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ワリスの公式

$$\lim_{n\to\infty} \frac{2\cdot 2\cdot 4\cdot 4\cdots 2n\cdot 2n}{1\cdot 3\cdot 3\cdot 5\cdots(2n-1)(2n+1)} = \frac{\pi}{2}$$

証明

$$S_n = \int_0^{\pi/2} \sin^n x\,dx \quad とおく$$

$$= \left[-\cos x\cdot\sin^{n-1} x\right]_0^{\frac{\pi}{2}} + (n-1)\int_0^{\pi/2} \sin^{n-2}\cdot\underset{(\sin^{n-2}-\sin^n x)}{\underset{\shortparallel}{\cos^2}}\,dx$$

$$= (n-1)S_{n-2} - (n-1)S_n = \frac{n-1}{n}S_{n-2}$$

$$\therefore S_{2n} = \frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}S_0 \longleftarrow \frac{\pi}{2}$$

$$S_{2n+1} = \frac{2n}{2n+1}\cdot\frac{2n-2}{2n-1}\cdots\frac{4}{5}\cdot\frac{2}{3}S_1 \longleftarrow 1$$

$$\therefore \frac{S_{2n+1}}{S_{2n}} = \frac{2n}{2n+1}\cdot\frac{2n}{2n-1}\cdot\frac{2n-2}{2n-1}\cdots\cdots\frac{4}{5}\cdot\frac{4}{3}\cdot\frac{2}{3}\cdot\frac{2}{1}\cdot\frac{2}{\pi} \lrcorner$$

$$0 < \sin^{2n+1} x < \sin^{2n} x < \sin^{2n-1} x,\quad 0 < \sin x < 1,\ 0 < x < \frac{\pi}{2}$$

$$0 < S_{2n+1} < S_{2n} < S_{2n-1} \qquad \int \bigcirc\,dx から$$

$$1 < \frac{S_{2n}}{S_{2n+1}} < \frac{S_{2n-1}}{S_{2n+1}} = \frac{S_{2n-1}}{\frac{2n}{2n+1}S_{2n-1}} = \frac{2n+1}{2n}$$

$$\therefore \lim_{n\to\infty} \frac{S_{2n}}{S_{2n+1}} = 1$$

$$\frac{\pi}{2}=\prod_{1}^{\infty}\left(\frac{2n\cdot 2n}{(2n-1)(2n+1)}\right)=\prod_{1}^{\infty}\left(\frac{(2n)^2}{(2n)^2-1}\right)=\prod_{1}^{\infty}\left(\frac{1}{1-\left(\frac{1}{2n}\right)^2}\right)$$

$$\therefore \prod_{1}^{\infty}\left(1-\frac{1}{(2n)^2}\right)=\frac{2}{\pi}$$

命題 5.2.

$$\lim_{n\to\infty}\frac{2^{2n}}{\sqrt{n}\binom{2n}{n}}=\sqrt{\pi}$$

証明)

$$S_{2n}\cdot S_{2n+1}=\frac{1}{2n+1}\cdot\frac{\pi}{2}$$

$$\sqrt{n}\,S_{2n+1}\underbrace{\sqrt{\frac{S_{2n}}{S_{2n+1}}}}_{\downarrow\ 1}=\underbrace{\sqrt{\frac{1}{2n+1}}}_{\downarrow\ \frac{1}{2}}\sqrt{\frac{\pi}{2}}\qquad \lim_{n\to\infty}$$

$$\therefore \lim_{n\to\infty}\sqrt{n}\,S_{2n+1}=\frac{\sqrt{\pi}}{2}$$

$$\lim_{n\to\infty}S_{2n+1}=\sqrt{\frac{\pi}{2n}}$$

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$$\int_0^{\frac{\pi}{2}} \sin^{2n+1} x \, dx$$

$$S_{2n+1} = \frac{2n\cdot(2n-2)\cdots 4\cdot 2}{(2n+1)(2n-1)\cdots 5\cdot 3} = \frac{(2n)^2(2n-2)^2\cdots 4^2\cdot 2^2}{(2n+1)!}$$

$$= \frac{(n!)^2\, 2^{2n}}{(2n+1)(2n)!} =$$

$$= \frac{2^{2n}}{(2n+1)\frac{(2n)!}{n!\,n!}} = \frac{2^{2n}}{(2n+1)\binom{2n}{n}}$$

$$\lim_{n\to\infty} \sqrt{n}\, \frac{2^{2n}}{(2n+1)\binom{2n}{n}} = \frac{\sqrt{\pi}}{2}$$

$$\sqrt{\pi} = \lim_{n\to\infty} \frac{2\sqrt{n}}{2n+1} \cdot \frac{2^{2n}}{\binom{2n}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{n}} \cdot \frac{2^{2n}}{\binom{2n}{n}}$$

$$\therefore \sqrt{\pi} = \frac{2^{2n}}{\sqrt{n}\binom{2n}{n}}$$

5.3 スターリングの公式

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2n\pi}\,n^n e^{-n}} = 1$$

$$\lim_{n\to\infty}\left(n! = \sqrt{2n\pi}\cdot \underbrace{n^n e^{-n}}\right)$$

$$= \sqrt{2\pi}\cdot\sqrt{n}\cdot n^n e^{-n}$$

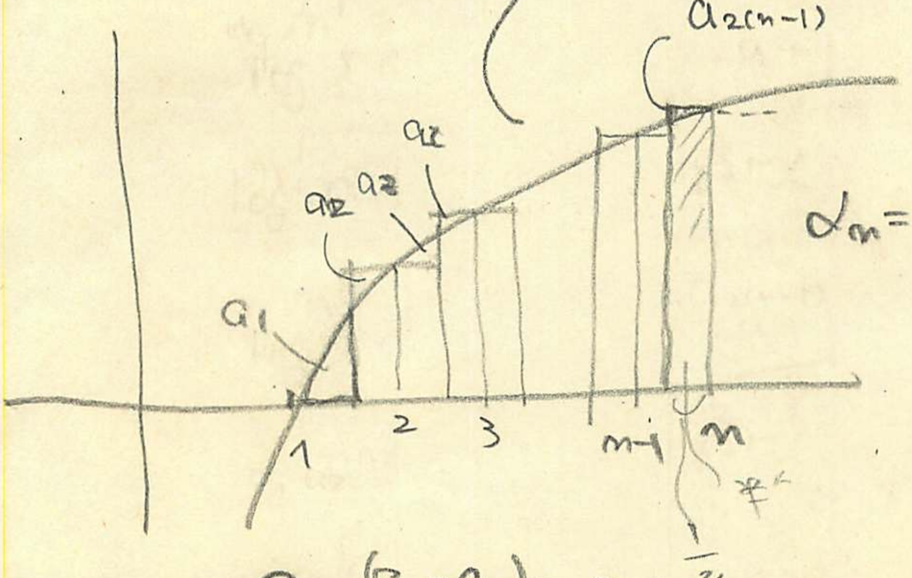
証）$\lim_{n\to\infty} n!$ の状態をみるため

$$\log n! = \sum_{k=1}^{n}\log k$$

$\int \log x\cdot dx = x\log x - x$ なので

$$\int_1^n \log x\,dx = n\log n - n + 1$$

$$= \log 2 + \log 3 + \cdots + \log(n-1) + \frac{1}{2}\log n + \alpha_n$$

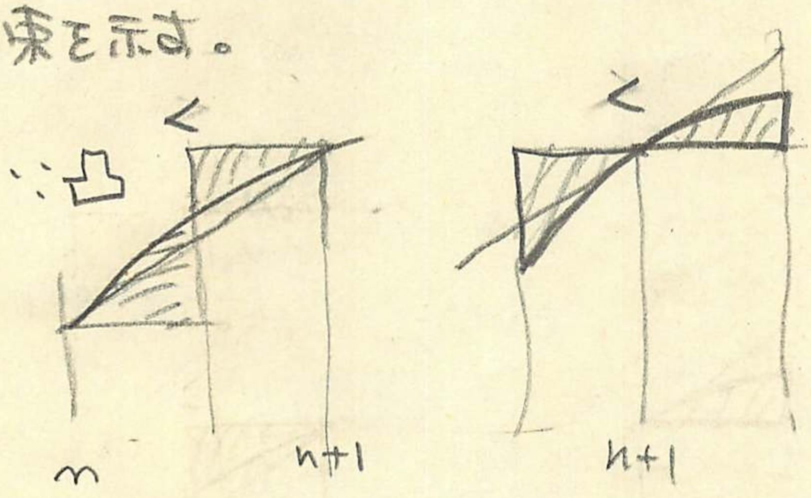


$$\alpha_n = +a_1 \mp a_2 \cdots - a_{2(n-1)}$$

$$a_1 - (a_2 - a_3) - \cdots$$

$d_n = a_1 - a_2 + a_3 - a_4 + a_5 \cdots - a_{2(n-1)}$

d 収束を示す。



$a_{2n-1} < a_{2n}$ $\qquad$ $a_{2n} > a_{2n+1}$

a_nは単調減少

$$a_{2n} < \frac{1}{2}\{\log n - \log(n-1)\} \to 0 \quad (n \to +\infty)$$

∴ d_nは収束する. $\lim_{n\to\infty} d_n = d$ とする

$$\log(n-1)! = \left(n - \frac{1}{2}\right)\log n - n + 1 - d_n$$

$$(n-1)! = n^{(n-\frac{1}{2})} \cdot e^{-n} \cdot e^{1-d_n}$$

$$\lim_{n\to\infty} \frac{(n-1)!}{n^{(n-\frac{1}{2})} \cdot e^{-n}} = e^{1-d}$$

$$\frac{n!}{n^{(n+\frac{1}{2})} \cdot e^{-n}} = e^{1-d}$$

$$\lim_{n\to\infty} \frac{(n!)^2}{n^{2n+1} e^{-2n}} = \left(e^{1-\alpha}\right)^2 \quad \text{—— ①}$$

e^{-2n}をつけ

$$(2n-1)! = (2n)^{(2n-1)/2} \cdot e^{-2n} \cdot e_n^{1-\alpha}.$$

もう一つの e^{-2n}

$$2n! = (2n)^{(2n+1)/2} \cdot e^{-2n} \cdot \frac{e^{1-\alpha}}{A}$$

$$\frac{1}{e^{1-\alpha}} = \frac{(2n)^{\frac{2n+1}{2}} \cdot e^{-2n}}{(2n)!} \quad \text{—— ②}$$

① × ②

$$\lim_{n\to\infty} \left(e^{1-\alpha} = \frac{(n!)^2 (2n)^{\frac{2n+1}{2}}}{(2n)!\; n^{2n+1}} = \frac{(n!)^2}{(2n)!} \frac{\sqrt{2}}{\sqrt{n}} (2)^{2n} \right)$$

$$\sqrt{2\pi} \qquad = e^{1-\alpha}$$

$$\therefore \lim_{n\to\infty} \frac{n!}{n^{(n+\frac{1}{2})} e^{-n}} = \sqrt{2\pi}$$

$$\therefore \lim_{n\to\infty} \frac{n!}{\sqrt{2n\pi}\, n^n e^{-n}} = 1$$

$$\lim_{n\to\infty} \left(n! = \sqrt{2\pi n} \cdot n^n e^{-n} \right)$$

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5-§2 中心極限定理

$$f(t) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-m)^2/2\sigma^2}$$

平均 m
分散 σ の正規分布

$N(m, \sigma^2)$

$$I = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(t-m)^2/2\sigma^2} dt$$ を考える $$x = \frac{t-m}{\sqrt{2}\sigma}$$

$$I = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = 1$$

$t = \sqrt{2}\sigma x + m$
$dt = \sqrt{2}\sigma\, dx$

$$m_i = \int_{-\infty}^{\infty} t \cdot f(t)\, dt = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\sqrt{2}\sigma x + m) e^{-x^2} dx = m$$

奇

$$\sigma_i^2 = \int_{-\infty}^{\infty} (t-m)^2 \cdot f(t)\, dt = \int_{-\infty}^{\infty} (\sqrt{2}\sigma x)^2 \cdot \frac{1}{\sqrt{2\pi}\,\sigma} e^{-(\quad x)^2} \sqrt{2}\sigma\, dx$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} x^2 e^{-x^2} dx$$

$\sqrt{\pi}$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \left[-\frac{x}{2} e^{-x^2} \right]_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} dx = \sigma^2$$

0

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命題5.5

$B_{N,p}(r)$ において $t=\dfrac{r-m_r}{\sigma_r}$ とおき $N\to\infty$

にすると t の分布は $N(0,1)$ に近づく　$\mu=Np$　$\sigma^2=Npq$

証) $N\to\infty$ では t は連続変数になるので

$$\Delta r=\sigma_r\Delta t=\underline{\sqrt{Npq}}\cdot\Delta t$$

t の分布を $f_N(t)$ とすると

$$\underbrace{f_N(t)}\Delta t=B_{N,p}(r)\underline{dr}$$

$$f_N(t)^{(\infty)}=\sqrt{Npq}\cdot\binom{N}{r}p^r q^{N-r}$$

(rの関数)

$$=\frac{N!\,N^{1/2}}{r!\,(N-r)!}\cdot p^{r+1/2}\cdot q^{N-r+1/2}$$

!を消す

$N\to\infty$ のとき

$$N!\simeq\sqrt{2\pi}\,N^{N+1/2}\cdot e^{-N}\quad(\text{スターリングの公式})$$

$$f_N(t)^{(r)}\approx\frac{\sqrt{2\pi}\,N^{N+1/2}\cdot e^{-N}\cdot N^{1/2}\cdot p^{r+1/2}\cdot q^{N-r+1/2}}{\sqrt{2\pi}\,r^{r+1/2}\cdot e^{-r}\,\sqrt{2\pi}\,(N-r)^{N-r+1/2}\cdot e^{-(N-r)}}$$

$$=\frac{1}{\sqrt{2\pi}}\cdot\left(\frac{Np}{r}\right)^{r+1/2}\cdot\left(\frac{Nq}{N-r}\right)^{N-r+1/2}$$

rも十分大きければ

こうできるが……

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$$t = \frac{r - m_r}{\sigma_r} = \frac{r - Np}{\sqrt{Npq}} \rightarrow r = Np + \sqrt{Npq}\cdot t$$

$$\frac{r}{Np} = 1 + \sqrt{\frac{q}{Np}}\cdot t$$

[illegible]

$$N - r = N - Np - \sqrt{Npq}\cdot t = N(1-p) - \sqrt{Npq}\cdot t$$
$$= Nq - \sqrt{Npq}\cdot t$$

$$\therefore \frac{N-r}{Nq} = 1 - \sqrt{\frac{p}{Nq}}\cdot t$$

N→∞の時に0になるものを除く

$$\therefore f_N(t) \approx \frac{1}{\sqrt{2\pi}} \underbrace{\boxed{\left(1 + \sqrt{\frac{q}{Np}}t\right)^{-Np-\sqrt{Npq}t-1/2} \left(1 - \sqrt{\frac{p}{Nq}}t\right)^{-Nq+\sqrt{Npq}t-1/2}}}_{A と [illegible]?}$$

≪1

≪1

$$\sqrt{\frac{q}{Np}}\cdot t$$

$$-\log A = \left(Np + \sqrt{Npq}t + \frac{1}{2}\right)\log\left(1 + \sqrt{\frac{q}{Np}}t\right)$$
$$+ \left(Nq + \sqrt{Npq}\cdot t + \frac{1}{2}\right)\log\left(1 - \sqrt{\frac{p}{Nq}}t\right)$$

$$\log(1+x) = x - \frac{x^2}{2} + O(x^3)$$

$$= \left(Np + \sqrt{Npq}\cdot t + \frac{1}{2}\right)\left(\sqrt{\frac{q}{Np}}\cdot t - \frac{1}{2}\frac{q}{Np}t^2 + O(N^{-\frac{3}{2}})\right)$$
$$+ \left(Nq - \sqrt{Npq}\cdot t + \frac{1}{2}\right)\left(-\sqrt{\frac{p}{Nq}}\cdot t + \frac{1}{2}\frac{p}{Nq}t^2 + O(N^{-\frac{3}{2}})\right)$$

$$= \sqrt{Npq}\cdot t - \frac{1}{2}qt^2 + qt^2 + O(N^{-1/2})$$
$$- \sqrt{Nqp}\cdot t - \frac{1}{2}pt^2 + pt^2 + O(N^{-1/2})$$
$$= \frac{1}{2}(p + q)t^2 + O(N^{-1/2})$$

$$\therefore \lim_{N\to\infty} \log A = -\frac{1}{2}t^2$$

$$\therefore \lim_{N\to\infty} f_N(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

系　$N\to\infty$ の時, $B_{N,p}(r)$

$$\sim \frac{1}{\sqrt{2\pi}\underbrace{\sqrt{Npq}}_{\sigma}} e^{-\overbrace{(r-Np)}^{r-\mu}{}^2/\underbrace{Npq}_{\sigma^2}}$$

証　$t = \dfrac{r-m_r}{\sigma_r} = \dfrac{r-Np}{\sqrt{Npq}}$ と置いた t の分布

$$t \sim \frac{1}{\sqrt{2\pi}} e^{-t^2/2}$$ なので

$$B_{N,p}(r) = \frac{1}{\sigma_r} f_N(t) \sim \frac{1}{\sigma_r}\frac{1}{\sqrt{2\pi}} e^{-(r-m_r)^2/2\sigma_r^2}$$

正規分布

誤差の分布

金貨投げ $\begin{cases} 表 \ +\varepsilon & 1/2 \\ 裏 \ -\varepsilon & 1/2 \end{cases}$

n回の試行後、

$-n\varepsilon$ .... $+n\varepsilon$

のどこかに居る

r回裏で(n−r)回表がでると

$(n-2r)\varepsilon$ の位置に

いる確率は

$\sim B_{n,\frac{1}{2}}(r)$　B(n, p

$n \gg 1$

$$B_{n,1/2}(r) \sim \frac{1}{\sqrt{2\pi}\sigma_r} e^{-(r-m_r)^2/2\sigma_r^2}$$

$m_r = n/2$

$\sigma_r = \sqrt{npq} = \frac{1}{2}\sqrt{n}$

$-(n-2r)\varepsilon = t$ とおくと

$r = \frac{1}{2}(n + \frac{t}{\varepsilon})$, $m_r = \frac{n}{2}$

ガウス積分

$$I = \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

証

$$1 - x^2 < e^{-x^2} < \frac{1}{1+x^2} \qquad x > 0$$

Taylorの初めの項

$$\int_0^1 (1-x^2)^n dx < \int_0^\infty e^{-nx^2} dx < \int_0^\infty \frac{dx}{(1+x^2)^n}$$

$x = \cos\theta$ とおいて

$$\int_0^1 (1-x^2)^n dx = \int_0^{\pi/2} \sin^{(2n+1)}\theta \, d\theta = S_{2n+1}$$

$x = \cot\theta$ とおく

$$\int_0^\infty \frac{dx}{(1+x^2)^n} = \int_{\pi/2}^0 \sin^{2n}\theta \frac{-d\theta}{\sin^2\theta} = S_{2n-2}$$

$\sqrt{n}x = y$ とおくと

$$\int_0^\infty e^{-nx^2} dx = \frac{1}{\sqrt{n}} \int_0^\infty e^{-y^2} dy$$

$$\sqrt{n} S_{2n+1} < I < \sqrt{n} S_{2n-2}$$

$$\underbrace{\lim_{n\to\infty} \sqrt{n} S_{2n+1}}_{\frac{\sqrt{\pi}}{2}} < I < \lim_{n\to\infty} \frac{S_{2n+1}}{S_{2n}} \cdot \frac{S_{2n}}{S_{2n-2}} \qquad \frac{\sqrt{\pi}}{2}$$

$$\therefore I = \frac{\sqrt{\pi}}{2}$$

フビニの定理

$$\int_0^\infty \int_0^\infty e^{-x^2-y^2} dx\,dy = \left(\int_0^\infty e^{-x^2} dx\right) \cdot \left(\int_0^\infty e^{-y^2} dy\right) = I^2$$

$x = rc_\theta,\ y = rs_\theta \qquad dx\,dy = r\,dr\,d\theta$

$$I^2 = \int_0^{\pi/2} d\theta \cdot \int_0^\infty re^{-r^2} dr$$

$$= \int_0^{\pi/2} d\theta \left[-\frac{1}{2}e^{-r^2}\right]_0^\infty$$

$$= \int_0^{\pi/2} \frac{1}{2} d\theta$$

$$= \frac{\pi}{2}$$

$\cos\theta \quad -r$

$dx = \cos\theta \overset{dr}{} - rs_\theta \overset{d\theta}{}$

$dy = s_\theta \overset{dr}{} + rc_\theta\, d\theta$

$dx\,dy = (\overset{dr}{c_\theta} - r\overset{d\theta}{s_\theta})(\overset{dr}{s_\theta} + r\overset{d\theta}{c_\theta})$

$= s_\theta c_\theta \overset{(dr)^2}{} - (rs_\theta^2 + rc_\theta^2) - r^2 s_\theta r_\theta)\, d\theta^2$

$dr\,d\theta$

$s_\theta \quad c_\theta dr - rs_\theta\, d\theta = dx$

$c_\theta \quad s_\theta dr + rc_\theta\, d\theta = dy$

$-rs_\theta^2\, d\theta = s_\theta dx$

$-rc_\theta^2\, d\theta =$ Jacobi.

$$\begin{pmatrix} dx \\ dy \end{pmatrix} = \begin{pmatrix} c_\theta & -rs_\theta \\ s_\theta & rc_\theta \end{pmatrix} \begin{pmatrix} dr \\ d\theta \end{pmatrix}$$

$r(c_\theta^2 + s_\theta^2)$