On minimal projective varieties with vanishing 2nd Chern classes

(Masataka Iwai Osaka Univ)

j. w. w. Shin-ichi Matsumura
(Tohoku Univ.)

Notation

- (X, ω) X cpt Kähler manifold $n = \dim X$.
  ω = Kähler form

- Ω_X^1 holomorphic cotangent bundle
  $K_X = \det \Omega_X^1$ canonical line bundle

- $c_2(X) = c_2(\Omega_X^1) \in H^{2,2}(X, \mathbb{R})$
  $c_1(X) = -c_1(\Omega_X^1) \in H^{1,1}(X, \mathbb{R})$ ..
  $\left(- c_1(K_X) \right)$

Thm (Miyaoka, Yau)

Assume. K_X is ample. $\left(\overset{\text{def ?}}{\Longleftarrow} \right)$ smooth metric with positive curvature

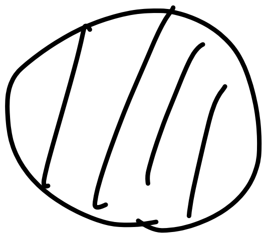
$\left(\Leftarrow \right) c_1(X) = -c_1(K_X) < 0 ,)$

Then

$$\left\{ c_2(X) - \frac{n}{2(n+1)} c_1(X)^2 \right\} \cdot c_1(K_X)^{n-2} \geq 0 \qquad (\star)$$

If "$=0$" holds in $(\star)$,

then the universal cover is a unit ball in $\mathbb{C}^n$

$\mathbb{B}^n$  $\longrightarrow X = \mathbb{B}^n / \Gamma$

Thm (Yau)

(ω: Kähler form)

Assume that $c_1(X) = 0$.

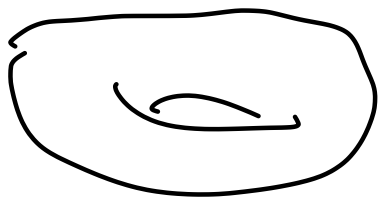
Then $c_2(X) \cdot \{\omega\}^{n-2} \geq 0$. ☆☆

If "$= 0$" holds in (☆☆),

Then, by taking a finite étale covering,

(finite covering no branched point)

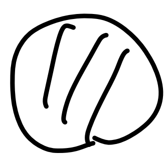
X is a torus.

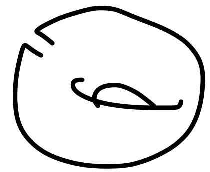


$\xrightarrow[\text{covering}]{\text{finite}} X$

Two Theorem (MY, Y) says

If "$=0$" holds for some "good" inequality w.r.t. c_2,
Then X has a "good" structure.

(MY) $c_1(X) < 0$, $c_2 - \frac{n}{2(n+1)} c_1^2 = 0$ $\implies$ $X_{univ} = B^n$,  $\longrightarrow X$

(Y) $c_1 = 0$, $c_2 = 0$ $\implies$ Torus $\to X$  $\longrightarrow X$

(finite covering)

Thm (Miyaoka 87, Enoki 93, J. Cao 13)

projective case. cpt Kähler case.

Assume that K_X is nef ($\Leftarrow \exists$ smooth semipositive metric)

$\Updownarrow$

$\left(\forall \varepsilon>0, \exists h_\varepsilon \text{ smooth metric on } K_X \text{ s.t. } \sqrt{-1}\,\Theta_{h_\varepsilon} \geq -\varepsilon\omega\right) \Leftrightarrow$ [Projective case] $\left(K_X\cdot C \geq 0 \ \forall C \subseteq X \text{ curve.}\right)$

(Chern curvature.)

Then $$\boxed{c_2(X)\cdot\left(c_1(X)+\varepsilon\{\omega\}\right)^{n-2} \geq 0 \quad 0<\forall\varepsilon \ll 1.} \qquad (\#)$$

Question

If "$=0$" in #, then what is structure of X??

Thm A [Structure Theorem] (I.-Matsumura 22)

Assume that K_X is nef & $c_2(X)\cdot(c_1(K_X)+\varepsilon\{\omega\})^{n-2}=0$

$(0<{}^{\exists}\varepsilon\ll 1.)$

$\uparrow$ in $H^{2,2}(X,\mathbb{R})$

($c_2(X)=0$ in $H^{2,2}(X,\mathbb{R})$)

Then • $c_1(K_X)^2=0$.

• By taking a finite étale cover, one of the follwings holds.

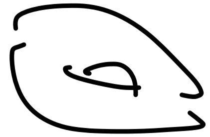
① ($c_1(K_X)=0$ case) X is a torus.

② ($c_1(K_X)\neq 0$ case) X is a torus fibration onto a curve with genus ≥ 2.

i.e. $\exists f: X \longrightarrow C$ smooth morphism (holomorphic submersion)

- C: curve with $g(C)\geq 2$.
- Any fiber is a torus.

Roughly Speaking -- (finite cover)

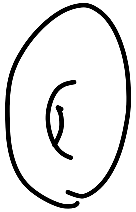
(iii) K_X nef, $c_2 = 0$ $\Longrightarrow$ ① Torus  $\to X$

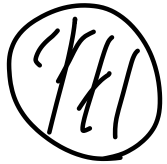
or

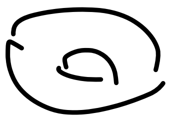
② Torus fibration onto a curve with genus ≥ 2

$X \longrightarrow$ 

$\downarrow$

Fiber = 

(iv) $c_1(X) < 0$, $\left(c_2 - \frac{n}{2(n+1)} c_1^2\right) = 0$ $\Longrightarrow$ $X_{univ} \simeq$ Ball  $\to X$

(v) $c_1 = 0$, $c_2 = 0$ $\Longrightarrow$ Torus  $\to X$

Application

(Big open problem in Birational geometry)

Conj [Abundance Conj]

If K_X is nef, then K_X is semi ample

($\exists m \in \mathbb{N}_{>0}$, $\forall x \in X$, $\exists s \in H^0(X, K_X^{\otimes m})$ s.t. $s(x) \neq 0$ (Basepointfree))

[History]

- $\dim X \leq 2$ Ok.
- $\dim X = 3$ & X: projective. Ok [Miyaoka 88, Kawamata 92]
- $\dim X = 3$ & X cpt Kähler open. [cf. Campana-Höring-Peternell 16]
- $c_1(X) = 0$ Ok [Kawamata 85]
- $c_1(X)^n \neq 0$ Ok Kawamata-Shokurov's basepointfree thm
- $\dim X \geq 4$ Open.

Thm B. [Abundance] (I.-Matsumura 22)

If K_X is nef & $c_2(X) = 0$,

then K_X is semiample.

i.e. Abundance conjecture holds for $c_2 = 0$

Sketch of Proof

"too technical proof"!

P1 K_X nef + $c_1 = 0$

$\Longrightarrow c_1^2 = 0$, Ω_X^1 nef $\Longleftrightarrow$ $\forall C \subset X$ curve $\Omega_X^1|_C$ nef

($\Leftarrow$ Griffith semipositive)

P2 $c_1 \neq 0$, Ω_X^1 nef $\Longrightarrow$ K_X semiample (Thm B)

P3 K_X semiample, Ω_X^1 nef $\Longrightarrow$ Thm A (structure theorem)

(Already proved by Höring B)

P1 $\Leftarrow$ W. Ou's classification (W. Ou '17)

P2 $\Leftarrow$ ETS. Find $\exists f : X \to C$ onto curve with $g(C) \geq 2$

$\Leftarrow$ by using
- Shafarevich map (Campana-Claudon-Eyssidieux '15)
- Campana's core map (Pereira-Rousseau-Touzet '22)

$\boxed{\text{Important!}}$ Cotangent bundle Ω_X^1 has "good" positivity

Outlook

Question If Ω_X^1 has "good" positivity, then (nef, ...)

Ⓐ [Structure] What is str of X??

Ⓑ [Abundance] Is K_X semi ample??

I-M22 $\boxed{\Omega_X^1 \text{ nef}}$ (K_X nef) $\Longrightarrow$ Ⓐ & Ⓑ holds

$c_1^2 = 0$

Thm (H.-H. Wu–F. Zheng 02, G. Liu 14)

Assume that X admits a Kähler metric with seminegative biholomorphic sectional curvature.

$\left(\forall \xi, \eta \in \mathbb{C}^n, \quad \underline{R_{i\bar{j}k\bar{l}}} \, \xi^i \bar{\xi}^j \eta^k \bar{\xi}^l \leq 0 \right.$ (In short. $BSC \leq 0$)

(Rieman curvature tensor)

Then Ⓐ by taking a finite (étale) cover,

$\exists f : X \longrightarrow Y$ submersion

$\begin{cases} \cdot\, Y : \text{ projective.} & (\text{Kyraple}) \ c_1(Y) < 0 \\ \cdot\, \forall \text{ Fiber is a torus.} \end{cases}$

Ⓑ K_X is semiample.

<u>Recall</u> $BSC \leq 0 \Rightarrow \Omega_X^1$ nef.

$\Updownarrow$

K_X nef.

<u>Conj</u> How about

<u>seminegative holomorphic sectional curvature</u> case??

$\left(\forall \xi \in \mathbb{C}^n \; R_{i\bar{j}k\bar{\ell}}\, \xi^i \bar{\xi}^j \xi^k \bar{\xi}^\ell \le 0\right)$ (HSC ≤ 0)

str?? Abundance??

$\left(\begin{array}{ccc} BSC \le 0 & \Rightarrow & \Omega_X^1 \text{ nef} \\ \Downarrow & & \Downarrow \\ HSC \le 0 & \Rightarrow & K_X \text{ nef} \end{array}\right)$

<u>Thm</u> [Tossatti-Yang 17]

HSC ≤ 0 $\Longrightarrow$ K_X nef

<u>Thm</u> [Heier-Lu-Wong-Zheng. 18] by taking a finite etale cov.

HSC ≤ 0
(B) K_X semiample $\Longrightarrow$ $X \simeq (\text{Torus}) \times \left(\begin{array}{l} Y \text{ proj} \\ c_1(Y) < 0 \end{array}\right)$
X proj

$BSC \le 0$

(A) str

(B) Abundance

ok

[Wu-Zheng, Liu]

Ω^1_X nef

(A) str

(B) Abundance

open

[Höring] (B) $\Rightarrow$ (A)

[I.-Matsumura] $c_1^2 = 0 \Rightarrow$ (A) (B) ok.

[Tossati-Yang]

$HSC \le 0$

(A) str

(B) Abundance

open

[Heier-Lu-Wong-Zheng]

(B) $\Rightarrow$ (A)

proj

K_X nef

(A) str

(B) Abundance

open.