Positivity of tangent sheaves of projective klt varieties.

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Str of KLT var. with a "semipositive tangent sheaf"

Today's theme

(Sun, KLT) ...
1st 2nd

Motivation

Figure out the structure of complex varieties X when T_X is "semipositive".

holomorphic tangent sheaf

($\exists$ metric ≥ 0, nef, ...)

Roughly Speaking ...

$\exists\ \hat{X} \to X$ finite cover.

T_X semipositive $\Longrightarrow$ $\exists\ \hat{X} \to AV$ (Abelian Variety)

"good" fibration (Sun, locally trivial)

s.t F (Fiber) is "Fano like variety"

(Fano, Rationally Connected)

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when T_X is "semipositive". ($\exists$ metric ≥ 0, nef, ...)

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Roughly Speaking ...

T_X semipositive $\Rightarrow$ $\exists\, \tilde{X} \to X$ finite cover.

$\exists\, \tilde{X} \to AV$ (Abelian Variety)

"good" fibration (Sm, locally trivial)

s.t. F (Fiber) is "Fano like variety"

(Fano, Rationally Connected)

Thm (Haward-Smyth-Wu81, Mok88) <u>DG side</u>

X: sm proj var. $/\mathbb{C}$

X has semipositive bihulomorphic sectional curvature

(i.e. T_X has <u>smooth</u> "semipositive" metric) "$T_X \geq 0$"

Then: $\exists\, \tilde{X} \longrightarrow X$ finite étale cover.

$\exists\, \tilde{X} \longrightarrow AV$ locally trivial. ($f^{-1}(U) \overset{\text{biholo}}{\simeq} U \times F$, $U \subseteq Y$ small analytic open)

s.t F (Fiber) is Fano ($-K_F$ ample)

Singular metric case

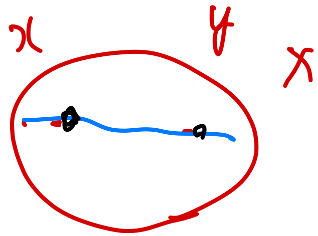
Thm (Hosono–I.–Matsumura 22)

T_X has "singular" semipositive metric

$\Rightarrow$ · $\tilde{X} \longrightarrow X$ finite étale.

· $f : \tilde{X} \longrightarrow AV$. locally trivial

s.t. F is Rationally Connected (RC)



$\forall x, y \in X$ general, $\exists$ Rat curve $\ni x, y$

Thm (I. 22)

T_X is almost nef

nef for very general curve.

($C \subseteq X$ very general curve., $\nu : \tilde{C} \longrightarrow C$ normalization

$\nu^* T_X$ is nef on C

($\nu^* T_X \twoheadrightarrow {}^{\forall} Q$, $\deg Q \geq 0$ ($\tilde{C}$ is curve)))

$\Rightarrow$ · $\tilde{X} \longrightarrow X$ finite étale.

· $f : \tilde{X} \longrightarrow AV$. smooth.

s.t. F (very general) is Rationally Connected (RC)

Summary

X sm proj var

DG side

Thm (HSW81/Mok88)
T_X has smooth semipositive metric
$\Rightarrow$
- $\hat{X} \to X$ f.e.
- $\hat{X} \to AV$ locally trivial
- F Fano

Fano $\Rightarrow$ RC (KMM)

Thm (HIM22)
T_X has singular semipositive metric
$\Rightarrow$
- $\hat{X} \to X$ f.e.
- $\hat{X} \to AV$ locally trivial
- F RC

AG side

Thm (CP91/DPS94)
T_X nef
$\Rightarrow$
- $\hat{X} \to X$ f.e.
- $\hat{X} \to AV$ smooth
- F Fano

Fano $\Rightarrow$ RC (KMM)

Thm (I. 22)
T_X almost nef
$\Rightarrow$
- $\hat{X} \to X$ f.e.
- $\hat{X} \to AV$ smooth
- F RC

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locally trivial $\Rightarrow$ sm

smooth metric ≥ 0 $\longrightarrow$ nef
$\downarrow$ $\qquad\qquad\qquad$ $\downarrow$
singular metric ≥ 0 $\longrightarrow$ almost nef

locally trivial $\Rightarrow$ smooth

§2 Singular Case — KLT case —

KLT: Kawamata Log Terminal ($\pi : \tilde{X} \to X$ resol, $K_{\tilde{X}} \sim_{\mathbb{Q}} \pi^* K_X + \sum a_i E_i$, $a_i > -1$) normal proj var

$$\boxed{\text{Not enough to consider only ``finite \underline{étale} cover'' in the singular case}}$$

$\boxed{\text{Ex}}$ (Ueno '75, Campana '10)

$A = \mathbb{C}^3 / (\mathbb{Z} \oplus i\mathbb{Z})^3$ AV 3-fold

$\mathbb{Z}_4 (= \mathbb{Z}/4\mathbb{Z})$ action $A \longrightarrow A$

$(x, y, z) \longmapsto (ix, iy, iz)$

$X := A/\mathbb{Z}_4$ $\underline{\text{KLT}}$, R.C.

$\boxed{\text{Should } X \text{ be "Fano like variety" or "AV like variety"}}$

- X RC $\leftarrow$ "Fano like variety"

hut

- $c_1(T_X) = 0 \quad c_2(T_X) = c_1(T_X)^2 = 0$

  T_X is semistable.

- T_X has "flat" singular metric.

  (T_X & T_X^* has semipositive singular metric)

AV like variety

Approach "Consider quasi-étale cover"

(Theorems supporting this approach)

Def $\pi : \hat{X} \to X$ finite morphism, normal proj var

quasi-étale $\Leftrightarrow$ étale in codim 1

$\Leftrightarrow$ $\exists Z \subseteq X$ $\mathrm{codim}_X Z \geq 2$

s.t. $\pi : \hat{X} - \pi^{-1}(Z) \to X \setminus Z$ étale.

Thm Greb-Kebekus-Peternell 16

- X KLT sm in codim 2.

T_X semistable, $c_1(T_X) H^{n-1} = 0$ $\Rightarrow$ $\exists$ AV $\to X$ quasi étale

$c_2(T_X) H^{n-2} = c_1(T_X)^2 H^{n-2} = 0$ (H ample Cartier div)

$\boxed{\text{Rem}}$ Quasi-étale cover of RC is not nes RC (Ueno, Campana)

Thm (J.-Matsumura-Zhong 23)

X KLT proj var. $/\mathbb{C}$ (not nes $\mathbb{Q}$-factorial)

① T_X has singular semipositive metric

$\Rightarrow$ $\exists\ \hat{X} \to X$ quasi-étale "strong RC"

$\exists f: \hat{X} \to AV$ locally trivial.

s.t F is RC & $\forall \hat{F} \to F$ quasi-étale. $\hat{F}$ is RC

② T_X almost nef

$\Rightarrow$ $\exists\ \hat{X} \to X$ quasi-étale

$\exists f: \hat{X} \to AV$ smooth

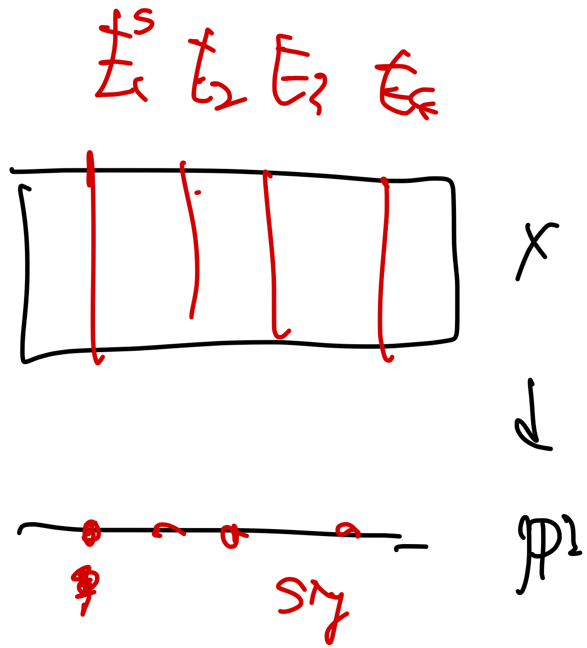
s.t F is RC & $\forall \hat{F} \to F$ quasi-étale. $\hat{F}$ is RC

very general fibre

Example (Greb–Kebekus–Peternell 14, Ou 13, Matsuzawa–Yoshikawa 21)

E : elliptic curve, $\mathbb{Z}_2$ action $E \to E$, $x \mapsto -x$; $\mathbb{P}^1 \to \mathbb{P}^1$, $(x:y) \to (y:x)$

$$X := E \times \mathbb{P}^1 / \mathbb{Z}_2$$



Property

- X KLT (canonical (8 pt sing with A_1 sing))
- $\varphi : X \to \mathbb{P}^1 = E/\mathbb{Z}_2$ branched 4 pt, $E_1, \ldots, E_4$ singular fib. multiplicity $= 2$.
- X RC ($\forall x, y$ general, x, y joined by chain of rat curv (RCC) $\overset{\text{normal}}{\Longrightarrow}$ RC)
- T_X has singular semipositive metric. ($T_{E\times\mathbb{P}^1}$ metric $\rightsquigarrow T_X$)

2:

$$\begin{tikzcd} \mathrm{Ex}\mathbb{P}^1 \arrow[r, "\text{quasi}", "\text{étale}"'] \arrow[d, "\text{locally}"', "\text{triv.}"] & X = \mathrm{Ex}\mathbb{P}^1/\mathbb{Z}_2 \arrow[d] \\ E \arrow[r, "\text{finite}"] & \mathbb{P}^1 = E/\mathbb{Z}_2 \end{tikzcd}$$

Focus

$$\begin{tikzcd} \mathrm{Ex}\mathbb{P}^1 \arrow[r, "\text{quasi-étale}"] \arrow[d, "f"', "\text{locally trivial}"] & X \\ E & \end{tikzcd}$$

Main Thm

Then $\mathbb{P}^1$: RC & $\forall$ quasi-étale cover is RC

$\boxed{\text{Cor}}$ (IMZ23) X KLT proj var.

T_X has singular semipositive metric or T_X almost nef

Then the following are equivalent

① $\forall$ quasi-étale cover is RC

② $T_X \to {}^{\forall}Q$, $\underline{c_1(Q) \neq 0}$

(generically surj torsion-free) ($(\det Q) H_1 \cdots H_{n-1} > 0$ H_i ample Q-Cartier)

This Cor is usefull for defining a "strong" RC (②)

$\exists\, T_X \to Q$ $c_1(Q) = 0$ $\Longrightarrow$ $\exists\, \tilde{X} \to X$ quasi-étale
$\exists\, \tilde{X} \to AV$ nontrivial

Example ① (Ueno, Campana)

$X = AV3fold/\mathbb{Z}_4 \rightsquigarrow c_1(T_X) = 0$

Indeed $AV \to X$ quasi étale

② (GKP, Ou, MY) $(\varphi : X \to \mathbb{P}^2)$

$X = E \times \mathbb{P}^1 / \mathbb{Z}_2 \rightsquigarrow T_X \twoheadrightarrow G_X(D) \quad c_1(D) = 0$

$D = \varphi^* K_{\mathbb{P}^2} + \sum_{i=1}^{4} (2-1) E_i \sim_{\mathbb{Q}} 0$

· E_i: singular f

- φ has multiple fiber on E_i

mult $= 2$

Indeed · $E \times \mathbb{P}^1 \to X$ quasi étale

· $E \times \mathbb{P}^1 \to Ec\,AV$.

$\boxed{\text{Proof}}$ Very technical proof

- Adapt the pf of smooth case to KLT case.

$\boxed{\text{Smooth Case}}$ (HIM, I)

- $f: X \dashrightarrow Y$ MRC fibration
- $\Rightarrow K_Y = \det T_Y^*$ psef (BHS, BDPP)

$\Longrightarrow$ Foliation Theory (Höring OP) "Reeb-stability"

$\mathcal{F} =$ "$T_{X/Y}$" relative tangent gives.

$f: X \longrightarrow Y'$ MRC fibration as a smooth morphism

Why Y' is AV?

"$T_X \geq 0$" $\Longrightarrow$ "$T_{Y'} \geq 0$"

$K_{Y'}$ psef $\quad$ $K_{Y'}$ psef $\quad \stackrel{\text{HIM}}{\Longrightarrow} \begin{cases} c_1(T_{Y'}) = c_2(T_{Y'}) = 0 \\ T_{Y'} \text{ semistable} \end{cases}$ ($T_{Y'}$ flat)

$\overset{?}{\Longrightarrow}$ $\exists$ AV $\to Y'$ finite étale

Difficulties (KLT case)

① Foliation Theory

← Use Druel's Foliation Theory

(However, this theory requires $\mathbb{Q}$ Factoriality ...)

(Difficulty 1)

② How to prove

$T_X^{(1)} \geq 0$, K_X psef $\Longrightarrow$ $\exists$ AV $\to Y$ quasi-étale ?

Smooth case

T_X ">0" $\Rightarrow$ $\left\{\begin{array}{l} \cdot\ c_1(T_X) = c_2(T_X) = 0 \\ \cdot\ T_X \text{ semistable} \end{array}\right.$ $\Rightarrow \exists$ AV $\longrightarrow X$

K_X psef HIM $\boxed{1}$ — Yau $\boxed{2}$ — quasi-étale

KLT case "Difficulties"

We can not well defined c_2

(X is not nes. smooth in codim 2.

& T_X is not locally free.)

$\leftarrow$ Use. Mumford's $\mathbb{Q}$-Chern class $\hat{c}_2$,

(Orbifold)

As for $\hat{c}_2$ in KLT.

$\boxed{2}$ is already proved by Lu-Taji 18

cf. (Greb-Kebekus-Peternell-Taji 19)

$\boxed{1}$ We proved it in IMZ23

(based on Gachet 22)

Thank you for your attention!!