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5. 確率変数の関数の分布（概観）

5.1 変数変換

5.3 $\overline{X}$ と S^2 の独立性

5.2.1 t−分布

$X_1 \sim N(0,1)$

$X_2 \sim \chi^2_n$

$T = \dfrac{X_1}{\sqrt{\frac{X_2}{n}}}$ の分布

5.2.2. F−分布

5.1 変数変換

<u>関心</u>

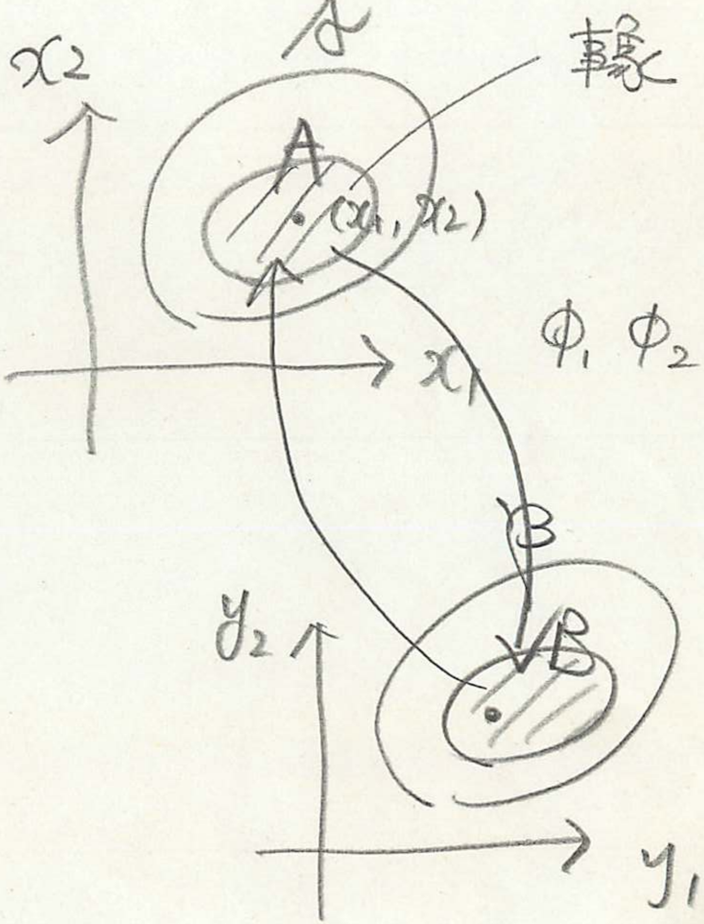
・X_1, X_2 の同時密度 f

∘ $f(x_1, x_2)$

∘ $A = \{(x_1, x_2) \mid f(x_1, x_2) > 0\}$

∘ $\left.\begin{array}{l} y_1 = \phi_1(x_1, x_2) \\ y_2 = \phi_2(x_1, x_2) \end{array}\right\}$ $A \to B$ への変換

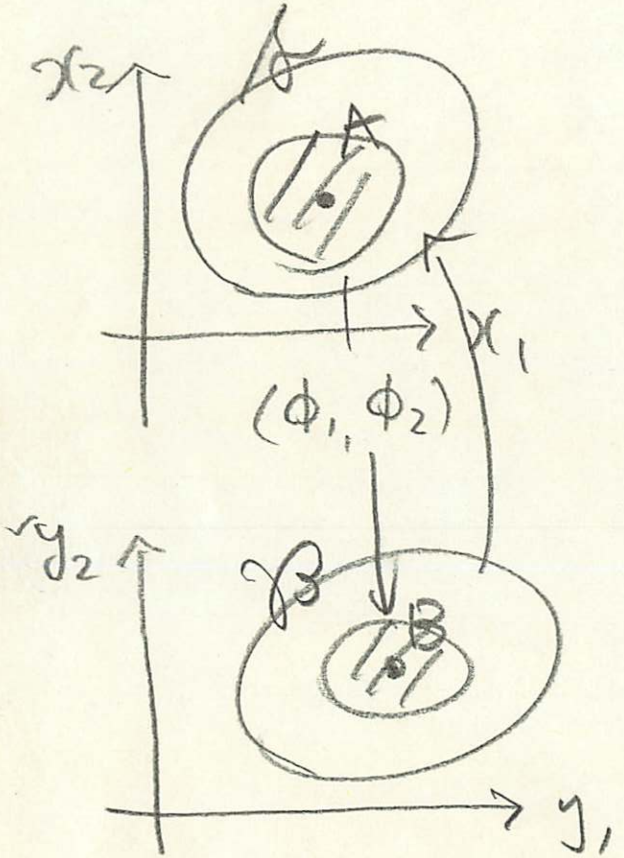
1対1



この時

$Y_1 = \phi_1(X_1, X_2)$
$Y_2 = \phi_2(X_1, X_2)$ のどちらか一方の

確率分布を求めたい.



5.1 変数変換 2

$\vec{X}_1 \in A \qquad \vec{Y} \in B$

$$(X_1, X_2) \in A \rightleftarrows (Y_1, Y_2) \in B$$

$$P\{\overset{\vec{x}}{(X_1, X_2)} \in A\} = P\{\overset{\vec{Y}}{(Y_1, Y_2)} \in B\}$$

$$= \iint_A \overset{\vec{x}}{f(x_1, x_2)}\, \overset{d\vec{x}}{dx_1 dx_2}$$

$y_1 = \phi_1(x_1, x_2)$, $y_2 = \phi_2(x_1, x_2)$ を通す

逆

$$= \iint_B f(\psi_1(y_1, y_2), \psi_2(y_1, y_2)) \cdot |J|\, dy_1 dy_2$$

逆変換

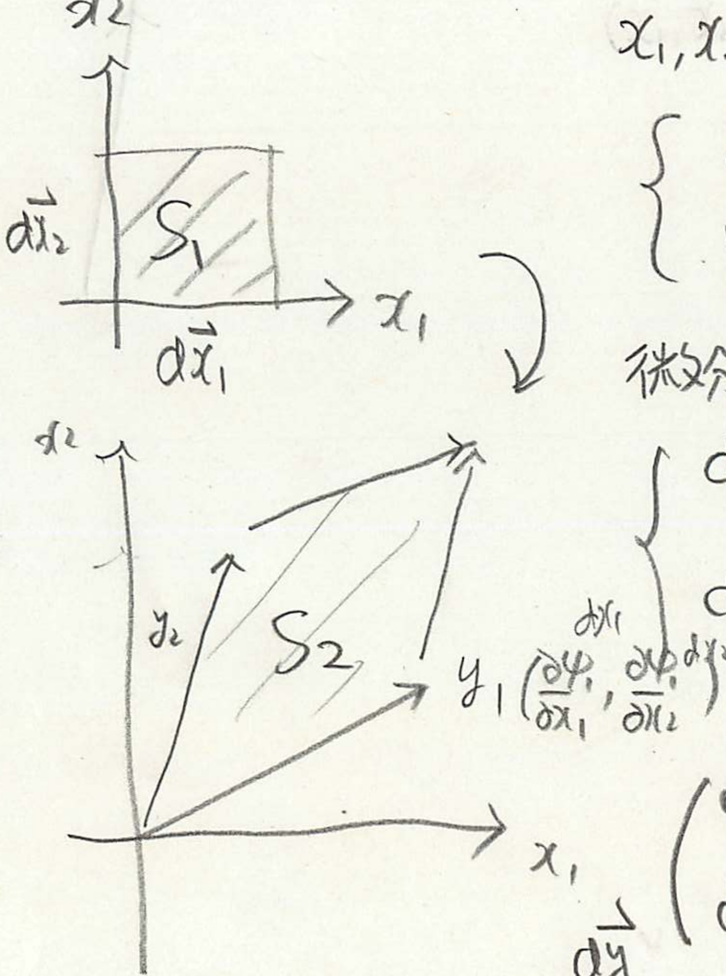
$x_1 = \psi_1(y_1, y_2)$

$x_2 = \psi_2(y_1, y_2)$

$$J = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \dfrac{\partial x_1}{\partial y_2} \\ \dfrac{\partial x_2}{\partial y_1} & \dfrac{\partial x_2}{\partial y_2} \end{vmatrix}$$

ヤコビアンの導出

$$\underbrace{dy_1 \cdot dy_2}_{S_1} = \overbrace{|J|\, dx_1 \cdot dx_2}^{S_2}$$



x_1, x_2座標系 → y_1, y_2 〃

$$\begin{cases} y_1 = \psi_1(x_1, x_2) \\ y_2 = \psi_2(x_1, x_2) \end{cases}$$

微分を考えると

$$\begin{cases} d\vec{y}_1 = \dfrac{\partial\psi_1}{\partial x_1} d\vec{x}_1 + \dfrac{\partial\psi_1}{\partial x_2} d\vec{x}_2 \\ d\vec{y}_2 = \dfrac{\partial\psi_2}{\partial x_1} d\vec{x}_1 + \dfrac{\partial\psi_2}{\partial x_2} d\vec{x}_2 \end{cases}$$

$$\underbrace{\begin{pmatrix} dy_1 \\ dy_2 \end{pmatrix}}_{d\vec{y}} = \underbrace{\begin{pmatrix} \dfrac{\partial\psi_1}{\partial x_1} & \dfrac{\partial\psi_1}{\partial x_2} \\ \dfrac{\partial\psi_2}{\partial x_1} & \dfrac{\partial\psi_2}{\partial x_2} \end{pmatrix}}_{J} \underbrace{\begin{pmatrix} dx_1 \\ dx_2 \end{pmatrix}}_{d\vec{x}}$$

同時

$$g(y_1, y_2) = f(\psi_1(y_1, y_2), \psi_2(y_1, y_2)) \cdot |J| \quad (y_1, y_2) \in \gamma$$

$$= 0 \quad \text{その他}.$$

周辺

$$g(y_1) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_2$$

$$g(y_2) = \int_{-\infty}^{\infty} g(y_1, y_2)\, dy_1$$

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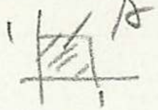
例11

X_1, X_2 は 一様分布 $[0,1]$ からの
大きさ2の random sample.

$$Y_1 = X_1 + X_2,\quad Y_2 = X_1/X_2$$

$$X_i \sim f(x_i) = \begin{cases} 1 & 0 < x_i < 1 \\ 0 & \text{その他} \end{cases}$$

$$f(x_1, x_2) = \begin{cases} 1 \\ 0 \end{cases}$$



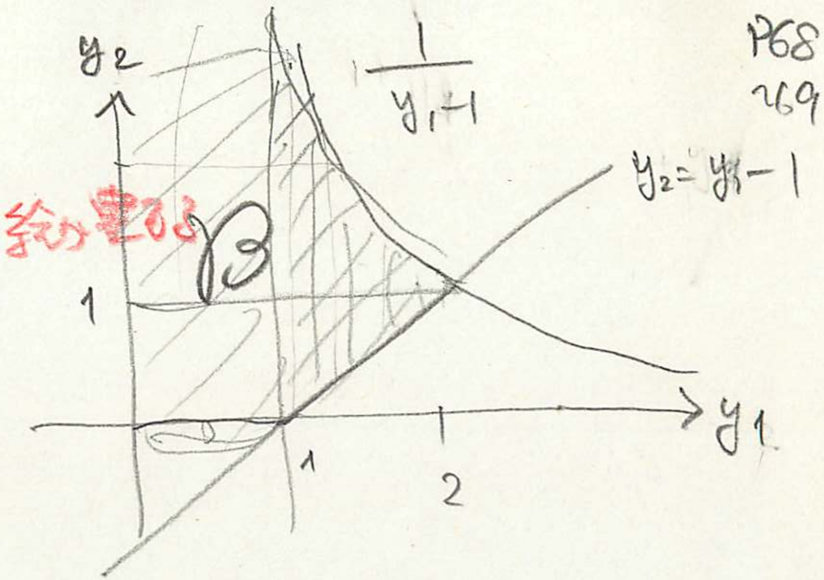
$$y_1 = x_1 + x_2$$
$$y_2 = x_1/x_2$$
$$x_1 = x_2 y_2$$
$$y_1 = x_2 y_2 + x_2$$
$$\therefore x_2 = \frac{y_1}{y_2+1}$$
$$\therefore x_1 = \frac{y_1 y_2}{y_2+1}$$

$$\begin{cases} 0 < \dfrac{y_1}{y_2+1} < 1 \\ 0 < \dfrac{y_1 y_2}{y_2+1} < 1 \end{cases}$$

$$y_1 > 0,\quad y_2 > 0$$

$$\begin{cases} 0 < y_1 < y_2 + 1 \\ 0 < y_1 y_2 < y_2 + 1 \end{cases}$$
$$0 < y_1 < 1 + \frac{1}{y_2}$$

$\int dy_1$ と
$\int dy_2$ で 絵が書ける



$$J = \begin{vmatrix} \dfrac{y_2}{y_2+1} & \dfrac{y_1(1+y_1) - y_1 y_2}{(y_1+1)^2} = \dfrac{y_1}{(1+y_1)^2} \\ \dfrac{1}{y_2+1} & -\dfrac{y_1}{(y_2+1)^2} \end{vmatrix}$$

$$= \frac{-y_1 y_2}{(y_2+1)^3} - \frac{y_1}{(1+y_2)^3}$$

$$= \frac{-y_1\cancel{(y_2+1)}}{(1+y_2)^{\cancel{3}\,2}} = \frac{-y_1}{(1+y_2)^2} \neq 0$$

例11の続き

$$g_1(y_1, y_2) = f\left(\frac{y_1 y_2}{y_2+1}, \frac{y_1}{y_2+1}\right)|J|$$

$$\begin{cases} = \dfrac{y_1}{(1+y_2)^2}, & (y_1, y_2) \in B \\ = 0 & その他 \end{cases}$$

周辺密度f

$$g_1(y_1) = \int_0^\infty \frac{y_1}{(1+y_2)^2} dy_2 \quad 0 < y_1 \leq 1$$
$$= y_1$$

$$\frac{-y_1}{1+y_2} =$$

$$= \int_{y_1-1}^{\frac{1}{y_1-1}} \frac{y_1}{(1+y_2)^2} dy_2 \quad 0 < y_1 < 2$$
$$= 2-y_1$$

$$\frac{-y_1}{1+y_1-1} - 1$$

$$\frac{-y_1}{1+\frac{1}{y_1-1}} = \frac{-y_1}{\frac{y_1}{y_1-1}} = -(y_1-1) = 2-y_1$$

$$\frac{1}{(1+y_2)^2}\left(\frac{1+y_2}{2}\right)^2 = \frac{1}{2}$$

$$g_2(y_2) = \int_0^{\frac{1+y_2}{2}} \frac{y_1}{(1+y_2)^2} dy_1 = \frac{1}{2}$$

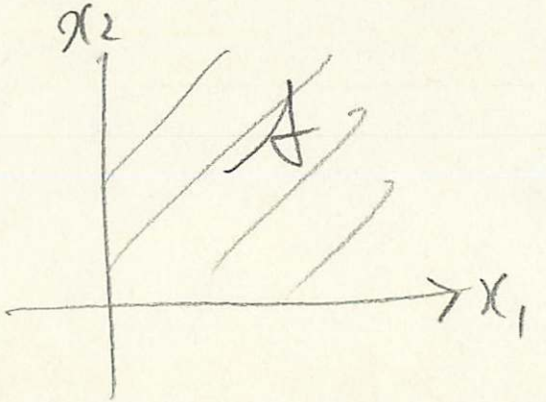
$$\int_0^{\frac{1+y_2}{y_2}} \frac{y_1}{(1+y_2)^2} dy_1$$

$$\frac{1}{(1+y_2)^2} \cdot \frac{1}{2}\left(\frac{1+y_2}{y_2}\right)^2 = \frac{1}{2y_2^2}$$

例12　X_1, X_2 は λ-指数分布からの
大きさ2の random sample

$Y = \lambda(X_1 - X_2)$ の $g(y)$

$X_i \sim \lambda e^{-\lambda x_i} \quad (0 < x_i < \infty)$



X_1, X_2 の同時密度 f,

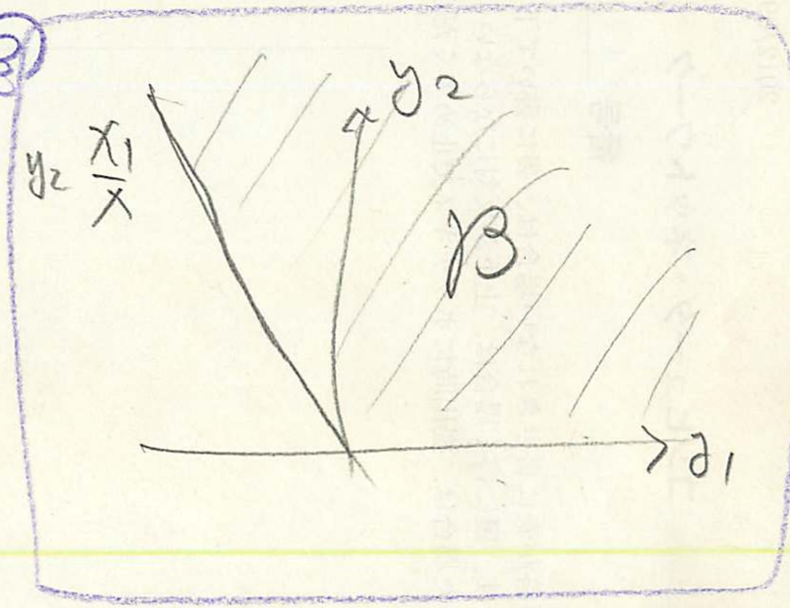
① $f(x_1, x_2) = \lambda^2 e^{-\lambda(x_1 + x_2)}$

変換　$Y_1 = \lambda(X_1 - X_2)$　とおると
$Y_2 = X_2$　③

$y_2 + \frac{y_1}{\lambda}$

$$\begin{cases} y_1 = \lambda(x_1 - x_2) \\ y_2 = x_2 \end{cases} \rightarrow \boxed{\begin{cases} x_1 = \dfrac{y_1 + \lambda y_2}{\lambda} > 0 \\ x_2 = y_2 > 0 \end{cases}}$$

$\frac{y_1}{\lambda} + y_2 > 0 \rightarrow y_2 > -\frac{y_1}{\lambda}$

③


例12の続き

$$\iint_{B} f(\varphi_1(y_1, y_2), \varphi_2(y_1, y_2)) |J| dy_1 dy_2$$

$$\parallel$$

$$P\{(Y_1, Y_2) \in B\}$$

$$|J| = \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix}$$

$$x_1 = \frac{1}{\lambda} y_1 + y_2$$

$$x_2 = y_2$$

$$\begin{vmatrix} \frac{1}{\lambda} & 1 \\ 0 & 1 \end{vmatrix} = \frac{1}{\lambda} > 0$$

$$g(y_1, y_2) = f\left(\frac{1}{\lambda} y_1 + y_2, y_2\right) \cdot |\lambda|$$

$$= \lambda^2 e^{-\lambda\left(\left(\frac{1}{\lambda} y_1 + y_2\right) + y_2\right)} \cdot \frac{1}{\lambda}$$

$$= \lambda e^{-(y_1 + 2\lambda y_2)}$$

$$g_1(y_1) =$$

5.2 t-, F-分布

同時密度 f を求める

$$\begin{cases} f_1(x_1) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x_1^2}{2}\right) \\ \qquad -\infty < x_1 < \infty \\ f_2(x_2) = \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x_2^{\frac{n}{2}-1} \cdot e^{-\frac{x_2}{2}} \\ \qquad 0 < x_2 < \infty \end{cases}$$

t分布

$$\boxed{\begin{array}{l} \left.\begin{array}{l} X_1 \sim N(0,1) \\ X_2 \sim \chi_n^2 \end{array}\right\} \text{互独立} \\ T = \dfrac{X_1}{\sqrt{\frac{X_2}{n}}} \text{ の分布} \end{array}}$$

$$f(x_1, x_2) = \frac{1}{\sqrt{2\pi}} \frac{1}{\Gamma\left(\frac{n}{2}\right) 2^{\frac{n}{2}}} x_2^{\frac{n}{2}-1} \cdot e^{-\frac{1}{2}(x_1^2 + x_2)}$$

$$(x_1, x_2) \in A$$

$$A = \{(x_1, x_2) \mid -\infty < x_1 < \infty,\ 0 < x_2 < \infty\}$$

逆変換を求める

$$\begin{cases} Y_1 = \dfrac{X_1}{\sqrt{\frac{X_2}{n}}} \\ Y_2 = X_2 \end{cases}$$

変換

$$y_1 = \frac{x_1}{\sqrt{\frac{x_2}{n}}}$$

$$y_2 = x_2$$

逆変換

$$\begin{cases} x_1 = y_1 \sqrt{\frac{y_2}{n}} \\ x_2 = y_2 \end{cases}$$

$$B = \{(y_1, y_2) \mid -\infty < y_1 < \infty,\ 0 < y_2 < \infty\}$$

t-分布の導き

Jを求める

$$\frac{\partial x_1}{\partial y_1} = \sqrt{\frac{y_2}{n}},\ \frac{\partial x_1}{\partial y_2} = \frac{y_1}{2\sqrt{n}}\frac{1}{\sqrt{y_2}}$$

$$\frac{\partial x_2}{\partial y_1} = 0 \quad \frac{\partial x_2}{\partial y_2} = 1$$

$$\begin{vmatrix} \sqrt{\frac{y_2}{n}} & \frac{y_1}{2\sqrt{n}\sqrt{y_2}} \\ 0 & 1 \end{vmatrix} = \sqrt{\frac{y_2}{n}} \neq 0$$

Y_1, Y_2の同時密度fを求める

$$g(y_1, y_2) = f\left(y_1\sqrt{\frac{y_2}{n}}, y_2\right)\cdot|J|$$

$$= \frac{1}{\sqrt{2n\pi}}\frac{1}{\Gamma\left(\frac{n}{2}\right)2^{\frac{n}{2}}} y_2^{\frac{n-1}{2}} e^{-\frac{1}{2}\left(1+\frac{y_1^2}{n}\right)\cdot y_2}$$

周辺密度f $g(y_1) = \frac{〃}{〃}\int_0^\infty dy_2$ 〃

$z = \frac{1}{2}\left(1+\frac{y_1^2}{n}\right)y_2$ とおくと

$$= \frac{1}{〃}\cdot\frac{1}{\left(1+\frac{y_1^2}{n}\right)^{\frac{n+1}{2}}}\underbrace{\int_0^\infty z^{\frac{n+1}{2}-1}\cdot e^{-z}dz}_{\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{1}{\sqrt{n\pi}}\cdot\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\left(1+\frac{y_1^2}{n}\right)^{-\frac{n+1}{2}}$$

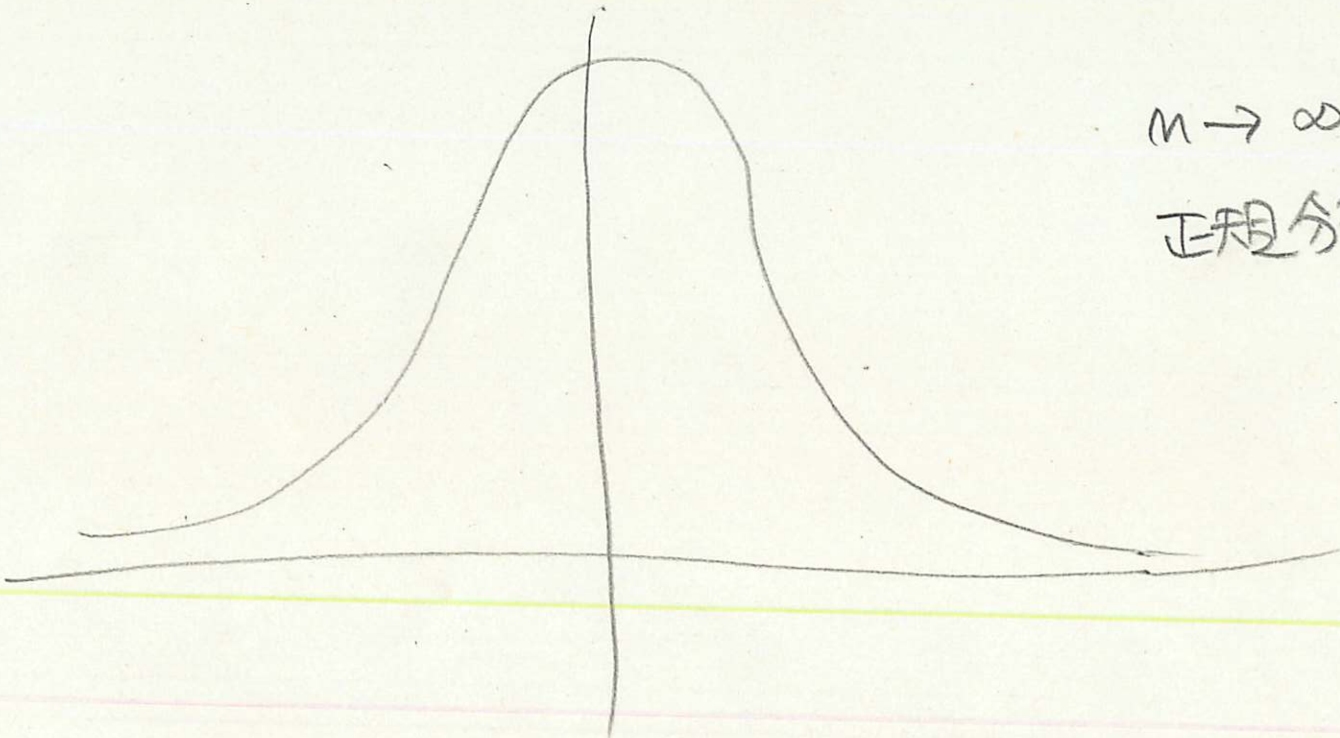
$-\infty < y_1 < \infty$

t-分布の続き

$$g(t) = \frac{1}{\sqrt{n\pi}} \cdot \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)} \cdot \left(1 + \frac{t^2}{n}\right)^{-\frac{n+1}{2}}$$

← 自由度 n の t-分布

$$-\infty < t < \infty$$



$n \to \infty$

正規分布

t-分布

定理1. X_1, X_2 は互独立,

$$X_1 \sim N(0,1)$$

$$X_2 \sim \chi^2_n$$

$$T = \frac{X_1}{\sqrt{\frac{X_2}{n}}} \sim t_n$$

t-分布

系、$X_1, \cdots, X_n \sim N(\mu, \sigma^2)$

からの大きさnの無作為標本

$$\left\{ \begin{array}{ll} \overline{X} = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i & \text{標本平均} \\ \overline{S}^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} (X_i - \overline{X})^2 & \\ & \text{標本分散} \end{array} \right\}$$ とするとき、

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n-1}} \sim t_{n-1}$$

t一分布
式の証明

① $\bar{X}$ と $\bar{S}^2$ が独立か?

$$T=\frac{\bar{X}-\mu}{S/\sqrt{n-1}}=\frac{\dfrac{(\bar{X}-\mu)}{\sigma/\sqrt{n}}}{\sqrt{\dfrac{\frac{1}{\sigma^2}\sum(X_i-\bar{X})^2}{\sqrt{n-1}}}}$$

～ $N(0,1)$ 条件をみたす

$\sqrt{\frac{\sigma^2}{n}}$

～自由度 n の χ^2_n か?

③ ～χ^2_n か

1. $\sum_1^n (X_i-\mu)^2=\sum\{(X_i-\bar{X})+(\bar{X}-\mu)\}^2$ 展開

$$=\sum(X_i-\bar{X})^2+n(\bar{X}-\mu)^2$$

$$\therefore \underbrace{\frac{\sum_1^n(X_i-\mu)^2}{\sigma^2}}_{\chi^2_n}=\frac{nS^2}{\sigma^2}+\underbrace{\left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2}_{\chi^2_1}$$

独立?

$$\frac{(\bar{X}-\mu)^2}{\frac{\sigma^2}{n}}$$

$\bar{X}$: 標本平均

μ: 母平均

σ^2: 母分散

t-分布 寻の証明の続き

$$\frac{\sum (X_i-\mu)^2}{\sigma^2} = \frac{nS^2}{\sigma^2} + \left(\frac{\bar{X}-\mu}{\frac{\sigma}{\sqrt{n}}}\right)^2$$

χ^2_n χ^2_1 $+$

積率母f

$$(1-2t)^{-\frac{n}{2}} = E\left\{\exp\left(t\frac{nS^2}{\sigma^2}\right)\right\}(1-2t)^{-\frac{1}{2}}$$

$$t<\frac{1}{2}$$

$$E\{\ 〃\ \} = (1-2t)^{-\frac{n-1}{2}},\ t<\frac{1}{2}$$

$$\frac{nS^2}{\sigma^2} = \frac{1}{\sigma^2}\sum (X_i-\bar{X})^2$$

F分布

$X_1, X_2 \sim \chi^2(r_i)$ のとき

$$F = \frac{X_1/r_1}{X_2/r_2} \text{ の分布}$$

$$f(x_1, x_2) = \frac{x_1^{\frac{r_1}{2}-1} \cdot x_2^{\frac{r_2}{2}-1} \cdot e^{-\frac{1}{2}(x_1+x_2)}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2}) 2^{\frac{r_1+r_2}{2}}}$$

$$\mathcal{A} = \{(x_1, x_2) \mid 0<x_1<\infty, 0<x_2<\infty\}$$

$Y_1 = \dfrac{X_1/r_1}{X_2/r_2}$, $Y_2 = X_2$ という変換

$$y_1 = \frac{x_1/r_1}{x_2/r_2}, \quad y_2 = x_2$$

$$\frac{x_1}{r_1} = y_1 \frac{y_2}{r_2} = \frac{r_1}{r_2} y_1 y_2,$$

$$x_2 = y_2$$

$$J = \begin{vmatrix} \frac{r_1}{r_2} y_2 & \frac{r_1}{r_2} y_1 \\ 0 & 1 \end{vmatrix}$$

$$|J| = \frac{r_1}{r_2} y_2$$

F分布の続き

$$g(y_1, y_2) = f\left(\frac{r_1}{r_2}y_1 y_2, y_2\right)\cdot|J|$$

$$= \frac{\left(\frac{r_1}{r_2}\right)^{\frac{r_1}{2}} y_1^{\frac{r_1}{2}-1} y_2^{\frac{r_1+r_2}{2}-1}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})2^{\frac{r_1+r_2}{2}}} e^{-\frac{1}{2}(1+\frac{r_1}{r_2}y_1)y_2}$$

$$B = \{(y_1, y_2) \mid 0<y_1<\infty, 0<y_2<\infty\}$$

$$g(y) = \int_{B_{y_2}} g(y_1, y_2)\,dy_2$$

がFの密度f.

$$g(f) = \frac{\Gamma(\frac{r_1+r_2}{2})\left(\frac{r_1}{r_2}\right)^{\frac{r_2}{2}}}{\Gamma(\frac{r_1}{2})\Gamma(\frac{r_2}{2})} f^{\frac{r_1}{2}-1}\cdot\left(1+\frac{r_1}{r_2}f\right)^{-\frac{r_1+r_2}{2}}$$

$$\frac{1}{B(\frac{n}{2}, \frac{m}{2})}$$

$$0<f<\infty$$