Appendix A
Complete Proof of Theorem 1

Theorem 1 (NP-hardness). *AIRM problem is NP-hard under HyLT model*

Proof. Consider a special case for a hypergraph G where all of the hyperedges include exactly two vertices. Then, G can be regarded as a non-hyper graph, and each type of restriction measures can be regarded as a deletion of an edge. Let $e_{v,u}$ be an edge from v to u and $w_{v,u}$ be its weight so that $w_{v,u} := IF(v) \cdot IF(e_{v,u})$. Then, the HyLT model can be regarded as the LT model where each edge's weight is represented by $w_{v,u}$.

Thus, the influence blocking maximization when applied to edges and the LT model are the special case of AIRM and HyLT, respectively. According to [10], the influence blocking maximization when applied to edges is NP-hard under the LT model. Therefore, the AIRM problem is also NP-hard under HyLT model. □

Appendix B
Complete Proof of Theorem 2

Theorem 2 (Submodularity). *For a hypergraph $G(V,H)$ and a measure m, $O_m(X)$ is submodular over $X \subseteq H$ under HyLT model. That is, for $S \subseteq T \subseteq H$ and $h \in H \setminus T$*

$$O_m(S \cup \{h\}) - O_m(S) \geq O_m(T \cup \{h\}) - O_m(T)$$

Proof. Given a hypergraph $G(V,H)$, we consider its transformation to a non-hyper directed graph $G^{non}(V, E_H)$ generated as follows:

- For each $h \in H$, $|V(h)|(|V(h)|-1)$ directed non-hyper edges are generated by connecting each vertex in $V(h)$ to the rest ones (thus, $V(h)$ becomes a clique).
- An edge $e_{v,u}$ generated from h has a weight $w_{v,u}$ which is equal to $IF(h) \cdot IF(v)$ ($e_{v,u}$ is a directed edge from a vertex v to u).

Then, we can obtain an one-to-one correspondence from $G(V,H)$ to $G^{non}(V,E_H)$. The behavior of LT model on $G^{non}(V,E_H)$ is equivalent to that of HyLT model on $G(V,H)$.

Let G_x^{non} be a *live-edge* graph [24] generated from $G^{non}(V,E_H)$ as discussed in Section IV-A. Let also the realization graph of $G(V,H)$ be G_x. Due to the aforementioned one-to-one correspondence, G_x is equivalent to G_x^{non}, i.e.,

$$Pr[G_x | G^{non}(V,E_H)] = Pr[G_x | G(V,H)] \tag{7}$$

Next, we discuss the correspondence of each restriction measure m_{can}, m_{sh}, m_{sp} on the non-hyper graph $G^{non}(V,E_H)$. For $m_{can(h)}$, the corresponding operation that can be applied to $G^{non}(V,E_H)$ is to delete a set of non-hyper edges $E_{can} \subseteq E_H$ s.t. $E_{can} = \{e_{v,u} |\ \forall v,u \in V(h)\}$. For $m_{sh(h)}$, the corresponding operation that can be applied to $G^{non}(V,E_H)$ is to delete a set of non-hyper edges E_{sh}. Let h_0 be the hyperedge after shrinking h. Then, $E_{sh} = \{e_{v,u} |\ \forall v,u \in V(h)\} \setminus \{e_{v,u} |\ \forall v,u \in V(h_0)\}$. Finally, for $m_{spl(h)}$, the corresponding operation that can be applied to $G^{non}(V,E_H)$ is to delete a set of non-hyper edges E_{sp}. Let $\{h_0, h_1, ..., h_{n-1}\}$ be the set of hyperedges after splitting h. Then, $E_{sp} = \{e_{v,u} |\ \forall v,u \in V(h)\} \setminus \left(\bigcup_{i \in [0,n)} \{e_{v,u} |\ \forall v,u \in V(h_i)\}\right)$. Therefore, any restriction measure applied to a non-hyper graph corresponds to the deletion of a certain set of edges.

Let $I'(v,E)$ be the influence speed from a vertex v under LT model on the graph $G^{non}(V,E)$. Let also $E_{H,m}(X)$ be a set of non-hyper edges to be deleted when applying $m(X)$ ($X \subseteq H$ and $E_{H,m}(X) \subseteq E_H$). Then, according to Eg. 7,

$$\begin{aligned} &I(v, H \setminus m(X) \cup X) \\ &\quad = \sum_{G_x \in \mathcal{G}} \Pr[G_x | G(V, H \setminus m(X) \cup X)] \cdot r(v, G_x) \\ &\quad = \sum_{G_x \in \mathcal{G}^{non}} \Pr[G_x | G^{non}(V, E_H \setminus E_{H,m}(X)))] \cdot r(v, G_x) \\ &\quad = I'(v, E_H \setminus E_{H,m}(X)) \end{aligned} \tag{8}$$

where $\mathcal{G}$ and $\mathcal{G}^{non}$ are a set of all realization graphs for $G(V, H \setminus m(X) \cup X)$ and $G^{non}(V, E_H \setminus E_{H,m}(X))$, respectively; $r(v, G_x)$ is the number of reachable vertices from v in G_x.

According to [10], the influence speed function of LT model is supermodular when considering edge deletions. Thus, for a non-hyper graph $G^{non}(V,E)$, any set of non-hyper edges $S^{non} \subseteq T^{non} \subseteq E$, and $e \in E \setminus T^{non}$, the following is established:

$$\begin{aligned} &I'(v, E \setminus (S^{non} \cup \{e\})) - I'(v, E \setminus S^{non}) \\ &\leq I'(v, E \setminus (T^{non} \cup \{e\})) - I'(v, E \setminus T^{non}) \end{aligned}$$

Therefore, for any $X^{non} \subseteq E \setminus T^{non}$,

$$\begin{aligned} &I'(v, E \setminus (S^{non} \cup X^{non})) - I'(v, E \setminus S^{non}) \\ &\leq I'(v, E \setminus (T^{non} \cup X^{non})) - I'(v, E \setminus T^{non}) \end{aligned} \tag{9}$$

On the other hand, based on Eq.(8), for any $S \subseteq T \subset H$ and $h \in H \setminus T$,

$$\begin{aligned} &O_m(S \cup \{h\}) - O_m(S) \\ &= \tfrac{1}{n} \sum_{v \in V} \{I(v, H \setminus S \cup m(S)) - I(v, H \setminus (S \cup \{h\}) \cup m(S \cup \{h\}))\} \\ &= \tfrac{1}{n} \sum_{v \in V} \{I'(v, E_H \setminus E_{H,m}(S)) - I'(v, E_H \setminus E_{H,m}(S \cup \{h\}))\} \quad (\because (8)) \\ &= \tfrac{1}{n} \sum_{v \in V} \{I'(v, E_H \setminus E_{H,m}(S)) - I'(v, E_H \setminus (E_{H,m}(S) \cup E_{H,m}(\{h\})))\} \\ &\qquad (\because E_{H,m}(S \cup \{h\}) = E_{H,m}(S) \cup E_{H,m}(\{h\})) \\ &\geq \tfrac{1}{n} \sum_{v \in V} \{I'(v, E_H \setminus E_{H,m}(T)) - I'(v, E_H \setminus (E_{H,m}(T) \cup E_{H,m}(\{h\})))\} \\ &\qquad (\because (9) \text{ and } S \subseteq T \Rightarrow E_{H,m}(S) \subseteq E_{H,m}(T)) \\ &= \tfrac{1}{n} \sum_{v \in V} \{I'(v, E_H \setminus E_{H,m}(T)) - I'(v, E_H \setminus E_{H,m}(T \cup \{h\}))\} \\ &= O_m(T \cup \{h\}) - O_m(T) \end{aligned}$$

Therefore,

$$O_m(S \cup \{h\}) - O_m(S) \geq O_m(T \cup \{h\}) - O_m(T),$$

which completes the proof of Theorem 2. □

APPENDIX C
PROOF OF COROLLARY 1

Corollary 1. Suppose $\widehat{X}$ is a solution obtained by Algorithm 1, and X^{OPT} is the optimum solution. Then,

$$O_m(\widehat{X}) \geq \left(1 - \frac{1}{e}\right) O_m(X^{OPT})$$

where e is Napier's constant.

Proof. According to [39], a general optimization problem for a non-decreasing function $z(\cdot)$,

$$\max_{S \subseteq N}\{z(S) : |S| \leq k, z(S) \text{ is submodular}, z(\varnothing) = 0\},$$

can approximately be solved by a greedy algorithm within a factor of $(1 - 1/e)$ from the optimum solution. Therefore, the approximate solution $\widehat{X}$ obtained by Algorithm 1 results in a $(1 - 1/e)$ approximation ratio for the maximization problem of $O_m(X)$, which is a non-decreasing submodular function and $O_m(\varnothing) = 0$ due to Theorem 2. □

APPENDIX D
COMPLETE PROOF OF LEMMA 1

Lemma 1. *For a set of θ random RR paths R,*

$$\mathbb{E}[O_m(X)] \approx \frac{1}{\theta} \sum_{r \in R} \Psi_X(r)$$

when θ is large enough.

Proof. Let r be a random RR path generated from a hypergraph $G(V, H)$ via HyLT model. Then, according to [25], the following equation holds true:

$$\mathbb{E}[I(v, H)] = n \cdot \Pr[r \text{ includes } v]$$

Let X be a set of restriction hyperedges and m be its restriction measure. r'_X is r's sub-path from the origin to the closest restriction hyperedge. Similarly,

$$\mathbb{E}\left[I(v, H \setminus X \cup m(X))\right] = n \cdot \Pr[r'_X \text{ includes } v]$$

Therefore,

$$\begin{aligned}
\mathbb{E}[O_m(X)] &= \mathbb{E}\left[\frac{1}{n} \sum_{v \in V} \{I(v, H) - I\left(v, H \setminus X \cup m(X)\right)\}\right] \\
&= \frac{1}{n} \sum_{v \in V} \{\mathbb{E}\left[I(v, H)\right] - \mathbb{E}[I\left(v, H \setminus X \cup m(X)\right)]\} \\
&= \sum_{v \in V} \{\Pr[r \text{ includes } v] - \Pr[r' \text{ includes } v]\} \\
&= \sum_{v \in V} \Pr[r \text{ includes } v \wedge r' \text{ does not include } v]
\end{aligned}$$

We define a random variable $\psi_{v,X}(r)$ as follows:

$$\psi_{v,X}(r) := \begin{cases} 1 & \text{if } r \text{ includes } v \wedge r'_X \text{ does not include } v \\ 0 & \text{otherwise} \end{cases}$$

For θ random RR paths R, the approximate value for $\Pr[\psi_{v,X}(r) = 1]$ is represented as follows:

$$\Pr[\psi_{v,X}(r) = 1] \approx \frac{\sum_{r \in R} \psi_{v,X}(r)}{\theta},$$

when θ is large enough.

Since $\sum_{v \in V} \psi_{v,X}(r) = \Psi_X(r)$, the following is established:

$$\begin{aligned}
\mathbb{E}[O_m(X)] &= \sum_{v \in V} \Pr[\psi_{v,X}(r) = 1] \\
&\approx \sum_{v \in V} \frac{\sum_{r \in R} \psi_{v,X}(r)}{\theta} \\
&= \frac{1}{\theta} \sum_{r \in R} \sum_{v \in V} \psi_{v,X}(r) \\
&= \frac{1}{\theta} \sum_{r \in R} \Psi_X(r)
\end{aligned}$$

Thus, the lemma holds true. □

APPENDIX E
COMPLETE PROOF OF LEMMA 2

Lemma 2. *Efficiency of Algorithm 3 is $O(k(|H| + \theta|V|))$.*

Proof. The dominant part of Algorithm 3 is the greedy selection (Lines 8–11). For each iteration of the loop, $O(|H|)$ time is required to obtain the arguments of maxima (Line 9). Scanning θ samples requires $O(\theta)$ time and each sample has $O(|V|)$ vertices (Line 11). Thus, the total time complexity is $O(k(|H| + \theta|V|))$. □

APPENDIX F
COMPLETE PROOF OF COROLLARY 2

Corollary 2. Algorithm 4 returns $(1 - 1/e - \epsilon)$-approximate solution with at least $1 - 1/n^l$ probability.

Proof. Basic steps of Algorithm 4 is the same as IMM [15], [26]. Some of the parameters in IMM are changed in our algorithm due to the difference of the search spaces. Our algorithm selects edges while IMM vertices. Thus, k-combination of a vertex set in IMM parameters is modified into that of an edge set in our algorithm, i.e., $\binom{n}{k}$ is into $\binom{|H|}{k}$. Therefore, the parameters of IMM,

$$\begin{aligned}
\alpha_{IMM} &= \sqrt{l \ln n + \ln 2} \\
\beta_{IMM} &= \sqrt{(1 - 1/e)(\ln \tbinom{n}{k} + l \ln n + \ln 2)} \\
\lambda^*_{IMM} &= 2n((1 - 1/e)) \cdot \alpha + \beta)^2 \cdot \epsilon^{-2} \\
\lambda'_{IMM} &= (2 + 2\epsilon'/3)(\ln \tbinom{n}{k} + l \ln n + \ln \log_2 n) \cdot n \cdot \epsilon'^2
\end{aligned}$$

are respectively modified into

$$\begin{aligned}
\alpha &= \sqrt{l \ln n + \ln 2} \quad \text{(no difference)} \\
\beta &= \sqrt{(1 - 1/e)(\ln \tbinom{|H|}{k} + l \ln n + \ln 2)} \\
\lambda^* &= 2n((1 - 1/e)) \cdot \alpha + \beta)^2 \cdot \epsilon^{-2} \quad \text{(no difference)} \\
\lambda' &= (2 + 2\epsilon'/3)(\ln \tbinom{|H|}{k} + l \ln n + \ln \log_2 n) \cdot n \cdot \epsilon'^2
\end{aligned}$$

[15] results in the fact that IMM achieves an $(1 - 1/e - \epsilon)$ approximation ratio with at least $1 - 1/n^l$ probability. This is based on the following: (i) the submodular influence speed

function; (ii) the RR-set-based sampling method; and (iii) the aforementioned IMM's parameters. Because (i) we have already proved that $O_m()$ is submodular; (ii) its approximation value is computed by using RR path that is functionally based on RR set; and (iii) the above IMM's parameters do not affect the probabilistic approximation ratio, then we conclude that Algorithm 4 achieves an $(1 - 1/e - \epsilon)$ approximation ratio with at least $1 - 1/n^l$ probability.

□