6/11

「・Boucksom-Diverio
・ときます 3? , Campana.
- Fano, schmmich, local simply
2 Campana-Brubene.
3 Campana Zhey π_1 general
4 Capure Peternel.
5 Kollár-Shafarevich map の $\boxed{\S 1 (6.1) \ 3.1}$-
6 Kawamata.
7 Höring-Peternell

「・Kollar より (Cap の次は)
・~~Gromov の話~~
CP の中の話 のとき→Lの幾何
Kob hyperbolic space.
2.9 10 C(d)
Kähler

「・$\chi(X)=0$
・π_1 general type につ...

Conj 1 (cf. Ueno's conjecture K.) Kollár.

① $\kappa(X)=0$, X sm proj var.

$\Rightarrow$ $\pi_1(X)$ is almost Abelian?

($\exists H \subset \pi_1(X)$ Abel. $[H : \pi_1(X)]$ finite index)

② $\kappa(X)=0$, X: sm proj var.

$\Rightarrow$ $\exists \tilde{X} \cong_{bir} AV \times$ Simply connected,

$\tilde{X} \downarrow$ finite étale

X

Kollár: X terminal & $K_X \equiv 0$ & $\pi_1(X)=$ Abelian

なら ②, ok.

・こか ① $\Rightarrow$ ② は いえるのかな? (Kollárの本 みる!)

② $\Rightarrow$ ① は clear.

~~ここで~~ $\widetilde{X}_{univ}$: X の universal cover.

$\pi: \widetilde{X}_{univ}$

X has generically large fundamental group.

$\Leftrightarrow$ $\exists \{D_i\}_{i \in \mathbb{N}}$, $D_i \subsetneq X$ closed sub

$\forall Z \not\subset \bigcup D_i$, closed sub.

$\mathrm{Im}(\pi_1(\overline{Z}) \to \pi_1(X))$ infinite.

$\widetilde{X}_{univ} \xrightarrow{\pi} X$

$\pi^{-1}(D_i)$ を除いて

$\Leftrightarrow$ $\widetilde{X}_{univ}$ の very general point を通る compact submanifold は存在しない

X has large fundamental group

$\Leftrightarrow$ $D_i = \phi$ としてよい

$\Leftrightarrow$ $\widetilde{X}_{univ}$ に compact submfd が存在しない

$\widetilde{X}_{univ}$: Stein

X : π_1 general type. (has generically - quasi-large fundamental group)

$\Leftrightarrow$ $\widetilde{X}_{univ} = \bigcup Z_i$, Z_i is positive dimensional cpt submfd.

を満たす

Rem

$\boxed{\text{reduction}}$

Shafarevich map のことばでいうと

$\dim Sh(X) = \dim X$ のこと

$$\begin{array}{ccc} Sh : X & \dashrightarrow & Sh(X) \text{ s.t.} \\ \cup & & \cup \\ X^{\circ} & \xrightarrow[f^{\circ}]{} & Z^{\circ} \end{array}$$

Zariski open set

① f° : proper, topological fibration with connected fibres

② $\forall z \in Z^{\circ}$, $\pi_1(X_z)$ finite.

③ $\forall z \in Z^{\circ}$ very general, $w : W \to X$ finite.

$\pi_1(W)$ finite & $w(W) \cap X_z^{\circ} \neq \emptyset$

$\Rightarrow w(W) \subset X_z$

$$\widetilde{X}_{univ} = \bigcup_i Z_{z^i}$$

$\Leftrightarrow$ $X = \bigcup_i \pi(Z_{z^i})$ $\pi(Z_{z^i})$ cpt. positive dim

$\pi_1(\pi(Z_{z^i}))$ finite

$\Leftrightarrow$ $\forall z$, X_z positive dim

$\Leftrightarrow$ $\dim Sh(X) < \dim X$

Cy2 (Campana, Kollar, Iitaka, etc...)

① X : π_1 general type

$\Rightarrow \exists \tilde{X}$

$K(X) \geq 0$ &, $\tilde{X} \downarrow$ finite étale X, $\tilde{X} \xrightarrow[\text{Iitaka}]{\text{submersion}} \exists Y$ Y general type & π_1-general

$\cup$

F.

F : AV.

② $\tilde{X}_{univ} \simeq \mathbb{C}^n$

$\Rightarrow \exists \tilde{X}$: AV, $\tilde{X} \downarrow$ finite étale X

Rem, ① $\Rightarrow$ ② Kobayashi-Oshima

2. ① is 3 dim までok.

K_X nef, & Abundance

$\Rightarrow$ ① ok

上こで Conjecture は { Abundance / nMMP をみとめるということ?

Conj 1 については そういう条件はない..

Conj 2 についても Kx nef をもってもいける?

こういうのは Abundance & nMMP から成立する,っての法
いってる感じよ..

(Kx

本講座の目標

1　Conj 1は Abundance と MMPを みとめればできる？

まつむらさんがのばなし

2　Conj 1は $h^1(X, \mathcal{O}_X) \neq 0$ (or $\chi(\mathcal{O}_X) \neq 0$) とかなどの条件つけてできる？

3　Conj 2は Abundance と MMP を みとめればできる (Kx nef を仮定)

4　Conj 2は $\widetilde{X_{univ}}$ が Stein. (もしくはそれに近いもの) にすればできる？

Q1 1=7112

Beauville-Bogomolov decomposition of singular space.

[Thm] (Höring-Peternell, Druel)

models. Theorem 1.1 allows to complete this challenge.

1.5. Theorem. *Let X be a normal projective variety with at most klt singularities such that $K_X \equiv 0$.*

Then there exists a projective variety $\tilde{X}$ with at most canonical singularities, a finite cover $f : \tilde{X} \to X$, étale in codimension one, and a decomposition

$$\tilde{X} \simeq A \times \prod_{j \in J} Y_j \times \prod_{k \in K} Z_k$$

into normal projective varieties with trivial canonical bundles, such that

- *A is an abelian variety;*
- *the Y_j are (singular) Calabi-Yau varieties;*
- *the Z_k are (singular) irreducible symplectic varieties.*

[GKP16b], let us explain the notions of singular Calabi-Yau and singular irreducible symplectic varieties.

1.4. Definition. *Let X be a normal projective variety of dimension $n \geq 2$ with at most canonical singularities such that $\omega_X \simeq \mathcal{O}_X$.*

- *X is a Calabi-Yau variety if $h^0(Y, \Omega_Y^{[q]}) = 0$ for all integers $1 \leq q \leq n-1$ and all finite covers $Y \to X$, étale in codimension one;*

reflexive.

- *X is irreducible symplectic if there exists a reflexive holomorphic 2-form $\sigma \in H^0(X, \Omega_X^{[2]})$ such that for all finite covers $\gamma : Y \to X$, étale in codimension one, the exterior algebra of holomorphic reflexive forms is generated by the reflexive pull-back $\gamma^{[*]}(\sigma)$.*

ここでは MMP 使われている $\pi_1(X) \simeq \pi_1(X')$ (たかだか)

$X \dashrightarrow X' \xleftarrow[\text{étale}]{\text{finite}} X'' \simeq A \times CY \times$ 対称

$\kappa(X) = 0$ eventually ok だろう

$\left\{ \begin{array}{l} K_X \text{ ref} \\ \kappa(X) = 0 \end{array} \right\} \Rightarrow$ Abundance $K_X \equiv 0$

SgCY ; Strsympl が存在するか? ok だろう ⇒ しらべたら できそう?

$\boxed{\text{〜が たくさんある}}$

[たとえば、Campana
Campana-Bruhn-Kollár
Gromov

~~一般化した~~ 例えば、たとえば X_{univ} の
L_2-index theorem

⇓

L^2 holo function ある

⇓

Poincaré 級数をつくる

⇓

X の下におとす。

⇓ $\pi_1(X)$ 大きい

Poincaré / Poincaré が 収束

nonconst meromorphic がいっぱいある

たとえば $K^{+}(X) \geq \dim sh(X)$ をしめす

それがしたい

Poinc
これにたよって 関数 $X \longrightarrow \mathbb{CP}^N$
をつくるから

Q2 これはどうしだ

$\tilde{q}(X) := \sup\{ h^1(\tilde{X}, \mathcal{O}_{\tilde{X}}) \mid \tilde{X} \to X \text{ finite étale} \}$?

Kawamata

$\kappa(X) = 0$ のとき $\alpha: X \to Alb(X)$

は surj algebraic fiberspace

(surj with connected fibre)

finite

THEOREM 13: *Let $f: X \to A$ be a finite morphism from a complete normal algebraic variety to an abelian variety. Then $\kappa(X) \geq 0$ and there are an abelian subvariety B of A, etale covers $\tilde{X}$ and $\tilde{B}$ of X and B, respectively, and a complete normal algebraic variety $\tilde{Y}$ such that:*

(1) *$\tilde{Y}$ is finite over A/B.*

(2) *$\tilde{X}$ is isomorphic to $\tilde{B} \times \tilde{Y}$.*

(3) $\kappa(\tilde{Y}) = \dim \tilde{Y} = \kappa(X)$.

$X \to A$ $\quad \tilde{X} \simeq \tilde{B} \times \tilde{Y}$

$\tilde{X} \simeq \tilde{B} \times \tilde{Y} \quad B \subset A$

$0 \le \hat{q}(X) \le \dim X$

$$\begin{tikzcd} F \subset Y \arrow[r] \arrow[d] & A^{\dim \hat{q}(X)} \\ X & \end{tikzcd}$$

このFはいつ simply connected?

$\boxed{\hat{q}(X)=0,\ \kappa(X)=0 \Rightarrow \text{simply connected?}}$

$\boxed{\kappa(X)=0\ \&\ \chi(\mathcal{O}_X)\neq 0 \Rightarrow \text{simply connected?}}$

ですよね?

$\pi_1(X)$ の有限性

3 ちょっとわからん

Kolla

4 X_{univ} : Stein

(Campana–Peternell)

THEOREM 0.4. — *Let X be a normal n-dimensional projective variety with at most rational singularities.*

(1) *Suppose that the universal cover of X is not covered by its positive-dimensional compact subvarieties. Then X is of general type if $\chi(\mathcal{O}_X) \neq 0$.*

(2) *If X has at most terminal singularities and $\tilde{X}$ does not contain any compact subvariety of positive dimension (eg. X is Stein), then either K_X is ample, or K_X is nef, $K_X^n = 0$, and $\chi(\mathcal{O}_X) = 0$.*

$(\chi(\mathcal{O}_X) \neq 0 \Rightarrow \kappa^+(X) \geq \dim Sh(X) = \dim X$

$\Rightarrow \kappa^+(X) = \dim X$

$\Rightarrow \kappa(X) = \dim X$

$\kappa^+(X) = \sup\{\kappa(\det \mathcal{F}) \mid \forall \mathcal{F} \subset \Omega^p_X\}$ torsion-free reflexive

We shall need the following generalization

THEOREM 1.6. — *Let X be a connected projective manifold, and $\alpha \in \overline{ME}(X)$ of the form*

$$\alpha = \pi_*(H_1 \cap \cdots \cap H_{n-1})$$

with $\pi : X' \to X$ a modification and H_j very ample on X'. If there exists a torsion free quotient sheaf

$$(\Omega^1_X)^{\otimes m} \to Q \to 0$$

for some $m \in \mathbb{N}$, such that $c_1(Q) \cdot \alpha < 0$, then X is uniruled.

$\forall f$ X uniruled

$T_X|_{C_i}$ for $\forall i$ nef not anti

$\bigcup_i C_i = Y$ ~ ample sub

(2) $\tilde{X}$ is general type $\Rightarrow$ ① $n = \chi^f(X) \Rightarrow \kappa(X) = \dim X$

or

② $\chi(\mathcal{O}_X) = 0$

$\tilde{X}$ Stein $\Rightarrow$ X has no rat curve

$\Rightarrow$ $\boxed{K_X \text{ nef}}$.

① K_X ample.

② K_X nef, $\chi(\mathcal{O}_X) = 0$, K_X not big

Bouckson-Diverio. X cpt Kähler

X_{univ} has bounded strictly psh

$\Rightarrow$

① Kobayashi hyperbolic (Brody) ($f: \mathbb{C} \to X \Rightarrow f$ const)

② X has large fundamental group

③ X proj, K_X ample

④ $\forall Z \subset X$ sub. K_Z big

①, ② は すぐわかる

($\varphi|_Z$ is strictly psh $\Rightarrow \varphi = 0 \Rightarrow$ 矛盾)

($Z \subseteq X$ sub, cpt)

$\boxed{(3),(4) について}$

$M := \widehat{X}_{univ}$ では

K_M の Bergman metric h_M を考える

Aut(M) 不変

ここに h_M は X におちる

$$h_M = h\, e^{-\varphi}$$

(K_M)

$$\varphi(x) = \log \sum_j |\eta_j|^2_h(x)$$

η_j: $H^0(M, K_M)$ の h による正規直交 ONB

$$= \log \sup_{\|\eta_j\| = 1} |\eta_j(x)|^2_h$$

$\boxed{\text{Kobayashi}}$ h_M は Zariski open 上で nondegenerate C^∞ metric

x で nondegenerate

$\Longleftrightarrow$ K_M が x で 1 jet generation

(すなわち $K_M \twoheadrightarrow \mathcal{O}_{M,x}/m^2_{M,x}$ つこと)

$\tilde{M}$ has bdd strictly psh φ at som pont

$\Rightarrow$ $\forall x \in \tilde{M}$, $\psi(z) = A\varphi(z) + (n+1)\log|z-x|$

$|z-x| \ll 1$ $(A \gg 1)$

L^2 estimate

$\Rightarrow \exists f \in H^0(M, K_M)$, f が x の 近傍で...

$\Rightarrow h_M$ smth on M & strictly psh ?

on Zariski open set

$\Rightarrow K_X \in \{?\} \Rightarrow$ ample

sm

K_X は Zariski open 上で strictly psh になる metric を持つ

(Non Amp(K_X) $\neq X \Rightarrow$ big)

$$
\begin{tikzcd}
Z \arrow[r, "\text{resol}"] & Y \arrow[r, phantom, "\subset"] & X \\
\widetilde{Z} \arrow[u] \arrow[r] \arrow[ur, phantom, "\text{P.b}"] & \widetilde{Y} \arrow[u] \arrow[r, phantom, "\subset"] & \widetilde{X}_{univ} \arrow[u]
\end{tikzcd}
$$

$Z \times_Y \widetilde{Y} = \widetilde{Z}$

sub

$\varphi \subset$ ——— ψ

bdd psh, strictly psh at some point

$\rightarrow$ K_Z big. $\exists K_Y$ big

Conj (Boucksom–Diverio)

Gromov [illegible]

$W = \mathbb{C}$ (bounded)

no positive dimensional set

6/25 or 7/2

Cao-Höring — Lazić-…

[illegible]の Höring

Druel, Höring-Peternell B.B decomposition

Abundance を[illegible]

Druel nef anti canonical

semipositive =

regular foliation を示す.

課題へ

alg map がいつ morphism になるか?

Druel, Höring Peternell を読む.

Foliation について

・ $F \subset T_X$ がいつ foliation になるか

→ 1つの判定法に Slope を使ったものがある.

・ foliation の leaf がいつ algebraic になるか.

→ Druel (Bost) による判定法 (mod p reduction を使う)

Campana-Păun による判定法 (F^* not psef ならば[illegible])

$$\begin{tikzcd} & \Gamma \arrow[dl, "\mathrm{bir}"'] \arrow[dr] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

map がないとき (多い)

◎ algebraic foliation による map はいつ morphism になるか?

→ Höring, Kebekus-Solá Conde… RC. compact leaf を使う.

何にでもなる? (Druel で示される)

X sm proj mfd

<Foliation>

$F \subset T_X$ coherent sub sheaf.

① $[F,F] \subset F$. integrable Lie bracketで閉じる

② F saturated (T_X/F torsion free.

$\mathrm{Sing}(F) = \{$small Zariski set of F is Not locally free$\}$

$\quad \| \; X \setminus X_F$

$r := \mathrm{rk}\, F$. $q := \dim X - r$

Leaf L is leaf ($\dim L = r$)

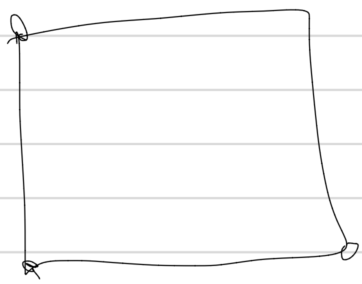
$\iff L \subset X_F$ & $T_L = F|_L$.

maximaly connected locally closed

Rem Leafは閉とは限らない?

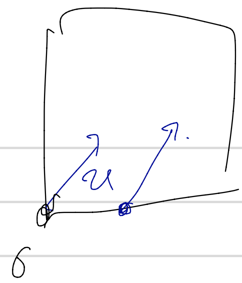
real mfdの例

$X = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}$.



$T_X = \underbrace{\boxed{T_{X,0}}}_{\cong\,\mathbb{C}} \otimes X$.

$T_X \leftarrow T_{X,0} \otimes X$



$(T_{X,p}$ στο $u) \leftarrow (u, p)$

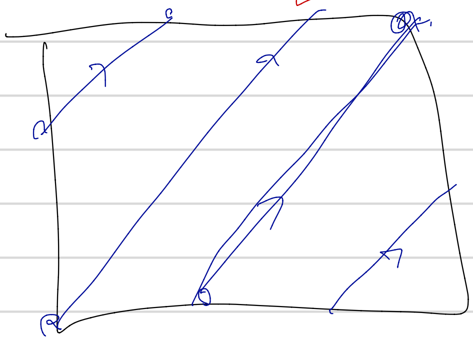
$T \ni u = 1 + \alpha\sqrt{-1}$ για

$F := \mathbb{R}u \otimes X \subset T_X$ για

foliation εξετάζω

Ο εξετάζω leaf για?

$\alpha = \frac{q}{p} \in \mathbb{Q}$ αν

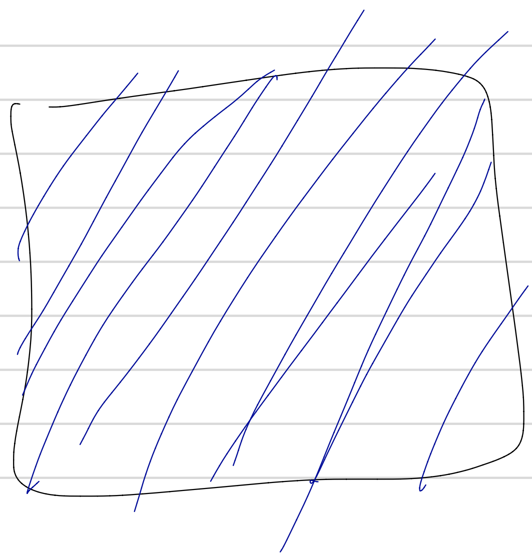


τότε $L \subset X$ Zariski closed

$$\boxed{F|_L = T_L}$$

$$\boxed{\overline{L}^{Zar} = L}$$

$\alpha \in \mathbb{R} \setminus \mathbb{Q}$ αν



$L \subseteq X$

Not Zariski closed

$\overline{L}^{Zar} = X \supsetneq L$

Def Algebraic leaf

$^\forall L$: general point を通るleafに対し

$\overline{L}^{Zar}$ は常に open ($\overline{L}^{Zar} \supset L$ Zariski open)

Rem Frobenius theorem

$\mathcal{F}$ foliation

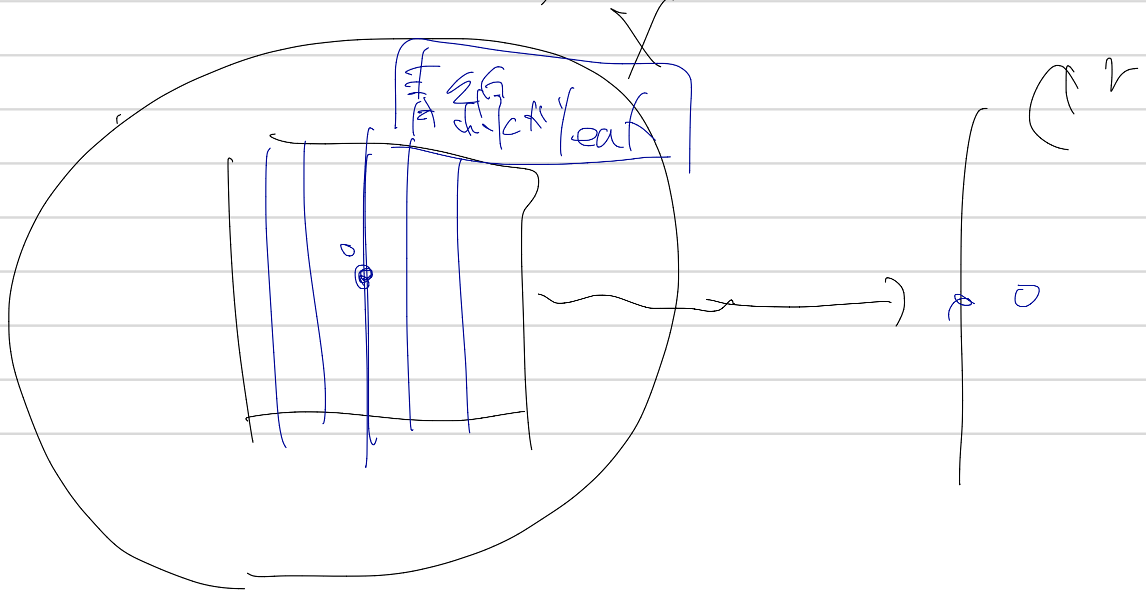
$\forall x \in X - \mathrm{Sing}(\mathcal{F})$. $x \in {}^\exists U$ (Euclid open) $\simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$

$\overset{\Phi}{\sim} \mathbb{C}^r \times \mathbb{C}^{n-r} \xrightarrow{pr_2} \mathbb{C}^r$

$\phi : U \to \mathbb{C}^r$

$u_1, \ldots, u_r$ は $\mathcal{F}|_U$ a basis

$u_i := \phi^*\left(\frac{\partial}{\partial x^i}\right)$ $\quad \frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^r}$



Foliation 及び [illegible]

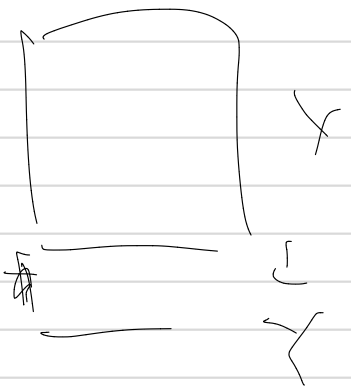
例(1) $\phi : X \longrightarrow Y$ $\boxed{\text{Smooth morphism}}$ (Smooth morphism ならば ...)

$$0 \to T_{X/Y} = \ker d\phi_* \to \boxed{TX \xrightarrow{d\phi_*} \phi^* TY} \to 0$$

Vector Bundle morphism になる (sheaf morphism になる ...)

$F = T_{X/Y}$ is Foliation

($\phi^* TY$ torsion free,



local

$T_{X/Y} = C(u_1, \dots, u_r)$ なる

$V = \sum a_i u_i$

$W = \sum b_i W_i$

$$[V, W] = \sum_{i,j} a_i (u_i(b_j)) W_j - b_j (W_j(a_i)) u_i$$

$$d\phi_*(f(x) u) = f(\phi(x))\, d\phi_*(u)$$

これは成立 ...

Smooth でなくても

$$0 \to \operatorname{Ker} d\phi \to TX \xrightarrow{d\phi} TY$$

が成立 ....

例12 Foliation と q-form

Foliation $\Longleftrightarrow$ $\omega \in H^0(\Omega_X^q \otimes \mathcal{N})$

$\mathcal{N} := (T_X/\mathcal{F})^{**}$ $\quad q = n - \mathrm{rk}\,\mathcal{F}$.

$T_X \to T_X/\mathcal{F} \leftarrow \det \mathcal{N}^* \hookrightarrow \Omega_X^{[q]}$

$\to \omega \in H^0(X, \Omega_X^{[q]} \otimes \det \mathcal{N})$

locally $\mathcal{O}\text{-}\mathcal{F} = \langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^r} \rangle \hookrightarrow \langle \frac{\partial}{\partial z^1}, \dots, \frac{\partial}{\partial z^n} \rangle = T_X$

$0 \to \mathcal{N}^* \to \Omega_X^1$

$dz^i \to dz^i \qquad i = r+1 \sim n$

$0 \to \det \mathcal{N}^* \to \Omega_X^q$

$\bigwedge_{i=r+1}^{n} dz^i \hookrightarrow$

$\omega = \bigwedge_{i=r+1}^{n} dz^i \qquad \omega^i = dz^i$

$d\omega^i = 0$ は可積分.

3.3 (Foliations defined by q-forms) Let $\mathcal{G}$ be a codimension q foliation on an n-dimensional normal variety X. The *normal sheaf* of $\mathcal{G}$ is $\mathcal{N} := (T_X/\mathcal{G})^{**}$. The q-th wedge product of the inclusion $\mathcal{N}^* \hookrightarrow \Omega_X^{[1]}$ gives rise to a non-zero global section $\omega \in H^0\big(X, \Omega_X^q \boxtimes \det(\mathcal{N})\big)$ whose zero locus has codimension at least two in X. Moreover, ω is *locally decomposable and integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \dots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathcal{G}$ is closed under the Lie bracket.

Conversely, let $\mathcal{L}$ be a reflexive sheaf of rank 1 on X, and let $\omega \in H^0(X, \Omega_X^q \boxtimes \mathcal{L})$ be a global section whose zero locus has codimension at least two in X. Suppose that ω is locally decomposable and integrable. Then the kernel of the morphism $T_X \to \Omega_X^{q-1} \boxtimes \mathcal{L}$ given by the contraction with ω defines a foliation of codimension q on X. These constructions are inverse of each other.

$\mathcal{N}^* \hookrightarrow \Omega_X^{[1]} \Longrightarrow \exists\, \omega \in H^0(X, \Omega_X^1$

3.3 (Foliations defined by q-forms) Let $\mathscr{G}$ be a codimension q foliation on an n-dimensional normal variety X. The *normal sheaf* of $\mathscr{G}$ is $\mathscr{N} := (T_X/\mathscr{G})^{**}$. The q-th wedge product of the inclusion $\mathscr{N}^* \hookrightarrow \Omega_X^{[1]}$ gives rise to a non-zero global section $\omega \in H^0(X, \Omega_X^q \boxtimes \det(\mathscr{N}))$ whose zero locus has codimension at least two in X. Moreover, ω is *locally decomposable and integrable*. To say that ω is locally decomposable means that, in a neighborhood of a general point of X, ω decomposes as the wedge product of q local 1-forms $\omega = \omega_1 \wedge \cdots \wedge \omega_q$. To say that it is integrable means that for this local decomposition one has $d\omega_i \wedge \omega = 0$ for every $i \in \{1, \ldots, q\}$. The integrability condition for ω is equivalent to the condition that $\mathscr{G}$ is closed under the Lie bracket.

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$\omega \in H^0(X, \Omega_X^q \boxtimes \mathscr{L})$

$d\omega \wedge \omega = 0$

$d\omega \wedge \omega (u_1, u_2, u_3) = 0$

$N^* \hookrightarrow \Omega_X^{[q]} \Rightarrow \exists\, \omega \in H^0(X, \Omega_X$

$\lrcorner\, \omega \quad T_X \to \Omega_X^{q-1} \otimes \mathcal{L}$ a kernel to foliation

18:42 7月1日(月) 23%

en.m.wikipedia.org

Interior product

Not to be confused with Inner product.

In mathematics, the **interior product** (a.k.a. **interior derivative**, **interior multiplication**, **inner multiplication**, **inner derivative**, **insertion operator**, or **inner derivation**) is a degree −1 (anti)derivation on the exterior algebra of differential forms on a smooth manifold. The interior product, named in opposition to the exterior product, should not be confused with an inner product. The interior product $\iota_X\omega$ is sometimes written as $X \lrcorner\, \omega$.[1]

Contents

Definition

The interior product is defined to be the contraction of a differential form with a vector field. Thus if X is a vector field on the manifold M, then

$$\iota_X : \Omega^p(M) \to \Omega^{p-1}(M)$$

is the map which sends a p-form ω to the $(p-1)$-form $\iota_X\omega$ defined by the property that

$$(\iota_X\omega)(X_1, \ldots, X_{p-1}) = \omega(X, X_1, \ldots, X_{p-1})$$

for any vector fields $X_1, \ldots, X_{p-1}$.

The interior product is the unique antiderivation of degree −1 on the exterior algebra such that on one-forms α

$$\iota_X\alpha = \alpha(X) = \langle \alpha, X \rangle,$$

where $\langle\ ,\ \rangle$ is the duality pairing between α and the vector X. Explicitly, if β is a p-form and γ is a q-form, then

$$\iota_X(\beta \wedge \gamma) = (\iota_X\beta) \wedge \gamma + (-1)^p \beta \wedge (\iota_X\gamma).$$

The above relation says that the interior product obeys a graded Leibniz rule. An operation satisfying linearity and a Leibniz rule is called a derivation.

Properties

By antisymmetry of forms,

$$\iota_X\iota_Y\omega = -\iota_Y\iota_X\omega$$

and so $\iota_X \circ \iota_X = 0$. This may be compared to the exterior derivative d, which has the property $d \circ d = 0$.

The interior product relates the exterior derivative and Lie derivative of differential forms by the **Cartan formula** (a.k.a. **Cartan identity**, **Cartan homotopy formula**[2] or **Cartan magic formula**):

$$\mathcal{L}_X\omega = d(\iota_X\omega) + \iota_X d\omega.$$

This identity defines a duality between the exterior and interior derivatives. Cartan's identity is important in symplectic geometry and general relativity: see moment map.[3] The Cartan homotopy formula is named after Élie Cartan.[4]

The interior product with respect to the commutator of two vector fields X, Y satisfies the identity

$$\iota_{[X,Y]} = [\mathcal{L}_X, \iota_Y].$$

See also

$\omega = \omega_1 \wedge \cdots \wedge \omega_q$

$d\omega_i \wedge \omega = 0$

$T_X|_U \to \Omega_X^{q-1} \otimes \mathcal{L}$

$u \to (u \lrcorner\, \omega)$

$(u \lrcorner\, \omega)(u_1 \wedge \cdots \wedge u_{q-1}) := \omega(u \wedge u_1 \wedge \cdots \wedge u_{q-1})$

$\lrcorner\, \omega \quad T_X^q \to \mathcal{O}_X$

$u_1 \wedge \cdots \wedge u_q \to \omega(u_1 \wedge \cdots \wedge u_q)$

$$\left[\begin{array}{l} \omega = \omega_1 \wedge \cdots \wedge \omega_q \\ d\omega_i \wedge \omega = 0 \end{array}\right.$$

$T_X/\mathcal{F}$ が torsion free

$\Leftrightarrow$ $\boxed{\forall u \in T_{X,x}\ \ au \in \mathcal{F}_x,\ 0 \neq a \in \mathcal{O}_{X,x} \Rightarrow u \in \mathcal{F}_x}$

$$\begin{array}{ccc} T_X|_U & \xrightarrow{\varphi\,(i_\omega)} & \Omega_X^{q-1} \otimes \mathcal{L} \\ \rotatebox{90}{$\in$} & & \\ u & \longmapsto & (u \lrcorner \omega) \end{array}$$

$au \in \ker\varphi$ とする.

$\boxed{q \geq 2}$

$$(u \lrcorner \omega)(u_1 \wedge \cdots \wedge u_{q-1}) := \omega(u \wedge u_1 \wedge \cdots \wedge u_{q-1})$$

$$(au \lrcorner \omega)(u_1) = (\omega_1 \wedge \omega_2)(au \wedge u_1)$$

$$= \begin{vmatrix} \omega_1(au) & \omega_1(u_2) \\ \omega_2(au) & \omega_2(u_2) \end{vmatrix}$$

$$= a^2 \begin{vmatrix} \omega_1(u) & \omega_1(u_2) \\ \omega_2(u) & \omega_2(u_2) \end{vmatrix} = 0$$

$\therefore$ $\ker\varphi$ saturated $\Rightarrow$ $u \in \ker\varphi$.

$a \neq 0$ なら

$$[u_1, u_2] \lrcorner \omega = i_{[u_1,u_2]}(\omega)$$

$$= [\mathcal{L}_{u_1}, i_{u_2}](\omega)$$

(なぜなら $i_{u_2}(\omega) = 0$.)

$$= -i_{u_2}(\mathcal{L}_{u_1}(\omega))$$

$(\mathcal{L}_X \omega = d(i_X \omega) + i_X(d\omega))$

$$= -i_{u_2}(i_{u_1}(d\omega))$$

$$\omega = \omega_1 \wedge \omega_2$$

$$d\omega = d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2.$$

$d\omega_i \wedge \omega_i = 0$ $$d\omega_1 \wedge \omega_1 \wedge \omega_2 = d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$$

とくに $$d\omega \wedge \omega_1 = d\omega \wedge \omega_2 = 0.$$

$$i_{u_2} \circ i_{u_1} (d\omega)(u_3)$$

$$= d\omega(u_1, u_2, u_3)$$

$$\omega([u_1, u_2], u_3)$$

$$= \begin{vmatrix} \omega_1([u_1, u_2]) & \omega_1(u_3) \\ \omega_2([u_1, u_2]), & \omega_2(u_3) \end{vmatrix}$$

$$d\omega(V_0, V_1) = V_0\, \omega(V_1) - V_1\, \omega(V_0) - \omega([u_1, u_2])$$

In terms of invariant formula

Alternatively, an explicit formula can be given for the exterior derivative of a k-form ω, when paired with $k+1$ arbitrary smooth vector fields $V_0, V_1, \ldots, V_k$:

$$d\omega(V_0, \ldots, V_k) = \sum_i (-1)^i V_i \left(\omega\left(V_0, \ldots, \hat{V}_i, \ldots, V_k\right)\right) + \sum_{i<j} (-1)^{i+j} \omega\left([V_i, V_j], V_0, \ldots, \hat{V}_i, \ldots, \hat{V}_j, \ldots, V_k\right)$$

$$w([u_1,u_2],u_3)$$

$$= \begin{vmatrix} w_1([u_1,u_2]) & w_1(u_3) \\ w_2([u_1,u_2]) & w_2(u_3) \end{vmatrix}$$

$$dw(V_0,V_1) = V_0\,w(V_1) - V_1\,w(V_0) - w([u_1,u_2])$$

$$w_1([u_1,u_2]) = u_1 w_1(u_2) - u_2 w_1(u_1) - dw_1(u_1,u_2)$$

$$w_2([u_1,u_2]) = u_1 w_2(u_2) - u_2 w_2(u_1) - dw_1(u_1,u_2)$$

$$\begin{vmatrix} u_1 w_1(u_2) - u_2 w_1(u_1) - dw_1(u_1,u_2), & w_1(u_3) \\ u_1 w_2(u_2) - u_2 \underbrace{w_2(u_1)}_{u_2\in\ker\varphi} - dw_2(u_1,u_2), & w_2(u_3) \end{vmatrix}$$

$$= u_1^2 \begin{vmatrix} w_1(u_2) & \underbrace{w_1(u_3)}_{=0} \\ w_2(u_2) & w_2(u_3) \end{vmatrix} + u_2^2 \begin{vmatrix} \underbrace{u_2 \in \ker \ldots}_{=0} & \\ & \end{vmatrix} \quad u_1 \in \ker\varphi$$

$$+ \begin{vmatrix} dw_1(u_1,u_2) & w_1(u_3) \\ dw_2(u_1,u_2) & w_2(u_3) \end{vmatrix}$$

?

$$\begin{vmatrix} a+b & c \\ d+e & g \end{vmatrix} = (a+b)g - c(d+e)$$

$$= \begin{vmatrix} a & c \\ d & g \end{vmatrix} + \begin{vmatrix} b & c \\ e & g \end{vmatrix}$$

$= d\omega_1(u_1, u_2)\cdot\omega_2(u_3) - d\omega_2(u_1, u_2)\,\omega_1(u_3)$

$\omega(u_1) = \omega(u_2) = 0$

$[u_1, u_2]\,\omega = \omega([u_1, u_2])$

$= -d\omega(u_1, u_2) = 0$

$\omega\wedge d\omega = 0$ $\quad d\omega\wedge\omega(u_1, u_2, u_3)$

$0 = d\omega(u_1, u_2)\,\omega(u_3)$

$- d\omega(u_2, u_3)\,\omega(u_1)$

$+ d\omega(u_3, u_1)\,\omega(u_2)$

$= d\omega(u_1, u_2)\,\omega(u_3)$

u_3 を $\omega \neq 0$ とすれば $d\omega(u_1, u_2) = 0$

とかで いける

注意 ω は codim 2

を orc... globally generated

(local では いけるが 注意 &

codim 2 なら ...

...)

(예11) Induced $\quad f : X \dashrightarrow Y$

g : Y 上 a foliation

$(f_s^{-1}(g))^{an} \subset T_X$ 上의 foliation?

f가 induce 하는 foliation 인가?

Q なぜ foliation を考えるのか?

"Foliation" = "解 と"...

Thm (Druel? ...)

$\mathcal{F}$: Foliation with algebraic leaf

$\exists!$ (unique)

$$\begin{array}{ccc} Z & \xrightarrow{\alpha} & Y \\ \beta \downarrow & & \\ X & & \end{array}$$

- Y, Z normal proj.
- $\forall y \in Y$ general, $\beta(\alpha^{-1}(y)) \subset X$ が $\mathcal{F}$ の leaf の closure.

ただし Y: Chow(X) の normalization で一般点が $\mathcal{F}$ の general leaf の closure.

$Z \to Y \times X$ = universal cycle の closure.

There is a unique normal complex projective variety Y contained in the normalization of the Chow variety of X whose general point parametrizes the closure of a general leaf of $\mathscr{G}$ (viewed as a reduced and irreducible cycle in X). Let $Z \to Y \times X$ denotes the normalization of the universal cycle. It comes with morphisms

$$\begin{array}{ccc} Z & \xrightarrow[\text{bir}]{\beta} & X \\ \psi \downarrow & & \\ Y & & \end{array}$$

where $\beta\colon Z \to X$ is birational and, for a general point $y \in Y$, $\boxed{\beta(\psi^{-1}(y))} \subset X$ is the closure of a leaf of $\mathscr{G}$. The variety Y is called the *family of leaves* of $\mathscr{G}$.

$\forall y \in Y$ general,

Suppose furthermore that $K_{\mathscr{G}}$ is **Q**-Cartier. There is a canonically defined effective Weil **Q**-divisor B on Z such that

$$K_{Z/Y} - R(\psi) + \underbrace{\boxed{B}}_{\beta\text{-exceptional}} \sim_{\mathbb{Q}} \beta^* K_{\mathscr{G}}, \qquad (3.1)$$

where $R(\psi)$ denotes the ramification divisor of ψ.

Remark 3.11 In the setup of 3.10, notice that B is β-exceptional. This is an immediate consequence of Example 3.9.

$(\det \mathscr{G}^\vee \simeq 0)$

The following result proves a kind of a converse:

Lemma 4.12. *Let X be a smooth projective variety and let $\mathscr{F}$ an algebraically integrable foliation on X. Then there is a unique irreducible closed subvariety W of* Chow(X) *whose general point parametrizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). In other words, if $U \subseteq W \times X$ is the universal cycle with projections $\pi\colon U \to W$ and $e\colon U \to X$, then e is birational and $e(\pi^{-1}(w)) \subseteq X$ is the closure of a leaf of $\mathscr{F}$ for a general point $w \in W$.*

$$\begin{array}{ccc} U & \xrightarrow{\pi} & W \\ \text{birat}\;\; e \downarrow & & \\ X & & \end{array}$$

Then there exists a foliation $\widehat{\mathscr{F}}$ on the normalisation $\nu\colon U^\nu \to U$ induced by $\pi \circ \nu$ and which coincides with $\mathscr{F}$ on $(e \circ \nu)^{-1}(X^\circ)$, where X° is a big open subset of X.

Proof. Since $\mathscr{F}$ has uncountably many leaves and Chow(X) has countably many irreducible components, there exists a closed subvariety W of Chow(X) such that:

(a) the universal cycle over W dominates X, and

(b) the subset W' of points in W parametrizing leaves of $\mathscr{F}$ (viewed as reduced and irreducible cycles in X) is Zariski dense in W.

Let $U \subseteq W \times X$ be the universal cycle over W, denote by $p\colon W \times X \to W$ and $q\colon W \times X \to X$ the projections, and set $\pi = p|_U$ and $e = q|_U$. It is clear that e is birational.

Lemma 3.2. *Let X be normal projective variety, and $\mathscr{F}$ an algebraically integrable foliation on X. There is a unique irreducible closed subvariety W of* Chow(X) *whose general point parametrizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). In other words, if $U \subset W \times X$ is the universal cycle, with universal morphisms $\pi : U \to W$ and $e : U \to X$, then e is birational, and, for a general point $w \in W$, $e(\pi^{-1}(w)) \subset X$ is the closure of a leaf of $\mathscr{F}$.*

Notation 3.3. We say that the subvariety W provided by Lemma 3.2 is *the closure in* Chow(X) *of the subvariety parametrizing general leaves of $\mathscr{F}$.*

Proof of Lemma 3.2. First of all, recall that Chow(X) has countably many irreducible components. On the other hand, since we are working over $\mathbb{C}$, $\mathscr{F}$ has uncountably many leaves. Therefore, there is a closed subvariety W of Chow(X) such that

(1) the universal cycle over W dominates X, and
(2) the subset of points in W parametrizing leaves of $\mathscr{F}$ (viewed as reduced and irreducible cycles in X) is Zariski dense in W.

Let $U \subset W \times X$ be the universal cycle over W, denote by $p : W \times X \to W$ and $q : W \times X \to X$ the natural projections, and by $\pi = p|_U : U \to W$ and $e = q|_U : U \to X$ their restrictions to U. We need to show that, for a general point $w \in W$, $e(\pi^{-1}(w)) \subset X$ is the closure of a leaf of $\mathscr{F}$.

Rem $-K_{\mathcal{F}} := (\det(\mathcal{F})^{\wedge\wedge})$ etc.

$\varphi : X \to Y$ f.s. induce $\mathcal{F}_{\varphi}$ foliation

$\mathcal{G} := \mathcal{F}_{\varphi}$ $\quad K_{\mathcal{G}} = K_{X/Y} - \mathrm{Ram}(\varphi)$

$\mathrm{Ram}(\varphi) := \sum_{D \subset Y \text{ prime}} \left(f^*D - (f^*D)_{red}\right)$

Idea:

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y \\ \beta \downarrow & & \\ X & & \end{array}$$

$\beta^* K_{\mathcal{F}} \cong K_{Z/Y} - \mathrm{Ram}(f) + B$

B : β-exceptional eff divisor

[2] Classical Ehresmann Theorem

allows to deduce properties of the universal covering from our structure results.

3.17. Theorem. [CLN85, V.,§2,Prop.1 and Thm.3] *Let $\phi : X \to Y$ be a submersion of manifolds with an integrable connection, i.e., an integrable subbundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$. Suppose furthermore one of the following:*

1.) *ϕ is proper.*
2.) *the restriction of ϕ to every V-leaf is a (not necessarily finite) étale map.*

Then $\phi : X \to Y$ is an analytic fibre bundle with typical fibre F. More precisely, if $\tilde{Y} \to Y$ is the universal cover, there is a representation $\rho : \pi_1(Y) \to \mathrm{Aut}(F)$ such that X is isomorphic to $(\tilde{Y} \times F)/\pi_1(Y)$. Denote by $\tilde{F} \to F$ the universal cover of F; then the map $\mu : \tilde{Y} \times \tilde{F} \to \tilde{Y} \times F \to (\tilde{Y} \times F)/\pi_1(Y) \simeq X$ is the universal cover of X and $\mu^* V \simeq p_{\tilde{Y}}^* T_{\tilde{Y}}$ and $\mu^* T_{X/Y} \simeq p_{\tilde{F}}^* T_{\tilde{F}}$.

$\phi : X \to Y$ submersion

& $\quad T_X = V \oplus T_{X/Y}$

$V \subset T_X$. integrable subbundle.

(1) ϕ = proper

$\Rightarrow$ ・ $\tilde{X} \simeq \tilde{Y} \times \tilde{F}$ $\quad \Rightarrow \rho : \pi_1(Y) \to \mathrm{Aut}(F)$

・ $X \simeq \tilde{Y} \times F / \pi_1(Y)$

(Rem $\mathcal{F} \oplus \mathcal{F}^{\perp} = T_X$ $\quad \mathcal{F} = T_{X/Y}$ $\quad \phi : X \rightsquigarrow Y$ で

$\mathcal{F}, \mathcal{F}^{\perp}$ foliation でも こういう分解は できるっぽい)

③ Fiber RCの定義

(cf. Höring-Ou, Solá Conde-Kebekus, Toma)

$V \subset T_X$ integrable subbundle

1つのleafが RC & cpt

Then $\exists\ f : X \to Y$ submersion

Y = smth, $T_{X/Y} = V$

Rem Reeb's stability Theorem

$\exists$ leaf が cpt & $\pi_1(F)$ が有限

$\Rightarrow$ すべてのleafが $\cong$

Foliation について Problems(?)

$$\begin{tikzcd} & \Gamma \arrow[dl, "bir"'] \arrow[dr] & \\ X \arrow[rr, dashed] & & Y \end{tikzcd}$$

map がつくれる (らしい)

① $\mathcal{F} \subset T_X$ がいつ foliation になるか

→ 1つの解答として slope を使ったものがある.

② foliation の leaf がいつ algebraic になるか.

→ Druel (Bost) による判定法 (mod p reduction を使う)

Campana-Păun による判定法 ($\mathcal{F}^*$ not psef ならばなる!)

③ algebraic foliation による map はいつ morphism になるか

→ Höring, Kebekus-Solacarde... RC. compact leaf だけ示した.

何にでもなる! (Druel が示した)

一般に　$\mathcal{F} \subset T_X$ が foliation を判定する方法はすくない

⇝ $[\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$ が むずかしい

$\mathcal{F}$ = saturated はどうにかなる

(saturation をとればよい.

Remark 4.2. Let X be a smooth variety and let $\mathscr{F} \subseteq T_X$ be a saturated subsheaf. The map $[\cdot,\cdot]\colon \mathscr{F} \times \mathscr{F} \to T_X$ is not $\mathscr{O}_X$-bilinear. Indeed, for two local sections $f_1, f_2 \in \mathscr{F}$ and a local section $s \in \mathscr{O}_X$ we have

$$[sf_1, f_2] = s[f_1, f_2] - f_2(s)f_1 \quad \text{and} \quad [f_1, sf_2] = s[f_1, f_2] + f_1(s)f_2.$$

$T_X/\mathcal{F}$ で 0 となる

This implies that the induced map $[\cdot,\cdot]\colon \mathscr{F} \times \mathscr{F} \to T_X/\mathscr{F}$ is $\mathscr{O}_X$-bilinear and anti-commutative, hence induces an $\mathscr{O}_X$-linear map $\wedge^2 \mathscr{F} \to T_X/\mathscr{F}$, hence an $\mathscr{O}_X$-linear map $(\wedge^2 \mathscr{F})/(\text{torsion}) \to T_X/\mathscr{F}$. Therefore, $\mathscr{F}$ is a foliation if and only if this map is the zero map.

int condition

$[\cdot,\cdot]\ \mathcal{F} \times \mathcal{F} \to T_X$

$[sf_1, f_2] = sf_1(f_2) - f_2(sf_1)$

$= s[f_1, f_2] \boxed{- f_2(s) f_1}$

$[f_1, f_2 s] = s[f_1, f_2] + \boxed{f_1(s) f_2}$

iいける

これから

$[\cdot, \cdot]^{\sim} : \mathcal{F} \times \mathcal{F} \to T_X \to T_X/\mathcal{F}$

- $\mathcal{O}_X$ bilinear
- anticommutative

induce

これが $\wedge^2 \cancel{\mathcal{F}} \to T_X/\mathcal{F} \Leftarrow \mathrm{Hom}(\wedge^2\mathcal{F}/\mathrm{tor}, T_X/\mathcal{F})$

torsion

$= 0$

2つの判定法 $\mathrm{Hom}(\Lambda^2 F/\mathcal{E}_W, T_X/F) = 0$

ということでこの条件になるのをもっとわかりやすいにかえたい

$\Longrightarrow$ slope

Def

$\cdot\ \overline{\mathrm{Mov}}(X) = \overline{\mathbb{R}\left\{ \pi_* C \;\middle|\; \begin{array}{l} \pi : X' \to X \text{ modification} \\ C = \bigcap_{i=1}^{n-1} D_i \quad D_i \in |m_i H_i'| \quad \text{complete intersection curve} \\ H_1, \ldots, H_{n-1} \text{ ample on } X' \end{array} \right\}}$

$\cap$

$N_1(X)$

movable cone.

$\cdot\ \mu_\alpha(\mathcal{E}) = \dfrac{c_1(\mathcal{E}) \cdot \alpha}{\mathrm{rk}\,\mathcal{E}} \qquad \alpha \in \overline{\mathrm{Mov}}(X)$

($\alpha = \pi_* C$ なら $c_1(\mathcal{E})\,\alpha = c_1(\pi^*\mathcal{E}) H_1 \cdots H_{n-1}$)

$\cdot\ \mathcal{E}$: α (semi)stable

$\Longleftrightarrow$ $^\forall 0 \neq F \subset \mathcal{E}$, $\mu_\alpha(F) \underset{(\leq)}{<} \mu_\alpha(\mathcal{E})$

$\mu_\alpha^{\max}(\mathcal{E}) = \sup\{\mu_\alpha(F) \mid \text{of } F \subseteq \mathcal{E}\}$ coherent subsheaf

$= \mu_\alpha^{\max}(F_1)$

$\mu_\alpha^{\min}(\mathcal{E}) = \inf\{\mu_\alpha(Q) \mid \mathcal{E} \to Q \neq 0\}$ coherent subsheaf

$= \mu_\alpha^{\min}(Q_1)$

attained

(1.3.11)を満たすこと

Remark 2.8. Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$. If we have a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{E} \to 0$$

of torsion-free coherent $\mathscr{O}_X$-modules, then clearly

$$\operatorname{rk}\mathscr{F}\,\mu_\alpha(\mathscr{F}) = \operatorname{rk}\mathscr{G}\,\mu_\alpha(\mathscr{G}) + \operatorname{rk}\mathscr{E}\,\mu_\alpha(\mathscr{E}).$$

Proposition 2.9. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$. Let $\mathscr{E}$ and $\mathscr{F}$ be torsion-free coherent sheaves on X.*

(i) *If $\mathscr{F} \subseteq \mathscr{E}$ and $\operatorname{rk}\mathscr{E} = \operatorname{rk}\mathscr{F}$, then $\mu_\alpha(\mathscr{F}) \le \mu_\alpha(\mathscr{E})$. In particular, $\mu_\alpha(\mathscr{F}) \le \mu_\alpha(\mathscr{F}_{\mathrm{sat}})$.*

(ii) *An α-stable sheaf is α-semistable.*

(iii) *We have $\mu_\alpha^{\max}(\mathscr{E}) = -\mu_\alpha^{\min}(\mathscr{E}^*)$.*

Proof. We have $\det\mathscr{F} \subseteq \det\mathscr{E}$, and since $\det\mathscr{E}$ and $\det\mathscr{F}$ are line bundles, there exists an effective Cartier divisor D such that

$$\det\mathscr{F} \otimes \mathscr{O}_X(D) \simeq \det\mathscr{E}.$$

In particular, we have $c_1(\mathscr{F}) + D = c_1(\mathscr{E})$. Since α is movable, we have that $D \cdot \alpha \ge 0$, and (i) follows. The claim (ii) is an obvious consequence of (i).

For (iii), since the functor $\mathscr{H}om(\cdot, \mathscr{O}_X)$ is contravariant left-exact, every torsion-free quotient $\mathscr{E}^* \twoheadrightarrow \mathscr{Q}$ gives rise to an inclusion $\mathscr{Q}^* \hookrightarrow \mathscr{E}^{**}$. Therefore,

$$\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha(\mathscr{Q}) = \mu_\alpha^{\max}(\mathscr{E}^{**}) - \mu_\alpha(\mathscr{Q}^*) \ge 0 \quad \text{for every } \mathscr{Q},$$

hence $\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha^{\min}(\mathscr{E}^*) \ge 0$. Conversely, every inclusion $\mathscr{F} \subseteq \mathscr{E}$ of torsion-free sheaves gives rise to a map $\mathscr{E}^* \to \mathscr{F}^*$, which is surjective on a big open subset of X on which both $\mathscr{E}$ and $\mathscr{F}$ are locally free. Therefore, by (i) we have

$$\mu_\alpha^{\min}(\mathscr{E}^*) + \mu_\alpha(\mathscr{F}) = \mu_\alpha^{\min}(\mathscr{E}^*) - \mu_\alpha(\mathscr{F}^*) \le 0 \quad \text{for every } \mathscr{F},$$

hence $\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha^{\min}(\mathscr{E}^*) \le 0$, which proves (iii). □

Proposition 2.10. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$. If $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are torsion-free coherent sheaves on X, then the following holds:*

(i) *if $\mathscr{F}$ is an α-semistable sheaf and if $\gamma: \mathscr{F} \to \mathscr{E}$ is a morphism of torsion-free $\mathscr{O}_X$-modules, then $\mu_\alpha(\gamma(\mathscr{F})) \ge \mu_\alpha(\mathscr{F})$,*

α-semistable

(ii) *if $\mathscr{F}$ is α-semistable and if $\mu_\alpha(\mathscr{F}) > \mu_\alpha^{\max}(\mathscr{E})$, then $\operatorname{Hom}(\mathscr{F}, \mathscr{E}) = 0$,*

(iii) *if $\mathscr{F}$ and $\mathscr{E}$ are torsion free coherent sheaves such that $\mu_\alpha^{\min}(\mathscr{F}) > \mu_\alpha^{\max}(\mathscr{E})$, then $\operatorname{Hom}(\mathscr{F}, \mathscr{E}) = 0$,*

(iv) *if $\mathscr{F} = \mathscr{E} \oplus \mathscr{G}$, then $\mathscr{F}$ is α-semistable if and only if $\mathscr{E}$ and $\mathscr{G}$ are α-semistable with $\mu_\alpha(\mathscr{E}) = \mu_\alpha(\mathscr{G})$,*

(v) *if $\mathscr{F}$ is a saturated subsheaf of $\mathscr{E}$ and if $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\max}(\mathscr{E})$, then*

$$\mu_\alpha^{\max}(\mathscr{E}/\mathscr{F}) \le \mu_\alpha(\mathscr{F}) = \mu_\alpha^{\min}(\mathscr{F}).$$

E/F torsion free

$\mathfrak{q}$CF

(2) $0 \to \operatorname{Ker}\gamma \to F \to \gamma(F) \to 0$

$0 \to \mathcal{E} \to F \to G \to 0$

$\operatorname{rk}\mathcal{E} + \operatorname{rk}G = \operatorname{rk}F$

$c_1(\mathcal{E}) + c_1(G) = c_1(F)$

(1) $\operatorname{rk}F\,\mu_\alpha(F) = \operatorname{rk}G\,\mu_\alpha(G) + \operatorname{rk}\mathcal{E}\,\mu_\alpha(\mathcal{E})$

F: α-semistable $\Leftrightarrow \mu_\alpha(F) \ge \mu_\alpha(\mathcal{E}) \Rightarrow \mu_\alpha(F) \le \mu_\alpha(G)$

(2) $\operatorname{Hom}(F, G) \ni \gamma \neq 0 \Rightarrow \mu_\alpha(F) \le \mu_\alpha(\gamma(F)) \le \mu_\alpha^{\max}(\mathcal{E})$

(3) $\operatorname{Hom}(F, \mathcal{E}) \ni \gamma$, $\mu_\alpha^{\min}(F) \le \mu_\alpha(\gamma(F)) \le \mu_\alpha^{\max}(\mathcal{E})$

Rem $\mathcal{F} \subset \mathcal{G}$ α-semistable

$$\Longrightarrow (\mathcal{F} \otimes \mathcal{G})^{\wedge\wedge} \ \alpha\text{-semistable}$$

これはめっちゃむずい…

(proof) ① $\mu_\alpha^{max}((\mathcal{F} \otimes \mathcal{G})^{\wedge\wedge}) = \mu_\alpha^{max}(\mathcal{F}) + \mu_\alpha^{max}(\mathcal{G})$

$$\mu_\alpha^{max}((\otimes^{a}\mathcal{F})^{\wedge\wedge}) = a\,\mu_\alpha^{max}(\mathcal{F})$$

② $T_X \supset \mathcal{F}$: saturated coherent sheaf とする

$$\mu_\alpha^{min}(\mathcal{F}) > \frac{1}{2}\mu_\alpha^{max}(T_X/\mathcal{F})$$

ならば $\mathcal{F}$ は foliation

$(Hom(\wedge^2\mathcal{F}/\mathrm{tor}, T_X/\mathcal{F}) = 0$ より…

$c_1(\det(\mathcal{F}_r)) = 0$

↑

何が存在するところ

$\mathbb{P}^1_C$

μ

$\xrightarrow{\ \mathcal{E}\ }$ $\ker r \subseteq \mathcal{E}$

$(\mathcal{F} \xrightarrow{r} Q$, $\mu_\alpha(\ker \mathcal{F}) \le \mu_\alpha(\mathcal{F})$

$\mu_\alpha(Q) \ge \mu_\alpha(\mathcal{F})$

$\Longrightarrow$ $\ge$ から言える ($\subseteq \mathcal{E}$)

↓

(5) $\mu_\alpha(\mathcal{F}) = \mu_\alpha^{max}(\mathcal{E})$ $\quad \mu_\alpha^{max}(\mathcal{E}/\mathcal{F}) \le \mu_\alpha(\mathcal{F}) = \mu_\alpha^{min}(\mathcal{F})$

$\mathcal{G} \subseteq \mathcal{E}/\mathcal{F} \longleftrightarrow \mathcal{E} \supset \mathcal{G}'$ $\quad \mu_\alpha(\mathcal{G}) \le \mu_\alpha(\mathcal{F})$

maxより

Thm $\mathcal{F} \subseteq T_X$ s.t. $\mu_\alpha^{max}(T_X) = \mu_\alpha(\mathcal{F})$!

If $\mu_\alpha^{max}(T_X) > 0 \Rightarrow \mathcal{F}$ foliation

pf. $\mathcal{F}$ = α semistable wrt ($\mathcal{F}$ is saturated)

$$2\mu_\alpha^{min}(\mathcal{F}) = 2\mu_\alpha(\mathcal{F}) \gneq \mu_\alpha(\mathcal{F}) \geq \mu_\alpha^{max}(T_X/\mathcal{F})$$

wrt

② $\mathcal{F}$: foliation

$\mathcal{F}$ の leaf は いつ algebraic になるか?

$\mathcal{F}$: foliation, $\Delta \subset X_\mathcal{F} \times X_\mathcal{F}$

$x \in X_\mathcal{F}$. $x \in W \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$. $X_\mathcal{F} \times X_\mathcal{F}$

$\cup$

$W \times W$

$\cup$

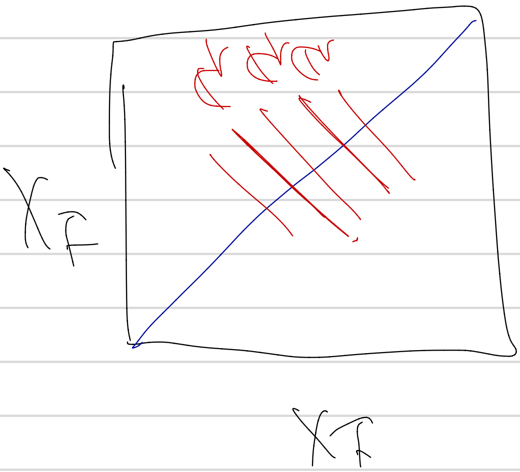
$V \subset X_\mathcal{F} \times X_\mathcal{F}$

$\{(x,z,y,z) \mid x \in \mathbb{C}^r, (y,z) \in W\} \simeq \Delta_{W\times W} \times \mathbb{C}^r$

$(x,z,y,z) \longrightarrow ((y,z),(y,z),x-y)$

leaf が [illegible]

$r < 3$



Lemma 4.7. *Let $\mathscr{F}$ be a foliation of rank r on a smooth variety X of dimension n and denote $X^\circ = X \setminus \mathrm{Sing}(\mathscr{F})$. Let $\Delta \subseteq X^\circ \times X^\circ$ be the diagonal, and let p_1 and p_2 be the projections of $X^\circ \times X^\circ$ onto the factors.*

Then there exists a smooth locally closed analytic submanifold $V \subseteq X^\circ \times X^\circ$ containing Δ such that $p_2|_V$ is smooth and such that its fibres are analytic open subsets of the leaves of the foliation $p_1^*\mathscr{F}|_{X^\circ}$ passing through points of Δ. Moreover, $N_{\Delta/V} \simeq \mathscr{F}|_{X^\circ}$. The analytic germ of V along Δ is the analytic graph *of the foliation $\mathscr{F}$.*

$\Delta \subset V \subset X^\circ_\mathcal{F} \times X_\mathcal{F}$ locally closed analytic subsp

$p_2|_V$ smooth

X°

$\boxed{N_{\Delta/V} \simeq \mathcal{F}|_{X^\circ}}$

Δ と交わる $p_1^*\mathcal{F}|_{X_\mathcal{F}}$ の foliation の leaf $=$ $p_2|_V$ の fiber

$\boxed{\text{Thm}}$ $\dim \overline{V}^{2ar} = n + r \Rightarrow$ leaf alg

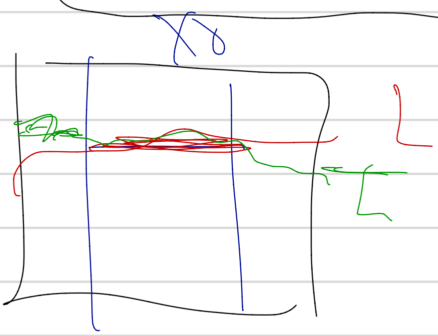
$\pi : \overline{V}^{2ar} \longrightarrow X$ $\quad \overline{V} \supset \Delta$ sing

$\cup$ general fiber $\dim \overline{V}^{2ar} - n$

$\pi^{-1}(X^0) = V^0 \longrightarrow X^0$ (fiber dim r)

$L := V^0 \cap \pi^{-1}(X^0)$ の連結成分 leaf

$\overline{L}^{2ar}$ in $\overline{V^0 \cap L}$ $\quad$ r dim ← これは 示すべきこと



$\underset{\text{大きいけど}}{\text{Chow}(X)} \supset^{\exists} W$

$$\begin{array}{ccc} W \times X & & \\ \cup & & \\ U & \xrightarrow{e} & W \\ r \downarrow & & \\ X & & \end{array}$$

leaf fiber = いっち

$\forall F \cap r(e^{-1}(w))$ が leaf?

(Thm 4.1 を示すのに使っている)

Thm

$\dim \bar{V}^{2a} = n+r$ （2はいえる $\leq$ Siegel）

$\Leftarrow$ $\mathcal{L}$: $X \times X$ ample,

$\dim H^0(\bar{V}^{2a}, \mathcal{L}^{\otimes k}) \leq C k^{n+r}$

をいう

$\left(\underline{\text{restriction}} \; H^0(\bar{V}^{2a}, \mathcal{L}^{\otimes k}) \to H^0(V, \mathcal{L}^{\otimes k}) \right.$

$\left. \quad f \longmapsto f|_V \right)$

$\boxed{\text{injective}}$

$\Leftarrow \dim H^0(V, \mathcal{L}^{\otimes k}) \leq C k^{n+r}$

$(0 \to \mathcal{J} \to \mathcal{O}_V \to \mathcal{O}_V / \mathcal{J} \to 0)$ $\quad \mathcal{J} \subset \mathcal{O}_V$ (the variety の ideal)

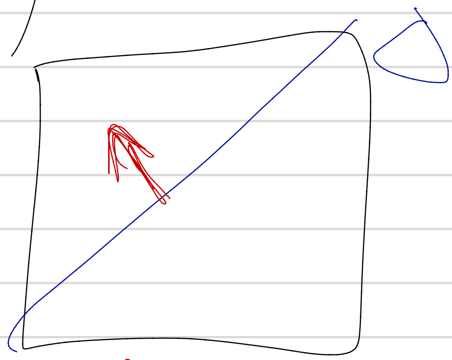
$0 \to \mathcal{L}^{\otimes k} \otimes \mathcal{J}^{m+1} \to \mathcal{L}^{\otimes k} \otimes \mathcal{J}^m \to \mathcal{L}^{\otimes k} \otimes \mathcal{J}^m / \mathcal{J}^{m+1} \to 0$

$V \supset \Delta$ $\quad N$

$$\mathcal{J}^m/\mathcal{J}^{m+1} = \mathrm{Sym}^m(\mathcal{J}/\mathcal{J}^2) = \mathrm{Sym}^m(N^*_{V/\Delta}) = \mathrm{Sym}^m(F^*)$$

$\mathcal{I}_1 \ne a$

$0 \to T_\Delta \to T_X|_\Delta \to N_{V/\Delta} \to 0$



$\mathcal{J}/\mathcal{J}^2$ a cotangent

(2nd excercise?)

$$h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{J}^{m+1}) \le h^0(V, \mathcal{L}^{\otimes k} \otimes \mathcal{J}^{m}) + h^0(\Delta, \mathcal{L}^{\otimes k} \otimes \mathrm{Sym}^m(F^*))$$

$$h^0(V, \mathcal{L}^{\otimes k}) \le \sum_{m=0}^{\infty} h^0(X, \mathcal{L}^{\otimes k} \otimes \mathrm{Sym}^m(F^*))$$

$$\Leftarrow \boxed{\sum_{m=0}^{\infty} h^0(X, L^{\otimes k_2} \otimes \mathrm{Sym}^m(F^*)) \le C k^{n}}$$

$\forall L$: ample on X

まとめると $\forall L$ ample. $\exists C$, $\forall k \geq 0$. (十分大きな k でない)

$$\sum_{m=0}^{\infty} h^0(X, L^{\otimes k} \otimes S^m \mathcal{F}^*) \leq C k^{n+1}$$

が成立

$\Rightarrow$ $\mathcal{F}$: has algebraic leaf.

Thm L ample fix. Assume.

$\exists D > 0$, $\forall k$, $\forall m \geq Dk$,

$$h^0(X, L^{\otimes k} \otimes S^m \mathcal{F}^*) = 0.$$

Then $\mathcal{F}$ has algebraic leaf

$\mathbb{P}\mathcal{F}$

$\mathbb{P}(\mathcal{F})$ の X を dominate する component

$\mathcal{F}$ reflexive

$\mathbb{P}(\mathcal{F}) \supset \mathbb{P}_{red}(S^m \mathcal{F}) \longrightarrow X$

X_0 (X sm. $\mathcal{F}|_{X_0}$ locally free)

resolving & $\pi^{-1}(X - X_0)$ divisor

P

つまり

$\pi : P \longrightarrow X \quad D = \pi^{-1}(X - X_0)$ divisor

$\mathcal{O}_P(1) := r^* \nu^* \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1)$ とする

$\exists \Delta$ eff $\operatorname{supp}(\Delta) \subseteq D$ で

then $\pi_*(\mathcal{O}_P(m) \otimes \mathcal{O}_P(m\Delta)) \simeq (S^m \mathcal{E})^{\wedge\wedge}$

2.2. Definition. *Let X be a normal variety, and let $\mathcal{E}$ be a reflexive sheaf on X.*

- *Denote by*
$$\nu : \mathbb{P}'(\mathcal{E}) \to \mathbb{P}(\mathcal{E})$$
the normalization of the unique component of $\mathbb{P}(\mathcal{E})$ that dominates X.
- *Set $\mathcal{O}_{\mathbb{P}'(\mathcal{E})}(1) := \nu^*(\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.*
- *Let $X_0 \subset X$ be the locus where X is smooth and $\mathcal{E}$ is locally free, and let*
$$r : P \to \mathbb{P}'(\mathcal{E})$$
be a birational morphism from a manifold P such that the complement of $(p \circ \nu \circ r)^{-1}(X_0) \subset P$ is a divisor D.
- *Set $\pi := p \circ \nu \circ r$ and $\mathcal{O}_P(1) := r^*(\mathcal{O}_{\mathbb{P}'(\mathcal{E})}(1))$, where $p : \mathbb{P}(\mathcal{E}) \to X$ is the projection.*
- *By* [Nak04, III.5.10.3] *there exists an effective divisor Λ supported on D such that*
$$\pi_*(\mathcal{O}_P(m) \otimes \mathcal{O}_P(m\Lambda)) \simeq S^{[m]}\mathcal{E} \qquad \forall\, m \in \mathbb{N}.$$
- *We call $\zeta := c_1(\mathcal{O}_P(1) \otimes \mathcal{O}_P(\Lambda)) \in N^1(P)$ a tautological class of $\mathcal{E}$.*

Using the defining property of a tautological class, the arguments of [Dru18, Lemma 2.7] apply literally to show the following:

2.3. Lemma. *Let X be a normal projective variety. Let $\mathcal{E}$ be a reflexive sheaf on X, and let ζ be a tautological class on $\pi : P \to X$ (cf. Definition 2.2). Then ζ is pseudoeffective if and only if $\mathcal{E}$ is pseudoeffective.*

$$\zeta := \mathcal{O}_{P(V)} \otimes \mathcal{O}(\Delta) \ni \tau_0^3$$

$$\sum_m h^0(X, L^{\otimes k} \otimes S^m F^{*})$$

$$= \sum_m h^0(P, (\pi^* L)^{\otimes k} \otimes \zeta^m)$$

A very ample on $\underline{P}$ ($n+r-1$ dim) s.t. $\pi^* L^{-1} \otimes A$, $\zeta^{\otimes -1} \otimes A$ ample

$$\sum_{m \geq 0} h^0(P, (\pi^* L)^{\otimes k} \otimes \zeta^m)$$

$$= \sum_{m=0}^{kD} h^0(P, (\pi^* L)^{\otimes k} \otimes \zeta^m)$$

$$\leq \sum_{m=0}^{kD} h^0(P, A^{m+k})$$

$$\leq (kD+1)\, h^0(P, A^{k(D+1)})$$

$$\leq C\,(kD+1)\, k^{n+r-1}$$

//

Thm (Druel 18, Campana–Paun)

① $(\star) \Leftrightarrow$ ($\mathcal{F}$: foliation) ξ is Not pseff effective ($\because P = \mathbb{P}(\mathcal{F}^*)$, $\xi = \mathcal{O}_{\mathbb{P}}(1)$)

($\mathcal{F}$ は locally free とする)

つまり $\mathcal{O}_{\mathbb{P}(\mathcal{F}^*)}(1)$ is Not pseff $\Leftrightarrow$ $\mathcal{F}$ has alg leaf

② $\mu_\alpha^{min}(\mathcal{F}) > 0 \Rightarrow (\star)$

Pf ① $\exists D > 0$, $\forall k, m$ s.t. $m > kD$

$(\star) \Leftarrow$ $H^0(X, L^{\otimes k} \otimes (\mathrm{Sym}^m \mathcal{F}^*)) = 0$ $\quad P = \mathbb{P}(\mathcal{F}^*)$

L ample

$H^0(P, \pi^* L^{\otimes k} \otimes \xi^{\otimes m})$

ξ pseff $\Rightarrow$ $\xi \otimes \pi^*(k_0 L)$ ample.

$\Rightarrow$ $\forall m$ $(\xi \otimes \pi^* k_0 L) \otimes \xi^m$ has section

$\Rightarrow$ not $(\star)$

ξ is Not pseff $\Rightarrow \exists D$. $\pi^* L \otimes \xi^{\otimes D}$ is Not pseff

$\Rightarrow m > kD$.

$\pi^* L^{\otimes k} \otimes \xi^m$ is Not pseff

② $\mu_\alpha^{min}(F) > 0$

$$H^0(X, L^{\otimes b} \otimes \mathrm{Sym}^m F^*)$$

$$= \mathrm{Hom}(\mathcal{O}_X, L^{\otimes b} \otimes \mathrm{Sym}^m F^*)$$

$$\mu_\alpha^{max}(L^{\otimes b} \otimes \mathrm{Sym}^m F^*)$$

$$= b\,\mu_\alpha^{max}(L) + (-m)\,\mu_\alpha^{min}(F)$$

$$= b\,\mu_\alpha(L) - m\,\mu_\alpha^{min}(F)$$

$$\mu_\alpha^{min}(\mathcal{O}_X) = 0$$

$$\mu_\alpha^{min}(\mathcal{O}_X) > \mu_\alpha^{max}(L^{\otimes b} \otimes \mathrm{Sym}^m F^*)$$

$$\Rightarrow H^0(\sim\sim) = 0$$

$d > 2$. $\quad b\,\mu_\alpha(L) - m\,\mu_\alpha^{min}(F) < 0$

$$\left\lceil \frac{\mu_\alpha(L)}{\mu_\alpha^{min}(F)} \right\rceil = D \quad \{ \}$$

Thm (Campana–Păun)

$$\boxed{\begin{gathered} F \subset T_X \text{ Foliation}, \exists \alpha \in \mathrm{Mov}\ \mu_\alpha^{\min}(F) > 0 \\ \Rightarrow F \text{ has algebraic leaf \& leaf is RC.} \end{gathered}}$$

$\boxed{\text{Cor}}$ $\exists \alpha \in \mathrm{Mov}\ \mu_\alpha^{\max}(T_X) > 0 \Leftrightarrow X$ uniruled

pf $\Leftarrow$

X Not Uniruled $\Leftrightarrow \forall \alpha \in \mathrm{Mov}(X),\ \mu_\alpha^{\max}(T_X) \leq 0.$

$(K_X$ psef$)$

$\left(\Lambda^m \Omega_X^p \otimes \ldots\right)$

$\left(\Lambda^n \Omega_Y, \Omega_X^{\otimes m} \ldots\right)$

$\forall$

$\Omega_X^{(\otimes m)} \twoheadrightarrow Q \to 0$

$$\mu_\alpha^{\min}(\otimes^m \Omega') \leq \mu_\alpha(Q)$$

$$0 \leq m\ \mu_\alpha^{\min}(\Omega_X)$$

$$(\mu_\alpha^{\min}(\Omega_X) = -\mu_\alpha^{\max}(T_X))$$

$\Rightarrow \forall \alpha,\ \mu_\alpha(Q) = c_1(Q) \cdot \alpha \geq 0$

$\Rightarrow c_1(Q)$ psef.

pf $\boxed{C_{ar}}$ ($\Rightarrow$) $\mu_\alpha^{max}(T_X)>0 \Rightarrow \exists \mathcal{F},\ \mu_\alpha^{max}(T_X)=\mu_\alpha^{min}(\mathcal{F})$ ... $\mu_\alpha^{max}(\mathcal{F})$

$\Rightarrow \mathcal{F}$ foliation has $\overset{RC}{alg}$ leafes.

$\Rightarrow \forall x \in X$ general point, $\exists C \ni x$ rat. curve. $=$ particular (?)

($\Leftarrow$) X uniruled $\Rightarrow K_X$ Not psef

$\Rightarrow \exists \alpha,\quad c_1(K_X)(\alpha) < 0.$

$\quad\quad -c_1(T_X)\alpha$

$\Rightarrow 0 < \mu_\alpha(T_X) \le \mu_\alpha^{max}(T_X)$

Pf of Cp

relative MRC of $f\ \tilde{X} \to Y$

(Z_y は point ではない) by general pt. $\tilde{X}_y \dashrightarrow Z_y$ (MRC)

$$\begin{array}{ccc} \tilde{X} & \overset{\pi}{\dashrightarrow} & Z \\ \varphi\downarrow & \searrow^{f} & \downarrow g \\ X & & Y \end{array}$$

π を morphism にとれる (blowup をとることで)

blowupで

π morphism, π flat over open set on Z

$f_{exc} \subset \varphi_{exc}$

$\pi_{exc} \subset \varphi_{exc}$

$$\begin{array}{ccc} X & \xrightarrow{r} & Y \\ \mu' \uparrow & & \uparrow \text{birat} \\ X' & \xrightarrow{r'} & Y' \end{array}$$

$(r'_{exc}) \subset \mu'_{exc}$

$\hat{\mathcal{F}}$ をつぎでていぎする

$$0 \to \hat{\mathcal{F}} \to T_{\tilde{X}} \xrightarrow{df} f^* T_Y$$

[by open set]

$\hat{\mathcal{F}}$ と $\pi^* \mathcal{F}$ は φ_{exc} 上で一致

Since they are both reflexive by (a), they coincide on the whole X. □

If X is a projective variety and if $\pi\colon \tilde{X} \to X$ is any resolution of singularities, then it is easy to see that a pullback of a movable class on X is again movable on $\tilde{X}$.

Lemma 2.25. *Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$ be a movable class. If $\mathscr{F}$ and $\mathscr{G}$ are torsion-free coherent sheaves of positive rank on X. Let $\pi\colon \tilde{X} \to X$ be any resolution of singularities. Then the following holds.*

(i) The sheaf $\mathscr{F}$ is α-stable if and only if $\pi^{[*]}\mathscr{F}$ is $\pi^*\alpha$-stable.

(ii) The sheaf $\mathscr{F} \boxtimes \mathscr{G}$ is α-semistable if and only if $\pi^{[*]}\mathscr{F} \boxtimes \pi^{[*]}\mathscr{G}$ is $\pi^*\alpha$-semistable.

Proof. For (i), let $\mathscr{E}$ be a torsion-free subsheaf of $\pi^{[*]}\mathscr{F}$. Then $\mathscr{E}$ and $\pi^*\pi_*\mathscr{E}$ agree away from the exceptional locus, hence if E_i are the prime exceptional divisors on $\tilde{X}$, there integers λ_i such that

$$c_1(\mathscr{E}) = \pi^* c_1(\pi_*\mathscr{E}) + \sum \lambda_i E_i.$$

Therefore $\mu_{\pi^*\alpha}(\mathscr{E}) = \mu_\alpha(\pi_*\mathscr{E})$. Conversely, let $\mathscr{G}$ be a torsion-free subsheaf of $\mathscr{F}$. Then $\mu_{\pi^*\alpha}(\pi^{[*]}\mathscr{G}) = \mu_\alpha(\pi_*\pi^{[*]}\mathscr{G})$ by what we just proved. Since $\pi_*\pi^{[*]}\mathscr{G}$ and $\mathscr{G}$ agree on a big open subset of X, we have $c_1(\pi_*\pi^{[*]}\mathscr{G}) = c_1(\mathscr{G})$, and hence $\mu_\alpha(\pi_*\pi^{[*]}\mathscr{G}) = \mu_\alpha(\mathscr{G})$. Now (i) follows (exercise), and (ii) is similar. □

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$$c_1(\hat{\mathcal{F}}) = c_1(\pi^*\mathcal{F}) + \sum \lambda_i E_i, \quad E_i \subset \varphi_{exc}$$

$$\boxed{c_1(\hat{\mathcal{F}}) \cdot \varphi^*\alpha = c_1(\mathcal{F}) \cdot \alpha}$$

$$\mu^{min}_{\varphi^*\alpha}(\hat{\mathcal{F}}) = \mu^{min}_{\alpha}(\hat{\mathcal{F}}) > 0$$

$(\forall \Sigma \subset \hat{F} \quad (\varphi_* \Sigma)$

$$\mu_{\varphi^*\alpha}(\Sigma) = \mu_\alpha(\varphi_*\Sigma) \le \mu_\alpha$$

$$\mu_\alpha((\varphi_*\Sigma) \cap F) = \mu_\alpha(\varphi^*\varphi_*\Sigma \cap \varphi^* F)$$

$$= \mu_\alpha(\varphi^*\varphi_*\Sigma) \int_{big}$$

$$\ge \mu_\alpha(\Sigma)$$

$\mathcal{G} \subset F$

$$\mu_\alpha(\mathcal{G}) = \mu_\alpha(\pi^* \mathcal{G} \cap \hat{F})$$

$$= \mu_{\pi^*\alpha}(\pi^* \mathcal{G}$$

$$0 \to \mathcal{G} \to T_Z \xrightarrow{dg} g^* T_Y \quad \text{foliation}$$

$$\begin{tikzcd} \hat{X} \arrow[r, "\pi"] \arrow[d, "\varphi"'] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

$\boxed{0 \neq}$ ~~Z positive dim~~

$$\mathcal{G}' = d\pi(\hat{F}) \subset \pi^* T_Z$$

$\left(T_X \xrightarrow{d\pi} \pi^* T_Z \right)$ on $\hat{X}$)

$$\mu_{\varphi^*\alpha}(\mathcal{G}') \ge \mu^{min}_{\varphi^*\alpha}(\hat{F}) > 0$$

$\mathcal{G}' \subset \pi^* \mathcal{G}$ b/c π exc $\subset$ Z ...

$\hat{\varphi}$ exc.

$$\Rightarrow \mu_{\varphi^*\alpha}(\mathcal{G}') = \mu_{\varphi^*\alpha}(\pi^{*c}\mathcal{G}) > 0.$$

$$0 < \mu_{\varphi^*\alpha}(\pi^*\mathcal{G}) = \frac{c_1(\pi^*\mathcal{G})\cdot\varphi^*\alpha}{\mathrm{rk}\,\mathcal{G}'}$$

$$= \frac{-\pi^* K_{\mathcal{G}}\cdot\varphi^*\alpha}{\mathrm{rk}\,\mathcal{G}'}$$

($\pi^*\det\mathcal{G}' = \det \pi^*\mathcal{G}'$)

π exc で一致 $\Rightarrow$ φ exc で一致

($K_{\mathcal{G}} = (\det\mathcal{G})^{-1}$.

ここで $\mathcal{G}$ は g から induced されたもの

$$K_{\mathcal{G}} = \underline{K_{Z/Y} - \mathrm{Ram}(g)} + \underline{E}$$

(Z が positive dim なら

(g-exc (blowup ででてくる)

$$-\pi^* K_{\mathcal{G}}\cdot\varphi^*\alpha$$

$$= -\pi^*\big(\boxed{K_{Z/Y} - \mathrm{Ram}(g)}\big)\cdot\varphi^*\alpha \le 0$$

psef

で矛盾

Corollary 3.1. *Let X be a smooth projective variety and let $f : X \to Z$ be a morphism with connected fibred such that Z is smooth and such that K_F is pseudoeffective for a general fibre F of f. Then the divisor*

$$K_{X/Z} - \mathrm{Ram}(f)$$

is pseudoeffective.

K_F psef

(MRC から

Application (the) Lazić–Peternell

X : minimal terminal n-fold, $\nu(X, K_X) = 1$.

$$\chi(X, \mathcal{O}_X) \neq 0 \implies \kappa(X, K_X) \geq 0$$

Plan

① K_X nef & $\nu(X, K_X) = 1$ & $K_X \equiv_{\mathbb{Q}} D + F$, $D \geq 0$, F psef $\implies K_X \equiv E$, $\kappa(K_X) \geq 0$

$h^0(mK_X) \geq h^0(mK_X + P)$, $c_1(P) = 0$ & $h^0(mK_X + P) \neq 0$ ($P = E - K_X$)

② $\pi : Y \to X$ resol of sing.

$\kappa(Y, K_X) = -\infty$

$\implies \forall p \leq n$, $\forall \{m_i\}_{i=1}^{\infty}$ sufficiently divisible

$$H^0(Y, \Omega^p \otimes \mathcal{O}(m_i \pi^* K_X)) = 0$$

$\exists p \leq n$, $\exists \{m_i\}_{i=1}^{\infty}$ suff divi.

$$H^0(Y, \Omega^{p_i} \otimes \mathcal{O}(m_i \pi^* K_X)) \neq 0$$

$\implies \kappa(X, K_X) \geq 0$

① $f^*F = \underbrace{P_\sigma(f^*F)}_{\boxed{P}\ \text{very ample} \ \text{of hyperplane}} + \underbrace{N_\sigma(f^*F)}_{N}$ eff

$\dim X = 3 = n$

sections is nef

$P \equiv 0 \Rightarrow F' = N + D'$.

$\Rightarrow K_X = E'$

$P \not\equiv 0$ $S = H_1 \cap \cdots \cap H_{n-2}$ $\subset X$

$$f^*F|_S = \underbrace{P|_S}_{\text{nef}} + \underbrace{N|_S}_{\text{eff}}$$

$$0 = ((f^*K_X)|_S)^2 = \underbrace{(f^*K_X \cdot P)|_S}_{\geq 0} + \underbrace{f^*K_X N|_S + (f^*K_X \cdot D)}_{\geq 0 \quad \geq 0}$$

$\Rightarrow (f^*K_X \cdot P) = 0 =$

$\Rightarrow$ $P|_S \equiv \lambda f^*K_X|_S$ $(P \equiv \lambda f^*K_X$

Hodge index thm

$\Rightarrow$ $P \equiv \lambda f^*K_X$

Lefschetz

$$f^*K_X \equiv \frac{1}{1-\lambda}(N + D') \quad ((\lambda K_X)\ \text{eff})$$

② $\Gamma = \{ m \in \mathbb{Z} \mid H^0(Y, \Omega_Y^p \otimes \pi^* \mathcal{O}_X(mK_X)) \neq 0 \}$

とする。

$K_X \not\equiv 0$ とにしない ($K_X \equiv 0 \Rightarrow K_X \sim_{\mathbb{Q}} 0$.) Kawamata

$p \geq 1$ としない

$\underline{K_X \text{ nef}}$ の $H^0(\mathcal{O}_{\mathbb{P}}(\Omega_Y^p), \mathcal{O}_{\mathbb{P}}(1) \otimes f^* \pi^* \mathcal{O}_X(mK_X))$

$(f: \mathcal{O}_{\mathbb{P}}(\Omega_Y^p) \to Y)$

$\forall m$

$\mathcal{O}_{\mathbb{P}}(1) - m F$ nef (psef) でないとする

psef $\Rightarrow$ H psef $\Rightarrow H \equiv 0$)

よって $\#\{\Gamma \cap \mathbb{N}\} = +\infty$

$\Gamma = \Gamma \cap \mathbb{N}$ としない

$\boxed{\bigoplus_{m \in \Gamma} \mathcal{O}_X(-mK_X)} \to \boxed{\mathcal{F}} \subset \Omega_X^p$ とすし

こいつらで生成されたもの

さらに #$\{(m_1,\dots,m_r)\in \mathbb{N}^r \mid \bigoplus_{i=1}^{r} \mathcal{O}_Y(-m_i\pi^*(K_X)) \to \mathcal{F}$ の像が has rank r な $\mathcal{F}\} = +\infty$ となるようにする

(もしそうならない場合

背理 $\boxed{}$ について m_i をすべて $\mathbb{N}$ から選び

→ n_2 について → $\mathcal{F}$ は有限のものしか

ないのでこのようになるが)

ある $\{m_i\}\in\mathbb{N}^r$ について

$\mathcal{O}_Y(-(m_1+\dots+m_r)\pi^*(K_X)) \xrightarrow{\text{nonzero}} \det\mathcal{F} \hookrightarrow \bigwedge^r \mathcal{R}_Y^p$ non zero map.

F_Y: Cartier div s.t. $((\det \mathcal{F})_{sat}) = \mathcal{O}_Y(-F_Y)$

$\therefore \exists I \subseteq \mathbb{N}$, $H^0(Y, \pi^*(mK_X) - F_Y) \neq 0$
#$I = +\infty$, ${}^\forall m \in I$.

$\therefore {}^\forall m \in I$, $\pi^*(mK_X) - F_Y \underset{\mathbb{Q}}{\sim} N_m^{\geq 0}$

$$0 \to G_{Y}(-F_Y) \hookrightarrow \Lambda^r \Omega^p_Y \to Q \to 0$$

$\boxed{K_X \text{ psef}}$ $\quad \hat{F}_Y = G_1(Q)$ psef

$\Rightarrow \quad \ell K_Y \sim_{\mathbb{Q}} \hat{F}_Y - F_Y$

$\forall m \in \mathbb{I}$

$\Rightarrow \quad m\,\pi^* K_X + \ell K_Y \sim_{\mathbb{Q}} \underline{N_m} + \hat{F}_Y$

π_* $\qquad mK_X \sim_{\mathbb{Q}} \underbrace{\boxed{\pi_* N_m}}_{\text{eff}} + \underbrace{\boxed{\pi^* \hat{N}_Y}}_{\text{psef}}$

$\Rightarrow$ ①

$$\left[\begin{array}{l} mK_X \equiv_{\mathbb{Q}} E \geq 0 \\ \kappa(X, K_X) \geq 0 \end{array}\right.$$

$\}$

Pf of Non vanishing

Step 1 $\forall h$: K_X^{-1} semipositive as Hm, $\forall x \in X$, $\nu(h, x) = 0$ なら

$\kappa(X, -K_X) = -\infty$ に矛盾を示す

$\kappa(X, -K_X) = -\infty \Rightarrow \forall p, \exists \{m_i\}_{i=1}^{\infty}$. $\pi: Y \to X$

$$H^0(Y, \Omega_Y^p \otimes \pi^*(mK_X)) = 0$$

DPS01

Theorem 5.13. *Let X be a compact Kähler manifold of dimension n with a Kähler form ω. Let $\mathscr{L}$ be a pseudoeffective line bundle on X with a singular hermitian metric h such that $\Theta_h(\mathscr{L}) \geq 0$. Then for every non-negative integer q the morphism*

$$H^0(X, \Omega_X^{n-q} \otimes \mathscr{L} \otimes \mathscr{I}(h)) \xrightarrow{\omega^q \wedge \bullet} H^q(X, \Omega_X^n \otimes \mathscr{L} \otimes \mathscr{I}(h))$$

is surjective.

surj

H

$\therefore$ $H^{n-p}(Y, \Omega_Y^n \otimes \pi^*(mK_X)) = 0$

$H^{n-p}(Y, K_Y \otimes \pi^*(mK_X)) = 0$

$\cong$

$H^p(Y, \pi^*(-mK_X))$ $\quad 0 \leq p \leq n$ $\forall m$.

$\therefore$ $\chi(Y, \pi^*(mK_X)) = 0$ $\quad \forall m$ $\{m_i\}$

$\Rightarrow$ $\chi(Y, \pi^*(-mK_X)) = 0$ $\forall m$

$H^{p,0}(X, -K_X) \simeq H^{n, n-p}(X,$

$\Rightarrow$ $(m=0)$ $\quad \chi(X, \mathcal{O}_X) = \chi(Y, \pi^*\mathcal{O}_X) = 0$

ここで矛盾

semipositive

Step 2 $\exists h$ $\sqrt{-1}\Theta \geq 0$ $\exists x$, $\nu(h, x) > 0$ $\frac{\text{def}}{}$

$\exists m \in \mathbb{N}$, $h_0 := \pi^* h^m$. $\pi : Y \to X$ on $\pi^*(mK_X)$

$$\mathcal{J}(h_0) \subsetneq \mathcal{O}_Y \quad (2.1.3)$$

$\exists y \in Y$, $\mathcal{J}(h_0)_y \subseteq I_y$ $(2.1.3)$

$\mu : \tilde{Y} \to Y$ $\hat{h} = \mu^* h_0$ etc

$E \to y$, $L = (\mu \circ \pi)^* mK_X$

on $\sqrt{-1}\Theta$

$$\mathcal{J}(\hat{h}) \subseteq \mathcal{O}_{\tilde{Y}}(-E)$$

(2.3.2) Siu's uniform global generation.

$\exists G$ ample, $\forall k$, $\mathcal{O}_{\tilde{Y}}(L^{\otimes k} \otimes G) \otimes \mathcal{J}(\hat{h}^k)$ is globally generated

$\Rightarrow$ $\forall k$, $0 \neq H^0(\tilde{Y}, \mathcal{O}_{\tilde{Y}}(kL + G) \otimes \mathcal{O}(-kE))$

$\Rightarrow$ $kL + G - kE$ psef $\quad (\mu \circ \pi)^* mK_X$ $\quad \overset{\text{eff}}{E} + \overset{\text{psef}}{F}$

$\Rightarrow$ $L - E$ psef $\Rightarrow$ $L = E + F$, $\kappa(X) \geq 0$

$\Rightarrow L \equiv E'_{\text{eff}} \Rightarrow K_X \equiv E'_{\text{eff}}$