Bauer.– Pignatelli 18の解説

わかったとこだけ!

①

M cpt cpx mfd.

Def M : infinitisimaly rigid $\Leftrightarrow$ $H^1(M, \Theta_M) = 0$

(Θ_M : holomorphic tangent bundle)

M rigid

$\Leftrightarrow$ $^\forall \pi : \mathcal{M} \to B_\varepsilon \subset \mathbb{C}^d$ deformation of M $\left\{ \begin{array}{l} \pi \text{ submersion} \\ 0 \in B_\varepsilon \subset \mathbb{C}^d \ \varepsilon\text{unit-ball } \varepsilon > 0 \\ \pi^{-1}(0) = M \end{array} \right.$

$0 \overset{\exists}{\in} U \underset{\text{open}}{\subset} B$ s.t. $\pi^{-1}(0) \underset{\text{biholo.}}{\simeq} M \times U$

Thm (Kodaira-Spencer. (Nirenberg) $\left[\begin{array}{l} \text{MK. P.45} \\ \text{Thm 3.2.} \end{array} \right]$)

infinitisimaly rigid $\Rightarrow$ rigid

M: cpt cpx mfd.

Def M: infinitisimaly rigid (Morrow-Kodaira P.45)

$\Leftrightarrow$ $H^1(M, \Theta_M) = 0$

(Θ_M holomorphic tangent bundle)

M rigid (へんけいがおきない)

$\Leftrightarrow$ $^\forall \pi : \mathcal{M} \to B$ deformation of M

(π submersion, $B \subset \mathbb{C}^m$ unit ball, $\pi^{-1}(0) = M$)

$0 \overset{\exists}{\in} U \underset{\text{open}}{\subset} B$ s.t. $\pi^{-1}(U) \simeq U \times M$.

Thm (Kodaira-Spencer (Nirenberg?))

(Morrow Kodaira P.45)

Infinitisimaly rigid $\Rightarrow$ rigid

Problem [MK P. 45]

Find an example of an M which is rigid, but $H^1(M, \Theta_M) \neq 0$ (Not easy?)

Thm [BP18]

$^\forall n \geq 8$ s.t. $3 \nmid n$ & $2 | n$.

$\exists S_n$ smooth surface

s.t. rigid & $H^1(S_n, \Theta_{S_n}) \neq 0$

Question M.K P.45

Find an example of M which is rigid

but $H^1(M, \Theta_M) \neq 0$ (Not easy?)

Thm (Bauer-Pignatelli 18)

$^\forall n \geq 8$ s.t. $3 \nmid n$, and $2 | n$

$\exists S_n$: minimal smooth surface (of general type)

s.t. $H^1(S_n, \Theta_{S_n}) \neq 0$ & rigid

$\boxed{\text{方針}}$ ① $G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$ とおく　いわれたらわかる　うつしの

② $C \longrightarrow \mathbb{P}^1$ で $\{0, 1, \infty\}$ で分岐し、
$C/G \simeq \mathbb{P}^1$ なる C をとる。

③ $C_1 = C_2 = C$ として $X_n = C_1 \times C_2 / G$ とおく
C_2 には G の作用をちょっとかえておる

④ $S_n \longrightarrow X_n$ resol of sing とする

ラフ版　${}^{\exists}C$ curve　$C/G \simeq \mathbb{P}^1$　2.5

$G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$　　$C \longrightarrow \mathbb{P}^1$ $\{0, 1, \infty\}$ で分岐

$\rightsquigarrow$　$X_n = C \times C / G$.　　$C_1 = C_2 = C$
$C_2 \curvearrowleft G$ をちょっとかえる

$\longrightarrow$　$S_n \longrightarrow X_n$ resol of sing
$\uparrow = F_n$

ふくそこうぞう

M cpt cpx mfd.

$M = \bigcup_{j=1}^{N} U_j \qquad (U_j, z_j^1, \ldots, z_j^n)$ ざひょう

M' Mに べつの ふくそ構造をいれた複素多様体

$\rightsquigarrow$ $T^{0,1}_{M'}$ (0,1) anti holomorphic は U_j 上 $\dfrac{\partial}{\partial \bar{z}_j^\alpha} + \sum_\beta \varphi^\beta_\alpha(z) \dfrac{\partial}{\partial z_j^\beta}$ $\alpha = 1, \ldots, n$ ではられる

$\left(T_{M'} \otimes \mathbb{C} = T^{1,0}_{M'} \oplus T^{0,1}_{M'} \right)$

$T_{M'}$ $\mathbb{C}^{2n} \dim$ $\mathbb{C}^n \dim$ $\mathbb{C}^n \dim$

($\mathbb{R}$-mfd としてのせっせん)

$\left(\dfrac{\partial}{\partial w} \quad \dfrac{\partial}{\partial \bar{w}} \text{ たち} \right)$

$\rightarrow$ $\varphi := \sum \varphi^\beta_\alpha \dfrac{\partial}{\partial z_j^\beta} d\bar{z}^\alpha$ とする.

C^∞ Θ_M valued (0,1) form

④

さらにつぎをみたす　$\bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0$

（zによる微分：$\bar{\partial}$）

逆に　φ C^∞ (T)$_M$値 (0,1)form で $\uparrow$ をみたすと

Mに新しく複素構造が入る

これを M_φ とかくことにする

複素構造全体が入る器がほしい　⇝ Kuranishi family

X

KS equation を示す(中の式?)

$$\frac{\partial}{\partial \overline{w}_j^\alpha} = \frac{\partial}{\partial \overline{z}_j^\alpha} - \sum_{\beta=1}^{n} \varphi_{\bar\alpha}^{\beta} \frac{\partial}{\partial z_j^\beta}$$

$$\bar\partial\left(\varphi_\alpha^\beta \frac{\partial}{\partial z_j^\beta} d\bar z^\alpha\right) = \partial_{\bar r}\varphi_\alpha^\beta \frac{\partial}{\partial z_j^\beta} d\bar z^r \wedge d\bar z^\alpha$$

$$\boxed{\varphi_\alpha^\beta d\bar z^\alpha \frac{\partial w^r}{\partial z_\beta} - \frac{\partial \overline{w}^r}{\partial \bar z_\alpha} d\bar z^\alpha = 0}$$

$$0 = \frac{\partial w^r}{\partial \overline{w}^\alpha} d\bar z^\alpha = \frac{\partial w^r}{\partial \bar z^\alpha} d\bar z^\alpha - \varphi_\alpha^\beta d\bar z^\alpha \frac{\partial w^r}{\partial z^\beta}$$

$$0 = \bar\partial\left(\varphi_\alpha^\beta \frac{\partial w^r}{\partial z^\beta} d\bar z^\alpha\right)$$

$$= \frac{\partial \varphi_\alpha^\beta}{\partial \bar z^\delta} \frac{\partial w^r}{\partial z^\beta} d\bar z^\delta \wedge d\bar z^\alpha - \varphi_\alpha^\beta d\bar z^\alpha \wedge \frac{\partial w^r}{\partial \bar z^\delta \partial z^\beta} d\bar z^\delta$$

$$\frac{\partial \varphi_\alpha^\beta}{\partial \bar z^\delta} \frac{\partial w^r}{\partial z^\beta} d\bar z^r \wedge d\bar z^\alpha = \varphi_\alpha^\beta d\bar z^\alpha \wedge \frac{\partial w^r}{\partial \bar z^\delta \partial z^\beta} d\bar z^\delta$$

$$= \sum \varphi_\alpha^\beta d\bar z^\alpha \wedge \frac{\partial}{\partial z^\beta}\left(\varphi_\delta^\alpha d\bar z^\delta \frac{\partial w^r}{\partial z^\alpha}\right)$$

$$= \varphi_\alpha^\beta d\bar z^\alpha \wedge \frac{\partial \varphi_\delta^\alpha}{\partial z^\beta} dz^\beta \wedge d\bar z^\delta \frac{\partial w^r}{\partial z^\alpha}$$

$$+ \varphi_\alpha^\beta d\bar z^\alpha \wedge \varphi_\delta^\alpha d\bar z^\delta \wedge \frac{\partial^2 w^r}{\partial z^\beta \partial z^\alpha} dz^\beta$$

$$\sum_i \frac{\partial \varphi_\alpha^\beta d\bar z^\alpha}{\partial \bar z^\alpha}$$

§1 Preliminary [MK chapter 4.1〜4.3]

M cpt cpx mfd. g : Hermitian metric (Sobolev $\|\ \|_k$)

$A^p := \{ C^\infty\ \Theta_M$ valued $(0,p)$ fom$\}$ ← 内積が定まる

$\vartheta\ (\bar\partial^*) :=$ adjoint of $\bar\partial$ $((\varphi, \bar\partial\psi) = (\vartheta\varphi, \psi)$ なるもの

$\square := \vartheta\bar\partial + \bar\partial\vartheta$ Laplacian.

Thm Hodge-DeRham-Kodaira.

$$A^p = \mathbb{H}^p \oplus \square A^p$$

$$\mathbb{H}^p = \{\varphi \in A^p \mid \square\varphi = 0\} \simeq H^p(M, \Theta_M)$$

§1 Kuranishi Theory (Morrow-Kodaira Chapter 4.1〜4.3)

M cpt cpx mfd. , g : Hermitian metric.

$A^p := \{\Theta_M$値 $(0,q)$ fom$\}$. ← 内積が定まる.

(てきせつに Sobolev ノルム $\|\ \|_k$ をいれる ($k \geq 2n+2$))

$\vartheta\ (\bar\partial^*) := \bar\partial$ の adjoint

$((\varphi, \bar\partial\psi) = (\vartheta\varphi, \psi)$ となるやつ$)$

$\square = \vartheta\bar\partial + \bar\partial\vartheta$ Laplacial

Thm Hodge-DeRham-Kodaira.

$$\mathcal{L}^p = \mathbb{H}^q \oplus \square\mathcal{L}^q.$$

$$\mathbb{H}^q := \{\varphi \in \mathcal{L}^q \mid \square\varphi = 0\} \simeq H^q(M, \Theta_M)$$

(finite dim)

$\hat{H}$ (hamnic gentr) $: \mathcal{L}^q \to \mathbb{H}^q$ projecm

} かんたんにするため

$\forall \varphi \in \mathcal{L}^q\ \exists! \psi \in \mathcal{L}^q$ s.t

$$\varphi = H\varphi + \square\psi = H\varphi + \square G\varphi.$$

G : Green operator $\mathcal{L}^q \longrightarrow \mathcal{L}^q$

$$\varphi \longmapsto G\varphi := \psi.$$

ここで

・H (harmonic operator) : $A^p \longrightarrow \mathbb{H}^p$ projection

$\forall \varphi \in A^p \ \exists! \psi \in A^p$ s.t $\varphi = H\varphi + \Box\psi = H\varphi + \Box G_r\varphi$

ここで G_r (Green operator) $A^p \longrightarrow A^p$

$\varphi \longmapsto \psi$

§2 Kuranishi Theory (Morrow-Kodaira Chapter 4-1~4-3)

M cpt cpx mfd. , g : Hermitian metric.

$A^p := \{ \Theta_M$値 $(0,q)$ form $\}$ ← 内積が定まる。

(さまざまに Sobolev $\|\cdot\|_k$ norm を入れる ($k \geq 2n+2$))

$\vartheta \ (\bar\partial^*) :=$ $\bar\partial$ の adjoint

($(\varphi, \bar\partial\psi) = (\vartheta\varphi, \psi)$ となるもの)

$\Box = \vartheta\bar\partial + \bar\partial\vartheta$ Laplacian

Thm Hodge-DeRham-Kodaira.

$\mathcal{L}^p = \mathbb{H}^q \oplus \Box\mathcal{L}^q$.

$\mathbb{H}^q := \{ \varphi \in \mathcal{L}^q \mid \Box\varphi = 0 \} \cong H^q(M, \Theta_M)$

(finite dim)

$\hat{H}$ (harmonic operator) : $\mathcal{L}^q \longrightarrow \mathbb{H}^q$ projection

} かんたんにするため

$\forall \varphi \in \mathcal{L}^q \ \exists! \psi \in \mathcal{L}^q$ s.t

$\varphi = H\varphi + \Box\psi = H\varphi + \Box G\varphi$.

G : Green operator $\mathcal{L}^q \longrightarrow \mathcal{L}^q$

$\varphi \longmapsto G\varphi := \psi$.

H^1 のONB $\eta_1, \dots, \eta_d$ とする。 A^P に値をもった $t_1, \dots, t_d$ 変数として $\varphi_M(t)$ を下をみたす $t_1, \dots, t_d$ のべキ級数とする

$$\varphi_M(t) = \sum_{i=1}^{d} \eta_i t_i + \frac{1}{2} \partial G_r[\varphi_M(t), \varphi_M(t)], \quad \varphi_M(0) = 0$$

[MK, Prop 2.14, 3.2]

・$\varphi_M(t)$ は唯一存在する。

（小さいすで t について holomorphic）

（$t \in \mathbb{C}^d$, $|t|$ が小さい ⇒ $\varphi = \sum \eta_i t_i + \frac{1}{2} \partial G[\varphi, \varphi]$ の解は $\varphi(t)$ にかぎる）

（いれたら $\varphi_1 = \sum \eta_i t_i$

$\varphi_n(t) = \frac{1}{2} \partial G_r[\varphi_x, \varphi_{n-x}]$ n次解

$\varphi_M(t) := \sum_{n=1}^{\infty} \varphi_n$ とすればよい（収束する））

H^1 の basis $\gamma_1, \dots, \gamma_e$ とおく。

（H^2 の basis $\beta_1, \dots, \beta_d$）

$t_1, \dots, t_e$ へんすうで $\mathbb{C}$ 値 (0,1) fn

$t_1, \dots, t_e$ をへんすうとして $\varphi(t)$ をつぎをみたす解とする べき級数 （せいかんすいにて）

$$\varphi(t) = \sum_{i=1}^{e} \gamma_i t_i + \frac{1}{2} \partial G[\varphi(t), \varphi(t)]$$

これは $|t|$ が小さければ存在（$\forall t$ について $\|\varphi(t)\|_k < \varepsilon_k$ ならば $t \in B_\varepsilon \subset \mathbb{C}^e$ ε半径 open ball.

（$\|\varphi\|_k$ が小さければ unique

（$\varphi(t)$ が $\sum \gamma_i t_i$ の unique sol. ））

（つくりかた

$$\varphi_n(t) = \frac{1}{2} \partial G\left(\sum_{x=1}^{n-1} [\varphi_x, \varphi_{n-x}]\right)$$

そして $\varphi(t) = \sum_{n=1}^{\infty} \varphi_n$ とする）

Prop 2.5 $\bar{\partial} \varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$

$\Longleftrightarrow H[\varphi(t), \varphi(t)] = 0$

$H[\varphi(t), \varphi(t)] = 0$ だったら

$\varphi(t)$ は ふくそう構造を与える

(8)

Thm [MK Prop 2.5 Thm 3.1]

(1) $\bar\partial\varphi_M - \frac{1}{2}[\varphi_M, \varphi_M] = 0 \iff H[\varphi_M(t), \varphi_M(t)] = 0.$

(2) $S := \{t \in B_\varepsilon \mid H[\varphi_M(t), \varphi_M(t)] = 0\}$

($0 \in B_\varepsilon \subset \mathbb{C}^d$ 　　B_ε は0のεきんぼう) とするとき

- S: analytic set. ($\operatorname{codim} S \geq d - \dim H^2(M, \Theta_M)$ ($M_{\varphi(t)}$))
- $\forall t \in S$, $\varphi(t)$は複素多様体の構造を定める.

(3) $\forall\chi$: C^∞-Θ_M value $(0,1)$ fm

$\bar\partial\chi - \frac{1}{2}[\chi, \chi] = 0$ について.

$\exists F: M \longrightarrow M$ diffeo $\exists t \in S$.

s.t. $\chi \circ F = \varphi_M(t)$ &

$F: M_\chi \longrightarrow M_{\varphi_M(t)}$ biholomorphic

つまり $t \in B_\varepsilon$ & $H[\varphi(t), \varphi(t)] = 0$

$\Longrightarrow$ $M_{\varphi(t)}$は ふくそ多様体

$S := \{t \in B_\varepsilon \mid \underline{H[\varphi(t), \varphi(t)] = 0}\}$

analytic set

(∵ H^2のbasis $\beta_1, \ldots, \beta_d$ とおくと $H[\varphi(t), \varphi(t)]$

$= \sum_{i=1}^{d} ([\varphi, \varphi], \beta_i)\, \beta_i$ に

$\Leftrightarrow$ $\underline{([\varphi(t), \varphi(t)], \beta_i) = 0}$ tにかんして正則

Thm Kuranishi

(1) $S = \{t \in B_\varepsilon \mid H[\varphi(t), \varphi(t)] = 0\}$ なる

にみける αt について.

$M_{\varphi(t)}$ は M のふくそこうぞうを定める

(2). $\forall\chi$: Θ_M値 $(0,1)$ fm で

$\bar\partial\chi - \frac{1}{2}[\chi, \chi] = 0$ をみたすものについて

$\exists F: M \longrightarrow M$ diffeo, $\exists t \in S$.

s.t. $\chi \circ F = \varphi(t)$ かつ

$F: M_\chi \xrightarrow{\sim} M_{\varphi(t)}$ biholomorphic

⟨Def⟩ (Cat 88) $\mathrm{Def}(M) := \{ t \in B_\varepsilon \mid H[\varphi_M(t), \varphi_M(t)] = 0 \}$ Kuranishi family. 9

Prop [BC16, Thm 2.3] $\mathrm{Def}(M)$ 1 point $\Rightarrow$ M rigid.
~~(0 dim)~~

pf $\pi : \mathcal{M} \to B$ deformation

$\rightsquigarrow \exists \varphi(t)$ C^∞ Θ_M-valued $(0,1)$ fun.

$\pi^{-1}(t) \simeq M_{\varphi(t)}$

$\Rightarrow \forall t \in B$ $\exists F$ diffeo, $\exists t_0 \in \mathrm{Def}(M)$ s.t.

$F : M_{\varphi(t)} \xrightarrow{\simeq} M_{\varphi(t_0)} \underset{(なぜ)}{=} M$.
bihalo

$\Rightarrow \forall t \in B,\ \pi^{-1}(t) \simeq M \overset{[FG]}{\Rightarrow} M$ rigid
(π: analytic fiber bundle)

Cor [MK Thm 3.2, P.45]
infinitesimal rigid $\Rightarrow$ $\mathrm{Def}(M)$ 0dim $\Rightarrow$ M rigid
($d=0$) 大きな差

Def $\mathrm{Def}(M) := \{ t \in B_\varepsilon \mid H[\varphi(t), \varphi(t)] = 0 \}$
Kuranishi family

(Rem M が smoothでないばあいは $\rightarrow$ Categorical にていぎすれば)

Prop $\mathrm{Def}(M)$ 1point $\Rightarrow$ M rigid.

pf $\pi : \mathcal{M} \to B$ をとれば

$\Rightarrow \exists \varphi(t)$. C^∞ T_M値 $(0,1)$ fun s.t.

$\pi^{-1}(t) \simeq M_{\varphi(t)}$ bihalo)

(($\pi^{-1}(t)$ は M に $\varphi(t)$ で ふくそこうぞうを入れたもの)

$\underset{(\mathrm{thm})}{\Rightarrow}$ $\exists F$ diffeo $\exists t_0 \in \mathrm{Def}(M)$ s.t.

$F : M_{\varphi(t)} \to M_{t_0}$ bihalo

$M_{t_0} = M$ ← Def(M) 1点

$\Rightarrow \forall t \in B$ $\pi^{-1}(t) \underset{\text{bihalo}}{\simeq} M \Rightarrow M$ rigid
(π は analytic fiber bundle)
(Fischer–Grauert)

Cor infinitesimal rigid $\Rightarrow$ $\mathrm{Def}(M)$ 1point
($d=0$) $\Rightarrow M$ rigid

$\boxed{\text{大きな差がある}}$

性質

([BP] Lem 3.2)

① $H^1(M\times N, \Theta_{M\times N}) = H^1(M, \Theta_M) \oplus H^1(N, \Theta_N)$

$\Rightarrow \mathrm{Def}(M\times N) = \mathrm{Def}(M) \times \mathrm{Def}(N)$

$\left(\varphi_{M\times N}(t) = \varphi_M(t) + \varphi_N(t) \text{ より} \right)$

② $G \curvearrowright M$　　G: finite, faithfully action.

$$\Rightarrow \underbrace{\mathrm{Def}(M)^G}_{\{t\in B_\varepsilon \mid [\varphi(t), \varphi(t)]=0,\ g_*\varphi(t)=\varphi(t)\}} = \mathrm{Def}(M) \cap \underbrace{H^1(M, \Theta_M)^G}_{\{\eta\in H^1,\ g_*\eta=\eta\}}$$

$$\left(\begin{array}{ccc} \mathrm{Def}(M) \subset B_\varepsilon & \longrightarrow & H^1(M, \Theta_M) \\ t & \longmapsto & \sum_{i=1}^{m} \eta_i t_i \end{array} \text{ で同一視} \right)$$

Prop 1

$$H^1(M\times N, \Theta_{M\times N}) = H^1(M, \Theta_M) \oplus H^1(N, \Theta_N)$$ のとき

$$\mathrm{Def}(M\times N) = \mathrm{Def}(M) \times \mathrm{Def}(N)$$

Prof $\varphi_{M\times N}(t) = \varphi_M(t) + \varphi_N(t)$ だから。

$\nu_1, \dots, \nu_{t_M}$, $\nu'_1, \dots, \nu'_{t_N}$

$\bigcap$ $\bigcap$

H^1_M H^1_N

$\to \varphi_{M\times N}(t) = \sum \nu_i t_i + \sum_{i=1}^{t_N} \nu'_i t_{t_M+i}$

$+ \frac{1}{2}\sum \partial G[\varphi_{M\times N}(t), \varphi_{M\times N}(t)]$

の unique 解。

$\varphi_M + \varphi_N$ とう

Prop 3 $G \curvearrowright M$ faithful G : finite.

$(g \neq id \Rightarrow \exists x \in M \; gx \neq x)$

$$\Rightarrow \mathrm{Def}(M)^G = \mathrm{Def}(M) \cap H^1(M, \Theta_M)^G$$

$\{t \in B_\varepsilon \mid \forall g \in G,\ g_* \varphi(t) = \varphi(t),\ H[\varphi(t), \varphi(t)] = 0\}$

$(r_1, \ldots, r_d :$ basis of $\ker \bar\partial \cap H^{0,1})$ ... $(\forall g \in G,\ g_* \sigma = \sigma_{g^{-1}x})$

pf $\varphi(t) = \sum r_i t_i - \frac{1}{2} \partial^* G [\varphi(t), \varphi(t)]$

$\forall g$ (M - G inv metric ... (G finite)

$$g_* \varphi(t) = \sum g_* r_i t_i - \frac{1}{2} \partial^* G [g_* \varphi, g_* \varphi]$$

$\forall g \in G$ ... $\forall g \in G$

t s.t. $g_* \varphi(t) = \varphi(t) \Rightarrow \sum_{i=1}^{d} g_* r_i t_i = \sum r_i t_i$

$$\Rightarrow \sum_{i=1}^{d} r_i t_i = \frac{1}{|G|} \sum_{g \in G} \sum_{i=1}^{d} g_* r_i t_i$$

$\Rightarrow t \in H^1(M, \Theta_M)^G$ G inv

$t \in \mathrm{Def}(M)^G \Rightarrow t \in \mathrm{Def}(M) \cap H^1(M, \Theta_M)^G$

$$\boxed{\begin{aligned} \mathrm{Def}(M) &\hookrightarrow H^1(M, \Theta_M) \\ t &\longmapsto \sum_{i=1}^{m} \eta_i t_i \end{aligned}}$$

証: $\mathrm{Def}(M) \cap H^1(M, \Theta_M)^G \ni t$

$\Rightarrow \quad \varphi(t) = \sum_{i=1}^{d} \eta_i t_i - \frac{1}{2}\partial G\,[\varphi(t), \varphi(t)]$

$\forall g \in G \quad g_*\varphi(t) = \sum_{i=1}^{d} \eta_i t_i - \frac{1}{2}\partial G\,[g_*\varphi(t), g_*\varphi(t)]$

t 小さいと $\rightsquigarrow$ $g_*\varphi(t) = \varphi(t)$. (uniqueness)

$\rightsquigarrow \quad \forall g \quad g_*\varphi(t) = \varphi(t)$

$\rightsquigarrow \quad t \in \mathrm{Def}(M)^G$.

③ Catanese's Theorem [Cat89]

Z: smooth proj surface. $\mathrm{Def}(Z)$ = smooth

G = finite. $G \curvearrowright Z$ faithful

Assume $X = Z/G$ has Duval singularity

(†) Xにかんするある条件をみたすならば (A_n, D_n, E_6, E_7, E_8)

then. $S \to X$ minimal resolution. について (−1 curve がない minimal)

$$\mathrm{Def}(S) = \mathrm{Def}(Z)^G \times R$$

R 1 point scheme

($\mathbb{C}^2/G$ $G < SL(2,\mathbb{C})$ finite subgroup)

(Duval singularity 全部でてくるのは

A_n 型 → $\mathbb{C}^2/\mathbb{Z}/(n+1)\mathbb{Z}$ で みたいなの

$(x,y) \to (\eta x, \eta^{-1} y)$ η = 1の(n+1)乗根)

(Rem)

Prop3 Catanese thm

Z: smooth proj surface, $\mathrm{Def}(Z)$ smooth

G finite grp. $G \curvearrowright Z$ faithful.

$X := Z/G$ has Duval singularity (A_n, D_n, E_6, E_7, E_8)

then $S \to X$ minimal resolution, について.

$\mathrm{Def}(S) = \mathrm{Def}(Z)^G \times R$

(R 1 point scheme)

$\boxed{\text{Xにかんするある条件}}$

Local-to-global Ext spectral sequence [edit]

$\mathbb{Z}/n+1$

There is a spectral sequence relating the global Ext and the sheaf Ext: let F, G be sheaves of modules over a ringed space $(X, \mathcal{O})$; e.g., a scheme. Then

$$\boxed{E_2^{p,q} = \mathrm{H}^p(X; \mathcal{E}xt^q_{\mathcal{O}}(F,G)) \Rightarrow \mathrm{Ext}^{p+q}_{\mathcal{O}}(F,G).}\text{[1]}$$

This is an instance of the Grothendieck spectral sequence: indeed,

and complete discussions, see the sections below. For the examples in this section, it suffices to use this definition: one says a spectral sequence converges to H with an increasing filtration F if $E^\infty_{p,q} = F_pH_{p+q}/F_{p-1}H_{p+q}$. The examples below illustrate how one relates such filtrations with the E^2-term in the forms of exact sequences; many exact sequences in applications (e.g., Gysin sequence) arise in this fashion.

2 columns and 2 rows [edit]

Let $E^r_{p,q}$ be a spectral sequence such that $E^2_{p,q} = 0$ for all p other than 0, 1. The differentials on the second page have degree (-2, 1) and therefore they are all zero; i.e., the spectral sequence degenerates: $E^\infty = E^2$. Say, it converges to H with a filtration

$$0 = F_{-1}H_n \subset F_0H_n \subset \cdots \subset F_nH_n = H_n$$

such that $E^\infty_{p,q} = F_pH_{p+q}/F_{p-1}H_{p+q}$. Then $F_0H_n = E^2_{0,n}$, $F_1H_n/F_0H_n = E^2_{1,n-1}$, $F_2H_n/F_1H_n = 0$, $F_3H_n/F_2H_n = 0$, etc. Thus, there is the exact sequence:[1]

$$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0.$$

Next, let $E^r_{p,q}$ be a spectral sequence whose second page consists only of two lines $q = 0, 1$. This need not degenerate at the second page but it still degenerates at the third page as the differentials there have degree (-3, 2). Note $E^3_{p,0} = \ker(d : E^2_{p,0} \to E^2_{p-2,1})$, as the denominator is zero. Similarly, $E^3_{p,1} = \operatorname{coker}(d : E^2_{p+2,0} \to E^2_{p,1})$. Thus,

$$0 \to E^\infty_{p,0} \to E^2_{p,0} \xrightarrow{d} E^2_{p-2,1} \to E^\infty_{p-2,1} \to 0.$$

Now, say, the spectral sequence converges to H with a filtration F as in the previous example. Since $F_{p-2}H_p/F_{p-3}H_p = E^\infty_{p-2,2} = 0$, $F_{p-3}H_p/F_{p-4}H_p = 0$, etc., we have: $0 \to E^\infty_{p-1,1} \to H_p \to E^\infty_{p,0} \to 0$. Putting everything together, one gets:[2]

$$\cdots \to H_{p+1} \to \boxed{E^2_{p+1,0} \xrightarrow{d} E^2_{p-1,1} \to H_p \to E^2_{p,0} \xrightarrow{d} E^2_{p-2,1}} \to H_{p-1} \to \cdots.$$

We consider the low term exact sequence deriving from the "local to global" Ext - spectral sequence

$$0 \to H^1(\Theta_X) \to \mathrm{Ext}^1(\Omega^1_X, \mathcal{O}_X) \to \boxed{H^0(\mathcal{E}xt^1(\Omega^1_X, \mathcal{O}_X)) =: T_X \xrightarrow{\mathrm{ob}} H^2(\Theta_X).}$$

$P=2$

$\Rightarrow$ ob a dual to surjective

A_n sing $\quad x^2+y^2+z^{n+1}=0$

$\mathbb{C}[X,Y] \,/\, \mathbb{C}[X^n, Y^n, XY] \quad \mathbb{Z}/(n+1)$

$X \longmapsto \eta X$

$Y \longmapsto \eta^{-1} Y$

$S \longmapsto X^n \qquad ST - U^{n+1} = 0$

$T \longmapsto Y^n$

$U \longmapsto XY$

analytic

$$\boxed{x^2+y^2+z^{n+1}=0}$$

§2 Exampleの紹介.

$n \geq 8$ s.t. $3 \nmid n$, $2 \nmid n$.

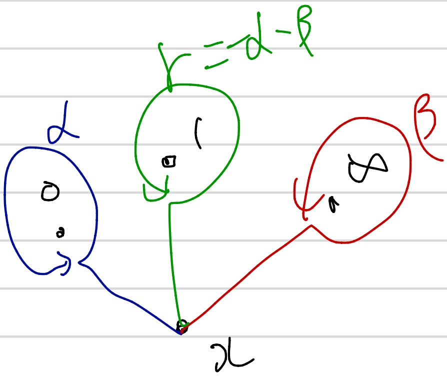
$x \in \mathbb{P}^1 \setminus \{0,1,\infty\}$ fix　　$G := \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$

$$\pi_1(\mathbb{P}^1 \setminus \{0,1,\infty\}, x) \longrightarrow G$$

$$F_2 = \langle \alpha, \beta \rangle$$

$$\alpha \longmapsto (1,0)$$

$$\beta \longmapsto (0,1)$$



Rieman ext thm

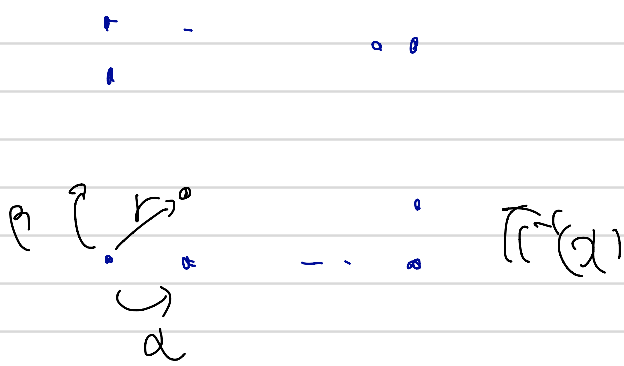
$\pi : \exists C$ curve $\longrightarrow \mathbb{P}^1$ finite $n^2 : 1$
$0, 1, \infty$で分岐

local Monodromy $\alpha = (1,0)$　$\beta = (0,1)$
$\gamma = (-1,-1)$ となる

$\#|\pi^{-1}(0)| = n$　　$n:1$.
$(1,\infty)$

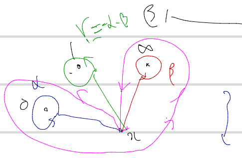
$\#|\pi^{-1}(x)| = n^2$

（αで n回まわすともどる，βで不変
→ $(\pi^{-1}(0)) = n$ & 全て分岐
→ $\pi^{-1}(0)$はn，nで分岐）



Bayer-P.の例に　　$G = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$　$3 \nmid n$, $2 \nmid n$, $n \geq 8$

$\pi_1(\mathbb{P}^1 - \{0,1,\infty\}, x) \longrightarrow G$.
$F_2 = \langle \alpha, \beta \rangle$
$\alpha \longmapsto (1,0)$
$\beta \longmapsto (0,1)$



$\sigma : \exists C$: curve $\longrightarrow \mathbb{P}^1$,　$n^2 : 1$ 写像. finite
$0, 1, \infty$で分岐する

monodromy が $\alpha = (1,0)$　$\beta = (0,1)$になる

$\# \sigma^{-1}(x) = n^2$　（γの生成は $\gamma = (-1,-1)$）

$\sigma^{-1}(x) = \mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/n\mathbb{Z}$　　$\alpha(a,b) = (a+1, b)$
$\beta(a,b) = (a, b+1)$
s.t. $\alpha = (+1, 0)$
$\beta = (0, +1)$

(13)

$g(C) = 1 + \frac{n(n-3)}{2} \geq 2$ ($2g(C)-2 = 2g(\mathbb{P}^1)-2 + 3 \times \underline{(n-1)}\,\underline{n}$)

n+h 컴 ファイバー

$C_1 = C_2 = C \quad \Sigma = C_1 \times C_2.$

$\pi : \Sigma \to X = C_1 \times C_2 / G_1 \quad S \to X$ minimal resol とおく

$G_1 \ni (a, b)$ について $(a, b)(x_1, x_2)$

$= ((a, b)x_1,\ (a', b')x_2)$

$\underbrace{\begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}}_{A} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ なる a', b' をとる

($\det A = -3$ より $A \in GL(2, \mathbb{Z}/n\mathbb{Z})$ $n \nmid 3$)

$g(C) = 1 + \frac{n(n-3)}{2} \geq 2$

$C = C_1 = C_2$ とし $\Sigma = C_1 \times C_2.$

$X = C_1 \times C_2 / G_1$ とおく。ここで

$G_1 \ni$

$(a, b) : C_1 \times C_2 \longrightarrow C_1 \times C_2.$

$(x_1, x_2) \longmapsto ((a, b)x_1, (a', b')x_2)$

$\underbrace{\begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}}_{A} \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$ なる (a', b') をとる

($\det = -1 + 4 = 3$ より逆行列が $\mathbb{Z}/n\mathbb{Z}$ にはとれる $[3 \nmid n]$)

$S \longrightarrow X$ minimal resolution とする。

14

Prop X は 6点の A_2 型 Duval singularity をもつ.

pf Notation $p \in \{0,1,\infty\}$ について $g_p \in \pi^{-1}(p)$ の点をとる

$\sigma : Z = C_1 \times C_2 \to X = C_1 \times C_2 / G$ について、

① $\sigma(g_p, g'_{p'})$ が特異点 $\Rightarrow p = p'$

対偶 $p \neq p'$ のとき $(a,b)(g_p, g'_{p'}) = (g_p, g'_{p'}) \Rightarrow (a,b) = 0$ となるから

($p=0, p'=\infty$ のとき $(a,b)\, g_0 = g_0 \Rightarrow a = 0.$

$(a',b')\, g'_\infty = g'_\infty \Rightarrow b' = 0$

$\Rightarrow b = 0.$

$$\begin{pmatrix} a' \\ b' \end{pmatrix} = A^{-1} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \frac{-a+2b}{3} \\ \frac{-2a+b}{3} \end{pmatrix}$$)

② $p = p'$ のとき $\#\,\mathrm{Stab}(g_p, g'_p) = 2$

(上の式で $a' = 0 \Rightarrow 2b = 0 \Rightarrow (a,b) = (0,0), (0, \frac{n}{2})$ のどちらか)

ということは 特異点は $\forall p \in \{0,1,\infty\}$ の各 $\pi^{-1}(p) \times \pi^{-1}(p)$ n^2 個のうち

半分が 1つにいき もう半分は もう1つにいく.

($\#(\mathrm{Orbit}(g_p, g_p)) = \frac{n^2}{2}$ から)

$3 \times 2 = 6$ コ

Singularity $\# \mathrm{Stab}(g_{p_1}, g'_p) = 2(d-1)$

localには と考えれば $\mathbb{C}^2/\mathbb{Z}/2\mathbb{Z}$ となっている A_1

$p = p'$ なので

$(a, b)\begin{pmatrix} g_p & g_{p'} \end{pmatrix} = (g_p, g_{p'})$ となる (a, b) は 2つしかない

(もし $\frac{1}{3}a' = 0$ となる

$\frac{2b}{3} = 0 \Rightarrow b = 0$ or $\frac{n}{2}$

$(2 \mid n)$)

代わりに…

よって X の

元の数は $\pi(g_p, g_{p^*})$ なので $p = p'$,

g_p, g_p' を fix すると

$p \in \{0, 1, \infty\}$ fix して $\#(\mathrm{Stab}(g_p, g_p')) = 2$

$\Downarrow$

$\boxed{n^2 \text{ ではない}}$ $\frac{n^2}{2}$ $\mathrm{Orbit}(g_p, g_p') = \frac{n^2}{2}$

n n

$\Downarrow$

$\frac{n^2}{2}$

$p \in$ fix すると

これは2つ

2つ ある

Localには Stabilizer 2つの A_2 singularity

$(\mathbb{C}^2 / S_2$ と同じ$)$

16

Thm S is rigid

$\boxed{\text{Pf}}$

① $\mathrm{Def}(Z) = \mathrm{Def}(C) \times \mathrm{Def}(C)$

② $\mathrm{Def}(C)^G = \mathrm{Def}(C) \cap H^1(C, \Theta_C)^G$

$= \cap$ $\quad \because H^1(\mathbb{P}, \Theta_{\mathbb{P}}(-\sum n_i)) = 0$

Catanese thm

$\mathrm{Def}(S) = \mathrm{Def}(Z)^G \times \mathbb{R}$

$= (\mathrm{Def}(C)^G \times \mathrm{Def}(C)^G) \times \mathbb{R}$

$=$ $0 \times \mathbb{R}$ $\quad$ 0 dim

$\to$ rigid ₍₍

(14)

Thm $H^1(S, \Theta_S) \neq 0$ $\quad S \to X$

$E_1, \ldots, E_6 \to x_1, \ldots, x_6$

Pf $E = \sum_{i=1}^{6} E_i$ Exceptional divisors $(E_i^2 = -2)$

Local cohomology Exact sequence ($\Gamma_E(X, -)$ exist functor a derived functor.)

[Hartshorne Chapter 3-2. Exercise]

$$0 \to H^0_E(S, \Theta_S) \to H^0(S, \Theta_S) \to H^0(S \setminus E, \Theta_S)$$
$$\to H^1_E(S, \Theta_S) \to H^1(S, \Theta_S) \to \cdots$$

$$H^0(S \setminus E, \Theta_S) \simeq H^0(X - \{x_1, \ldots, x_6\}, \Theta_X)$$
$$\simeq H^0(Z \setminus \{P_i's\}, \Theta_Z)^G$$
$$= H^0(Z, \Theta_Z)^G = 0$$
$$(H^0(C, \Theta_C) = 0)$$

$$\therefore H^1_E(S, \Theta_S) \hookrightarrow H^1(S, \Theta_S)$$

[Burns-Wahl §4 Prop 1.10] (A_1型なら6つよい) 6dim

$$H^1_E(S, \Theta_S) = \bigoplus_{i=1}^{6} H^1_{E_i}(S, \Theta_S) \neq 0$$

(local cohomology より local にわかることなので

1つについて

$$\begin{array}{ccc} V & \xrightarrow{\text{minimal resol.}} & \operatorname{Spec} \mathbb{C}[X,Y,Z]/(X^2+Y^2+Z^2=0) \\ E & \longrightarrow & 0 \quad A_1\text{ で} \end{array}$$

かんたんに計算される)

$$\therefore \quad H^1(S, \Theta_S) \neq 0$$

(じつは 6じげん (特に同型))