The Artin Wedderburn theorem

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Statement of the theorem

- Noncommutative ring

Statement of the theorem

- Noncommutative ring
- Division ring

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- Module

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- Noncommutative ring
- Division ring
- Module
- Semisimple ring

Classical proof

- $\mathrm{End}(R) \cong \prod_{i=n}^{n} M_n(D_i)$

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- Problems with Mathlib library

Further definitions

- Corner ring

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- Ideal

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- Prime ring

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- Ideal
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- Artinian ring

Further definitions

- Corner ring
- Ideal
- Prime ring
- Artinian ring
- Matrix units

Proving the theorem

Artin Wedderburn's theorem

If R is a nonzero prime left artinian ring, then there exist a division ring D and a positive integer n such that $R \cong M_n(D)$.

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Lemma

If a prime ring R contains pairwise orthogonal idempotents $e_1, \ldots, e_n$ such that $\sum_{i=0}^{n} e_i = 1$ and $e_i R e_i$ is a division ring for each i then $R \cong M_n(e_i R e_i)$.

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Lemma 3

If e and $f \neq 0$ are orthogonal idempotents of a ring R then $R(1-e-f) \subset R(1-e)$.

Why alternative proof?

- Easier to work in parallel,
- more incremental,
- formalization of corner rings and matrix units.

Further work

- Generalization to semisimple rings,
- uniqueness of division rings up to isomorphism.

References

Matej Brešar.
The wedderburn-artin theorem, 2024.