}) + 15(3^{n-1})$$

$$= 2^n + 5(3^1 . 3^{n-1})$$

$$= 2^n + 5(3^n) = a_n$$

$$= LHS$$

- By using iterative approach find a solution to each of these recurrence relations with the given initial condition

(i) $a_n = a_{n-1} + 2$, $a_0 = 3$

(ii) $a_n = a_{n-1} + n$, $a_0 = 1$

(iii) $a_n = a_{n-1} + 2n + 3$, $a_0 = 4$

(iv) $a_n = 3a_{n-1} + 1$, $a_0 = 1$

(i) $a_n = a_{n-1} + 2 \longrightarrow (1)$

Take $n = 1$, $a_1 = a_0 + 2 = 3 + 2$

Take $n = 2$, $a_2 = a_1 + 2 = (3+2) + 2 = 3 + 2($

Take $n=3$, $a_3 = a_2+2 = (3+2)+2+2$

$= 3+2(3)$

Take $n=4$, $a_4 = a_3+2 = 3+2(3)+2$

$= 3+2(4)$

----- $n$, $a_n = 3+2(n)$

(ii) $a_n = a_{n-1}+n \rightarrow$ (1)

Take $n=1$, $a_1 = a_0+1$

$= 1+1 = 1+(1)$.

Take $n=2$, $a_2 = a_1+2$

$= (1+1)+2 = 1+(1+2)$

Take $n=3$, $a_3 = a_2+3$

$= (1+1)+2+3 = 1+(1+2+3)$

Take $n=4$, $a_4 = a_3+4$

$= 1+(1+2+3+4)$

----- $n$, $a_n = 1+\Sigma(n)$

(iii) $a_n = a_{n-1}+2n+3 \rightarrow$ (1)

Take $n=1$ $a_1 = a_0+2+3$

$= 4+2+3 = 9 = (1+2)^2$

Take $n=2$ $a_2 = a_1+4+3$

$= (2+4+3)+4+3 = (2+2)^2$

Take $n=3$

$a_3 = a_2+2(3)+3 = (3+2)^2$

$= 25$

Take $n=4$

$$a_4 = a_3 + 2(4) + 3$$

$$= 25 + 11 = 36 = (4+2)^2$$

---- $a_n = (n+2)^2$

(iv) $a_n = 3a_{n-1} + 1 \longrightarrow (1)$

$n=1$ $\quad a_1 = 3(1) + 1$

$$= 3+1$$

$n=2$

$$a_2 = 3(3+1)+1 = (3^2+3)+1$$

$n=3$

$$a_3 = 3(12+1)+1$$

$$= (3^3+3^2+3)+1$$

$n=4$

$$a_4 = 3(40)+1$$

$$= (3^4+3^3+3^2+3)+1$$

---- $a_n = (3^1+3^2+3^3+\dots+3^n)+1$

$$= \left(\frac{3^{n+1}-1}{2}\right)$$

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. A person deposits 1000/- in an account that yields 9% interest & compounded yearly.

i) set a recurrence relation for the amount at the end of n years

ii) Find an explicit formula for the amount in the account at the end of n years

iii) How much money will the account contain after 100 years

(i) Let $S_n$ denoted the amount in the account after 'n' years.

But the amount in the account after n years is equal to The amount in the account after (n-1) years + interest for the nth year

i.e $S_n = S_{n-1} + (0.09) S_{n-1}$

since the interest is 9% per year

i.e

$$S_n = (1+0.09) S_{n-1} = (1.09) S_{n-1}$$

$$S_n = (1.09) S_{n-1} \rightarrow (1)$$

This is the recurrence relation for the amount in the account at the end of n years.

(ii) Explicit Formula for $S_n$

$n=0 \Rightarrow S_0 = 1000$

$n=1 \Rightarrow S_1 = (1.09)\, S_0 = (1.09) \times 1000$

$n=2 \Rightarrow S_2 = (1.09)\, S_1 = (1.09)^2 \times 1000$

$n=3 \Rightarrow S_3 = (1.09)\, S_2 = (1.09)^3 \times 1000$

$S_n = (1.09)\, S_{n-1}$

$S_n = (1.09)(1.09)^{n-1} \times 1000$

$S_n = (1.09)^n \times 1000 \longrightarrow$ (2)

using Mathematical Induction we can prove the validity of eq (2)

When $n=0$

$S_0 = (1.09)^0 \times 1000 = 1000$

Therefore the result is true for $n=0$

We assume that eq (2) is true for $n=k$

$S_k = (1.09)^k \times 1000$

We need to prove that $S_{k+1} = (1.09)^{k+1} \times 1000$ is true

From the recurrence relation we have

$S_{k+1} = (1.09)\, S_k$ (from eq (1))

$= (1.09)(1.09)^k \times 1000$

$= (1.09)^{k+1} \times 1000$

$\therefore$ $S_{K+1}$ is true

Thus By the principle of Mathematical Induction $S_n$ is true $\forall$ values of $n$

$\therefore$ The explicit formula is $\boxed{S_n = (1.09)^n \times 1000}$

(iii) When $n = 100$

we have $S_{100} = (1.09)^{100} \times 1000$

$=$

$\therefore$ Money in the account after 100 years is Rs $(1.09)^{100} \times 1000$

- Suppose the number of bacteria in a colony triples every hour

(i) Set up a recurrence relation for the number of bacteria after $n$ hours have elapsed.

(ii) If 100 bacteria are used to begin a new colony, How many bacteria will be there in the colony in 10 hours

(i)

Let $a_n$ be the number of bacteria at the end of $n$ hours

$\therefore$ $a_{n-1}$ is the number of bacteria at the end of $(n-1)$ hours

$\because$ the number of bacteria in a colony triples every hour $a_n = 3 \times a_{n-1} \rightarrow$ ①

This is true whenever $n$ is a positive integer.

Hence the recurrence relation for number of bacteria after $n$ years is

$$\boxed{a_n = 3 \times a_{n-1}}$$

(iii) Let $a_0 = 100$

Take $n=1 \Rightarrow a_1 = 3a_0 = 3 \times 100$

$n=2 \Rightarrow a_2 = 3a_1 = 3^2 \times 100$

$n=3 \Rightarrow a_3 = 3a_2 = 3^3 \times 100$

$\vdots$

$n$ hours

$a_n = 3a_{n-1}$

$= 3(3^{n-1})100$

$= 3^n(100)$

$a_n = 3^n \times 100 \rightarrow (2)$

Mathematical Induction Theorem

Take eq (2) is true for $n=0$

$a_0 = 3^0 \times 100 = 100$

Assume eq (2) is true for $n=k$

$a_k = 3^k \times 100$

We have to prove that it is true for $n=k+1$

For this using Recursion Relation i.e

eq ① $a_{k+1} = 3a_k$

$= 3(3^k)100$

$= 3^{k+1}(100)$

$\therefore$ For $n=k+1$ is true

By principle of Mathematical Induction it is true for all +ve integers (m)

$\therefore$ The explicit Form is

$$a_n = 3^n \times 100$$

Now calculate for $n=10$ hours

$a_{10} = 3^{10} \times 100$

$= 5904900$

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- Fibinocci Sequence

The sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, ---- is called Fibinocci sequence. The recurrence relation for the fibinocci sequence is $F_n = F_{n-1} + F_{n-2}$, $n \geq 2$

Here initial values $F_0 = 0$ and $F_1 = 1$

(or)

The sequence 1, 1, 2, 3, 5, 8, 13, ---- is also called a Fibinocci sequence. Then the recurrence relation of Fibinocci

Sequence is $F_n = F_{n-1} + F_{n-2}$ , $n \geq 2$

Here Initial conditions $F_0 = 1$ , $F_1 = 1$

- Find the Recurrence relation and basis for the sequence $\{1, 3, 3^2, 3^3, \ldots\}$

Let us denote the terms as follows

$a_0 = 1$

$a_1 = 3$

$a_2 = 3^2$

$a_3 = 3^3$ ------ $a_n = 3^n$

Now the recurrence relation for given sequence $a_n = 3a_{n-1}$ and the base is or initial condition $a_0 = 1$

- Solution of Linear Homogeneous recurrence Relation with constant coefficients

The recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}$ where $c_1, c_2, \ldots c_k$ are real numbers and $c_k \neq 0$ is called a linear Homogeneous recurrence relation of degree k with constant coefficients.

The recurrence relation is linear since each $a_i$ has the power 1 and no terms of the type $a_i a_j$ occurred.

The degree of the recurrence relation is $k$ since $a_n$ is expressed in terms of the previous $k$ terms of the sequence that is degree is the difference between greatest and lowest subscript of the member of the sequence occurring in the recurrence relation.

The coefficients of the sequence are all constants. They are not functions of $n$. If $f(n)=0$ then the recurrence relation is said to be Homogeneous otherwise it is said to be non Homogeneous

Eg) Determine whether the following Recurrence relations are Linear Homogeneous recurrence relation with constant coefficients and also find their degree

(i) $a_n = 2a_{n-4} + a_{n-2}^2$

(ii) $H_n = 2H_{n-1} + 2$

(iii) $B_n = nB_{n-1}$

(iv) $a_n = 2na_{n-1} + a_{n-2}$

(v) $a_n = 3a_{n-1} + 4a_{n-2}^2 + 5a_{n-3}$

(vi) $a_n = a_{n-1} + 2$

(vii) $a_n = a_{n-1} + a_{n-4}$

(viii) $a_n = a_{n-1} + n$

(ix) $a_n = a_{n-2}$

(x) $a_n = a_{n-1}^2 + a_{n-2}$

(i) $a_n = 2a_{n-4} + \underline{a_{n-2}^2}$

The given recurrence relation is not an Linear Homogeneous Recurrence Relation

(ii) $H_n = 2H_{n-1} + \underline{2}$

Since constant is there

not LHR relation

(iii). $B_n = \underline{n B_{n-1}}$

Product of n

So not LHR relation

(iv) $a_n = \underline{2n a_{n-1}} + a_{n-2}$

product of 2n

So not LHR relation

(v) $a_n = 3a_{n-1} + \underline{4a_{n-2}^2} + 5a_{n-3}$

power = 2

So not LHR relation

(vi) $a_n = a_{n-1} + \underline{2}$

sum of 2

So not LHR relation

(vii) $a_n = a_{n-1} + a_{n-4}$

given recurrence relation is LHR relation ; degree $= n-(n-4)$

$= 4$

(viii) $a_n = a_{n-1} + \underline{n}$

sum of n

So not LHR relation

(ix) $a_n = a_{n-2}$

given relation is LHR relation

$\therefore$ degree $= n-(n-2) = 2$

(x) $a_n = a^2_{n-1} + \underline{n}$

sum with n

So not LHR relation

- Method of characteristic Roots

In this method the solution of recurrence relation is given $a_n = a_n^{(H)} + a_n^{(P)}$

where $a_n^{(H)}$ is the Homogeneous Solution and $a_n^{(P)}$ is the particular solution of the recurrence relation where f(n) has any one of the form i.e, polynomial in n or constant or powers of constant

- characteristic equation and characteristic roots

If the recurrence relation $a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_K a_{n-K} = 0$ is of degree K then the characteristic equation of Recurrence relation is given by $r^k + c_1 r^{k-1} + c_2 r^{k-2} + \cdots + c_k = 0$

The solution of characteristic equation are called characteristic roots

## To find $a_n^{(H)}$

1. If characteristic roots are distinct $r_1, r_2, r_3, \cdots r_k$ then

$$a_n^{(H)} = c_1 r_1^n + c_2 r_2^n + \cdots + c_k r_k^n$$

where $c_1, c_2, \cdots c_k$ are constants

2. If characteristic equation has equal roots i.e $r_1 = r_2 = r_3 = r_4 = r, r_5, r_6, \cdots \cdots r_k$ then

$$a_n^{(H)} = [c_1 + n c_2 + n^2 c_3 + n^3 c_4] r^n + c_5 r_5^n + c_6 r_6^n + \cdots + c_k r_k^n$$

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3. If the characteristic equation has complex root $a \pm ib$ then $a_n^{(H)}$ is given as

$$a_n^{(H)} = (r)^n [c_1 \cos n\theta + c_2 \sin n\theta]$$

where $r = \sqrt{a^2+b^2}$

$\theta = \tan^{-1}\left(\frac{b}{a}\right)$

. Solve $a_n = a_{n-1} + 2a_{n-2}$ ; $n \geq 2$ with $a_0 = 0$ , $a_1 = 1$

given recurrence relation is $[a_n - a_{n-1} - 2a_{n-2}] = 0$ , $n \geq 2$ with $a_0 = 0$ , $a_1 = 1$ is a Linear Homogeneous Recurrence Relation with constant coefficients of order 2

The characteristic equation is

$$r^2 - r - 2 = 0$$

$$r^{n-(n-2)} - r^{n-(n-1)} - 2r^{n-n} = 0$$

$$r^2 - r - 2 = 0$$

$r = -1, 2$ (Real and distinct)

$$\therefore a_n^{(H)} = c_1(-1)^n + c_2(2)^n \longrightarrow (1)$$

intials are $a_0 = 0$, $a_1 = 1$

Substitute in eq (1)

$$n=0 \Rightarrow a_0 = c_1(-1)^0 + c_2(2)^0$$

$$0 = c_1 + c_2 \longrightarrow (2)$$

$$n = 1 \Rightarrow a_1 = c_1(-1)^1 + c_2(2)^1$$

$$1 = -c_1 + 2c_2 \longrightarrow (3)$$

solving (2), (3) we have

$$c_1 = -1/3 , \quad c_2 = 1/3$$

substitute $c_1, c_2$ in eq (1)

$$\therefore \left\{ a_n^{(H)} = \frac{-1}{3}(-1)^n + \frac{1}{3}(2)^n \right\}$$

- Solve the Recurrence Relation

(i) $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$,

$a_0 = 1$, $a_1 = 0$

(ii) $a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$ for $n = 3, 4,$

where $a_0 = 3$, $a_1 = 6$ and $a_2 = 0$

(iii) Find an explicit formula for the fibinocci numbers

(iv) $a_n = 2a_{n-1} - 2a_{n-2}$, $a_0 = 1$, $a_1 = 2$

(iii) Let Fibinocci series $\{0, 1, 1, 2, 3, 5, 8, \ldots$

$F_n = F_{n-1} + F_{n-2}$; $F_0 = 0$, $F_1 = 1$

The characteristic equation is

$$r^2 - r - 1 = 0$$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$r = \frac{1 \pm \sqrt{5}}{2}$$

$$r = \frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$$

$$\therefore F_n^{(H)} = c_1\left(\frac{1+\sqrt{5}}{2}\right)^n + c_2\left(\frac{1-\sqrt{5}}{2}\right)^n \longrightarrow (1)$$

$F_0 = 0, F_1 = 1$ substituted in eq ①

$\Rightarrow n = 0$

$$F_0 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^0 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^0$$

$$0 = c_1 + c_2 \quad \Rightarrow \quad c_1 = -c_2$$

$n = 1$

$$F_1 = c_1\left(\frac{1+\sqrt{5}}{2}\right)^1 + c_2\left(\frac{1-\sqrt{5}}{2}\right)^1$$

substitute $c_1 = -c_2$

$$1 = -c_2\left(\frac{1+\sqrt{5}}{2}\right) + c_2\left(\frac{1-\sqrt{5}}{2}\right)$$

$$1 = c_2\left[-\frac{1}{2} - \frac{\sqrt{5}}{2} + \frac{1}{2} - \frac{\sqrt{5}}{2}\right]$$

$$1 = c_2(-\sqrt{5})$$

$$\Rightarrow c_2 = \frac{-1}{\sqrt{5}} \quad \therefore \quad c_1 = \frac{1}{\sqrt{5}}$$

$$\therefore \left\{ F_n^{(H)} = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n \right\}$$

(iv) Given Recurrence relation is

$$a_n - 2a_{n-1} + 2a_{n-2} = 0 \text{ with}$$

$$a_0 = 1, \quad a_1 = 2$$

The characteristic equation is

$$r^2 - 2r + 2 = 0$$

$$r = 1 \pm i$$

$$r = \sqrt{a^2+b^2} = \sqrt{1^2+1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = \frac{\pi}{4}$$

Now $a_n^{(H)} = (\sqrt{2})^n \left[c_1 \cos\frac{n\pi}{4} + c_2 \sin\frac{n\pi}{4}\right]$ $\longrightarrow$ ①

Since $a_0 = 1$, $a_1 = 2$

Substitute $n=0$ in ①

$$a_0 = (\sqrt{2})^0 \left[c_1 \cos 0 + c_2 \sin 0\right]$$

$$1 = c_1$$

$n = 1$ in ①

$$a_1 = (\sqrt{2})^1 \left[c_1 \cos\frac{\pi}{4} + c_2 \sin\frac{\pi}{4}\right]$$

$$2 = \sqrt{2}\left[(1)\frac{1}{\sqrt{2}} + c_2\left(\frac{1}{\sqrt{2}}\right)\right]$$

$$\frac{2}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left[c_2 + 1\right]$$

$$c_2 = 1$$

Substitute $c_1$, $c_2$ in eq ①

$$\left\{ a_n^{(H)} = (\sqrt{2})^n \left[\cos\frac{n\pi}{4} + \sin\frac{\pi n}{4}\right] \right\}$$

(i) Given Recurrence Relation is

$$a_n = 5a_{n-1} - 6a_{n-2} \text{ with}$$

$$a_0 = 1, \; a_1 = 0$$

The characteristic equation is

$$r^2 - 5r + 6 = 0$$

$$r = 3, 2$$

$$\therefore \; a_n^{(H)} = c_1(2)^n + c_2(3)^n \rightarrow (1)$$

$n=0$ in eq (1)

$$a_0 = c_1(2)^0 + c_2(3)^0$$

$$1 = c_1 + c_2 \rightarrow (2)$$

$n=1$ in eq (1)

$$a_1 = c_1(2)^1 + c_2(3)^1$$

$$0 = 2c_1 + 3c_2 \rightarrow (3)$$

solving (2), (3) we have

$$c_1 = 3, \; c_2 = -2$$

substitute $c_1, c_2$ in eq (1)

$$\therefore \left\{ a_n^{(H)} = 3(2)^n - 2(3)^n \right\}$$

(ii) Given Recurrence Relation is

$$a_n = 2a_{n-1} + a_{n-2} - 2a_{n-3}$$

with $a_0 = 3$, $a_1 = 6$, $a_2 = 0$

The characteristic equation is

$$r^3 - 2r^2 - r + 2 = 0$$

$$r = -1, 2, 1$$

$$\therefore a_n^{(H)} = c_1(-1)^n + c_2(2)^n + c_3(1)^n \rightarrow (2)$$

Take $n = 0$

$$a_0 = c_1(-1)^0 + c_2(2)^0 + c_3(1)^0$$

$$3 = c_1 + c_2 + c_3 \rightarrow (3)$$

$n = 1$; $a_1 = c_1(-1)^1 + c_2(2)^1 + c_3(1)^1$

$$6 = -c_1 + 2c_2 + c_3 \rightarrow (4)$$

$n = 2$; $a_2 = c_1(-1)^2 + c_2(2)^2 + c_3(1)^2$

$$0 = c_1 + 4c_2 + c_3 \rightarrow (5)$$

From (3), (4), (5) $c_1 = \frac{-6}{43}$, $c_2 = \frac{-123}{43}$,

$c_3 = \frac{498}{43}$ Substitute $c_1, c_2, c_3$ in (2)

$$\therefore \left\{ a_n^{(H)} = \frac{-6}{43}(-1)^n - \frac{123}{43}(2)^n + \frac{498}{43}(1)^n \right\}$$

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- solve the recurrence Relation

$$a_n = 2a_{n-1} - a_{n-2}, \quad a_0 = 25, \ a_1 = 16$$

$a_n = 2a_{n-1} - a_{n-2}$ is given $\rightarrow (1)$

Recurrence relation with $a_0 = 25$, $a_1 = 16$

The characteristic eq of (1) is

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0 \quad \therefore r = 1, 1$$

$$\therefore a_n^{(H)} = (c_1 + c_2 n)(1)^n \rightarrow (2)$$

Take $n = 0$ ; $a_0 = [c_1 + c_2(0)](1)^0$

$$\{25 = c_1\}$$

Take $n = 1$ ; $a_1 = (25 + c_2)$

$$16 = 25 + c_2 \quad \therefore c_2 = -9$$

Substitute $c_1$ and $c_2$ in (2)

$$\therefore \left\{ a_n^{(H)} = (25 - 9n)(1)^n \right\}$$

## * Solution of Non Homogeneous Recurrence relation or InHomogeneous Recurrence relation

- A Linear non Homogeneous Recurrence relation with constant coefficient of degree k is a recurrence relation of the form $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n)$

where $c_1, c_2, c_3, \ldots c_k$ are real numbers and $f(n)$ is a function not identically zero depending only on "n".

To find $a_n^{(P)}$

| S.No | $f(n)$ | $a_n^{(P)}$ |
|---|---|---|
| 1 | A constant, c | $a_n^{(P)} = d$ |
| 2 | $c_0 = c_1 n$<br>$n = 0 + 1 \cdot n$<br>$n^2 = 0 + 0 \cdot n + n^2$ | $d_0 + d_1 k$<br>$d_0 + d_1 k$<br>$d_0 + d_1 k + d_2 k^2$ |
| 3 | $r^n$ | $d r^n$ |

Solve the Recurrence relation

$a_n = 3a_{n-1} + 2^n$ , $a_0 = 1$

The non Homogeneous recurrence relation

is $a_n - 3a_{n-1} = 2^n$ , $a_0 = 1$

$\longrightarrow$ ①

(1) The Associated Homogeneous recurrence relation is

$a_n - 3a_{n-1} = 0$

$\longrightarrow$ ②

It's characteristic equation is

$r - 3 = 0 \Rightarrow r = 3$

$\therefore$ The Homogeneous solution is

$a_n^{(H)} = c_1 3^n \longrightarrow$ ③

(2) Since the RHS of the Recurrence relation is $2^n$ and 2 is not the characteristic root, Let the particular Solution of the recurrence relation be

$a_n^{(P)} = d 2^n$

using this eq in given recurrence relation we get

eq ① $\Rightarrow d2^n - 3d2^{n-1} = 2^n$

$\Rightarrow d2^n - 3d2^n \cdot 2^{-1} = 2^n$

$\Rightarrow 2^n[d - 3d/2] = 2^n$

$$\therefore \quad d - \frac{3d}{2} = 1$$

$$\Rightarrow \quad \boxed{d = -2}$$

$$\therefore \quad a_n^{(P)} = -2(2)^n$$

$$= -(2)^{n+1}$$

Hence the general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$= c_1 3^n + \left[-(2)^{n+1}\right]$$

$$a_n = c_1 3^n - (2)^{n+1}$$

put $n=0$ in above eq

$$a_0 = c_1 3^0 - (2)^{0+1}$$

Since we know that $a_0 = 1$

$$1 = c_1 - 2 \quad \Rightarrow \quad c_1 = 3$$

substitute $c_1$ in general solution

$$a_n = 3(3)^n - (2)^{n+1}$$

$$\therefore \left\{ a_n = 3^{n+1} - 2^{n+1} \right\}$$

- Solve the Recurrence Relation

$$a_n = 2a_{n-1} + 2^n, \quad a_0 = 2$$

The given non Homogeneous recurrence Relation is $a_n - 2a_{n-1} = 2^n \rightarrow (1)$, $a_0 = 2$

(1) The Associated Homogeneous recurrence relation is $a_n - 2a_{n-1} = 0$

The characteristic equation is

$$r - 2 = 0$$

$$r = 2$$

$\therefore$ The Homogeneous solution is

$$a_n^{(H)} = c_1 (2)^n$$

(2) Since the RHS of the recurrence relation is $2^n$ and 2 is the characteristic root, let the particular solution of recurrence relation be

$$a_n^{(P)} = d n 2^n \rightarrow (2)$$

Substitute eq (2) in eq (1)

$$dn2^n - 2[d(n-1)2^{n-1}] = 2^n$$

$$dn2^n - 2^{n-1+1} d(n-1) = 2^n$$

$$2^n [nd - nd + d] = 2^n$$

$$\boxed{d = 1}$$

$$\therefore \quad a_n^{(P)} = n(2^n)$$

The general solution of eq ① is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = c_1(2)^n + n2^n \longrightarrow ③$$

Substitute $n=0$ in eq ③

$$a_0 = c_1(2)^0 + 0(2^0)$$

$\boxed{c_1 = 2}$ substitute $c_1$ in eq ③

$$a_n = 2(2)^n + n2^n$$

$$\{a_n = 2^n(n+2)\}$$

- Solve the Recurrence Relation

$$a_k - 7a_{k-1} + 10a_{k-2} = 6+8k, \quad a_0 = 1,$$

$$a_1 = 2$$

The given non Homogeneous Recurrence relation is

$$a_k - 7a_{k-1} + 10a_{k-2} = 6+8k;$$

$$a_0 = 1, \quad a_1 = 2 \longrightarrow ①$$

(1) The Associated Homogeneous eq is

$$a_k - 7a_{k-1} + 10a_{k-2} = 0$$

The characteristic equation is

$$r^2 - 7r + 10 = 0$$

$$r = 2, 5$$

$$a_K^{(H)} = c_1 (2)^K + c_2 (5)^K \longrightarrow ①$$

(2) We know that the particular solution of RHS of eq ① is

$$a_K^{(P)} = d_0 + d_1 k$$

substitute $a_K$ value in eq ①

$$d_0 + d_1 K - 7\left[d_0 + d_1 (K-1)\right] + 10\left[d_0 + d_1 (K-2)\right] = 6 + 8K$$

$$\Rightarrow d_0 + d_1 K - 7d_0 - 7d_1 K + 7d_1 + 10d_0 + 10d_1 k - 20d_1 = 6 + 8K$$

$$\Rightarrow 4d_0 - 13d_1 + 4d_1 K = 6 + 8K$$

comparing both sides k terms and constant terms, we have

K terms : $4d_1 = 8 \Rightarrow \boxed{d_1 = 2}$

constant terms : $4d_0 - 13d_1 = 6$

$$\Rightarrow \boxed{d_0 = 8}$$

$\therefore$ The general solution of eq ①

is $a_K = a_K^{(H)} + a_K^{(P)}$

$$a_K = c_1(2)^K + c_2(5)^K + 8 + 2k \rightarrow ④$$

Substitue $K=0$ in eq ④

$$a_0 = c_1(2)^0 + c_2(5)^0 + 8 + 2(0)$$

$$1 = c_1 + c_2 + 8 \Rightarrow c_1 + c_2 = -7 \rightarrow ⑤$$

$K = 1$

$$a_1 = c_1(2) + c_2(5) + 8 + 2(1)$$

$$-10 + 2 = 2c_1 + 5c_2$$

$$\Rightarrow 2c_1 + 5c_2 = -8 \rightarrow ⑥$$

⑤, ⑥ $\Rightarrow$ $c_1 = -9$, $c_2 = 2$

Substitue $c_1$, $c_2$ in eq ④

$$\{ a_K = -9(2)^K + 2(5)^K + 8 + 2K \}$$

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- Solve the Recurrence Relation

$$a_n = 4a_{n-1} - 4a_{n-2} + 3n + 2^n ;$$

$a_0 = 1$, $a_1 = 1$

Given non Homogeneous Recurrence Relation

is $a_n - 4a_{n-1} + 4a_{n-2} - 3n - 2^n = 0$, $a_0 = 1$, $a_1 = 1$ $\longrightarrow$ ①

(1) The Associated Homogeneous equation

is

$$a_n - 4a_{n-1} + 4a_{n-2} = 0 \longrightarrow ⓪$$

The characteristic equation is

$$r^2 - 4r + 4 = 0$$

$$\Rightarrow r = 2, 2$$

$$\therefore a_n^{(H)} = (c_1 + c_2 n)(2^n)$$

(2) To Find Particular Solution of RHS

of eq ① is

$$a_n^{(P)} = \underbrace{3n}_{a_n^{(P_1)}} + \underbrace{2^n}_{a_n^{(P_2)}}$$

consider $a_n^{(P_1)} = 3n$ $(n = 0 + 1 \cdot n = d_0 + d_1 n)$

$$\therefore a_n^{(P_1)} = d_0 + d_1 n \longrightarrow ②$$

substitute in eq ①

eq ① $\Rightarrow$

$$(d_0 + d_1 n) - 4(d_0 + d_1(n-1)) + 4(d_0 + d_1(n-2)) = 3n$$

$\Rightarrow d_0 + d_1 n - 4d_0 - 4d_1 n + 4d_1 + 4d_0 + 4d_1 n$

$- 8d_1 = 3n$

$\Rightarrow (d_0 - 4d_1) + d_1 n = 3n$

comparing both sides constant terms and n terms

$d_0 - 4d_1 = 0$ | $d_1 = 3$

$d_0 = 4(3)$

$d_0 = 12$

Substitute $d_0, d_1$ values in eq ②

$a_n^{(P_1)} = 12 + 3n$

consider $a_n^{(P_2)} = 2^n$

The particular solution for $2^n$, characteristic root is repeated twice

$\therefore a_n^{(P_2)} = dn^2(2)^n \rightarrow$ ③ substitute in eq①

$(dn^2(2)^n - 4(d(n-1)^2 2^{n-1} + 4(d(n-2)^2 2^{n-2}) = 2^n$

$\Rightarrow (dn^2 2^n - 4d(n^2+1-2n)2^n \cdot 2^{-1} +$

$4d(n^2+4-4n)2^n \cdot 2^{-2}) = 2^n$

$\Rightarrow 2^n[dn^2 - 4d(n^2+1-2n)\cdot 2^{-1} + 4d(n^2+4-4n)2^{-2}]$

$= 2^n$

$\Rightarrow dn^2 - 2d(n^2+1-2n) + d(n^2-4n+4) = 1$

$\Rightarrow dn^2 - 2dn^2 - 2d + 4dn + dn^2 - 4dn + 4d = 1$

$\Rightarrow 2d = 1 \Rightarrow d = 1/2$

substitute in eq ③

$$a_n^{(P_2)} = \frac{1}{2}(n^2)(2^n)$$

$$a_n^{(P_2)} = n^2(2)^{n-1}$$

substitute $a_n^{(P_1)}$ and $a_n^{(P_2)}$ in eq

$$a_n^{(P)} = a_n^{(P_1)} + a_n^{(P_2)}$$

$$a_n^{(P)} = 12 + 3n + n^2(2)^{n-1}$$

∴ The general solution of eq ① is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = (c_1+c_2n)2^n + 12 + 3n + n^2(2)^{n-1}$$

Take $n=0$

$a_0 = (c_1+0) + 12$

$(a_0 = 1) \Rightarrow 1 = c_1 + 12 \Rightarrow c_1 = -11$

Take $n=1$

$$a_1 = (c_1+c_2)2+12+3+1$$

$$1-1 = (c_2-11)2+15$$

$$c_2 = \frac{-15}{2}+11 = 7/2$$

$$\therefore \left\{ a_n = \left[-11+\left(\frac{7}{2}\right)n\right]2^n+12+3n+n^2(2)^{n-1} \right\}.$$

- Solve the recurrence relation

$$a_n - 2a_{n-1} + a_{n-2} = 2, \text{ with } a_0 = 25,$$

$$a_1 = 16$$

given $a_n - 2a_{n-1} + a_{n-2} = 2 \longrightarrow (1)$

The Associated Homogeneous equation

is $a_n - 2a_{n-1} + a_{n-2} = 0$

The characteristic equation is

$$r^2 - 2r + 1 = 0$$

$$r = 1, 1$$

$$\therefore a_n^{(H)} = (c_1 + c_2 n)(1)^n$$

since RHS of eq (1) is '2' then

the particular solution is

$a_n^{(P)} = d$

substitute in eq ①

$d - 2d + d = 2$

$0 = 2$

Let $a_n^{(P)} = nd$

substitute in eq ①

$nd - 2(n-1)d + (n-2)d = 2$

$nd - 2nd + 2d + nd - 2d = 2$

$0 = 2$

Let $a_n^{(P)} = n^2 d \rightarrow$ ②

substitute in eq ①

$n^2d - 2((n-1)^2 d) + (n-2)^2 d = 2$

$n^2d - 2[(n^2+1-2n)d] + [(n^2+4-4n)d] = 2$

$n^2d - 2n^2d - 2d + 4nd + n^2d + 4d - 4nd = 2$

$\Rightarrow 2d = 2$

$\Rightarrow d = 1$

substitute d value in eq ②

$a_n^{(P)} = n^2$

∴ The general solution is

$$a_n = a_n^{(H)} + a_n^{(P)}$$

$$a_n = [c_1 + c_2 n](1)^n + n^2 \quad \longrightarrow (3)$$

Substitute $n = 0$

$$a_0 = [c_1 + c_2(0)](1)^0 + 0^2$$

$$25 = c_1$$

Substitute $n = 1$

$$a_1 = [c_1 + c_2] + 1$$

$$c_2 = 16 - 25 - 1$$

$$c_2 = -10$$

Substitute $c_1$, $c_2$ in eq (3)

$$\therefore \left\{ a_n = [25 - 10(n)](1)^n + n^2 \right\}$$