REMARKS ON MINIMAL COMPACT KÄHLER MANIFOLDS WITH VANISHING SECOND CHERN CLASS

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1. Remarks on minimal compact Kähler manifolds with vanishing second Chern class

Theorem 1.1. *Let $X \to C$ be a smooth torus fibration onto a curve C of genus ≥ 2. Then, the numerical Kodaira dimension $\nu(K_X)$ is equal to one and the cotangent bundle Ω_X is nef. In particular, $c_2(\Omega_X) = 0$.*

The following proof is based on an idea by Xiaojun Wu.

Proof. Since f is a smooth torus fibration, the curvature of $K_{X/C}$ is equal to the pull-back of the Weil-Peterson form as follows (c.f. [BCS20, Equation 1.1]):

$$2\pi c_1(K_{X/C}) = \frac{1}{\mathrm{vol} X_c} f^* \omega_{\mathrm{WP}}.$$

The dimension of C is one, so we conclude that $c_1(\Omega_X)^2 = 0$.

Since $f^*\Omega_C$ is nef, it is enough to show that $\Omega_{X/C}$ is also nef. This is directly follows from [Gri70, Theorem 5.2 and Corollary 7.8] and [Kra97, Proposition 5]. Indeed, by [Gri70], $R^1 f_* \mathcal{O}_X$ is Griffith seminegative.[1] Thus $f_*\Omega_{X/C}$ is Griffith semipositive, so is $\Omega_{X/C} = f^* f_* \Omega_{X/C}$. □

Since the condition "$c_2(\Omega_X) = 0$" is invariant under finite étale covers, we obtain the following corollary from [IM22].

Corollary 1.2 (cf. [IM22])**.** *Let X be a compact Kähler manifold with nef canonical divisor K_X. Then $c_2(\Omega_X) = 0$ in $H^{2,2}(X, \mathbb{R})$ if and only if there exists a finite étale cover $X' \to X$ such that one of the following holds depending on the Kodaira dimension:*

- (i) *In the case where $\nu(K_X) = \kappa(K_X) = 0$, the variety X' is isomorphic to a complex torus.*
- (ii) *In the case where $\nu(K_X) = \kappa(K_X) = 1$, the variety X' admits a smooth torus fibration $X \to C$ onto a curve of genus ≥ 2.*

Similarly, we obtain the following corollary.

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[1] In Griffith's paper, the fibration is assumed to be projective. However, this result holds even in the case when f is proper Kähler submersion. For more details, please refer to [MT07, Section 2.3]. Using the same notation as in [MT07, Section 2.3], setting $d = 1$ and $p = 0$, it follows that $R^1 f_* \mathcal{O}_X$ is Griffith seminegative.

Corollary 1.3 (cf. [IMM24]). *Let X be a projective klt variety of dimension n with nef canonical divisor K_X. Then the Miyaoka's equality holds for some ample divisors H_i on X:*

$$\left(3\widehat{c}_2(\Omega_X^{[1]}) - \widehat{c}_1(\Omega_X^{[1]})^2\right) H_1 \cdots H_{n-2} = 0$$

if and only if there exists a finite quasi-étale cover $X' \to X$ such that one of the following holds depending on the Kodaira dimension:

(*i*) *In the case where $\nu(K_X) = \kappa(K_X) = 0$, the variety X' is isomorphic to an abelian variety.*

(*ii*) *In the case where $\nu(K_X) = \kappa(K_X) = 1$, the variety X' admits the structure of an abelian group scheme $X' \to C$ over a curve C of general type.*

(*iii*) *In the case where $\nu(K_X) = \kappa(K_X) = 2$, the variety X' is isomorphic to the product $A \times S$ of an abelian variety A and a smooth surface S whose universal cover is an open ball in $\mathbb{C}^2$.*

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