Plan of the talk

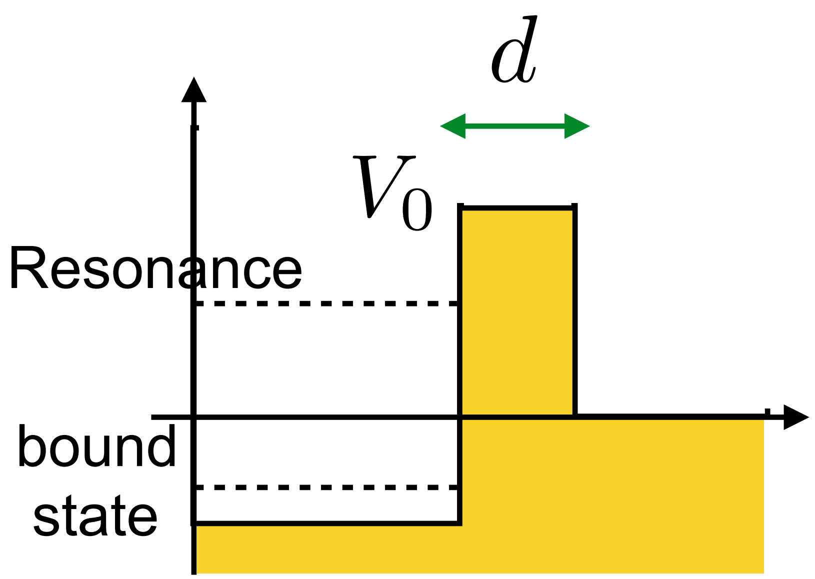
(1)Hamiltonian system: SYK and Majorana chains

(2)symmetry in open systems

(3)dinamics in open SYK

Resonance and Open systems

Consider a particle trapped in a potential, but decay by quantum tunneling.



Gamov' factor $\gamma_i \sim e^{-\frac{2d}{\hbar}\sqrt{2m(V_0 - E_i)}}$

bound state: $e^{-\kappa r} = e^{i(i\kappa)r}$ outgoing wave: pure imaginary momentum

Resonance: $e^{(\gamma + ip)r} = e^{i(p - i\gamma)r}$ outgoing wave: complex momentum

= connection to the environment

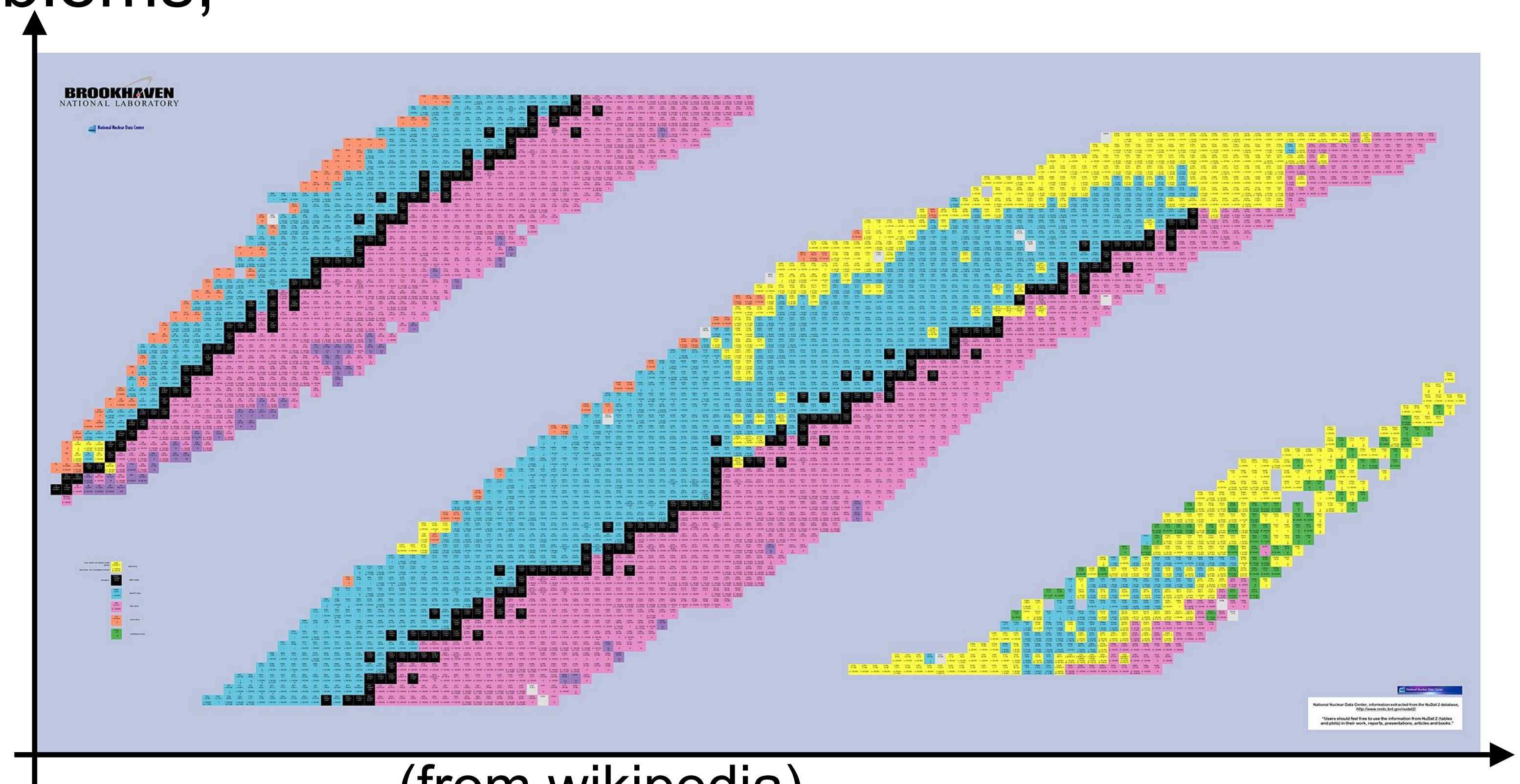
The operator acting ***outside of the Hilbert space*** becomes non-Hermitian!

Compatible with the characterization as complex poles in the green functions

$$G(E) = \mathrm{Tr}\frac{1}{E - H}$$

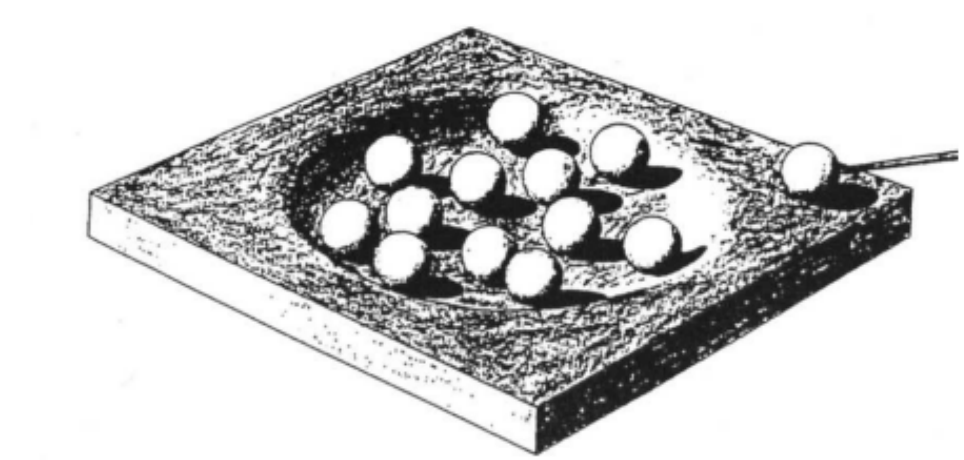
Open Sachdev-Ye-Kitaev

Going back to the nuclear problems,
many nuclei are resonances!



(from wikipedia)

It is natural to think about SYK model coupled to an environment and study the universality of level statistics



many body: $\sum_{i<j;k<l} J_{ij;kl} c_i^\dagger c_j^\dagger c_k c_l$ + dissipation

Quantum channels

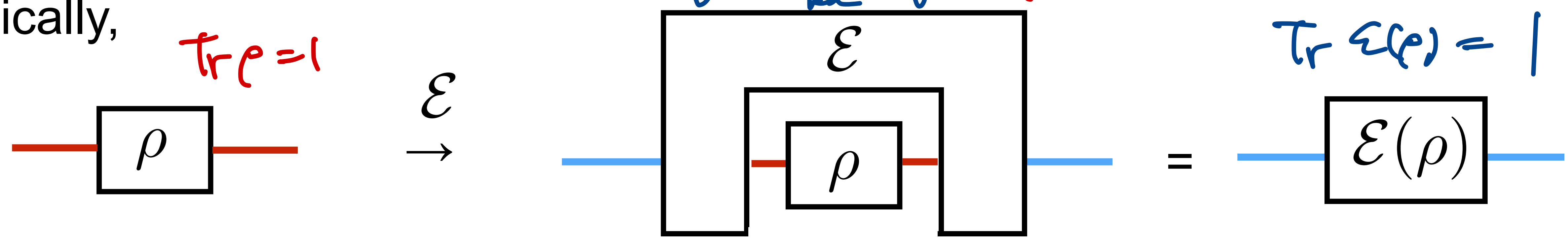
Completely Positive (CP) and Trace Preserving (TP) (CPTP) map
state ρ in system A (with $\mathcal{H}_A$) to a state in system B (with $\mathcal{H}_B$)

$\mathcal{E}(\rho)_{ij} = \sum_{kl} \mathcal{E}_{ij,kl}\, \rho_{kl}$

Graphically,

$\mathrm{Tr}\,\rho = 1$

$\mathrm{Tr}\,\mathcal{E}(\rho) = 1$



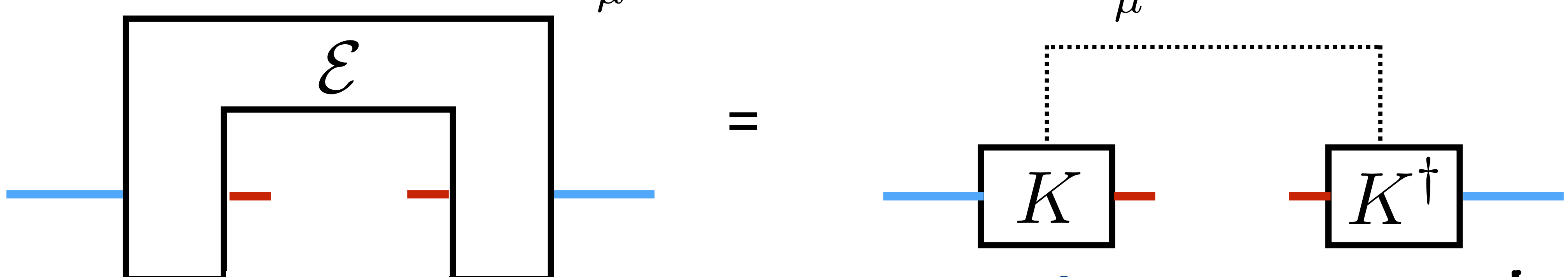
Trace preserving is required since finally we obtain a density matrix.

Complete positivity is needed to guarantee that we get positive operators even when the state is entangled with an environment.

They have so-called Kraus representation:

unitary $U\rho U^\dagger$

$$\mathcal{E}(\rho) = \sum_\mu K_\mu \rho K_\mu^\dagger \qquad \sum_\mu K_\mu^\dagger K_\mu = I$$



$\mathcal{E}_{ij,kl} = \sum_\mu K^\mu_{ik} K^{\mu *}_{jl}$

Steinspring representation

We can realize any CPTP map as a unitary (isometry) conjugation for the system+ environment E

$$\mathcal{E}(\rho) = \mathrm{Tr}_E(V\rho V^\dagger)$$

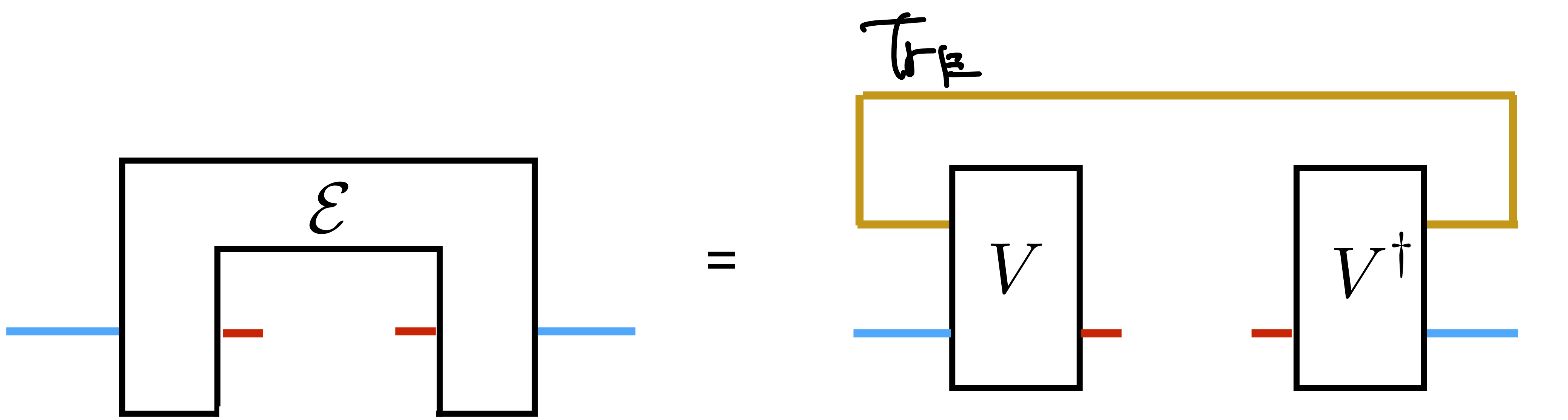
$V^\dagger V = 1$

$VV^\dagger = P$

We can realize all the quantum channels as unitary maps.

$V|\psi\rangle$

Graphically,



The choice of environments is *not unique (like gauge symmetry)*.

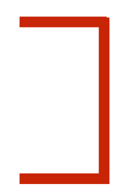
Schwinger-Keldysh formalism

Transposing bra vector, we obtain a entanlged state from density matrix:

$$\rho = \sum_{i,j} |i\rangle \, \rho_{ij} \, \langle j| \qquad \rightarrow \qquad |\rho\rangle = \sum_{i,j} \rho_{ij} \, |i\rangle \otimes |j\rangle$$

To do more systematically, we first prepare a reference maximally entangled state $|I>$ in a doubled Hilbert space $\mathcal{H}_{\mathrm{double}} = \mathcal{H}_{\mathrm{bra}} \otimes \mathcal{H}_{\mathrm{ket}}$

$$\frac{1}{\sqrt{\dim \mathcal{H}}} \sum_i |i\rangle_{\mathrm{bra}} |i\rangle_{\mathrm{ket}} = |I\rangle \in \mathcal{H}_{\mathrm{bra}} \otimes \mathcal{H}_{\mathrm{ket}}$$

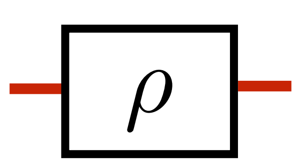


$$\rho^t = |i\rangle_{\mathrm{bra}} \rho_{ij} \langle j|_{\mathrm{bra}} \otimes Id_{\mathrm{ket}}$$

And then apply the density

$$\rho^t \otimes I$$

$$\rho \quad \rightarrow \quad \rho^+ |I\rangle \quad = \quad |\rho\rangle \in \mathcal{H}_{\mathrm{bra}} \otimes \mathcal{H}_{\mathrm{ket}}$$



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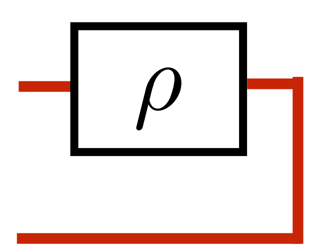

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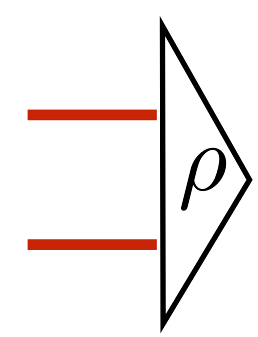
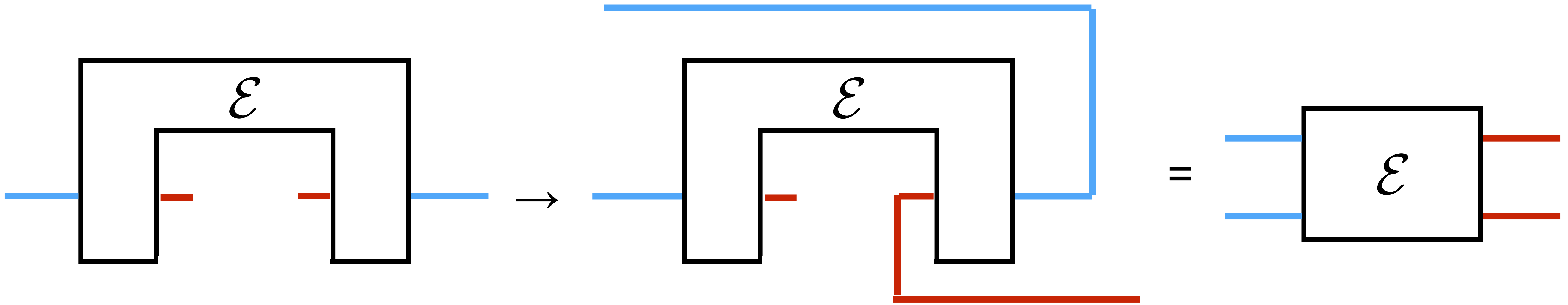


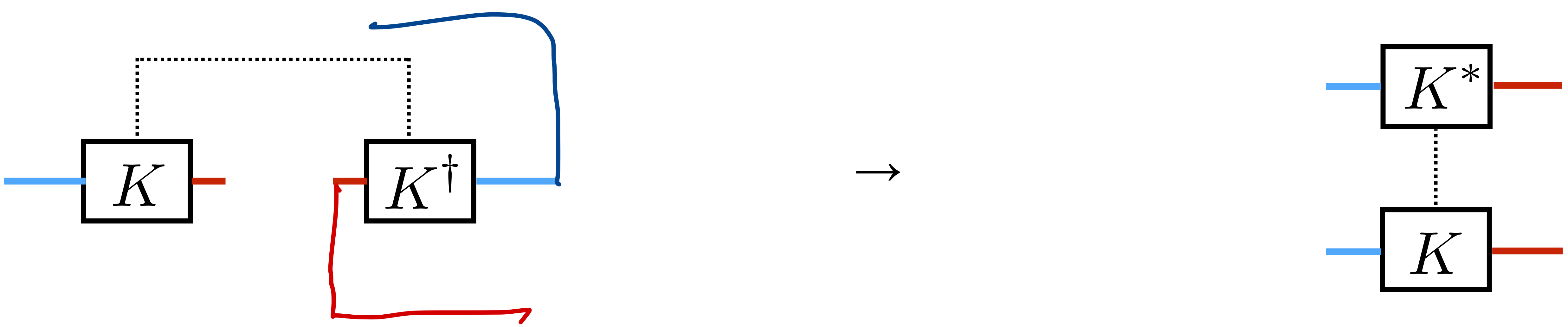
Schwinger-Keldysh formalism

For quantum channels,

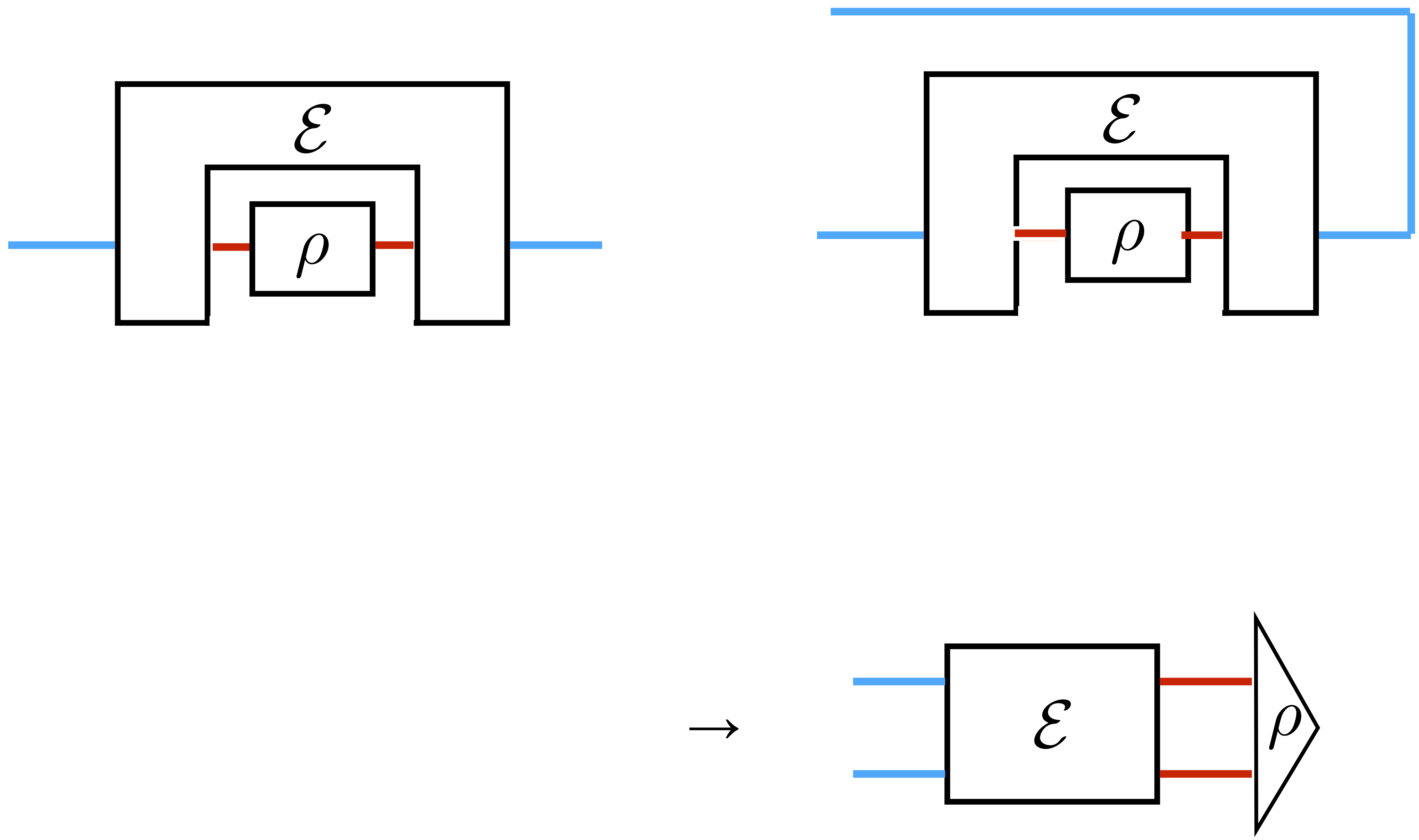


$$\mathcal{E} : \mathcal{H}_A \otimes \mathcal{H}_A^* \to \mathcal{H}_B \otimes \mathcal{H}_B^*$$

In the Kraus representation,



Schwinger-Keldysh formalism



Strong and weak symmetry

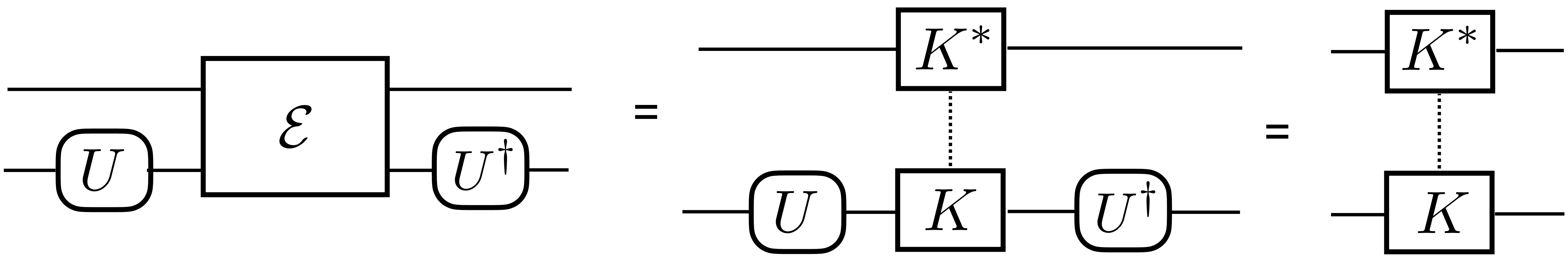
assume $\mathcal{H}_A = \mathcal{H}_B$

$U\rho = e^{i\theta}\rho$

$UAU^\dagger$

- strong symmetry

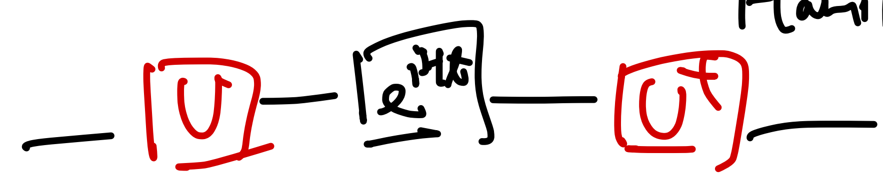
Symmetry of a Hamiltonian is $UHU^\dagger = H$. From this perspective,



is a natural symmetry for quantum channels

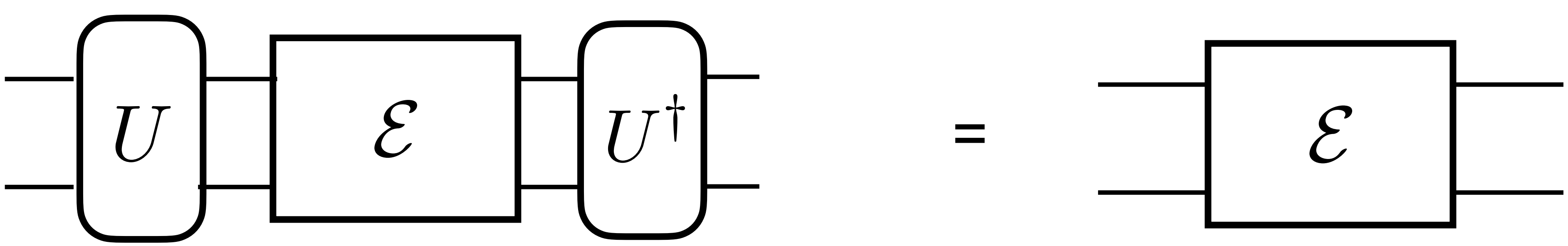
This is called strong symmetry.

Hamiltonian



- weak symmetry

On the other hand, when we think of quantum channels as a single operator



is a symmetry. This is called weak symmetry.

Modular conjugation symmetry

- After ~~vectorization~~ Schwinger-Keldysh, hermitian conjugate is now represented by the modular conjugation operator $\mathcal{J}$

- This is *anti-unitary* operator:

$$\mathcal{J}(\underset{\text{bra}}{A} \otimes \underset{\text{ket}}{B})\mathcal{J}^{-1} = B^* \otimes A^*$$

- hermiticity = invariance under modular conjugation

$$\mathcal{J}\,|\rho\rangle = \sum_{i,j} \mathcal{J}\rho_{ij}\,|i\rangle\,|j\rangle = \sum_{i,J} \rho^*_{ij}\,|j\rangle\,|i\rangle = |\rho^\dagger\rangle$$

- Any quantum channels are symmetric under the modular conjugation:

$$\mathcal{J}\sum_{\mu}(K_\mu \otimes K^*_\mu)\mathcal{J}^{-1} = \sum_{\mu}(K_\mu \otimes K^*_\mu)$$

Quantum Master equation (Lindblad equation, GKSL):

Assuming the following expansion for Kraus representation :

$$K_\mu = L_\mu \sqrt{\Delta t} \qquad K_0 = I + (K - iH)\Delta t$$

$K_0^\dagger K_0 + \sum_\mu K_\mu^\dagger K_\mu = 1$

$K = -\frac{1}{2}\sum_k L_k^\dagger L_k$

Then, we get the Lindblad (GKSL) equation

$$\boxed{\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2}\rho(t) L_k^\dagger L_k \right)}$$

These are called Lindblad equation (GKSL or quantum master equation)

[Lindblad, 76] [Gorini-Kossakowski-Sudarshan, 76]

L_k : Jump operators, gives a non-Hamiltonian evolution

Quantum Master equation (Lindblad equation, GKSL):

Applying the Schwinger Keldysh formalism,

$$\frac{d}{dt}\rho(t) = -i[H,\rho(t)] + \sum_k \left(L_k \rho(t) L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho(t) - \frac{1}{2}\rho(t) L_k^\dagger L_k \right)$$

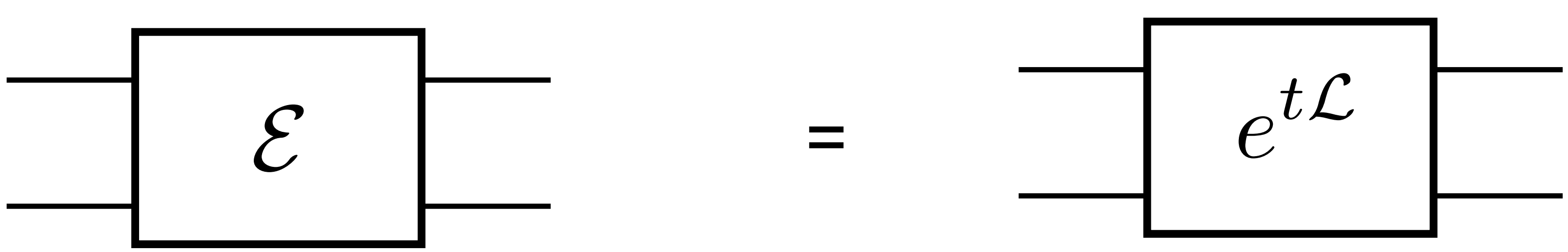
$$\underset{\times |I\rangle}{\rightarrow} \qquad \underline{\frac{d}{dt}|\rho(t)\rangle = \mathcal{L}|\rho(t)\rangle}$$

$$\boxed{\mathcal{L} = -iH_+ + iH_- + \sum_k L_{k+} \otimes L_{k-}^* - \frac{1}{2}\sum_k L_{k+}^\dagger L_{k+} \mathbb{I}_- - \frac{1}{2}\mathbb{I}_+ \otimes \sum_k L_{k-}^T L_{k-}^*}$$

$\mathcal{L}$:Non-Hermitian Hamiltonian on doubled Hilbert space $\mathcal{H}_+ \otimes \mathcal{H}_-$

$\mathcal{H}_{bra} \otimes \mathcal{H}_{ket}$

Lindbladians are quantum channels described by the exponential of a generator $\mathcal{L}$



Example; free fermion with dissipation

Consider the fermionic oscillator

$$\mathcal{H} = \mathrm{span}\{|0\rangle, c^\dagger |0\rangle\}$$

$\dim \mathcal{H} = 2$

$$H = \omega c^\dagger c \qquad L = \sqrt{\Gamma} c$$

Lindblad equation

$$\frac{d}{dt}\rho(t) = -i[\omega c^\dagger c, \rho(t)] + \Gamma c \rho(t) c^\dagger - \frac{\Gamma}{2} c^\dagger c \rho(t) - \frac{\Gamma}{2} \rho(t) c^\dagger c$$

Heisenberg operator $\mathrm{Tr}(\rho A(t)) \equiv \mathrm{Tr}(\rho(t) A)$ is

$A(t)B(t) \neq AB(t)$

$$c(t) = e^{-i\omega t - \frac{\Gamma}{2} t} c \qquad c^\dagger(t) = e^{i\omega t - \frac{\Gamma}{2} t} c^\dagger$$

The number operator decays $n(t) = e^{-\Gamma t} n$

$\rightarrow$ Losing the fermion excitation, approaching to the vacuum $|0\rangle$

Γ : Decay rate

Example; free fermion with dissipation

Enlarge the Hilbert space $\mathcal{H}_{\mathrm{double}} = \mathcal{H}_{\mathrm{bra}} \otimes \mathcal{H}_{\mathrm{ket}}$

$\dim \mathcal{H}_{\mathrm{double}} = 4$

The maximally entangled state is

$$|I\rangle = e^{-ic_+^\dagger c_-^\dagger} |00\rangle \;\in \mathcal{H}_{\mathrm{double}}$$

The Liouvillian operator is

$$\mathcal{L} = \underbrace{-i\omega c_+^\dagger c_+ + i\omega c_-^\dagger c_-}_{\substack{\text{Hamiltonian term} \\ \text{anti-Hermitian}}} \underbrace{- i\Gamma c_+ c_- - \frac{\Gamma}{2} c_+^\dagger c_+ - \frac{\Gamma}{2} c_-^\dagger c_-}_{\substack{\text{Dissipation term} \\ \text{non-(anti)Hermitian}}}$$

SYK Lindbladian:

[Kulkarni- **TN** - Ryu, 21]

$$H = H_{SYK}$$

$$L_m = \sum_{i_1 < \cdots < i_p} K_{m,i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p} \qquad K_{m,i_1 \cdots i_p} \in \mathbb{C}$$

: p-body Jump operators.

We have to double the Hilbert space to vertorize:

$\psi_i^+ , \ \psi_i^-$: in total $N + N = 2N$ Majorana fermions.

↳ Jordan - Wigner transformation

The reference maximally entangled state (= infinite temp state):

$$\psi_+^i \left| I \right\rangle = -i \psi_-^i \left| I \right\rangle \quad \xrightarrow{\psi_+^i} \quad |I\rangle = \underbrace{-2i\psi_+^i \psi_-^i}\, |I\rangle$$

$$(-2i\psi_+^i\psi_-^i)^2 = 1$$

For example, for $L_i = \sqrt{\mu}\psi^i$

$$\mathcal{L} = -iH_{SYK}^{+} + i(-1)^{\frac{q}{2}} H_{SYK}^{-} - i\mu \sum_i \psi_+^i \psi_-^i - \mu \frac{N}{2} \mathbb{I}_+ \otimes \mathbb{I}_-$$

Symmetries of SYK Lindbladians

$+$ -- bra

$-$ -- ket

Weak fermion parity

$$(-1)^{\mathcal{F}}\psi_i^{\pm}(-1)^{\mathcal{F}} = -\psi_i^{\pm}$$

Strong fermion parity

$$(-1)^{\mathcal{F}_+}\psi_i^{+}(-1)^{\mathcal{F}_+} = -\psi_i^{+}$$

$$(-1)^{\mathcal{F}_+}\psi_i^{-}(-1)^{\mathcal{F}_+} = \psi_i^{-}$$

Modular conjugation

$$\mathcal{J}\psi_i^{+}\mathcal{J}^{-1} = \psi_i^{-}$$

$$\mathcal{J}\psi_i^{-}\mathcal{J}^{-1} = \psi_i^{+}$$

anti-unitary

Time reversal

$$\mathcal{R}\psi_i^{\pm}\mathcal{R}^{-1} = \psi_i^{\pm}$$

anti-unitary

Weak fermion parity

bra --- N Majoranas
ket --- N Majoranas
total --- 2N Majoranas

Action of weak fermion parity is

$$(-1)^{\mathcal{F}}\psi_i^{\pm}(-1)^{\mathcal{F}} = -\psi_i^{\pm}$$

N: even

$$(-1)^{\mathcal{F}} = \prod_{i=1}^{\frac{N}{2}}(2i\psi_{2i-1}^{+}\psi_{2i}^{+})(2i\psi_{2i-1}^{-}\psi_{2i}^{-})$$

N: odd

$$(-1)^{\mathcal{F}} = 2i\psi_N^{+}\psi_N^{-}\prod_{i=1}^{\frac{N-1}{2}}(2i\psi_{2i-1}^{+}\psi_{2i}^{+})(2i\psi_{2i-1}^{-}\psi_{2i}^{-})$$

Strong fermion parity

Action of strong fermion parity is

$$(-1)^{\mathcal{F}_+}\psi_i^+(-1)^{\mathcal{F}_+} = -\psi_i^+$$

$$(-1)^{\mathcal{F}_+}\psi_i^-(-1)^{\mathcal{F}_+} = \psi_i^-$$

N: even

$$(-1)^{F_+} = \prod_{i=1}^{\frac{N}{2}}(2i\psi_{2i-1}^+\psi_{2i}^+)$$

N: odd

$$(-1)^{F_+} = \sqrt{2}\psi_N^- \prod_{i=1}^{\frac{N-1}{2}}(2i\psi_{2i-1}^-\psi_{2i}^-)$$

(Constructed using the opposite Hilbert space)

Modular conjugation

Action of modular conjugation

$$\mathcal{J}\psi_i^+\mathcal{J}^{-1} = \psi_i^-$$

$$\mathcal{J}\psi_i^-\mathcal{J}^{-1} = \psi_i^+$$

N: even

$$\mathcal{J} = \prod_{i=1}^{\frac{N}{2}}(\psi_{2i-1}^+ - \psi_{2i-1}^-)(i\psi_{2i}^+ + i\psi_{2i}^-)K$$

complex conjugation

N: odd

$$\mathcal{J} = (\psi_N^+ + \psi_N^-)\prod_{i=1}^{\frac{N-1}{2}}(\psi_{2i+1}^+ + \psi_{2i+1}^-)(i\psi_{2i}^+ - i\psi_{2i}^-)K$$

Time reversal symmetry

Action of anti unitary symmetry is

$$\mathcal{R}\psi_i^{\pm}\mathcal{R}^{-1} = \psi_i^{\pm}$$

$$\mathcal{R}(aA + bB)\mathcal{R}^{-1} = a^*\mathcal{R}A\mathcal{R}^{-1} + b^*\mathcal{R}B\mathcal{R}^{-1}$$

N: even

$$\mathcal{R} = \prod_{i=1}^{\frac{N}{2}} 2(i\psi_{2i}^{+})(i\psi_{2i}^{-})K$$

N: odd

$$\mathcal{R} = \prod_{i=1}^{\frac{N-1}{2}} 2(i\psi_{2i}^{+})(i\psi_{2i}^{-})K$$

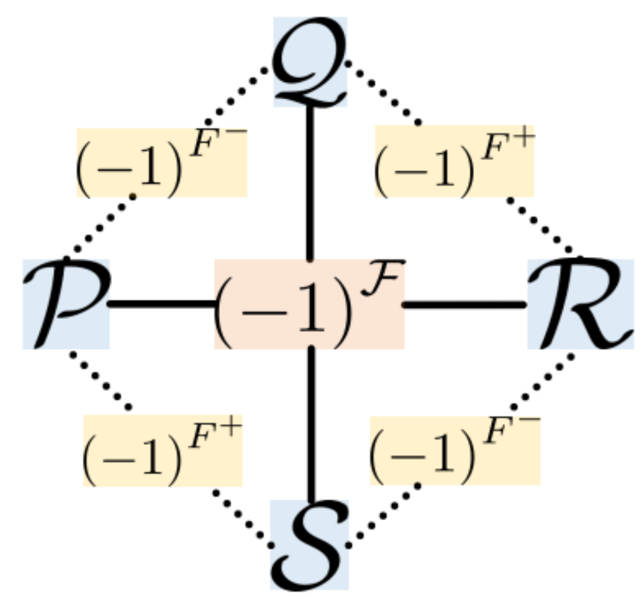
Other time reversal symmetreis

Similarly to the SYK time reversal, we can combine the fermion parity symmetries to form another anti-unitary symmetry:

$$\mathcal{P} = \mathcal{R}(-1)^{\mathcal{F}}$$

$$\mathcal{Q} = \mathcal{R}(-1)^{F_+}$$

$$\mathcal{S} = \mathcal{R}(-1)^{F_-}$$



't Hooft anomalies for open SYK

[Kawataba, Kulkarni, **TN,** Li, Ryu 22]

Possible phases are

$$\mathcal{R}^2 = \pm 1$$

$\mathbb{Z}_2^{\mathcal{R}} \times \mathbb{Z}_2^{\mathcal{F}} \times \mathbb{Z}_2^{\mathcal{J}}$

$$\mathcal{R}(-1)^{\mathcal{F}} = a(-1)^{\mathcal{F}}\mathcal{R} \qquad \mathcal{J}\mathcal{R} = b\mathcal{R}\mathcal{J}$$

The algebra only depends on $N \mod 4$, in contrast to $N \mod 8$ in SYK

$N \pmod 4$	0	1	2	3
a	$+1$	-1	$+1$	-1
b	$+1$	$+1$	-1	-1
$\mathcal{R}^2$	$+1$	$+1$	-1	-1

Table of Symmetry classification of Lindbladian

p-body dissipation: $$L_m = \sum_{1\le i_1<\cdots<i_p\le N} K_{m;i_1\cdots i_p}\psi_{i_1}\cdots\psi_{i_p}$$

$$p = 1$$

$N \pmod 4$	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{P} = \mathcal{R}(-1)^{\mathcal{F}}$	$+1$	0	-1	0
$\mathcal{Q} = \mathcal{R}(-1)^{F^-}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{R}$	$+1$	0	-1	0
$\mathcal{S} = \mathcal{R}(-1)^{F^+}$	$+1$	$+1$	$+1$	$+1$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \notin \mathbb{R}]$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \in \mathbb{R}]$	$\mathrm{BDI}^\dagger + \mathcal{S}_{++}$	$\mathrm{BDI}^\dagger$	$\mathrm{CI}^\dagger + \mathcal{S}_{+-}$	$\mathrm{BDI}^\dagger$
$q \equiv 2 \pmod 4$	BDI	$\mathrm{AI} = \mathrm{D}^\dagger$	CI	$\mathrm{AI} = \mathrm{D}^\dagger$

Symmetry and Dissipative many body Chaos: edge level statistics

eigenvalue of $\mathcal{L}$ on $\mathcal{H}_{\text{double}}$

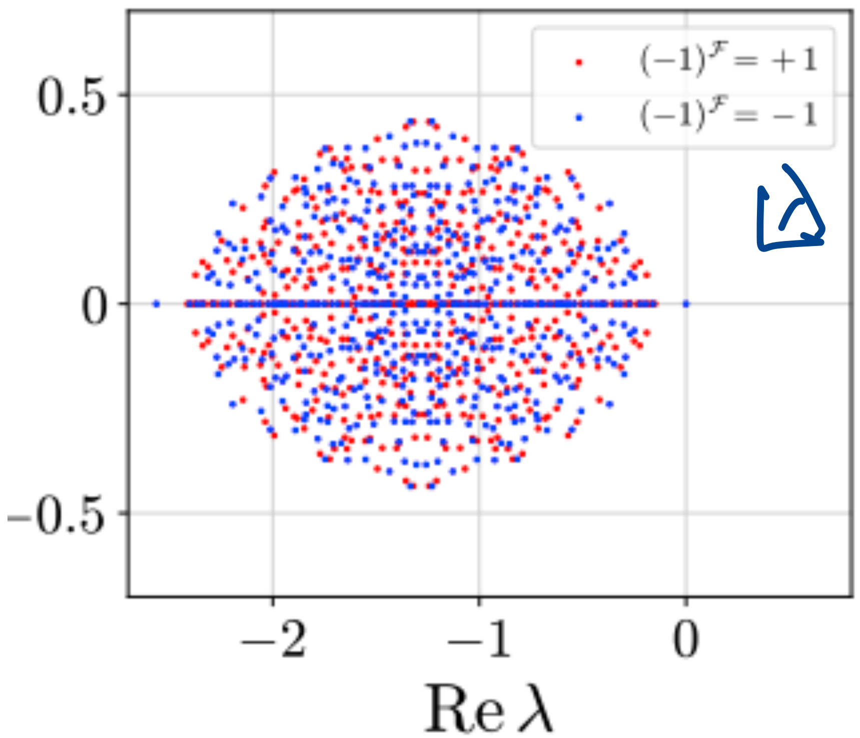
$L_n = \Sigma k_{n,i} \psi^i$

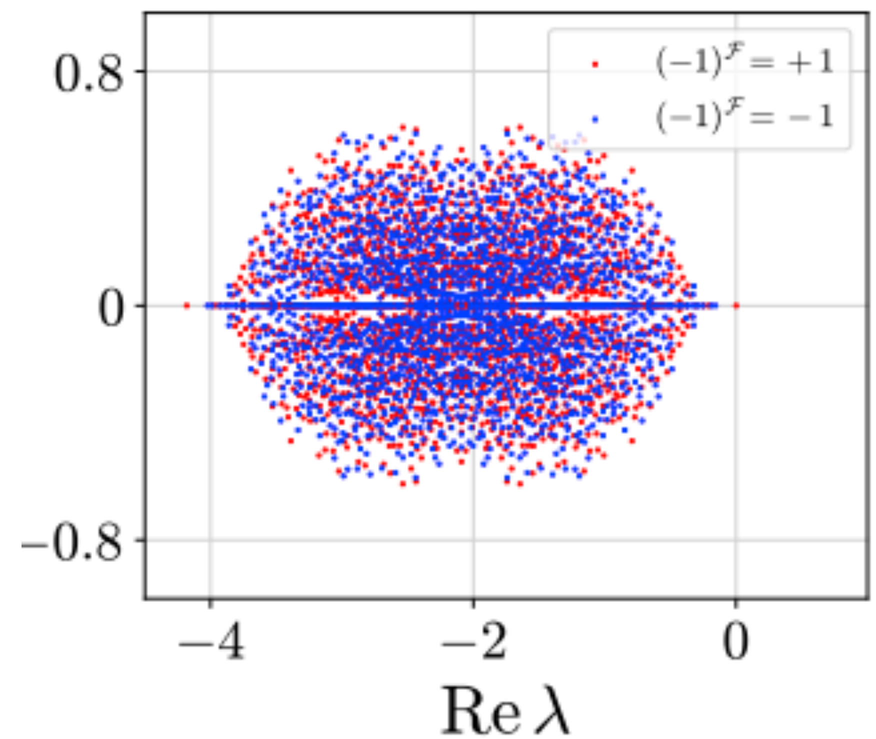
N=10, p=1

N=12, p=1

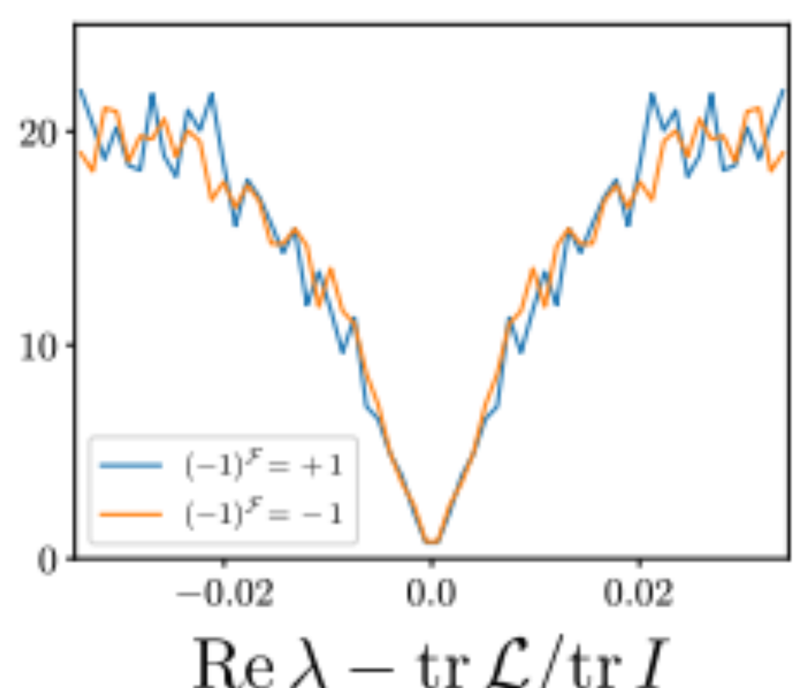
DoS

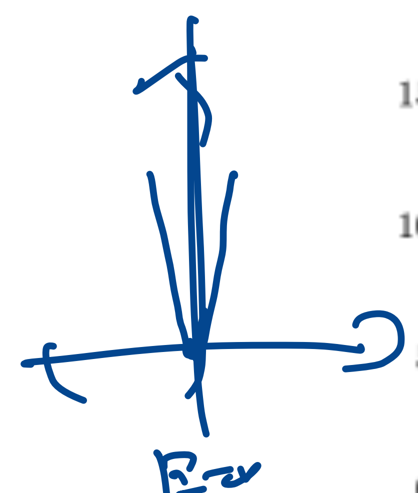
eigenvalue: complex

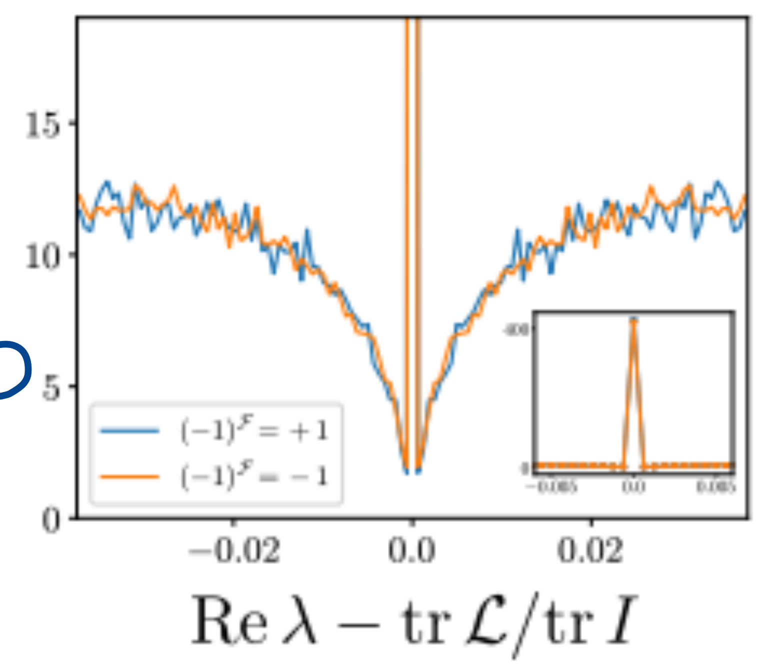




real axis







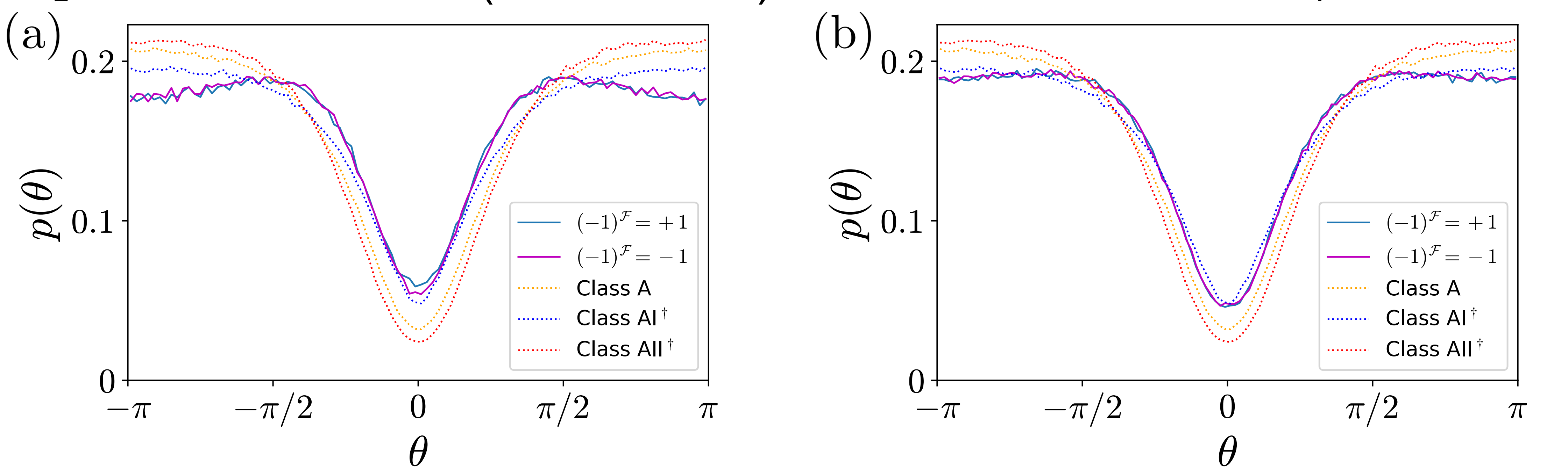
Symmetry and Dissipative many body Chaos: bulk level statistics

complex spectral gap ratio: $$z := \frac{\lambda - \lambda^{\mathrm{NN}}}{\lambda - \lambda^{\mathrm{NNN}}}$$ [Sa,Ribeiro,Prosen 19]

distribution $p(r,\theta)$ of $z = re^{i\theta}$

angle distribution $p(\theta) := \int p(r,\theta)dr$

$p=1$ $N=10$ (bulk class A) $p=1$ $N=12$ (bulk class A)



discrepancy good agreement

perhaps because Lindbladian is not completely random and see the transition

[cf: Garcia-Garcia Loureiro Romero-Bermudez Tezuka, 17]

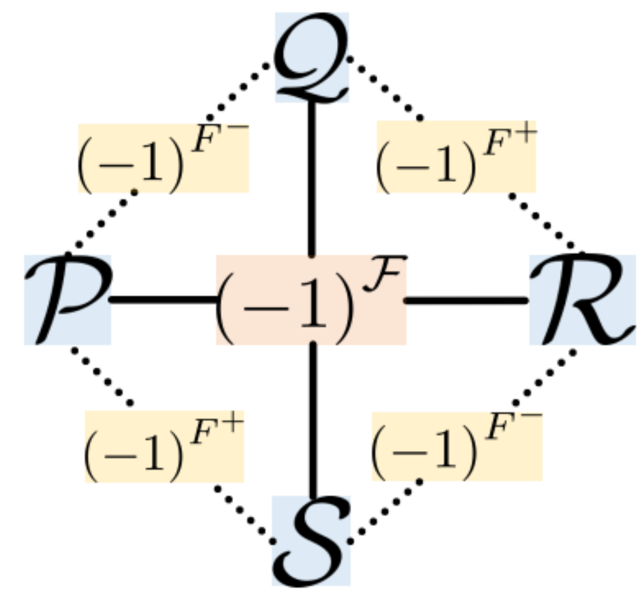
Summary

doubled

Symmetry of SYK Lindbladian

$$\mathcal{R}(-1)^{\mathcal{F}} = a(-1)^{\mathcal{F}}\mathcal{R} \qquad \mathcal{J}\mathcal{R} = b\mathcal{R}\mathcal{J}$$

N (mod 4)	0	1	2	3
a	$+1$	-1	$+1$	-1
b	$+1$	$+1$	-1	-1
$\mathcal{R}^2$	$+1$	$+1$	-1	-1



Table

$p = 1$

N (mod 4)	0	1	2	3
fermion parity $(-1)^{\mathcal{F}}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
modular conjugation $\mathcal{J}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{P}$	$+1$	0	-1	0
$\mathcal{Q}$	$+1$	$+1$	$+1$	$+1$
$\mathcal{R}$	$+1$	0	-1	0
$\mathcal{S}$	$+1$	$+1$	$+1$	$+1$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \notin \mathbb{R}]$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$	$\mathrm{AI} = \mathrm{D}^\dagger$
$q \equiv 0 \pmod 4$ $[K_{m;i}K^*_{m;j} \in \mathbb{R}]$	$\mathrm{BDI}^\dagger + \mathcal{S}_{++}$	$\mathrm{BDI}^\dagger$	$\mathrm{CI}^\dagger + \mathcal{S}_{+-}$	$\mathrm{BDI}^\dagger$
$q \equiv 2 \pmod 4$	BDI	$\mathrm{AI} = \mathrm{D}^\dagger$	CI	$\mathrm{AI} = \mathrm{D}^\dagger$