12/17

12月17日（木）
10:00–11:00 岩井雅崇（大阪市立大学、京都大学）
相対的反標準因子のasymptotic base lociについて

①

On Asymptotic base loci of relative anticanonical divisors.

(Joint work with Sho Ejiri (Osaka Univ.)
· Shin-ichi Matsumura (Tohoku Univ.)

§1 Main result.
§2. Proofs.

Notation.

- X, Y smooth projective variety $/\mathbb{C}$
- $f: X \to Y$ surjective morphism with connected fiber.
  (algebraic fiber space)
- $-K_{X/Y} := -(K_X - f^*K_Y)$ $K_X = \det \Omega_X$
  relative anticanonical divisor.
- divisor = line bundle.

"$-K_{X/Y}$ is hard to have positivity"
(ample, nef, - - - -)

"$-K_{X/Y}$は正値性をもちづらい"
(ample, nef, - - -)

D: divisor on X (Positivity) 12/27 ②

D ample $\Leftrightarrow$ $\exists h$ smooth metric on D

s.t $\sqrt{-1}\,\Theta_h > 0$ (positive curvature)

D: nef $\Leftrightarrow$ $\forall C \subset X$ curve, $D \cdot C \geq 0$

$\overset{\text{Demailly}}{\Longleftrightarrow}$ $\forall \varepsilon > 0$, $\exists h_\varepsilon$ smooth metric on D

s.t $\sqrt{-1}\,\Theta_{h_\varepsilon} \geq -\varepsilon\omega$ (ω = Kähler form)

D big $\Leftrightarrow$ $h^0(X, mD) \sim O(m^{\dim X})$ $m \gg 0$

$\overset{\text{Demailly}}{\Longleftrightarrow}$ $\exists h$ singular metric, $\sqrt{-1}\,\Theta_h \geq \varepsilon\omega$

$\exists \varepsilon > 0$ (in the sense of current)

D Pseudo-effective (Psef) $\Leftrightarrow$ $\forall A$ ample $\forall m \in \mathbb{N}_{>0}$, $mD + A$ big

$\overset{\text{Demailly}}{\Longleftrightarrow}$ $\exists h$ singular metric

s.t $\sqrt{-1}\,\Theta_h \geq 0$ (in the sense of current)

$$\begin{array}{ccc}
\text{Ample} & \Longrightarrow & \text{Nef} \\
\Downarrow & & \Downarrow \\
\text{Big} & \Longrightarrow & \text{Psef}
\end{array}$$

Example

- $\mathcal{O}_{\mathbb{P}^n}(1)$ is ample.
  (Fubini-Study metric)
- Zero divisor is Not big
  $(\because h^0(X, m0) = 1 \quad \forall m >> 0)$

D: divisor on X (Positivity)

D ample $\overset{\text{def}}{\iff} \exists h$ smooth metric on D s.t. $\sqrt{-1}\Theta_h > 0$ (positive curvature)

D: nef $\overset{\text{def}}{\iff} \forall C \subset X$ curve, $D \cdot C \geq 0$

$\underset{\text{Demailly}}{\iff} \forall \varepsilon > 0, \exists h_\varepsilon$ smooth metric on D s.t. $\sqrt{-1}\Theta_{h_\varepsilon} \geq -\varepsilon\omega$ (ω = Kähler form)

D big $\overset{\text{def}}{\iff} h^0(X, mD) \sim O(m^{\dim X})$ $m >> 0$

$\underset{\text{Demailly}}{\iff} \exists h$ singular metric, $\exists \varepsilon > 0$ $\sqrt{-1}\Theta_h \geq \varepsilon\omega$ (in the sense of current)

D Pseudo-effective (Psef) $\overset{\text{def}}{\iff} \forall A$ ample $\forall m \in \mathbb{N}_{>0}$, $mD + A$ big

$\underset{\text{Demailly}}{\iff} \exists h$ singular metric s.t. $\sqrt{-1}\Theta_h \geq 0$ (in the sense of current)

$$\begin{tikzcd}
\text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\
\text{Big} \arrow[r, Rightarrow] & \text{Psef}
\end{tikzcd}$$

- $e \in \mathbb{N}_{\geq 0}$, $E_e := \mathcal{O}_{\mathbb{C}\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(-e)$.

$\therefore F_e = \mathbb{P}(E_e)$ Hirzebruch Surface.

$\pi : F_e \to \mathbb{C}\mathbb{P}^1$,

$$D = -K_{F_e/\mathbb{C}\mathbb{P}^1} = -(K_{F_e} - \pi^* K_{\mathbb{C}\mathbb{P}^1}).$$

$$h^0(F_e, mD) = h^0(\mathbb{C}\mathbb{P}^1, \mathrm{Sym}^m(E_e) \otimes \mathcal{O}_{\mathbb{C}\mathbb{P}^1}(me))$$
$$= \cdots = \frac{1}{2}(me+2)(m+1)$$

$\therefore$ D is big on $F_e \iff e > 0$.

$$\begin{array}{ccc} \text{Ample} & \longrightarrow & \text{Nef} \\ \Downarrow & & \Downarrow \\ \text{Big} & \longrightarrow & \text{Psef} \end{array}$$

§1 Main Result

Thm (Kollár–Miyaoka–Mori '92)

$$\boxed{\text{If } -K_{X/Y} \text{ is ample, then } \dim Y = 0.}$$

Thm (Cao '19, Cao–Höring '19, Campana–Cao–Matsumura '19, Patakfalvi–Zdanowicz '19 (appendix with Codogni))

If $-K_{X/Y}$ is nef, then f is locally trivial (analytic fiber bundle)

($\forall y \in Y$, $\exists U \subset Y$ Euclid open near y s.t. $f^{-1}(U) \underset{\text{biholo}}{\simeq} U \times F$, $F := f^{-1}(y)$ fiber)

Aim Extend above result in case of big or psef.

Example $\pi : F_e = \mathbb{P}^1(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-e)) \to \mathbb{CP}^1$.

$-K_{F_e/\mathbb{CP}^1}$ is big if $e > 0$.

Use Asymptotic base loci.

Aim 上の結果をbigやpsefのときに拡張したい!! (きりっ)

Asymptotic base loci.

D: divisor on X, A ample on X.

- Base locus

$$\mathrm{Bs}(D) := \{x \in X \mid \forall s \in H^0(X, D)\ s(x) = 0\}$$

- Stable base locus.

$$B(D) = \bigcap_{m \in \mathbb{N}_{>0}} \mathrm{Bs}(mD)$$

Augumented base locus. (= Non-ample locus)

$$B_+(D) = \bigcap_{m \in \mathbb{N}_{>0}} B(mD - A)$$

Restricted base locus. (= Non-nef locus)

$$B_-(D) = \bigcup_{m \in \mathbb{N}_{>0}} B(mD + A)$$

<u>Property</u>. $B_-(D) \subset B(D) \subset B_+(D) \subset X$

$$\begin{array}{ccc} \boxed{D \text{ ample} \Leftrightarrow B_+(D) = \phi} & \longrightarrow & \boxed{D \text{ nef} \Leftrightarrow B_-(D) = \phi} \\ \downarrow & & \downarrow \\ \boxed{D \text{ big} \Leftrightarrow B_+(D) \neq X} & \longrightarrow & \boxed{D \text{ psef} \Leftrightarrow B_-(D) \neq X} \end{array}$$

Main thm (Ejiri - I. - Matsumura 20)

① If $f(B_+(-K_{X/Y})) \neq Y$, then $\dim Y = 0$.

② If $f(B_-(-K_{X/Y})) \neq Y$, then $B_-(-K_{X/Y}) = \phi$
($-K_{X/Y}$ is nef).

In particular, f is locally trivial.

③ If $f(B(-K_{X/Y})) \neq Y$, then $B(-K_{X/Y}) = \phi$.

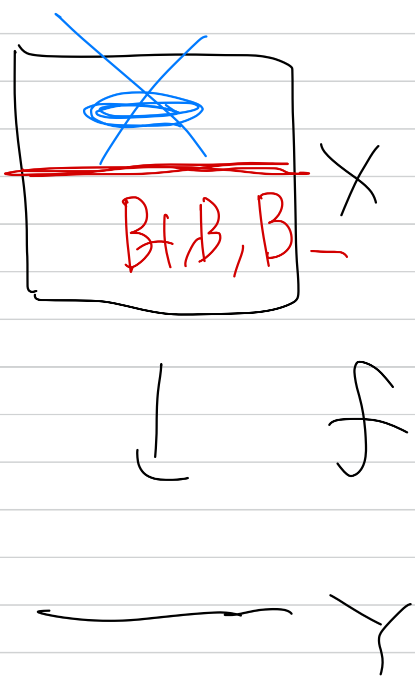
③' If $B(-K_{X/Y}) = \phi$, then

$\exists \pi : Y' \to Y$ finite étale.

s.t. $X \times_Y Y' \underset{\text{biholo}}{\simeq} F \times Y'$

(F : fiber of f)

$$\begin{tikzcd}
F \times Y' \arrow[d, phantom, "\simeq" rotate=90] & \\
X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "\pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$



Property. $B_-(D) \subset B(D) \subset B_+(D) \subset X$

$$\begin{tikzcd}
\boxed{D \text{ ample} \Leftrightarrow B_+(D) = \phi} \arrow[r] \arrow[d] & \boxed{D \text{ nef} \Leftrightarrow B_-(D) = \phi} \arrow[d] \\
\boxed{D \text{ big} \Leftrightarrow B_+(D) \neq X} \arrow[r] & \boxed{D \text{ psef} \Leftrightarrow B_-(D) \neq X}
\end{tikzcd}$$

Example. $\pi : F_e = \mathbb{P}^1(\mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(-e)) \to \mathbb{CP}^1$.
$-K_{F_e/\mathbb{CP}^1}$ is big if $e > 0$.

$$f(B_+(-K_{F_e/\mathbb{CP}^1})) = \mathbb{CP}^1 \text{ //.}$$

Main thm (Ejiri - I. - Matsumura 20)

① If $f(B_+(-K_{X/Y})) \neq Y$, then $\dim Y = 0$.

② If $f(B_-(-K_{X/Y})) \neq Y$, then $B_-(-K_{X/Y}) = \phi$ ($-K_{X/Y}$ is nef).

In particular, f is locally trivial.

③ If $f(B(-K_{X/Y})) \neq Y$, then $B(-K_{X/Y}) = \phi$

③′ If $B(-K_{X/Y}) = \phi$, then $\exists \pi : Y' \to Y$ finite étale. s.t. $X \times_Y Y' \underset{\text{biholo}}{\simeq} F \times Y'$

$$\begin{tikzcd} F \times Y' \\ X' \arrow[u, phantom, "\cong" description] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y' \arrow[d, "\pi"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

(F : fiber of f)

Rem · ③′ is already proved by [Ambro05].

· In the case of pairs ----??

Δ effective $\mathbb{Q}$-divisor on X

We consider "$-(K_{X/Y} + \Delta)$" instead of $-K_{X/Y}$.

① O.K. if (X, Δ) is lc $\mathbb{Q}$-pair.

②, ③ O.K. if (X, Δ) is klt $\mathbb{Q}$-pair.

· In the case of singular varieties ---??

① O.K if X is lc (lc on a general fiber)

Y is canonical.

(Recently, Chang proved ①

if X is lc

(without assuming Y is canonical))

§2 Proofs of Main thm ①.

(By singular Hermitian metrics of direct image sheaves)

ω_Y = Kähler form on Y., $\dim Y \neq 0$.

Thm (Cao–Păun 17 (Lem 5.4), Păun–Takayama 18, Berndtsson–Păun 08)

L divisor, h = singular metric on L, metric

Assume ① $\sqrt{-1}\,\Theta_h \geq \varepsilon f^*\omega_Y$, $\varepsilon > 0$

② $\mathcal{J}(h^{\frac{1}{m}}|_F) = \mathcal{O}_F$, generic fiber F.

③ $f_*(mK_{X/Y} + L) \neq 0$.

Then $\exists H$ singular metric on $f_*(mK_{X/Y} + L)$.

s.t. $\sqrt{-1}\,\Theta_{\det H}(\det f_*(mK_{X/Y} + L)) \geq r\varepsilon\omega_Y$

$r := \operatorname{rk} f_*(mK_{X/Y} + L)$

Asymptotic base loci.
D: divisor on X, A ample on X.

- Base locus
$Bs(D) := \{x \in X \mid {}^{\forall} s \in H^0(X, D)\ s(x) = 0\}$
- Stable base locus.
$B(D) = \bigcap_{m \in \mathbb{N}_{>0}} Bs(mD)$

Augumented base locus. (=Non-ample locus)
$B_+(D) = \bigcap_{m \in \mathbb{N}_{>0}} B(mD - A)$

Proofs

Setting ${}^{\exists}A$ ample on X
${}^{\exists}h_A$ sm metric

s.t. $\begin{cases} \omega_X := \sqrt{-1}\,\Theta_{h_A} \text{ Kähler form} \\ \omega_X \geq f^*\omega_Y. \end{cases}$

Assume $f(B_+(-K_{X/Y})) \neq Y$ & $\dim Y \neq 0$.

${}^{\exists}m \in \mathbb{N}_{>0}$, $f(B(-mK_{X/Y} - A)) \neq Y$.

$\Rightarrow$ ${}^{\exists}h$ singular metric on $-mK_{X/Y} - A$.

s.t. $\begin{cases} h|_F \text{ is smooth on general fiber } F. \\ \sqrt{-1}\,\Theta_h \geq 0. \end{cases}$

$\Rightarrow$ $\tilde{h} := h \cdot h_A$ singular metric on $-mK_{X/Y}$

s.t. $\begin{cases} \tilde{h}|_F \text{ is smooth on general fiber } F \\ \sqrt{-1}\,\Theta_{\tilde{h}} \geq \omega_X \geq f^*\omega_Y. \end{cases}$

Thm (Cao–Păun 17 (Lem 5.4)
Păun–Takayama 18, Berndtsson–Păun 08)

L divisor, h = singular metric, $m \in \mathbb{Z}_{>0}$

Assume ① $\sqrt{-1}\Theta_h \geq \varepsilon f^*\omega_Y$, $\varepsilon > 0$

② $\mathcal{J}(h^{\frac{1}{m}}|_F) = \mathcal{O}_F$, F = generic fiber F.

③ $f_*(mK_{X/Y} + L) \neq 0$.

Then $\exists H$: singular Hermitian metric on $f_*(mK_{X/Y}+L)$

s.t. $\sqrt{-1}\Theta_{\det H}(\det f_*(mK_{X/Y}+L)) \geq r\varepsilon\omega_Y$

$r := \mathrm{rk}\, f_*(mK_{X/Y}+L)$

$\Rightarrow$ Zero divisor on Y

$\quad\quad \shortparallel$

$f_*(mK_{X/Y} - mK_{X/Y})$

has singular metric H

s.t. $\sqrt{-1}\Theta_{\det H} \geq \omega_Y$.

$\Rightarrow$ Zero divisor on Y is big.

<u>Contradiction!</u>

§ Application of Păun-Takayama.

Thm (Deng 17)

① $-K_X$ big & $f(B_+(-K_X)) \neq Y$
$\Rightarrow$ $-K_Y$ big

② $-K_X$ psef & $f(B_-(-K_X)) \neq Y$
$\Rightarrow$ $-K_Y$ is psef

Thm (Cao-Păun 17 (Lem 5.4), Păun-Takayama 18, Berndtsson-Păun 08)

L divisor. h = singular metric, $m \in \mathbb{N}_{>0}$

Assume ① $\sqrt{-1}\Theta_h \geq \varepsilon f^*\omega_Y$, $\varepsilon > 0$

② $\mathcal{J}(h^{\frac{1}{m}}|_F) = \mathcal{O}_F$ generic fiber F.

③ $f_*(mK_{X/Y} + L) \neq 0$.

Then $\exists H$ singular Hermitian metric on $f_*(mK_{X/Y}+L)$.

s.t. $\sqrt{-1}\Theta_{\det H}(\det f_*(mK_{X/Y}+L)) \geq r\varepsilon\omega_Y$

$r := \mathrm{rk}\, f_*(mK_{X/Y}+L)$

$\boxed{\text{Pf}}$ • $f(B_+(-K_X)) \neq Y$

$\Rightarrow \exists m \in \mathbb{N}_{>0}$, $\exists h$ on $-mK_X$.

s.t. $\begin{cases} \sqrt{-1}\Theta_h \geq f^*\omega_Y. \\ \mathcal{J}(h|_F^{\frac{1}{m}}) = \mathcal{O}_F \quad \text{general fiber } F. \end{cases}$

$\Rightarrow -mK_Y = f_*(mK_{X/Y} - mK_X)$.

has sHm H s.t. $\sqrt{-1}\Theta_H \geq \omega_Y$.

$\Rightarrow -mK_Y$ big.

Rem

- Thm (Kollár–Miyaoka–Mori 92, Fujino–Gongyo12)
  f smooth & $-K_X$ ample $\Rightarrow$ $-K_Y$ ample.
- Thm (Miyaoka 93, Fujino–Gongyo 14)
  f smooth & $-K_X$ nef $\Rightarrow$ $-K_Y$ nef

Q. $\exists$ proofs of above thms.
by using "Păun–Takayama's method" ?

$\to$ Yes!

Thm (Păun–Takayama 18, Thm 5.1.2 & Cor 5.2.2)

- Y_0 : the set of regular values of f.
- L : Divisor on X, h : singular metric on L.

Assume. ① $\sqrt{-1}\Theta_h \geq 0$ ($\sqrt{-1}\Theta_h \geq \varepsilon f^*\omega_Y$)
② $\mathcal{J}(h^{\frac{1}{m}}|_F) = \mathcal{O}_F$ general fiber F
③ $f_*(mK_{X/Y}+L) \neq 0$.
④. h is conti on $f^{-1}(Y_0)$

Then $\exists H$ singular metric on $f_*(mK_{X/Y}+L)$

s.t. $\begin{cases} H \text{ is continuous on } Y_0. \\ \sqrt{-1}\Theta_{\det H} \geq 0 \end{cases}$ ($\sqrt{-1}\Theta_{\det H} \geq r\varepsilon\omega_Y$)

Pf) $-K_X$ ample

$\overset{\exists m \in \mathbb{N}_{>0}}{\Rightarrow}$ $-mK_X - A$ has semipositive metric h

$\Rightarrow$ $-mK_X$ has smooth metric

s.t. $\sqrt{-1}\Theta_h \geq f^*\omega_Y$

$\Rightarrow$ $-mK_Y = f_*(mK_{X/Y} - mf^*K_X)$

has singular metric H.

s.t. $\sqrt{-1}\Theta_H \geq \omega_Y$ & H is conti on Y.

$\Rightarrow$ $-K_Y$ ample.

(じ[illegible])

Fact Siu's uniform global generation.

$\exists A_Y$ ample, $\forall D$ divisor, h: singular metric s.t. $\sqrt{-1}\,\Theta_h \geq 0$,

$\mathcal{O}_Y(A_Y + D) \otimes \mathcal{J}(h)$ is globally generated.

$\Updownarrow$

$\left(Bs(A+D) = \phi \right)$

- Base locus
  $Bs(D) := \{x \in X \mid {}^{\forall} s \in H^0(X, D)\ s(x) = 0\}$
- Stable base locus
  $B(D) = \bigcap_{m \in \mathbb{N}_{>0}} Bs(mD)$

Augumented base locus (= Non-ample locus)
$B_+(D) = \bigcap_{m \in \mathbb{N}_{>0}} B(mD - A)$

$-K_Y$ has singular metric H.
H cont. on Y & $\sqrt{-1}\,\Theta_H \geq \omega_Y$.

A_Y ample (as above)
h_Y positive metric.

$\exists m > 0$, s.t. $m\sqrt{-1}\,\Theta_H - 2\sqrt{-1}\,\Theta_{h_Y} \geq 0$.

$\Rightarrow D = -mK_Y - 2A_Y$ has singular metric $h = H^m h_Y^{-2}$
s.t. $\sqrt{-1}\,\Theta_h \geq 0$.

$\Rightarrow \quad Bs(A_Y + D) = Bs(-mK_Y - A_Y) = \phi$

$\subset$

$B_+(-K_Y) \quad \therefore \ -K_Y$ ample.

Rem

By Takayama's method (in [Takayama16])
If f is semistable & $-K_X$ ample (resp. nef)
then $-K_Y$ ample (resp. nef)

f: semistable
$\Leftrightarrow$ $f: X \longrightarrow Y$ + (α - - -)

$$\begin{array}{ccc} X & \longrightarrow & Y \\ \cup & & \cup \\ (U, z_1, \ldots z_n) & \mapsto & (V, w_1, \ldots w_m) \end{array}$$

s.t. $w_i = \prod_{j=1}^{m_i} z_{ij}$