A COMPLETE PROOF OF THEOREM 3

Theorem 3 (NP-hardness). *The graph edge ordering problem is NP-hard if $|E|$ is much larger than k_{max} so that less than k_{max} edges do not affect the optimized result.*

Proof. We first show that the graph edge ordering problem is NP-hard for single k, i.e., $k_{min} = k_{max}$. We then prove the general case of multiple k, i.e., $k_{min} < k_{max}$.

(*Case of Single* k): Suppose $k_{min} = k_{max} = k$. The objective of the graph edge ordering problem is represented as follows:

$$\min_{\phi \in \Phi} \frac{1}{|V|} \sum_{p=0}^{k-1} \left| V\left(E_{ch}^{\phi}\left(\sum_{x=0}^{p-1} \left\lfloor \frac{|E|+x}{k} \right\rfloor, \left\lfloor \frac{|E|+p}{k} \right\rfloor \right) \right) \right|. \tag{5}$$

Now, we define a function to convert the edge order into the partition, $\texttt{ID2P}_k$: $i \mapsto p$, as Algorithm 2. By using $\texttt{ID2P}_k$, we can generate new edge partitions from the edge orders in linear time.

Algorithm 2: Conversion from Edge ID to Partition

Input : i – Ordered Edge ID
Output: p – Partition ID

```
ID2P_k(i)
    p ← 0; cur ← ⌊(|E|+p)/k⌋
    while i < cur do
        p ← p + 1; cur ← cur + ⌊(|E|+p)/k⌋
    return p
```

Suppose the order ϕ_{opt} is the optimal solution for the graph edge ordering problem. Then, the edge partitions converted from ϕ_{opt} via $\texttt{ID2P}_k$ is also the optimal solution for the edge partitioning problem in a case when $\epsilon \approx 0$ in Definition 1.

The reason is as follows. If the edge partitions converted from ϕ_{opt} via $\texttt{ID2P}_k$ is *not* the optimal solution (more specifically, more than k_{max} edges are in the different partitions from the optimal partitions), then there exist another optimal edge partitions, $\mathcal{E}_k^{opt} := \{\mathcal{E}_k^{opt}[p] \mid 0 \leq p < k\}$, which provides a better solution for the edge partitioning problem than ϕ_{opt}. Based on $\mathcal{E}_k^{opt}$, we can generate new edge ordering ϕ' in such a way that for p

$$\mathcal{E}_k^{opt}[p] = \left\{ E^{\phi'}[b], E^{\phi'}[b+1], ..., E^{\phi'}\left[b + \lfloor \frac{|E|+p}{k} \rfloor - 1 \right] \right\},$$

where $b := \sum_{x=0}^{p-1} \left\lfloor \frac{|E|+x}{k} \right\rfloor$. Since $\mathcal{E}_k^{opt}$ provides the optimal solution,

$$RF(\mathcal{E}_k^{opt}) := \frac{1}{|V|} \sum_{p=0}^{k-1} |V(\mathcal{E}_k^{opt}[p])|$$

$$= \frac{1}{|V|} \sum_{p=0}^{k-1} \left| V\left(E_{ch}^{\phi'}\left(\sum_{x=0}^{p-1} \left\lfloor \frac{|E|+x}{k} \right\rfloor, \left\lfloor \frac{|E|+p}{k} \right\rfloor \right) \right) \right|$$

is the optimal value. On the other hand, ϕ_{opt} provides the optimal value of Eq. (5) as follows:

$$\frac{1}{|V|} \sum_{p=0}^{k-1} \left| V\left(E_{ch}^{\phi_{opt}}\left(\sum_{x=0}^{p-1} \left\lfloor \frac{|E|+x}{k} \right\rfloor, \left\lfloor \frac{|E|+p}{k} \right\rfloor \right) \right) \right|.$$

This is a contradiction to the assumption that $\mathcal{E}_k^{opt}$ provides the better solution than ϕ_{opt}. Thus, ϕ_{opt} can provide the optimal solution for the edge partitioning problem as well.

Therefore, the problem (5) is reducible to the balanced k-way edge partitioning problem, which is an NP-hard problem as proved in [80].

(*Case of* $k_{min} < k_{max}$): We explain the case when $k_{min} = 2$ and $k_{max} = 3$. The following discussion can be straightforwardly generalized to any k_{min} and k_{max}.

According to Definition 5, we define a function, $Num(k, p)$, for the normalized number of vertices involved in the chunk of edges as follows:

$$Num(k, p) := \frac{1}{|V|} \left| V\left(E_{ch}^{\phi}\left(\sum_{x=0}^{p-1} \left\lfloor \frac{|E|+x}{k} \right\rfloor, \left\lfloor \frac{|E|+p}{k} \right\rfloor \right) \right) \right|.$$

Suppose $k_{min} = 2$ and $k_{max} = 3$, we will show the NP-hardness of the optimization problem as follows:

$$\min_{\phi \in \Phi} \sum_{k=2}^{3} \sum_{p=0}^{k-1} N(k, p) = \min_{\phi \in \Phi} \{Num(2, 0) + Num(2, 1)$$
$$+ Num(3, 0) + Num(3, 1) + Num(3, 2)\}. \tag{6}$$

Here, based on the above discussion of the single k, the following optimization problems are already proved to be NP-hard:

$$\min_{\phi \in \Phi} \{Num(2, 0) + Num(2, 1)\} \tag{7}$$

$$\min_{\phi \in \Phi} \{Num(3, 0) + Num(3, 1) + Num(3, 2)\}. \tag{8}$$

Suppose ϕ_{opt} is the optimal order for (6), then the order can be also the optimal for (7) and (8). Thus, if (6) is not NP-hard, it is a contradiction to the NP-hardness of (7) and (8). Therefore, (6) is also NP-hard. To summarize, the graph edge ordering problem is NP-hard. ■

B COMPLETE PROOF OF LEMMA 2

Lemma 2. *Suppose* $|E|$ *is much larger than* k_{max} *such that* $w := \left\lfloor \frac{|E|}{k} \right\rfloor = \left\lfloor \frac{|E| + \texttt{ID2P}_k(\cdot)}{k} \right\rfloor$ *and* $D[v] < \frac{|E|}{k_{max}}$ *for* $\forall v \in V$. *Then,*

$$\forall v, u \in V_{rest} \cap V(X^{\phi}), \quad p(v) > p(u) \Rightarrow F_v > F_u,$$

where F_v *and* F_u *are the value of Eq.* (3) *for* $X^{\phi} + N(v)$ *and* $X^{\phi} + N(u)$ *respectively, as shown in Line 10 of Algorithm 3.*

Proof. Suppose $Xv^{\phi} := X^{\phi} + N(v)$, $Xu^{\phi} := X^{\phi} + N(u)$.

$$F_v > F_u \quad \Leftrightarrow \quad F_v - F_u > 0$$

$$\Leftrightarrow \sum_{k=k_{min}}^{k_{max}} \sum_{i=0}^{|E|-1} \left\{ f\left(Xv^{\phi}, i, w \right) - f\left(Xu^{\phi}, i, w \right) \right\} > 0$$

$$\Leftarrow \sum_{k=k_{min}}^{k_{max}} \sum_{i=0}^{|E|-1} \left\{ \left| V\left(Xv_{ch}^{\phi}(i-w+1, w) \right) \right| - \left| V\left(Xu_{ch}^{\phi}(i-w+1, w) \right) \right| \right\} > 0$$

$$\Leftrightarrow \sum_{k=k_{min}}^{k_{max}} \sum_{i=|X|^{\phi}}^{|E|-1} \{\Delta V(v, i) - \Delta V(u, i)\} > 0, \tag{9}$$

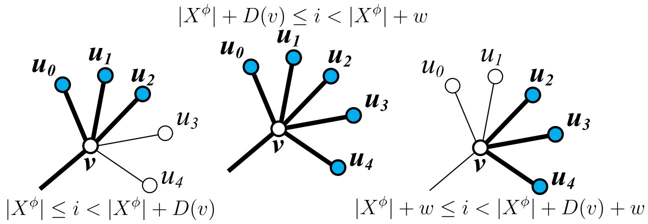

Figure 14: The value of $n(i)$: # of the additional vertices derived from $N(v) = \{u_0, u_1, ..., u_4\}$. Blue vertices are the additional when v is selected for the expansion.

where $\Delta V(v, i) := |V(Xv^{\phi}_{ch}(i{-}w{+}1, w))| - |V(X^{\phi}_{ch}(i{-}w{+}1, w))|$.

Next, we will calculate $\Delta V(v, i)$ for $i \geq |X^{\phi}|$. Intuitively, $\Delta V(v, i)$ means the number of additional replicated vertices in a chunk when we select v to expand the ordered edges. For each chunk determined by i, each additional replicated vertex comes from v or $N(v)$. Thus, $\Delta V(v, i)$ can be represented by the sum of two functions:

$$\Delta V(v, i) = \chi(i) + n(i),$$

where $\chi(i)$ is the number of replicated vertices caused by v; $n(i)$ is caused by $N(v)$.

First, $\chi(i)$ is the indicator function. If $X^{\phi}_{ch}(i - w + 1, w)$ already involves v, then the number of replicated vertices does not increase due to the additional v. Therefore, $\chi(i)$ is 0. Specifically, this case appears if $i > M[v]+w$, because $X^{\phi}_{ch}(i{-}w{+}1, w)$ involves an edge e whose order is $M[v]$ (i.e., $\phi(e) = M[v]$). Otherwise, v's replication is newly added to the chunk $Xv^{\phi}_{ch}(i - w + 1, w)$, and thus $\chi(i)$ is 1. Therefore, $\chi(i)$ is represented as follows:

$$\chi(i) = \begin{cases} 1 & \text{if } i \in [M[v] + w, |X^{\phi}| + D[v] + w) \\ 0 & \text{if } i \notin [M[v] + w, |X^{\phi}| + D[v] + w) \end{cases}$$

where we also consider a case that i is larger so that $Xv^{\phi}_{ch}(i - w + 1, w)$ is empty. In this case, $\chi(i)$ is obviously 0.

Second, $n(i)$ is the number of the additional vertices derived from $N(v)$. Its value can be represented as follows:

$$n(i) = \begin{cases} i - |X^{\phi}| & (|X^{\phi}| \leq i < |X^{\phi}| + D[v]) \\ D[v] & (|X^{\phi}| + D[v] \leq i < |X^{\phi}| + w) \\ D[v]{-}i{+}|X^{\phi}|{+}w & (|X^{\phi}| + w \leq i < |X^{\phi}| + D[v] + w) \\ 0 & (|X^{\phi}| + D[v] + w < i) \end{cases}$$

Figure 14 shows an example of these cases. Suppose v is selected in the greedy algorithm and new edge orders are assigned to v's neighbor edges, e_{v,u_0}, e_{v,u_1}, e_{v,u_2}, e_{v,u_3}, and e_{v,u_4}. Then, if $|X^{\phi}| \leq i < |X^{\phi}| + D[v]$, a part of $N(v)$ are added (e.g., u_0, u_1, u_2 in Figure 14). If $|X^{\phi}| + D[v] \leq i < |X^{\phi}| + w$, all vertices in $N(v)$ are added (e.g., u_0, u_1, u_2, u_3, u_4 in Figure 14). If $|X^{\phi}| + w \leq i < |X^{\phi}| + D[v] + w$, also a part of $N(v)$ are added (e.g., u_2, u_3, u_4 in Figure 14). If $|X^{\phi}| + D[v] + w \leq i$, then $Xv^{\phi}_{ch}(i - w + 1, w)$ involves no vertices.

Therefore,

$$\begin{aligned} \sum_{i=|X^{\phi}|}^{|E|-1} \Delta V(v, i) = & \sum_{i=|X^{\phi}|}^{|E|-1} \chi(i) + \sum_{i=|X^{\phi}|}^{|X^{\phi}|+D[v]-1} \left\{ i - |X^{\phi}| \right\} + \sum_{i=|X^{\phi}|+D[v]}^{|X^{\phi}|+w-1} D[v] \\ & + \sum_{i=|X^{\phi}|+w}^{|X^{\phi}|+w+D[v]-1} \left\{ D[v] - i + |X^{\phi}| + w \right\} \\ = & \quad wD[v] + |X^{\phi}| + D[v] - M[v] \end{aligned}$$

Let $\Delta D := D[v] - D[u]$ and $\Delta M := M[v] - M[u]$.

$$\begin{aligned} \sum_{i=|X^{\phi}|}^{|E|-1} \{\Delta V(v, i) - \Delta V(u, i)\} &= w\Delta D + \Delta D - \Delta M \\ &\sim w\Delta D - \Delta M \quad (\because w > \frac{|E|}{k_{max}} \gg 1) \end{aligned}$$

Therefore,

$$\begin{aligned} & p(v) > p(u) \\ \Leftrightarrow \quad & \alpha \cdot D[v] - \beta \cdot M[v] > \alpha \cdot D[u] - \beta \cdot M[u] \\ \Leftrightarrow \quad & \sum_{k=k_{min}}^{k_{max}} (w \cdot D[v] - M[v]) > \sum_{k=k_{min}}^{k_{max}} (w \cdot D[u] - M[u]) \\ \Leftrightarrow \quad & \sum_{k=k_{min}}^{k_{max}} (w\Delta D - \Delta M) > 0 \\ \Leftrightarrow \quad & \sum_{k=k_{min}}^{k_{max}} \sum_{i=|X^{\phi}|}^{|E|-1} \{\Delta V(v, i) - \Delta V(u, i)\} > 0 \\ \Rightarrow \quad & F_v > F_u. \quad (\because (9)) \end{aligned}$$

Thus, the lemma is proved. ■