Thm Campana-$\cancel{P}$Păun 15

K_X psef, X sm proj n-dim var$/\mathbb{C}$

$\Rightarrow$ ${}^\forall m \in \mathbb{N}_{>0}$, ${}^\forall Q$ torsion-free

s.t. $(\Omega_X^1)^{\otimes m} \longrightarrow Q \longrightarrow 0$. について

$\det Q$ is psef

どうか

Thm (Campana-Păun)

X sm proj, K_X psef.

$\Rightarrow$ ${}^\forall m \in \mathbb{N}$, $(\Omega_X^1)^{\otimes m} \longrightarrow {}^\forall Q \to 0$

Q: torsion free,

$\det Q$ is psef

Miyaoka 87.

K_X psef $\Rightarrow$ Ω_X^1 generically nef

(i.e. $A_1, \dots, A_{n-1}$ very ample.
$C = A_1 \cap \dots \cap A_{n-1}$ について
$\Omega_X^1|_C$ nef)

CPIS $\Rightarrow$ Miyaoka 87.

(そんなに自明ではない…)

Notation

- X: smooth proj n-dim variety $/\mathbb{C}$
- $\mathcal{F}, \mathcal{G}$ torsion free sheaf とする

$X_{\mathcal{F}}$: maximal Zariski open set s.t. $\mathcal{F}$ is localy free

(とくに $\operatorname{codim}(X \setminus X_{\mathcal{F}}) \geq 2$)

$$\mathcal{F} \boxtimes \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{\vee\vee}$$

($\vee$ は dual をいみする)

$$\mathrm{Sym}^{[m]}\mathcal{F} = (\mathrm{Sym}^m \mathcal{F})^{\vee\vee}, \quad \bigwedge^{[m]} \mathcal{F} = \left(\bigwedge^m \mathcal{F}\right)^{\vee\vee} \text{ とする}$$

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf.

$\mathcal{F} \subset T_X$ is foliation

$\overset{\text{def}}{\Longleftrightarrow}$ ① $\mathcal{F}$ saturated (i.e. $T_X/\mathcal{F}$ is torsion-free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Lie bracket で閉じている)

Notation

X: sm proj $n$dim variety $/\mathbb{C}$.

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf.

$\mathcal{F} \subseteq T_X$ (singular) Foliation

$\overset{\text{def}}{\Longleftrightarrow}$ ① $\mathcal{F}$: saturated ($T_X/\mathcal{F}$ torsion free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Lie bracket で閉じている)

$\left(\left[a_i \frac{\partial}{\partial z^i}, b_j \frac{\partial}{\partial z^j}\right] = a_i \frac{\partial b_j}{\partial z^i} \frac{\partial}{\partial z^j} - b_j \frac{\partial a_i}{\partial z^j} \frac{\partial}{\partial z^i}\right)$

$X_{\mathcal{F}} :=$ maximal Zariski open set s.t. $\mathcal{F}$ is localy free.

(とくにことわらなければ $r = \operatorname{rk}\mathcal{F}$, $q := n - r$ とする.)

$L \subset X_{\mathcal{F}}$ leaf

$\Leftrightarrow$ L is maximal connected locally closed submfd $L \subseteq X_{\mathcal{F}}$ & $T_L = \mathcal{F}|_L$.

$L \subset X_{\mathcal{F}}$ leaf

$\overset{\text{def}}{\iff}$ $T_L = \mathcal{F}|_L$ なる locally closed connected submfd で極大のもの.

/ℂ

?

Notation

X : sm proj variety / ℂ. (n dim)

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf

$\mathcal{F} \subseteq T_X$ (singular) Foliation

$\overset{\text{def}}{\iff}$ ① $\mathcal{F}$: saturated ($T_X/\mathcal{F}$ torsion-free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Lie bracket でとじている

$\left(\left[a_i \frac{\partial}{\partial z^i}, b_j \frac{\partial}{\partial z^j}\right] = a_i \frac{\partial b_j}{\partial z^i} \frac{\partial}{\partial z^j} - b_j \frac{\partial a_i}{\partial z^j} \frac{\partial}{\partial z^i} \right)$

$X_{\mathcal{F}}$:= maximal Zariski open set s.t. $\mathcal{F}$ is localy free.

(とくにことわらなければ $r = \mathrm{rk}\,\mathcal{F}$, $q := n - r$ とする.)

$L \subset X_{\mathcal{F}}$ leaf

$\iff$ L is maximal connected locally closed submfd $L \subseteq X_{\mathcal{F}}$ & $T_L = \mathcal{F}|_L$.

Rem 4.2 $[\ ,\ ] : T_X \times T_X \longrightarrow T_X$

$a, b \longmapsto [a,b]$

$\}$ induced

$[\ ,\ ]^{\sim} : \mathcal{F} \times \mathcal{F} \longrightarrow T_X \longrightarrow T_X/\mathcal{F}.$

$(a, b) \longmapsto [a,b]$

$\}$ induced $\qquad \mathcal{O}_X$ bilinear map

$\pi : \Lambda^2\mathcal{F}/\mathrm{torsion} \longrightarrow T_X/\mathcal{F}.$

$\Big(\because\ [sa, b] = s[a,b] - b(s)a \equiv s[a,b] \pmod{\mathcal{F}}$

$[a, sb] \equiv -s[a,b] \Big)$

$\mathcal{F}$ is foliation $\Longleftrightarrow \pi$ is zero map

$\Leftarrow \mathrm{Hom}(\Lambda^2\mathcal{F}/\mathrm{tor.}, T_X/\mathcal{F}) = 0.$

$\quad \| \quad H^0(X, (\Lambda^2\mathcal{F}/\mathrm{tor})^{\vee} \otimes T_X/\mathcal{F})$

(dualは $\vee$ であらわすことにする)

Rem 4.3

$$[\ ,\ ] : T_X \times T_X \longrightarrow T_X$$
$$(f_1, f_2) \longmapsto [f_1, f_2]$$

$\}$ induced

$$\pi : \mathcal{F} \times \mathcal{F} \xrightarrow{[\ ,\ ]} T_X \longrightarrow T_X/\mathcal{F}$$
$$(f_1, f_2) \longmapsto [f_1, f_2]$$

π is $\mathcal{O}_X$-anti bilinear. 交代形式に

$$\Big([sf_1, f_2] = s[f_1, f_2] - f_2(s)f_1 \equiv s[f_1, f_2] \mod \mathcal{F}.$$

逆も同じ. $\Big)$

$$\therefore\ \exists\, \tilde{\pi} : \Lambda^2\mathcal{F} \longrightarrow T_X/\mathcal{F}.$$

$\mathcal{F}$ is foliation $\Longleftrightarrow \tilde{\pi}$ is zero map

$$\Longleftarrow \quad \mathrm{Hom}(\Lambda^{[2]}\mathcal{F}, T_X/\mathcal{F}) = 0$$

leaf について

Thm 4.4 Frobenius について

$\mathcal{F} \subset T_X$ foliation rank r

${}^\forall x \in X_{\mathcal{F}}$, $x \in {}^\exists U \subset X_{\mathcal{F}}$ Euclid open

s.t. $P : U \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$.

& $P_1 := Pr_1 \circ P : U \overset{P}{\simeq} \mathbb{C}^r \times \mathbb{C}^{n-r} \xrightarrow{Pr_1} \mathbb{C}^r$,

$\mathbb{C}^r$ の座標を $z_1, \ldots z_r$ とすると

$P_1^*\left(\frac{\partial}{\partial z^1}\right), \ldots P_1^*\left(\frac{\partial}{\partial z^r}\right)$ が $\mathcal{F}|_U$ の basis となる

$\Updownarrow$

$P_2 := Pr_2 \circ P : U \overset{P}{\simeq} \mathbb{C}^r \times \mathbb{C}^{n-r} \xrightarrow{Pr_2} \mathbb{C}^{n-r}$ として

${}^\forall y \in U$ について y の $\mathcal{F}$ の leaf を L_y として

$$L_y \cap U = Pr_2^{-1}(Pr_2(y))$$

③ Theorem 4.4 Frobenius について

$\mathcal{F} \subset T_X$ foliation $r = \mathrm{rk}\,\mathcal{F}$

${}^\forall x \in X_{\mathcal{F}}$, $x \in {}^\exists U \subset X_{\mathcal{F}}$ Euclid open

s.t. $P : U \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$ biholo ($Pr_1 : \mathbb{C}^r \times \mathbb{C}^{n-r} \to \mathbb{C}^r$ 1st proj)

& $P_1 := Pr_1 \circ P : U \to \mathbb{C}^r$.

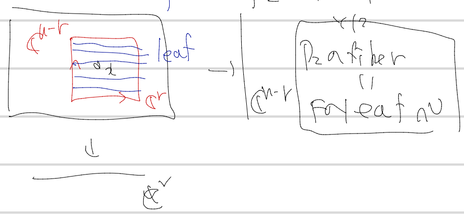
$z_1, \ldots, z_r$: $\mathbb{C}^r$ の座標とすると

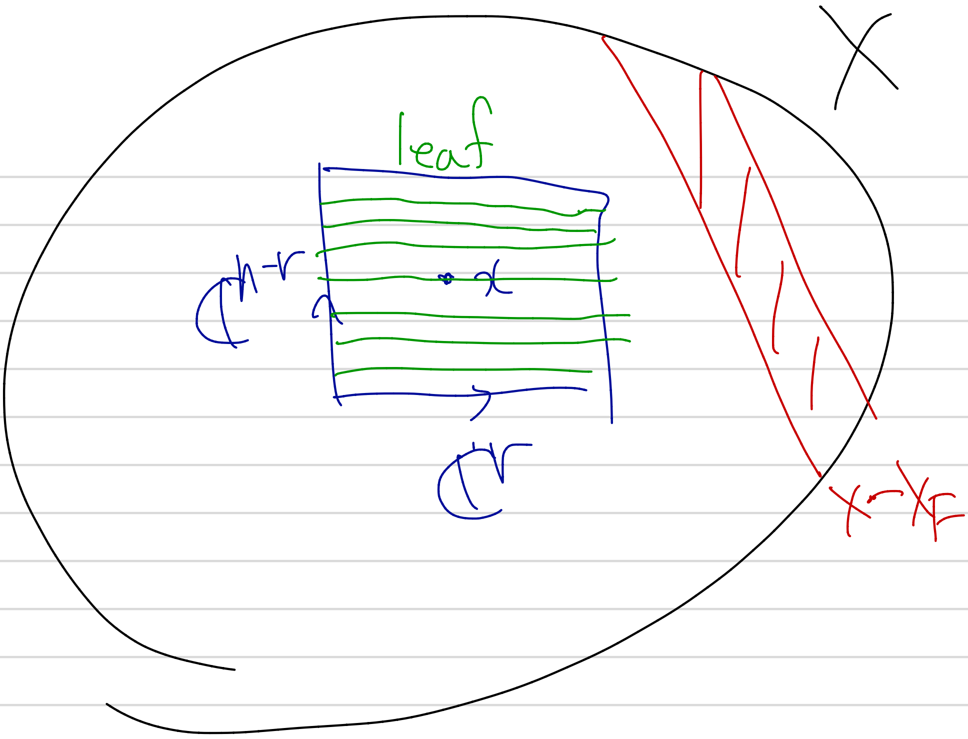
$P_1^*(z_1), P_1^*(z_2), \ldots P_1^*(z_r)$ が

$\underline{\mathcal{F}|_U}$ の basis となる

($U \subset X_{\mathcal{F}}$ より $\mathcal{F}|_U$ は vector bundle)

$P_2 = Pr_2 \circ P : U \to \mathbb{C}^{n-r}$




X
leaf
$\mathbb{C}^{n-r}$
x
$\mathbb{C}^r$
$X - X_F$

Example

X, Y sm proj

$f : X \to Y$ surj morphism

$(r := \dim X - \dim Y)$

$T_X \xrightarrow{df} f^* T_Y$ 微分写像.

$(df(p) : T_{X,p} \longrightarrow f^* T_{Y,p} = T_{Y,f(p)})$

$\mathcal{F} := \ker df$. $\left(df(T_X) \subset f^* T_Y \right.$ torsion free $\rightsquigarrow$ $\left. \mathcal{F} \text{ saturated} \right)$

$\Big(\ker df \ni f_1, f_2$. Lie bracket がうつる.

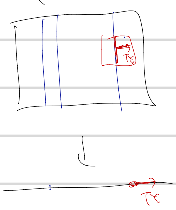
$df[f_1, f_2] = [df(f_1), df(f_2)] = 0$. $\Big)$

<u>f の general fiber は $\mathcal{F}$ の leaf</u>

Example: $f : X \to Y$. (Y = normal proj.)

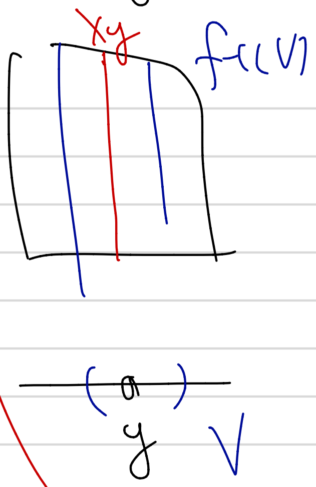
$\mathcal{F} := \ker df$ を、f からていぎされる foliation という.

$\Big(\ker df \to \underbrace{T_X \xrightarrow{df} f^* T_Y}_{\text{sheaf の射}} \Big)$



このとき general fiber は leaf となる.

∵ $y \in Y$ regular value をとる.

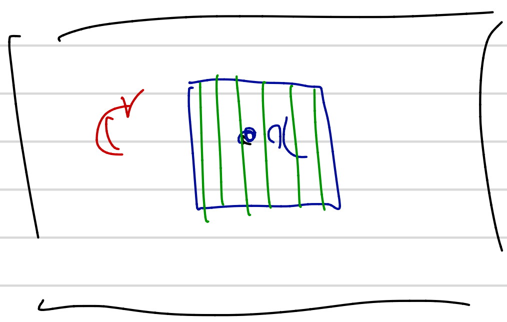


$f^{-1}(V)$ 上で

$$0 \to F \to T_X \xrightarrow{df} f^*T_Y \to 0$$

$$F|_{X_y} = T_{X_y}$$ 示せば

∵ $x \in X$. smooth point of f, $y = f(x)$ とおく

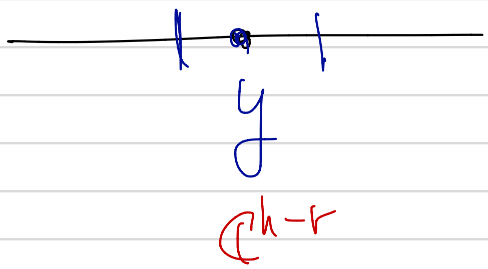


$\Rightarrow {}^\exists x \in U \subset X$ $(U, z_1, \dots, z_n)$ 座標,

$y \in {}^\exists V \subset Y$ $(V, w_1, \dots, w_{n-r})$

s.t. $f : U \longrightarrow V$

$(z_1, \dots, z_n) \to (z_1, \dots, z_{n-r})$



T_X は $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}$ で生成

f^*T_Y は $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{n-r}}$ で

∴ $\mathrm{Ker}\, df$ は $\frac{\partial}{\partial z^{n-r+1}}, \dots, \frac{\partial}{\partial z^n}$ で生成.

$T_{X_y} = F|_{X_y}$ on U. $\rightsquigarrow$ 明らかに $T_{X_y} = F|_{X_y}$

極限操作は ここであらわる.
それいらない

Def $\mathcal{F} \subset T_X$ foliation

$\mathcal{F}$ is algebraic foliation

$\overset{\text{まず}}{\Longleftrightarrow}$ (CP II) $\exists f : X \dashrightarrow Z$ dominant rational map s.t. $\mathcal{F} = \ker df$ generically on X

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists f : X \dashrightarrow Z$ dominant rational map

$\exists \pi : \tilde{X} \to X$ birational map, $\exists X_0 \subseteq X$ Zariski open s.t. $\operatorname{codim}(X \setminus X_0) \geq 2$

s.t. $\tilde{f} := f \circ \pi : \tilde{X} \to Z$ is morphism

$\pi^* \mathcal{F} = \ker d\tilde{f}$ on $\pi^{-1}(X_0)$

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"'] & & Z \end{tikzcd}$$

Def (4.35.4) $\mathcal{F} \subset T_X$: Foliation

$\mathcal{F}$ is algebraic.

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists Z$ sm proj var.,

$\exists \pi : \tilde{X} \to X$ birational map

$\exists f : \tilde{X} \to Z$ surjective morphism

s.t. $\pi^* \mathcal{F} = \ker df$ on some (nonempty) Zariski open set.

Codimension 2 をのぞいて成立

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "\pi"] & Z \\ X & \end{tikzcd}$$

つまりは algebraic foliation は

ほぼ"手でつくられたもの"

↳ うれしい

$\boxed{\text{Caution}}$ Lazić の定義とはちょっとちがう

$\mathcal{F} \subset T_X$ foliation L : $\mathcal{F}$ a leaf. ($L \subset X$)

L is algebraic $\overset{\text{def}}{\iff}$ $\begin{cases} L \text{ is Zariski open on } \overline{L}^{\text{Zar}} \\ \dim L = \dim \overline{L}^{\text{Zar}} \end{cases}$

$\mathcal{F}$ is algebraic foliation

$\overset{\text{def}}{\iff}$ general point $\forall x \in X$, $x \in {}^{\forall}L$ leaf について

L is algebraic.

Lem 4.12. Laziの定義とさっきの定義は一致する

[Sketch pf $\mathrm{Chow}_{r,\delta}(X) = \{ r\text{dim subvariety of } X, \text{degree } \delta \}$

Chow variety.

$$p : X_0 \longrightarrow \bigcup_{\delta \geq 0} \mathrm{Chow}_{r,\delta}(X)$$

$$x \longmapsto [x\text{をとおる}\mathcal{F}\text{ a leaf}]$$

$$W = \overline{p(X_0)}^{\mathrm{Zar}}.$$

$$U = \overline{\{ (x,w) \in X_0 \times W \mid p(x) = w \}}^{\mathrm{Zar}} \subset X \times W$$

$\leadsto$

$$\begin{tikzcd} & U \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X \arrow[rr, dashed, "p"'] & & W \end{tikzcd}$$

general ptをとおる $pr_1^* \mathcal{F}$ a leaf $=$ pr_2 a general fiber]

Question $\mathcal{F} \subseteq T_X$ torsion-free

① $\mathcal{F}$ はいつ foliation になるか?

② $\mathcal{F}$ はいつ algebraic foliation になるか

$\rightsquigarrow$

"slope" をもちいた判定法

(②については数論的な判定法もあり)

Question. $\mathcal{F} \subseteq T_X$ $\mathcal{F}$: torsion-free

① $\mathcal{F}$ はいつ foliation になるか?

② $\mathcal{F}$ が foliation のとき、$\mathcal{F}$ はいつ algebraic になるか

$\rightsquigarrow$

一般にはムズかしい

1つの方法として slope をもちいた判定法がある

(②については数論的な判定法もあるけど…)

$\boxed{\text{Slope.}}$

$$N_1(X)_{\mathbb{R}} = \left\{ \sum_{i=1}^{\ell} a_i C_i \;\middle|\; a_i \in \mathbb{R},\ C_i : \text{irr reduced proper curve} \right\}$$

$$\mathrm{Mov}(X) = \left\{ \alpha \in N_1(X)_{\mathbb{R}} \;\middle|\; \begin{array}{l} \forall D \text{ eff div} \\ \alpha \cdot D \geq 0 \end{array} \right\}$$

by [BDPP]

① $$SMC(X) = \left\{ \alpha \in N_1(X)_{\mathbb{R}} \;\middle|\; \begin{array}{l} \exists \pi : \tilde{X} \to X \text{ birational morphism} \\ \exists C = A_1 \cap \cdots \cap A_{n-1} \text{ complete intersection} \\ \quad \text{by very ample divisors } A_1 \ldots A_{n-1} \\ \text{s.t. } \alpha = \pi_* C \end{array} \right\}$$

とおく

$$\mathrm{Mov}(X) = \overline{SMC(X)} \subset N_1(X)_{\mathbb{R}}$$

② L: line bundle

$$L \text{ is psef} \iff \forall \alpha \in \mathrm{Mov}(X),\ \alpha \cdot L \geq 0$$

6.3. **Higher Fujita's decomposition**

Definition 6.3.1. [**KM98**] Let X be a smooth projective manifold.

(1) A *1-cycle* is a formal linear combination of irreducible reduced and proper curves $C = \sum a_i C_i$.
(2) Two 1-cycle C, C' is *numerically equivalent* if $D.C = D.C'$ for any Cartier divisor D.
(3) $N_1(X)_{\mathbb{R}}$ is a $\mathbb{R}$-vector space of 1-cycles with real coefficients modulo numerical equivalence.

2.2 $\boxed{\text{Slope.}}$

$N_1(X)_{\mathbb{R}} = \{\sum a_i C_i \mid a_i \in \mathbb{R},\ C_i : \text{irr reduced proper curve}\}$ / $\equiv_{\text{num}}$

$\mathrm{Mov}(X) = \{\alpha \in N_1(X)_{\mathbb{R}} \mid \forall D \text{ eff},\ \alpha \cdot D \geq 0\}$

$=$ by [BDPP]

$=$ ① $SMC(X) = \{\alpha \in N_1(X)_{\mathbb{R}} \mid \exists \pi : \tilde{X} \to X$ birat morphism, $\exists C := A_1 \cap \cdots \cap A_{n-1}$ complete intersection, $\exists A_i$ very ample, s.t. $\alpha = \pi_* C\}$

とおく

$\mathrm{Mov}(X) = \overline{SMC(X)}$

② L: line bundle.

L psef $\iff \forall \alpha \in \mathrm{Mov}(X),\ L \cdot \alpha \geq 0$

Surface αをとる(??)

Def 2.8 $\alpha \in \mathrm{Mov}(X)$ をとっておく

$0 \neq \mathcal{E}$ torsion free coh sheaf について

① $\mathcal{E}$ の α による slope $\mu_\alpha(\mathcal{E}) := \dfrac{c_1(\mathcal{E})\cdot\alpha}{\mathrm{rk}\,\mathcal{E}}$

$c_1(\mathcal{E}) := \left(\bigwedge^{\mathrm{rk}\,\mathcal{E}}\mathcal{E}\right)^{\vee\vee}$

② $\mathcal{E}$ is α semistable (α-s.s.) $\quad$ α-stable

$\iff {}^\forall 0 \neq \mathcal{F} \subseteq \mathcal{E}$, $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ $\quad (<)$

③ $\mu_\alpha^{max}(\mathcal{E}) = \sup\{\mu_\alpha(\mathcal{F}) \mid 0 \neq \mathcal{F} \subset \mathcal{E}$ coherent$\}$

$\mu_\alpha^{min}(\mathcal{E}) = \inf\{\mu_\alpha(\mathcal{Q}) \mid \mathcal{E} \twoheadrightarrow \mathcal{Q}$ torsionfree coh$\}$

Rem 1

これは exact seq. $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$ について

$c_1(\mathcal{F}) + c_1(\mathcal{Q}) = c_1(\mathcal{E})$

$\therefore \boxed{\mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F}) + \mathrm{rk}\,\mathcal{Q}\,\mu_\alpha(\mathcal{Q}) = \mathrm{rk}\,\mathcal{E}\,\mu_\alpha(\mathcal{E})}$

その2 $\mathcal{E}$ - αss $\iff \mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{E}) \iff \mu_\alpha^{min}(\mathcal{E}) = \mu_\alpha(\mathcal{E})$

Def 2.8 $\alpha \in \mathrm{Mov}(X)$ とする.

$\mathcal{E} \neq 0$ torsionfree coh sheaf について. $\quad (\bigwedge^{\mathrm{rk}\,\mathcal{F}})^{\vee\vee}$

① $\mathcal{E}$ の slope w.r.t α $\overset{def}{\iff}$ $\mu_\alpha(\mathcal{E}) := \dfrac{c_1(\mathcal{E})\cdot\alpha}{\mathrm{rk}\,\mathcal{E}}$

② $\mathcal{E}$ is α-semistable (α stable)

$\iff {}^\forall 0 \neq \mathcal{F} \subsetneq \mathcal{E}$. coherent (torsionfreeにしてもよい)

$\mu_\alpha(\mathcal{F}) \leq \mathcal{E}$ $\quad (<)$

③ $\mu_\alpha^{max}(\mathcal{E}) = \sup\{\mu_\alpha(\mathcal{F}) \mid 0 \neq \mathcal{F} \subset \mathcal{E}$ coherent$\}$

$\mu_\alpha^{min}(\mathcal{E}) = \inf\{\mu_\alpha(\mathcal{Q}) \mid \mathcal{E} \twoheadrightarrow \mathcal{Q}$ torsionfree coherent$\}$

Rem. $\mathcal{E}$ α semistable $\iff \mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{E})$

$(\Leftarrow$ 大事$)$ $\mu_\alpha(\mathcal{E}) = \mu_\alpha^{min}(\mathcal{E})$

・ $\mathcal{F} \hookrightarrow \mathcal{E}$. $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{E}$

$\Rightarrow \det \mathcal{F} \hookrightarrow \det \mathcal{E}$

とくに saturation とることにより $\Rightarrow {}^\exists D$ eff $\det \mathcal{F} \otimes \mathcal{O}_X(+D) = \det \mathcal{E}$

$\Rightarrow \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ //

Rem2 $\mathcal{F} \subset \mathcal{E}$ & $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{E}$

$\Rightarrow$ $\det \mathcal{F} \xhookrightarrow{\text{inclusion}} \det \mathcal{E}$ より

$\exists D$ eff. $\det \mathcal{F} \otimes \mathcal{O}_X(-D) = \det \mathcal{E}$.

$\Rightarrow$ $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$.

とくに $\mathcal{E}$ locally free として $\forall \mathcal{F} \subset \mathcal{E}$ coherent subsheaf として

$\mathcal{E} \supset \mathcal{F}_{sat}$: $\mathcal{F}$ のsaturation ($\mathrm{rk}\,\mathcal{G} = \mathrm{rk}\,\mathcal{F}$ で、かつ $\mathcal{F} \subset \mathcal{G}$ saturated sheaf のなかで一番大きいもの.)

とおいて $\mathcal{F} \subset \mathcal{F}_{sat}$ より $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{F}_{sat})$

Rem3 $\mathcal{F}, \mathcal{G}$ torsion free coherent.

$$\mu_\alpha^{max}(\mathcal{F} \boxtimes \mathcal{G}) = \mu_\alpha^{max}(\mathcal{F}) + \mu_\alpha^{max}(\mathcal{G})$$

(これはかなり難しい)

($\mathcal{F}, \mathcal{G}$ αss $\Rightarrow$ $\mathcal{F} \boxtimes \mathcal{G}$ αss を示すとよい ...)

とくに $\mu_\alpha^{max}(\mathrm{Sym}^{[m]} \mathcal{F}) = m\,\mu_\alpha^{max}(\mathcal{F})$

$\mu_\alpha^{max}(\bigwedge^{[m]} \mathcal{F}) = m\,\mu_\alpha^{max}(\mathcal{F})$

Lem (Lazic Lem 2.9, 2.10) $\mathcal{E}$ torsion free coherent

① $\mu_\alpha^{max}(\mathcal{E}) = -\mu_\alpha^{min}(\mathcal{E}^\vee)$

② $\mathcal{F} = \alpha$ ss とする

$0 \neq {}^\forall r : \mathcal{F} \to \mathcal{E}$, $\mu_\alpha(r(\mathcal{F})) \geq \mu_\alpha(\mathcal{F})$

とくに $\mu_\alpha(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$

③ $\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$

④ $\mathcal{F} \subset \mathcal{E}$ $\mathcal{F}$ saturated & $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$

$\Rightarrow \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \leq \mu_\alpha(\mathcal{F})$ & $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

⑤ $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ とすれば

$\mathcal{E} = \alpha$-s.s $\Leftrightarrow$ $\mathcal{F}, \mathcal{G}$ α-s.s. & $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$

とくに L line bundle について $L^{\oplus N}$ is α-ss ${}^\forall N \in \mathbb{N}_{>0}$

Lemma (Lazic Prop 2.9, Lem 2.10)

① $\mu_\alpha^{max}(\mathcal{E}) = -\mu_\alpha^{min}(\mathcal{E}^\vee)$

② $\mathcal{F}$: α semistable.

$0 \neq {}^\forall r : \mathcal{F} \to \mathcal{E}$, $\mu_\alpha(r(\mathcal{F})) \geq \mu_\alpha(\mathcal{F})$.

とくに $\mu_\alpha(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$

③ $\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathrm{Hom}(\mathcal{F}, \mathcal{E}) = 0$

④ $\mathcal{F} \subset \mathcal{E}$. $\mathcal{F}$ saturated, $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$ ($\mathbb{Q}$①は $\mathcal{E}$ α-semistable $\Leftrightarrow \mathcal{E}^\vee$ α semistable)

$\Rightarrow \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \leq \mu_\alpha(\mathcal{F})$ &

$\mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

⑤ $\mathcal{E} := \mathcal{F} \oplus \mathcal{G}$ とする

$\mathcal{F}, \mathcal{G}$ α-semistable & $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$ $\Leftrightarrow$ $\mathcal{E}$ α-semistable

とくに L : line bundle とり $L^{\oplus N}$ は α-semistable pf ① $\forall \mathcal{F} \subset \mathcal{E}$ について $\mathcal{E}^\vee \twoheadrightarrow \mathcal{F}^\vee$ より

$$\mu_\alpha(\mathcal{F}) = -\mu_\alpha(\mathcal{F}^\vee) \leq -\mu_\alpha^{min}(\mathcal{E}^\vee)$$

$$\therefore \mu_\alpha^{max}(\mathcal{E}) \leq -\mu_\alpha^{min}(\mathcal{E}^\vee) \text{ を得る}$$

② $0 \to {}^\exists K \to \mathcal{F} \to r(\mathcal{F}) \to 0$ より

$$\mathrm{rk}(r(\mathcal{F}))\mu_\alpha(\mathcal{F}) \underset{\alpha\text{-ss}}{\leq} \mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F}) - \mathrm{rk}\,K\,\mu_\alpha(K)$$

$$= \mathrm{rk}(r(\mathcal{F}))\,\mu_\alpha(r(\mathcal{F}))$$

$$\therefore \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F})) \quad //$$

③ $0 \neq {}^\forall r\ \mathcal{F} \to \mathcal{E}$ について $\mathcal{F} \twoheadrightarrow r(\mathcal{F}) \hookrightarrow \mathcal{E}$ より

$$\mu_\alpha^{min}(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F})) \leq \mu_\alpha^{max}(\mathcal{E}) \text{ より}$$

pf ① $0 \neq {}^\forall \mathcal{F} \subset \mathcal{E}$ について. $\mathcal{E}^\vee \to \mathcal{F}^\vee \to 0$ より

$$\mu_\alpha(\mathcal{F}^\vee) = -\mu_\alpha(\mathcal{F}) \geq -\mu_\alpha^{max}(\mathcal{E})$$

$$\Rightarrow \mu_\alpha^{min}(\mathcal{E}^\vee) \geq -\mu_\alpha^{max}(\mathcal{E}) \text{ を得る}$$

② $0 \to {}^\exists K \to \mathcal{F} \to r(\mathcal{F}) \to 0$ と

$c_1(\mathcal{F}) - c_1(K) = c_1(r(\mathcal{F}))$ より

$$\mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F}) - \mathrm{rk}\,K\,\mu_\alpha(K) = \mathrm{rk}(r(\mathcal{F}))\mu_\alpha(r(\mathcal{F}))$$

VI $\mathcal{F}$: α-semistable.

$$\mathrm{rk}(r(\mathcal{F}))\mu_\alpha(\mathcal{F}) \quad \Rightarrow \quad \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F}))$$

③ ${}^\forall r \neq 0 : \mathcal{F} \to \mathcal{E}$ について.

$\mathcal{F} \twoheadrightarrow r(\mathcal{F}) \hookrightarrow \mathcal{E}$ から

$\mu_\alpha^{min}(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F})) \leq \mu_\alpha^{max}(\mathcal{E})$ となる為

④

④ $0 \neq {}^{\forall}\mathcal{G} \subset \mathcal{E}/\mathcal{F}$ について $\mu_\alpha(\mathcal{G}) \le \underset{\overset{\shortparallel}{\mu_\alpha(\mathcal{F})}}{\mu_\alpha^{\max}(\mathcal{E}/\mathcal{F})}$ をいう

$\exists \mathcal{G}' \subset \mathcal{E}$ s.t. $\mathcal{F} \subset \mathcal{G}'$ & $\mathcal{G}'/\mathcal{F} = \mathcal{G}$

$(\mu_\alpha(\mathcal{G}') \le \mu_\alpha(\mathcal{F}))$

$0 \to \mathcal{F} \to \mathcal{G}' \to \mathcal{G} \to 0$ より

$$\mathrm{rk}\,\mathcal{G}\,\mu_\alpha(\mathcal{G}) = \mathrm{rk}\,\mathcal{G}'\,\mu_\alpha(\mathcal{G}') - \mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F})$$
$$\le \mathrm{rk}\,\mathcal{G}\,\mu_\alpha(\mathcal{F}) \qquad \therefore\ \mu_\alpha(\mathcal{G}) \le \mu_\alpha(\mathcal{F}).$$

$\mathcal{F} \twoheadrightarrow {}^{\forall}\mathcal{Q}$ について $\mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$ をいう

$0 \to {}^{\exists}\mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$ より

② [illegible] $\quad \mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$ //

---

④ $\mathcal{F} \subset \mathcal{E}$, $\mathcal{F}$ saturated, $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{\max}(\mathcal{E})$
$\Rightarrow \mu_\alpha^{\max}(\mathcal{E}/\mathcal{F}) \le \mu_\alpha(\mathcal{F})$ &
$\mu_\alpha(\mathcal{F}) = \mu_\alpha^{\min}(\mathcal{F})$

④ of ${}^{\forall}\mathcal{G} \to \mathcal{E}/\mathcal{F}$ (saturatedより) torsion free

$\Rightarrow \mathcal{G}' \subset \mathcal{E}$ s.t. $\mathcal{F} \subset \mathcal{G}'$ & $\mathcal{G}'/\mathcal{F} = \mathcal{G}$

$\Rightarrow \mu_\alpha(\mathcal{G}') \le \mu_\alpha^{\max}(\mathcal{E}) = \mu_\alpha(\mathcal{F})$

$0 \to \mathcal{F} \to \mathcal{G}' \to \mathcal{G} \to 0$ $(c_1(\mathcal{G}) = c_1(\mathcal{G}') - c_1(\mathcal{F})$

$\Rightarrow$ (② [illegible]) $\quad \mu_\alpha(\mathcal{F}) \ge \mu_\alpha(\mathcal{G}) \Rightarrow \mu_\alpha(\mathcal{F}) \ge \mu_\alpha^{\max}(\mathcal{E}/\mathcal{F})$
maxで

$\mathcal{F} \to {}^{\forall}\mathcal{Q} \to 0$ について

$0 \to {}^{\exists}\mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$ $(c_1(\mathcal{Q}) = c_1(\mathcal{F}) - c_1(\mathcal{K})$

$\Rightarrow$ ② [illegible] $\quad \mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$

$\Rightarrow \quad \mu_\alpha^{\min}(\mathcal{F}) \ge \mu_\alpha(\mathcal{F}) \ge \mu_\alpha^{\min}(\mathcal{F})$.
ごおい [illegible] より.

$\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$

⑤ $\mathcal{F}, \mathcal{G}$ α-semistable & $\mathcal{E}$ NOT α-semistable

$\Rightarrow \exists$ of $\mathcal{E}' \subset \mathcal{E}$ & $\mu_\alpha(\mathcal{E}') > \mu_\alpha(\mathcal{E})$

$\mathcal{E}'$-αsemistable $\mathcal{F} \oplus \mathcal{G}$

$\Rightarrow \mathcal{E}' \cap \mathcal{F} \neq 0$ により ($\mu_\alpha(\mathcal{E}') > \mu_\alpha(\mathcal{E})$ で $\mathcal{E}'$ は α semistable なものでmax)

$\Rightarrow \exists r : \mathcal{E}' \to \mathcal{F}_{(\text{or } \mathcal{G})}$ $r \neq 0$

$\Rightarrow \mu_\alpha(\mathcal{F}) \geq \mu_\alpha(r(\mathcal{E}')) \geq \mu_\alpha(\mathcal{E}') > \mu_\alpha(\mathcal{E})$

$\mathcal{F}$: α-semistable　　$\mathcal{E}'$: α semistable

$\mu_\alpha(\mathcal{E})$ ($\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$ & $C_1(\mathcal{F}) + C_1(\mathcal{G}) = C_1(\mathcal{E})$)

で矛盾

$\mathcal{E}$ - α s.s. $\Rightarrow 0 \neq Pr$ $\mathcal{E} \to \mathcal{F}$.

$\Rightarrow \underset{\mu_\alpha(\mathcal{E})}{\underset{\shortparallel}{\mu_\alpha^{max}(\mathcal{E})}} \geq \mu_\alpha^{max}(\mathcal{F}) \geq \mu_\alpha(\mathcal{E})$ ($\mathcal{E}$ α s.s)

$\Rightarrow \mu_\alpha(\mathcal{E}) = \mu_\alpha^{max}(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{G})$

(1) $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$

($C_1(\mathcal{F}) = C_1(\mathcal{E}) - C_1(\mathcal{G})$

$\Rightarrow$ rk $\mathcal{F}$ $\mu_\alpha(\mathcal{F}) \geq$ rk $\mathcal{F}$ $\mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{E})$)

⑤ $\mathcal{E}$ - - α.s.s　　$\mathcal{E} = \mathcal{F} \oplus \mathcal{G} \to \mathcal{F}$.

$\Rightarrow 0 \neq Pr_1 : \mathcal{E} \longrightarrow \mathcal{F}$

(2) $\Rightarrow \underset{\mu_\alpha(\mathcal{E})}{\underset{\shortparallel}{\mu_\alpha^{max}(\mathcal{E})}} \geq \mu_\alpha^{max}(\mathcal{F}) \geq \mu_\alpha(\mathcal{E})$

$\Rightarrow \mu_\alpha(\mathcal{E}) = \mu_\alpha^{max}(\mathcal{F}) = \mu_\alpha(\mathcal{E})$

一方で $C_1(\mathcal{F}) = C_1(\mathcal{E}) - C_1(\mathcal{G})$ より　　$\mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{max}(\mathcal{E})$ //

$\mathcal{F}, \mathcal{G}$ α ss & $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$ とする. $\mathcal{E}$ が NOT α-ss とかって矛

$\mu_\alpha(\mathcal{E}) = \mu_\alpha(\mathcal{F})$

$\exists$ of $\mathcal{E}' \subset \mathcal{E}$. s.t $\mu_\alpha(\mathcal{E}') > \mu_\alpha(\mathcal{E})$

$\Rightarrow \mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ より　$\mathcal{E} \cap \mathcal{F} \neq \emptyset$ により

$0 \neq \exists Pr_2$ $\mathcal{E} \longrightarrow \mathcal{F}$.

$\mu_\alpha(\mathcal{E}) = \mu_\alpha(\mathcal{F}) \geq \mu_\alpha(r(\mathcal{E}')) \overset{(2)}{\geq} \mu_\alpha(\mathcal{E}') > \mu_\alpha(\mathcal{E})$

$\square$

Cor 2.14 (α-maximal destabilizing sheaf の存在)

$\alpha \in \mathrm{Mov}(X)$ $\mathcal{E}$ torsion-free coh

$\exists\, \mathcal{E}_{max} \subset \mathcal{E}$ α-s.s. saturated

$$\begin{cases} \mu_\alpha(\mathcal{E}_{max}) = \mu_\alpha^{max}(\mathcal{E}) \\ \forall\, \mathcal{F} \subset \mathcal{E},\ \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathcal{F} \subset \mathcal{E}_{max} \end{cases}$$

$\mathcal{E}_{max}$ α-maximal destabilizing sheaf

(Corollary 2.14) $\alpha \in \mathrm{Mov}(X)$, of $\mathcal{E}$ torsion-free coh sheaf

$\exists\, \mathcal{E}_{max} \subset \mathcal{E}$. α-s.s. saturated (⇒reflexive)

- $\mu_\alpha(\mathcal{E}_{max}) = \mu_\alpha^{max}(\mathcal{E})$
- $\forall\, \mathcal{F} \subseteq \mathcal{E}$ $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathcal{F} \subset \mathcal{E}_{max}$.

we call $\mathcal{E}_{max}$ by "maximal destabilizing sheaf"

Sketch pf ①: $\mu_\alpha^{max}(\mathcal{E}) < +\infty$

$\left(\exists N \quad \mathcal{E} \hookrightarrow \mathcal{H}^{\oplus N} \quad \mathcal{H}^{\oplus N} \text{ semistable より} \right)$

② of $\exists \mathcal{F} \subset \mathcal{E}$ s.t. $\mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{F})$

はいりほう $\forall \mathcal{F} \subset \mathcal{E},\ \mu_\alpha(\mathcal{F}) < \mu_\alpha^{max}(\mathcal{E})$ とする)

$k = \max\left\{ 1 \le k \le rk\,\mathcal{E} \;\middle|\; \exists \{\mathcal{F}_i\}_{i=1}^{\infty},\ rk\,\mathcal{F}_i = k,\ \mu_\alpha(\mathcal{F}_1) \le \mu_\alpha(\mathcal{F}_2) \le \cdots,\ \mathcal{F}_i \text{ saturated},\ \lim_{i \to \infty} \mu_\alpha(\mathcal{F}_i) = \mu_\alpha^{max}(\mathcal{E}),\ \mathcal{F}_i \ne \mathcal{F}_j \right\}$ (☆)

とおく $k < rk\,\mathcal{E}$ である

しかし、$\mathcal{G}_i = \mathcal{F}_{i+1} + \mathcal{F}_i$ とおくと $rk\,\mathcal{G}_i > k$ で

$\{\mathcal{G}_i\}$ が (☆) をみたす

③ $l = \max\left\{ 1 \le l \le rk\,\mathcal{E} \;\middle|\; \exists \mathcal{F} \subset \mathcal{E},\ \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E}),\ \mathcal{F} \text{ saturated} \right\}$

とおく。$\mathcal{E}_{max}$ を l を attain する $\mathcal{F}$ とする

はいりほう

$\exists \mathcal{G} \subset \mathcal{E},\ \mu_\alpha(\mathcal{G}) = \mu_\alpha^{max}(\mathcal{E})\ \&\ \mathcal{G} \not\subset \mathcal{F}$

($\mathcal{F}$ も $\mathcal{G}$ は saturated とする)

$\Rightarrow rk(\mathcal{G} + \mathcal{F}) > l.$ ($rk\,\mathcal{F} = rk\,\mathcal{G}$ なら saturation の定義)

$\left(rk(\mathcal{G} + \mathcal{F}) = l \Rightarrow \mathcal{F}_{sat} \supset \mathcal{G} + \mathcal{F} \supset \mathcal{F} \Rightarrow \mathcal{G} \subset \mathcal{F} \right)$

$$0 \to \mathcal{F} \cap \mathcal{G} \to \mathcal{F} \oplus \mathcal{G} \to \mathcal{F} + \mathcal{G} \to 0$$

$$\mathrm{rk}(\mathcal{F}+\mathcal{G})\,\mu_\alpha(\mathcal{F}+\mathcal{G}) = \mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F}) + \mathrm{rk}\,\mathcal{G}\,\mu_\alpha(\mathcal{G}) - \mathrm{rk}(\mathcal{F} \cap \mathcal{G})\,\mu_\alpha(\mathcal{F} \cap \mathcal{G})$$

$$\geq \mathrm{rk}(\mathcal{F}+\mathcal{G})\,\mu_\alpha^{max}(\mathcal{E})$$

$$\therefore \quad \mu_\alpha(\mathcal{F}+\mathcal{G}) = \mu_\alpha^{max}(\mathcal{E})$$

Problem ① の問い: $\mathcal{F} \subset T_X$ saturated はいつ foliation になるか

1つの Answer

$$\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F}) \Rightarrow \mathcal{F} \text{ foliation}$$

とくに $\mathcal{F} \subset T_X$ s.t. $\mu_\alpha^{max}(T_X) = \mu_\alpha(\mathcal{F})$ をみたす

$$\mu_\alpha^{max}(T_X) > 0 \Rightarrow \mathcal{F} \text{ foliation}$$

つまり $\mu_\alpha^{max}(T_X) > 0$ なら α-maximal destabilizing sheaf T_X^{max} は foliation

Problem 1 $\mathcal{F} \subset T_X$ saturated

$\mathcal{F}$ はいつ foliation になるか

1つの answer

$$\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F})$$

$$\Rightarrow \mathcal{F} : \text{foliation}$$

$$\because \mu_\alpha^{min}(\wedge^{[2]}\mathcal{F}) = 2\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(T_X/\mathcal{F})$$

$$\Rightarrow 0 = \mathrm{Hom}(\wedge^{[2]}\mathcal{F}, T_X/\mathcal{F}) \text{ より}$$

とくに α-maximal destabilizing sheaf $\mathcal{F}$ について

$$\mu_\alpha^{max}(T_X) > 0 \qquad \mathcal{F} \text{ foliation}$$

pf $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X) > 0$

$$\overset{⑤}{\Rightarrow} 2\mu_\alpha^{min}(\mathcal{F}) = 2\mu_\alpha(\mathcal{F}) > \mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{max}(T_X/\mathcal{F})$$

より

Analytic Graph of $\mathcal{F}$

$r = \mathrm{rk}\,\mathcal{F}$

Frobenius より

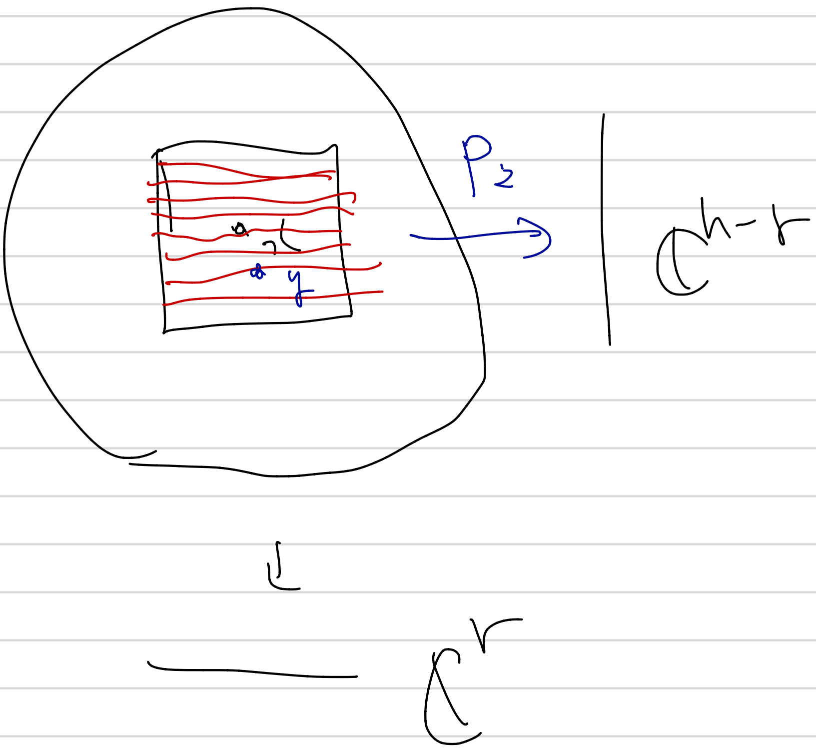
$\forall x \in X_{\mathcal{F}}$ ${}^{\exists}U_x \subset X_{\mathcal{F}}$

s.t. $\begin{cases} p: U_x \simeq \mathbb{C}^r \times \mathbb{C}^{n-r} \end{cases}$

$p_2 := pr_2 \circ p : U_x \longrightarrow \mathbb{C}^{n-r}$ として、

$\forall y \in U_x$, L_y: y を通る $\mathcal{F}$ の leaf

$$L_y \cap U_x = p_2^{-1}(p_2(y))$$



Analytic Graph のこころ　CP 4.1 Lazic Lem 4.1

Frobenius の定理 より、

$\forall x \in X_{\mathcal{F}}$ ${}^{\exists}U_x \subset X_{\mathcal{F}}$ Euclid open

s.t. ${}^{\exists}p_2: U_x \longrightarrow \mathbb{C}^{n-r}$.

s.t. p_2 の fiber $= \mathcal{F}$ の leaf $\cap U_x$

($\forall y \in U_x$ として p_2 の $p_2(y)$ の fiber $=$ y を通る leaf $\cap U_x$)

ここで

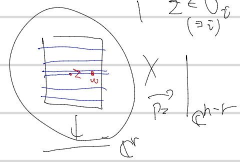
$\bigcup_{x \in X_{\mathcal{F}}} U_x = X_{\mathcal{F}}$

$\Rightarrow$ $\exists \{U_i\}_{i=1}^{\infty}$, locally finite cover of $X_{\mathcal{F}}$

$\Rightarrow$ $\widetilde{U} = \bigcup_{i=1}^{\infty} U_i \times U_i \subset X_{\mathcal{F}} \times X_{\mathcal{F}}$. Euclid open.

$\widetilde{U} \supset \cancel{A}\, V := \{(z, w) \in X \times X \mid z \in U_i\ ({}^{\exists}i),\ w \in U_i \cap \boxed{\begin{matrix} z \text{を通る} \\ \mathcal{F}\text{ leaf} \end{matrix}}\}$ とおく

$\boxed{p_2(z) \text{ の } p_2 \text{ の fiber}}$



Analytic Graph のこうせい

CP 4.1 Lazic Lemma 4.1

Frobeniusのていりより、

$\forall x \in X_F \ \exists_x\ U_x \subset X_F$ Euclid open

s.t. $\exists\, p_2 : U_x \to \mathbb{C}^{n-r}$.

s.t. p_2 のfiber $= F$ の leaf $\cap U_x$

($\forall y \in U_x$ とすると p_2 の $p_2(y)$ のfiber $=$ y を通る leaf $\cap U_x$)

ここで

$\bigcup_{x \in X_F} U_x = X_F$

$\Rightarrow \exists \{U_i\}_{i=1}^{\infty}$, locally finite cover of X_F

$\Rightarrow \tilde{U} = \bigcup_{i=1}^{\infty} U_i \times U_i \subset X_F \times X_F$ Euclid open.

$\tilde{U} \supset V := \{(z,w) \in X \times X \mid z \in U_i\ (\exists i),\ w \in U_i \cap$ (zを通る F の leaf) ($p_2(z)$ の p_2 の fiber)$\}$

とおく

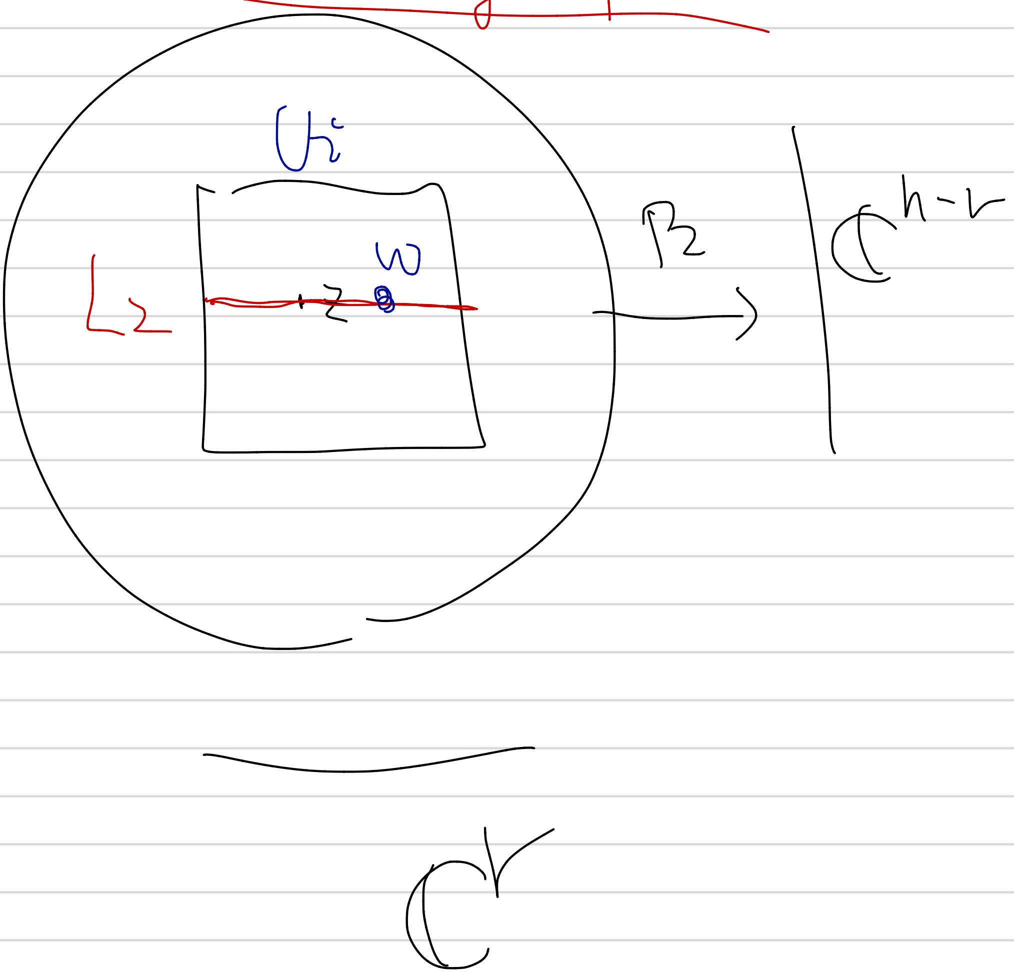
$$\bigcup_{x \in X_F} U_x = X_F$$

$$\rightsquigarrow \exists \{U_{x_i}\}_{i=1}^{\infty} \quad \bigcup_{i=1}^{\infty} U_{x_i} = X_F$$

$$V = \{(z,w) \in X \times X \mid \exists i,\ z \in U_i,\ w \in U_i \cap L_z\}$$

とおく

analytic graph



Vについて ・V is locally closed analytic subset

・$V \supset \Delta := \{(x,x) \mid x \in X_F\}$ diagonal set

・$\dim V = n+r$

ここで

~~$\overline{V}^{zar} \subset X \times X$ とおく~~

$\overline{V}^{zar} \subset X\times X$ をかんがえる

Rem $N_{\Delta/V} = F|_{X_F}$ $\left(N_{\Delta/V} = T_V/T_\Delta\right)$

($\Delta \cong X_F$ の同一視のもと

これは Frobenius $P_r : U_x \longrightarrow \mathbb{C}^r \times \mathbb{C}^{n-r}$

$(z_1, \dots, z_r)(z_{r+1}, \dots, z_n)$ とおくと

$F|_{U_x}$ は $D_i^* = (P_r \circ p)^* z_i$ $(1 \le i \le r)$ で生成

一方 T_V は $D_i^*(z_2^i)$ $1 \le i \le r$ と $D_2^*(z_2^i)$ $1 \le i \le r$

T_Δ は $D_1^*(z_2^i)$ $1 \le i \le r$.

で書かれる (D2 をくつつける)

Rem

・V : localy closed analytic subset

・$\dim V = n+r$

・$V \supset \Delta := \{(x,x) \mid x \in X_F\}$

・$N_{\Delta/V} \left(:= T_V / T_\Delta\right) = F|_{X_F}$

($\Delta \underset{pr_2}{\xrightarrow{\sim}} X_F$ の同一視において)

$X \times X \supset U_x \times U_x$ の座標を

$(z_1^1, z_1^2, \dots, z_1^n)$ $(z_2^1, z_2^2, \dots, z_2^n)$

$pr_2^* F|_{U_x}$ $\quad \dfrac{\partial}{\partial z_2^1}, \dfrac{\partial}{\partial z_2^2}, \dots, \dfrac{\partial}{\partial z_2^r}$ で生成

T_V $\quad \dfrac{\partial}{\partial z_1^1} \cdots \dfrac{\partial}{\partial z_1^n} \quad \dfrac{\partial}{\partial z_2^1} \cdots \dfrac{\partial}{\partial z_2^r}$

T_Δ $\quad \dfrac{\partial}{\partial z_1^1} \cdots \dfrac{\partial}{\partial z_1^n}$

$\underline{\text{Lem 4.9}}$ $\overline{V}^{Zar} \subset X \times X$

$$\dim \overline{V}^{Zar} = n + r$$

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation

$\underline{\text{Lem 4.9}}$ $\mathcal{F}$: rank r, Foliation

$V \subset X \times X$ closed analytic manifold

analytic germ is analytic graph of $\mathcal{F}$.

$\dim \overline{V}^{Zar} = n+r$

$\Rightarrow$ $\mathcal{F}$ is algebraic integrable

$\boxed{\text{Pf}}$ $V \subset \overline{V}^{Zar}$ open (何 open?)

$pr_2|_{\overline{V}^{Zar}} : \underset{n+r}{\overline{V}^{Zar}} \to \underset{n}{X}$ をかんがえる

general fiber $=$ $\underline{\text{irreducible}, r\dim}$ (~~Fa leaf の Zariski closure.~~)

($X_F \ni x$ の fiber $=$ L_x: leaf $\subset X_F$ xをとおる

$L_x \subset \overline{V}^{Zar} \cap pr_1^{-1}(x)$

$\underset{r\dim の 閉包}{\overline{L_x}^{Zar}} \subset \underset{r\dim}{\overline{V}^{Zar} \cap pr_1^{-1}(x)}$)

$\underline{\text{Pf}}$ $\pi = pr_2|_{\overline{V}^{Zar}} : \underset{n+r}{\overline{V}^{Zar}} \longrightarrow \underset{n}{X}$ は第2射影

$$X_0 = \{ x \in X_F \mid \dim \pi^{-1}(x) = r \}$$

non empty Zariski open

$\pi^{-1}(x)$

$\pi := p_2|_{\overline{V}^{Zar}} \quad \overline{V}^{Zar} \to X$ surj

π の fiber $=$ r dim

$X_0 := \{x \in X \mid \dim(\pi \text{ の fiber}) = r\}$ Zariski open, nonempty $\subset X_F$ //

[x_0 の F を通る leaf は algebraic を示す

$\forall x_0 \in \pi^{-1}(X_0)$, $\forall L_{x_0}$: $p_1^* F$ の x_0 を通る leaf. / $X \times X \xrightarrow{p_1} X$ が algebraic を示せばよい

$x_0 = (z_0, w_0)$

$L_x \cap \pi^{-1}(X_0)$

$V = \{X \times X \ni (z, w) \mid z \in X_F,\ w \in (F \text{ の } z \text{ を通る leaf} \cap X_F)\}$

$L_x \cap \pi^{-1}(X_0) = \{X \times X \ni (z, w_0) \mid z \in X_F \cap (F \text{ の } w_0 \text{ を通る leaf})\}$ rdim

$\subset \pi^{-1}(w_0)$

$\therefore \overline{L_x \cap \pi^{-1}(X_0)}^{Zar}$ rdim

$\forall (z_0, w_0) \in \pi^{-1}(X_0)$,

$L_{(z_0, w_0)}$: を通る $p_1^* F$ の leaf.

$L \cap \pi^{-1}(X_0) = \{(z, w_0) \in X \times X \mid z \in X_F \cap F \text{ の } w_0 \text{ を通る leaf}\}$ rdim

$\subset \pi^{-1}(w_0)$ rdim

$\therefore \overline{L \cap \pi^{-1}(X_0)}^{Zar}$ rdim

$\therefore \overline{(w_0 \text{ を通る } F \text{ の leaf})}^{Zar}$ rdim

$\cup$ open

$(w_0 \text{ を通る } F \text{ の leaf})$

$(\overline{\{w_0\}}) \cap X_F = (w_0 \text{ を通る } F \text{ の leaf})$

不満がある気に…

$$P : X_0 \longrightarrow \bigcup_{\delta>0} C_{\mathrm{hom}, r, \delta}(X \times X)$$

$$x \longmapsto [\pi^{-1}(x)]$$

$$W = \overline{P(X_0)}^{\mathrm{Zar}}$$

$$U = \overline{\{(x,w) \in X' \times W \mid P(x) = w\}}^{\mathrm{Zar}} \subset X \times W$$

$$\begin{tikzcd}
X \arrow[r, dashed, "P"] & W \\
U \arrow[u, "\mathrm{pr}_1"] \arrow[ur, "\mathrm{pr}_2"'] &
\end{tikzcd}$$

$\mathrm{Cor}\,\mathrm{pr}_2$.

Lem $\mathcal{F} \subset T_X$ foliation

Assume $\forall L$ line bundle $\exists C \in \mathbb{N}_{>0}$ (☆)
$\forall k \in \mathbb{N}$ $\forall m \in \mathbb{N}$ $m \geq Ck$,
$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{F}^\vee) = 0$

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation

Pf 以下 $f, g : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ について

$f \lesssim g \overset{\mathrm{def}}{\iff} \exists M \in \mathbb{N}, \forall k \gg 0, f(k) \leq M g(k)$ とする

$\dim \overline{V}^{\mathrm{Zar}} = n + r$ を示す

$\mathcal{L} \to \overline{V}^{\mathrm{Zar}}$ ample とし、

$$h^0(\overline{V}^{\mathrm{Zar}}, \mathcal{L}^{\otimes k}) \lesssim k^{n+r} \text{ を示す}$$

$V \subset \overline{V}^{\mathrm{Zar}}$ より $h^0(V, \mathcal{L}^{\otimes k}) \lesssim k^{n+r}$ を示せばよい

Lem Assume $\forall L$ line bundle $\exists D \in \mathbb{N}_{>0}$ $\forall k, \forall m \geq Dk$,
$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^m \mathcal{F}^\vee) = 0$ (☆)

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation.

Pf $\dim \overline{V}^{\mathrm{Zar}} = n + r$ を示せばよい

$\mathcal{L} = \overline{V}^{\mathrm{Zar}}$ ample divisor とするとき

$h^0(\overline{V}^{\mathrm{Zar}}, \mathcal{L}^{\otimes k}) \approx O(k^{n+r})$ を示せば

$\overline{V}^{\mathrm{Zar}} \supseteq V$ on Zariski dense

与 $H^0(\overline{V}^{\mathrm{Zar}}, \mathcal{L}^{\otimes k}) \to H^0(V, \mathcal{L}^{\otimes k})$ を考えるとこれは inj

よって $h^0(V, \mathcal{L}|_V^{\otimes k}) = O(k^{n+r})$ を示せばよい

ここで $\forall 1 \leq m$ について $0 \to \mathcal{I}_\Delta^m \hookrightarrow \mathcal{I}_\Delta^{m-1} \to \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m \to 0$,

($\Delta \hookrightarrow V$ より $\mathcal{I}_\Delta \subset \mathcal{O}_V$ を ideal sheaf とする)

$\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m = \mathrm{Sym}^m \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ ← 一般論 (Wiki に書いてる)

$= \mathrm{Sym}^m N_{\Delta/V}^\vee$

$= \mathrm{Sym}^m \mathcal{F}^\vee$ に行けばよいと

$\forall \ell \in \mathbb{N}_{\geq 0}$ $\mathcal{I}_\Delta \subset \mathcal{O}_V$ ideal sheaf

を考えるとこれは triv

$h^0(V, \mathcal{L}|_V^{\otimes k}) = O(k^{n+r})$ を示せば

ここで $\forall 1 \leq m$ について $0 \to \mathcal{I}_\Delta^m \hookrightarrow \mathcal{I}_\Delta^{m-1} \to \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m \to 0$,

($\Delta \hookrightarrow V$ より $\mathcal{I}_\Delta \subset \mathcal{O}_V$ を ideal sheaf とする)

$$\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m = \mathrm{Sym}^m \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$$ ← 一般論 (wikiより)

$$= \mathrm{Sym}^m N_{\Delta/V}^\vee$$

$$= \mathrm{Sym}^m \mathcal{F}^\vee$$ により

$$0 \to \mathcal{I}_\Delta^\ell \to \mathcal{I}_\Delta^{\ell-1} \to \mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^\ell \to 0$$

$$\mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^\ell = \mathrm{Sym}^{\ell-1} \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$$

$$= \mathrm{Sym}^{\ell-1} N_{\Delta/V}^\vee$$

$$= \mathrm{Sym}^{\ell-1} \mathcal{F}^\vee.$$

$$h^0(V, \mathcal{L}^{\otimes k}) \leq h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{O}_V/\mathcal{I}_\Delta) + h^0(V \otimes \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta)$$

$\Delta \simeq X_{\mathcal{F}}$ な(2)-花

$$= h^0(X_{\mathcal{F}}, \mathcal{L}|_{X_{\mathcal{F}}}^{\otimes k}) + h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta)$$

$$h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^\ell)$$

$$= h^0(X_{\mathcal{F}}, (\mathcal{L}|_{X_{\mathcal{F}}})^{\otimes k} \otimes \mathrm{Sym}^{\ell-1} \mathcal{F}^\vee),$$

$$h^0(U, \mathcal{L}^{\otimes k}|_U) \leq \sum_{m=1}^{\infty} h^0(U, \mathcal{L}^{\otimes k}|_U \otimes I_\Delta^{m-1}/I_\Delta^{m})$$

$$= \sum_{m=1}^{\infty} h^0(X_{\mathcal{F}}, \mathcal{L}|_{X_{\mathcal{F}}}^{\otimes k} \otimes \mathrm{Sym}^{m}\mathcal{F}^{\vee})$$

($\Delta \subseteq X_{\mathcal{F}}$ の開近傍にとり)

$$L = (\mathcal{L}|_{X_{\mathcal{F}}})^{\vee\vee} \quad = \sum_{m=1}^{\infty} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]}\mathcal{F}^{\vee})$$

$$\boxed{\text{かてい}} \quad = \sum_{m=1}^{Dk} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]}\mathcal{F}^{\vee}) \quad (\star)$$

$$\boxed{\text{Claim} \quad h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]}\mathcal{F}^{\vee}) = O(k^{n+r}}$$

これをみとめると

$$(\star) = \sum_{m=1}^{Dk} O(k^{n+r-1}) = O(k^{n+r}) \quad //$$

$f(k) \lesssim g(k) \iff \exists C$ const があって $\forall k \gg 0,\ f(k) \leq C g(k)$

これを ていぎすれば よい

$$h^0(U, \mathcal{L}^{\otimes k})$$

$$\leq \sum_{\ell=0}^{\infty} h^0(X_{\mathcal{F}}, \mathcal{L}|_{X_{\mathcal{F}}}^{\otimes k} \otimes \mathrm{Sym}^{\ell}\mathcal{F}^{\vee})$$

$$= \sum_{\ell=0}^{\infty} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]}\mathcal{F}^{\vee})$$

(かてい)

$$= \sum_{\ell=0}^{Ck} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]}\mathcal{F}^{\vee}) \qquad (L = (\mathcal{L}|_{X_{\mathcal{F}}})^{\vee\vee})$$

$$\left(\underline{\text{Claim}} \quad h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]}\mathcal{F}^{\vee}) \lesssim O(k^{n+r-1})\right)$$

これより,

$$\lesssim k^{n+r-1} \cdot Ck \lesssim k^{n+r} \quad //$$

(じかんがなければ)

$$\left(\mathcal{F} \text{ locally free のとき} \quad h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{\ell}\mathcal{F}^{\vee}) = h^0(\mathbb{P}(\mathcal{F}), \pi^* L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(\ell)) \lesssim k^{n+r-1} \quad //\right)$$

Claim

~~$\mathcal{E} = \mathcal{F}^{\vee}$~~ とする　一般の torsion free $\mathcal{E}$ について

$$\mathbb{P}(\mathcal{E}) = \operatorname{Proj}\left(\widehat{\bigoplus_{\ell \geq 0}} \operatorname{Sym}^{\ell} \mathcal{E}\right)$$

$$\pi : \mathbb{P}(\mathcal{E}) \longrightarrow X.$$

$\exists\, \nu : P \longrightarrow \mathbb{P}(\mathcal{E}')$ bir.

s.t. $\begin{cases} P \text{ : smooth,} \\ (\nu \circ \pi)^{-1}(X_{\mathcal{E}}) \text{ is divisor.} \end{cases}$

$$\mathcal{O}_P(1) := \nu^* \mathcal{O}_{\mathbb{P}'}(1).$$

[Nakayama] $\exists\, D$ divisor s.t. $\pi \circ \nu^{-1}(D) \subset X_{\mathcal{E}}$

s.t. $\forall \ell > 0$, $(\pi \circ \nu)_* (\mathcal{O}_P(1)^{\otimes \ell} \otimes \mathcal{O}_P(\ell D)) = \operatorname{Sym}^{[\ell]} \mathcal{E}$

$$\xi_{\mathcal{E}} := \mathcal{O}_P(1) \otimes \mathcal{O}_P(D).$$

($\mathcal{E}$ が locally free の時 $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ に一致)

Claim の証明

$\mathcal{E}$: rk n torsion free coherent

$\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\widehat{\bigoplus_{\ell \geq 0}} \operatorname{Sym}^{\ell} \mathcal{E}) \leftarrow$ variety とは限らない.

$\pi : \mathbb{P}(\mathcal{E}) \to X$

$\nu : \mathbb{P}(\mathcal{E})' \to \mathbb{P}(\mathcal{E})$: $\mathbb{P}(\mathcal{E})$ の "X を dominate する component" の normalization.

$\mu : P \longrightarrow \mathbb{P}'(\mathcal{E})$

s.t. $\begin{cases} P \text{ is smooth} \\ \mu \text{ birational} \\ P_{\cdot}(\pi \circ \nu \circ \mu)^{-1}(X_{\mathcal{F}}) \text{ is divisor.} \end{cases}$

$$P \xrightarrow{\mu} \mathbb{P}(\mathcal{E})' \xrightarrow{\nu} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X$$

$\tilde{\pi}$

これをつかって

$$h^0(X, L^{\otimes p_2} \otimes \mathrm{Sym}^{[\ell]} F^\vee)$$

$$= h^0(P, (\pi_C)^* L^{\otimes p_2} \otimes \xi_{F^\vee}^{\otimes \ell})$$

$$\cong k^{n+r-1},$$

Thm 4.10 (+ Druel)

$\mathcal{F}$ = Foliation

① $\mu_\alpha^{\min}(\mathcal{F}) > 0$

$\Rightarrow$ (☆) をみたす

② $\xi_{\mathcal{F}^\vee}$ is NOT psef

$\Leftrightarrow$ (☆) をみたす

時間がないとき

(($\mathcal{F}$ locally free かつ $\mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$ is NOT psef) とする)

Thm 4.10 (+ Druel) $\mathcal{F}$ = Foliation

① $\mu_\alpha^{\min}(\mathcal{F}) > 0 \Rightarrow$ ☆

$\mathcal{F}$ is algebraic foliation

② ξを前の証明でのものとする

(つまり $P \to \mathbb{P}'(\mathcal{F}^\vee) \to \mathbb{P}(\mathcal{F}^\vee) \to X$, π

ξは $\pi_*(\ell\xi) = \mathrm{Sym}^{[\ell]}\mathcal{F}^\vee$ となるもの)

ξ is NOT psef on P

$\Leftrightarrow$ ☆をみたす

とくに $\mathcal{F}$ が vector bundle のとき

$\mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$ is NOT psef

$\Leftrightarrow$ ☆

Pf ①

$$H^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee) = \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k})$$

$C := \left\lceil \dfrac{\mu_\alpha(L)}{\mu_\alpha^{min}(F)} \right\rceil + 1$ とすると

$$m \geq Ck \Rightarrow \mu_\alpha^{min}(\mathrm{Sym}^{[m]} F) - \mu_\alpha^{max}(L^{\otimes k}) = m\,\mu_\alpha^{min}(F) - k\,\mu_\alpha(L) > 0$$

より

② から

Pf: $H^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee) = \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k})$

よって $D = \left\lceil \dfrac{\mu_\alpha(L)}{\mu_\alpha^{min}(F)} \right\rceil + 1$ とすると

$$m \geq Dk \Rightarrow \mu_\alpha^{min}(\mathrm{Sym}^{[m]} F) - \mu_\alpha^{max}(L^{\otimes k}) = m\,\mu_\alpha^{min}(F) - k\,\mu_\alpha(L) > k\,\mu_\alpha(L) - k\,\mu_\alpha(L) = 0$$

③ $\therefore \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k}) = 0$

② L fix する. $\xi_{\mathcal{F}^\vee}$ is NOT psef

$\pi : \mathbb{P}(\mathcal{F}^\vee) \to X$

$\Longleftrightarrow \exists C > 0,\ \forall \ell > C,$

$$\pi^* L \otimes \underbrace{\xi_{\mathcal{F}^\vee}^{\otimes \ell}}_{G_{\mathbb{P}(\mathcal{F}^\vee)}(\ell)} \text{ is NOT psef}$$

($\xi_{\mathcal{F}^\vee} = G_{\mathbb{P}(\mathcal{F}^\vee)}(1)$)

よって $m > Ck$

$$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{F}^\vee) = h^0(P, \pi^* L^k \otimes \underbrace{\xi_{\mathcal{F}^\vee}^m}_{G_{\mathbb{P}(\mathcal{F}^\vee)}(m)}) = 0$$

② $\Leftrightarrow$ $\forall L$ line bundle $\exists D > 0$ $\forall k,$ $\forall m > Dk,$

$$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{E}^\vee) = 0$$

$$\| $$

$$h^0(P, \pi^*(L^{\otimes k}) \otimes \xi^{\otimes m}) \text{ を示す}$$

ξ is NOT psef

$\Longleftrightarrow \forall \tilde{L}$ line bundle on P, $\exists D > 0,\ \forall \ell \geq D$ に対して

$\underline{\tilde{L} \otimes \xi^\ell \text{ is NOT psef}}$

$(h^0(P, \tilde{L} \otimes \xi^\ell) = 0$

$\tilde{L} = \pi^* L^{\otimes ?}$ として

よって $m > Dk$

$\Rightarrow h^0(P, \tilde{L}^{\otimes k} \otimes \xi^{\otimes m}) = 0$

逆に $\xi_{\mathcal{F}^\vee}$ psef のとき

$\pi : \mathbb{P}(\mathcal{F}^\vee)$, $\xi_{\mathcal{F}^\vee} = G_{\mathbb{P}(\mathcal{F}^\vee)}(1)$

$\exists L$ big. $\forall C > 0$, $\pi^* L \otimes \xi_{\mathcal{F}^\vee}^{\otimes C}$ is big

$\exists L$ ample $\forall C > 0$ $\pi^* L \otimes G_{\mathbb{P}(\mathcal{F}^\vee)}(C)$ is big

よって $\exists k,\ h^0(X, L^k \otimes \mathrm{Sym}^{\otimes Ck} \mathcal{F}^\vee) \neq 0$

$$\| $$

$$h^0(\mathbb{P}(\mathcal{F}^\vee), \pi^* L^{\otimes k} \otimes \xi_{\mathcal{F}^\vee}^{Ck})$$

$$\left(\exists k,\ h^0(X, L^k \otimes \mathrm{Sym}^{Ck} \mathcal{F}^\vee) \neq 0 \right.$$

$$\| $$

$$h^0(\mathbb{P}(\mathcal{F}^\vee), \pi^* L^k \otimes G_{\mathbb{P}(\mathcal{F}^\vee)}(Ck)))$$

Campana–Păun Thm 1.4

$\mathcal{F} \subset T_X$ saturated

$\mu_\alpha^{min}(\mathcal{F}) > 0$ & $\mu_\alpha^{min}(\mathcal{F}) > \frac{\mu_\alpha^{max}(T_X/\mathcal{F})}{2}$

$\Rightarrow$ $\mathcal{F}$ algebraic foliation

$\Leftarrow$:= $\mu_\alpha^{max}(T_X) > 0$

$\Rightarrow$ α-maximal destabilizing sheaf

is algebraic foliation.

まずに Corollary を示しておく

Cor 4.22 X uniruled $\iff$ $\exists \alpha \in \mathrm{Mov}(X)$ $\mu_\alpha^{max}(T_X) > 0$

pf: X uniruled $\iff$ K_X is NOT psef

$\iff \exists \alpha \in \mathrm{Mov}(X)$. $K_X \cdot \alpha < 0$ である.

$(\Rightarrow)$ $\mu_\alpha^{max}(T_X) \geq \mu_\alpha(T_X) = -\dfrac{K_X \cdot \alpha}{n} > 0$.

$(\Leftarrow)$ $\mathcal{F}$ を α-maximal destabilizing sheaf とする

$\mu_\alpha^{min}(\mathcal{F}) \overset{(5)}{=} \mu_\alpha(\mathcal{F}) > 0$

$(\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X)$ より$)$

$\Rightarrow$ $\mathcal{F}$ has RC leaves.

$\Rightarrow$ X の general point を通る leaf は RC

$\Rightarrow$ Uniruled

($\forall x$ general point に対して, それを通る rational curve がある)

Cor 4.22.

X is uniruled $\iff$ $\exists \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{max}(T_X) > 0$.

(K_X is NOT psef)

pf

$(\Rightarrow)$ $\exists \alpha \in \mathrm{Mov}(X)$. $K_X \cdot \alpha < 0$

$\Rightarrow$ $\mu_\alpha^{max}(T_X) \geq \mu_\alpha(T_X) > 0$.

$(\Leftarrow)$ $\mathcal{F}$ = α-maximal destabilizing sheaf とする.

$$\mu_\alpha^{min}(\mathcal{F}) = \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X) > 0$$

$\Rightarrow$ $\mathcal{F}$ is alg foliation with RC leaves

$\Rightarrow$ $\forall x$ general point, $\exists L$ leaf

$x \in L$ & L is RC (つまり x を通る rational curve がある)

$\Rightarrow$ Uniruled.

Thm (Campana–Păun)

K_X psef

$\Rightarrow \forall m \in \mathbb{N}_{>0}$,

$\forall Q$ torsion free s.t. $(\Omega_X^1)^{\otimes m} \to Q \to 0$,

$\det Q$ is psef.

pf K_X psef $\Rightarrow \forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{min}(T_X) \le 0$

m, Q 를 fix 하자

$\forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha(Q) \ge \mu_\alpha^{min}((\Omega_X^1)^{\otimes m})$

$= m\,\mu_\alpha^{min}(\Omega_X^1) \ge 0$

$\Rightarrow \forall \alpha \in \mathrm{Mov}(X)$, $(\det Q)\cdot\alpha \ge 0$

$\Rightarrow \det Q$ psef //

Thm (Campana–Păun)

X smproj, K_X psef.

$\Rightarrow \forall m \in \mathbb{N}$, $(\Omega_X^1)^{\otimes m} \to \forall Q \to 0$

Q: torsion free.

$\det Q$ is psef

pf K_X psef $\Leftrightarrow X$ is Not uniruled

$\Leftrightarrow \forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{max}(T_X) \le 0$

$\Leftrightarrow \forall \alpha \in \mathrm{Mov}(X)$ $\mu_\alpha^{min}(\Omega_X) \ge 0$

[illegible] Q, $(\Omega_X^1)^{\otimes m} \to \forall Q \to 0$ [illegible]

$\forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha(Q) \ge 0$

$\Rightarrow \det Q$ is psef //

いくつか仮定 Prelim

$F \subset T_X$ foliation について $K_F := \left(\bigwedge^{rk} F\right)^{\vee\vee}$ とする。

とくべつな F の場合では K_F が計算できる

<u>Lem 4.15</u> $f: X \to Y$, surj

$F := \ker df$ とする。 $(F \subset T_X/C$ f a Mori flat locus$)$

$\exists$ <u>E f-exceptional divisor.</u>

s.t. $K_F \underset{\mathbb{Q}}{\sim} K_{X/Y} - \mathrm{Ram}(f) + E$

ここで $\mathrm{Ram}(f) := \sum (\mathrm{ord}_Q f^*(f(Q)) - 1) Q$

Q: prime div. on X

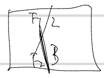
s.t. $f(Q)$ is prm on Y

Claim $\exists E$ g-exc divisor.

s.t. $K_g \sim K_{Z/Y} - \mathrm{Ram}(g) + E$

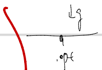
$\mathrm{Ram}(g) = \sum (\mathrm{ord}_W g^*(g(W)) - 1)[W]$

$\left(\begin{array}{l} W \subset Z \text{ prime div} \\ g(W) \text{ prime div} \end{array}\right)$



F_1 2, F_2 3

$\mathrm{Ram}(g) = F_1 + 3F_2$



K_g

pt Mosmth

(codim $(Y - Y_0) \geq 2$)

Pf $g: g^{-1}(Y_0) \to Y_0$ is flat とする。

$g^{-1}(Y_0)$ 上で $K_g \sim K_{Z/Y} - \mathrm{Ram}(g)$ を示せばよい

codim 2 以上で外せるので

$\pi_* \mathrm{Ram}(g) = \bigcup_{i=1}^{l} P_i$

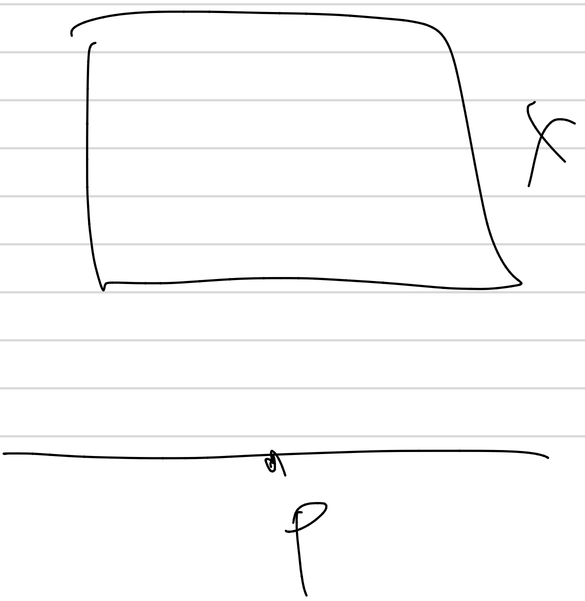
prime divisor, disjoint union としてよい

かんたんのため $l = 1$ とする。

$f^* P_i = \sum w_j Q_j$ $w_j \in \mathbb{N}_{>0}$ とおく。

$\mathrm{Ram}(f)$ について。 Y = curve. $f^* P = \sum w_j Q_j$ とおくと

X-pt smth



X

Y

P

<u>$\mathrm{Ram}(f) = \sum (w_j - 1) Q_j$</u>

以下 $f = f_U : X_0 \to Y_0$ s.t. f flat としてよい

X_0 上 $K_{\mathcal{F}} \cong K_{X/Y} - \mathrm{Ram}(f)$ を示す

$df : T_X \to f^*T_Y$ より $Q := df(T_X)$ とする

$0 \to \mathcal{F} \to T_X \to Q \to 0$ より $K_{\mathcal{F}} = K_X + \det Q$ より

$\det Q = -(f^*K_Y + \mathrm{Ram}(f))$ を示せばよい。

codim 2 は無視できるので

$f(\mathrm{Supp}(\mathrm{Ram}(f))) = \sum_{j=1}^{l} P_j$ P_j divisor としてよい

簡単のため $l = 1$ とする

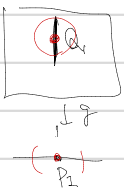
$f^*P_1 = \sum_{j=1}^{k} w_j Q_j$ とする.

$dg : T_Z \to g^*T_Y$ より $dg(T_Z) = Q$ とおく

$0 \to \mathcal{F} \to T_X \to Q \to 0$ である

$(\det \mathcal{F}^\vee) = K_Z \otimes \det Q$ より

$\det Q = -f^*(K_Y) + \sum (w_j - 1) Q_j$ を示せばよい.



$\forall z \in Q_1$ general.

$y = g(z)$ とおくと

$(z \in U, z_1, \dots, z_l)$ coordinate of Σ in Z

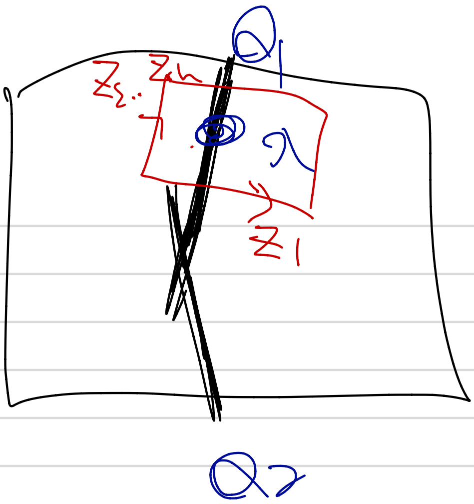
$(V, y_1, \dots, y_k)$: of y in Y

s.t. $(z_1 = 0) = Q_1 \cap U$

$(y_1 = 0) = P_1 \cap V$

$g : U \to V$

$(z_1, \dots, z_k) \to (z_1^{w_1}, z_2, \dots, z_k)$ とできる.



$x \in Q_1 - \bigcup_{i=2}^{k} Q_i$ generalとし $y = f(x)$ とする。

$\exists x \in (U, z_1, \dots, z_n)$ 原点がxに移る
近傍　　　　座標

$\exists y \in (V, w_1, \dots, w_{n-r})$; y :

s.t. $U \cap Q_1 = (z_1 = 0)$

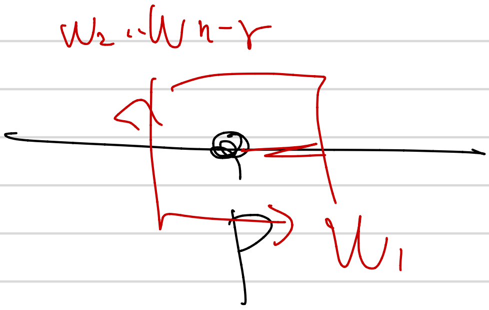
$V \cap P = (w_1 = 0)$

$f : U \longrightarrow V$

$(z_1, \dots, z_n) \longmapsto (z_1^{w_1}, z_2, \dots, z_{n-r})$

$\qquad\qquad\qquad\quad\ \ \| \qquad \| \qquad\quad \|$

$\qquad\qquad\qquad\quad\ w_1 \quad w_2 \ \cdots \ w_{n-r}$



$T_X|_U$ は $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ で生成されているより、

$Q|_U \ni \left(df\left(\frac{\partial}{\partial z_1}\right), \dots, df\left(\frac{\partial}{\partial z_n}\right) \right)$

$= \left(z_1^{w_1 - 1} f^*\left(\frac{\partial}{\partial w_1}\right), f^*\frac{\partial}{\partial w_2}, \dots, f^*\left(\frac{\partial}{\partial w_{n-r}}\right) \right)$ で生成。

$\therefore \det Q|_U$ は $z_1^{w_1 - 1} f^*\left(\frac{\partial}{\partial w_1} \wedge \dots \wedge \frac{\partial}{\partial w_{n-r}}\right)$ で生成される

よって $\det Q|_U \sim (w_1 - 1) Q_1 - f^* K_Y$.

$Q_2, \dots, Q_k$ もたしあわせて //

$\boxed{\text{Pre 2}}$ BP

$m \in \mathbb{N}_{>0}$ fix $f : X \to Y$ with connected fibre surj

$L \to X$ line bundle

h_L s.h.m on X, s.t. $\sqrt{-1}\Theta_{h_L} \geq 0$ &

Assume $\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$ y general.

$f_*(mK_{X/Y}+L) \neq 0$

$\Rightarrow \exists\, h_m$ m-Bergman metric on $mK_{X/Y}+L$

s.t.
$$\begin{cases} \sqrt{-1}\Theta_{h_m} \geq m\,\mathrm{Ram}(f) \geq 0 \quad (\Leftarrow mK_{X/Y}+L \text{ psef}) \\ \forall y \text{ general}, \ \forall x \in X_y, \ \forall u \in H^0(X, mK_{X/Y}+L), \\ \underbrace{|u|_h^{2/m}(x)}_{C^0 \text{ bounded}} \leq \int_{X_y} |u|_{h_L}^{\frac{2}{m}} < +\infty \end{cases}$$

Thm 3 [Campana]

$f : X \to Y$ surj with connected fibre

F : general fiber, Assume K_F is psef

$\Rightarrow$ $K_{X/Y} - \mathrm{Ram}(f)$ psef.

pf Fix A very ample on X

$\forall m \in \mathbb{N}_{>0}$, $m(K_{X/Y} - \mathrm{Ram}(f)) + A$ psef & π-ef

Fix m. $\exists n$, $h^0(F, n(mK_F + A|_F)) \neq 0$

$\rightsquigarrow$ $\exists h_{mn}$ Bergman metric on $n(mK_{X/Y} + A)$

s.t. $\sqrt{-1}\Theta_{h_{mn}} \geq mn\,\mathrm{Ram}(f)$.

$\rightsquigarrow$ $n(mK_{X/Y} + A) - mn\,\mathrm{Ram}(f)$ psef.

//

$\rightsquigarrow$

Pr.3 Raynaud Flattening

Lem 4.16 $f: X \to Y$ mphis

$\exists\ \tau: Y' \to Y$ biratl.

$\exists\ X'$: $X \times_Y Y'$ a desingularization,

$$\begin{tikzcd} X' \arrow[r, "\exists f'"] \arrow[d, "\exists \tau'"'] \arrow[dr, phantom, "\circlearrowleft"] & Y' \arrow[d, "\tau"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

s.t. f' exc

$\Rightarrow \tau'$ exc.

④ Relative MRC

Thm [Kollár] (Chapter 5?)

$f : X \to Y$ morphism

$\exists\ \pi : X \dashrightarrow Z$ dominant rational map

$\exists\ g : Z \to Y$ morphism

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr] & Z \arrow[d] \\ & Y \end{tikzcd}$$

s.t. $\forall y$: general point,

π_y $X_y \dashrightarrow Z_y$ is MRC.

$\mathrm{ch}\ k = 0$, K_{Z_y} is psef,

$\boxed{\dim Z_y > 0 \text{ for } \forall y \text{ (} \forall y \text{ general, } X_y \text{ is NOT RC)}}$

pf of Thm 4.21

$\mathcal{F} \subset T_X$ foliation, $\mu_\alpha^{min}(\mathcal{F}) > 0$ とする

$\boxed{\text{leaf が RC であることを示す.}}$

$\mathcal{F}$: algebraic foliation なので

$\exists$ and $\ker df = \varphi^* \mathcal{F}$ on $\varphi^{-1}(X_{\mathcal{F}})$.

$$\begin{tikzcd}
\exists \tilde{X} \arrow[rr, dashed, "\pi"] \arrow[d, "\varphi"] \arrow[drr, "f"] & & Z \arrow[d, "g"] \\
X & & \exists Y
\end{tikzcd}$$

f は以下に注意すれば f の relative MRC を考えると

$\dim Z_y > 0$ である
$y \in Y$

resolution をとり,
Prop 4 と Stein Factorization を
組み合わせて示す

$$
\begin{tikzcd}
\exists\, \tilde{X} \arrow[r, "\pi"] \arrow[d, "\varphi"'] \arrow[dr, "f"] & \exists\, Z \arrow[d, "g"] \\
X & \exists\, Y
\end{tikzcd}
$$

s.t.

① π ← morphism

② f exc $\Rightarrow$ φ exc

③ π-exc $\Rightarrow$ φ-exc

④ φ with connected fibre, $\dim Z_y \neq 0$, K_{Z_y} is psef ($y \in Y$ general)

⑤ $\ker df = \varphi^* \mathcal{F}$ on $\tilde{X}_0 \subset \tilde{X}$ open & $\operatorname{codim} \tilde{X} \setminus \tilde{X}_0 \geq 2$

$$\tilde{\mathcal{F}} = \ker df \subset T_{\tilde{X}}$$

$$\mu^{\min}_{\alpha}(\mathcal{F}) = \mu^{\min}_{\varphi^*\alpha}(\tilde{\mathcal{F}}) > 0$$

$\therefore$ $$\mu_{\varphi^*\alpha}(d\pi(\tilde{\mathcal{F}})) > 0 \,/\!/$$

$$\mathcal{G} = \ker dg \subset T_Z$$

$\xrightarrow{\pi^* T_Z}$ $d\pi(\tilde{\mathcal{F}}) \subset \pi^*(\ker dg)$ $\exists$

($g \circ \pi = f$) $\qquad$ π flat ⟸ ...

$\longrightarrow$ ③ $$d\pi(\tilde{\mathcal{F}}) = \pi^*(\ker dg)$$ on $\exists X'$ s.t. $\tilde{X} \setminus X'$ is φ-exc

$\therefore\ N\varphi^*\alpha\,(\pi^*(\cancel{\mathrm{Ker}(g)}))>0$

$G(f)$

$\rightsquigarrow\quad \underline{\pi^*(K_g)\cdot\varphi^*\alpha<0}$

$\xrightarrow{\text{Pref}}\quad K_g\sim K_{Z/Y}-\mathrm{Ram}(g)+E$

$\exists E$ g-exc.

Pre2.1(f)より $K_{Z/Y}-\mathrm{Ram}(g)$ psef

③より $\pi^*K_g\cdot\varphi^*\alpha$

$=(K_{Z/Y}-\mathrm{Ram}(g))\varphi^*\alpha>0$

で矛盾する。