3.2.4
ガンマ分布

ランダムな事象が n 回起こるまでの時間の分布。

$Ex(\lambda)$

1. 期間 μ に1回起こるランダムな事象で、
1回起こるまでの時間の分布
⇒ 指数分布 $f(x) = \lambda e^{-\lambda x}$ 　　$\lambda = \frac{1}{\mu}$ 頻度
(μ 時間に1回起こる)

$\Gamma(n, \lambda)$

2. $X_{i=1 \sim n} \sim Ex(\lambda)$
$\sum X_i$ が従う分布（ガンマ分布）

$$f(\vec{x}) = \prod_1^n \lambda \cdot e^{-\lambda x_i} \qquad f_i(x_i) = \lambda e^{-\lambda x_i}$$

$Y = \sum_1^n X_i$ とおると

たたみこみ:

$$f_1(y) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_1(y - \sum_2^n x_i)\, f_2(x_2) \cdots f_n(x_n)\, d\vec{x}_i^{(1)}$$

$0 \leq x_i \leq y$ なので

$$= \int_0^{\vec{(1)}\,y} \lambda^n e^{-\lambda y}\, d\vec{x}_i^{(1)}$$

$$= \frac{y^{n-1}}{(n-1)!} \lambda^n e^{-\lambda y}$$

ガンマ関数

$$\Gamma(z)=\int_0^\infty t^{z-1}\cdot e^{-t}dt$$

$$\begin{aligned}\Gamma(z+1)&=\int_0^\infty t^z e^{-t}dt\\&=\int_0^\infty t^z(-e^{-t})'dt\\&=\left[-t^z e^{-t}\right]_0^\infty+z\int_0^\infty t^{z-1}e^{-t}dt\\&=z\int_0^\infty t^{z-1}e^{-t}dt\end{aligned}$$

$$\Gamma(1)=1$$

$$\Gamma(n)=(n-1)!$$

$$\Gamma(1/2)=\sqrt{\pi}$$ ガウス積分

$$\Gamma(1/2+n)=\frac{(2n-1)!!}{2^n}\sqrt{\pi}$$

3.4.4 Γ-分布の統計量

$$X \sim \Gamma(\alpha, \beta) = \overbrace{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}}^{f(x)} \quad (x>0)$$

$$E[X] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} \frac{\beta^\alpha}{\Gamma(\alpha)} x^\alpha e^{-\beta x} dx$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int x^{(\alpha+1)-1} e^{-\beta x} dx$$

$\beta x = t$
$x = \frac{t}{\beta}$,
$dx = \frac{1}{\beta} dt$

$$= \int \left(\frac{t}{\beta}\right)^{\alpha+1-1} e^{-t} \left(\frac{1}{\beta}\right) dt$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{1}{\beta} \cdot \frac{1}{\beta^\alpha} \overbrace{\int t^\alpha e^{-t} dt}^{\Gamma\text{関数}}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+1)}{\beta^{\alpha+1}} \quad \alpha!$$

$(\alpha-1)!$

$$= \frac{\alpha}{\beta} = \alpha\left(\frac{1}{\beta}\right) = n\lambda$$

回数

Γ-関数の分散

$$E[X^2] = \int_0^\infty \frac{\lambda^n}{\Gamma(n)} x^{n+1} e^{-\lambda x} dx$$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n+1} e^{-\lambda x} dx$$

・$\lambda x = y$ とおく，$x = y/\lambda$, $dx = \frac{dy}{\lambda}$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^\infty \left(\frac{y}{\lambda}\right)^{n+1} e^{-y} \cdot \frac{dy}{\lambda}$$

$$= \frac{\lambda^n}{\Gamma(n)} \cdot \frac{1}{\lambda^{n+2}} \underbrace{\int_0^\infty y^{n+1} e^{-y} dy}_{\Gamma(n+2)}$$

$$= \lambda^2 n \cdot (n+1)$$

$$V[X] = E[X^2] - \{E[X]\}^2$$

$$= \lambda^2 n(n+1) - (n\lambda)^2$$

$$= n\lambda^2$$

Γ-分布の　モーメント母関数　　$X \sim \Gamma(n;\lambda)$

$$E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \cdot \frac{\lambda^n}{\Gamma(n)} \cdot x^{n-1} \cdot e^{-\lambda x}\,dx$$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^{\infty} x^{n-1} e^{-(\lambda-t)x}\,dx$$

$(\lambda-t)x = y \quad x = \frac{y}{\lambda-t}$

$(\lambda-t)dx = dy$

$$= \frac{\lambda^n}{\Gamma(n)} \int_0^{\infty} \left(\frac{y}{\lambda-t}\right)^{n-1} e^{-y} \frac{1}{\lambda-t}\,dy$$

$$= \frac{\lambda^n}{\Gamma(n)} \cdot \frac{1}{(\lambda-t)^n} \underbrace{\int_0^{\infty} y^{n-1} e^{-y}\,dy}_{\Gamma(n)}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^n$$

3.2.5 カイ二乗分布とt−分布

$X_{i=1(n)} \sim N(0,1)$

$\chi = \sum X_i^2 \sim \Gamma(n/2, 1/2)$

$\sim \chi_n^2$ 自由度nのχ^2分布

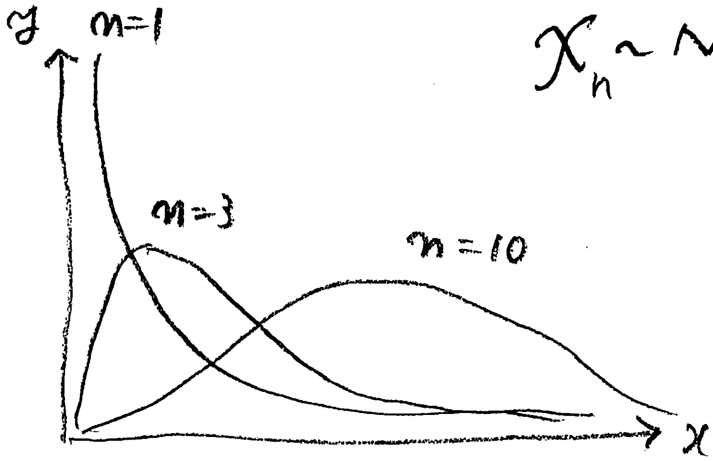
$$\chi \sim f(x) = \frac{1}{2^{n/2}\Gamma(n/2)} x^{\frac{n}{2}-1} e^{-x/2} \quad (x>0)$$

$\mu = n$

$\sigma^2 = 2n$

$n \to \infty$

$\chi_n \sim N(n, 2n)$



自由度 n の　t 分布

$X \sim N(0,1)$
$Y \sim \chi_n^2$
X と Y は独立

$\Downarrow$

$T = X / \sqrt{Y/n}$

$$\sim f(t) = \frac{\Gamma((n+1)/2)}{\sqrt{n\pi}\,\Gamma(n/2)} \left(1+\frac{t^2}{n}\right)^{-(n+1)/2}$$

$$(-\infty < t < \infty)$$

$T \sim t_n$

n が十分大きければ $N(0,1)$

3.2.5 カイ二乗分布

独立 $X_{i=1:n} \sim N(0,1)$

$\Downarrow$

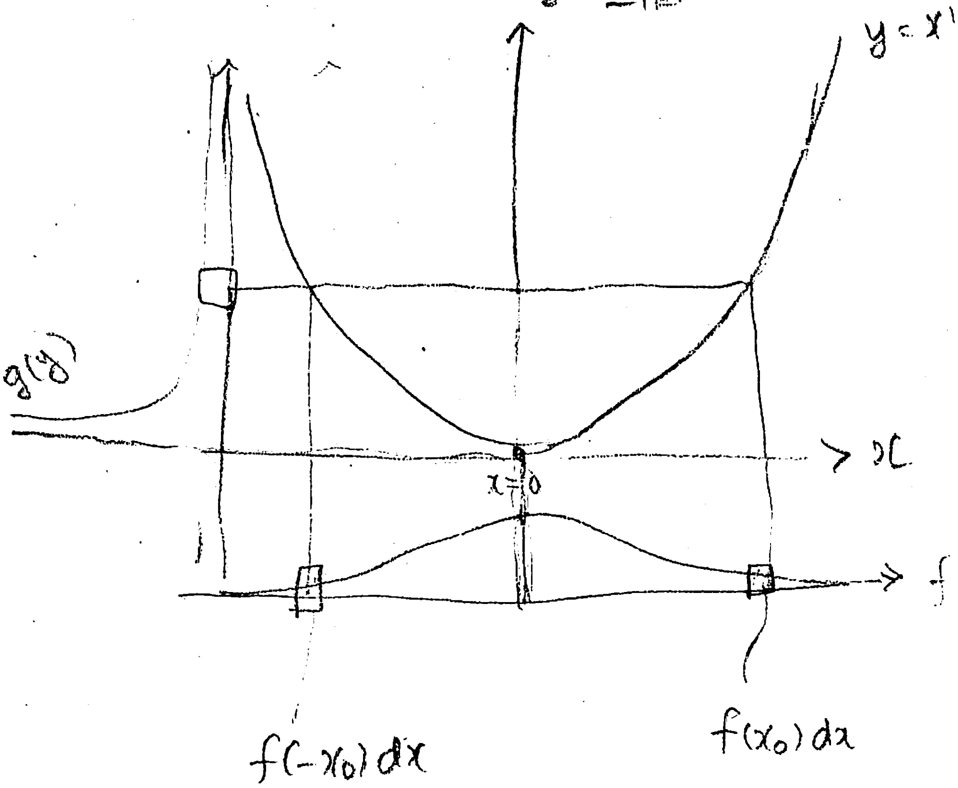
$$Y = X_1^2 + \cdots + X_n^2 \sim \Gamma(n/2, 1/2)$$

$Y = X^2$

$$X \sim N(0,1), \quad f(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$

$$Y = X^2, \quad y = x^2, \quad x = \pm\sqrt{y}, \quad dx = \frac{1}{2\sqrt{y}} dy$$

y 二価



$$\int_{-\infty}^{\infty} g(y)\,dy = 1$$

$$g(y)\,dy = f(x)\,dx$$

$$f_i(x_i) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x_i^2}{2}}$$

$$\left[f(\vec{x}) = \prod_{i=1}^{n} f_i(x_i) \right].$$

$Y = X_i^2$ の分布

$$y = u(x) = x^2 \qquad y > 0$$

$$u^{-1}(y) = \pm x \qquad u_1^{-1} = +\sqrt{y} \qquad \begin{cases} x_1 = \sqrt{y} \\ x_2 = -\sqrt{y} \end{cases} \text{の二つが対応.}$$

$$\sqrt{y} = \pm x \qquad u_2^{-1} = -\sqrt{y}$$

$$g(y)\,dy = f(x)\,dx + f(-x)\,dx \qquad f(x)\text{は対称.}$$

$$= 2f(x)\,dx \qquad y>0,\ x>0$$

$$= 2f(\sqrt{y}) \cdot \frac{1}{2\sqrt{y}}\,dy$$

$$= f(\sqrt{y}) \cdot y^{-\frac{1}{2}}\,dy$$

$$\therefore f(y) = \frac{1}{\sqrt{2\pi}} e^{-y/2}\, y^{-1/2}$$

3.2.5 カイ二乗分布

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$\Downarrow$

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$Y = X^2$

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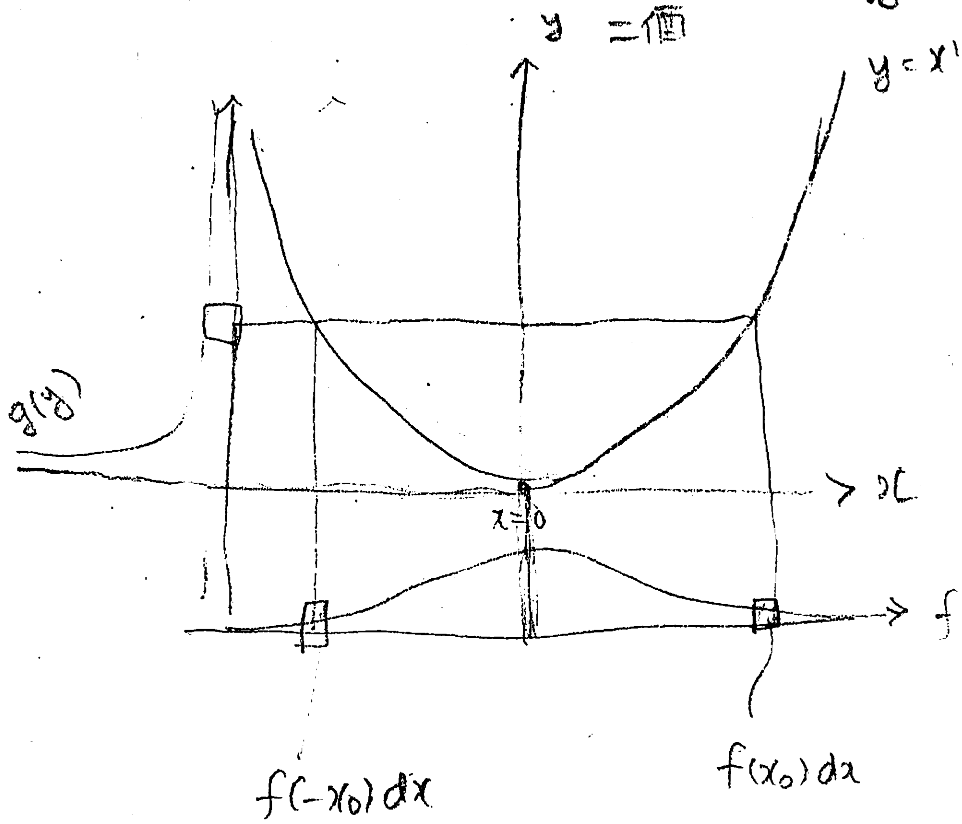

$$\int_{-\infty}^{\infty} g(y) dy = 1$$

$$g(y) dy = f(x) dx$$