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Open Majorana system

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Plan of the talk

(1)Hamiltonian system: SYK and Majorana chains

(2)symmetry in open systems

(3)dinamics in open SYK

Motivations

- theoretically interesting and simple
- less symmetry than ordinary (complex) fermions
- related to qubits and quantum computations ↙ Bravyi Kitaev
  2 body . 4 body
- open Majorana system may be a new window for lattice fermions
  Majorana

Today I want to introduce the recent development to generalize notion of symmetry, it’s breaking and anomaly for open systems

Majorana fermions

Complex fermions

$c^\dagger c + c c^\dagger = 1$

Canonical anti-ommutation relation: $\{c^\dagger, c\} = 1$ $\quad c^2 = 0.\ (c^\dagger)^2 = 0$

Fock Vacuum: $c\,|0\rangle = 0$

Occupied state: $c^\dagger\,|0\rangle$ $\qquad c^\dagger(c^\dagger\,|0\rangle) = (c^\dagger)^2\,|0\rangle = 0$

$H = \varepsilon c^\dagger c$

Majorana fermions

$\psi_1 \quad \psi_2$ $\qquad \psi_1 = \frac{1}{2}(c + c^\dagger)$

$$c = \psi_1 + i\psi_2$$

2 Majoranas $\qquad \psi_2 = \frac{1}{2i}(c - c^\dagger)$

$$\{\psi_i, \psi_j\} = \delta_{ij} \qquad \psi_i^\dagger = \psi_i$$

Majorana fermions

N number's of Majorana fermions

$$\{\psi_i, \psi_j\} = \delta_{ij} \qquad \psi_i^\dagger = \psi_i$$

$$i = 1, \cdots, N$$

Equivalent to N/2 complex fermions:

$C_a = f_{2a-1} + i f_{2a}$

$$\{c_a^\dagger, c_b\} = \delta_{ab} \qquad (c_a^\dagger)^2 = c_a^2 = 0$$

$$a = 1 \cdots N/2$$

Hilbert space: $2^{\frac{N}{2}}$ dimensions

$C_a|0\rangle = 0$

Spanned by $c_{i_1}^\dagger \cdots c_{i_p}^\dagger |0\rangle$

Majorana fermions

Generic Hamiltonians

$$H = \sum_{p:even} \sum_{p} c_{i_1 i_2 \cdots i_p} \psi_{i_1} \psi_{i_2} \ldots \psi_{i_p}$$

$i_1 < \cdots < i_p$

- SYK

$$H_{SYK} = i^{\frac{q}{2}} \sum_{i_1 < i_2 < \cdots < i_q} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$$

- Majorana Chain

$$H_{\mathrm{KC}} = -i \sum_{\ell=1}^{N} \psi_\ell \psi_{\ell+1}$$

Jordan-Wigner transformation

Majorana fermions are represented as matrices using the Jordan-Wigner transformation;

$i-1$ $\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

$\sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

<u>even N</u>

$$\psi_{2i-1} = \frac{1}{\sqrt{2}} \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^x} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$

$$\psi_{2i} = \frac{1}{\sqrt{2}} \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{i-\text{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

$2^{\frac{N}{2}}$

Then they satisfy $\{\psi_i, \psi_j\} = \delta_{ij}$. Equivalent to *qubits*!

The same algebra with the γ matrices (in Euclidean signature)

$\chi_i \equiv \sqrt{2}\psi_i$:gamma matrix normalization $\{\chi_i, \chi_j\} = 2\delta^{ij}$

Odd N Majorana fermions

odd N

$N-1 = \text{even}$

$\underline{\psi_1 \cdots \psi_{N-1}}$

(1) Represent ψ_N using N-1 Majorana fermions as

$2^{\frac{N-1}{2}}$

$$\psi_N = \pm \frac{1}{\sqrt{2}}(2i\psi_1\psi_2)(2i\psi_3\psi_4)\cdots(2i\psi_{N-2}\psi_{N-1})$$

$\{\psi_i, \psi_N\} = 0 \quad i = 1 \sim N-1$

$\psi_N^2 = 1$

(2) Use $\psi_1, \cdots, \psi_N$ out of N+1 Majorana fermions

$N+1 = \text{even}$

$\underbrace{\psi_1 \cdots \psi_N}\psi_{N+1}$

$2^{\frac{N+1}{2}}$

(3) $\sqrt{2}$

Jordan-Wigner transformation: example

Example; 4 Majorana fermions $\quad 2^{\frac{4}{2}} = 4$

$$\sqrt{2}\,\gamma_1 = \sigma^x \otimes I = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 \\ \multicolumn{2}{c}{0} & 1 & 0 \end{pmatrix}$$

$$\sqrt{2}\,\gamma_2 = \sigma^y \otimes I$$

$$\sqrt{2}\,\gamma_3 = \sigma^z \otimes \sigma^x$$

$$\sqrt{2}\,\gamma_4 = \sigma^z \otimes \sigma^y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i & & \\ i & 0 & \multicolumn{2}{c}{0} \\ & & 0 & i \\ \multicolumn{2}{c}{0} & -i & 0 \end{pmatrix}$$

Example; 5 Majorana fermions

(ii) $$\gamma_5 = \frac{1}{\sqrt{2}}(-2i\gamma_1\gamma_2)(-2i\gamma_3\gamma_4) = \frac{1}{\sqrt{2}}(\sigma^z \otimes I)(I \otimes \sigma^z)$$

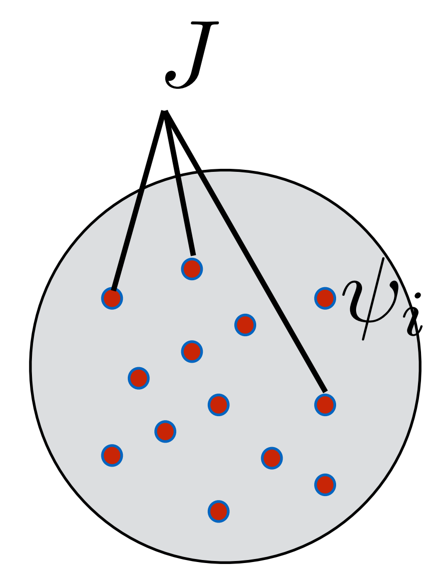
$$= \frac{1}{\sqrt{2}}(\sigma^z \otimes \sigma^z)$$

Sachdev-Ye-Kitaev

[Sachdev-Ye 93] [Kitaev 15]

N Majorana fermion $\{\psi_i, \psi_j\} = \delta_{ij}$ ($\dim\mathcal{H} = 2^{\frac{N}{2}}$)

Hamiltonian: $H_{SYK} = i^{\frac{q}{2}} \sum_{i_1 < i_2 < \cdots < i_q} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$



q : even

$J_{i_1 i_2 \cdots i_q}$: Random coupling constant

$\langle J_{i_1 i_2 \cdots i_q} \rangle_J = 0$ and $\langle J^2_{i_1 i_2 \cdots i_q} \rangle_J = \dfrac{\mathcal{J}^2 (q-1)!}{q(2N)^{q-1}}$

Some motivations to the SYK

- Relation to quantum gravity: [Maldacena-Stanford, 16]
  [Maldacena-Stanford-Yang, 16]

  Low energy theory is the same with 2d dilaton gravity

- As strongly correlated systems

  Similar to the multi-band hubbard model

- relation to quantum chaos $2^{\frac{N}{2}}$

  The SYK is a very sparse random matrix.

  random matrix = characterization of quantum chaos

  (Energy level statistics)

  0+1d *'t Hooft anomalies* determine the level statistics

Example: q=2 SYK

$$H_{q=2} = i \sum_{i<j} J_{ij}\psi_i\psi_j$$

$J \to \quad O J O^T = \begin{pmatrix} 0 & \lambda_1 & & \\ -\lambda_1 & 0 & & \\ & & 0 & \lambda_2 \\ & & -\lambda_2 & 0 \\ & & & & \ddots \end{pmatrix}$

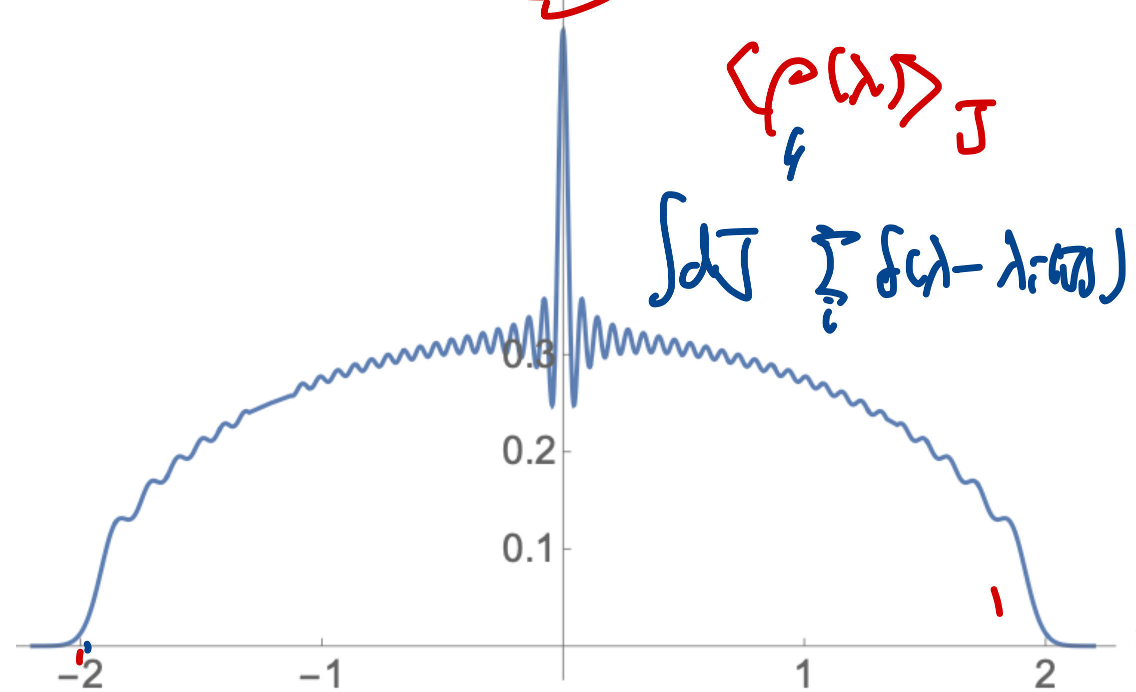
↓ orthogonal

Using orthogonal matrices $\psi_i \to O_{ij}\psi_j = \psi'_i$ we can diagonalize

$$H_{q=2} = i \sum_{k=1}^{\frac{N}{2}} \lambda_i \psi'_{2i-1}\psi'_{2i}$$

λ_i : random mass

$\sum_i \delta(\lambda - \lambda_i(J))$



$\langle \rho(\lambda) \rangle_J$

$\int dJ \sum_i \delta(\lambda - \lambda_i(J))$

Symmetry of q=4 SYK

$$H_{q=4} = \sum_{i<j<k<l} J_{ijkl}\psi^i\psi^j\psi^k\psi^l$$

$T c = c^* T$

$T(aA + bB)T^{-1}$

$= a^* TAT^{-1} + b^* TBT^{-1}$

$T H_{q=4} T^{-1} =$

$\sum_{i<j<k<l} J_{ijkl}\, T\psi^i\psi^j\psi^k\psi^l T^{-1}$

$= H_{q=4}$

Is invariant under $\mathbb{Z}_2^T \times \mathbb{Z}_2$ symmetry:

- $\mathbb{Z}_2$ Fermion parity $(-1)^F$

$$(-1)^F\psi^i(-1)^F = -\psi^i$$

- $\mathbb{Z}_2^T$ Time reversal symmetry $\mathcal{T}$ ← anti-unitary symmetry.

$$\mathcal{T}\chi_i\mathcal{T}^{-1} = \chi_i$$

$T\psi_i T^{-1} = \psi_i$

Depending on N mod 8, there is an t' Hooft anomalies

$$\mathcal{T}(-1)^F = \pm(-1)^F\mathcal{T}$$

$$\mathcal{T}^2 = \pm 1$$

(Projective representations)

fermion parity symmetry

$\chi_i := \sqrt{2}\,\gamma_i$

$(-1)^F$

even N

$$(-1)^F = (i\chi_1\chi_2)(i\chi_3\chi_4)\cdots(i\chi_{N-1}\chi_N)$$ anti-commutes with all the fermions.

Using the Jordan-Wigner transformation,

$$\chi_{2i-1} = \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^x} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\chi_{2i} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{i-\text{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

$$\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$i\chi_{2i-1}\chi_{2i} = \underbrace{I \otimes \cdots \otimes I}_{(i-1)\text{products}} \otimes \overset{i-\text{th}}{\sigma^z} \otimes \underbrace{I \otimes \cdots \otimes I}_{(N/2-i)\text{products}}$$ spin at site i.

$$(-1)^F = \underbrace{\sigma^z \otimes \cdots \otimes \sigma^z}_{\frac{N}{2}}$$: product of all the spins

fermion parity symmetry

odd N

We add $(-1)^F \equiv \chi_N$ to the algebra at N-1.

We do not have fermion parity any more.

Time reversal symmetry : even N

[cf: Stanford Witten 19]

even N

The time-reversal symmetry acts on Majorana fermions as

$$\mathcal{T}\chi_i\mathcal{T}^{-1} = \chi_i$$

In our conventions, the complex conjugate flips the sign of the even fermions:

$$K\chi_{2i-1}K = \chi_{2i-1} \qquad\qquad K\chi_{2i}K = -\chi_{2i}$$

Therefore to ensure the commutativity, we need a modification.

The correct one is

$$\mathcal{T} = \begin{cases} \chi_2\chi_4\chi_6\cdots\chi_N K & \text{for N = 0 mod 4} \\ \chi_1\chi_3\chi_5\cdots\chi_{N-1}K & \text{for N = 2 mod 4} \end{cases}$$

$$= (\text{phase}) \times \underline{\sigma^y \otimes \sigma^x \otimes \sigma^y \otimes \cdots K}$$

Another anti-unitary operators are given by

$$\mathcal{T}' = \mathcal{T}(-1)^F$$
$$= (\text{phase}) \times \sigma^x \otimes \sigma^y \otimes \sigma^x \otimes \cdots K$$

which anti-commutes with the fermion: $\mathcal{T}'\chi_i\mathcal{T}'^{-1} = -\chi_i$

Time reversal symmetry : even N

property of the time reversal for even N

$N=2\quad \sigma^y K$
$N=4\quad \sigma^x \sigma^y K$
$N=6\quad \sigma^x \sigma^y \sigma^x K$
$N=8\quad \sigma^x \sigma^y \sigma^x \sigma^y K$

Explicitly calculating, we obtain

$$\underline{\mathcal{T}}^2 = \begin{cases} +1 & \text{for N = 0,}\cancel{2}\text{6 mod 8} \quad \text{(number of } \sigma^y \text{ : even)} \\ \underline{-}1 & \text{for N = } \cancel{4,6}\ \text{2,4 mod 8} \quad \text{(number of } \sigma^y \text{ : odd)} \end{cases}$$

and

$$\mathcal{T}(-1)^F = \begin{cases} +(-1)^F \mathcal{T} & \text{for N = 0,4 mod 8} \quad \text{(number of } \sigma \text{ : even)} \\ -(-1)^F \mathcal{T} & \text{for N = 2,6 mod 8} \quad \text{(number of } \sigma \text{ : odd)} \end{cases}$$

From those two, we can deduce

$$\sigma^y K \sigma^y K = \sigma^y(-\sigma^y)\underbrace{KK}_{1}$$
$$= -1$$

$$\mathcal{T}'^2 = \begin{cases} +1 & \text{for N = 0,6 mod 8} \\ -1 & \text{for N = 2,4 mod 8} \end{cases}$$

N $\rightarrow$ 8-N mod 8 exchanges $\mathcal{T}$ and $\mathcal{T}'$

Time reversal symmetry : odd N

odd N   we do have $(-1)^F$

not

What we have for odd N is

$$\widehat{\mathcal{T}} = \chi_1 \chi_3 \cdots \chi_{N-2} \chi_N K$$

This satisfies

$$\widehat{\mathcal{T}} \chi_i \widehat{\mathcal{T}}^{-1} = \begin{cases} +\chi_i & \text{for N = 1 mod 4} \\ -\chi_i & \text{for N = 3 mod 4} \end{cases}$$

$\widehat{\mathcal{T}} = \mathcal{T}$

$\widehat{\mathcal{T}} = \mathcal{T}'$

and

$$\widehat{\mathcal{T}}^2 = \begin{cases} +1 & \text{for N = 1,7 mod 8} \\ -1 & \text{for N = 3,5 mod 8} \end{cases}$$

(# of σ^y = even)

(# of σ^y = odd)

The property of the time reversal is symmetric under N → 8-N mod 8

Time reversal symmetry :

summary

$N \bmod 8$	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$[\mathcal{T}, (-1)^F]_\pm$
0	$+1$	$+1$	1
1	$+1$		
2	$+1$	-1	-1
3		-1	
4	-1	-1	1
5	-1		
6	-1	$+1$	-1
7		$+1$	

T

T'

$$\mathcal{T}_+ \chi_i \mathcal{T}_+^{-1} = \chi_i$$

$$\mathcal{T}_- \chi_i \mathcal{T}_-^{-1} = -\chi_i$$

$$(-1)^F \chi_i = -(-1)^F \chi_i$$

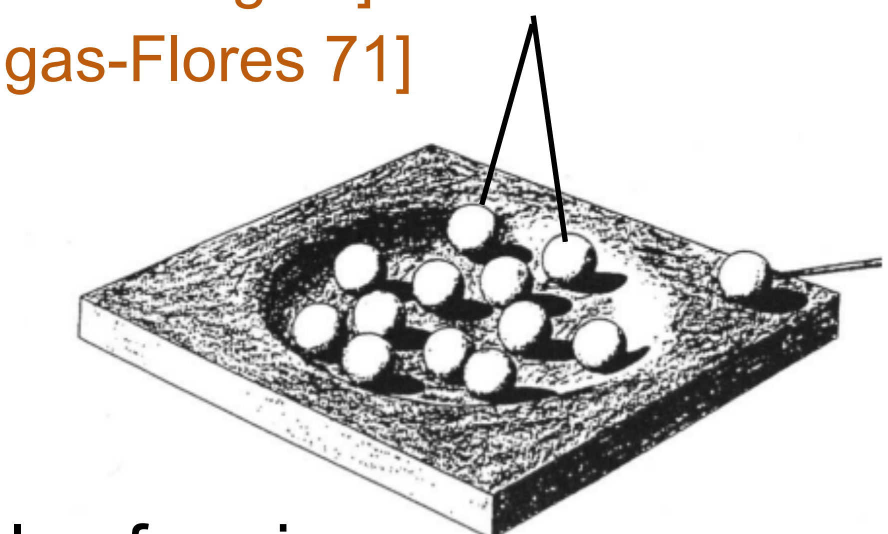
SYK Hamiltonian is invariant under both $\mathcal{T}_+$ and $\mathcal{T}_-$

Quantum chaos and SYK

Before Kitaev, and even more before Sachdev and Ye, it was introduced in the context of nuclear physics to study the interaction effect on the level statistics of the nuclear spectrum!

[French-Wong 71]
[Bohigas-Flores 71]

$$\sum_{i<j;k<l} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$$



$\{c_i^\dagger, c_j\} = \delta_{ij}$ $\quad$ $\{c_i^\dagger, c_j^\dagger\} = \{c_i, c_j\} = 0$:complex fermions

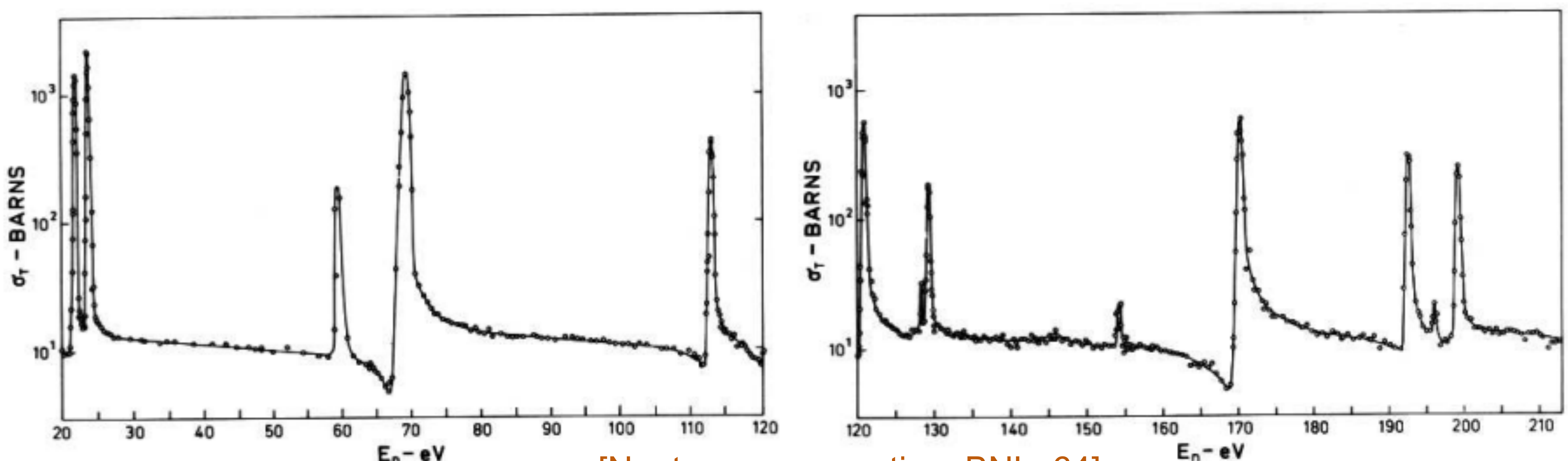
Therefore the SYK is a natural setup to study the level statistics.
Kitaev's Majorana version reproduce all the Wigner-Dyson ensemble

(The supersymmetric version realizes all the Altland-Zirnbauer ensembles)

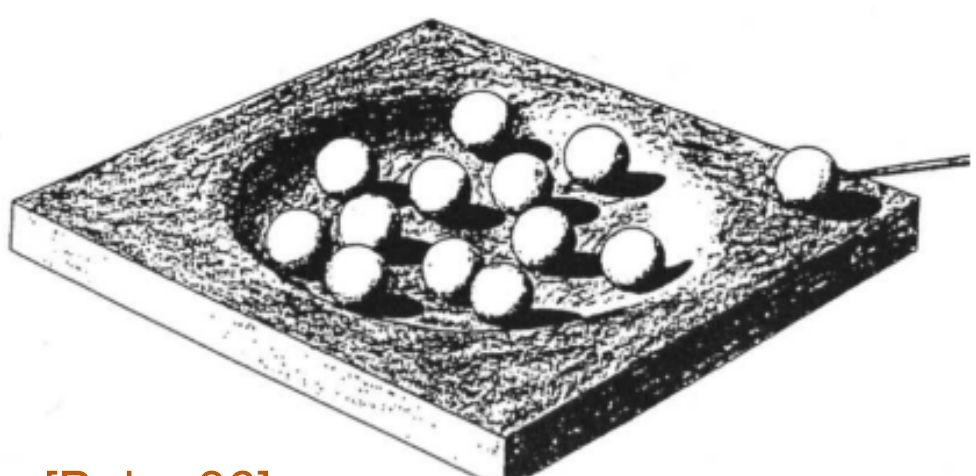
Nuclei spectrum

Random matrices are first applied to physics by Wigner.

Motivated by the complicated nuclear spectrum.



[Neutron cross section, BNL, 64]



[Bohr, 36]

Bohr’s wooden toy model.

compounded nuclei are strongly interacting

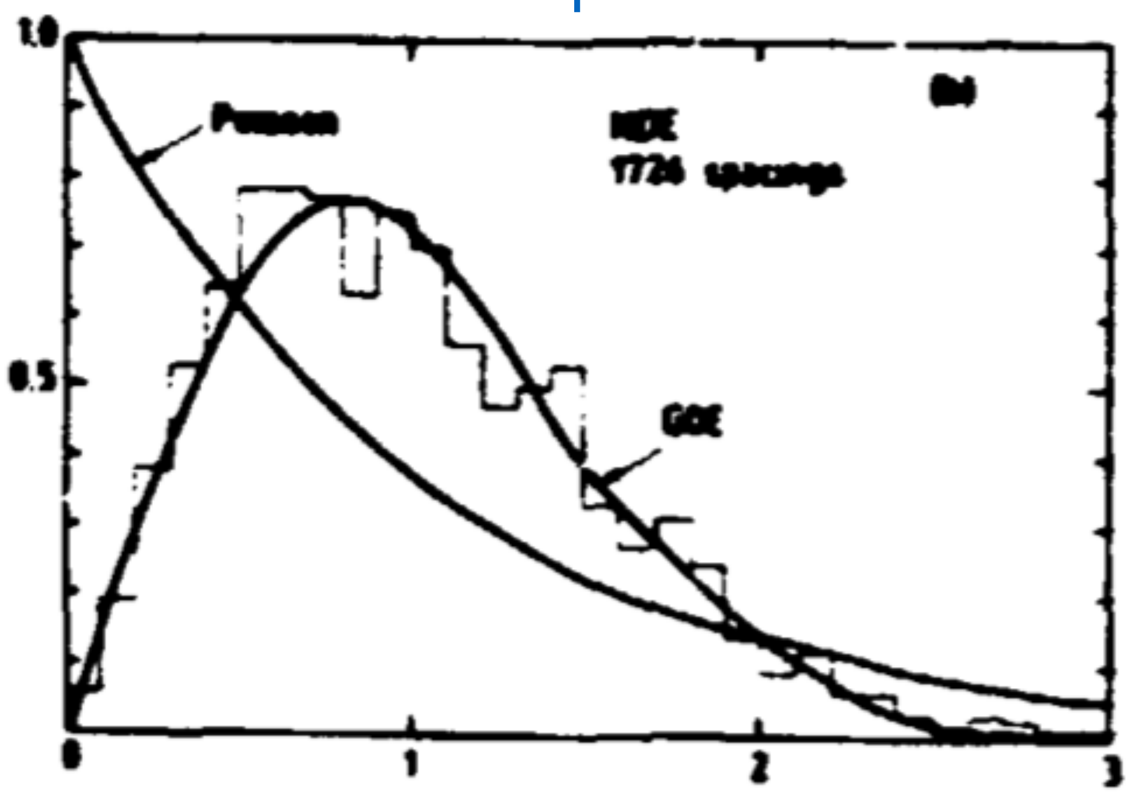
Random matrices

$E_{n+1}-E_n$

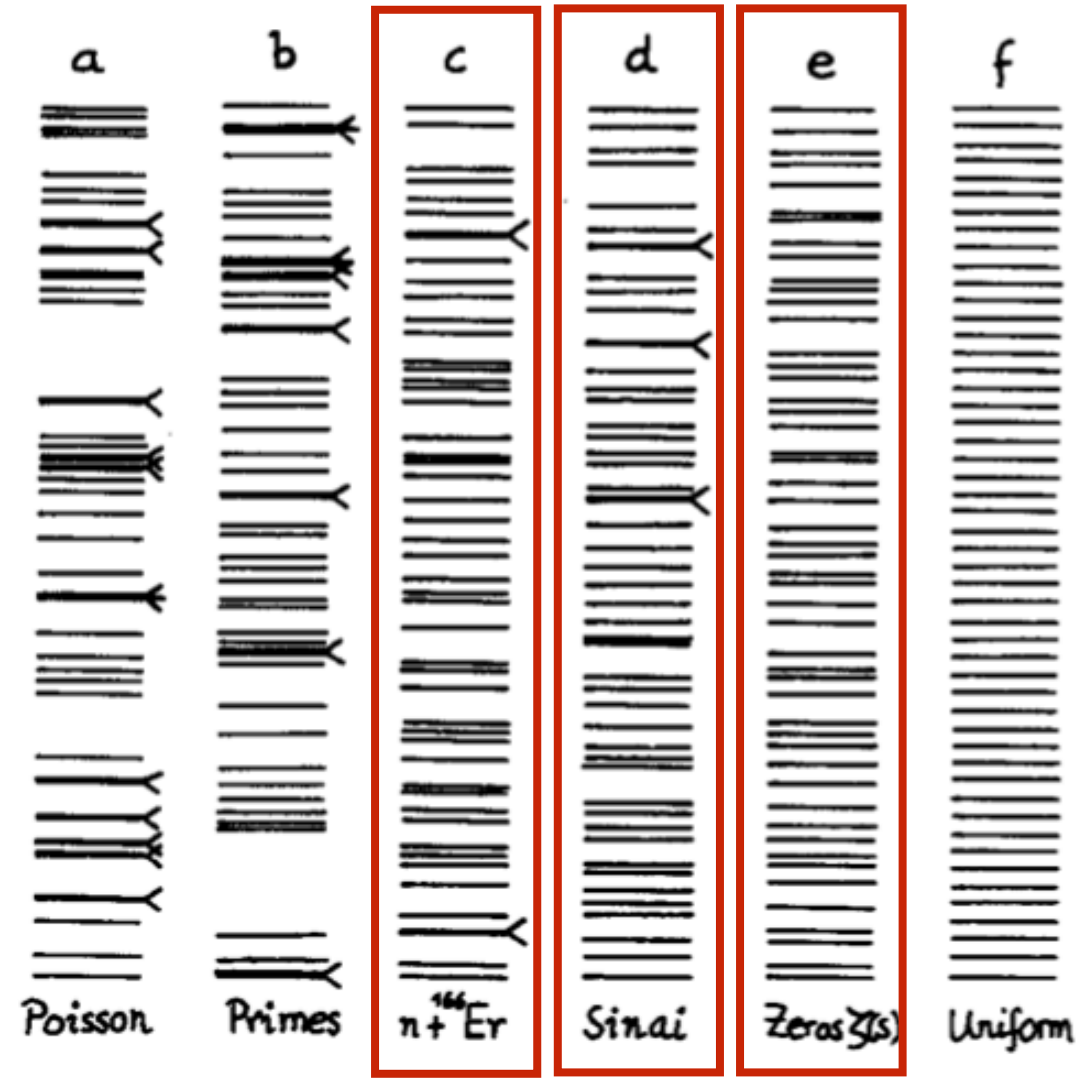
The philosophy of a random matrix approach :

focus on the statistical property of the spectrum rather than understanding the specific eigenvalues themselves.

Nuclear spectrum



[Bohigas Haq Pandey, 83]



[Bohigas Giannoni, 84]

Nuclear spectrum, Sinai billiard and zero's of the Riemann zeta have the same statistical property! Very universal.

Time reversal and ensemble

No time reversal $\rightarrow$ GUE

unitary

time reversal with $\mathcal{T}^2 = 1$ $\rightarrow$ GOE

orthogonal

time reversal with $\mathcal{T}^2 = -1$ $\rightarrow$ GSE

symplectic

The spin 1/2 should transform $\mathcal{T}\boldsymbol{s}\mathcal{T}^{-1} = -\boldsymbol{s}$ $\quad \boldsymbol{s} = \frac{1}{2}\boldsymbol{\sigma}$

$\mathcal{T}\,\sigma_{x,y,z}\,\mathcal{T}^{-1}$

$= -\sigma_{x,y,z}$

We can choose $\mathcal{T} = i\sigma^y K \;= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K$

Now the time reversal satisfies $\mathcal{T}^2 = -1$

N mod 8 classification of ordinary SYK

[Xu,Ludwig,You 16]

$\mathcal{T}_{\pm}$: time reversal

$(-1)^F$: fermion parity

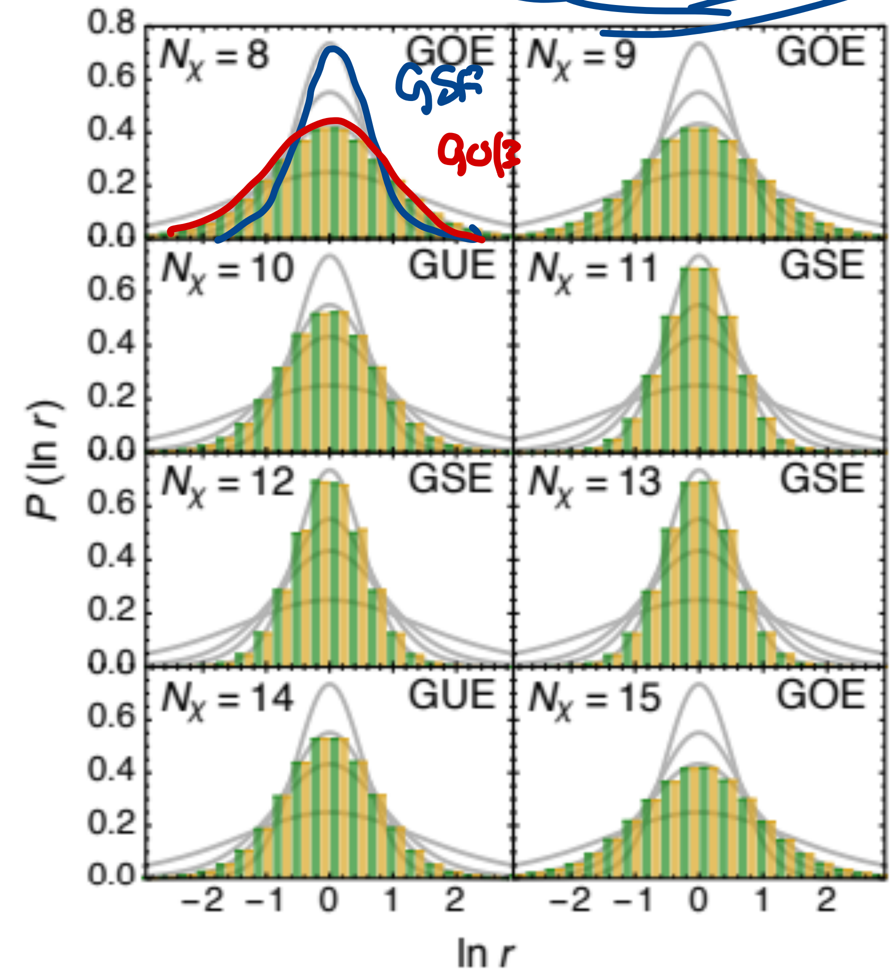
$\Sigma\, \chi\chi\chi\chi$

$+ \Sigma\, \chi\chi\chi\chi\, \chi\chi\chi\chi$

Gap Ratio $r_n = \dfrac{E_{n+1} - E_n}{E_n - E_{n-1}}$

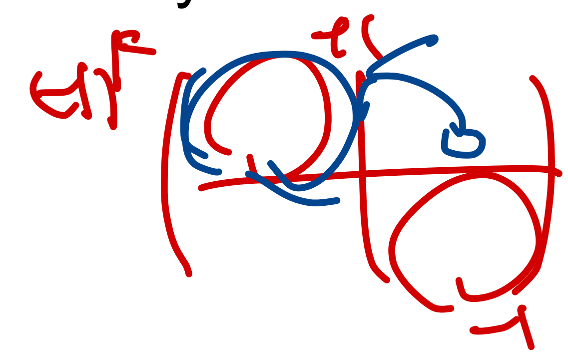
Table of N mod 8 dependence

N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$(-1)^F$	$\mathcal{T}(-1)^F = a(-1)^F\mathcal{T}$	Level Stat	qdim
0	$+1$	$+1$	Yes	1	GOE	1
1	$+1$		No		GOE	$\sqrt{2}$
2	$+1$	-1	Yes	-1	GUE	2
3		-1	No		GSE	$2\sqrt{2}$
4	-1	-1	Yes	1	GSE	2
5	-1		No		GSE	$2\sqrt{2}$
6	-1	$+1$	Yes	-1	GUE	2
7		$+1$	No		GOE	$\sqrt{2}$



SYK realize all the Wigner-Dyson statistics!

When $\mathcal{T}$ anti-commutes with $(-1)^F$, $\mathcal{T}$ shuffles two chirality sectors.



To distinguish N and 8-N, we need additional ingredient

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$$

Relation to properties of gamma matrices

$$\gamma^{2k-1} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{k-\mathrm{th}}{\sigma^x} \otimes I \otimes \cdots \otimes I$$

$$\gamma^{2k} = \sigma^z \otimes \cdots \otimes \sigma^z \otimes \overset{k-\mathrm{th}}{\sigma^y} \otimes I \otimes \cdots \otimes I$$

chirality: $\Gamma = (i\gamma^1\gamma^2)(i\gamma^3\gamma^4)\cdots(i\gamma^{N-1}\gamma^N)$ (for even d)

Charge conjugation matrices

$$C_\eta \gamma^\mu C_\eta^{-1} = \eta(\gamma^\mu)^T$$

$$C_\eta^\dagger = C_\eta \quad \eta = \pm 1$$

SYK		**Gamma matrices**
$\mathcal{T}_\eta$	$\leftrightarrow$	C_η
$\mathcal{T}_\eta^2$	$\leftrightarrow$	$C_\eta^* C_\eta$
$(-1)^F$	$\leftrightarrow$	Γ

Corresponds to Level statistics

d mod 8	$C_+^* C_+$	$C_-^* C_-$	$[C,\Gamma]_\pm$	spinor	reality
0	$+1$	$+1$	1	MW	Real
1	$+1$			M	Real
2	$+1$	-1	-1	M,W	Complex
3		-1			Pseudo
4	-1	-1	1	W	Pseudo
5	-1				Pseudo
6	-1	$+1$	-1	M,W	Complex
7		$+1$		M	Real

M = Majorana
W = Weyl

Summary of SYK

SYK

N Majorana fermion $\{\psi_i, \psi_j\} = \delta_{ij}$ ($\dim\mathcal{H} = 2^{\frac{N}{2}}$)

Hamiltonian: $H_{SYK} = i^{\frac{q}{2}} \sum_{i_1 < i_2 < \cdots < i_q} J_{i_1 i_2 \cdots i_q} \psi_{i_1} \psi_{i_2} \cdots \psi_{i_q}$

q : even

with $\langle J_{i_1 i_2 \cdots i_q} \rangle_J = 0$ and $\langle J^2_{i_1 i_2 \cdots i_q} \rangle_J = \frac{\mathcal{J}^2 (q-1)!}{q(2N)^{q-1}}$

$q = 4k$

symmetry Time reversal $\mathcal{T}$ and fermion parity $(-1)^F$

N mod 8	$\mathcal{T}_+^2$	$\mathcal{T}_-^2$	$(-1)^F$	$\mathcal{T}(-1)^F = a(-1)^F\mathcal{T}$	Level Stat	qdim
0	+1	+1	Yes	1	GOE	1
1	+1		No		GOE	$\sqrt{2}$
2	+1	−1	Yes	−1	GUE	2
3		−1	No		GSE	$2\sqrt{2}$
4	−1	−1	Yes	1	GSE	2
5	−1		No		GSE	$2\sqrt{2}$
6	−1	+1	Yes	−1	GUE	2
7		+1	No		GOE	$\sqrt{2}$