

$X$  cpt Kähler nfd ( $n = \dim X$ ) .  $L$ : line ldl. (= divisor)

$$K(X, L) := \limsup_{m \rightarrow +\infty} \frac{\log h^0(X, L^m)}{\log m} \in \{-\infty, 0, 1, \dots, n\}$$

(Kodaira-Iitaka 級数).

⇒ いじくと,  $h^0(X, L^m) \sim O(m^n)$ .

$$K(X, L) = -\infty \iff h^0(X, L^m) = 0 \quad \forall m \geq 0.$$

•  $L$ : big (E大)  $\iff K(X, L) = n$ .

$X$ : general type  $\xrightleftharpoons[\text{def}]{\iff} K(X, K_X) = n$   
 $(X, D)$ : (log) general type  $K(X, K_X + D) = n$ .

例)  $n=1$

$K(K_X)$	$-\infty$	$0$	$1$ (general type)
$X$	$\mathbb{CP}^1$	複用曲線	$g \geq 2$ .
$K_X$ の曲率	$< 0$	$= 0$	$> 0$ .
$\pi_1(X)$	$1$ (Abel)	$\mathbb{Z}^2$ (Abel)	自由群を含む
$d_{\text{Koh}}$	$\equiv 0$	$\equiv 0$	正反対.

Campana の Special variety は.

$\pi_1(X)$  が Abel とか  $d_{\text{Koh}} \equiv 0$  とかと 関連する もの.

$K(K_X)$  では どうもされない もの.

$n=2$ . (bimeric 分類).

$K(K_X)$	$-\infty$	0	1	2
$\pi_1(X)$ Almost abel $(\exists H \triangleleft \pi_1(X), H \text{Abel} \text{ かつ } H \neq \{1\})$	なる or なる。 $\pi_1(X) \neq \{1\}$	なる. $\pi_1(X) \neq \{1\}$	なる or $\pi_1(X) \neq \{1\}$	?

$\rightarrow$  例 (1)  $\mathbb{P}^2$

$\forall$  cpt Kähler mfld

"special"  $\Leftrightarrow$

"log general" は 1 つ ある で す る

$f: X \dashrightarrow (Y, \Delta)$

$F$

Def  $X: \text{Special} \stackrel{\text{def}}{\iff} \forall p \in \{1, \dots, n\}, \forall L \subset \Omega_X^1$  line hdll  
 $K(X, L) < p$ .

Rem ① Bogomolov-Sommese vanishing.

$$L \text{ が } (\star) \text{ を 持 つ } \implies K(X, L) \leq p.$$

② Special は

- Bimeromorphic で 不変.
- $X$  special,  $X \dashrightarrow Y$  dominate  $\implies Y$  special.
- $X, X'$  special  $\implies X \times X'$  そし.
- $X \rightarrow Y$  finite \'etale cover,  $X$ : special  $\iff Y$ : special

例 Fano  $(-K_X \text{ ample}) \implies \text{Special}$

たとえ 強い 構造連結  $\implies$  Special

(Rationally connected)

$\Leftrightarrow \forall x, y \in X$  general  
 $\exists f: \mathbb{C}\mathbb{P}^1 \rightarrow X$  (有理曲線) s.t.  $x, y \in f(\mathbb{C}\mathbb{P}^1)$ .

proof) ( $\Rightarrow$ ) 有理連結  $\Rightarrow H^0(X, \Omega_X^{\mathbb{P}}) = 0$ .

$\Downarrow$  (Riemann 本)  
 $\pi_1(X) = 1$ .

$x \in X$  terms of special.

例2.  $C_1(K_X) = 0 \Rightarrow$  Special

もしくは  $K(X, K_X) = 0 \Rightarrow$  Special

(Itaka Num Conj の証明でも使う).

proof)  $C_1(K) = 0 \Rightarrow \forall L \subset \Omega_X^{\mathbb{P}}, C_1(L) \cdot [n]^{n-1} \leq 0$   
 KE 存在  $(n\text{-K\"ahler})$

$\Downarrow$   
 $\Omega_X^{\mathbb{P}}$  poly stable  $\Rightarrow K(X, L) \leq 0$ .

例.  $X$  general type  $\Rightarrow$  Not special ( $L = K_X \subseteq \Omega_X^{\mathbb{P}}$  も)

引の def

Prop

$X$  が non-special  $\Leftrightarrow$

$\lim_{\substack{\leftarrow \\ \text{min}}} \tilde{X}$   $\xrightarrow{\sim} \tilde{Y}$   $\lim_{\substack{\rightarrow \\ \text{min}}}$

$\tilde{X}, \tilde{Y}$  cpt K\"ahler

$\dim Y > 0$

s.t.  $\tilde{f}: \tilde{X} \rightarrow \tilde{Y}$  fibration

+ "good" condition  
 (neat prepared.)

&

$\pi: \text{holomorphic}$

s.t.  $\Delta(\tilde{f}): \mathbb{Q}\text{-div on } \tilde{Y}$  (ある)

$\therefore \pi^* \Delta(\tilde{f}) \in \mathbb{Z}$ .

$K(Y + \Delta(\tilde{f})) = \dim Y$ .

$D$ : prime div on  $Y$

$$f^* D = \sum_{i=1}^n m_i E_i + R$$

$\in \mathbb{N}$

$\text{codim } f(R) \geq 2$  の.

$$M_D = \inf m : \exists i \in I. \quad A(f) = \sum_{\substack{D \subset X \\ \text{prime div}}} \left(1 - \frac{1}{m_D}\right) D.$$

$$E_1 = (\gamma_i = 0) \xrightarrow{f} D = (w_i = 0)$$

$$(U, \gamma_1, \dots, \gamma_n) \longrightarrow (\gamma_1^{m_i}, \gamma_2, \dots, \gamma_n).$$

例  $C$  (Hyperelliptic of)  $g=2$ . ( $\exists h: C \rightarrow \mathbb{P}^1$ ,  $\exists \iota: \text{involution}$   
 $2:1$  ( $2 \circ \iota = \text{id}$ ))

$E$ : elliptic curve ( $a: \text{位数 } 2 \text{ の元}$ )

$$\begin{array}{l} j: E \rightarrow E \\ x \mapsto x+a \end{array} \quad (j \circ j = \text{id})$$

$$\begin{array}{l} \text{ex)} \quad E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z}) \\ a = \sqrt{2} \end{array}$$

$$X = C \times E / (i(x)) \xleftarrow{\text{etale}} C \times E. \quad ? \quad f: X \rightarrow \mathbb{C} \cong \mathbb{CP}^1 \text{ 和形}$$

今.  $h: C \xrightarrow{2:1} \mathbb{CP}^1$  は 6 點  $(p_1, \dots, p_6)$  を 交わる.

$$\begin{array}{l} (g_1, g_2) \mapsto \mathcal{E} \\ a \longmapsto p \end{array} \quad (\mathcal{E}_p = 2)$$

$$\implies D = \mathbb{P}[], \quad p \in \mathbb{CP}^1 \text{ は } 12 \text{ 個}.$$

$$f^* D = \begin{cases} 2 f^*(p)_{\text{red}}, & p = p_1, \dots, p_6 \\ E_1 + E_2, & \text{other} \end{cases}$$

$$\implies A(f) = \sum_{D \text{ div}} \left(1 - \frac{1}{m_D} D\right) = \boxed{\frac{1}{2} \sum_{i=1}^6 [\mathbb{P}:].}$$

$$\deg(K_{\mathbb{CP}^1} + A(f)) = -2 + \frac{1}{2} 6 = 1.$$

$\implies (\mathbb{CP}^1, A(f))$ : log general type

$f: X \rightarrow \mathbb{CP}^1$  が elliptic  $\implies$  special

Thm [Core map].

$\forall X: \text{cpt K\"ahler}, \exists C(X) \text{ s.m. proj}.$

$\exists C_x: X \dashrightarrow C(X)$  filtration

s.t.  $\oplus C_x$ : gen filer special

② Almost hol  $\hookleftarrow$  (不確定な C(X) 全体は  
→ うまい).

③

$$\begin{array}{ccc} \hat{X} & & \\ \pi \searrow & \swarrow f & \\ X & \dashrightarrow C_x & C(X) \end{array}$$

217 log general

$$K(K_{C(X)} + A(f))$$
$$\dim C(X)$$

special  $\hookrightarrow$  ④  $\forall x \in X$  very general  
割り当てる

$\forall Z \subset X$  special

$$Z \cap C_x(C_x(x)) \neq \emptyset \Rightarrow Z \subset C_x(C_x(x))$$

例  $\mathbb{P}^n \rightarrow X \Rightarrow$  special

$x \in \mathbb{P}^n$ . Oka  $\Rightarrow$  special

(Kohayashi - Ochiai).

$X$ : special  $\pi$  と つき.

- Alb filtration
- $\pi_1(X) \rightarrow GL(r, \mathbb{C})$  の 像が almost Abel.

CW15 h-principle

Def  $X$  cpt sp

$X$  の h-principle  $\Leftrightarrow H\mathcal{T}_2 \nparallel$  (BOP)

$\Leftrightarrow \forall S: \text{Stein mfd}, f: S \rightarrow X: \text{cont'}$

$\exists F: S \rightarrow X$  hol.,  $F \sim f$   
homotopic

Oka  $\implies$  h-principle CWIS proj. twiz.  
 $\rightsquigarrow$  special

例.  $X$ : 可解  $\implies$  h-principle

[CW15]  $X$  Brody hyperbolic ( $\forall C \rightarrow X$ : const)  
 $X$ : 可解  $\iff$  h-principle

例.  $D$ : h-principle  $\infty$  stft.

$C: g \geq 2$  の 1-2 曲線. h-principle  $\not\in$  Hkt.( $\infty$ )

$$D \xrightarrow[\text{étale}]{} C$$

Thm CW15

①  $X$ : proj mfld, h-principle  $\implies$  special

②  $X$  irr proj & h-principle

$\implies \forall Y$ : Brody hyperbolic Kähler  
 $\forall f: X \rightarrow Y$ : const.

Thm (Jouanolou's trick) Stem

$X$ : proj ~~mfld~~,  $\exists M$  affine,  $\tau: M \rightarrow X$ : hol.

s.t. ①  $M \cap X$  ~~正规~~-Afd.

②  $\tau \circ \text{fiber} \cong \mathbb{C}^r$

③  $\tau$  is  $\mathbb{C}^n$  fiber 束.

④  $\exists \sigma: X \rightarrow M$   $C^\infty$  section

proof)  $X = \mathbb{C}\mathbb{P}^n$   $n \geq 2$ .  $P = \mathbb{C}\mathbb{P}^n \times \mathbb{C}\mathbb{P}^n$ ,  $D = \{(z, w) \in P \mid \sum_i z_i w_i = 0\}$

$M = P \setminus D$ ,  $\iota: M \rightarrow X$ ,  $\sigma: \mathbb{C}\mathbb{P}^n \rightarrow M$   
 $(z, w) \mapsto z$        $z \mapsto (z, \bar{z})$

$$\begin{array}{ccc} M_x & \longrightarrow & M \\ \downarrow & \lrcorner & \downarrow \\ X & \hookrightarrow & \mathbb{C}\mathbb{P}^n \end{array}$$

一般化.

Def  $X$ :  $n$  dim cpx str.  $J: X$  の 極複素構造

$\bar{X} = (X_R, -J)$  opposite cpx str

例.  $X = \mathbb{C}^n$ ,  $(z_1, \dots, z_n)$ : coord       $(T_R X_R \otimes \mathbb{C}) = TX \oplus T'X$   
 $\bar{X} = \mathbb{C}^n$ ,  $(\bar{z}_1, \dots, \bar{z}_n)$ : coord.       $(X_R, J)$   $(X_R, -J)$

$$\begin{aligned} h: X &\xrightarrow{\text{Hermitian metric}} w_h \\ \sim \bar{h}: \bar{X} &= \sim w_{\bar{h}} \\ \implies w_{\bar{h}} &= -w_h \quad (h \text{ K\"ahler} \iff \bar{h} \text{ K\"ahler}) \end{aligned}$$

Lem  $(X, w_h)$ : cpt K\"ahler

s.t. ①  $\iota: \bar{X} \rightarrow X$ :  $C^\infty$  map,  $\iota \sim \text{id}_X$

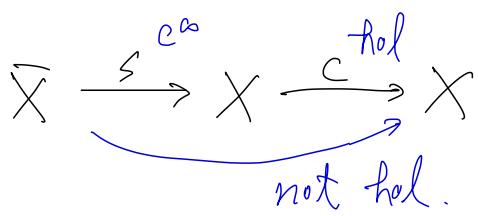
②  $c: X \dashrightarrow Y$ ,  $Y$  cpt K\"ahler,  $c$ : meromorphic map

$$\forall d, \int_{\bar{X}} (w_h)^{h-d} \wedge (c \circ \iota)^* w_Y^d = (-1)^d \int_X w_h^{h-d} \wedge c^* w_Y^d$$

$X \times \bar{X}$  の  $\text{rk } \Omega \leq (-1)^n + d$ .

$\Sigma \subset \bar{X}$   $c$  dominate  $\implies c \circ \iota$  は meromorphic でない.

$$\text{例 } \zeta = \text{id}_X = C$$



$X : h \text{ principle} \Rightarrow \forall Y \text{ Brody K\"ahler}$

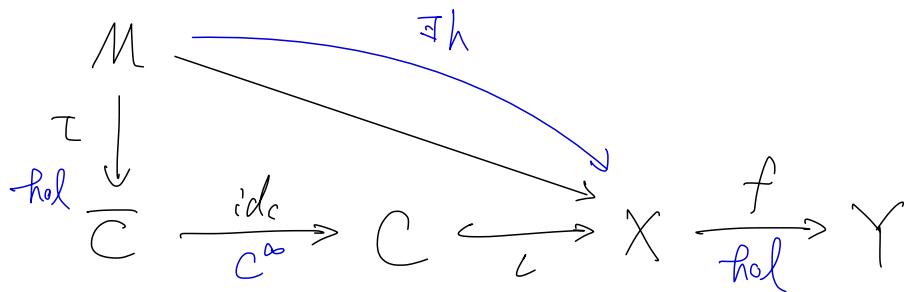
$f : X \rightarrow Y : \text{const}$

背理法

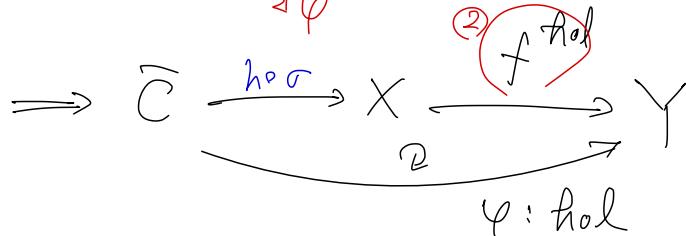
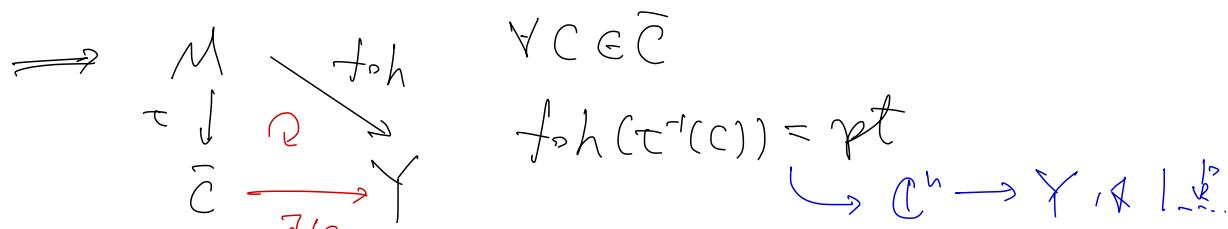
proof  $C \subset X$  curve s.t.  $f|_C : C \rightarrow Y$  not const

$\Rightarrow \exists M \rightarrow \bar{C}, M \text{ affine } \mathbb{C}^n \text{ ball}$ .

$\textcircled{1} \sim \textcircled{2} \text{ 之矛盾}$



$\Rightarrow \exists h : \text{hol } M \rightarrow X, h \sim \text{id} \circ \tau$ .



$$\varphi \sim f \circ l \quad \text{[!]} \quad 0 < \int_{\bar{C}} \varphi^* \omega_Y = \int_{\bar{C}} (f \circ l)^* \omega_Y \\ = (-1) \int_C f^* \omega_Y \leq 0.$$

由 Lem 13 之推  $\square$

proof of h principle  $\Rightarrow$  special .

反证法  $X$  not special

$\Rightarrow C_X: X \dashrightarrow C(X)$  core map .

$\Rightarrow \exists M \rightarrow \widehat{X}$  affine

$\Rightarrow$   $M$   
 $(\sharp, \#)$   $\begin{array}{c} \pi \\ \text{hol} \downarrow \end{array} \sim \begin{array}{c} \exists h \text{ hol} \\ \widehat{X} \longrightarrow X \dashrightarrow C(X) \end{array}$

$\Rightarrow \exists \varphi: \widehat{X} \dashrightarrow C(X)$  meromorphic .

$\varphi \sim \text{id}_C \circ C_X$  矛盾 .