❀ UNTT - 1 ❀

❀ Mathematical Logic ❀

Mathematical Logic is one of the concept of Discrete mathematical structures that deals with the method of reasoning. It provides rules and techniques for determining whether a given argument or mathematical proof or conclusion is valid or not. Logic is concerned with studying arguments and conclusion. Logic used in computer science to verify the correctness of a program. Mathematical logic is divided into two components

- propositional calculus
- predicate calculus

Proposition

A proposition or statement is a declarative sentence which is either true or false but not both.

The truth or falsi of a proposition is called the truth value. These truth values true and false are denoted by symbols T and F respectively. Sometimes these are also denoted by the symbols 1 and 0 respectively.

Eg) consider the following sentences :

- Delhi is the capital of India
- Kolkata is a country
- 5 is a prime number
- $2+3=4$

These are propositions or statements because they are either true or false

consider the following sentences :

- How beautiful are you?
- Wish you a happy new year!
- $x+y=z$
- Take a book

These are not propositions as they are not declarative in nature i.e they do not declare a definite truth value T or F.

→ Propositional calculus is also known as a statement calculus. It is the branch of mathematics that is used to describe a logical system or structure.

→ A Logical system consists of a universal proposition, truth tables for the logical operators and definitions that explain equivalence and implication of proposition.

→ Connectives:

The word or phrase or symbols which are used to make a sentence or proposition by two or more propositions are called logical connectives or simply connectives. There are 5 basic connectives called negation (~), conjunction (∧), disjunction (∨), conditional (→), biconditional (⇒)

→ Negation:

The negation of a statement is generally formal by writing the word 'not' at a proper place in the statement or by prefixing the statement with phrase. If the truth value of 'p' is T then the truth value of '~p' is F. Also if the truth value of p is false then the truth value of '~p' is true. Here 'p' denotes the statement then the '~p' is written as '~p' or '¬p' and read as "not p"

Truth table for negation }

P	~P
T	F
F	T

Note) In general if there are 'n' distinct statements, we need to consider 2^n possible combinations of truth values in order to obtain the truth table.

→ conjunction :

conjunction of two statements p and q is denoted by '$p \wedge q$' and read as "p and q"

Truth table for conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

→ Disjunction :

Disjunction of two statements p and q is denoted by '$p \vee q$' and read as "p or q"

Truth table for disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

→ Conditional :

The two statements p and q, the conditional statement can be written as '$p \to q$' which is read as "if p then q" or "p implies q".

Truth table for conditional statement

p	q	$p \to q$
T	T	T
T.	F	F
F	T	T
F	F	T

→ Biconditional :

If p and q are two statements the biconditional statement can be written as '$p \leftrightarrow q$' and read as "p if and only if q".

Truth table for biconditional proposition

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Eg) Translate the following statements

⊙ "Jack and Jill went up the hill" into symbolic form using conjunction.

let p : Jack went up the hill

q : Jill went up the hill

∴ The symbolic form of Jack and Jill went up the hill is '$p \wedge q$'

⊙ "The crop will be destroyed if there is a flood" into symbolic form using conditional connective

Let p : The crop will be destroyed

q : There is a flood

∴ The crop will be destroyed if there is a flood in symbolic form is $q \rightarrow p$

⊙ If "p: It is cold" and

"q: It is raining". Write the simple

verbal sentences which describes each of the following statements.

(i) $\sim p$ (ii) $p \wedge q$ (iii) $p \vee q$ (iv) $p \vee \sim q$

sol) $\sim p$: It is not cold

$p \wedge q$: It is cold and raining

$p \vee q$: It is cold or raining

$p \vee \sim q$: It is cold or not raining

→ Statement Formula :

The statement formula which contains one or more statements and atleast one connective are called compound statements.

Eg) $\sim p$, $p \vee q$, $p \to q$, $\sim(p \vee q)$

The above compound statements are also called as statement formula derived from the statement variables p and q.

→ Well Formed Formula :

A well formed formula can be generated by the following rules.

- A statement variable standing alone is a well formed formula.
- If 'A' is a well formed formula then '$\sim A$' is also a well formed formula.
- If 'A' and 'B' are well formed formulae then $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \leftrightarrow B)$ are well formed formulas

- A string of variables containing statement variables, connectives and paranthesis is a well formed formula if and if it can be obtained by finite applications of rules 1,2,3.

Eg) $(P\to(q\to r))$ is a well formed formula

- $P\to q\to r$ is not a well formed formula
- $(P\vee(q\wedge\neg P))$ is a well formed formula
- $P\vee(P\wedge\neg P)$ is not a well formed formula

* converse, inverse and contrapositive

If $P\to q$ is a conditional statement, then

- $q\to P$ is called converse
- $\sim P\to\sim q$ is called inverse
- $\sim q\to\sim P$ is called contrapositive

→ construct the truth table for each compound preposition

(a) $(P\wedge(\sim q\vee q))$

(b) $(\sim(P\vee q)\vee(\sim P\wedge\sim q))$

(c) $((P\to q)\wedge(q\to P))$

(d) $(((\sim(P\wedge q))\vee r)\to(\sim P))$

(a) $(P\wedge(\sim q\vee q))$

p	q	$\sim q$	$\sim q \vee q$	$p \wedge (\sim q \vee q)$
T	T	F	T	T
T	F	T	T	T
F	T	F	T	F
F	F	T	T	F

(b) $(\sim(p \vee q) \vee (\sim p \wedge \sim q))$

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$	$(\sim(p \vee q) \vee (\sim p \wedge \sim q))$
T	T	F	F	T	F	F	F
T	F	F	T	T	F	F	F
F	T	T	F	T	F	F	F
F	F	T	T	F	T	T	T

(c) $((p \rightarrow q) \wedge (q \rightarrow p))$

p	q	$p \rightarrow q$	$q \rightarrow p$	$((p \rightarrow q) \wedge (q \rightarrow p))$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

(d) $((\sim(p \wedge q) \vee r) \rightarrow (\sim p))$

p	q	r	$p \wedge q$	$\sim p$	$\sim(p \wedge q)$	$(\neg(p \wedge q) \vee r)$	$((\neg(p \wedge q) \vee r) \rightarrow (\neg p))$
T	T	T	T	F	F	T	F
T	T	F	T	F	F	F	T
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	F	T	T	T	T
F	T	F	F	T	T	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

→ construct the truth table for the following statement formulas

(a) $p \wedge (\sim q)$

(b) $((\sim p) \vee q)$

(c) $(q \wedge ((\sim r) \rightarrow p))$

(d) $(p \rightarrow (q \rightarrow r))$

(e) $(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q))$

(f) $((p \vee q) \wedge r)$

(g) $(p \vee (q \wedge r))$

(h) $(p \vee \sim q) \rightarrow p$

(a) $p \wedge (\sim q)$

p	q	$\sim q$	$p \wedge (\sim q)$
T	T	F	F
T	F	T	T
F	T	F	F
F	F	T	F

(b) $((\sim p) \vee q)$

p	q	$\sim p$	$((\sim p) \vee q)$
T	T	F	T
T	F	F	F
F	T	T	T
F	F	T	T

(c) $(q \wedge ((\sim r) \rightarrow p))$

p	q	r	$\sim r$	$((\neg r) \rightarrow p))$	$(q \wedge ((\neg r) \rightarrow p))$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	T	F
T	F	F	T	T	F
F	T	T	F	T	T
F	T	F	T	F	F
F	F	T	F	T	F
F	F	F	T	F	F

(d) $(p \to (q \to r))$

p	q	r	$(q \to r)$	$(p \to (q \to r))$
T	T	T	T	T
T	T	F	F	F
T	F	T	T	T
T	F	F	T	T
F	T	T	T	T
F	T	F	F	T
F	F	T	T	T
F	F	F	T	T

(e) $\left.(p \wedge q) \vee (\neg p \wedge q) \vee (p \wedge \neg q) \vee (\neg p \wedge \neg q)\right\} A$

p	q	$\neg p$	$\neg q$	$p \wedge q$	$\neg p \wedge q$	$p \wedge \neg q$	$\neg p \wedge \neg q$	A
T	T	F	F	T	F	F	F	T
T	F	F	T	F	F	T	F	T
F	T	T	F	F	T	F	F	T
F	F	T	T	F	F	F	T	T

(h) $(p \vee \sim q) \to p$

p	q	$\sim q$	$(p \vee \sim q)$	$(p \vee \sim q) \to p$
T	T	F	T	T
T	F	T	T	T
F	T	F	F	T
F	F	T	T	F

(f) $((p \vee q) \wedge r)$

p	q	r	$(p \vee q)$	$((p \vee q) \wedge r)$
T	T	T	T	T
T	T	F	T	F
T	F	T	T	T
T	F	F	T	F
F	T	T	T	T
F	T	F	T	F
F	F	T	F	F
F	F	F	F	F

(g) $(p \vee (q \wedge r))$

p	q	r	$(q \wedge r)$	$(p \vee (q \wedge r))$
T	T	T	T	T
T	T	F	F	T
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	F	F
F	F	T	F	F
F	F	F	F	F

* Tautalogy and Contradiction

Tautology

Each entry in the final column of a truth table of a statement formula is 'T' alone, then it is called as tautology

Contradiction

Each entry in the final column of the truth table of a statement formula is 'F' alone then it is called as contradiction

→ If statement formula is neither tautology nor contradiction is called a contigency

- Verify whether $P \vee (\sim P)$ is a tautology

P	$\neg P$	$P \vee (\sim P)$
T	F	T
F	T	T

$\therefore$ The final column of statement formula is 'T' alone so the given statement formula is tautology

• verify $P \wedge (\sim P)$ is a tautology.

P	$\neg P$	$P \wedge (\neg P)$
T	F	F
F	T	F

$\because$ The final column of statement formula is 'F' alone and not 'T', the given statement is a contradiction.

→ construct the truth table & verify whether tautology or not for the following

(a) $\sim(\sim P \wedge \sim q)$ (b) $(\sim P \leftrightarrow \sim q) \leftrightarrow (q \rightarrow r)$

(c) $(P \wedge (P \leftrightarrow q)) \rightarrow q$ (d) $(\sim P \wedge (P \rightarrow q)) \rightarrow \neg q$

(b) $\left.(\sim P \leftrightarrow \sim q) \leftrightarrow (q \rightarrow r)\right\} A$

P	q	r	$\sim P$	$\sim q$	$\neg P \leftrightarrow \neg q$	$q \rightarrow r$	A
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

∵ Final column is not 'T' or 'F' alone, hence it is contigency

(a) $\sim(\sim p \wedge \neg q)$

p	q	$\sim p$	$\sim q$	$\neg p \wedge \neg q$	$\sim(\sim p \wedge \sim q)$
T	T	F	F	F	T
T	F	F	T	F	T
F	T	T	F	F	T
F	F	T	T	T	F

∵ the given statement formula is contigency as the final column contains both T and F

(c) $(p \wedge (p \leftrightarrow q)) \rightarrow q$

p	q	$(p \leftrightarrow q)$	$(p \wedge (p \leftrightarrow q))$	$(p \wedge (p \leftrightarrow q)) \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	T	F	T

∵ the final column of the statement formula is 'T' alone, the given statement is tautology

(d) $\underbrace{(\sim p \wedge (p \rightarrow q)) \rightarrow \neg q}_{B}$

p	q	~p	~q	(p→q)	(¬p∧(p→q))	B
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	T	T	F	T	F	T
F	F	T	T	T	T	T

∵ the final column of the statement formula is 'T' alone, the given statement is tautology

- verify (b) $(\sim p \wedge (p \rightarrow q)) \rightarrow q$

  (a) $(p \rightarrow q) \rightarrow q$ is a tautology

(a) $(p \rightarrow q) \rightarrow q$

p	q	(p→q)	(p→q)→q
T	T	T	T
T	F	F	T
F	T	T	T
F	F	T	F

The given statement is not a Tautology

It is a contigency

(b) $(\sim p \wedge (p \to q)) \to q \} A$

p	q	$\sim p$	$(\sim p \wedge (p \to q))$	$(p \to q)$	A
T	T	F	F	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	F

The given statement is contigency but not a tautology

* Equivalence of Formulas

Two formulas A and B are said to be equivalent to each other if and only if $A \leftrightarrow B$ is tautology. If $A \leftrightarrow B$ is a tautology we write '$A \Leftrightarrow B$' which is read as "A is equivalent to B".

Note :

(i) "$\Leftrightarrow$" is only a symbol but not connective

(ii) "$A \leftrightarrow B$" is a tautology if and only if truth tables of A and B are the same

(iii) Equivalence relation is symmetric and transitive

* Method 1 (Truth Table Method)

one method to determine whether any two statement formulas are equivalent is to construct their truth tables.

- prove $p \vee q \Leftrightarrow \sim(\sim p \wedge \sim q)$

$p \vee q : A$

$\sim(\sim p \wedge \sim q) : B$

p	q	$\sim p$	$\sim q$	$p \vee q$ Ⓐ	$\sim p \wedge \sim q$ ①	B : $\sim$①	$A \leftrightarrow B$
T	T	F	F	T	F	T	T
T	F	F	T	T	F	T	T
F	T	T	F	T	F	T	T
F	F	T	T	F	T	F	T

$\therefore$ $(p \vee q) \leftrightarrow \sim(\sim p \wedge \sim q)$ is a tautology

$(p \vee q) \Leftrightarrow \neg(\neg p \wedge \neg q)$

* Equivalence Formulas

- Idempotent Laws

(a) $p \vee p \Leftrightarrow p$ (b) $p \wedge p \Leftrightarrow p$

- Associative Laws

(a) $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

(b) $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

- Commutative Laws.

(a) $p \vee q \Leftrightarrow q \vee p$ (b) $p \wedge q \Leftrightarrow q \wedge p$

- Identity Laws

(a) (i) $p \vee F \Leftrightarrow p$ (b) (i) $p \wedge T \Leftrightarrow p$

(ii) $p \vee T \Leftrightarrow T$ (ii) $p \wedge F \Leftrightarrow F$

- Distributive Laws

(a) $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

(b) $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

- Component laws

(a) (i) $(p \vee (\sim p)) \Leftrightarrow T$ (b) (i) $\sim(\sim p) \Leftrightarrow p$

(ii) $p \wedge \sim p \Leftrightarrow F$ (ii) $\sim T \Leftrightarrow F$

(iii) $\sim F \Leftrightarrow T$

- Absorption Laws

(a) $p \vee (p \wedge q) \Leftrightarrow p$ (b) $p \wedge (p \vee q) \Leftrightarrow p$

- DeMorgan Laws

(a) $\sim(p \vee q) \Leftrightarrow \sim p \wedge \sim q$

(b) $\sim(p \wedge q) \Leftrightarrow \sim p \vee \sim q$

(c) $(p \rightarrow q) \Leftrightarrow \sim p \vee q$

(d) $p \leftrightarrow q \Leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$

* Method 2 (Replacement process)

consider the formula $A:(P \to (q \to r))$

The formula $(q \to r)$ is a part of the formula A. If we replace $(q \to r)$ by an equivalent formula $(\sim q \vee r)$ in A we get another formula B

$B: (P \to (\sim q \vee r))$

One can easily verify that the formulae A and B are equivalent to each other.

This process of obtaining B from A is known as the replacement process.

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- Show that (a) $(P \to q) \wedge (P \to r)$ and $P \to (q \wedge r)$.

(b) $P \to (q \to r)$ and $(P \wedge q) \to r$

(c) $\sim(P \vee (\sim P \wedge q))$ and $(\sim P \wedge \sim q)$

(d) $(\sim P \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (P \wedge r)$ and r

are logically equivalent without using truth tables

(a) $(P \to q) \wedge (P \to r)$

$\Leftrightarrow (\neg P \vee q) \wedge (\neg P \vee r)$ $(\because P \to q \Leftrightarrow \sim P \vee q)$

$\Leftrightarrow \sim P \vee (q \wedge r)$ (Distributive law)

$\Leftrightarrow P \to (q \wedge r)$ $(\because \sim P \vee q \Leftrightarrow P \to q)$

$\therefore (P \to q) \wedge (P \to r) \Leftrightarrow P \to (q \wedge r)$

(b) $p \to (q \to r)$

$\Leftrightarrow p \to (\neg q \vee r)$ $(\because q \to r \Leftrightarrow \sim q \vee r)$

$\Leftrightarrow \sim p \vee (\sim q \vee r)$

$\Leftrightarrow (\sim p \vee \sim q) \vee r$ $(\because (p \vee (q \vee r)) \Leftrightarrow (p \vee q) \vee r)$

$\Leftrightarrow \sim (p \wedge q) \vee r$ $(\because \sim p \vee \sim q \Leftrightarrow \sim (p \wedge q))$

$\Leftrightarrow (p \wedge q) \to r$

$\therefore p \to (q \to r) \Leftrightarrow (p \wedge q) \to r$

(c) $\sim (p \vee (\sim p \wedge q))$

$\Leftrightarrow \sim ((p \vee \sim p) \wedge (p \vee q))$ ($\because$ Distributive Law)

$\Leftrightarrow \sim (T \wedge (p \vee q))$ (component Law)

$\Leftrightarrow \sim (p \vee q)$ ($T \wedge p \Leftrightarrow p$; identity law)

$\Leftrightarrow \sim p \wedge \sim q$

$\therefore \sim (p \vee (\sim p \wedge q)) \Leftrightarrow \sim p \wedge \sim q$

(d) $(\sim p \wedge (\sim q \wedge r)) \vee (q \wedge r) \vee (p \wedge r)$

$\Leftrightarrow (\sim p \wedge (\sim q \wedge r)) \vee ((q \vee p) \wedge r)$ (distributive law)

$\Leftrightarrow ((\sim p \wedge \sim q) \wedge r) \vee ((q \vee p) \wedge r)$ (Associative)

$\Leftrightarrow ((\sim p \wedge \sim q) \vee (q \vee p)) \wedge r$ (distributive Law)

$\Leftrightarrow (\sim (p \vee q) \vee (p \vee q)) \wedge r$ (De Morgan Law)

$\Leftrightarrow (T \wedge r)$ (component Law)

$\Leftrightarrow r$ (identity Law)

* Duality Law

Two formulas A and A^* are said to be duals of each other if either one can be obtained from the other by replacing $\vee$ by $\wedge$ and $\wedge$ by $\vee$.

The connectives $\vee$ and $\wedge$ are called duals of each other. If the formula A contain the special variable T or F then A^* it's dual is obtained by replacing T by F and F by T; in addition to the above mentioned interchanges.

Eg) Write the duals of the following formulas

(a) $(P \vee q) \wedge r$ (b) $(P \wedge q) \vee T$ (c) $(P \wedge q) \vee F$

(d) $(P \wedge q) \vee (P \vee \sim(q \wedge \sim S))$

The duals of the formulas are

(a) $(P \wedge q) \vee r$

(b) $(P \vee q) \wedge F$

(c) $(P \vee q) \wedge T$

(d) $(P \vee q) \wedge (P \wedge \sim(q \vee \sim S))$

Note)

(i) The negation of the formula is equivalent to its dual in which every variable is replaced by its negation i.e

$$\sim A(P_1, P_2, \dots P_n) \Leftrightarrow A^*(\sim P_1, \sim P_2, \dots \sim P_n)$$

(ii) If $A \Leftrightarrow B$ then the dual is also equivalent

$$A^* \Leftrightarrow B^*$$

Eg) Prove that (a) $\sim(P \wedge q) \rightarrow (\sim P \vee (\sim P \vee q)) \Leftrightarrow \sim P \vee q$

(b) $(P \vee q) \wedge (\sim P \wedge (\sim P \wedge q)) \Leftrightarrow \sim P \wedge q$

(a) $\sim(P \wedge q) \rightarrow (\sim P \vee (\sim P \vee q))$

$\Leftrightarrow \sim(\sim(P \wedge q)) \vee (\sim P \vee (\sim P \vee q))$

[$P \rightarrow q \Leftrightarrow \sim P \vee q$]

$\Leftrightarrow (P \wedge q) \vee ((\sim P \vee \sim P) \vee q)$

[Associative Law]

$\Leftrightarrow (P \wedge q) \vee (\sim P \vee q)$ [Distributive Law]

$\Leftrightarrow (P \vee \sim P \vee q) \wedge (q \vee \sim P \vee q)$

$\Leftrightarrow (T \vee q) \wedge (q \vee \sim P)$

$\Leftrightarrow T \wedge (q \vee \sim P)$ [Identity Law]

$\Leftrightarrow q \vee \sim P$

$\Leftrightarrow \sim P \vee q$ = RHS

$\therefore \sim(P \wedge q) \rightarrow (\sim P \vee (\sim P \vee q)) \Leftrightarrow \sim P \vee q$

(b) From (a)

$\sim(p\wedge q) \rightarrow (\sim p \vee (\sim p \vee q)) \Leftrightarrow \sim p \vee q$

$\sim(\sim(p\wedge q) \vee (\sim p \vee (\sim p \vee q)) \Leftrightarrow \sim p \vee q$

$(p\wedge q) \vee (\sim p \vee (\sim p \vee q)) \Leftrightarrow \sim p \vee q \rightarrow ①$

dual of eq ①:

$(p\vee q) \wedge (\sim p \wedge (\sim p \wedge q)) \Leftrightarrow \sim p \wedge q$

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* Tautological Implications

A statement x is said to tautologically implied to a statement y if and only if "$x \rightarrow y$" is a tautology.

- We shall denote this idea by "$x \Rightarrow y$" which is read as "x implies y" that is x implies y means $x \rightarrow y$ is a tautology

* Implication Formulas

- $p\wedge q \Rightarrow p$
- $p\wedge q \Rightarrow q$
- $p \Rightarrow p\vee q$
- $q \Rightarrow p \vee q$
- $\sim p \Rightarrow p \rightarrow q$
- $q \Rightarrow p \rightarrow q$
- $\sim(p\rightarrow q) \Rightarrow p$
- $\sim(p \rightarrow q) \Rightarrow \sim q$
- $p\wedge(p\rightarrow q) \Rightarrow q$
- $\sim q \wedge (p\rightarrow q) \Rightarrow \sim p$
- $\sim p \wedge (p\vee q) \Rightarrow q$
- $(p\rightarrow q)\wedge(q\rightarrow r) \Rightarrow p\rightarrow r$
- $(p\vee q)\wedge(p\rightarrow r)\wedge(q\rightarrow r) \Rightarrow r$

- Write the equivalent formula for $P \wedge (q \leftrightarrow r) \vee (r \leftrightarrow P)$. Thus does not contain the biconditional statement

$P \wedge (q \leftrightarrow r) \vee (r \leftrightarrow P)$

$\Leftrightarrow P \wedge ((q \rightarrow r) \wedge (r \rightarrow q)) \vee ((r \rightarrow P) \wedge (P \rightarrow r))$

$\Leftrightarrow P \wedge ((\sim q \vee r) \wedge (\sim r \vee q)) \vee ((\sim r \vee P) \wedge (\sim P \vee r))$

which is required equivalent formula

- Write the formulas which are equivalent to the formulas given below and contain the connectives $\wedge, \sim$

a) $((P \vee q) \wedge r) \rightarrow (P \vee r)$

b) $\sim (P \leftrightarrow (r \vee P))$

(a) $((P \vee q) \wedge r) \rightarrow (P \vee r)$

$\Leftrightarrow \sim [((P \vee q) \wedge r)] \vee [(P \vee r)]$

$\Leftrightarrow \sim [(P \wedge r) \vee (q \wedge r)] \vee (P \vee r)$

$\Leftrightarrow [\sim (P \wedge r) \wedge \sim (q \wedge r)] \vee (P \vee r)$

$\Leftrightarrow ((\sim P \vee \sim r) \wedge (\sim q \vee \sim r)) \vee (P \vee r)$

$\Leftrightarrow ((\sim P \wedge \sim q) \vee \sim r) \vee (P \vee r)$

$\Leftrightarrow (\sim P \wedge \sim q) \vee T \vee P \quad [\because \sim r \vee r \Leftrightarrow T]$

$\Leftrightarrow (\sim P \wedge \sim q) \vee T$

$\Leftrightarrow (\sim p \vee T) \wedge (\sim q \vee T)$

$\Leftrightarrow T \wedge T \qquad [T \Leftrightarrow p \vee \sim p \text{ or } q \vee \sim q]$

$\Leftrightarrow (p \vee \sim p) \wedge (q \vee \sim q)$

$\Leftrightarrow \sim(\sim p \wedge \sim(\sim p)) \wedge \sim(\sim q \wedge \sim(\sim q))$

$\Leftrightarrow \sim(\sim p \wedge p) \wedge \sim(\sim q \wedge q)$

which is required equivalent formula

(b) $\sim(p \leftrightarrow (r \vee p))$

$\Leftrightarrow \sim[(p \rightarrow (r \vee p)) \wedge ((r \vee p) \rightarrow p)]$

$\Leftrightarrow \sim[(\sim p \vee (r \vee p)) \wedge (\sim(r \vee p) \vee p)]$

$\Leftrightarrow \sim[(\sim p \vee (p \vee r)) \wedge (\sim(r \vee p) \vee p)]$

$\Leftrightarrow \sim[((\sim p \vee p) \vee r) \wedge (\sim r \wedge \sim p \vee p)]$

$\Leftrightarrow \sim[(T \vee r) \wedge (\sim r \wedge T)]$

$\Leftrightarrow \sim[T \wedge (\sim r)]$

$\Leftrightarrow \sim[\sim(\sim p \wedge p) \wedge (\sim r)]$

* Other connectives

- Exclusive OR

Let P and Q be any two propositions, the exclusive OR (or) XOR of P and Q denoted by "$P \oplus Q$" or "$P \bar{\vee} q$" is the proposition that is true when exactly one of P and Q is true but not both and is false otherwise

Truth table for XOR

P	Q	$P \bar{\vee} q$
T	T	F
T	F	T
F	T	T
F	F	F

- The Exclusive OR is also called the Exclusive disjunction
- The following are some of the important results of XOR

  - $P \bar{\vee} q \Leftrightarrow q \bar{\vee} P$ (symmetric)
  - $(P \bar{\vee} q) \bar{\vee} r \Leftrightarrow P \bar{\vee} (q \bar{\vee} r)$ (Associative)
  - $P \wedge (q \bar{\vee} r) \Leftrightarrow (P \wedge q) \bar{\vee} (P \wedge r)$ (Distributive)
  - $P \bar{\vee} q \Leftrightarrow (P \wedge \sim q) \bar{\vee} (\sim P \wedge q)$
  - $P \bar{\vee} q \Leftrightarrow \sim (P \leftrightarrow q)$

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- NAND

The word NAND is combination of NOT and AND where NOT stands for negation and AND stands for conjunction. The connective NAND is denoted by the symbol "$\uparrow$" or "$|$". For any two formulas P and q "$P \uparrow q \Leftrightarrow \sim(P \wedge q)$"

Truth Table for NAND

P	q	$P \wedge q$	$\sim(P \wedge q) \Leftrightarrow P \uparrow q$
T	T	T	F
T	F	F	T
F	T	F	T
F	F	F	T

- NAND Equalences :

- $P \uparrow P \Leftrightarrow \sim(P \wedge P) \Leftrightarrow \sim P \vee \sim P \Leftrightarrow \sim P$

- $(P \uparrow q) \uparrow (P \uparrow q) \Leftrightarrow \sim(P \uparrow q) \Leftrightarrow \sim(\sim(P \wedge q))$

$$\Leftrightarrow P \wedge q$$

- $(P \uparrow P) \uparrow (q \uparrow q) \Leftrightarrow \sim P \uparrow \sim q \Leftrightarrow \sim(\sim P \wedge \sim q)$

$$\Leftrightarrow P \vee q$$

- NAND commutative : let P and q be any 2 statements

$$P \uparrow q \Leftrightarrow \sim(P \wedge q)$$

$$\Leftrightarrow \sim(q \wedge P)$$

$$\Leftrightarrow q \uparrow P$$

- NAND is not associative :

$$p \uparrow (q \uparrow r) = \sim(p \wedge (q \uparrow r))$$

$$= \sim(p \wedge \sim(q \wedge r))$$

$$= \sim p \vee \sim(\sim(q \wedge r))$$

$$= \sim p \vee (q \wedge r)$$

$$(p \uparrow q) \uparrow r = \sim((p \uparrow q) \wedge r)$$

$$= \sim(\sim(p \wedge q) \vee \sim r)$$

$$= (p \wedge q) \vee \sim r$$

$$\therefore \; p \uparrow (q \uparrow r) \neq (p \uparrow q) \uparrow r$$

- NOR

The word NOR is a combination of NOT and OR where NOT stands for negation and OR stands for disjunction. The connective NOR is denoted by the symbol "$\downarrow$" For any two formulas p and q "$p \downarrow q \Leftrightarrow \sim(p \vee q)$"

Truth Table for NOR

p	q	$p \vee q$	$\sim(p \vee q)$
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

. NOR equivalance

- $p \downarrow p \Leftrightarrow \sim(p \vee p) \Leftrightarrow \sim p \wedge \sim p \Leftrightarrow \sim p$

- $(p \downarrow q) \downarrow (p \downarrow q) \Leftrightarrow \sim(p \downarrow q) \Leftrightarrow \sim(\sim(p \vee q))$

  $\Leftrightarrow p \vee q$

- $(p \downarrow p) \downarrow (q \downarrow q) \Leftrightarrow \sim p \downarrow \sim q \Leftrightarrow \sim(\sim p \vee \sim q)$

  $\Leftrightarrow p \wedge q$

- NOR commutative : let p and q be any two statements

$$p \downarrow q \Leftrightarrow \sim(p \vee q)$$

$$\Leftrightarrow \sim(q \vee p)$$

$$\Leftrightarrow q \downarrow p$$

- NOR is not associative :-

$$p \downarrow (q \downarrow r) = \sim(p \vee (q \downarrow r))$$

$$= \sim(p \vee \sim(q \vee r))$$

$$= \sim p \wedge \sim(\sim(q \vee r))$$

$$= \sim p \wedge (q \vee r)$$

$$(p \downarrow q) \downarrow r = \sim((p \downarrow q) \vee r)$$

$$= \sim(\sim(p \vee q)) \wedge \sim r$$

$$= (p \vee q) \wedge \sim r$$

$$\therefore (p \downarrow q) \downarrow r \neq p \downarrow (q \downarrow r)$$

- $p \uparrow q$ } Write it in terms of "$\downarrow$" only

$$p \uparrow q = \sim(p \wedge q)$$

$$= (p \wedge q) \downarrow (p \wedge q)$$

$$= (p \downarrow p) \downarrow (q \downarrow q) \downarrow (p \downarrow p) \downarrow (q \downarrow q)$$

- Express $p \downarrow q$ in terms of "$\uparrow$"

$$p \downarrow q = \sim(p \vee q)$$

$$[\equiv \sim p \wedge \sim q] = (p \vee q) \uparrow (p \vee q)$$

$$= (p \uparrow p) \uparrow (q \uparrow q) \uparrow (p \uparrow p) \uparrow (q \uparrow q)$$

- Show that $(A \oplus B) \vee (A \downarrow B) \Leftrightarrow A \uparrow B$

$$\underbrace{(A \bar{\vee} B) \vee (A \downarrow B)}_{P} \Leftrightarrow \underbrace{A \uparrow B}_{Q}$$

A	B	$A \bar{\vee} B$	$A \downarrow B$	$(A \bar{\vee} B) \vee (A \downarrow B)$	$A \uparrow B$	$P \Leftrightarrow Q$
T	T	F	F	F	F	T
T	F	T	F	T	T	T
F	T	T	F	T	T	T
F	F	F	T	T	T	T

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- Normal Forms / Canonical Forms :

  - The problem of determining in a finite number of steps whether a given statement formula is a tautology or a contradiction or at least contigency is called a decision problem.

  - So solving these decision problems by constructing and comparing the truth tables may be difficult even with the help of computers. Therefore reduce the given statement formula to Normal formula and find whether a given statement formula is a tautology or a contradiction or at least contigency.

  - In our current discussion we use the word product is in the place of conjunction and sum is in the place of disjunction.

- Elementary product :

A product of the variable and their negation in a formula is called elementary product

Eg) $P \wedge Q$, $\sim P \wedge Q$, $P \wedge \sim Q$ etc

- Elementary Sum

A sum of the variables and their negations in a formula is called an elementary sum.

Eg) $\sim q \vee P$, $P \vee q$, $P \vee \sim q$ etc

- Disjunctive Normal Form (DNF)

A formula which is equivalent to a given formula and which consists of a sum of elementary product is called DNF of the given formula

- Conjunctive Normal Form (CNF)

A formula which is equivalent to a given formula and consists of a product of elementary sum is called CNF of the given formula.

- Obtain the DNF and CNF of $P \wedge (P \rightarrow q)$

DNF

$P \wedge (P \rightarrow q)$

$\Leftrightarrow P \wedge (\sim P \vee q)$ $[\because P \rightarrow q \Leftrightarrow \sim P \vee q]$

$\Leftrightarrow (P \wedge \sim P) \vee (P \wedge q)$

[Distributive Law
$P \vee (q \wedge r) \Leftrightarrow$
$(P \vee q) \wedge (P \vee r)$]

CNF

$P \wedge (P \rightarrow q)$

$\Leftrightarrow P \wedge (\sim P \vee q)$

- obtain DNF of $\sim(P \vee q) \leftrightarrow (P \wedge q)$

$\sim(P \vee q) \leftrightarrow (P \wedge q)$

$\Leftrightarrow (\sim(P \vee q) \wedge (P \wedge q)) \vee (\sim(\sim(P \vee q)) \wedge \neg(P \wedge q))$

$[\because P \leftrightarrow q \Leftrightarrow (P \wedge q) \vee (\sim P \wedge \sim q)]$

$\Leftrightarrow (\sim P \wedge \sim q \wedge P \wedge q) \vee ((P \vee q) \wedge (\sim P \vee \sim q))$

$\Leftrightarrow (\sim P \wedge \sim q \wedge P \wedge q) \vee ((P \wedge \sim P) \vee (P \wedge \sim q) \vee (q \wedge \sim P) \vee (q \wedge \sim q))$

DNF

CNF :

$\sim(P \vee q) \leftrightarrow (P \wedge q)$

$\Leftrightarrow (\sim(P \vee q) \longrightarrow (P \wedge q)) \wedge ((P \wedge q) \longrightarrow \sim(P \vee q))$

$\Leftrightarrow (\sim(\sim(P \vee q)) \vee (P \wedge q)) \wedge ((\sim(P \wedge q)) \vee \sim(P \vee q))$

$\Leftrightarrow ((P \vee q) \vee (P \wedge q)) \wedge ((\sim P \vee \sim q) \vee \quad))$

$\Leftrightarrow ((P \vee q) \vee (\sim(\sim P \vee \sim q))) \wedge ((\sim P \vee \sim q) \vee (\sim P \vee \sim q))$

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- obtain CNF and DNF of the following

(i) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

(ii) $p \rightarrow [(p \rightarrow q) \wedge \sim(\sim q \vee \sim p)]$

(i) CNF :

$\sim(p \vee q) \leftrightarrow (p \wedge q)$

$(\because p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p))$

$\Leftrightarrow [\sim(p \vee q) \rightarrow (p \wedge q)] \wedge [(p \wedge q) \rightarrow \sim(p \vee q)]$

$(\because p \rightarrow q \Leftrightarrow \sim p \vee q)$

$\Leftrightarrow [\sim(\sim(p \vee q)) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee \sim(p \vee q)]$

$\Leftrightarrow [(p \vee q) \vee (p \wedge q)] \wedge [\sim(p \wedge q) \vee (\sim p \wedge \sim q)]$

(ii) $p \rightarrow [(p \rightarrow q) \wedge \sim(\sim q \vee \sim p)]$

DNF :

$\Leftrightarrow p \rightarrow [(p \rightarrow q) \wedge (q \wedge p)]$ ($\because$ Double negation)

$\Leftrightarrow p \rightarrow [(\sim p \vee q) \wedge (q \wedge p)]$ ($\because p \rightarrow q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow \sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\Leftrightarrow \sim p \vee [(\sim p \wedge q \wedge p) \vee (q \wedge q \wedge p)]$

$[\because p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)]$

$\Leftrightarrow \sim p \vee [(F \wedge q) \vee (q \wedge p)]$ ($\because p \wedge \sim p \Leftrightarrow F$)

$\Leftrightarrow \sim p \vee [F \vee (p \wedge q)]$ ($\because F \vee p \Leftrightarrow p$)

$\Leftrightarrow \sim p \vee (p \wedge q)$ // DNF

$\Leftrightarrow (\sim p \vee p) \wedge (\sim p \vee q)$ $[\because p \vee \sim p \Leftrightarrow T]$

$\Leftrightarrow T \wedge (\sim p \vee q)$

$\Leftrightarrow \sim p \vee q$ // CNF

- Principle Disjunctive Normal Form (PDNF)

Minterm

For a given number of variables, the minterm consists of conjunction in which each variable or it's negations but not both appears only once.

- Let p and q be two statement variables construct all possible formulas that consists of conjuctions of p or it's negation and conjunction of q or it's negation. None of the formulas should contain both variable and it's negation. These formulas are called minterms or boolean conjunction of p and q. For 2 variables p and q $(p \wedge q)$, $(p \wedge \sim q)$, $(\sim p \wedge q)$, $(\sim p \wedge \sim q)$ are called minterms. For 3 variables p, q and r the minterms are $(p \wedge q \wedge r)$, $(\sim p \wedge q \wedge r)$, $(p \wedge \sim q \wedge r)$, $(p \wedge q \wedge \sim r)$, $(\sim p \wedge \sim q \wedge r)$, $(\sim p \wedge q \wedge \sim r)$, $(p \wedge q \wedge \sim r)$, $(\sim p \wedge \sim q \wedge \sim r)$

- Definition (PDNF)

For a given formula an equivalent formula consisting of disjunction of minterms only is known as it's PDNF. Such a normal form is also called the "Sum of products canonical Form".

- Principle Conjunctive Normal Form (PCNF)

Maxterm

For a given number of variables the maxterm consists of disjunction in which each variable or it's negation but not both appears only once.

The maxterms are the dual of the minterms. For example for any two variables p and q the maxterms are $(\sim p \vee \sim q)$, $(\sim p \vee q)$, $(p \vee \sim q)$, $(p \vee q)$. For any 3 variables p, q and r the maxterms are $(p \vee q \vee r)$, $(p \vee q \vee \sim r)$, $(p \vee \sim q \vee r)$, $(\sim p \vee q \vee r)$, $(\sim p \vee \sim q \vee r)$, $(p \vee \sim q \vee \sim r)$, $(\sim p \vee q \vee \sim r)$, $(\sim p \vee \sim q \vee \sim r)$

Definition

For a given formula an equivalent formula consisting of conjunction of maxterms only is known as it's PCNF. Such a normal form is also called "Product of Sums canonical Form".

- obtain PDNF of $\sim p \vee q$

p	q	minterms	$\sim p$	$\sim p \vee q$
T	T	$p \wedge q$	F	T
T	F	$p \wedge \sim q$	F	F
F	T	$\sim p \wedge q$	T	T
F	F	$\sim p \wedge \sim q$	T	T

$\therefore$ PDNF $\}$ $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

method (ii)

$\sim p \vee q$ $\quad (\because p \wedge T \Leftrightarrow p)$

$\Leftrightarrow (\sim p \wedge T) \vee (q \wedge T)$ $\quad (\because T \Leftrightarrow p \vee \sim p)$

$\Leftrightarrow (\sim p \wedge (q \vee \sim q)) \vee (q \wedge (p \vee \sim p))$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (q \wedge p) \vee (q \wedge \sim p)$

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge q)$

- Obtain PDNF of the formulas

  (i) $(p \to q)$ (iii) $\sim(p \wedge q)$

  (ii) $(p \vee q)$

p	q	minterms	$(p \to q)$	$(p \vee q)$	$(p \wedge q)$	$\sim(p \wedge q)$
T	T	$p \wedge q$	T	T	T	F
T	F	$p \wedge \sim q$	F	T	F	T
F	T	$\sim p \wedge q$	T	T	F	T
F	F	$\sim p \wedge \sim q$	T	F	F	T

$\therefore$ PDNF of

$(p \to q) : (p \wedge q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

$(p \vee q) : (p \wedge q) \vee (p \wedge \sim q) \vee (\sim p \wedge q)$

$\sim(p \wedge q) : (p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

- Obtain PDNF of $p \to [(p \to q) \to \sim(\sim q \to \sim p)]$

p	q	minterm	$\sim p$	$\sim q$	(A) $p \to q$	① $\sim q \to \sim p$	② $\sim$①	(B) $A \to$②	$p \to B$
T	T	$p \wedge q$	F	F	T	T	F	F	F
T	F	$p \wedge \sim q$	F	T	F	F	T	T	T
F	T	$\sim p \wedge q$	T	F	T	T	F	F	T
F	F	$\sim p \wedge \sim q$	T	T	T	T	F	F	T

$\therefore$ PDNF of

$p \to [(p \to q) \to \sim(\sim q \to \sim p)]$:

$(p \wedge \sim q) \vee (\sim p \wedge q) \vee (\sim p \wedge \sim q)$

method (ii)

$p \to [(p \to q) \to \sim(\sim q \to \sim p)]$

$\Leftrightarrow p \to [(p \to q) \to \sim(\sim(\sim q) \vee \sim p)]$ ($\because p \to q \Leftrightarrow \sim p \vee q$)

$\Leftrightarrow p \to [(\sim p \vee q) \to (\sim q \wedge p)]$

$\Leftrightarrow p \to [\sim(\sim p \vee q) \vee (\sim q \wedge p)]$

$\Leftrightarrow \sim p \vee [(p \wedge \sim q) \vee (\sim q \wedge p)]$

$\Leftrightarrow \sim p \vee [(p \wedge \sim q) \vee (p \wedge \sim q)]$ ($p \vee p \Leftrightarrow p$)

$\Leftrightarrow \sim p \vee [p \wedge \sim q]$

$\Leftrightarrow (\sim p \wedge T) \vee (p \wedge \sim q)$ ($\because T \Leftrightarrow q \vee \sim q$)

$\Leftrightarrow (\sim p \wedge (q \vee \sim q)) \vee (p \wedge \sim q)$ ($\because p \wedge T \Leftrightarrow p$)

$\Leftrightarrow (\sim p \wedge q) \vee (\sim p \wedge \sim q) \vee (p \wedge \sim q)$

- obtain PDNF of

(i) $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$

(ii) $\sim[(p \to (\sim p \to (\sim q \to r)))]$

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• obtain PCNF of

(a) $\sim(p \vee q) \leftrightarrow (p \wedge q)$

(b) $(\sim p \wedge (p \rightarrow q)) \rightarrow \sim q$

a) $\underbrace{\sim(p \vee q) \leftrightarrow (p \wedge q)}_{A}$

p	q	Maxterm	$p \vee q$	$\sim(p \vee q)$	$p \wedge q$	A
T	T	$\sim p \vee \sim q$	T	F	T	F
T	F	$\sim p \vee q$	T	F	F	T
F	T	$p \vee \sim q$	T	F	F	T
F	F	$p \vee q$	F	T	F	F

$\therefore$ PCNF is $(\sim p \vee \sim q) \wedge (p \vee q)$

method (ii)

$\sim(p \vee q) \leftrightarrow (p \wedge q)$ $\quad (p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\sim p \wedge \sim q))$

$\Leftrightarrow [(\sim(p \vee q)) \wedge (p \wedge q)] \vee [\sim(\sim(p \vee q)) \wedge \sim(p \wedge q)]$

$\Leftrightarrow [(\sim p \wedge \sim q) \wedge (p \wedge q)] \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$

$\Leftrightarrow [\sim p \wedge \sim q \wedge p \wedge q] \vee [(p \vee q) \wedge (\sim p \vee \sim q)]$

$\Leftrightarrow [F \vee [(p \vee q) \wedge (\sim p \vee \sim q)]]$ $\quad (\because p \wedge \sim p \Leftrightarrow F$, $q \wedge \sim q \Leftrightarrow F)$

$\Leftrightarrow (p \vee q) \wedge (\sim p \vee \sim q)$ $\quad (F \vee P \Leftrightarrow P)$

b) $(\sim p \wedge (p \rightarrow q)) \rightarrow \sim q$

p	q	maxterms	① $\sim p$	$\sim q$	② $p \rightarrow q$	(B) ① $\wedge$ ②	$B \rightarrow \sim q$
T	T	$\sim p \vee \sim q$	F	F	T	F	T
T	F	$\sim p \vee q$	F	T	F	F	T
F	T	$p \vee \sim q$	T	F	T	T	F
F	F	$p \vee q$	T	T	T	T	T

$\therefore$ PCNF is $(p \vee \sim q)$

method 2

- obtain the PDNF of $\sim\{p \vee (\sim p \wedge \sim q \wedge r)\}$

p	q	r	$\sim p$	$\sim q$	$\sim p \wedge \sim q \wedge r$ (A)	$p \vee (A)$ ①	$\sim$①
T	T	T	F	F	F	T	F
T	T	F	F	F	F	T	F
T	F	T	F	T	F	T	F
T	F	F	F	T	F	T	F
F	T	T	T	F	F	F	T
F	T	F	T	F	F	F	T
F	F	T	T	T	T	T	F
F	F	F	T	T	F	F	T

$\therefore$ PDNF is $(\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge \sim r)$

- obtain PCNF of the following formulas

(i) $\sim(p \vee q)$

(ii) $\sim(p \to q)$

(iii) $\sim(p \leftrightarrow q)$

(iv) $[p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]]$

also find PDNF

p	q	maxterms	$p \vee q$	$\sim(p \vee q)$	$p \to q$	$\sim(p \to q)$	$p \leftrightarrow q$	$\sim(p \leftrightarrow q)$
T	T	$\sim p \vee \sim q$	T	F	T	F	T	F
T	F	$\sim p \vee q$	T	F	F	T	F	T
F	T	$p \vee \sim q$	T	F	T	F	F	T
F	F	$p \vee q$	F	T	T	F	T	F

$\therefore$ PCNF of

(i) $\sim(p \vee q)$ is $(\sim p \vee \sim q) \wedge (\sim p \vee q) \wedge (p \vee \sim q)$

(ii) $\sim(p \rightarrow q)$ is $(\sim p \vee \sim q) \wedge (p \vee \sim q) \wedge (p \vee q)$

(iii) $\sim(p \leftrightarrow q)$ is $(\sim p \vee \sim q) \wedge (p \vee q)$

(iv)

p	q	minterm	Maxterm	$\sim p$	$\sim q$	② $p \rightarrow q$	① $\sim q \vee \sim p$	$\sim$①	③ ② $\wedge \sim$①	$p \rightarrow$ ③
T	T	$p \wedge q$	$\sim p \vee \sim q$	F	F	T	F	T	T	T
T	F	$p \wedge \sim q$	$\sim p \vee q$	F	T	F	T	F	F	F
F	T	$\sim p \wedge q$	$p \vee \sim q$	T	F	T	T	F	F	T
F	F	$\sim p \wedge \sim q$	$p \vee q$	T	T	T	T	F	F	T

$\therefore$ PCNF of $[p \rightarrow [(p \rightarrow q) \wedge \sim(\sim q \vee \sim p)]]$

is : $(\sim p \vee q)$

PDNF is : $(p \wedge q) \vee (\neg p \wedge q) \vee (\sim p \wedge \sim q)$

* Theory of Inference for a statement Formula

The main aim of logic is to provide rules of inference to infer a conclusion from certain premises. The theory associated with these rules of inference is known as inference theory.

- If a conclusion is derived from a set of premises by using the accepted rules of reasoning then such a process of derivation is called a deduction or a formal proof and the argument or conclusion is called a valid argument or a valid conclusion

- Rules of Inference

The following two important rules of inference

- Rule P

A premises may be introduced at any point in the derivation

- Rule T

A Formula "S" may be introduced in a derivation if S is tautologically implied by one or more of the preceeding formulas in the derivation

- Demonstrate r is a valid inference from the premises $p \to q$, $q \to r$ and p

{1}	(1)	$p \to q$	Rule P
{2}	(2)	$q \to r$	Rule P
{1, 2}	(3)	$p \to r$	Rule T, $(p \to q) \wedge (q \to r) \Rightarrow p \to r$
{4}	(4)	p	Rule P
{1, 2, 4}	(5)	r	Rule T, $p \wedge (p \to q) \Rightarrow q$

- Implication Formulas

I_1 : $P \wedge Q \Rightarrow P$

I_2 : $P \wedge Q \Rightarrow Q$

I_3 : $P \Rightarrow P \vee Q$

I_4 : $Q \Rightarrow P \vee Q$

I_5 : $\sim P \Rightarrow P \rightarrow Q$

I_6 : $Q \Rightarrow P \rightarrow Q$

I_7 : $\sim (P \rightarrow Q) \Rightarrow P$

I_8 : $\sim (P \rightarrow Q) \Rightarrow \sim Q$

I_9 : $P, Q \Rightarrow P \wedge Q$

I_{10} : $\sim P, P \vee Q \Rightarrow Q$

I_{11} : $P, P \rightarrow Q \Rightarrow Q$

I_{12} : $\sim Q, P \rightarrow Q \Rightarrow \sim P$

I_{13} : $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$

I_{14} : $P \vee Q, P \rightarrow R, Q \rightarrow R \Rightarrow R$

- Show that $S \vee R$ is tautologically implied by $(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

$(P \vee Q) \wedge (P \to R) \wedge (Q \to S)$

$H_1 : P \vee Q \quad H_2 : P \to R \quad H_3 : Q \to S$

$\{1\}$	(1)	$P \vee Q$	Rule P
$\{1\}$	(2)	$\sim P \to Q$	Rule T, equivalence formula $P \vee Q \Leftrightarrow \neg P \to Q$
$\{3\}$	(3)	$Q \to S$	Rule P
$\{1,3\}$	(4)	$\sim P \to S$	Rule T, I : $P \to Q, Q \to R \Rightarrow P \to R$
$\{1,3\}$	(5)	$\sim S \to P$	Rule T, $\sim P \to q \Leftrightarrow \sim q \to P$
$\{6\}$	(6)	$P \to R$	Rule P
$\{1,3,6\}$	(7)	$\sim S \to R$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
$\{1,3,6\}$	(8)	$S \vee R$	Rule T, $\sim S \to R \Leftrightarrow S \vee R$

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- Show that $R \wedge (P \vee Q)$ is valid conclusion from the premises $P \vee Q$, $Q \to R$, $P \to M$ and $\sim M$

$\{1\}$	(1)	$P \to M$	Rule P
$\{2\}$	(2)	$\sim M$	Rule P
$\{1,2\}$	(3)	$\sim P$	Rule T, $\sim M, P \to M \Rightarrow \sim P$
$\{4\}$	(4)	$P \vee Q$.	Rule P

{1, 2, 4}	(5)	Q	Rule T, $\sim P, P \vee Q \Rightarrow Q$
{6}	(6)	$Q \rightarrow R$	Rule P
{1,2,4,6}	(7)	R	Rule T, $Q, Q \rightarrow R \Rightarrow R$
{1,2,4,6}	(8)	$R \wedge (P \vee Q)$	Rule T, (4), (7) $\Rightarrow P, Q \sim P \wedge Q$

- Test the validity of the following arguments

1: if you work hard, you will pass the exam.

2: you did not pass; Therefore "you did not work hard"

Let P : you work hard

Q : you pass the exam

$\sim P$: you didnot work hard

$\sim Q$: you did not pass the exam

Premises : $P \rightarrow Q$, $\sim Q$, "$\sim P$"

{1}	(1)	$P \rightarrow Q$	Rule P
{2}	(2)	$\sim Q$	Rule P
{1,2}	(3)	$\sim P$	Rule T, $\sim Q, P \rightarrow Q \Rightarrow \sim P$

- Test the validity of the following arguments

'Sonia is watching TV'

'If sonia is watching TV, and then she is not studying.'

'If she is not studying then her father will not buy her a scooty.'

Therefore "sonia's father will not buy a scooty"

Let P : Sonia is watching TV

Q : Sonia is not studying

R : Her father will not buy a scooty

premises : P, $P \to Q$, $Q \to R$

conclusion : R

$\{1\}$	(1)	$P \to Q$	Rule P
$\{2\}$	(2)	$Q \to R$	Rule P
$\{1, 2\}$	(3)	$P \to R$	Rule T, $P \to Q, Q \to R \Rightarrow P \to R$
$\{4\}$	(4)	P	Rule P
$\{1,2,4\}$	(5)	R	Rule T $P, P \to R \Rightarrow R$

- Rule of conditional Proof | Deduction Theorem

Another important rule used in logic is the rule of conditional proof or Rule CP. This rule is defined as follows

Rule CP : If we can derive S from R and a set of premises then we can derive $R \to S$ from a set of premises alone

• Rule CP is generally used if the conclusion is of the form $R \to S$, in such cases R is taken as an additional premisive and S is derived from the given premises and R

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• Show that $r \to s$ can be derived from the premises $p \to (q \to s)$, $\sim r \vee p$ and q

Instead of deriving $r \to s$ we shall include r as an additional premises and show s first

{1}	(1)	r	Rule P, (additional premise)
{2}	(2)	$\sim r \vee p$	Rule P
{1, 2}	(3)	p	Rule T, (1), (2), $P, \sim P \vee q \Rightarrow q$
{4}	(4)	$p \to (q \to s)$	Rule P
{1, 2, 4}	(5)	$q \to s$	Rule T, $P, P \to (q \to s) \Rightarrow q \to s$
{6}	(6)	q	Rule P
{1, 2, 4, 6}	(7)	s	Rule T, (5), (6), $q \to s, q \Rightarrow s$
{1, 2, 4, 6}	(8)	$r \to s$	Rule CP, (1), (7), $r, s \Rightarrow r \to s$

Show that $p \to s$ can be derived from the premises $\sim p \vee q$, $\sim q \vee r$, $r \to s$

Let us take additional premise p

{1}	(1) p	Rule P	(Additional premise)
{2}	(2) $\sim p \vee q$	Rule P	
{2}	(3) $p \to q$	Rule T,	$\sim p \vee q \Leftrightarrow p \to q$
{1,2}	(4) q	Rule T,	(1), (3), $p, p \to q \Leftrightarrow q$
{5}	(5) $\sim q \vee r$	Rule P	
{5}	(6) $q \to r$	Rule T,	$\sim q \vee r \Leftrightarrow q \to r$
{1,2,5}	(7) r	Rule T,	(4), (6), $p, p \to q \Rightarrow q$
{8}	(8) $r \to s$	Rule P.	
{1,2,5,8}	(9) s	Rule T	, $r, r \to s \Rightarrow s$
{1,2,5,8}	(10) $p \to s$	Rule CP,	(1), (9), $p, q \Rightarrow p \to q$

* Indirect Method of Proof:

To show that a conclusion C follows logically from the premises $H_1, H_2, H_3, \ldots H_N$ we assume that C is false and $\sim C$ is an additional premise. If the new set of premises is inconsistent then they implied a contradiction $\therefore$ the assumption $\sim C$ is true does not hold. Hence C is true whenever $H_1, H_2, H_3, \ldots H_N$ are true. Thus C follows logically from the premises H_1, $H_2, \ldots H_N$.

- Show that $\sim(P \wedge q)$ follows from $\sim P \wedge \sim q$ using indirect method of proof

To prove the above statement we will introduce the negation of conclusion i.e $\sim(\sim(P \wedge q))$ as an additional premise then we will show that this leads to a contradiction

{1}	(1)	$\sim(\sim(P \wedge q))$	Rule P (Additional premise)
{1}	(2)	$P \wedge q$	Rule T, $\sim\sim P \Leftrightarrow P$
{1}	(3)	P	Rule T; $P \wedge q \Rightarrow P$
{4}	(4)	$\sim P \wedge \sim q$	Rule P
{4}	(5)	$\sim P$	Rule T, $P \wedge q \Rightarrow P$
{1,4}	(6)	$P \wedge \sim P$	Rule T, (3), (5), $P, q \Rightarrow P \wedge q$

Thus $P \wedge \sim P \Leftrightarrow F$. Here our assumption is wrong. Thus $\sim(P \wedge q)$ follows from $\sim P \wedge \sim q$

. using Indirect method show that $p \to q$, $q \to r$, $p \vee r \Leftrightarrow r$

{1}	(1)	$\sim r$	Rule P (Additional premise)
{2}	(2)	$q \to r$	Rule P
{2}	(3)	$\sim r \to \sim q$	Rule T, $q \to r \Leftrightarrow \sim r \to \sim q$
{1,2}	(4)	$\sim q$	Rule T, (1),(3), P, $P \to q \Rightarrow q$
{5}	(5)	$p \to q$	Rule P
{5}	(6)	$\sim q \to \sim p$	Rule T, $P \to q \Leftrightarrow \sim q \to \sim p$
{1,2,5}	(7)	$\sim p$	Rule T, $\sim P$, $\sim P \to \sim q \Rightarrow \sim q$
{8}	(8)	$p \vee r$	Rule P
{8}	(9)	$\sim p \to r$	Rule T, $P \vee q \Leftrightarrow \sim P \to q$
{1,2,5,8}	(10)	r	Rule T, (7),(9), P, $P \to q \Rightarrow q$
{1,2,5,8}	(11)	$\sim r \wedge r$	Rule T, (1),(10), P, q $\Rightarrow P \wedge q$

Thus $\sim r \wedge r \Leftrightarrow F$. Here our assumption is wrong. Thus r is logically follows from the given premises

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- Consistency of premises :

A set of formulas H1, H2, H3, ---- HN is said to be consistent if their conjunction has truth value T for some assignment of truth values to the atomic variables appearing in H1, H2, ---- HN if, for the every assignment of the truth value to the atomic variables atleast one of the formula H1, H2, --- Hn is false, so that their conjunction is identically false, then the formulas H1, H2, --- HN are called inconsistent i.e a set of formulas $H_1, H_2, H_3, --- H_n$ are called inconsistent if their conjunction implies a contradiction

- Show that the following premises

$a \to (b \to c)$,

$d \to (b \wedge \sim c)$,

and

are inconsistent

(P.T.O)

{1}	(1)	$a \wedge d$	Rule P
{1}	(2)	a	Rule T, $a \wedge d \Rightarrow a$
{1}	(3)	d	Rule T, $a \wedge d \Rightarrow d$
{4}	(4)	$a \to (b \to c)$	Rule P
{1,4}	(5)	$b \to c$	Rule T; (2), (4), $a, a \to (b \to c) \Rightarrow b \to c$
{1,4}	(6)	$\sim b \vee c$	Rule T, $P \to q \Leftrightarrow \sim P \vee q$
{7}	(7)	$d \to (b \wedge \sim c)$	Rule P
{7}	(8)	$\sim (b \wedge \sim c) \to \sim d$	Rule T, $P \Rightarrow q \Leftrightarrow \sim q \to \sim P$
{7}	(9)	$(\sim b \vee c) \to \sim d$	Rule T, Negation
{1,4,7}	(10)	$\sim d$	Rule T, (6), (9), $P, P \to q \Rightarrow q$
{1,4,7}	(11)	$d \wedge \sim d$	Rule T, (3), (10), $P, q \Rightarrow P \wedge q$

Since $d \wedge \sim d$ is false, the given premises are inconsistent.

PREDICATE CALCULUS

• **Predicate:**

A part of a declarative sentence describing the property of an object is called a predicate. The logic based upon the analysis of predicate in any statement is called predicate Logic.

Eg) If you consider 2 statements

Tom is a graduate

Bob is a graduate

In each statement "is a graduate" is a predicate. If P stands for predicate "is a graduate" then $P(x)$ stands for "x is a graduate" where x is a predicate variable

- Note that $P(x)$ not a statement but just an expression. Once a value is assigned to x, $P(x)$ becomes a statement and has the truth value

- Quantifiers :

Quantifiers are words that refer to Quantities such as some or All. There are mainly 2 types of quantifiers

(i) universal Quantifiers :

The phrase for All (denoted $\forall$) is called the universal Quantifiers

Eg) • All birds have wings,

• Every apple is red

. Any integer either positive or negative

The above statements can be written in symbolic forms

(i) All Birds have wings.

Let $B(x)$: x is a bird

$W(x)$: x have wings

Now, All Birds have wings can be written as

- For all x, if x is a bird then x have wings

$(\forall x)[B(x) \rightarrow W(x)]$

(ii) Every Apple is red

Let $A(x)$: x is an apple

$R(x)$: x is Red

Now, Every Apple is Red can be written as

- For all x, if x is an apple then x is red

$(\forall x)[A(x) \rightarrow R(x)]$

(iii) Any integer either positive or negative

Let $I(x)$: x is an integer

$P(x)$: x is Positive

$N(x)$: x is negative

Now, Any integer either positive or negative can be written as : For all x, if x is an integer then x is positive or x is negative

$(\forall x)[I(x) \rightarrow (P(x) \vee N(x))]$

(ii) Existential Quantifiers :

The phrase there exist (denoted by $\exists$) is called the existential quantifiers.

Eg) • There exist a man

• Some real numbers are rational

• Some students are clever

The above statements can be written in symbolic forms

(i) There exist a man

Let $M(x)$: x is a man

Now, there exist a man $\Rightarrow (\exists x)[M(x)]$

(ii) Some real numbers are rational

Let $R1(x)$: x is a real number

$R2(x)$: x is a rational number

Now, Some real numbers are rational $\Rightarrow$

$(\exists x)[R1(x) \wedge R2(x)]$

(iii) Some students are clever.

Let $S(x)$: x is a student

$C(x)$: x is clever

Now, Some students are clever $\Rightarrow$

$(\exists x)[S(x) \wedge C(x)]$

. Write the following statements in symbolic form

$G(x)$: x is good

b) Something is good $(\exists x)[G(x)]$

c) Everything is good $(\forall x)[G(x)]$

d) Nothing is good $(\forall x)[\sim G(x)]$

e) Something is not good $(\exists x)[\sim G(x)]$

. Free and Bound Variables :

Given a formula containing a part of the form $(\forall x)(P(x))$ or $(\exists x)(P(x))$, such a part is called bounded part of the formula and x is called bounded variable otherwise x is called free variable

Eg) $(\forall x)\, P(x,y)$ – x is a bounded variable
– y is a free variable

$(\forall x)(P(x) \rightarrow (\exists y)\, R(x,y))$

– both x and y are bounded variables

• Inference Theory of predicate calculus :

To understand the inference Theory of predicate calculus it is important to be familiar with the following Rules

Rule US (universal specification)

$$(\forall x) A(x) \Rightarrow A(y)$$

Rule ES (Existential specification)

$$(\exists x) A(x) \Rightarrow A(y)$$

Rule UG (universal generalisation)

$$A(x) \Rightarrow (\forall y) A(y)$$

Rule EG (Existential generalisation)

$$A(x) \Rightarrow (\exists y) A(y)$$

- Equivalance Formulas

(1) $(\exists x)[A(x) \vee B(x)] \Leftrightarrow (\exists x) A(x) \vee (\exists x) B(x)$

(2) $(\forall x)[A(x) \wedge B(x)] \Leftrightarrow (\forall x) A(x) \wedge (\forall x) B(x)$

(3) $\sim(\exists x) A(x) \Leftrightarrow \sim A(x)$

(4) $\sim(x) A(x) \Leftrightarrow (\exists x) \sim (A(x))$

(5) $(\forall x)(A \vee B(x)) \Leftrightarrow A \vee (\forall x) B(x)$

(6) $(\exists x)(A \wedge B(x)) \Leftrightarrow A \wedge (\exists x) B(x)$

(7) $(\forall x) A(x) \rightarrow B \Leftrightarrow (\exists x)(A(x) \rightarrow B)$

(8) $(\exists x) A(x) \rightarrow B \Leftrightarrow (\forall x)(A(x) \rightarrow B)$

(9) $A \rightarrow (\forall x) B(x) \Leftrightarrow (\forall x)(A \rightarrow B(x))$

(10) $A \rightarrow (\exists x) B(x) \Leftrightarrow (\exists x)[A \rightarrow B(x)]$

(11) $(\exists x)(A(x) \rightarrow B(x)) \Leftrightarrow (\forall x)A(x) \rightarrow (\exists x)B(x)$

(12) $(\exists x)A(x) \rightarrow (x)B(x) \Leftrightarrow (\forall x)[A(x) \rightarrow B(x)]$

- Implication Formulas

(1) $(\forall x)A(x) \vee (\forall x)B(x) \Rightarrow (\forall x)(A(x) \vee B(x))$

(2) $(\exists x)(A(x) \wedge B(x)) \Rightarrow (\exists x)A(x) \wedge (\exists x)B(x)$

- show that

$$(\forall x)(P(x) \rightarrow Q(x)) \wedge (\forall x)(Q(x) \rightarrow R(x)) \Rightarrow (\forall x)(P(x) \rightarrow R(x))$$

{1}	(1)	$(\forall x)(P(x) \rightarrow Q(x))$	Rule P
{1}	(2)	$P(y) \rightarrow Q(y)$	Rule US
{3}	(3)	$(\forall x)(Q(x) \rightarrow R(x))$	Rule P
{3}	(4)	$Q(y) \rightarrow R(y)$	Rule US
{1, 3}	(5)	$P(y) \rightarrow R(y)$	Rule T, $P \rightarrow Q, Q \rightarrow R \Rightarrow P \rightarrow R$
{1,3}	(6)	$(P(x) \rightarrow R(x))$	Rule UG

- Verify the validity of the following arguments "All men are mortal. Socrates is a man. Therefore Socrates is mortal"

Let $M1(x)$: x is man

$M2(x)$: x is mortal

s : socrates

$(\forall x)[M1(x) \rightarrow M2(x)] \wedge M1(S) \Rightarrow M2(S)$

{1}	(1)	$(\forall x)(M1(x) \rightarrow M2(x))$	Rule P
{1}	(2)	$M1(S) \rightarrow M2(S)$	Rule US
{3}	(3)	$M1(S)$	Rule P
{1,3}	(4)	$M2(S)$	Rule T, $P, P \rightarrow Q \Rightarrow Q$

- Verify "Every living thing is a plant or an animal". "Joes gold fish is alive and it is not a plant". "All animals have hearts". "Therfore Joes gold fish has a heart".

Let $P(x)$: x is a plant

$A(x)$: x is an animal

$H(x)$: x have a heart

F : Joes gold fish

Then the given statements become

$(\forall x)(P(x) \vee A(x))$, $\sim P(F)$, $(\forall x)[A(x) \rightarrow H(x)]$

$\Rightarrow H(F)$

{1}	(1)	$(\forall x)[P(x) \vee A(x)]$	Rule P
{1}	(2)	$P(f) \vee A(f)$	Rule US
{1}	(3)	$\sim P(f) \rightarrow A(f)$	Rule T, (2), $P \vee q \Leftrightarrow \sim P \rightarrow q$

$\{4\}$	(4)	$\sim P(f)$	Rule P
$\{1,4\}$	(5)	$A(f)$	Rule T, (3), (4), $P \to q$, P $\Rightarrow q$
$\{6\}$	(6)	$(\forall x)[A(x) \to H(x)]$	Rule P
$\{6\}$	(7)	$A(f) \to H(f)$	Rule US
$\{1,4,6\}$	(8)	$H(f)$	Rule T, P, $P \to q \Rightarrow q$

- Test the validity of the following arguments "Some intelligent boys are lazy; Ravi is an intelligent boy : Ravi is lazy"

Let $I(x)$: x is an intelligent boy

R : Ravi

$L(x)$: x is Lazy

Now the given statements can be written as

$(\exists x)[I(x) \wedge L(x)]$, $I(R) \Rightarrow L(R)$

$\{1\}$	(1)	$(\exists x)(I(x) \wedge L(x))$	Rule P
$\{1\}$	(2)	$I(R) \wedge L(R)$	Rule ES
$\{3\}$	(3)	$I(R)$	Rule P
$\{1,3\}$	(4)	$L(R)$	Rule T, (2), (3), P, $P \wedge Q \Rightarrow Q$

- Establish the validity of the following arguments " All integers are rational numbers, some integers are powers of 2, Therefore some rational numbers are powers of 2".

Let

$I(x)$: x is an integer

$P(x)$: x is power of 2.

$R(x)$: x is a rational number.

Now given statement can be written as

$$(\forall x)[I(x) \wedge R(x)], \ (\exists x)[I(x) \wedge P(x)]$$

$$\Rightarrow \quad (\exists x)[R(x) \wedge P(x)]$$