D : divisor on X. (正値性, Positivity)

D ample $\overset{\text{(Kodaira)}}{\Longleftrightarrow}$ $\exists h$ smooth metric on D with positive curvature.

D semiample. $\Leftrightarrow$ $\exists m \in \mathbb{N}_{>0}$, $\forall x \in X$. $\exists s \in H^0(X, mD)$ s.t. $s(x) \neq 0$.

D nef. $\Leftrightarrow$ $\forall C \subset X$ curve., $D \cdot C \geq 0$.

D big $\Leftrightarrow$ $h^0(X, mD) \sim O(m^{\dim X})$ $m \gg 0$

D pseudo-effective (psef) $\Leftrightarrow$ $\forall A$ ample, $\forall m \in \mathbb{N}_{>0}$, $mL + A$ is big.

$$\begin{array}{ccccc} \text{ample.} & \Longrightarrow & \text{semiample.} & \Longrightarrow & \text{nef.} \\ \Downarrow & & & & \Downarrow \\ \text{big} & & \Longrightarrow & & \text{psef} \end{array}$$

Example · $\mathcal{O}_{\mathbb{P}^n}(1)$ ample.

Hirzebruch Surface

· $e \in \mathbb{N}_{\geq 0}$. $E_e = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)$, $F_e = \mathbb{P}(E_e)$

$$D = -K_{F_e/\mathbb{P}^1} = -(K_{F_e} - \pi^* K_{\mathbb{P}^1}) \quad (\pi : F_e \to \mathbb{P}^1)$$

$$h^0(X, mD) = h^0(\mathbb{P}^1, \mathrm{Sym}^m(E_e) \otimes \mathcal{O}_{\mathbb{P}^1}(me))$$

$$= \cdots\cdots = \frac{(me+2)(m+1)}{2}$$

D is big $\Leftrightarrow$ $e > 0$ //

Asymptotic base loci.

D: divisor on X, A: ample on X.

- Base locus.

$$\mathrm{Bs}(D) := \{x \in X \mid \forall s \in H^0(X, D),\ s(x) = 0\}.$$

- Stable base locus.

$$B(D) := \bigcap_{m \in \mathbb{N}_{>0}} \mathrm{Bs}(mD).$$

- Augumented base locus.

$$B_+(D) := \bigcap_{m \in \mathbb{N}_{>0}} B(mD - A)$$

- Restricted base locus.

$$B_-(D) := \bigcup_{m \in \mathbb{N}_{>0}} B(mD + A)$$

A=ampleのとり方によらない!

<u>Property</u>. • $B_-(D) \subset B(D) \subset B_+(D) \subset X$.

$$\begin{tikzcd}
 & \boxed{D \text{ semiample} \Leftrightarrow B(D) = \phi} \arrow[dr] & \\
\boxed{D \text{ ample} \Leftrightarrow B_+(D) = \phi} \arrow[ur] \arrow[rr] \arrow[d] & & \boxed{D \text{ nef} \Leftrightarrow B_-(D) = \phi} \arrow[d] \\
\boxed{D \text{ big} \Leftrightarrow B_+(D) \neq X} \arrow[rr] & & \boxed{D \text{ psef} \Leftrightarrow B_-(D) \neq X}
\end{tikzcd}$$

Thm (Cao19) [Ohsawa-Takegoshi L^2 extension] (relative version).

$f: X \to Y$ surj-morphism with connected fiber between sm proj varieties.

$\forall y \in V$ (regular point), $\exists A_Y$ ample divisor, 次をみたす。

(次)

$\forall m \in \mathbb{N}_{>0}$, $\forall L$: divisor.

$\forall h$ singular Hermitian metric on L.

s.t. ① $\sqrt{-1}\,\Theta_h \geq 0$ (in the sense of current)

② F = general fiber of f, $\mathcal{J}(h|_F^{\frac{1}{m}}) = \mathcal{O}_F$.

(multiplier ideal sheaf)

③ $\mathcal{J}(h|_{X_y}^{\frac{1}{m}}) = \mathcal{O}_{X_y}$ $\quad (X_y := f^{-1}(y))$

このとき制限写像

$$H^0(X, mK_{X/Y} + L + f^*A_Y) \to H^0(X_y, (mK_{X/Y} + L + f^*A_Y)|_{X_y})$$

は全射である。

Start

$f(B_-(-K_{X/Y})) \neq Y$.

↓ Cao-Höring

$\exists A$ ample on X

$\tilde{A} = A - \frac{1}{\mathrm{rk}(f_*A)} f^*(\det f_*A)$

is Cartier divisor.

↓ Cao (Viehweg's technique, O Text)

$\tilde{A}$ psef & f ample & $c_1 = 0$

Lu-Tu-Zhang-Zhang, Patakfalvi-Zdanowicz (use m-Bergman metric (Cao-Păun))

- f is flat. (fiber same dim)
- f is semistable. ($\forall y \in Y$, $f^{-1}(y)$ is reduced)

↓ Păun-Takayama

$\forall m, p \in \mathbb{N}_{\geq 0}$.
$f_*(-mK_{X/Y} + p\tilde{A})$ is weakly positively curved (psef) & $c_1 = 0$

↓ Campana-Cao-Matsumura.

$\forall m, p \in \mathbb{N}_{\geq 0}$, $f_*(-mK_{X/Y} + p\tilde{A})$ is numerically flat vector bundle (nef & $c_1 = 0$)

($m = 0$) ↓ Cao, Campana-Cao-Matsumura.

(f ample)

f is locally trivial.

($p = 1$) ↓ Campana-Cao-Matsumura (CM regularity)

$\exists B$ ample, $\forall m \in \mathbb{N}_{\geq 0}$ $f_*(-mK_{X/Y} + \tilde{A}) \otimes B$ is globally generated

↓

$-mK_{X/Y} + \tilde{A} + f^*B$ is globally generated

$-K_{X/Y}$ is nef

↓ $m \to +\infty$

Goal

$-K_{X/Y}$ is nef.

Start

$f(B_-(-K_{X/Y})) \neq Y$.

$\downarrow$ Cao-Höring

$\exists A$ ample on X

$\tilde{A} = A - \frac{1}{rk(f_*A)} f^*(\det f_*A)$

is Cartier divisor.

Lu-Tu-Zhang-Zhang
Patakfalvi-Zdanowicz
(Use m-Bergman metric (Cao-Păun))

$\rightarrow$

- f is flat. (fiber same dim)
- f is semistable. ($\forall y \in Y$, $f^{-1}(y)$ is reduced)

$\downarrow$ Cao (Viehweg's technique, O Text)

$\tilde{A}$ psef & f ample & $c_1=0$

$\downarrow$ Păun-Takayama

$\forall m,p \in \mathbb{N}_{\geq 0}$.
$f_*(-mK_{X/Y}+p\tilde{A})$ is weakly positively curved (psef) & $c_1=0$

$\downarrow$ Campana-Cao-Matsumura.

$\forall m,p\in\mathbb{N}_{\geq 0}$, $f_*(-mK_{X/Y}+p\tilde{A})$ is numerically flat vector bundle.
(nef & $c_1=0$)

($m=0$) $\downarrow$ Cao, Campana-Cao-Matsumura.

(f ample) f is locally trivial.

($p=1$) $\downarrow$ Campana-Cao-Matsumura (CM regularity)

$\exists B$ ample, $\forall m\in\mathbb{N}_{\geq 0}$, $f_*(-mK_{X/Y}+\tilde{A})\otimes B$ is globally generated

$\downarrow$

$-mK_{X/Y}+\tilde{A}+f^*B$ is globally generated

$-K_{X_y}$ is nef

$\downarrow$ $m\to+\infty$

Goal

$-K_{X/Y}$ is nef.

Start

$\boxed{f(B_-(-K_{X/Y})) \neq Y.}$

Lu-Tu-Zhang-Zhang, Patakfalvi-Zdanowicz (Use m-Bergman metric (Cao-Păun))

- f is flat. (fiber same dim)
- f is semistable. ($\forall y \in Y$, $f^{-1}(y)$ is reduced)

$\downarrow$ Cao-Höring

$\exists A$ ample on X
$\hat{A} = A - \frac{1}{rk(f_*A)} f^*(\det f_*A)$
is Cartier divisor.

$\downarrow$ Cao (Viehweg's technique, O Text)

$\hat{A}$ psef & f ample & $c_1=0$

$\downarrow$ Păun-Takayama

$\forall m, p \in \mathbb{N}_{\geq 0}$.
$f_*(-mK_{X/Y} + p\hat{A})$ is weakly positively curved (psef) & $c_1 = 0$

$\downarrow$ Campana-Cao-Matsumura.

$\forall m, p \in \mathbb{N}_{\geq 0}$, $f_*(-mK_{X/Y} + p\hat{A})$ is numerically flat vector bundle. (nef & $c_1=0$)

($m=0$) $\downarrow$ Cao, Campana-Cao-Matsumura.

(f ample) f is locally trivial.

($p=1$) $\downarrow$ Campana-Cao-Matsumura (CM regularity)

$\exists B$ ample, $\forall m \in \mathbb{N}_{\geq 0}$ $f_*(-mK_{X/Y} + \hat{A}) \otimes B$ is globally generated

$\downarrow$

$-mK_{X/Y} + \hat{A} + f^*B$ is globally generated

$-K_{X_y}$ is nef

$\downarrow$ $m \to +\infty$

Goal

$-K_{X/Y}$ is nef.