Artin Wedderburn's theorem
Formalization in Lean

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I was working on the project of formalizing Artin Wedderburn's theorem, together with Job Petrovčič and Matevž Miščič, students from University of Ljubljana. We were following Matej Brešar's article [1], which is also linked in the github repository. As oppose to the classical approach to the proof, which uses modules, the proof we followed uses corner rings instead.

The main reason for choosing to follow this article was the opportunity to also formalize corner rings and matrix units, which were not in the Mathlib library before. While working on the project we defined corner rings of idempotent elements as subsets of R, consisting of elements $e * r * e$ for $r \in R$. We proved that it forms a subring of R and some of its properties. Besides proving some smaller lemmas, I worked on coercing the ideals from corner rings to ideals of R. I defined lift and push of ideals and proved that performing them one after the other results in the initial ideal, which we later used to prove that the corner ring of (left) artinian ring is (left) artinian. Additionally, we proved that the corner ring of prime ring is prime and that every simple ring is prime. In the scope of the project matrix units were defined as elements $\{e_{i,j}\}$, indexed by i and j, which behave in a similar was as basis of $n \times n$ matrices $\{E_{i,j}\}$, meaning that they satisfy $\sum_{i=0}^{n} e_{i,i} = 1$ and $e_{i,j}e_{k,l} = \delta_{j,k}e_{i,l}$. It was proven before I joined the project that a ring with matrix units is isomorphic to a matrix ring over corner ring $e_{0,0}Re_{0,0}$. My main focus was the proof that nontrivial, simple, (left) artinian ring contains a set of idempotents, which sum up to 1 and are pairwise orthogonal. Furthermore, their corner rings are division rings. We then used those idempotents to construct matrix units of R, using lemma 2.19 from the article, which was also my part of the work. I defined a class of orthogonal idempotents (named "OrtIdem" and "OrtIdemDiv" in Lean) and property of ideal, which asserts that for every ideal, generated by an idempotent, the corner ring of the generator contains orthogonal idempotents, each of them defining corner ring, which is division ring (named "NiceIdeal" in Lean). The main result of this part of the project was the theorem, stating that if every ideal contained in ideal I is "nice", then I itself is a "nice" ideal. This enabled us to use the artinian property to conclude that nontrivial, prime, (left) artinian ring is "nice", and therefore contains orthogonal idempotents. After this I combined the formalized results into the proof of the main theorem, for which I also needed a lot of smaller lemmas about idempotent elements, that can be found in the file Idempotents. The main lemma of the file is lemma 2.19, which plays a crucial role in the construction of matrix units. Important lemma I formalized is also the one asserting that nontrivial prime (left) artinian ring contains an idempotent element, whose corner ring is a division ring.

Even though the theorem is proven, there are some sorry's. Since the other members working on the project have the deadline for submission in the end of February, the project is not finished yet. We are planning to add the characterization of prime ring via two-sided ideals and the generalization of the theorem to semisimple rings.

References

[1] Matej Brešar. The wedderburn-artin theorem, 2024.