$\boxed{\text{難しい}}$ Campana のとこが わかる

$K_X = (\det \Omega_X)$ $\qquad \Omega_X$ holomorphic cotangent bundle.

X の構造を調べたい

これらの関係は?

K_X の positivity —— Ω_X の positivity

弱い $\qquad\qquad\qquad\qquad$ 強い

general type

K_X big $\Longleftarrow$ Ω_X big

Ω_X ample.

$\overset{??}{\longleftrightarrow}$ hyperbolicity

$\Longrightarrow$??

K_X の positivity から Ω_X の positivity は言えるか?

(一般の Vector Bundle だと言えない)

($X = \mathbb{P}^1$ $\quad E = \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

$\det E$ ample. $\quad E$ psef (nef) でない)

Thm Campana-Păun '15

K_X psef, X sm proj n-dim var$/\mathbb{C}$

$\Rightarrow$ ${}^\forall m \in \mathbb{N}_{>0}$, ${}^\forall Q$ torsion-free

s.t. $(\Omega_X^1)^{\otimes m} \longrightarrow Q \longrightarrow 0$ とする

$\det Q$ is psef

どうか

Thm (Campana-Păun)

X sm proj, K_X psef.

$\Rightarrow$ ${}^\forall m \in \mathbb{N}$, $(\Omega_X^1)^{\otimes m} \longrightarrow {}^\forall Q \to 0$

Q: torsion free,

$\det Q$ is psef

Miyaoka 87.

K_X psef $\Rightarrow$ Ω_X^1 generically nef

(i.e. $A_1, \ldots, A_{n-1}$ very ample.
$C = A_1 \cap \cdots \cap A_{n-1}$ について
$\Omega_X^1|_C$ nef)

CP15 $\Rightarrow$ Miyaoka 87.

(そんなに自明ではない…)

Foliation をこいぎね

したいこと

- Foliation のりろんをせいりする

(algebraic foliation)

- Slope をつかった Foliation のとくちょうづけ

Notation

- X: smooth proj n-dim variety$/\mathbb{C}$
- $\mathcal{F}, \mathcal{G}$ torsion free sheaf とする

$X_{\mathcal{F}}$: maximal Zariski open set s.t. $\mathcal{F}$ is localy free

(とくに $\operatorname{codim}(X \setminus X_{\mathcal{F}}) \geq 2$)

$$\mathcal{F} \boxtimes \mathcal{G} := (\mathcal{F} \otimes \mathcal{G})^{\vee\vee}$$

($\vee$ は dual をいみする)

$$S^{[m]}\mathcal{F} = (\mathrm{Sym}^m \mathcal{F})^{\vee\vee}, \quad \bigwedge^{[m]} \mathcal{F} = (\bigwedge^m \mathcal{F})^{\vee\vee} \text{ とする}$$

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf.

$\mathcal{F} \subset T_X$ is foliation

$\overset{\text{def}}{\Longleftrightarrow}$ ① $\mathcal{F}$ saturated (i.e. $T_X/\mathcal{F}$ is torsion-free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Liebracket でとじている)

Notation

X: sm proj $n$dim variety $/\mathbb{C}$.

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf.

$\mathcal{F} \subseteq T_X$ (singular) Foliation

$\overset{\text{def}}{\Longleftrightarrow}$ ① $\mathcal{F}$: saturated ($T_X/\mathcal{F}$ torsion free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Liebracket でとじている)

$$\left(\left[a_i \frac{\partial}{\partial z^i}, b_j \frac{\partial}{\partial z^j}\right] = a_i \frac{\partial b_j}{\partial z^i} \frac{\partial}{\partial z^j} - b_j \frac{\partial a_i}{\partial z^j} \frac{\partial}{\partial z^i}\right)$$

$X_{\mathcal{F}} :=$ maximal Zariski open set s.t. $\mathcal{F}$ is localy free.

(とくにことわらなければ $r = \operatorname{rk} \mathcal{F}$, $q := n - r$ とする.)

$L \subset X_{\mathcal{F}}$ leaf

$\Leftrightarrow$ L is maximal connected locally closed submfd $L \subseteq X_{\mathcal{F}}$ & $T_L = \mathcal{F}|_L$.

$L \subset X_{\mathcal{F}}$ leaf

$\overset{\text{def}}{\iff}$ $T_L = \mathcal{F}|_L$ なる locally closed connected submfd で極大のもの.

Notation

X: sm proj variety $/\mathbb{C}$ (n dim)

Def 4.1 $\mathcal{F}$: torsion-free coherent sheaf

$\mathcal{F} \subseteq T_X$ (singular) Foliation

$\overset{\text{def}}{\iff}$ ① $\mathcal{F}$: saturated ($T_X/\mathcal{F}$ torsion free)

② $[\mathcal{F}, \mathcal{F}] \subset T_X$ (Lie bracket で閉じている)

$\left(\left[a_i \frac{\partial}{\partial z^i}, b_j \frac{\partial}{\partial z^j}\right] = a_i \frac{\partial b_j}{\partial z^i} \frac{\partial}{\partial z^j} - b_j \frac{\partial a_i}{\partial z^j} \frac{\partial}{\partial z^i} \right)$

$X_{\mathcal{F}} :=$ maximal Zariski openset s.t. $\mathcal{F}$ is localy free.

(とくにことわらなければ $r = \mathrm{rk}\,\mathcal{F}$, $q := n - r$ とする.)

$L \subset X_{\mathcal{F}}$ leaf

$\iff$ L is maximal connected locally closed submfd $L \subseteq X_{\mathcal{F}}$ & $T_L = \mathcal{F}|_L$.

Rem 4.2 $[\ ,\ ] : T_X \times T_X \longrightarrow T_X$

$\quad a, b \longrightarrow [a, b]$

$\}$ induced

$[\ ,\ ]^{\sim} \ \mathcal{F} \times \mathcal{F} \longrightarrow T_X \longrightarrow T_X/\mathcal{F}$

$\quad (a, b) \longrightarrow [a, b]$

$\}$ induced $\qquad \mathcal{O}_X$ bilinear map

$\pi : \Lambda^2 \mathcal{F}/\mathrm{torsion} \longrightarrow T_X/\mathcal{F}$

$$\left(\because \ [sa, b] = s[a, b] - b(s) a \equiv s[a, b] \pmod{\mathcal{F}}, \quad [a, sb] \equiv -s[a, b] \right)$$

$\mathcal{F}$ is foliation $\iff \pi$ is zero map

$\iff \mathrm{Hom}(\Lambda^2 \mathcal{F}/\mathrm{tor.}, T_X/\mathcal{F}) = 0.$

$\quad \parallel$

$H^0(X, (\Lambda^2 \mathcal{F}/\mathrm{tor})^{\vee} \otimes T_X/\mathcal{F})$

(dualは $\vee$ であらわすことにする)

Rem 4.3

$$[\ ,\ ] : T_X \times T_X \longrightarrow T_X$$
$$(f_1, f_2) \longrightarrow [f_1, f_2]$$

$\}$ induced

$$\pi : \mathcal{F} \times \mathcal{F} \xrightarrow{[\ ,\ ]} T_X \longrightarrow T_X/\mathcal{F}$$
$$(f_1, f_2) \longrightarrow [f_1, f_2]$$

π is $\mathcal{O}_X$-anti bilinear. 交代形式

$$\left([sf_1, f_2] = s[f_1, f_2] - f_2(s) f_1 \equiv s[f_1, f_2] \mod \mathcal{F}. \right.$$

逆も同じ $\left. \right)$

$$\therefore \ \exists \tilde{\pi} : \Lambda^2 \mathcal{F} \longrightarrow T_X/\mathcal{F}.$$

$\mathcal{F}$ is foliation $\iff \tilde{\pi}$ is zero map

$$\Leftarrow \quad \mathrm{Hom}(\Lambda^{[2]} \mathcal{F}, T_X/\mathcal{F}) = 0$$

leaf, 2　(小こやさんの ほうかい くれ(?))

leaf, て…

Thm 4.4 Frobenius について

$\mathcal{F} \subset T_X$ foliation rank r

${}^{\forall} x \in X_{\mathcal{F}}$, $x \in {}^{\exists} U \subset X_{\mathcal{F}}$
Euclid open

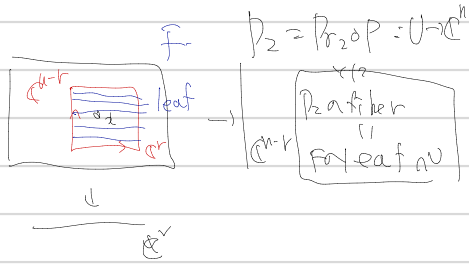
s.t. $P : U \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$.

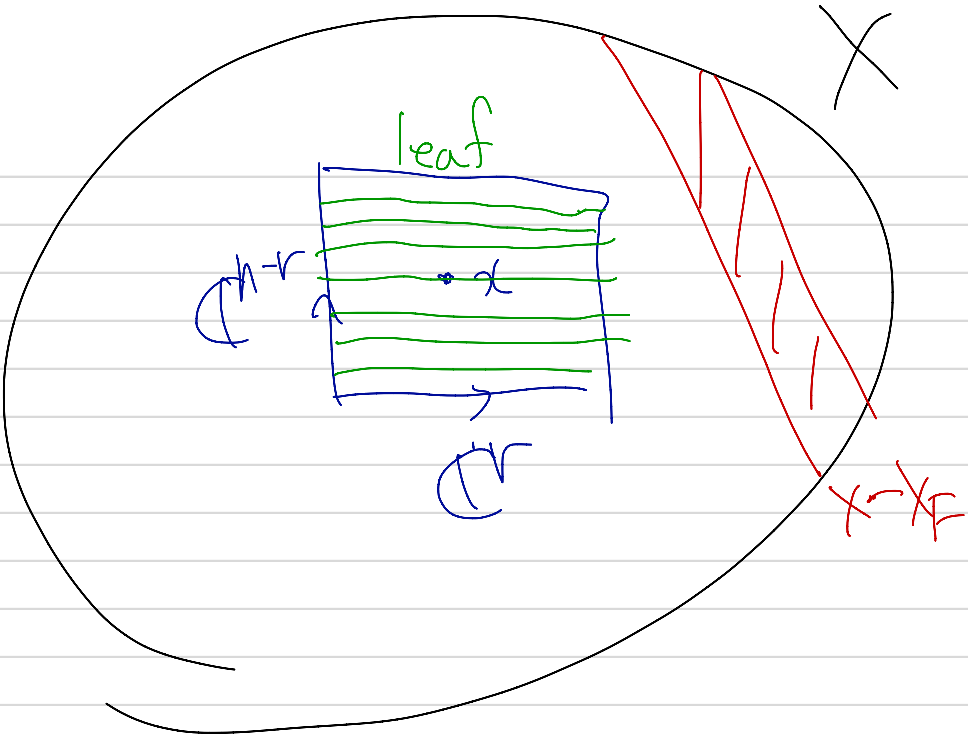
& $P_1 := Pr_1 \circ P : U \overset{P}{\simeq} \mathbb{C}^r \times \mathbb{C}^{n-r} \xrightarrow{Pr_1} \mathbb{C}^r$,
$\mathbb{C}^r$ の座標を $z_1, \dots z_r$ とすると
$P_1^*\left(\frac{\partial}{\partial z^1}\right), \dots P_1^*\left(\frac{\partial}{\partial z^r}\right)$ が $\mathcal{F}|_U$ の basis となる

$P_2 : Pr_2 \circ P : U \overset{P}{\simeq} \mathbb{C}^r \times \mathbb{C}^{n-r} \xrightarrow{Pr_2} \mathbb{C}^{n-r}$. として.
${}^{\forall} y \in U$ について y の $\mathcal{F}$ の leaf を L_y として.
$L_y \cap U = Pr_2^{-1}(Pr_2(y))$

③ Theorem 4.4 Frobenius について
$\mathcal{F} \subset T_X$ foliation $r = \mathrm{rk}\,\mathcal{F}$
${}^{\forall} x \in X_{\mathcal{F}}$. $x \in {}^{\exists} U \subset X_{\mathcal{F}}$
Euclid open.
s.t. $P : U \overset{\text{biholo}}{\simeq} \mathbb{C}^r \times \mathbb{C}^{n-r}$ ($Pr_1 : \mathbb{C}^r \times \mathbb{C}^{n-r} \to \mathbb{C}^r$ 1st proj)
& $P_1 := Pr_1 \circ P : U \to \mathbb{C}^r$.
$z_1, \dots z_r$: $\mathbb{C}^r$ の座標 とすると
$P_1^*(z_1), P_1^*(z_2), \dots P_1^*(z_r)$ が
$\mathcal{F}|_U$ の basis となる
($U \subset X_{\mathcal{F}}$ より $\mathcal{F}|_U$ は vectorbundle)

$P_2 = Pr_2 \circ P : U \to \mathbb{C}^{n-r}$

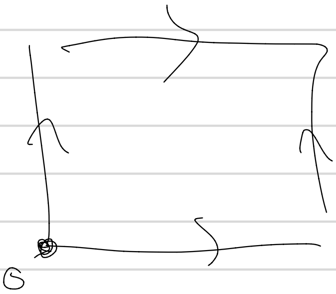



X
leaf
$\mathbb{C}^{n-r}$
x
$\mathbb{C}^r$
$X - X_F$

(Example)

自明な例 実だけど

$M = \mathbb{C}/2\pi\mathbb{Z}\oplus 2\pi\mathbb{Z}[\sqrt{-1}]$

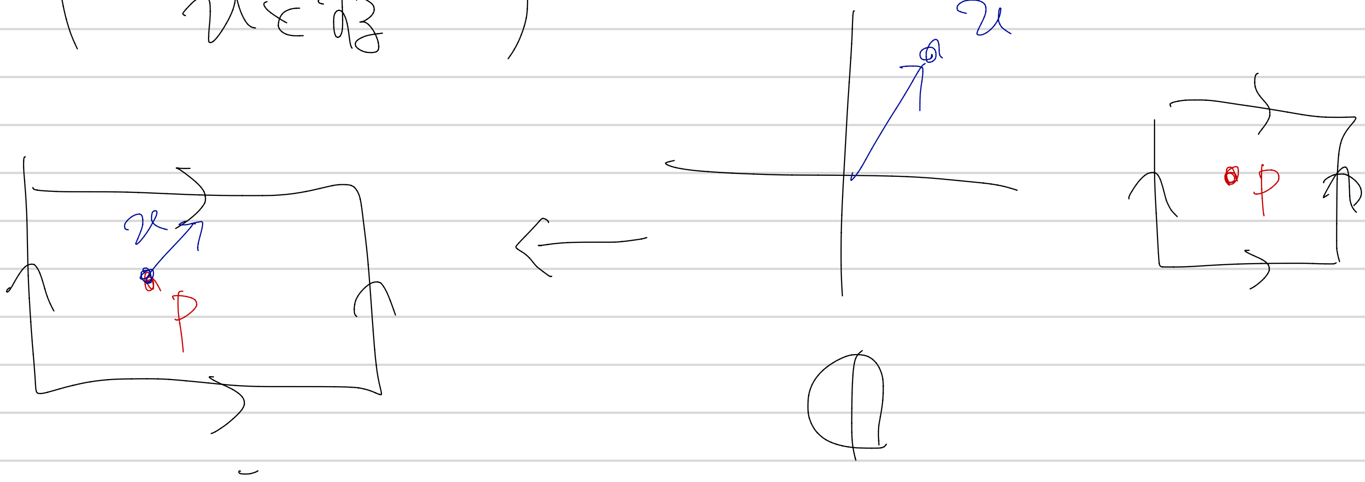


$T^{\mathbb{R}}_M \cong \mathbb{C}\times M$ 自明束。

$(T^{1,0}_M)$

対応は…　$T^{\mathbb{R}}_M \xleftarrow{\sigma} \mathbb{C}\times M$

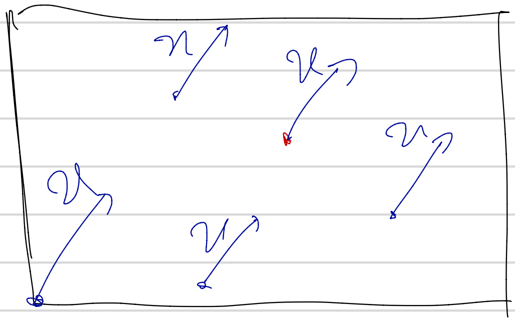
$\left(\begin{array}{c} T^{\mathbb{R}}_{M,p} \text{ での接ベクトルを} \\ v \text{ とする} \end{array}\right) \longleftarrow (v, p)$



この対応を用いて次のfoliationをつくる。

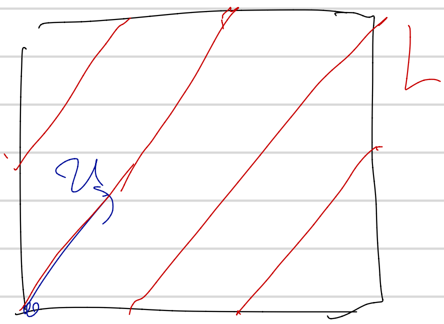
$v = 1 + \alpha\sqrt{-1}$ $(\alpha \in \mathbb{R})$ として

$$\mathcal{F} = \sigma(\mathbb{R}v \times M) \subset TM.$$



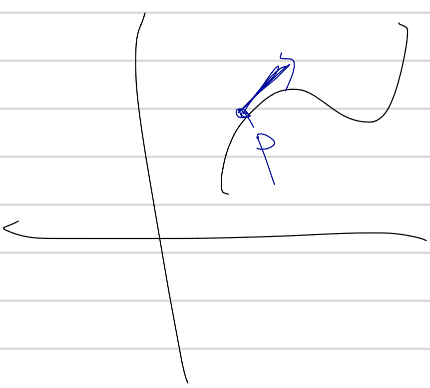
こんなん

0をとおるleaf。 $T_L = \mathcal{F}|_L$ をみたす極大……



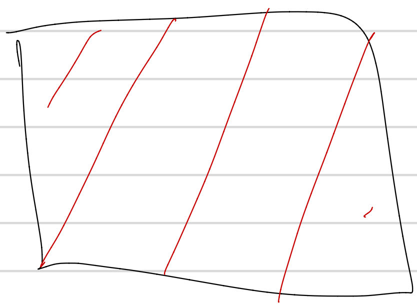
0からv方向上せんをかけば良い。できるわけ。

$$TM|_L = \mathbb{R}v \times L$$



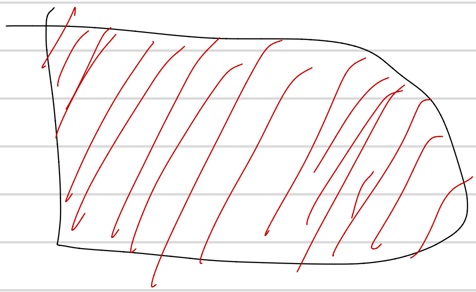
Case ① $\alpha \in \mathbb{Q}$

L leaf $\cong S^1$



Case ② $\alpha \in \mathbb{R} - \mathbb{Q}$

L leaf $\cong \mathbb{R}^1$



Example2 $p: X \dashrightarrow Z$ dominant rational map

$\mathrm{Ker}(dp)$ is foliation

上記で

$\exists X_1 \subset X, \exists Z_1 \subset Z$ Zariski open

$p: X_1 \to Z_1$ $p|_{X_1} =: p_1 : X_1 \to Z_1$ smooth である

$0 \to \mathrm{Ker}(dp_1) \to T_{X_1} \to p_1^* T_{Z_1} \to 0$ on X_1

$\mathrm{Ker}(dp) := \left(\mathrm{Ker}(dp_1)_{sat} \right)^{\vee\vee}$

saturation

$F \subset E$, $F_{sat} = \{ E \subset F_{sat} \subset E$ sat で一番大きいもの $\}$

$\mathrm{rk}\, E = \mathrm{rk}\, F_{sat}$ で

Example X, Y sm prj

$f: X \to Y$ surj morphism

$(r := \dim X - \dim Y)$

$$T_X \xrightarrow{df} f^*T_Y$$ 微分写像.

$(df(p) : T_{X,p} \to f^*T_{Y,p} = T_{Y,f(p)})$

$\mathcal{F} := \ker df$. $\left(df(T_X) \subset f^*T_Y \leftarrow \text{torsion free} \rightsquigarrow \mathcal{F} \text{ saturated} \right)$

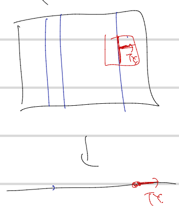
$\left(\ker df \ni f_1, f_2. \quad \text{Lie bracket が閉じる.} \quad df[f_1, f_2] = [df(f_1), df(f_2)] = 0. \right)$

f の general fiber は $\mathcal{F}$ の leaf

Example: $f: X \to Y$. (Y = normal proj.)

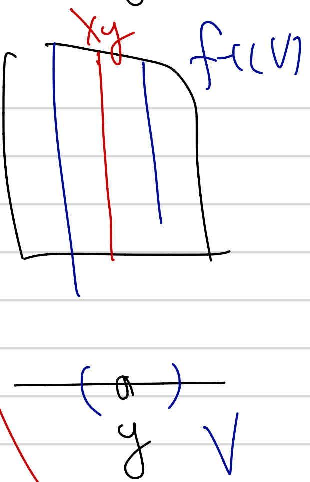
$\mathcal{F} := \ker df$ を, f から定まる foliation という.

$\left(\ker df \to \underbrace{T_X \xrightarrow{df} f^*T_Y}_{\text{sheaf の 射}} \right)$



このとき general fiber. は leaf となる.

∵ $y \in Y$ regular value をとる.

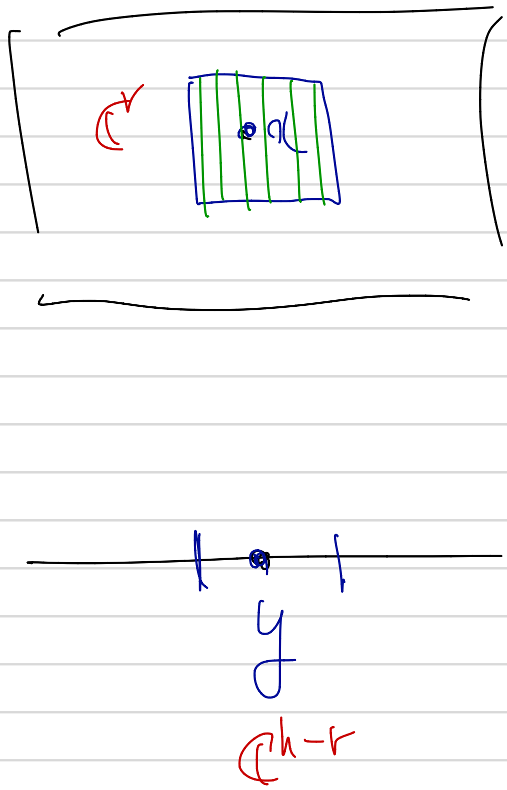


$f^{-1}(V)$ 上で

$$0 \to F \to T_X \xrightarrow{df} f^*T_Y \to 0$$

$$F|_{X_y} = T_{X_y} \quad \text{示せば}$$

∵ $x \in X$ smooth point of f, $y = f(x)$ とすと



$\Rightarrow \exists x \in U \subset X$ $(U, z_1, \dots, z_n)$ 座標

$y \in^{\exists} V \subset Y$ $(V, w_1, \dots, w_{n-r})$

s.t. $f : U \longrightarrow V$

$(z_1, \dots, z_n) \to (z_1, \dots, z_{n-r})$

T_X は $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n}$ で生成

f^*T_Y は $\frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^{n-r}}$ で

∴ $\mathrm{Ker}\, df$ は $\frac{\partial}{\partial z^{n-r+1}}, \dots, \frac{\partial}{\partial z^n}$ で生成.

$T_{X_y} = F|_{X_y}$ on U. $\rightsquigarrow$ 一般に, $T_{X_y} = F|_{X_y}$

極限性は ここからでる.
わからない

Def $\mathcal{F} \subset T_X$ foliation

$\mathcal{F}$ is algebraic foliation

57:

$\overset{(CP\,II)}{\Longleftrightarrow}$ $\boxed{{}^{\exists} f : X \dashrightarrow Z \text{ dominant rational map s.t. } \mathcal{F} = \ker df \text{ generically on } X}$

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists} f : X \dashrightarrow Z$ dominant rational map

${}^{\exists} \pi : \tilde{X} \to X$ birational map, ${}^{\exists} X_0 \subseteq X$ Zariski open. s.t. $\operatorname{codim}(X \setminus X_0) \geq 2$

s.t. $\tilde{f} := f \circ \pi : \tilde{X} \to Z$ is morphism

$\pi^* \mathcal{F} = \ker d\tilde{f}$ on $\pi^{-1}(X_0)$

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[rr, dashed, "f"'] & & Z \end{tikzcd}$$

Def [illegible] $\mathcal{F} \subset T_X$: Foliation

$\mathcal{F}$ is algebraic.

$\overset{\text{def}}{\Longleftrightarrow}$ ${}^{\exists} Z$ sm proj var.,

${}^{\exists} \pi : \tilde{X} \to X$ birational map

${}^{\exists} f : \tilde{X} \to Z$ surjective morphism

s.t. $\pi^* \mathcal{F} = \ker df$ on some nonempty Zariski open set.

codimension 2 をつけて成立

$$\begin{tikzcd} \tilde{X} \arrow[r, "f"] \arrow[d, "\pi"] & Z \\ X & \end{tikzcd}$$

つまりいうと

algebraic foliation は

ほぼ"写像でつくられたもの"

↳ うれしい

$\boxed{\text{Caution}}$ Lazić の定義とはちょっとちがう

$\mathcal{F} \subset T_X$ foliation L: $\mathcal{F}$ a leaf. ($L \subset X$)

L is algebraic $\overset{\text{def}}{\iff}$ $\begin{cases} L \text{ is Zariski open on } \overline{L}^{\text{Zar.}} \\ \dim L = \dim \overline{L}^{\text{Zar.}} \end{cases}$

$\mathcal{F}$ is algebraic foliation

$\overset{\text{def}}{\iff}$ general point $^{\forall}x \in X$, $x \in {}^{\forall}L$ leaf について

L is algebraic.

Lem 4.12. Laziの定義とさっきの定義は一致する

「Sketch pf

$$\underbrace{\mathrm{Chow}_{r,\delta}(X) = \left\{ \begin{array}{l} r\text{dim subvariety of } X \\ \text{degree } \delta \end{array} \right\}}_{\text{Chow variety}}$$

$$\begin{array}{rcl} p : X_0 & \dashrightarrow & \bigcup_{\delta \geq 0} \mathrm{Chow}_{r,\delta}(X) \\ x & \longmapsto & [x\text{を通る}\ \mathcal{F}\ \text{の leaf}] \end{array}$$

$$W = \overline{p(X_0)}^{\mathrm{Zar}}.$$

$$U = \overline{\{ (x,w) \in X_0 \times W \mid p(x) = w \}}^{\mathrm{Zar}} \subset X \times W$$

$\rightsquigarrow$

$$\begin{tikzcd} & U \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X \arrow[rr, dashed, "p"'] & & W \end{tikzcd}$$

$$\boxed{\begin{array}{l}\text{general pt を通る} \\ \quad pr_1^* \mathcal{F} \text{ の leaf} = pr_2 \text{ の general fiber}\end{array}}$$

」

Caution 今回ここまでは Lazic とはちょっとちがう (Def 4.8)

Lazic Def 4.8 $\mathcal{F} \subset T_X$ algebraic

$\Leftrightarrow$ $^\forall x \in X$ general point

$x \in {}^\forall L$ leaf,

s.t. L is Zariski open in $\overline{L}^{Zar}$

(? うつくしいてleafがvariety)

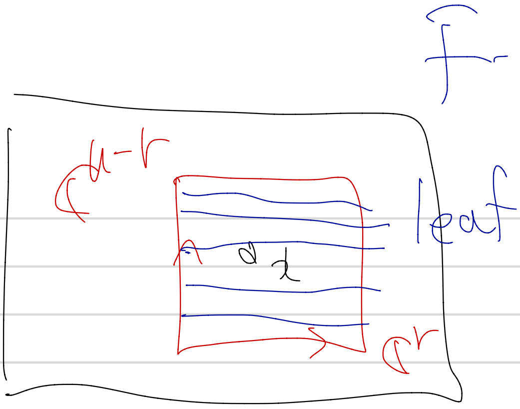
Lazic Lemma 12.

$\mathcal{F}$ algebraic foliation

$\Rightarrow$ (Def (φ) が成立する)

(Chow variety を使うこと)

(ここは省略)



F
leaf

Question $\mathcal{F} \subseteq T_X$ torsion-free

① $\mathcal{F}$ はいつ foliation になるか?

② $\mathcal{F}$ はいつ algebraic foliation になるか

⇓

"slope" をもちいた判定法.

(②については数論的な判定法もあり)

Question. $\mathcal{F} \subseteq T_X$ $\mathcal{F}$: torsion-free

① $\mathcal{F}$ はいつ foliation になるか?

② $\mathcal{F}$ が foliation のとき, $\mathcal{F}$ はいつ algebraic になるか

⇓

一般にはムズかしい.

1つの法として slope を用いた判定法有

(②については数論的な判定法もあるけど…)

6.3. Higher Fujita's decomposition

Definition 6.3.1. [**KM98**] Let X be a smooth projective manifold.

(1) A *1-cycle* is a formal linear combination of irreducible reduced and proper curves $C = \sum a_i C_i$.
(2) Two 1-cycle C, C' is *numerically equivalent* if $D.C = D.C'$ for any Cartier divisor D.
(3) $N_1(X)_{\mathbb{R}}$ is a $\mathbb{R}$-vector space of 1-cycles with real coefficients modulo numerical equivalence.

$\boxed{\text{Slope.}}$

$$N_1(X)_{\mathbb{R}} = \left\{ \sum_{i=1}^{\ell} a_i C_i \;\middle|\; a_i \in \mathbb{R},\ C_i : \text{irr reduced proper curve} \right\}$$

$$Mov(X) = \left\{ \alpha \in N_1(X)_{\mathbb{R}} \;\middle|\; \forall D \text{ eff div},\ \alpha \cdot D \geq 0 \right\}$$

by [BDPP]

① $$SMC(X) = \left\{ \alpha \in N_1(X)_{\mathbb{R}} \;\middle|\; \begin{array}{l} \exists \pi : \tilde{X} \to X \text{ birational morphism} \\ \exists C = A_1 \cap \cdots \cap A_{n-1} \text{ complete intersection} \\ \quad \text{by very ample divisors } A_1 \ldots A_{n-1} \\ \text{s.t. } \alpha = \pi_* C \end{array} \right\}$$

とくに

$$Mov(X) = \overline{SMC(X)} \subset N_1(X)_{\mathbb{R}}$$

② L : line bundle

L is psef $\iff \forall \alpha \in Mov(X),\ \alpha \cdot L \geq 0$

Def 2.8 $\alpha \in \mathrm{Mov}(X)$ をとっておく

$0 \neq \mathcal{E}$ torsion free coh sheaf について

① $\mathcal{E}$ の α による slope $\mu_\alpha(\mathcal{E}) := \dfrac{c_1(\mathcal{E}) \cdot \alpha}{\mathrm{rk}\,\mathcal{E}}$

$c_1(\mathcal{E}) := \left(\bigwedge^{\mathrm{rk}\,\mathcal{E}} \mathcal{E}\right)^{\vee\vee}$

② $\mathcal{E}$ is α semistable (α-s.s.) α-stable

$\iff \forall\, 0 \neq \mathcal{F} \subseteq \mathcal{E},\ \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ $(<)$

③ $\mu_\alpha^{\max}(\mathcal{E}) = \sup\{\mu_\alpha(\mathcal{F}) \mid 0 \neq \mathcal{F} \subset \mathcal{E} \text{ coherent}\}$

$\mu_\alpha^{\min}(\mathcal{E}) = \inf\{\mu_\alpha(\mathcal{Q}) \mid \mathcal{E} \twoheadrightarrow \mathcal{Q} \text{ torsionfree coh}\}$

Def 2.8 $\alpha \in \mathrm{Mov}(X)$ とする. $(\bigwedge^{\mathrm{rk}\,\mathcal{F}})^{\vee\vee}$

$\mathcal{E}$ を torsionfree coh sheaf とする.

① $\mathcal{E}$ の slope w.r.t. α $\overset{\mathrm{def}}{\iff}$ $\mu_\alpha(\mathcal{E}) := \dfrac{c_1(\mathcal{E}) \cdot \alpha}{\mathrm{rk}\,\mathcal{E}}$

② $\mathcal{E}$ is α-semistable (α stable)

$\iff \forall\, 0 \neq \mathcal{F} \subsetneq \mathcal{E}$ coherent (torsionfreeにはなる)

$\mu_\alpha(\mathcal{F}) \leq \mathcal{E}$ $(<)$

③ $\mu_\alpha^{\max}(\mathcal{E}) = \sup\{\mu_\alpha(\mathcal{F}) \mid 0 \neq \mathcal{F} \subset \mathcal{E} \text{ coherent}\}$

$\mu_\alpha^{\min}(\mathcal{E}) = \inf\{\mu_\alpha(\mathcal{Q}) \mid \mathcal{E} \twoheadrightarrow \mathcal{Q} \text{ torsionfree coherent}\}$

Rem: $\mathcal{E}$ α semistable $\iff \mu_\alpha^{\max}(\mathcal{E}) = \mu_\alpha(\mathcal{E})$

$(\Leftarrow$ 落ちて$)$ $\mu_\alpha(\mathcal{E}) = \mu_\alpha^{\min}(\mathcal{E})$

・ $\mathcal{F} \hookrightarrow \mathcal{E}$, $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{E}$

$\Rightarrow \det \mathcal{F} \hookrightarrow \det \mathcal{E}$

とくに saturate とるときに $\Rightarrow \exists D$ eff $\det \mathcal{F} \otimes \mathcal{O}_X(+D) = \det \mathcal{E}$

$\Rightarrow \mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ //

Rem 1

一般の exact seq. $0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{Q} \to 0$ について

$c_1(\mathcal{F}) + c_1(\mathcal{Q}) = c_1(\mathcal{E})$

$\therefore \boxed{\mathrm{rk}\,\mathcal{F}\,\mu_\alpha(\mathcal{F}) + \mathrm{rk}\,\mathcal{Q}\,\mu_\alpha(\mathcal{Q}) = \mathrm{rk}\,\mathcal{E}\,\mu_\alpha(\mathcal{E})}$

よって $\mathcal{E}$ - αss $\iff \mu_\alpha^{\max}(\mathcal{E}) = \mu_\alpha(\mathcal{E}) \iff \mu_\alpha^{\min}(\mathcal{E}) = \mu_\alpha(\mathcal{E})$

Rem 2 $\mathcal{F} \subset \mathcal{E}$ & $\mathrm{rk}\,\mathcal{F} = \mathrm{rk}\,\mathcal{E}$

$\Rightarrow$ $\det \mathcal{F} \hookrightarrow \det \mathcal{E}$ より
inclusion
$\exists D$ eff. $\det \mathcal{F} \otimes \mathcal{O}_X(-D) = \det \mathcal{E}$.

$\Rightarrow$ $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$.

とくに $\mathcal{E}$ locally free として $\forall \mathcal{F} \subset \mathcal{E}$ coherent subsheaf として

$\mathcal{E} \supset \mathcal{F}^{sat}$: $\mathcal{F}$ の saturation ($\mathrm{rk}\,\mathcal{G} = \mathrm{rk}\,\mathcal{F}$ かつ $\mathcal{F} \subset \mathcal{G}$ で saturated sheaf のなかで一番大きいもの.)

とおくと $\mathcal{F} \subset \mathcal{F}^{sat}$ より $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{F}^{sat})$

Rem 3 $\mathcal{F}, \mathcal{G}$ torsion free coherent.

$$\mu_\alpha^{max}(\mathcal{F} \boxtimes \mathcal{G}) = \mu_\alpha^{max}(\mathcal{F}) + \mu_\alpha^{max}(\mathcal{G})$$

(これはかなり難しい)

($\mathcal{F}, \mathcal{G}$ αss $\Rightarrow$ $\mathcal{F} \boxtimes \mathcal{G}$ αss を示すとよい...)

とくに $\mu_\alpha^{max}(\mathrm{Sym}^{[m]} \mathcal{F}) = m\,\mu_\alpha^{max}(\mathcal{F})$

$\mu_\alpha^{max}(\Lambda^{[m]} \mathcal{F}) = m\,\mu_\alpha^{max}(\mathcal{F})$

Lem (Lazic Lem 2.9, 2.10) $\mathcal{E}$ torsion free coherent

① $\mu_\alpha^{max}(\mathcal{E}) = -\mu_\alpha^{min}(\mathcal{E}^\vee)$

② $\mathcal{F}$: α ss とする

$0 \neq {}^\forall r : \mathcal{F} \to \mathcal{E}$, $\mu_\alpha(r(\mathcal{F})) \geq \mu_\alpha(\mathcal{F})$

とくに $\mu_\alpha(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow Hom(\mathcal{F}, \mathcal{E}) = 0$

③ $\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow Hom(\mathcal{F}, \mathcal{E}) = 0$

④ $\mathcal{F} \subset \mathcal{E}$ $\quad \mathcal{F}$ saturated & $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$

$\Rightarrow \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \leq \mu_\alpha(\mathcal{F})$ & $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

⑤ $\mathcal{E} = \mathcal{F} \oplus \mathcal{G}$ とすれば

$\mathcal{E} = \alpha$-s.s $\Leftarrow\Rightarrow$ $\mathcal{F}, \mathcal{G}$ α-s.s. & $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$

とくに L line budle について $L^{\oplus N}$ is α-ss ${}^\forall N \in \mathbb{N}_{>0}$

(Lazic Prop2.9, Lem2.10)

Lemma

① $\mu_\alpha^{max}(\mathcal{E}) = -\mu_\alpha^{min}(\mathcal{E}^\vee)$

② $\mathcal{F}$: α semistable.

$0 \neq {}^\forall r : \mathcal{F} \to \mathcal{E}$, $\mu_\alpha(r(\mathcal{F})) \geq \mu_\alpha(\mathcal{F})$.

とくに $\mu_\alpha(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow Hom(\mathcal{F}, \mathcal{E}) = 0$

③ $\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(\mathcal{E}) \Rightarrow Hom(\mathcal{F}, \mathcal{E}) = 0$

④ $\mathcal{F} \subset \mathcal{E}$. $\mathcal{F}$ saturated, (①より $\mathcal{E}$ α-semistable $\Leftarrow\Rightarrow$ $\mathcal{E}^\vee$ α semistable)

$\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$

$\Rightarrow \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \leq \mu_\alpha(\mathcal{F})$ &

$\mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

⑤ $\mathcal{E} := \mathcal{F} \oplus \mathcal{G}$ とする

$\mathcal{F}, \mathcal{G}$ α-semistable & $\mu_\alpha(\mathcal{F}) = \mu_\alpha(\mathcal{G})$ $\Leftarrow\Rightarrow$ $\mathcal{E}$ α-semistable

とくに L : line budle とし $L^{\oplus N}$ は α-semistable pf ① $\forall \mathcal{F} \subset \mathcal{E}$ について $\mathcal{E}^\vee \twoheadrightarrow \mathcal{F}^\vee$ より

$$\mu_\alpha(\mathcal{F}) = -\mu_\alpha(\mathcal{F}^\vee) \leq -\mu_\alpha^{\min}(\mathcal{E}^\vee)$$

$\therefore$ $\mu_\alpha^{\max}(\mathcal{E}) \leq -\mu_\alpha^{\min}(\mathcal{E}^\vee)$ を示せた

② $0 \to {}^\exists\mathcal{K} \to \mathcal{F} \to r(\mathcal{F}) \to 0$ より

$$\operatorname{rk}(r(\mathcal{F}))\,\mu_\alpha(\mathcal{F}) \underset{\alpha\text{-ss}}{\leq} \operatorname{rk}\mathcal{F}\,\mu_\alpha(\mathcal{F}) - \operatorname{rk}\mathcal{K}\,\mu_\alpha(\mathcal{K})$$

$$= \operatorname{rk}(r(\mathcal{F}))\,\mu_\alpha(r(\mathcal{F}))$$

$\therefore$ $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F}))$ //

③ ${}^\forall r$ $0 \neq \mathcal{F} \to \mathcal{E}$ について $\mathcal{F} \twoheadrightarrow r(\mathcal{F}) \hookrightarrow \mathcal{E}$ より

$$\mu_\alpha^{\min}(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F})) \leq \mu_\alpha^{\max}(\mathcal{E})$$ より

pf ① $0 \neq {}^\forall\mathcal{F} \subset \mathcal{E}$ について. $\mathcal{E}^\vee \to \mathcal{F}^\vee \to 0$ より

$$\mu_\alpha(\mathcal{F}^\vee) = -\mu_\alpha(\mathcal{F}) \geq -\mu_\alpha^{\max}(\mathcal{E})$$

$\Rightarrow$ $\mu_\alpha^{\min}(\mathcal{E}^\vee) \geq -\mu_\alpha^{\max}(\mathcal{E})$ を示せばよい.

② $0 \to {}^\exists\mathcal{K} \to \mathcal{F} \to r(\mathcal{F}) \to 0$ と

$c_1(\mathcal{F}) - c_1(\mathcal{K}) = c_1(r(\mathcal{F}))$ より

$$\operatorname{rk}\mathcal{F}\,\mu_\alpha(\mathcal{F}) - \operatorname{rk}\mathcal{K}\,\mu_\alpha(\mathcal{K}) = \operatorname{rk}(r(\mathcal{F}))\,\mu_\alpha(r(\mathcal{F}))$$

$\geq$ $\mathcal{F}$: α-semistable.

$\operatorname{rk}(r(\mathcal{F}))\,\mu_\alpha(\mathcal{F})$ $\Rightarrow$ $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F}))$

③ ${}^\forall r$ $0 \neq \mathcal{F} \to \mathcal{E}$ について,

$\mathcal{F} \twoheadrightarrow r(\mathcal{F}) \hookrightarrow \mathcal{E}$ から

$\mu_\alpha^{\min}(\mathcal{F}) \leq \mu_\alpha(r(\mathcal{F})) \leq \mu_\alpha^{\max}(\mathcal{E})$ となる為

④

④ $0 \neq {}^\forall \mathcal{G} \subset \mathcal{E}/\mathcal{F}$ について $\mu_\alpha(\mathcal{G}) \le \underbrace{\mu_\alpha^{max}(\mathcal{E}/\mathcal{F})}_{\le\,\mu_\alpha(\mathcal{F})}$ をいう

$\exists \mathcal{G}' \subset \mathcal{E}$ s.t. $\mathcal{F} \subset \mathcal{G}'$ & $\mathcal{G}'/\mathcal{F} = \mathcal{G}$

$(\mu_\alpha(\mathcal{G}') \le \mu_\alpha(\mathcal{F}))$

$0 \to \mathcal{F} \to \mathcal{G}' \to \mathcal{G} \to 0$ より

$$rk\,\mathcal{G}\,\mu_\alpha(\mathcal{G}) = rk\,\mathcal{G}'\,\mu_\alpha(\mathcal{G}') - rk\,\mathcal{F}\,\mu_\alpha(\mathcal{F})$$
$$\le rk\,\mathcal{G}\,\mu_\alpha(\mathcal{F}) \qquad \therefore \mu_\alpha(\mathcal{G}) \le \mu_\alpha(\mathcal{F}).$$

$\mathcal{F} \twoheadrightarrow {}^\forall \mathcal{Q}$ について $\mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$ をいう

$0 \to {}^\exists \mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$ より

②と同様に $\mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$ //

---

④ $\mathcal{F} \subset \mathcal{E}$, $\mathcal{F}$ saturated, $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$
$\Rightarrow \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \le \mu_\alpha(\mathcal{F})$ & $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

④ of ${}^\forall \mathcal{G} \to \mathcal{E}/\mathcal{F}$ (saturated 故) torsion free

$\Rightarrow \mathcal{G}' \subset \mathcal{E}$ s.t. $\mathcal{F} \subset \mathcal{G}'$ & $\mathcal{G}'/\mathcal{F} = \mathcal{G}$

$\Rightarrow \mu_\alpha(\mathcal{G}') \le \mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{F})$

$0 \to \mathcal{F} \to \mathcal{G}' \to \mathcal{G} \to 0$ $(c_1(\mathcal{G}) = c_1(\mathcal{G}') - c_1(\mathcal{F})$

$\Rightarrow$ (②の逆より) $\mu_\alpha(\mathcal{F}) \ge \mu_\alpha(\mathcal{G}) \Rightarrow \mu_\alpha(\mathcal{F}) \ge \mu_\alpha^{max}(\mathcal{E}/\mathcal{F})$
maxでも

$\mathcal{F} \to {}^\forall \mathcal{Q} \to 0$ について

$0 \to {}^\exists \mathcal{K} \to \mathcal{F} \to \mathcal{Q} \to 0$ $(c_1(\mathcal{Q}) = c_1(\mathcal{F}) - c_1(\mathcal{K})$

$\Rightarrow$ (②の手順) $\mu_\alpha(\mathcal{Q}) \ge \mu_\alpha(\mathcal{F})$

$\Rightarrow \mu_\alpha^{min}(\mathcal{F}) \ge \mu_\alpha(\mathcal{F}) \ge \mu_\alpha^{min}(\mathcal{F})$.
ごくい より

$\mu_\alpha(F) = \mu_\alpha(G)$

⑤ F, G α-semistable & Σ NOT α-semistable

$\Rightarrow \exists\, 0 \neq \Sigma' \subset \Sigma$ & $\mu_\alpha(\Sigma') > \mu_\alpha(\Sigma)$

Σ'-α-semistable $\quad F \oplus G$ $\quad$ ($\mu_\alpha(\Sigma') > \mu_\alpha(\Sigma)$ で Σ' α-semistable なものがとれる)

$\Rightarrow \Sigma' \cap F \neq 0$ として

$\Rightarrow \exists\, r: \Sigma' \to F_{(\text{or } G)}$, $r \neq 0$

$\Rightarrow \mu_\alpha(F) \geq \mu_\alpha(r(\Sigma')) \geq \mu_\alpha(\Sigma') > \mu_\alpha(\Sigma)$

$\quad \| \quad F$: α-semistable $\qquad \Sigma'$: α-semistable

$\mu_\alpha(\Sigma)$ $\quad (\mu_\alpha(F) = \mu_\alpha(G)$ & $C_1(F) + C_1(G) = C_1(\Sigma))$

で矛盾

Σ - α.s.s. $\Rightarrow 0 \neq p_r\ \Sigma \to F$.

$\Rightarrow \mu_\alpha^{max}(\Sigma) \geq \mu_\alpha^{max}(F) \geq \mu_\alpha(\Sigma)$ $\quad (\Sigma$ α.s.s$)$

$\quad \| \quad \mu_\alpha(\Sigma)$

$\Rightarrow \mu_\alpha(\Sigma) = \mu_\alpha^{max}(F) = \mu_\alpha^{max}(G)$

(1) $\mu_\alpha(F) = \mu_\alpha(G)$

$\left(C_1(F) = C_1(\Sigma) - C_1(G) \Rightarrow \text{rk}F\, \mu_\alpha(F) \geq \text{rk}F\, \mu_\alpha^{max}(\Sigma) = \mu_\alpha(\Sigma) \right)$

---

⑤ Σ … α.s.s $\qquad \Sigma = F \oplus G \to F$.

$\Rightarrow 0 \neq p_{r_1}: \Sigma \to F$

by (2) $\Rightarrow \mu_\alpha^{max}(\Sigma) \geq \mu_\alpha^{max}(F) \geq \mu_\alpha(\Sigma)$

$\quad \| \quad \mu_\alpha(\Sigma)$

$\Rightarrow \mu_\alpha(\Sigma) = \mu_\alpha^{max}(F) = \mu_\alpha(\Sigma)$

一方で $C_1(F) = C_1(\Sigma) - C_1(G)$ より $\quad \mu_\alpha(F) \geq \mu_\alpha^{max}(\Sigma)$

F, G α ss & $\mu_\alpha(F) = \mu_\alpha(G)$ として. Σ が NOT α-ss となる

$\mu_\alpha(\Sigma) = \mu_\alpha(F)$

$\exists\, 0 \neq \Sigma' \subset \Sigma$ s.t. $\mu_\alpha(\Sigma') > \mu_\alpha(\Sigma)$

$\Rightarrow \Sigma = F \oplus G$ より $\Sigma \cap F \neq 0$ として

$0 \neq \exists\, p_{r_2}\ \Sigma \to F$.

(2)

$\mu_\alpha(G) = \mu_\alpha(F) \geq \mu_\alpha(r(\Sigma')) \geq \mu_\alpha(\Sigma') > \mu_\alpha(\Sigma)$

□

Cor 2.14 (α-maximal destabilizing sheaf の存在)

$\alpha \in \mathrm{Mov}(X)$ $\mathcal{E}$ torsion-free coh

$\exists\, \mathcal{E}_{max} \subset \mathcal{E}$ α-s.s. saturated

$$\begin{cases} \mu_\alpha(\mathcal{E}_{max}) = \mu_\alpha^{max}(\mathcal{E}) \\ \forall \mathcal{F} \subset \mathcal{E},\ \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathcal{F} \subset \mathcal{E}_{max} \end{cases}$$

$\mathcal{E}_{max}$ α-maximal destabilizing sheaf

(Corollary 2.14) $\alpha \in \mathrm{Mov}(X)$, $\mathcal{E}$ torsion-free coh sheaf

$\exists\, \mathcal{E}_{max} \subset \mathcal{E}$. α-s.s. saturated (⇒reflexive)

- $\mu_\alpha(\mathcal{E}_{max}) = \mu_\alpha^{max}(\mathcal{E})$
- $\forall \mathcal{F} \subset \mathcal{E}$ $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E}) \Rightarrow \mathcal{F} \subset \mathcal{E}_{max}$.

We call $\mathcal{E}_{max}$ by "maximal destabilizing sheaf".

Sketch of ①: $\mu_\alpha^{max}(\mathcal{E})<+\infty$

($\exists N$ $\mathcal{E}\hookrightarrow \mathcal{H}^{\oplus N}$ $\mathcal{H}^{\oplus N}$ semistable より)

② of $\exists \mathcal{F}\subset\mathcal{E}$ s.t. $\mu_\alpha^{max}(\mathcal{E})=\mu_\alpha(\mathcal{F})$

背理法 $\forall \mathcal{F}\subset\mathcal{E}$, $\mu_\alpha(\mathcal{F})<\mu_\alpha^{max}(\mathcal{E})$ とする)

$k=\max\left\{1\le k\le rk\,\mathcal{E} \;\middle|\; \begin{array}{l}\exists\{\mathcal{F}_i\}_{i=1}^{\infty},\ rk\,\mathcal{F}_i=k\\ \mu_\alpha(\mathcal{F}_1)\le\mu_\alpha(\mathcal{F}_2)\le\cdots\\ \mathcal{F}_i \text{ saturated} \quad \lim_{i\to\infty}\mu_\alpha(\mathcal{F}_i)=\mu_\alpha^{max}(\mathcal{E})\\ \mathcal{F}_i\ne\mathcal{F}_j\end{array}\right\}$ ☆

とおく $k<rk\,\mathcal{E}$ である

しかし、$\mathcal{G}_i=\mathcal{F}_{i+1}+\mathcal{F}_i$ とおくと $rk\,\mathcal{G}_i>k$ で

$\{\mathcal{G}_i\}$ が (☆) をみたす

③ $l=\max\left\{1\le l\le rk\,\mathcal{E} \;\middle|\; \exists\mathcal{F}\subset\mathcal{E}\ \mu_\alpha(\mathcal{F})=\mu_\alpha^{max}(\mathcal{E}),\ \mathcal{F}\text{ saturated}\right\}$

とおく。$\mathcal{E}_{max}$ を l をattainする $\mathcal{F}$ とする

背理法

$\exists\mathcal{G}\subset\mathcal{E}$, $\mu_\alpha(\mathcal{G})=\mu_\alpha^{max}(\mathcal{E})$ & $\mathcal{G}\not\subset\mathcal{F}$

($\mathcal{F}$ と $\mathcal{G}$ は saturated とする)

$\Rightarrow rk(\mathcal{G}+\mathcal{F})>l$

(rk $\mathcal{F}$ = rk $\mathcal{G}$ なら saturationの時)

$\left(rk(\mathcal{G}+\mathcal{F})=l \Rightarrow \mathcal{F}_{sat}\overset{\mathcal{F}}{\underset{\shortparallel}{}} \supset\mathcal{G}+\mathcal{F}\supset\mathcal{F}\Rightarrow\mathcal{G}\subset\mathcal{F}\right)$

$$0 \to \mathcal{F} \cap \mathcal{G} \to \mathcal{F} \oplus \mathcal{G} \to \mathcal{F} + \mathcal{G} \to 0$$

$$\operatorname{rk}(\mathcal{F}+\mathcal{G})\,\mu_\alpha(\mathcal{F}+\mathcal{G}) = \operatorname{rk}\mathcal{F}\,\mu_\alpha(\mathcal{F}) + \operatorname{rk}\mathcal{G}\,\mu_\alpha(\mathcal{G})$$

$$- \operatorname{rk}(\mathcal{F}\cap\mathcal{G})\,\mu_\alpha(\mathcal{F}\cap\mathcal{G})$$

$$\geq \operatorname{rk}(\mathcal{F}+\mathcal{G})\,\mu_\alpha^{\max}(\mathcal{E})$$

$$\therefore \quad \mu_\alpha(\mathcal{F}+\mathcal{G}) = \mu_\alpha^{\max}(\mathcal{E})$$

Lem 2.12 $\mathcal{F}_1, \mathcal{F}_2$ saturated. $\mathcal{F}_1 \neq \mathcal{F}_2$.
$r = rk\mathcal{F}_1 = rk\mathcal{F}_2$

$\Rightarrow$ ① $\mathcal{F}_1 \not\subseteq \mathcal{F}_2$ & $\mathcal{F}_2 \not\subseteq \mathcal{F}_1$ & $rk(\mathcal{F}_1 + \mathcal{F}_2) > r$

② if $\mu_\alpha(\mathcal{F}_2) \geq \mu_\alpha^{max}(\mathcal{E}) - \frac{\delta}{2}$ $\exists \delta > 0$

$\Rightarrow \mu_\alpha(\mathcal{F}_1 + \mathcal{F}_2) \geq \mu_\alpha^{max}(\mathcal{E}) - \delta$

(証) ① $\mathcal{F}_1 \subseteq \mathcal{F}_2 \Rightarrow \underbrace{\mathcal{F}_2/\mathcal{F}_1}_{rk\,0} \subseteq \underbrace{\mathcal{E}/\mathcal{F}_1}$ torsion free

$\Rightarrow \mathcal{F}_2/\mathcal{F}_1 = 0 \Rightarrow \mathcal{F}_2 = \mathcal{F}_1$ 矛盾.

$rk(\mathcal{F}_1 + \mathcal{F}_2) = r \Rightarrow \mathcal{F}_2 \neq \mathcal{F}_1 + \mathcal{F}_2$

$\Rightarrow \mathcal{F}_1 + \mathcal{F}_2/\mathcal{F}_2 \hookrightarrow \mathcal{E}/\mathcal{F}$

$\Rightarrow \mathcal{F}_1 + \mathcal{F}_2/\mathcal{F}_2 = 0$ 矛盾

② $0 \to \mathcal{F}_1 \cap \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_2 \to \mathcal{F}_1 + \mathcal{F}_2 \to 0$

$\rightsquigarrow rk(\mathcal{F}_1 + \mathcal{F}_2)\mu_\alpha(\mathcal{F}_1 + \mathcal{F}_2)$

$= r\mu_\alpha(\mathcal{F}_1) + r\mu_\alpha(\mathcal{F}_2) - rk(\mathcal{F}_1 \cap \mathcal{F}_2)\mu_\alpha(\mathcal{F}_1 \cap \mathcal{F}_2)$

$\geq 2r\mu_\alpha^{max}(\mathcal{E}) - r\delta - rk(\mathcal{F}_1 \cap \mathcal{F}_2)\mu_\alpha^{max}(\mathcal{E})$

$= rk(\mathcal{F}_1 + \mathcal{F}_2)\mu_\alpha^{max}(\mathcal{E}) - r\delta$

$\Rightarrow \mu_\alpha(\mathcal{F}_1 + \mathcal{F}_2) \geq \mu_\alpha^{max}(\mathcal{E}) - \frac{r}{rk(\mathcal{F}_1 + \mathcal{F}_2)}\delta$ //

Prop 2.13 $\mathcal{E}$ torsion free $\mathcal{E}\neq 0 \Rightarrow \mu_\alpha^{max}(\mathcal{E})<\infty$

$\exists_1 \mathcal{F}$ coherent (saturated) $\mu_\alpha^{max}(\mathcal{E})=\mu_\alpha(\mathcal{F})$.

~~さらに $\exists \mathcal{F}$ $\mathcal{E}$の stable reflexive sheaf~~

(わかりにくい)

~~s.t. $\mu_\alpha(\mathcal{F})=\mu_\alpha^{max}(\mathcal{E})$~~

Pf) $\mu_\alpha^{max}(\mathcal{E})<\infty$ 示す $\exists \mathcal{H}$ very ample. $\mathcal{E}^\vee\otimes\mathcal{H}$ globally generated

$\Rightarrow$ $\mathcal{O}_X^{\oplus N}\to\mathcal{E}^\vee\otimes\mathcal{H}\to 0$ $\exists N$.

$\Rightarrow$ $(\mathcal{H}^\vee)^{\oplus N}\twoheadrightarrow\mathcal{E}^\vee$

$\Rightarrow$ $\mathcal{E}\subset\mathcal{E}^{\vee\vee}\hookrightarrow\mathcal{H}^{\oplus N}$

$\Rightarrow$ $\mu_\alpha^{max}(\mathcal{E})\leq\mu_\alpha^{max}(\mathcal{E}^{\vee\vee})\leq\mu_\alpha^{max}(\mathcal{H}^{\oplus N})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$ ($\mathcal{H}$ ss)

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \mu_\alpha(\mathcal{H}^{\oplus N})<+\infty$

$\boxed{2}$ 怪しい点 つまり $^\forall F \subsetneq \mathcal{E}$, $\mu_\alpha^{max}(\mathcal{E}) > \mu_\alpha(F)$ とする

$\exists (F_i)$ saturated, $\mu_\alpha(F_i) \uparrow \mu_\alpha^{max}(\mathcal{E})$ としてよい

$rk F_i$ はすべて同じとしてよい

$$\left(\begin{array}{l} rk F_1 < rk F_2 \Rightarrow F_2 \text{を1番目とする} \\ rk F_1 > rk F_2 \Rightarrow F_2' = F_1 + F_2 \text{ とおいて} \end{array} \right.$$

$$r := \max_{1 \le r \le rk \mathcal{E}} \left\{ \exists (F_i)_{i=1}^{\infty} \;\middle|\; \begin{array}{l} F_i \subset \mathcal{E} \ \& \\ rk F_i = r \ \& \\ \mu_\alpha^{max}(\mathcal{E}) = \lim_{i \to \infty} F_i \end{array} \right\}$$

右の集合は存在する

(F_i) saturated なる → rk ごとにふりわける

↓

どれかになるかにが有る

↓

ア列が有る.

0

1 2 3 — $rk \mathcal{E}$

F_i

今 $\{F_i\}_{i=1}^{\infty}$ $\mathrm{rk}\,F_i = r$, $F_i \subset \Sigma$
$\lim_{i\to\infty} \mu_\alpha(F_i) = \mu_\alpha^{max}(\Sigma)$

とする

$\mu_\alpha(F_1) < \mu_\alpha(F_2) < \cdots$ にしてよい . F_i saturated とれるから
(lem)

今 $\delta_n = \frac{1}{n} > 0$ とする.

i_n を $i_n \geq i_{n-1}$ かつ $\mu_\alpha(F_{i_n}) > \mu_\alpha^{max}(\Sigma) - \frac{1}{n}$ となるように
とる.

$$\mathcal{G}_n = F_{i_{n+1}} + F_{i_n} \quad \text{とおくと}$$

Lem2.12 より $F_n \neq F_{n+1}$ より $\mathrm{rk}(\mathcal{G}_n) > r$.

$$\lim \mu_\alpha(\mathcal{G}_n) = \mu_\alpha^{max}(\Sigma).$$

よってすっきりさせると ∃ $\{\mathcal{G}_n\}$, $\mathrm{rk}\,\mathcal{G}_n = \tilde{r} > r$

$\mathcal{G}_n \subset \Sigma$, $\mu_\alpha^{max}(\Sigma) = \lim_{n\to\infty} \mu_\alpha(\mathcal{G}_n)$

となる

これは r のとり方に矛盾.

③ $\mu_\alpha(\mathcal{F}) = \mu_L^{max}(\mathcal{E})$ なる $\mathcal{F}$ を stable reflexive (saturated?) とできる

$\boxed{\text{Pf}}$ $\exists \mathcal{F}_1 \subset \mathcal{E}$ $\mu_\alpha(\mathcal{F}_1) = \mu_\alpha^{max}(\mathcal{E})$ ととれる

$\mathcal{F}_1$ NOT α-stable $\Rightarrow \exists \mathcal{F}_2 \subset \mathcal{F}_1$.

$\mu_\alpha(\mathcal{F}_2) = \mu_\alpha^{max}(\mathcal{E})$.

$\overrightarrow{\text{これを続けていく}}$ (rk $\mathcal{F}$ は有限なので有限 step で止まる

(これが示される)

Pf of Cor 2.14　Prop 2.13 より、$\exists\, F_1 \subset \mathcal{E}$ で

~~α-stable~~ saturated, $\mu_\alpha(F_1) = \mu_\alpha^{max}(\mathcal{E})$ が存在。

ここで $r_1 = \mathrm{rk}\,F_1$ は上の F の中で一番大きいものとする

これが ② を満たす（$F_2 \subset \mathcal{E}$, $\mu_\alpha(F_2) = \mu_\alpha^{max}(\mathcal{E})$

$\Rightarrow F_2 \subset F_1$ をみたすことを示す）

~~と（$F_2 \not\subset F_1$ とする）~~ $\mu_\alpha(F) = \mu_\alpha^{max}(\mathcal{E})$

$\Rightarrow \quad \widetilde{F} := F_1 + F_2$

$$\mathrm{rk}\,\widetilde{F}\,\mu_\alpha(\widetilde{F}) = \mathrm{rk}F_1\,\mu_\alpha(F_1) + \mathrm{rk}F_2\,\mu_\alpha(F_2) - \mathrm{rk}(F_1 \cap F)\,\mu_\alpha(F_1 \cap F)$$

$$\geq (\mathrm{rk}F_1 + \mathrm{rk}F_2 - \mathrm{rk}(F_1 \cap F))\,\mu_\alpha^{max}(\mathcal{E})$$

$$\therefore \quad \mu_\alpha(\widetilde{F}) = \mu_\alpha^{max}(\mathcal{E})$$

$\Rightarrow \quad F_1 \subset \widetilde{F}\mathrm{sat}$ かつ $\mathrm{rk}\,\widetilde{F}\mathrm{sat} \overset{\text{なので}}{=} r_1$

$\Rightarrow \quad F_1 = \widetilde{F}\mathrm{sat} \supset F_2$ //

F_1 : saturated

Cor 2.10 (uni) HN-filtration $\alpha \in \mathrm{Mov}(X)$

$\mathcal{E}$: torsion free coh

$\Rightarrow \mathcal{E}_0 = 0 \subsetneq \mathcal{E}_1 \subset \cdots \subseteq \mathcal{E}_r = \mathcal{E}$

s.t. $\begin{cases} Q_i = \mathcal{E}_i / \mathcal{E}_{i-1} \ \alpha\text{-s.s. torsion free} \\ \mu_\alpha(Q_1) > \mu_\alpha(Q_2) > \cdots > \mu_\alpha(Q_r) \end{cases}$

(pf) $r k \mathcal{E}$ に対する帰納法 でok.

$\mathcal{E}_1$: maximal destabilizing sheaf of $\mathcal{E}$.

$\Rightarrow 0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \cdots \subsetneq \mathcal{G}_{r-1} = \mathcal{E}/\mathcal{E}_1$

$\Rightarrow \exists \mathcal{E}_{i+1} \subset \mathcal{E}$ s.t. $\mathcal{E}_{i+1}/\mathcal{E}_1 = \mathcal{G}_i$ とすると

$0 = \mathcal{E}_0 = \mathcal{E}_1 \subsetneq \mathcal{E}_2 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$.

$\mu_\alpha(\mathcal{E}_2/\mathcal{E}_1) > \mu_\alpha(\mathcal{E}_3/\mathcal{E}_2) > \cdots$

よって $\mu_\alpha(\mathcal{E}_1) > \mu_\alpha(\mathcal{E}_2/\mathcal{E}_1)$ を示せばよい。

$0 \to \mathcal{E}_1 \to \mathcal{E}_2 \to \mathcal{E}_2/\mathcal{E}_1 \to 0$ があるので)

$(\mu_\alpha(\mathcal{E}_1) = \mu_\alpha^{max}(\mathcal{E}_2)$ から

Uniqueness $\Sigma_\bullet$ $\Sigma'_\bullet$ が filtration であるとする

$$\mu_\alpha(\Sigma_1) \le \mu_\alpha(\Sigma_1') \quad \&$$

$\exists j$, $\Sigma_1' \subset \Sigma_j$ としてよい

(jはその最小とする)

$$\Rightarrow \underbrace{\Sigma_1'}_{\alpha ss} \to \Sigma_j \to \Sigma_j/\Sigma_{j-1} \text{ is Non zero map}$$

$$\Rightarrow \mu_\alpha^{max}(\Sigma_j/\Sigma_{j-1}) \underset{\alpha ss}{=} \mu_\alpha(\Sigma_j/\Sigma_{j-1})$$

$$\ge \mu_\alpha(\Sigma_1') \ge \mu_\alpha(\Sigma_1)$$

よって、 $\mu_\alpha(\Sigma_1) > \mu_\alpha(\Sigma_j/\Sigma_{j-1})$ ($j \ne 1$ なら)

より $j = 1$, $\Sigma_1' = \Sigma_1$

同様にして、 Σ/Σ_1 (Σ'/Σ_1) として行けば

uniqueness を示せる)

Cor 2.16 Jordan-Hölder filtration

$\mathcal{E}$ - α s.s. torsion free coh sheaf

$\Rightarrow 0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$

s.t. $Q_i := \mathcal{E}_i / \mathcal{E}_{i-1}$ torsion free

α stable. $\mu_\alpha(Q_i) = \mu_\alpha(\mathcal{E})$

$= \mu_\alpha^{max}(\mathcal{E})$ (α s.s.)

$\boxed{\text{pf}}$ Prop 2.13 より $\exists\, \mathcal{E}_1 \subset \mathcal{E}$. α stable, saturated. が

$\mu_\alpha(\mathcal{E}_1) = \mu_\alpha^{max}(\mathcal{E})$ が有る

$\Rightarrow 0 = \mathcal{G}_0 \subsetneq \mathcal{G}_1 \subsetneq \cdots \subsetneq \mathcal{E}/\mathcal{E}_1$ JH filtr.

$\Rightarrow$ $\mathcal{E}_{i+1}$ を $\mathcal{G}_i$ の preimage とすればよい

Rem HN + JN

$\Rightarrow \forall \mathcal{E}$ に対し

$\exists\ 0 = \mathcal{E}_0 \subsetneq \mathcal{E}_1 \subsetneq \cdots \subsetneq \mathcal{E}_r = \mathcal{E}$,

$Q_i = \mathcal{E}_i/\mathcal{E}_{i-1}$ torsion free α stable

& $\mu_\alpha(Q_1) \geq \mu_\alpha(Q_2) \geq \cdots\cdots$

Thm 2.24, Cor 2.28 F, g torsion free coherent sheaf

① $\mu_\alpha^{max}(F \boxtimes g) = \mu_\alpha^{max}(F) + \mu_\alpha^{max}(g)$

$(F \boxtimes g := (F \otimes g)^{\vee\vee}$

$F^{\boxtimes m}$ も同様

② $\mu_\alpha^{max}((\mathrm{Sym}^a F)^{\vee\vee}) = a\,\mu_\alpha^{max}(F)$

③ $\mu_\alpha^{max}((\Lambda^a F)^{\vee\vee}) = a\,\mu_\alpha^{max}(F)$.

Pf

略してよい

② $F^{\otimes q_n} = \mathrm{Sym}^{q_n} F \otimes F^{q_n} / \mathrm{Sym}^{q_n} F$

(관찰이)

~~$F^{\otimes q_n} = \Lambda^{q_n} F \otimes F^{q_n} / \Lambda^{q_n} F$~~

on sum Zariski openset (codim 2 outside)

$\Rightarrow F^{\otimes q_n} = \mathrm{Sym}^{[q]} F \otimes \left(F^{\otimes q_n} / \mathrm{Sym}^{q} F \right)^{\vee\vee}$

~~$= \Lambda^{[q]} F \otimes \left(F^{q} / \Lambda^{q} F \right)^{\vee\vee}$~~

$\Rightarrow \mu_\alpha^{max}(\mathrm{Sym}^{[q]} F) \leq \mu_\alpha^{max}(F^{\otimes q_n})$

~~$(\Lambda^{[q]} F)$~~ $= q_n \mu_\alpha^{max}(F)$

$A \subset F$ を α-maximal destabilizing sheaf とする

$$c_1(\mathrm{Sym}^{[a]} A) = \binom{r+a-1}{a-1} c_1(A)$$

$$\mathrm{rk}(\mathrm{Sym}^{[a]} A) = \binom{r+a-1}{a}$$

$$\mu_\alpha(\mathrm{Sym}^{[a]} A) = a\mu_\alpha(A) \quad //$$

従って

$$\mathrm{Sym}^{[a]} A \subseteq \mathrm{Sym}^{[a]} F \text{ より}$$

$$a\mu_\alpha^{max}(F) \geq \mu_\alpha^{max}(\mathrm{Sym}^{[a]} F) \geq \mu_\alpha(\mathrm{Sym}^{[a]} A)$$

$$= a\,\mu_\alpha(A) = a\mu_\alpha^{max}(F)$$

Problem ① の問い $\mathcal{F} \subset T_X$ saturated はいつ foliation になるか

1つの Answer

$$\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F})$$
$$\Rightarrow \mathcal{F} \text{ foliation}$$

とくに $\mathcal{F} \subset T_X$ s.t. $\mu_\alpha^{max}(T_X) = \mu_\alpha(\mathcal{F})$ とする
$\mu_\alpha^{max}(T_X) > 0 \Rightarrow \mathcal{F}$ foliation

つまり $\mu_\alpha^{max}(T_X) > 0$ なら α-maximal destabilizing sheaf T_X^{max} は foliation

Problem 1

$\mathcal{F} \subset T_X$ saturated

$\mathcal{F}$ はいつ foliation になるか

1つの answer

$$\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F})$$

$$\Rightarrow \mathcal{F} : \text{Foliation}$$

$$\because\ \mu_\alpha^{min}(\wedge^{[2]}\mathcal{F}) = 2\mu_\alpha^{min}(\mathcal{F}) > \mu_\alpha^{max}(T_X/\mathcal{F})$$

$$\Rightarrow 0 = \mathrm{Hom}(\wedge^{[2]}\mathcal{F}, T_X/\mathcal{F}) \text{ より}$$

とくに α-maximal destabilizing sheaf $\mathcal{F}$ について

$\mu_\alpha^{max}(T_X) > 0$ $\quad \mathcal{F}$ foliation

pf $\quad \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X) > 0$

⑤ $\Rightarrow 2\mu_\alpha^{min}(\mathcal{F}) = 2\mu_\alpha(\mathcal{F}) > \mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{max}(T_X/\mathcal{F})$

より

Pf (示式) $\Leftrightarrow 2\mu_\alpha^{\min}(\mathcal{F}) = \mu_\alpha^{\min}(\Lambda^{[2]}\mathcal{F}) > \mu_\alpha^{\max}(T_X/\mathcal{F})$

$$\overset{②}{\Longrightarrow} \quad \mathrm{Hom}(\Lambda^2\mathcal{F}, T_X/\mathcal{F}) = 0$$

$$\Longrightarrow \quad \mathcal{F} \text{ foliation}$$

$\mu_\alpha^{\max}(T_X) = \mu_\alpha(\mathcal{F})$ aとせよ

⑤ $\mu_\alpha^{\min}(\mathcal{F}) = \mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{\max}(T_X/\mathcal{F})$

よって $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{\max}(T_X) > 0$ なので

$\mu_\alpha(\Lambda^{[2]}\mathcal{F}) = 2\mu_\alpha(\mathcal{F}) > \mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{\max}(T_X/\mathcal{F})$

よいいけた

Problem② $\mathcal{F}$ foliation analytic graph をもつ

Lem 4.1 $\mathcal{F}$: foliation $\Delta \subseteq X_{\mathcal{F}} \times X_{\mathcal{F}}$ diagonal

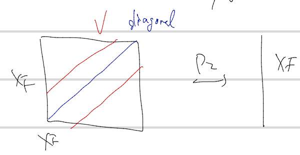
p_i: ith projection とする

$\exists\ V \subset X_{\mathcal{F}} \times X_{\mathcal{F}}$ smooth locally analytic submfd

s.t. $\begin{cases} \cdot\ \Delta \subset V \\ \cdot\ p_2|_V \text{ is smooth} \\ \cdot\ p_2|_V \text{の fiber は} \\ \quad p_1^*\mathcal{F}|_{X_{\mathcal{F}}} \text{の } \Delta \text{ の各点を通る leaf の analytic open submfd である} \end{cases}$

また $N_{\Delta/V} := i^* T_V / T_\Delta$ $\quad i: \Delta \hookrightarrow V$ $\boxed{\text{on } \Delta}$

について $N_{\Delta/V} = \mathcal{F}|_{X^\circ}$



Analytic Graph of $\mathcal{F}$

$r = \mathrm{rk}\,\mathcal{F}$

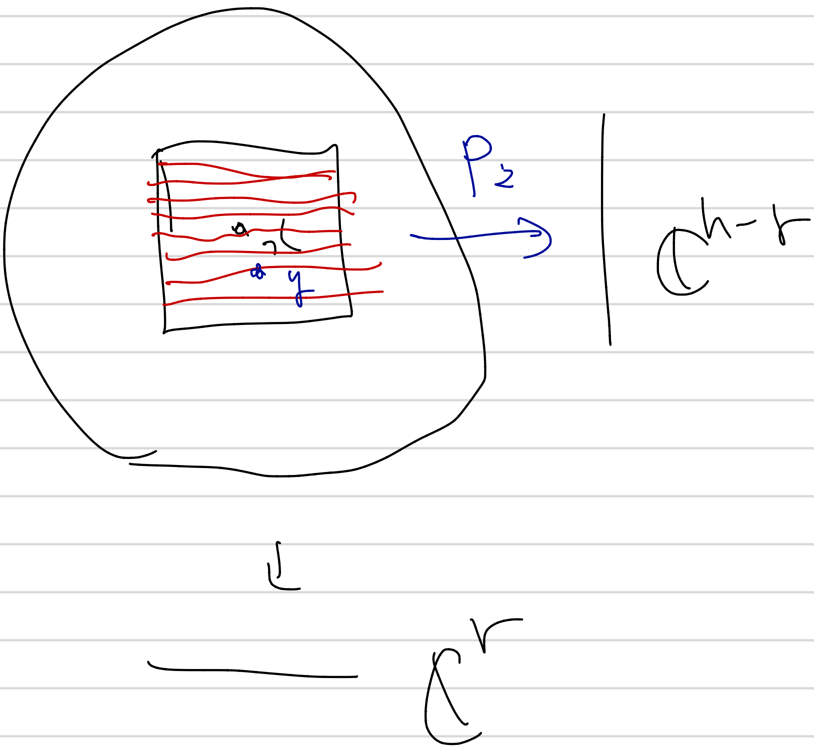
Frobeniusより

$\forall x \in X_{\mathcal{F}}$ $\exists U_x \subset X_{\mathcal{F}}$

s.t. $\left\{ \begin{array}{l} \exists p : U_x \simeq \mathbb{C}^r \times \mathbb{C}^{n-r} \\ P_2 := pr_2 \circ p : U_x \longrightarrow \mathbb{C}^{n-r} \text{ とする。} \end{array} \right.$

$\forall y \in U_x$, L_y : yを通る$\mathcal{F}$のleaf

$L_y \cap U_x = P_2^{-1}(P_2(y))$



Analytic Graph のこうせい

CP 4.1 Local Leaf 存在

Frobenius定理より、

$\forall x \in X_{\mathcal{F}}$ $\exists U_x \subset X_{\mathcal{F}}$ Euclid open

s.t. $\exists \psi, U_x \longrightarrow \mathbb{C}^{n-r}$

s.t. P_2のfiber $= \mathcal{F}$のleaf $\cap U_x$

($\forall y \in U_x$とし P_2の$P_2(y)$のfiber $=$ yを通るleaf $\cap U_x$)

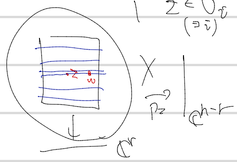
ここで

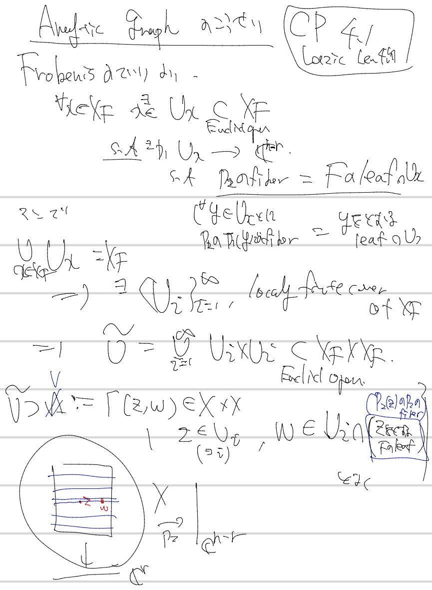
$\bigcup_{x \in X_{\mathcal{F}}} U_x = X_{\mathcal{F}}$

$\Rightarrow \exists \{U_i\}_{i=1}^{\infty}$, locally finite cover of $X_{\mathcal{F}}$

$\Rightarrow \tilde{U} = \bigcup_{i=1}^{\infty} U_i \times U_i \subset X_{\mathcal{F}} \times X_{\mathcal{F}}$. Euclid open

$\tilde{U} \supset \cancel{\Delta}\ V := \{(z, w) \in X \times X \mid z \in U_i\ (\exists i),\ w \in U_i \cap \boxed{(z\text{を通る}\mathcal{F}\text{のleaf})}\}$ ($P_2(z)$のP_2のfiber) とおく




Analytic Graph のこうせい
CP 4.1 Lemma 4.1
Frobenius の定理より、
$\forall x \in X_F$ $\exists U_x \subset X_F$ Euclid open
s.t. $\exists p_2 : U_x \to \mathbb{C}^{n-r}$
s.t. p_2 の fiber $=$ F の leaf $\cap U_x$
($\forall y \in U_x$ をとれば p_2 の $p_2(y)$ の fiber $=$ y を通る leaf $\cap U_x$)
ここで
$\bigcup_{x \in X_F} U_x = X_F$
$\Rightarrow \exists \{U_i\}_{i=1}^{\infty}$, locally finite cover of X_F
$\Rightarrow \tilde{U} = \bigcup_{i=1}^{\infty} U_i \times U_i \subset X_F \times X_F$. Euclid open
$\tilde{U} \supset V := \{(z,w) \in X \times X \mid z \in U_i\ (\exists i),\ w \in U_i \cap (z$ を通る F の leaf$)\}$ とおく
$X \xrightarrow{p_2} \mathbb{C}^{n-r}$
$\mathbb{C}^r$

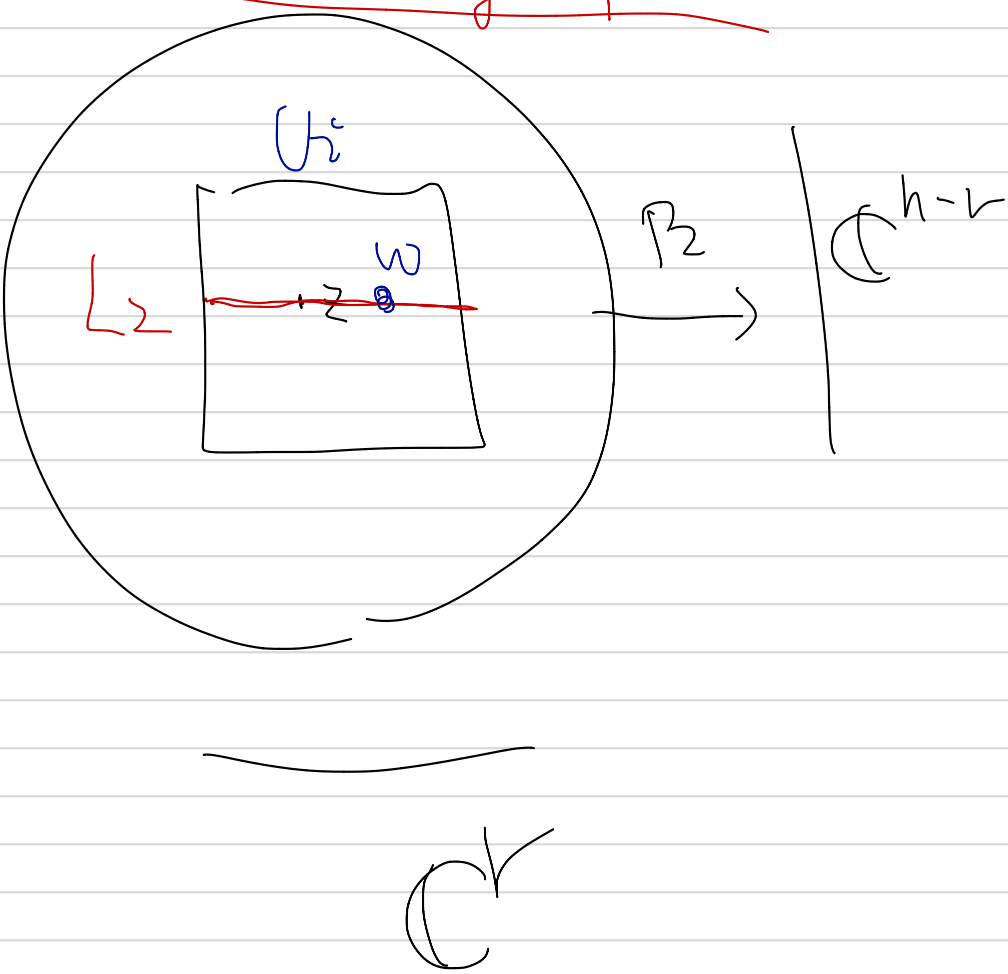
$$\bigcup_{x \in X_F} U_x = X_F$$

$$\rightsquigarrow \exists \{U_{x_i}\}_{i=1}^{\infty} \quad \bigcup_{i=1}^{\infty} U_{x_i} = X_F$$

$$V = \{(z,w) \in X \times X \mid \exists i,\ z \in U_i,\ w \in U_i \cap L_z\}$$

とおく

analytic graph



Rem

・ V : localy closed analytic sub set

・ $\dim V = n+r$

・ $V \supset \Delta := \{(x,x) \mid x \in X_F\}$

・ $N_{\Delta/V} \left(:= T_V / T_\Delta\right) = F|_{X_F}$

($\Delta \underset{pr_2}{\cong} X_F$ の同一視において)

$X \times X \supset U_x \times U_x$ の座標を

$(z_1^1, z_1^2, \ldots, z_1^n)\ (z_2^1, z_2^2, \ldots, z_2^n)$

$pr_2^* F|_{U_x}$ $\quad \dfrac{\partial}{\partial z_2^1}, \dfrac{\partial}{\partial z_2^2}, \ldots, \dfrac{\partial}{\partial z_2^r}$ で生成

T_V $\quad \dfrac{\partial}{\partial z_1^1} \cdots \dfrac{\partial}{\partial z_1^n}\ \dfrac{\partial}{\partial z_2^1} \cdots \dfrac{\partial}{\partial z_2^r}$

T_Δ $\quad \dfrac{\partial}{\partial z_1^1} \cdots \dfrac{\partial}{\partial z_1^n}$

Vについて ・ V is locally closed analytic subset

・ $V \supset \Delta := \{(x,x) \mid x \in X_F\}$ diagonal set

・ $\dim V = n+r$.

ここで ~~$\overline{V}^{zar} \subset X \times X$ とおく.~~

$\overline{V}^{zar} \subset X \times X$ をかんがえる

Rem $N_{\Delta/V} = F|_{X_F}$ $\left(N_{\Delta/V} = T_V / T_\Delta\right)$

($\Delta \cong X_F$ の同一視のもと

これは Frobenius $pr_1 : U_x \longrightarrow \mathbb{C}^r \times \mathbb{C}^{n-r}$ $(z_1, \ldots, z_r)(z_{r+1}, \ldots, z_n)$ とおく.

$F|_{U_x}$ は $p_1^* = (pr_1 \circ p)^* z_i$ $(1 \leq i \leq r)$ で生成 される

一方 T_V は $p_1^*(z_i)$ $1 \leq i \leq r$ と $p_2^*(z_i)$ $1 \leq i \leq r$

T_Δ は $p_1^*(z_i)$ $1 \leq i \leq r$.

[illegible] ([illegible])

<u>Lem 4.9</u> $\overline{V}^{Zar} \subset X \times X$

$\dim \overline{V}^{Zar} = n + r$

$\Rightarrow \mathcal{F}$ is algebraic foliation

<u>Pf</u> $\pi = pr_2|_{\overline{V}^{Zar}} \quad \underset{n+r}{\overline{V}^{Zar}} \longrightarrow \underset{n}{X}$ 第2射影

$X_0 = \{ x \in X_{\mathcal{F}} \mid \dim \pi^{-1}(x) = r \}$

non empty Zariski open

$\pi^{-1}(x)$

Lem 4.9 $\mathcal{F}$: rank r, Foliation

$V \subset X \times X$ closed analytic manifold

analytic germ is analytic graph of $\mathcal{F}$.

$\dim \overline{V}^{Zar} = n+r$

$\Rightarrow \mathcal{F}$ is algebraic integrable

$\boxed{\text{Pf}}$ $V \subset \overline{V}^{Zar}$ open (何 open?)

$pr_2|_{\overline{V}^{Zar}} : \underset{n+r}{\overline{V}^{Zar}} \to \underset{n}{X}$ をかんがえる

general fiber $=$ ~~irreducible, r dim~~ ($\mathcal{F}$ a leaf の Zariski closure (Sing)

($X_{\mathcal{F}} \ni x$ a fiber $=$ leaf $\subset X_{\mathcal{F}}$ L_x 次をとる

$L_x \subset \overline{V}^{Zar} \cap pr_1^{-1}(x)$

$\underset{r\dim の 既}{\overline{L_x}^{Zar}} \subset \underset{r\dim}{\overline{V}^{Zar} \cap pr_1^{-1}(x)}$)

$\forall (z_0, w_0) \in \pi^{-1}(X_0)$,

$L_{(z_0,w_0)}$ を通る $pr_1^* \mathcal{F}$ の leaf

$L \cap \pi^{-1}(X_0) = \{ (z, w_0) \in X \times X \mid z \in X_{\mathcal{F}} \cap$ $\mathcal{F}$ の w_0 を通る leaf $\}$

rdim

$\cap$

$\pi^{-1}(w_0)$ rdim

$\therefore \overline{L \cap \pi^{-1}(X_0)}^{Zar}$ rdim

$\therefore \overline{(w_0 \text{を通る } \mathcal{F} \text{ の leaf})}^{Zar}$ rdim

$\cup$ open

(w_0 を通る $\mathcal{F}$ の leaf)

$(\overline{w_0 \quad\quad}) \cap X_{\mathcal{F}} = (w_0$ を通る $\mathcal{F}$ の leaf$)$

$\pi = p_2|_{\overline{V}^{Zar}} \quad \overline{V}^{Zar} \to X$ surj

π の fiber $=$ rdim

$X_0 := \{ x \in X \mid \dim(\pi \text{ の fiber}) = r \}$ Zariski open, nonempty $\subset X_{\mathcal{F}}$.

[($X_0 \cap \mathcal{F}$) を通る leaf は algebraic 集合]

$\forall x_0 \in \pi^{-1}(X_0)$, $\forall L_{x_0}$: $pr_1^* \mathcal{F}$ の x_0 を通る leaf.
$x_0 = (z_0, w_0)$ $\quad X \times X \xrightarrow{pr_1} X$ が algebraic 集合ならば

$L_x \cap \pi^{-1}(X_0)$

$V = \{ X \times X \ni (z, w) \mid z \in X_{\mathcal{F}},\ w \in (\mathcal{F} \text{ の } z \text{ を通る leaf} \cap X_{\mathcal{F}}) \}$

$L_x \cap \pi^{-1}(X_0) = \{ X \times X \ni (z, w_0) \mid z \in X_{\mathcal{F}} \cap (\mathcal{F} \text{ の } w_0 \text{ を通る leaf}) \}$ rdim

$\cap$

$\pi^{-1}(w_0)$

$\therefore \overline{L_x \cap \pi^{-1}(X_0)}^{Zar}$ rdim

不満がある方に…

$$P : X_0 \longrightarrow \bigcup_{\delta>0} C_{hom, r, \delta}(X \times X)$$

$$x \longmapsto [\pi^{-1}(x)]$$

$$W = \overline{P(X_0)}^{Zar}$$

$$U = \overline{\{(x,w) \in X \times W \mid P(x) = w\}}^{Zar} \subset X \times W$$

$$\begin{tikzcd}
X \arrow[r, dashed, "P"] & W \\
U \arrow[u, "pr_1"] \arrow[ur, "pr_2"'] &
\end{tikzcd}$$

$\ker pr_2$

つまり

$\forall x_0 \in \pi^{-1}(x_0)$, $\overline{L_{x_0} \cap \pi^{-1}(x_0)}^{Zar}$ is r dim

$$p : X_0 \longrightarrow \coprod_{\delta \geq 0} \mathrm{Chow}_{r,\delta}\left(\overline{V}^{Zar}\right) = \mathrm{Chow}\left(\overline{V}^{Zar}\right)$$

$$x_1 \longmapsto \overline{L_{x_1} \cap \pi^{-1}(x_0)}^{Zar}$$

$$\begin{tikzcd}
X_0 \arrow[d, phantom, "\cap"] & \\
X \arrow[r, dashed] & \overline{p(X_0)}^{Zar} = Z \\
U \arrow[u, "pr_1\ \mathrm{bir}"] \arrow[ru, phantom] & \\
\end{tikzcd}$$

$$U = \overline{\left\{ (x, z) \in X_0 \times \mathrm{Chow}\left(\overline{V}^{Zar}\right) \ \text{s.t.}\ p(x) = z \right\}}^{Zar} \subset X \times Z, \qquad U \xrightarrow{pr_2} Z$$

もし、こうなる

もしくは Chowにうめこまない leaf が algebraic を示すですね、

$\forall x \in X_F$, Fをそなえる leaf L_x.

X

$$\begin{tikzcd}
X \arrow[rr, dashed, "p"] & & Z \\
U \arrow[u, "\pi"] \arrow[urr] & \supset \tilde{Z}\times U \arrow[r] & \tilde{Z} \arrow[u] \\
\tilde{U} \arrow[u] \arrow[ur] \arrow[urr, "\tilde{p}"'] \arrow[uu, bend left, "\tilde{\pi}"] & &
\end{tikzcd}$$

normalization & resolution

resolution

$\tilde{\pi}^{*}\mathcal{F} = \ker d\tilde{p}$ on $\tilde{\pi}^{-1}(X_0)$?

証明) $\forall x \in X_0$ に対して $\tilde{\pi}(\tilde{x}_a) = x$ となる $\tilde{x} \in \{\ \}$ (有限個)

$\tilde{x}$ の $\tilde{\pi}^{*}\mathcal{F}$ の leaf $\cap\, \pi^{-1}(X_0)$

$=$ $\tilde{p}$ の $\tilde{p}(U)$ での fiber

x く1個とる

Lem $\mathcal{F} \subset T_X$ foliation (☆)

$$\boxed{\text{Assume } \forall L \text{ line bundle } \exists C \in \mathbb{N}_{>0} \ \forall k \in \mathbb{N} \ \forall m \in \mathbb{N} \ m \geq Ck, \ h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{F}^\vee) = 0}$$

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation

Pf 一般に $f, g : \mathbb{N}_{>0} \to \mathbb{N}_{>0}$ に対して

$f \lesssim g \overset{\text{def}}{\iff} \exists M \in \mathbb{N}, \ \forall k \gg 0, \ f(k) \leq M g(k)$ とする

$\dim \overline{V}^{2ar} = n + r$ を示す

$\mathcal{L} \to \overline{V}^{2ar}$ ample とし、

$h^0(\overline{V}^{2ar}, \mathcal{L}^{\otimes k}) \lesssim k^{n+r}$ を示す

$V \subset \overline{V}^{2ar}$ より $h^0(V, \mathcal{L}^{\otimes k}) \lesssim k^{n+r}$ を示せばよい

Lem Assume $\boxed{\forall L \text{ linebundle } \exists D \in \mathbb{N}_{>0} \ \forall k, \forall m \geq Dk, \ h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^m \mathcal{F}^\vee) = 0}$ (☆)

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation.

Pf $\dim \overline{V}^{2ar} = n+r$ を示せばよい

$\mathcal{L} = \overline{V}^{2ar}$ ample divisor とするとき

$h^0(\overline{V}^{2ar}, \mathcal{L}^{\otimes k}) \simeq O(k^{n+r})$ を示せば

与 $H^0(\overline{V}^{2ar}, \mathcal{L}^{\otimes k}) \to H^0(V, \mathcal{L}^{\otimes k})$ を考えるとこれは inj ($\overline{V}^{2ar} = V$ on some open set)

よって $h^0(V, \mathcal{L}|_V^{\otimes k}) = O(k^{n+r})$ を示せばよい

ここで $\forall 1 \leq m$ に対して $0 \to \mathcal{I}_\Delta^m \hookrightarrow \mathcal{I}_\Delta^{m-1} \to \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m \to 0$,

($\Delta \hookrightarrow V$ より $\mathcal{I}_\Delta \subset \mathcal{O}_V$ を ideal sheaf とする)

$$\begin{aligned} \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m &= \mathrm{Sym}^m \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 \\ &= \mathrm{Sym}^m N_{\Delta/V}^\vee \\ &= \mathrm{Sym}^m \mathcal{F}^\vee \end{aligned}$$

一般論 (Wikiにある)

に子分けするとき

$\forall \ell \in \mathbb{N}_{\geq 0}$ $\mathcal{I}_\Delta \subset \mathcal{O}_V$ idealsheaf

を考えるとこれは triv

$h^0(V, \mathcal{L}|_V^{\otimes k}) = O(k^{n+r})$ を示せば

ここで $\forall 1 \leq m$ に対し $0 \to \mathcal{I}_\Delta^m \hookrightarrow \mathcal{I}_\Delta^{m-1} \to \mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m \to 0$,

($\Delta \hookrightarrow V$ より $\mathcal{I}_\Delta \subset \mathcal{O}_V$ を ideal sheaf とする)

$\mathcal{I}_\Delta^{m-1}/\mathcal{I}_\Delta^m = \mathrm{Sym}^m \mathcal{I}_\Delta/\mathcal{I}_\Delta^2$ ← 一般論 (wikiより)

$= \mathrm{Sym}^m N_{\Delta/V}^\vee$

$= \mathrm{Sym}^m \mathcal{F}^\vee$ にて

$$0 \to \mathcal{I}_\Delta^\ell \to \mathcal{I}_\Delta^{\ell-1} \to \mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^{\ell} \to 0$$

$$\mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^\ell = \mathrm{Sym}^{\ell-1} \mathcal{I}_\Delta/\mathcal{I}_\Delta^2 = \mathrm{Sym}^{\ell-1} N_{\Delta/V}^\vee = \mathrm{Sym}^{\ell-1} \mathcal{F}^\vee.$$

$$h^0(V, \mathcal{L}^{\otimes k}) \leq h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{O}_\Delta/\mathcal{I}_\Delta) + h^0(V \otimes \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta)$$

$\Delta \simeq X_\mathcal{F}$ [illegible]

$$= h^0(X_\mathcal{F}, \mathcal{L}|_{X_\mathcal{F}}^{\otimes k}) + h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta)$$

$$h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{I}_\Delta^{\ell-1}/\mathcal{I}_\Delta^\ell) = h^0(X_\mathcal{F}, (\mathcal{L}|_{X_\mathcal{F}})^{\otimes k} \otimes \mathrm{Sym}^{\ell-1} \mathcal{F}^\vee),$$

$h^0(V, \mathcal{L}^{\otimes k}|_V) \le \sum_{m=1}^{\infty} h^0(V, \mathcal{L}^{\otimes k}|_V \otimes I_\Delta^{m-1}/I_\Delta^{m})$

$= \sum_{m=1}^{\infty} h^0(X_F, (\mathcal{L}|_{X_F})^{\otimes k} \otimes \mathrm{Sym}^m F^\vee)$

($\Delta \simeq X_F$ の同一視において)

$L = (\mathcal{L}|_{X_F})^{\vee\vee}$ $= \sum_{m=1}^{\infty} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee)$

$\boxed{\text{かてい}}$ $= \sum_{m=1}^{Dk} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee)$ (☆)

<u>Claim</u> $h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee) = O(k^{n+r}$

これをみとめると

(☆) $= \sum_{m=1}^{Dk} O(k^{n+r-1}) = O(k^{n+r})$ //

($f(k) \lesssim g(k) \Leftrightarrow \exists C$ const があって $\forall k > 0$, $f(k) \le C g(k)$
これをていぎすればよい)

$h^0(V, \mathcal{L}^{\otimes k})$

$\le \sum_{\ell=0}^{\infty} h^0(X_F, (\mathcal{L}|_{X_F})^{\otimes k} \otimes \mathrm{Sym}^{\ell} F^\vee)$

$= \sum_{\ell=0}^{\infty} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]} F^\vee)$

(かてい)

$= \sum_{\ell=0}^{Ck} h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]} F^\vee)$ $\quad (L = (\mathcal{L}|_{X_F})^{\vee\vee})$

(<u>Claim</u> $h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]} F^\vee) \lesssim O(k^{n+r-1})$)

これより, $\lesssim k^{n+r-1} \cdot Ck \lesssim k^{n+r}$ //

(じかんがなければ)

(F locally free のとき
$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{\ell} F^\vee) = h^0(\mathbb{P}(F), \pi^* L^{\otimes k} \otimes \mathcal{O}_{\mathbb{P}}(\ell))$
$\lesssim k^{n+r-1}$ //)

Claim 一般の torsion free $\mathcal{E}$ について ~~$\mathcal{E} = \mathcal{F}^{\vee}$~~ とする

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\left(\bigoplus_{l \geq 0} \mathrm{Sym}^{l} \mathcal{E}\right)$$

$$\pi : \mathbb{P}('' \longrightarrow X.$$

$$\exists \nu : P \longrightarrow \mathbb{P}(\mathcal{E}') \ \text{bir}.$$

s.t. $\begin{cases} P : \text{smooth}, \\ (\nu \circ \pi)^{-1}(X_{\mathcal{E}}) \text{ is divisor}. \end{cases}$

$$\mathcal{O}_P(1) := \nu^{*} \mathcal{O}_{\mathbb{P}(\mathcal{E}')}(1).$$

[Nakayama] $\exists D$ divisor s.t. $\pi \circ \nu(D) \subset X_{\mathcal{E}}$

~~例~~. $\forall l > 0$, $(\pi \circ \nu)_{*}\left(\mathcal{O}_P(1)^{\otimes l} \otimes \mathcal{O}_P(lD)\right)$

$$= \mathrm{Sym}^{[l]} \mathcal{E}$$

$$\xi_{\mathcal{E}} := \mathcal{O}_P(1) \otimes \mathcal{O}_P(D).$$

↑

($\mathcal{E}$ が locally free の時 $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ と一致)

Claim の証明

$\mathcal{E}$: rk r torsion free coherent

$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\left(\bigoplus_{l \geq 0} \mathrm{Sym}^{l} \mathcal{E}\right)$ ← variety とは限らない

$\pi : \mathbb{P}(\mathcal{E}) \to X$

$\nu : \mathbb{P}'(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$: $\mathbb{P}(\mathcal{E})$ の "X を dominate する component" の normalization.

$\mu : P \longrightarrow \mathbb{P}'(\mathcal{E})$

s.t. $\begin{cases} P \text{ is smooth} \\ \mu \text{ birational} \\ P(\pi \circ \nu \circ \mu)^{-1}(X_{\mathcal{F}}) \text{ is divisor}. \end{cases}$

$$P \xrightarrow{\mu} \mathbb{P}'(\mathcal{E}) \xrightarrow{\nu} \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} X, \qquad \tilde{\pi} : P \to X$$

これをつかって

$$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[\ell]} F^{\vee})$$

$$= h^0(P, (\pi(\gamma))^{*} L^{\otimes k} \otimes \xi_{F^{\vee}}^{\otimes \ell})$$

$$\simeq k^{n+r-1},$$

$$G_P(1) := \mu^* \nu^* G_{\mathbb{P}(\mathcal{F})}(1)$$

このとき $\forall \ell \in \mathbb{N}_{>0}$,

$$(\tilde{\pi}_* \ell G_P(1)) = \mathrm{Sym}^{[\ell]} \mathcal{E}$$

$\|$ (Nakayama 04 Lem 5.2)

$$\tilde{\pi}_*(\ell G_P(1) + \ell D) \quad \because D \text{ の def.}$$

よって、$\zeta := G_P(1) + D$ とおくと

$$\tilde{\pi}_*(\ell \zeta) = \mathrm{Sym}^{[\ell]} \mathcal{E}$$

$$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{E})$$

$$= h^0(X, L^{\otimes k} \otimes \hat{\pi}_*(m\xi))$$

$$= h^0(P, \hat{\pi}^* L^{\otimes k} \otimes (m\xi))$$

$$= O(k^{n+r-1}) \quad \dim P = n+r-1$$

Thm 4.10 (+ Druel)

$\alpha \in \mathrm{Mov}$ $\quad \mathcal{F} = \text{Foliation}$

① $\mu_\alpha^{min}(\mathcal{F}) > 0$

$\Rightarrow$ ✪ をみたすか

② $\xi_{\mathcal{F}^\vee}$ is NOT psef

$\Leftrightarrow$ ✪ をみたす

時間がないとき

(($\mathcal{F}$ locally free かつ

$\mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$ is NOT psef たりる)

Thm 4.10 (+ Druel) $\quad \mathcal{F} = \text{Foliation}$

① $\mu_\alpha^{min}(\mathcal{F}) > 0 \Rightarrow$ ✪

$\mathcal{F}$ is algebraic foliation

② ξを前の証明での物にする

(つまり $P \to \mathbb{P}'(\mathcal{F}) \to \mathbb{P}(\mathcal{F}^\vee) \to X$, π'

ξは $\pi'_*(\ell\xi) = \mathrm{Sym}^{[\ell]}\mathcal{F}^\vee$ となるもの)

ξ is NOT psef on P (

$\Leftrightarrow$ ✪ をみたす

とくに $\mathcal{F}$ が vector bundle のとき

$\mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$ is NOT psef

$\Leftrightarrow$ ✪

Pf:
$H^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee)$
$= \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k})$

よって $D = \left\lceil \dfrac{\mu_\alpha(L)}{\mu_\alpha^{min}(F)} \right\rceil + 1$ とすると

$m \geq Dk \Rightarrow \mu_\alpha^{min}(\mathrm{Sym}^{[m]} F) - \mu_\alpha^{max}(L^{\otimes k})$
$= m\,\mu_\alpha^{min}(F) - k\,\mu_\alpha(L)$
$> k\,\mu_\alpha(L) - k\,\mu_\alpha(L) = 0$

③
$\therefore \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k}) = 0$

Pf ①

$$H^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} F^\vee)$$
$$= \mathrm{Hom}(\mathrm{Sym}^{[m]} F, L^{\otimes k})$$

$C := \left\lceil \dfrac{\mu_\alpha(L)}{\mu_\alpha^{min}(F)} \right\rceil + 1$ とすると

$$m \geq Ck \Rightarrow \mu_\alpha^{min}(\mathrm{Sym}^{[m]} F) - \mu_\alpha^{max}(L^{\otimes k})$$
$$= m\,\mu_\alpha^{min}(F) - k\,\mu_\alpha(L) > 0$$

より

②から

$\pi : \mathbb{P}(\mathcal{F}^\vee) \to X$

② L fixすると $\xi_{\mathcal{F}^\vee}$ is NOT psef

$\Leftrightarrow \exists C > 0,\ {}^\forall l > C,$

$\pi^* L \otimes \xi_{\mathcal{F}^\vee}^{\otimes l}$ is NOT psef

($\xi_{\mathcal{F}^\vee} = \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$, $\xi_{\mathcal{F}^\vee}^{\otimes l} = \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(l)$)

② $\Rightarrow$ ${}^\forall L$ linebundle $\exists D>0$ ${}^\forall k$, ${}^\forall m > Dk$,
$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{E}^\vee) = 0$
$\quad \| $
$h^0(P, \pi^*(L^{\otimes k}) \otimes \xi^{\otimes m})$ を示す

ξ is NOT psef
$\Leftrightarrow {}^\forall \tilde{L}$ linebundle on P $\exists D > 0$, ${}^\forall l \geq D$ に対し
$\tilde{L} \otimes \xi^l$ is NOT psef
$(h^0(P, \tilde{L} \otimes \xi^l) = 0)$

$\tilde{L} = \pi^* L^{\otimes k}$ として
よって $m > Dk$
$\Rightarrow h^0(P, \tilde{L}^{\otimes k} \otimes \xi^{\otimes m}) = 0$

よって $m > Ck$

$$h^0(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathcal{F}^\vee)$$
$$= h^0(\mathbb{P}(\mathcal{F}^\vee), \pi^* L^k \otimes \xi_{\mathcal{F}^\vee}^m) = 0$$

($\xi_{\mathcal{F}^\vee}^m = \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(m)$)

逆に $\xi_{\mathcal{F}^\vee}$ psef のとき ($\xi_{\mathcal{F}^\vee} = \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(1)$)

$\exists L$ big ${}^\forall c > 0$, $\pi^* L \otimes \xi_{\mathcal{F}^\vee}^{\otimes c}$ is big

$\exists L$ ample ${}^\forall c > 0$ $\pi^* L \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(c)$ is big

よって $\exists k$, $h^0(X, L^k \otimes \mathrm{Sym}^{\otimes ck} \mathcal{F}^\vee) \neq 0$
$\quad \|$
$h^0(\mathbb{P}(\mathcal{F}^\vee), \pi^* L^{\otimes k} \otimes \xi_{\mathcal{F}^\vee}^{ck})$

$\left(\exists k,\ h^0(X, L^k \otimes \mathrm{Sym}^{ck} \mathcal{F}^\vee) \neq 0 \right.$
$\quad \|$
$\left. h^0(\mathbb{P}(\mathcal{F}^\vee), \pi^* L^k \otimes \mathcal{O}_{\mathbb{P}(\mathcal{F}^\vee)}(ck)) \right)$

Campana-Păun Thm 1.4

$\mathcal{F} \subset T_X$ saturated

$\mu_\alpha^{min}(\mathcal{F}) > 0$ & $\mu_\alpha^{min}(\mathcal{F}) > \frac{\mu_\alpha^{max}(T_X/\mathcal{F})}{2}$

$\Rightarrow \mathcal{F}$ algebraic foliation

$\Leftarrow$ i.e. $\mu_\alpha^{max}(T_X) > 0$

$\Rightarrow$ α-maximal destabilizing sheaf

is algebraic foliation.

~~təmɛ?~~ $\exists L$ ample.

ξ is psef $\Longrightarrow$ $\tilde{\pi}^* L$ big

$\Longrightarrow$ $\forall D > 0$, $\tilde{\pi}^* L \otimes \zeta^{\otimes D}$ is big

$\Longrightarrow$ $\exists k$ $h^0(P, \tilde{\pi}^* L^{\otimes k} \otimes \zeta^{\otimes Dk}) \neq 0$

?OK

示すこと $\exists L$ ample $\forall D > 0$ $\exists k$, $\exists m > Dk$

s.t. $h^0(X, L^{\otimes k} \otimes S^{[m]}_{X} E) \neq 0$

$\quad \| $

$h^0(P, \tilde{\pi}^* L^{\otimes k} \otimes \zeta^{\otimes m})$

Campana-Păun Thm[14] $\mathcal{F} \subset T_X$ saturated

$\mu_\alpha^{min}(\mathcal{F}) > 0$ $\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F})$

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation

ex) $\mu_\alpha^{max}(T_X) > 0$

$\Rightarrow$ α-maximal destabilizing sheaf is algebraic foliation.

Campana-Păun

Thm 4.21 $\mathcal{F} \subset T_X$ foliation

$\exists \alpha \in \mathrm{Mov}(X)$ s.t. $\mu_\alpha^{\min}(\mathcal{F}) > 0$

$\Rightarrow$ $\mathcal{F}$ - algebraic foliation (葉が代数的)

with rationally connected leaves

Thm 4.21

$\mathcal{F} \subset T_X$ foliation

$\exists \alpha \in \mathrm{Mov}(X)$ s.t. $\mu_\alpha^{\min}(\mathcal{F}) > 0$.

$\Rightarrow$ $\mathcal{F}$ is algebraic foliation (葉が代数的)

with rationally connected leaves

(general point を通る leaf の Zariski closure が RC)

$K_{\mathcal{F}}$ pseff (で無い) $\Rightarrow$ $\exists \alpha$ $0 > K_{\mathcal{F}} \cdot \alpha = \mu_\alpha(\mathcal{F})$

$\Rightarrow$ $\mu_\alpha(\mathcal{F}) > 0$

$\Rightarrow$ $\exists \mathcal{G}$ maximal destabilizing sheaf $\mathcal{G} \subset \mathcal{F}$

$\mu_\alpha^{\min}(\mathcal{G}) > 0$

$\Rightarrow$

さらに Corollary を示しておく

Cor 4.22 X uniruled $\iff \exists \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{max}(T_X) > 0$

pf: X uniruled $\iff K_X$ is NOT psef

$\iff \exists\ \alpha \in \mathrm{Mov}(X)$, $K_X \cdot \alpha < 0$ である.

($\Rightarrow$) $\mu_\alpha^{max}(T_X) \geq \mu_\alpha(T_X) = -\dfrac{K_X \cdot \alpha}{n} > 0$.

($\Leftarrow$) $\mathcal{F}$ を α-maximal destabilizing sheaf とする.

$\mu_\alpha^{min}(\mathcal{F}) \overset{(5)}{=} \mu_\alpha(\mathcal{F}) > 0$

($\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X)$ より)

$\Rightarrow \mathcal{F}$ has RC leaves.

$\Rightarrow X$ の general point を通る leaf は RC

$\Rightarrow$ Uniruled

($\forall x$ general point に対して, それを通る rational curve がある)

Cor 4.22.

X is uniruled $\iff \exists \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{max}(T_X) > 0$.

(K_X is NOT psef)

pf

($\Rightarrow$) $\exists \alpha \in \mathrm{Mov}(X)$, $K_X \cdot \alpha < 0$

$\Rightarrow \mu_\alpha^{max}(T_X) \geq \mu_\alpha(T_X) > 0$.

($\Leftarrow$) $\mathcal{F}$ = α-maximal destabilizing sheaf とする.

$\mu_\alpha^{min}(\mathcal{F}) = \mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(T_X) > 0$

$\Rightarrow \mathcal{F}$ is alg foliation with RC leaves

$\Rightarrow {}^\forall x$ general point, $\exists L$ leaf

$x \in L$ & L is RC (つまり x を通る rational curve がある)

$\Rightarrow$ Uniruled //

Thm (Campana-Păun)

$$\boxed{\begin{array}{l} K_X \text{ psef} \\ \Rightarrow \forall m \in \mathbb{N}_{>0}, \\ \quad \forall Q \text{ torsion free s.t. } (\Omega_X^1)^{\otimes m} \to Q \to 0, \\ \quad \det Q \text{ is psef.} \end{array}}$$

pf K_X psef $\Rightarrow$ $\forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{\min}(T_X) \le 0$

m, Q를 fix하자

$\forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha(Q) \geq \mu_\alpha^{\min}((\Omega_X^1)^{\otimes m})$

$= m\, \mu_\alpha^{\min}(\Omega_X^1) \geq 0$

$\Rightarrow \forall \alpha \in \mathrm{Mov}(X)$, $(\det Q) \cdot \alpha \geq 0$

$\Rightarrow$ $\det Q$ psef //

$$\boxed{\begin{array}{l} \text{Thm (Campana-Păun)} \\ X \text{ smproj}, \ K_X \text{ psef.} \\ \Rightarrow \forall m \in \mathbb{N}, \ (\Omega_X^1)^{\otimes m} \to \forall Q \to 0 \\ \quad Q: \text{torsion free.} \\ \quad \det Q \text{ is psef} \end{array}}$$

$\boxed{\text{pf}}$ K_X psef $\Leftrightarrow$ X is Not uniruled

$\Leftrightarrow \forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha^{\max}(T_X) \le 0$

$\Leftrightarrow \forall \alpha \in \mathrm{Mov}(X)$ $\mu_\alpha^{\min}(\Omega_X) \geq 0$

[illegible] Q, $(\Omega_X^1)^{\otimes m} \to \forall Q \to 0$ 이라면

$\forall \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha(Q) \geq 0$

$\Rightarrow \det Q$ is psef //

いくつかじゅんび　Prelim

$\mathcal{F} \subset T_X$ foliation について $K_\mathcal{F} := (\wedge^{rk}\mathcal{F})^{\vee}$ とする.

とくべつな $\mathcal{F}$ の場合では $K_\mathcal{F}$ がけいさんできる

Lem 4.15 $f : X \to Y$, surj.

$\mathcal{F} := \ker df$ とする.

$\exists E$ f-exceptional divisor.　($f(E) \subset f$ の non flat locus)

s.t. $K_\mathcal{F} \sim_{\mathbb{Q}} K_{X/Y} - \mathrm{Ram}(f) + E$

ここで $\mathrm{Ram}(f) := \sum (\mathrm{ord}_Q f^*(f(Q)) - 1) Q$

Q : prime div. on X

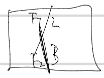
s.t. $f(Q)$ is prime on Y

Claim $\exists E$ g-exc divisor.

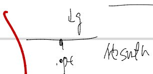
s.t. $K_g \sim K_{Z/Y} - \mathrm{Ram}(g) + E$

$\mathrm{Ram}(g) = \sum (\mathrm{ord}_W g^*(g(W)) - 1)[W]$

($W \subset Z$ prime div, $g(W)$ prime div)



$\mathrm{Ram}(g) = F_1 + 3F_2$



Pf $g : g^{-1}(Y_0) \to Y_0$ is flat とする ($\mathrm{codim}(Y - Y_0) \geq 2$)

$g^{-1}(Y_0)$ 上で $K_g \sim K_{Z/Y} - \mathrm{Ram}(g)$ を示せばよい

codim 2 以上は無視できるので

$\pi_* \mathrm{Ram}(g) = \bigcup_{i=1}^{l} P_i$

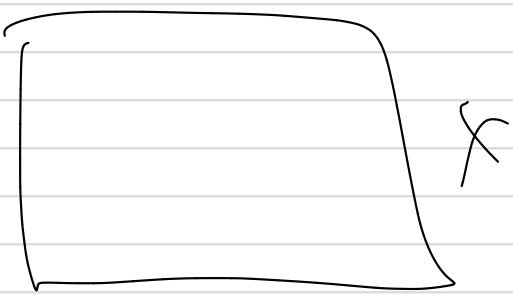
prime divisor の disjoint union としてよい

かんたんのため $l = 1$ とする.

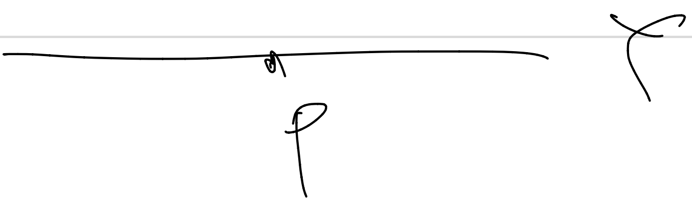
$f^* P_i = \sum w_j Q_j$　$w_j \in \mathbb{N}_{>0}$ とおく.

$\mathrm{Ram}(f)$ について. Y = curve.　$f^* P = \sum w_j Q_j$ とおくと

$X - P$ 上 smooth



$\mathrm{Ram}(f) = \sum (w_j - 1) Q_j$



$\exists f : f|_V = X_0 \to Y_0$ s.t. f flat としてよい

X_0 上 $K_{\mathcal{F}} \cong K_{X/Y} - \mathrm{Ram} f$ を示す

$d\mathcal{F} : T_X \to f^* T_Y$ より $Q := d\mathcal{F}(T_X)$ とする

$0 \to \mathcal{F} \to T_X \to Q \to 0$ より $K_{\mathcal{F}} = K_X + \det Q$ より

$\det Q = f^* K_Y + \mathrm{Ram}(f)$ を示せばよい。

codim2 はムシできるので

$f(\mathrm{Supp}(\mathrm{Ram}(f))) = \sum_{j=1}^{l} P_j$ P_j divisor としてよい

かんたんのため $l = 1$ とする

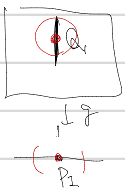
$$f^* P_1 = \sum_{j=1}^{k} w_j Q_j$$ とする.

$dg : T_Z \to g^* T_Y$ より $dg(T_Z) = Q$ とおく

$0 \to \mathcal{F} \to T_Z \to Q \to 0$ である

$(\det \mathcal{F}^\vee) = K_Z \otimes \det Q$ より

$\det Q = g^*(K_Y) + \sum (w_j - 1) Q_j$ を示せばよい。



$\forall z \in Q_1$ general.

$y = g(z)$ とおくと

$\exists (U, z_1, \dots, z_l)$ coordinate of z in Z

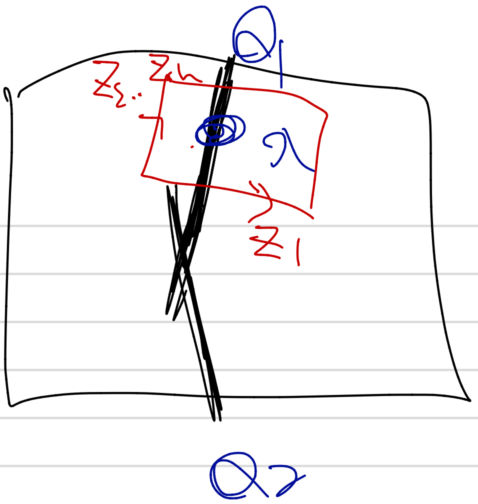
$(V, y_1, \dots, y_k)$: of y in Y

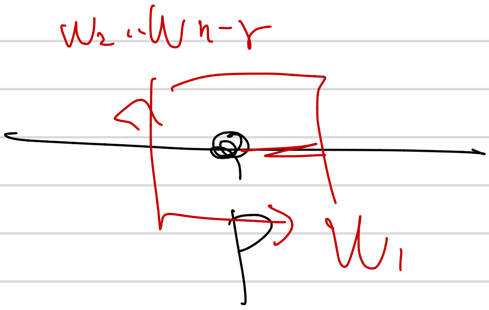
s.t. $(z_1 = 0) = Q_1 \cap U$

$(y_1 = 0) = P_1 \cap V$

$g : U \to V$

$(z_1, \dots, z_k) \to (z_1^{w_1}, z_2, \dots, z_k)$ とできる.





$x \in Q_1 - \bigcup_{i=2}^{k} Q_i$ general にとり $y = f(x)$ とする.

$x \in {}^{\exists}(U, z_1, \dots, z_n)$ 近傍 座標 ! 原点がxに移る

$y \in {}^{\exists}(V, w_1, \dots, w_{n-r})$; y :

s.t. $U \cap Q_1 = (z_1 = 0)$

$V \cap P = (w_1 = 0)$

$f : U \longrightarrow V$

$(z_1, \dots, z_n) \longmapsto (\underbrace{z_1^{m_1}}_{=w_1}, \underbrace{z_2}_{=w_2}, \dots, \underbrace{z_{n-r}}_{=w_{n-r}})$

$T_X|_U$ は $\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}$ で生成されているより,

$Q|_U$ は $\left(d\cancel{\pi}f\left(\frac{\partial}{\partial z_1}\right), \dots, d\cancel{\pi}f\left(\frac{\partial}{\partial z_n}\right) \right)$

$= \left(z_1^{m_1-1} \cancel{\pi}^{*}f\left(\frac{\partial}{\partial w_1}\right), \cancel{\pi}^{*}f\left(\frac{\partial}{\partial w_2}\right), \dots, \cancel{\pi}^{*}f\left(\frac{\partial}{\partial w_{n-r}}\right) \right)$ で生成.

∴ $\det Q|_U$ は $z_1^{m_1-1} \pi^{*}\left(\frac{\partial}{\partial w_1} \wedge \cdots \wedge \frac{\partial}{\partial w_{n-r}}\right)$ で生成される

∴ $\det Q|_U \sim (m_1 - 1) Q_1 - f^{*} K_Y$.

$Q_2, \dots, Q_k$ もたしあわせて //

$\boxed{\text{Pte 2}}$ BP

$m \in \mathbb{N}_{>0}$ fix $f : X \to Y$ with connected fibre surj

$L \to X$ line bundle

h_L s.t.m on X, s.t. $\sqrt{-1}\,\Theta_{h_L} \geq 0$ &

$\mathcal{J}(h_L^{\frac{1}{m}}|_{X_y}) = \mathcal{O}_{X_y}$

y general.

Assume

$f_*(mK_{X/Y} + L) \neq 0$

$\Rightarrow \exists\, h_m$ m-Bergman metric on $mK_{X/Y} + L$

s.t. $\begin{cases} \sqrt{-1}\,\Theta_{h_m} \geq m\,\mathrm{Ram}(f) \geq 0 \ (\text{i.e. } mK_{X/Y}+L \text{ psef}) \\ \forall y \text{ general}, \ \forall x \in X_y, \ \forall u \in H^0(X, mK_{X/Y}+L), \\ \underbrace{|u|_h^{2/m}(x)}_{C^0 \text{ bunded}} \leq \int_{X_y} |u|_{h_L}^{\frac{2}{m}} < +\infty \end{cases}$

Thm3 [Campana]

$f: X \to Y$ smooth with connected fibre

F: general fiber, Assume K_F is psef

$\Rightarrow$ $K_{X/Y} - \mathrm{Ram}(f)$ psef.

pf Fix A very ample on X

$\forall m \in \mathbb{N}_{>0}$, $m(K_{X/Y} - \mathrm{Ram}(f)) + A$ psef & nef

Fix m. $\exists n$, $h^0(F, n(mK_F + A|_F)) \neq 0$

$\rightsquigarrow$ $\exists h_{mn}$ Bergman metric on $n(mK_{X/Y} + A)$

s.t. $\sqrt{-1}\Theta_{h_{mn}} \geq mn\,\mathrm{Ram}(f)$.

$\rightsquigarrow$ $n(mK_{X/Y} + A) - mn\,\mathrm{Ram}(f)$ psef.

//

$\rightsquigarrow$

Pr3 Raynaud Flattening

Lem 4.16 $f: X \to Y$ mphis

$\exists\, \tau : Y' \to Y$ birat.

$\exists\, X' :$ $X \times_Y Y'$ a desingularization

$$\begin{tikzcd} X' \arrow[r, "\exists f'"] \arrow[d, "\exists\tau'"'] \arrow[dr, phantom, "@"] & Y' \arrow[d, "\tau"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

s.t. f' exc

$\Rightarrow \tau'$ exc.

④ Relative MRC

Thm [Kol Chapter 5]

$f: X \to Y$ morphism

$\exists\ \pi: X \dashrightarrow Z$ dominant rat'l map

$\Rightarrow g: Z \to Y$ morphism

$$\begin{tikzcd} X \arrow[r, dashed, "\pi"] \arrow[dr] & Z \arrow[d] \\ & Y \end{tikzcd}$$

st $\forall y$ = general pt,

π_y $X_y \dashrightarrow Z_y$

is MRC

$\Leftarrow$ K_{Z_y} is psef,

$\boxed{\dim Z_y > 0 \text{ for } y \ (\forall y \text{ general}, X_y \text{ is NOT RC})}$

pf of Thm 4.21

$\mathcal{F} \subset T_X$ foliation

$\mu_\alpha^{\min}(\mathcal{F}) > 0$ とする

$\boxed{\text{leaf が RC でないとする.}}$

$\mathcal{F}$: algebraic foliation より

$$\exists \quad \text{st} \quad \ker df = \varphi^*\mathcal{F} \text{ on } \varphi^{-1}(X_{\mathcal{F}}).$$

$$\begin{tikzcd}
\tilde{X} \arrow[rr, dashed, "\pi"] \arrow[d, "\varphi"] \arrow[drr, "f"] & & Z \arrow[d, "g"] \\
X & & \exists\, Y
\end{tikzcd}$$

φはいい注のかていより
f の relative MRC をとると
$\dim Z_y > 0$ である
$y \in Y$

remark として,
Prop 4 と Stein Factorization を
使ったよって?

$$
\begin{tikzcd}
\exists \tilde X \arrow[r, "\pi"] \arrow[d, "\varphi"'] \arrow[dr, "f"] & \exists Z \arrow[d, "g"] \\
X & \exists Y
\end{tikzcd}
$$

s.t.

① π ← morphism

② f exc $\Rightarrow$ φ exc

③ π exc $\Rightarrow$ φ exc

④ φ with connected fibre.

$\dim Z_y \neq 0$ K_{Z_y} is psef ($y \in Y$ genl)

⑤ $\ker df = \varphi^* \mathcal{F}$ on $\tilde X_0 \subset \tilde X$ open & codim $\tilde X \setminus \tilde X_0 \geq 2$

$\tilde{\mathcal F} = \ker df \subset T_{\tilde X}$

$\mu^{min}_{\alpha}(\mathcal F) = \mu^{min}_{\varphi^*\alpha}(\tilde{\mathcal F}) > 0$

so $\mu_{\varphi^*\alpha}(d\pi(\tilde{\mathcal F})) > 0$ //

$\mathcal G = \ker dg \subset T_Z$

$\xrightarrow{\pi^* T_Z}$ $d\pi(\tilde{\mathcal F}) \subset \pi^*(\ker dg)$ 由 ($g \circ \pi = f$)

π が flat ...

$\rightsquigarrow$ ③ $d\pi(\tilde{\mathcal F}) = \pi^*(\ker dg)$ on $\exists X'$ s.t. $\tilde X \setminus X'$ is φ-ex

$$\therefore\ \mathrm{Mult}_{\varphi^*\alpha}\left(\pi^*\left(\cancel{\ker(dg)}\ \mathcal{G}_f\right)\right) > 0$$

$$\rightsquigarrow\ \underline{\pi^*(K_{\mathcal{G}}) \cdot \varphi^*\alpha < 0}$$

$\underline{\text{Pref}}$

$$\to)\quad K_{\mathcal{G}} \sim K_{Z/Y} - \mathrm{Ram}(g) + E$$

$\exists E$ g-exc.

$\underline{\text{Pre2}}$ の(f)より　$K_{Z/Y} - \mathrm{Ram}(g)$ psef

③より　$\pi^*K_{\mathcal{G}} \cdot \varphi^*\alpha$

$$= (K_{Z/Y} - \mathrm{Ram}(g))\,\varphi^*\alpha > 0$$

で矛盾する。

$\mathcal{J}$Thm 4.23 $\mathcal{F}$: Foliation.

$K_{\mathcal{F}} := (\det \mathcal{F})^{\vee}$ is psef.

$\Leftrightarrow$ $\forall m$, $(\mathcal{F}^{\vee})^{\boxtimes m} \to {}^{\forall}Q \to 0 \Rightarrow \det Q$ psef.

e.g. K_X psef. $\Leftrightarrow$ $\forall m$, $(\Omega_X^1)^{\otimes m} \to {}^{\forall}Q \to 0$ (二つ?) $\det Q$ is psef.

Pf) $\exists m$, $\exists Q$. s.t. $\mathcal{F}^{\vee \otimes m} \to Q \to 0$ & $\det Q$ is Not psef

$\Rightarrow \exists \alpha \in \mathrm{Mov}(X)$, $\mu_\alpha(Q) < 0$.

$\Rightarrow \mu_\alpha^{\min}(\mathcal{F}^{\vee}) < 0$.

$\Rightarrow \mu_\alpha^{\max}(\mathcal{F}) > 0$

$\mathbin{\rotatebox[origin=c]{-45}{$\geq$}}$

$\mu_\alpha^{\max}(T_X)$

今, $\mathcal{G} \subset \mathcal{F}$ を α-maximal destabilizing sheaf of $\mathcal{F}$ とする。

$\mathcal{F}$ foliation より $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$

より $\mathrm{Hom}(\Lambda^2 \mathcal{G}, \mathcal{F}/\mathcal{G}) = 0$

$\Rightarrow$ $[\mathcal{G}, \mathcal{G}] \subset \mathcal{G}$ 成り立つ

今, $\mu_\alpha^{min}(\mathcal{G}) = \mu_\alpha(\mathcal{G}) > \mu_\alpha^{max}(\mathcal{F}/\mathcal{G})$

$\rightarrow$ $2\mu_\alpha^{min}(\mathcal{G})$ より から従う

$\mathcal{G}$ algebraic foliation with RC leaves ($\mu_\alpha^{min}(\mathcal{G}) > 0$)

$\mathcal{G} \subsetneq \mathcal{F}$.

alg leaves

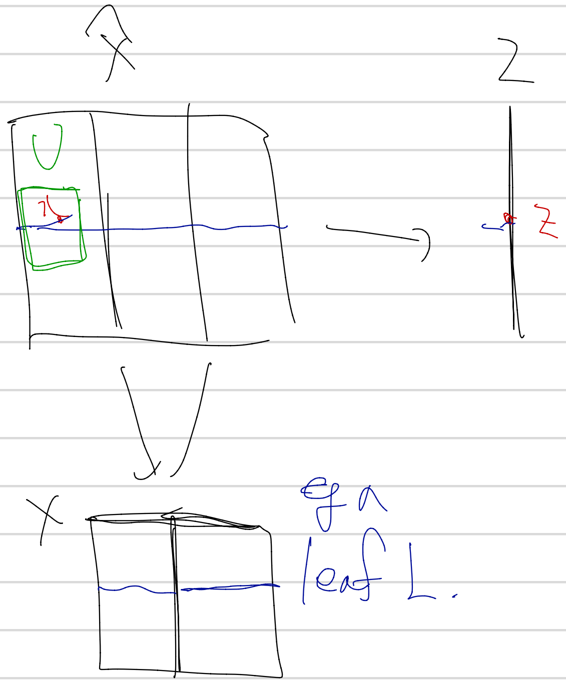
$\mathcal{G} \subset \mathcal{F} \subset T_X \quad \mathcal{G}$: algebraic foliation

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\hat{f}} & Z \\ {\scriptstyle \pi}\downarrow & \nearrow_{f} & \\ X & & \end{array}$$

$\exists x_0$;

$\tilde{\mathcal{G}} = \pi^*\mathcal{G}$, $\tilde{\mathcal{F}} = \pi^*\mathcal{F}$

$\tilde{\mathcal{G}} = \ker d\hat{f}$ on $\pi^{-1}(x_0)$



$z \in Z$ smooth point,

$\tilde{x} \in \hat{f}^{-1}(z) \cap \pi^{-1}(x_0) \neq \{ \}$

$x \in U$ $\mathcal{G}|_U = \ker p_1$

$p_1 : \mathbb{C}^n \to \mathbb{C}^{n-rk\mathcal{G}}$

Claim F general leaf of $\mathcal{G}$

$\Rightarrow$ $$K_{\mathcal{G}}|_F \sim K_{\mathcal{G}}|_F \sim K_F$$

F very general にとれば $K_{\mathcal{G}}|_F$ psef なので (leaf は almost きつから)

これを使えば K_F psef となるが

F は RC (uniruled 故に) 矛盾する

pf of Claim

$\mathcal{G} \subset \mathcal{F}$ $\quad \mathcal{G} = \ker df$

$f: X \to Y$

f with connected fibre.

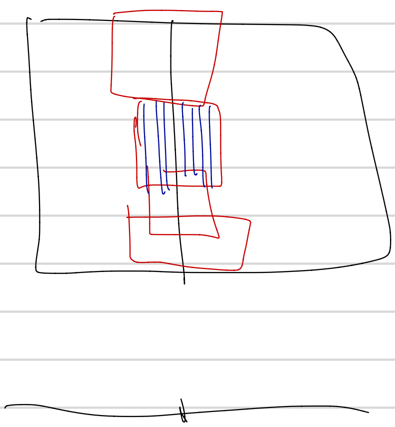
$f: X \to Y$

$\cup \qquad \cup$

$f|_{X^0 = f^{-1}(Y^0)}: X^0 \to Y^0$ smth $\qquad y \in Y^0$ fixed

$\forall x \in X_y$, $\exists x \in U_x \subset X$

$\because U_x \simeq \mathbb{C}^{\mathrm{rk}\mathcal{F}} \times \mathbb{C}^{n-\mathrm{rk}\mathcal{F}}$



$$\begin{array}{c} \\ \end{array} \quad U_x \xrightarrow{P_{\mathcal{F}2}} \mathbb{C}^{n-\mathrm{rk}\mathcal{F}}$$

$$G_x := P_{\mathcal{F}_2}^{-1}(P_{\mathcal{F}_2}(x))$$

$$\mathcal{G}|_{U_x} \subset \mathcal{F}|_{U_x} = \left(P_{\mathcal{F}_1}^{*}\frac{\partial}{\partial z^1}, \dots, P_{\mathcal{F}_1}^{*}\frac{\partial}{\partial z^{\mathrm{rk}\mathcal{F}}}\right)$$

$\boxed{\text{Pf of Thm 4.21}}$

~~$\mu_\alpha^{min}(\mathcal{F}) > 0$ for a generic foliation~~

$\boxed{\text{leaf が RC でないとする}}$

方針

$$\begin{tikzcd}
\tilde{X} \arrow[d, "\varphi"'] \arrow[dr, "f"] & \\
X \arrow[r, dashed] & Y
\end{tikzcd}$$

$\hat{\mathcal{F}} := \ker df = \varphi^* \mathcal{F}$ on $\pi^{-1}(X_{\mathcal{F}})$

$$\mu^{min}_{\varphi^*\alpha}(\hat{\mathcal{F}}) = \mu^{min}_{\alpha}(\mathcal{F}) > 0 \ (\text{仮定})$$

f の fiber もまた RC でない

relative MRC をとる

$$\begin{tikzcd}
\tilde{X} \arrow[r, dashed, "\mathrm{MRC}"] \arrow[d, "\varphi"'] \arrow[dr, "f"'] & Z \arrow[d, "g"] \\
X & Y
\end{tikzcd}$$

$f|_{L}$

$$\begin{tikzcd}
& \tilde{X}_2 \arrow[r, "m"] \arrow[d, "\text{flattening}"'] \arrow[dl] & Z_2 \arrow[d] \\
& \tilde{X}_1 \arrow[r] \arrow[d, "\text{resol of sing}"'] & Z_1 \arrow[d, equal] \\
X & \tilde{X}_0 \arrow[l] \arrow[r, dashed] & Z_0
\end{tikzcd}$$

$$\begin{tikzcd} \tilde{X} \arrow[r, "\pi"] \arrow[d, "\varphi"'] \arrow[dr, "f"] & Z \arrow[d, "g"] \\ X & Y \end{tikzcd}$$

π: ～ complisu

π exc $\Rightarrow \varphi$ exc

f exc $\Rightarrow \varphi$ exc

$$0 \to \hat{\mathcal{F}} \to T_{\tilde{X}} \xrightarrow{df} f^* T_Y \qquad \mathcal{G}' = d\pi(\hat{\mathcal{F}}) \subset \pi^* T_Z$$

$df = \pi^*(g) \circ d\pi$

$$0 \to \mathcal{G} \to T_Z \to g^* T_Y \qquad d\pi : T_{\tilde{X}} \to \pi^* T_Z$$

$$\hat{\mathcal{F}} \to d\pi(\hat{\mathcal{F}})$$

$$\pi^* \mathcal{G} \subset \pi^* T_Z, \quad \mathcal{G}' = d\pi(\hat{\mathcal{F}}) \subset \pi^* T_Z \text{ on } \tilde{X}$$

$$\mu_{\varphi^* \alpha}(\mathcal{G}') \geq \mu^{\min}_{\varphi^* \alpha}(\hat{\mathcal{F}}) > 0$$

$$\boxed{\begin{aligned} \mathcal{G} &= \operatorname{Ker} g \qquad \pi^* \mathcal{G} = \pi^*(\operatorname{Ker} g) \\ \mathcal{G}' &= d\pi(\operatorname{Ker} f) \subset \\ &= d\pi(\operatorname{Ker}(g \circ \pi)) \end{aligned}}$$

π exc φ exc でなければ

f exc でないから

π の flat [illegible] $\pi^* g$ と g [illegible]
[illegible]

$df = \pi^*(dg) \circ d\pi$

$f = g \circ \pi$

$\pi^*(z) \Rightarrow d\pi(\ker f) = \pi^*(\ker g)$

$d\pi_*\left(\frac{\partial}{\partial x}\right) \qquad \pi^*\left(\frac{\partial}{\partial z}\right)$

π が flat となる [illegible]

$$\begin{array}{ccc} T_X & \xrightarrow{d\pi} & \pi^* T_Z \\ & {\scriptstyle df}\searrow & \downarrow{\scriptstyle dg} \\ & & f^* T_Y \end{array}$$

π: flat なら

$\pi : \pi^{-1}(z_0) \to z_0$ flat である

open より

$\forall x \in X, x \in {}^{\exists} U$ (open), $\pi : U \to \pi(U)$

$\pi(U)$ is open

$(x_1, \dots, x_\ell) \to (f_1(x), \dots, f_\beta(x))$ と表せる

${}^\forall(\ker f) \ni {}^\forall\left(a_i \frac{\partial}{\partial x^i}\right)$

$d\pi\left(a_i \frac{\partial}{\partial x^i}\right) = \sum a_i \frac{\partial f_j}{\partial x^i} \frac{\partial}{\partial y^j} = \pi^*\left(a^i \frac{\partial}{\partial x^i}\right)$

$\pi^*\left(b_j \frac{\partial}{\partial y^j}\right)$

$\therefore \quad \mu_{\varphi^*\alpha}(\pi^*\mathcal{G}) > 0$

$\Big(\mathcal{G} : \quad 0 \to \mathcal{G} \to T_Z \to f^* T_Y$

$c_1(\pi^*\mathcal{G}) = \pi^* c_1 \mathcal{G}$ on codim 2 set $\Big)$

$= -\pi^* K_{\mathcal{G}}$

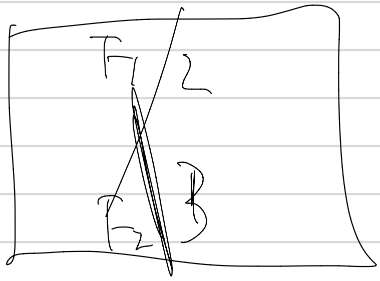
$$\frac{c_1(\pi^*\mathcal{G}) \cdot \varphi^*\alpha}{rk\, \mathcal{G}} = - \frac{\pi^* K_{\mathcal{G}} \cdot \varphi^*\alpha}{rk\, \mathcal{G}} > 0.$$

Claim $\exists E$ g-exc divisor

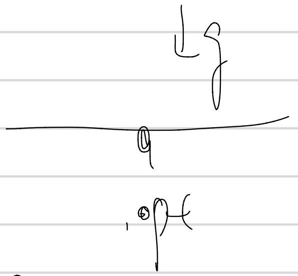
s.t. $K_{\mathcal{Z}} \sim K_{\mathcal{Z}/Y} - \mathrm{Ram}(g) + E$

$$\mathrm{Ram}(g) = \sum (\mathrm{ord}_W g^*(g(W)) - 1)[W]$$

$\left(\begin{array}{l} W \subset \mathcal{Z} \text{ prime div} \\ g(W) \text{ prime div} \end{array}\right)$



$\mathrm{Ram}(g) = F_1 + 3F_2$



他 smth

(codim $(Y - Y_0) \geq 2$)

pf $g: g^{-1}(Y_0) \to Y_0$ is flat とする.

$g^{-1}(Y_0)$ 上で $K_g \sim K_{\mathcal{Z}/Y} - \mathrm{Ram}(g)$

を示せばよい

(codim 2 以上は無視できるので)

$$\pi(\mathrm{Ram}(g)) = \bigcup_{i=1}^{l} P_i$$

prime divisor の disjoint union とする.

簡単のため $l = 1$ とする.

$$f^* P_i = \sum w_j Q_j \quad w_j \in \mathbb{N}_{>0} \text{ とする.}$$

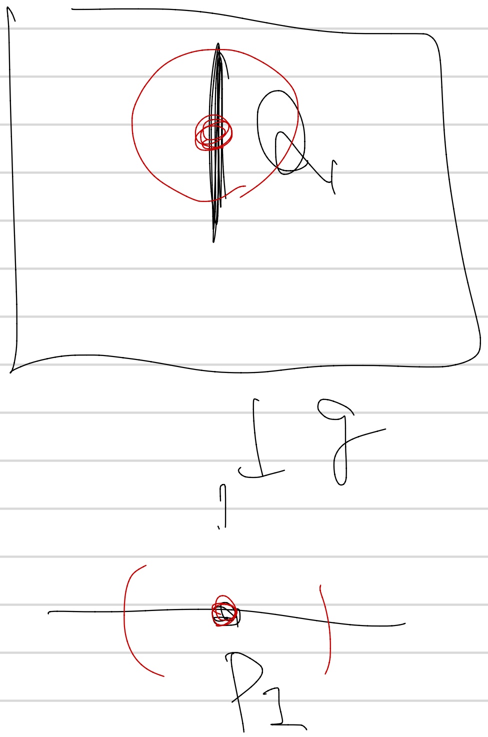
$dg : T_Z \to g^* T_Y$ より $dg(T_Z) = Q$ とおく

$0 \to F \to T_Z \to Q \to 0$ である

$(\det F^\vee) = K_Z \otimes \det Q$ より

$\det Q = g^*(K_Y) + \sum (W_j - 1) Q_j$

を示せばよい。



$\forall z \in Q_1$ general

$y = g(z)$ とおく

$(U, z_1, \dots, z_\ell)$ coordinate of Z in Z

$(V, y_1, \dots, y_k)$ " of y in Y

s.t. $(z_1 = 0) = Q_1 \cap U$

$(y_1 = 0) = P_1 \cap V$

$g : U \to V$

$(z_1, \dots, z_k) \mapsto (z_1^{W_1}, z_2, \dots, z_\ell)$ とできる.

よって、 $Q = dg(T_Z) \subset g^* T_X$ は、

$dg\left(\frac{\partial}{\partial z^1}\right)$, $dg\left(\frac{\partial}{\partial z^2}\right)$, …, $dg\left(\frac{\partial}{\partial z^k}\right)$

$\shortparallel$ $\shortparallel$ $\shortparallel$

$(z^1)^{m_1-1} \pi^*\left(\frac{\partial}{\partial y^1}\right)$ $\pi^*\left(\frac{\partial}{\partial y^2}\right)$ … $\pi^*\left(\frac{\partial}{\partial y^k}\right)$

で生成される。

$\det Q$ は U 上で $(z^1)^{m-1} \pi^*\left(\frac{\partial}{\partial y^1} \wedge \cdots \wedge \frac{\partial}{\partial y^\ell}\right)$

で生成される。

$\leadsto$ $\det Q|_U \simeq \mathcal{O}_X((m_j-1)Q_j) \otimes \pi^* K_Y^\vee|_U$

$\leadsto$ $\det Q \simeq \mathcal{O}_Z\left(\sum_{j=1}^{\ell} (m_j-1)Q_j\right) \otimes \pi^* K_Y^\vee$

$j=1 \sim \ell$ わたって

$d_{1,2}$ $K_g \sim K_{Z/Y} - \mathrm{Ram}(g') + E$

$$\varphi^*_\alpha(\pi^* K_g) = \varphi^*_\alpha(\pi^*(K_{Z/Y} - \mathrm{Ram}(g'))) \geq 0$$

2で係数。

$(K_{Z/Y} - \mathrm{Ram}(g'$ f-pseef d_{11} 系係$)$

Campana – Cao – Păun

Pereira – Touzet

$\mathcal{F}$ holo foliation. $c_1(\mathcal{F}) = 0$

$\Rightarrow$ $\mathcal{F}$ is locally free
$(X_{\mathcal{F}} = X)$

Conj $c_2(\mathcal{F}) \neq 0$ &

$\mathcal{F}$ is α-stable. $\left(\alpha = H^{n-1}, \ \exists H \text{ very ample}\right)$

$\Rightarrow$ $\mathcal{F}$ is algebraic.

Thm C.C.P. $\mathcal{F} \subset T_X$ foliation $r = \operatorname{rk} \mathcal{F}$

(1) $c_1(\mathcal{F}) = 0$, $c_2(\mathcal{F}) \neq 0$ $(\alpha = H^{n-1})$

(2) $\exists p \geq r$, $\operatorname{Sym}^{[p]}(\mathcal{F}^\vee)$ is α stable

$\Rightarrow$ $\mathcal{F}$ is algebraic.

$\mu_\alpha^{max}(T_X) > 0 \quad \forall \alpha \in Mov(X)$ $\quad \boxed{\exists \alpha \in Mov(X) \text{ s.t. } \mu_\alpha^{min}(T_X) > 0}$

$\mu_\alpha^{min}(T_X) > 0$
$\Rightarrow$ $f^*\Omega_Y \hookrightarrow \Omega_{\tilde{X}}$
$\mu_\alpha^{max}(\Omega_{\tilde{X}}) < 0.$
$\mu_\alpha(f^*\Omega_Y) = K_Y \cdot \alpha > 0$ で矛盾

$\alpha = \pi_* A^{n-1}$ にとる.

$$\begin{tikzcd} \tilde{X} \arrow[d, "\pi"'] \arrow[dr, "\tilde{f}"] & \\ X \arrow[r, dashed, "\mathrm{MRC}"'] & Y \end{tikzcd}$$

$$0 \to \tilde{\mathcal{F}} \to T_{\tilde{X}} \to f^*T_Y, \qquad T_{\tilde{X}} \to Q \to f^*T_Y, \quad Q \to 0$$

$$\mathrm{rk}\,Q\,\mu_\alpha(Q) = \mathrm{rk}\,T_{\tilde{X}}\,\mu_\alpha(T_{\tilde{X}}) - \mathrm{rk}\,\tilde{\mathcal{F}}\,\mu_\alpha(\tilde{\mathcal{F}})$$

$\exists \alpha \in Mov(X)$ s.t. $\mu_\alpha^{min}(T_X) > 0 \Rightarrow \mu_\alpha^{max}(\tilde{X}) > 0.$

$$\mu_\alpha^{max}(T_X) \geq \underbrace{\mu_\alpha(T_X)}_{K_X^{-1}\cdot\alpha > 0} \geq \mu_\alpha^{min}(T_X)$$

$\exists \alpha \quad \mu_\alpha^{min}(T_X) > 0 \underset{\text{rien}}{\Rightarrow} X : \mathrm{RC}$

$\mu_\alpha^{min}(T_X)$

T_X big $\Rightarrow$ $\forall \alpha \in Mov(X)$ $\mu_\alpha^{min}(T_X) > 0$

$\Downarrow$

$X \simeq \mathbb{P}^n$

Tx $h \cdot y \Rightarrow \overline{(x)} \forall \alpha$ $\det\alpha$ $b \cdot y$

$\Rightarrow \forall \alpha \in \mathrm{Mor}$ $\det\alpha \cdot \alpha > 0$

$E \to A$ $\bar{A} \cdot \alpha$