3.3 多次元確率分布

3.3.1 多項分布

二項分布の多次元化

コインの代りに、1〜kの目のサイコロ投げ

目のでる確率 p_i （iの目の出る確率）

$$\sum_{i=1}^{k} p_i = 1$$

n回サイコロを投げる。

それぞれの目の出た回数 X_i $i=1\sim k$

多次元確率変数 $\vec{X} = (X_1, \cdots, X_k)$

$\vec{X} \sim M_k(n; p_1, \cdots, p_k)$

$$f(\vec{x}) = P(\vec{X} = \vec{x}) = \frac{n!}{\prod_{j=1}^{k} x_j!} \prod_{j=1}^{n} p_j^{x_j}$$

$$(x_i = 0 \sim n; \ \sum x_i = n)$$

$\vec{\mu}$

二項の自然な拡張

σ^2

$$\vec{\mu} = E[\vec{X}] = n(p_1, \cdots, p_k) = (np_1, \cdots np_k)$$

$$\sigma^2 = V[\vec{X}] = n(p_1(1-p_1), \cdots)$$

$$\mathrm{Cov}[X_i, X_j] = \underbrace{-np_ip_j}_{\text{一方が増えると一方は減る}} \quad (i \neq j)$$

(σ_{ij}^2)

多項分布の続き.

周辺 pdf

$\vec{X} \sim M_k(n; p_1 \cdots p_k)$ の時

X_1の周辺確率分布は?　1の出る回数であるので $B(n; p)$であろう

$k=3$の時.

$$f_1(x_1) = \sum_{x_2\, x_3} f(x_1, x_2, x_3)$$

$x_1+x_2+x_3=n$

$x_3=(n-x_1)-x_2$

$$= \sum_{x_2=0}^{n-x_1} \frac{n!\, p_1^{x_1} p_2^{x_2} p_3^{n-x_1-x_2}}{x_1!\, x_2!\, (n-x_1-x_2)!}$$

$$= \frac{n!\, p_1^{x_1}}{x_1!\,(n-x_1)!} \sum_{x_2=0}^{n-x_1} \frac{(n-x_1)!}{x_2!\,(n-x_1-x_2)!} p_2^{x_2} p_3^{n-x_1-x_2}$$

$$= \frac{n!\, p_1^{x_1}}{x_1!\,(n-x_1)!} \cdot \underline{(p_2+p_3)}^{n-x_1}$$

$$= \frac{n!}{x_1!\,(n-x_1)!} p_1^{x_1} (1-p_1)^{n-x_1}$$

$X_1 \sim B(n; p_1)$

3.3.2 多次元正規分布

一次元正規分布の多次元化.

$$\vec{X} = (X_1, \cdots, X_k) \sim N_k(\vec{\mu}, \Sigma)$$

$$f(\vec{x}) = \frac{1}{\sqrt{(2\pi)^k}\,|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^t \Sigma^{-1} (\vec{x}-\vec{\mu})\right\}$$

$$\vec{x} \in \mathbb{R}^k$$

3.5 多次元正規分布の性質

(二次元)

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

$$|\Sigma| = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2$$

$$\Sigma^{-1} = \frac{1}{\sigma_1^2\sigma_2^2 - \sigma_{12}^2} \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}$$

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正規分布の期待値 $N(\mu, \sigma^2)$

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$= \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx \qquad \begin{cases} y = x - \mu, \\ dx = dy \end{cases}$$

$$= \int_{-\infty}^{\infty} (y+\mu) \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}} dy$$

$$= \underbrace{\int_{-\infty}^{\infty} y \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}}_{\text{奇関数}} dy + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{y^2}{2\sigma^2}} dy$$

奇関数 … 積分は0

$$z = \frac{y}{\sigma},\ dy = \sigma dz$$

$$= \mu \int_{-\infty}^{\infty} \underbrace{\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{y^2}{\sigma^2}}}_{N(0,\sigma^2)\text{の p.d.f}} dy$$

$N(0, \sigma^2)$の p.d.f

$$= \mu$$

正規分布 $X \sim N(\mu, \sigma^2)$ の分散

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$$V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) \cdot dx$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}} dx \qquad \begin{cases} y = \dfrac{x-\mu}{\sigma} \\ dx = \sigma \cdot dy \end{cases}$$

$$= \int_{-\infty}^{\infty} \sigma^2 y^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy$$

$$= \sigma^2 \int_{-\infty}^{\infty} y \cdot \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \right)' dy$$

$$= \sigma^2 \left[-y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \right]_{-\infty}^{\infty} + \sigma^2 \underbrace{\int \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{N(0,1) \text{の pdf}}$$

$$= \sigma^2$$

3.5 多次元正規分布の性質

(二次元) 確率変数のベクトル

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$$

変換行列

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$$

行列式

$$|\Sigma| = \sigma_1^2 \sigma_2^2 - \sigma_{12}^2$$ 変換の大きさ

$$\Sigma^{-1} = \frac{1}{|\Sigma|} \begin{pmatrix} \sigma_{12}^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix}$$

二次元正規分布の p.d.f

$$\phi(\vec{x}; \vec{\mu}, \Sigma) = \frac{1}{(2\pi)^{2/2} |\Sigma|^{1/2}}$$

$$\times \exp\left\{ -\frac{1}{2} (\vec{x} - \vec{\mu})^t \Sigma^{-1} (\vec{x} - \vec{\mu}) \right\}$$

3.5.1 周辺確率分布

$\vec{X} = (X_1, X_2) \sim N_2(\vec{\mu}, \Sigma)$ の時 $f_1(X_1)$ は?

expの中身

$$(\vec{x}-\vec{\mu})^t \Sigma^{-1} (\vec{x}-\vec{\mu})$$

$$= \frac{1}{|\Sigma|} (\underset{x_1'}{x_1-\mu_1}, \underset{x_2'}{x_2-\mu_2}) \begin{pmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{pmatrix} \begin{pmatrix} \overset{x_1'}{x_1-\mu_1} \\ \overset{x_2'}{x_2-\mu_2} \end{pmatrix}$$

$$= \frac{1}{|\Sigma|} (x_1', x_2') \begin{pmatrix} \sigma_2^2 x_1' - \sigma_{12} x_2' \\ -\sigma_{12} x_1' + \sigma_1^2 \cdot x_2' \end{pmatrix}$$

$$= \frac{1}{|\Sigma|} \left\{ \left(\sigma_2^2 x_1'^2 \underbrace{- \sigma_{12} x_1' x_2' - \sigma_{12} x_1' x_2'}_{2\sigma_{12} x_1' x_2'} + \sigma_1^2 x_2'^2 \right) \right\}$$

$$= 1/|\Sigma| \left\{ \sigma_1^2 \left\{ x_2'^2 - 2 \frac{\sigma_{12}}{\sigma_1^2} x_1' x_2' + \frac{\sigma_{12}^2}{\sigma_1^4} x_1'^2 \right\} + \sigma_2^2 x_1'^2 - \frac{\sigma_{12}^2}{\sigma_1^2} x_1'^2 \right\}$$

$$= \frac{1}{|\Sigma|} \left\{ \sigma_1^2 \left(x_2' - \frac{\sigma_{12}}{\sigma_1^2} x_1' \right)^2 + \left(\sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \right) x_1'^2 \right\}$$

$$= \frac{\sigma_1^2}{|\Sigma|} \left(x_2' - \frac{\sigma_{12}}{\sigma_1^2} x_1' \right)^2 + \frac{1}{\sigma_1^2} x_1'^2$$

$$\frac{\sigma_1^2}{|\Sigma|} = \frac{1}{|\Sigma|/\sigma_1^2} = \frac{1}{\eta^2}$$ と置く。

$$\eta^2 = \frac{|\Sigma|}{\sigma_1^2} = \frac{1}{\sigma_1^2} (\sigma_1^2 \sigma_2^2 - \sigma_{12}^2)$$

$$= \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2}$$

$$\frac{1}{|\Sigma|} = \frac{\sigma_1^2}{\eta^2}$$

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pdf

$$\phi(\vec{x};\vec{\mu},\Sigma) = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}}\exp\left\{-\frac{1}{2}(\vec{x}-\vec{\mu})^t\Sigma^{-1}(\vec{x}-\vec{\mu})\right\}$$

$\vec{x}' = \vec{x} - \vec{\mu}$ と置いて

$$= \frac{1}{\sqrt{2\pi\eta^2}}\exp\left\{-\frac{1}{2}\left(x_2' - \frac{\sigma_{12}}{\sigma_1^2}x_1'\right)\right\} \times$$

$$\frac{1}{\sqrt{2\pi\sigma_1^2}}\exp\left\{-\frac{1}{2}\frac{x_1'^2}{\sigma_1^2}\right\}$$

$$= \phi\left(x_2'; \frac{\sigma_{12}}{\sigma_1^2}x_1', \eta^2\right)\cdot\phi(x_1'; 0, \sigma_1^2)$$

(3.1)

$$= \underbrace{\phi_{2?}\left(x_2, \mu_2 + \frac{\sigma_{12}}{\sigma_1^2}x_1', \eta^2\right)}\cdot\underbrace{\phi_{1?}(x_1; \mu_1, \sigma_1^2)}$$

$f_1(x_1)$

$$f_1(x_1) = \int\phi(\vec{x};\vec{\mu},\Sigma)$$

行列

$$= \int\underline{\phi\left(x_2; \mu_2 + \frac{\sigma_{12}^2}{\sigma_1^2}x_1', \eta^2\right)}\phi(x_1; \mu_1, \sigma_1^2)\, dx_2$$

$$= \phi(x_1; \mu_1, \sigma_1^2)$$

$$\therefore\ X_1 \sim N(\mu_1, \sigma_1^2)$$

3.5.2 平均と分散

$$E[\vec{X}] = \vec{\mu},$$

$$V[\vec{X}] = E[\vec{X}^2] - (E[\vec{X}])^2 = \Sigma$$

確認
$k=2$ の時.

$f_1(x_1) \sim N(\mu_1, \sigma_1^2)$ が既知.
$f_2(x_2) \sim N(\mu_2, \sigma_2^2)$

$$\mathrm{Cov}[X_1, X_2] = E[(X_1-\mu_1)(X_2-\mu_2)]$$

$$= \iint (x_1-\mu_1)(x_2-\mu_2)\,\phi(\vec{x};\vec{\mu},\Sigma)\,d\vec{x}$$

$$= \iint \quad '' \quad \phi(x_2;\nu(x_1),\eta^2)\,\phi(x_1;\mu_1,\sigma_1^2)\,d\vec{x}$$

$$= \int (x_1-\mu_1)\Big[\underbrace{\int (x_2-\mu_2)\,\phi_2(\;''\;)\,dx_2}_{\nu_1(x_1)-\mu_2}\Big]\phi(x_1;\mu_1,\sigma_1^2)\,dx_1$$

$$= \int (x_1-\mu_1)(\nu(x_1)-\mu_2)\,\phi(x_1;\mu_1,\sigma_1^2)\,dx_1$$

$$(x_1-\mu_1)\left(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(x_1-\mu_1)\right)$$

$$\frac{\sigma_{12}}{\sigma_1^2}(x_1-\mu_1)^2 + \mu_2(x_1-\mu_1)$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{\sigma_{12}}{\sigma_1^2}\;\sigma_1^2 \qquad\qquad 0\text{になる}$$

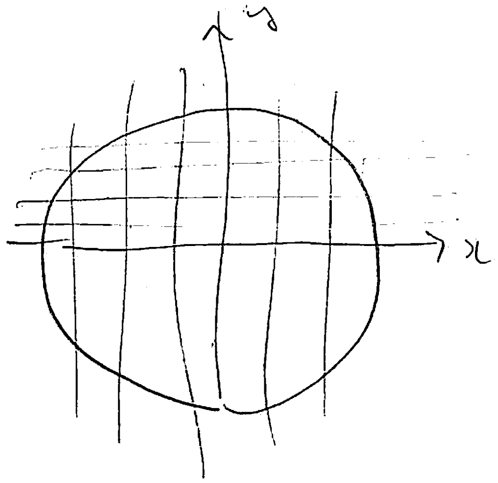
$$= \int \underbrace{(x_1 - \mu_1)\left(\mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(x_1 - \mu_1)\right)} \phi(x_1; \mu_1, \sigma_1^2)\, dx_1$$

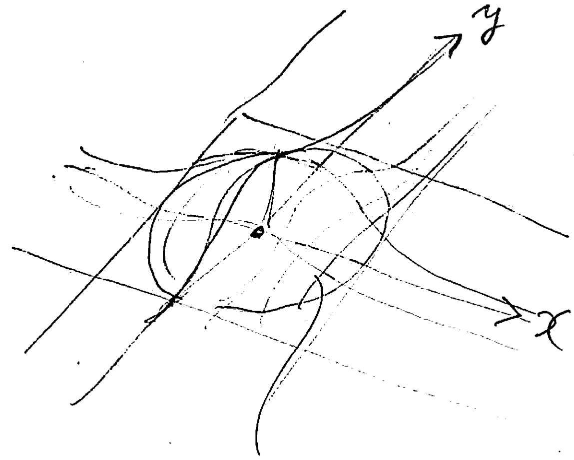
$$= \int \left(\underbrace{\frac{\sigma_{12}}{\sigma_1^2}(x_1 - \mu_1)^2}_{\frac{\sigma_{12}}{\sigma_1^2} \cdot (\sigma_1)^2} + \underbrace{\mu_2 (x_1 - \mu_1)}_{\text{積分して}\,0} \right) \quad 〃 \quad dx_1$$

$$= \sigma_{12}$$

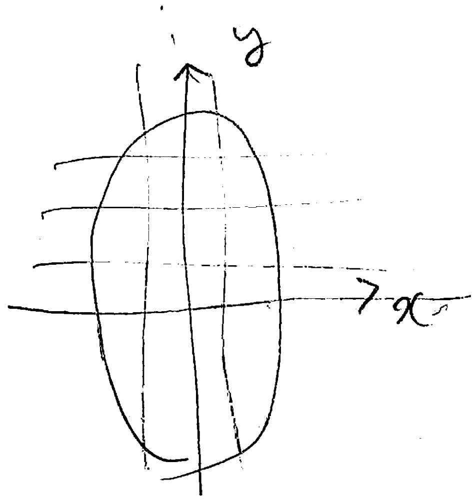
3.5.3 $\phi(\vec{x}; \vec{\mu}, \Sigma) \sim N_2(\vec{\mu}, \Sigma)$ のグラフ

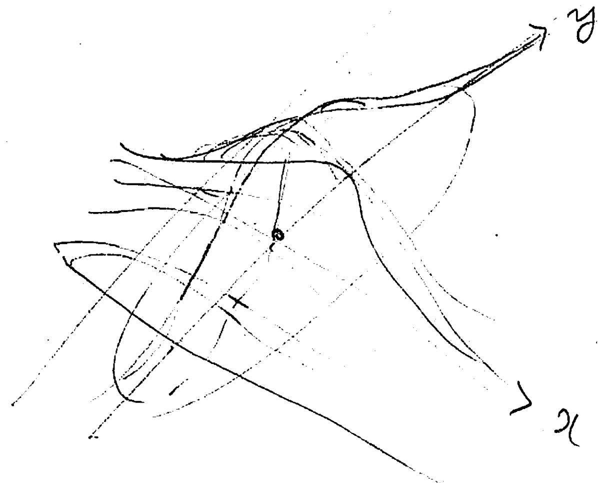
パラメーターは, $\sigma_1, \sigma_2, \rho = \dfrac{\sigma_{12}}{\sqrt{\sigma_1 \sigma_2}}$ (相関係数)





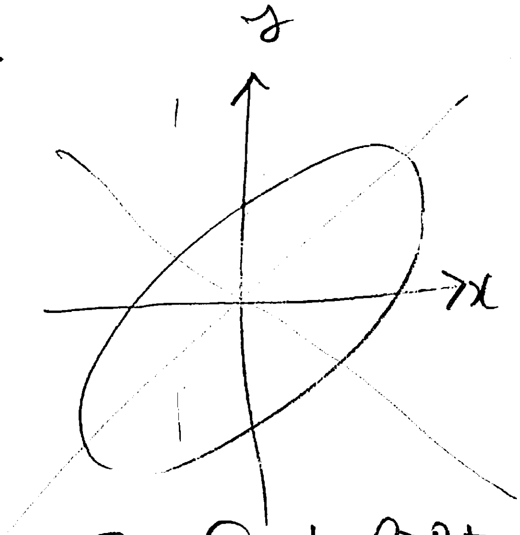
$\sigma_1 = \sigma_2 = 1,\ \rho = 0$



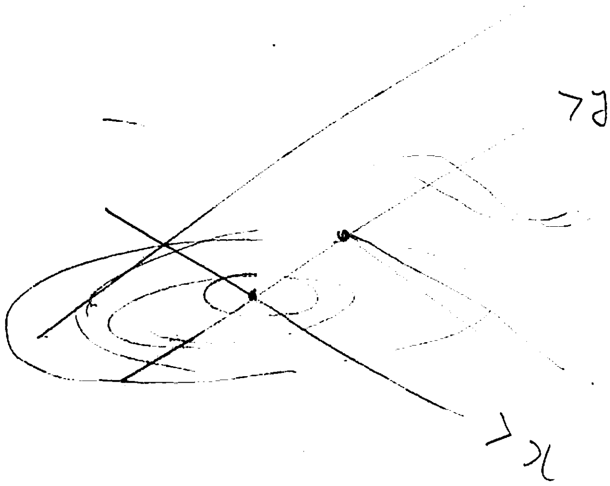


$\sigma_1 = 1,\ \sigma_2 = 2,\ \rho = 0$

グラフ続き



$\sigma_1 = \sigma_2 = 1,\ \rho = 0.5$



3.5.4 独立性と条件付確率分布

X_1とX_2が独立

$\Downarrow$

無相関である $\equiv \sigma_{12} = Cov[X_1, X_2] = 0$

二次元正規分布では

X_1とX_2が独立 $\rightleftarrows$ 無相関である

$\sigma_{12} = 0$を仮定する

$$\phi(\vec{x}; \vec{\mu}, \Sigma) = \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(\vec{x}-\vec{\mu})^t \Sigma^{-1}(\vec{x}-\vec{\mu})}$$

$$= \frac{1}{(2\pi)^{2/2}|\Sigma|^{1/2}} e^{\left\{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{\left\{-\frac{1}{2}\frac{(x_1-\mu_1)^2}{\sigma_1^2}\right\}} \cdot \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{\left\{-\frac{1}{2}\frac{(x_2-\mu_2)^2}{\sigma_2^2}\right\}}$$

独立

$$= \phi_1(x_1; \mu_1, \sigma_1^2)\, \phi_2(x_2; \mu_2, \sigma_2^2)$$

互いに独立な $X_j \sim N(\mu_j, \sigma_j^2)$ $\quad j = 1:k$

$$\Rightarrow \vec{X} \sim N_k\left(\vec{\mu}, \begin{pmatrix} \sigma_1^2 & & 0 \\ & \ddots & \\ 0 & & \sigma_k^2 \end{pmatrix}\right)$$

3.5.4 続き

条件付密度関数

$$f_{X_2|X_1}(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)}$$

$f(x_1, x_2)$ と $X_1 = x_1$
の交る曲線
$\propto f_2(x_2)$

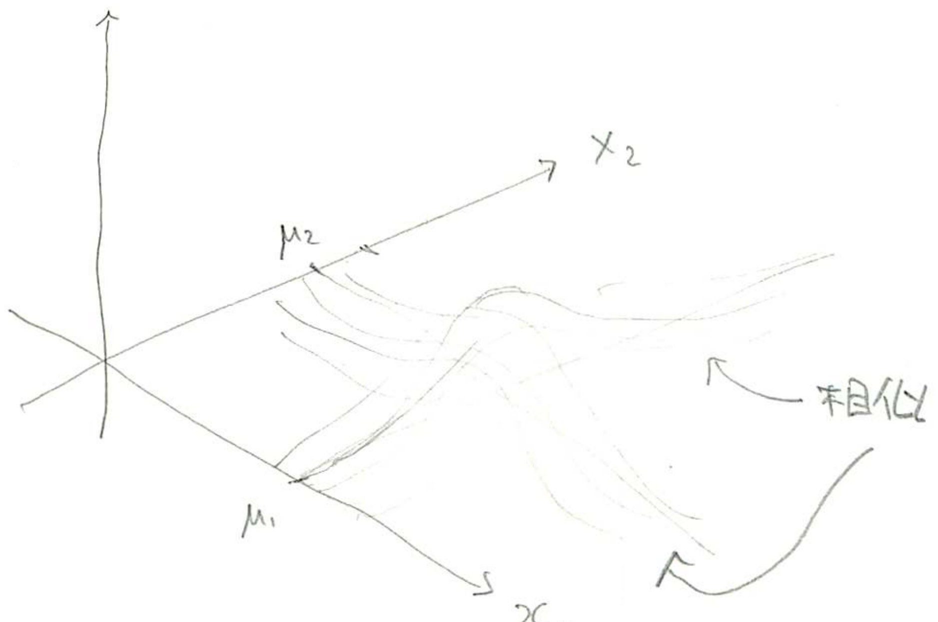
$$= \frac{\phi(\vec{x}; \vec{\mu}, \Sigma)}{\phi(x_1; \mu_1, \sigma_1^2)}$$

$$= \phi(x_2; \nu(x_1), \eta^2)$$

$$\nu(x_1) = \mu_2 + \frac{\sigma_{12}}{\sigma_1^2}(x_1 - \mu)$$

$$\eta^2 = \sigma_2^2 - \frac{\sigma_{12}^2}{\sigma_1^2} \quad \left(\frac{\sigma_{11}\sigma_{22} - \sigma_{12}^2}{\sigma_{11}} \right)$$

$$\frac{1}{\sigma_{11}} |\Sigma|$$



教科書3.6
モーメント母関数

モーメント母関数

確率分布を特徴づける関数(表現)

- $\int_{-\infty}^{\infty} x^k f(x)$ 　x^kの期待値 $(k=0\sim\infty)$ の情報を $\underbrace{\text{閉じ込めた}}_{\text{一つの式に}}$

- $1+\frac{x^1}{1!}+\frac{x^2}{2!}+\cdots = \sum_0^{\infty}\frac{x^n}{n!} = e^x$ なので

$$e^{tx} = \sum_0^{\infty}\frac{1}{n!}(tx)^n$$

とも書く

モーメント母関数 $\psi_X(t)$ 　$(M_X(t))$

$$\begin{cases} \psi_X(t) = E[e^{tX}] \\ \qquad\quad = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx \\ \psi_X(0) = 1 \end{cases}$$

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$X \sim N(\mu, \sigma^2)$ のモーメント母関数の導出

$$\psi_X(t) = E[e^{tX}]$$

$$= \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{x^2 - 2(\mu + t\sigma^2)x + \mu^2}{2\sigma^2}\right\} dx$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{\{x-(\mu+t\sigma^2)\}^2}{2\sigma^2}\right\} \cdot \underbrace{\exp\left\{\frac{(\mu+t\sigma^2)^2 - \mu^2}{2\sigma^2}\right\}}_{x\text{を含まない定数}} dx$$

$$= e^{\frac{(\mu+t\sigma^2)^2 - \mu^2}{2\sigma^2}} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{\{x-(\mu+t\sigma^2)\}^2}{2\sigma^2}} dx}_{N(\mu+t\sigma^2,\ \sigma^2)\text{ の pdf}}$$

$$= e^{\mu t + \frac{\sigma^2}{2} t^2}$$

$$((\mu + t\sigma^2) - \mu^2)/2\sigma^2$$

$$= (t^2\sigma^4 + 2\mu t\sigma^2)/2\sigma^2$$

$$= \mu t + \frac{\sigma^2}{2} t^2$$

教科書 3.6
のっている

モーメント母関数を使った
正規分布 $(N(\mu, \sigma^2))$ の平均と分散の導出。

$$E[X] = \frac{d}{dt}\psi_x(t)\Big|_{t=0}$$

$$= \left\{e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right\}'\Big|_{t=0}$$

$$= \left[(\mu + \sigma^2 t)\, e^{\mu t + \frac{1}{2}\sigma^2 t^2}\right]_{t=0}$$

$$= \mu$$

$$V[X] = E[X^2] - (E[X])^2 = E[X^2] - \mu^2$$

$$E[X^2] = \frac{d^2}{dt^2}\psi_x(t)\Big|_{t=0}$$

$$= \left|(\mu + \sigma^2 t)^2 e^{''} + \sigma^2 e^{''}\right|_{t=0}$$

$$= \mu^2 + \sigma^2$$

$$\therefore V[X] = E[X^2] - E^2[X]$$

$$= \mu^2 + \sigma^2 - \mu^2$$

$$= \sigma^2$$