Application of Quantum Computation to High Energy Physics

– QFT on quantum computer –

Masazumi Honda

(本多正純)

Intensive lectures @RESCEU（day 2）

Plan of the intensive lectures

Day 1

- Lecture 1: introduction, basics of quantum computation
- Lecture 2: quantum simulation of spin system
- Hands-on 1: Basics on IBM's qiskit

Day 2

- Lecture 3: quantum field theory (QFT)
- Lecture 4: QFT on quantum computer
- Hands-on 2: Time evolution of spin system

Day 3

- Lecture 5: quantum error correction
- Lecture 6: some advanced topics, future prospects
- Hands-on 3: Constructing ground state of spin system

What is meant by

"Application of Quantum Computation to Quantum Field Theory" ?

In general, it is

to replace (a part of) computations by quantum algorithm

Therefore,

physical meaning of qubits in quantum computer depends on contexts

Here,

qubits = states in physical system

Plan

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

“Regularization” of Hilbert space

Hilbert space of QFT is typically ∞ dimensional

$\longrightarrow$ Make it finite dimensional!

- **Fermion** is easiest (up to doubling problem)

  — Putting on spatial lattice, Hilbert sp. is finite dimensional

- scalar

  — Hilbert sp. at each site is ∞ dimensional
  (need truncation or additional regularization)

- gauge field (w/ kinetic term)

  — no physical d.o.f. in 0+1D/1+1D (w/ open bdy. condition)

  — ∞ dimensional Hilbert sp. in higher dimensions

Citation history of "Hamiltonian Formulation of Wilson's Lattice Gauge Theories" by Kogut-Susskind

(totally 2148 at this moment)

(1+1)d free Dirac fermion (continuum)

Lagrangian:

$$\mathcal{L} = \int dx\left[i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi\right] \qquad \{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}$$

$$\Big\Downarrow \qquad \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \bar{\psi}$$

Hamiltonian:

$$H = \int dx\left[-i\bar{\psi}\gamma^{1}\partial_{1}\psi + m\bar{\psi}\psi\right]$$

$$\{\psi(x),\bar{\psi}(y)\} = \delta(x-y)$$

(1+1)d free Dirac fermion (lattice)

Continuum:

$$\psi(x) = \begin{pmatrix} \psi_u(x) \\ \psi_d(x) \end{pmatrix} \qquad \gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2$$

$$H = \int dx \left[-i\bar{\psi}\gamma^1 \partial_1 \psi + m\bar{\psi}\psi \right]$$

$$= \int dx \left[-i(\psi_u^\dagger \partial_1 \psi_d + \psi_d^\dagger \partial_1 \psi_u) + m(\psi_u^\dagger \psi_u - \psi_d^\dagger \psi_d) \right]$$

$$\Downarrow$$

Lattice (w/ N sites and spacing a):

"Staggered fermion" [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{bmatrix} \psi_u \\ \psi_d \end{bmatrix} \begin{matrix} \rightarrow \text{odd site} \\ \rightarrow \text{even site} \end{matrix}$$

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$$= \int dx\left[-i(\psi_u^\dagger\partial_1\psi_d + \psi_d^\dagger\partial_1\psi_u) + m(\psi_u^\dagger\psi_u - \psi_d^\dagger\psi_d)\right]$$

$$\Downarrow$$

Lattice (w/ N sites and spacing a)**:** *"Staggered fermion"* [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix} \begin{matrix}\longrightarrow \text{odd site}\\ \longrightarrow \text{even site}\end{matrix}$$

$$H = -\frac{i}{2a}\sum_{n=1}^{N-1}\left(\chi_n^\dagger\chi_{n+1} - \chi_{n+1}^\dagger\chi_n\right) + m\sum_{n=1}^{N}(-1)^n\chi_n^\dagger\chi_n$$

$$\{\chi_m, \chi_n^\dagger\} = \delta_{\mathrm{mn}},\ \{\chi_m, \chi_n\} = 0$$

Jordan-Wigner transformation

$$\{\chi_m, \chi_n^\dagger\} = \delta_{\mathrm{mn}}, \ \{\chi_m, \chi_n\} = 0$$

This is satisfied by the operator: [Jordan-Wigner'28]

$$\chi_n = \frac{X_n - \mathrm{i}Y_n}{2} \left(\prod_{i=1}^{n-1} -\mathrm{i}Z_i \right)$$

(X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

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(X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

Then the system is mapped to the spin system:

$$\boxed{\hat{H} = \frac{w}{2}\sum_{n=1}^{N-1}\left(X_n X_{n+1} + Y_n Y_{n+1}\right) + \frac{m}{2}\sum_{n=1}^{N}(-1)^n Z_n}$$

Now we can apply quantum algorithms to QFT!

Scalar field theory (continuum)

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

$$\Downarrow \qquad \Pi(\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi$$

Hamiltonian:

$$\mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

$$[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = i\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

Scalar field theory (lattice)

Continuum Hamiltonian:

$$H = \int d^d\boldsymbol{x} \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)\right]$$

$$\int d^d x \to \ a^d \sum_n \ ,$$

$$\partial_\mu\phi(x) \to \Delta_\mu\phi(x_n) \equiv \frac{\phi(x_n + ae_\mu) - \phi(x_n)}{a}$$

Lattice Hamiltonian (simplest):

$$H = a^d \sum_n \left[\frac{1}{2}\Pi_{\mathrm{n}}^2 + \frac{1}{2}\sum_i (\Delta_i\phi_n)^2 + V(\phi_n)\right]$$

$$[\phi(\boldsymbol{x_m}), \Pi(\boldsymbol{x_n})] = i\delta_{\boldsymbol{m},\boldsymbol{n}}$$

technically the same as multi-particle QM

Regularization for single particle QM

$$\widehat{H} = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{x}^2 + V_{\text{int}}(\hat{x})$$

Most naïve approach = truncation in harmonic osc. basis:

$$\hat{a} = \sqrt{\frac{\omega}{2}}\,\hat{x} + \frac{i}{\sqrt{2\omega}}\,\hat{p} \;\; = \;\; \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle\langle n+1|$$

$$\xrightarrow{\textit{regularize!}} \;\; \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

Regularization for single particle QM

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Most naïve approach = truncation in harmonic osc. basis:

$$\hat{a} = \sqrt{\frac{\omega}{2}}\,\hat{x} + \frac{i}{\sqrt{2\omega}}\,\hat{p} \;=\; \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle\langle n+1|$$

$$\underset{\textit{regularize!}}{\Longrightarrow} \quad \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

Then replace $\hat{p}$ & $\hat{x}$ by

$$\hat{x}\Big|_{\text{regularized}} \equiv \frac{1}{\sqrt{2\omega}}(\hat{a}+\hat{a}^\dagger)\Big|_{\text{regularized}}$$

$$\hat{p}\Big|_{\text{regularized}} \equiv \frac{1}{i}\sqrt{\frac{\omega}{2}}\,(\hat{a}-\hat{a}^\dagger)\Big|_{\text{regularized}}$$

Regularization for single particle QM (Cont'd)

$$\hat{a}\Big|_{\text{regularized}} = \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

We can rewrite the Fock basis in terms of qubits:

$$|n\rangle = |b_{K-1}\rangle|b_{K-2}\rangle\cdots|b_0\rangle \qquad K \equiv \log_2 \Lambda$$

$$n = \mathrm{b}_{\mathrm{K}-1}2^{\mathrm{K}-1} + \mathrm{b}_{\mathrm{K}-2}2^{\mathrm{K}-2} + \cdots + b_0 2^0$$ (binary representation)

Regularization for single particle QM (Cont'd)

$$\hat{a}\Big|_{\text{regularized}} = \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

We can rewrite the Fock basis in terms of qubits:

$$|n\rangle = |b_{K-1}\rangle|b_{K-2}\rangle \cdots |b_0\rangle \qquad K \equiv \log_2 \Lambda$$

$$n = \mathrm{b}_{\mathrm{K}-1} 2^{\mathrm{K}-1} + \mathrm{b}_{\mathrm{K}-2} 2^{\mathrm{K}-2} + \cdots + b_0 2^0$$ (binary representation)

Then,

$$|n\rangle\langle n+1| = \otimes_{\ell=0}^{K-1} \underbrace{(|b'_\ell\rangle\langle b_\ell|)}_{\text{either one of}}$$

$$|0\rangle\langle 0| = \frac{1_2 - \sigma_z}{2}, \qquad |1\rangle\langle 1| = \frac{1_2 + \sigma_z}{2},$$

$$|0\rangle\langle 1| = \frac{\sigma_x + i\sigma_y}{2}, \qquad |1\rangle\langle 0| = \frac{\sigma_x - i\sigma_y}{2}$$

Pure Maxwell theory (continuum)

Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

$$\Downarrow$$

temporal gauge $A_0 = 0$

$$E^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}^i$$

Hamiltonian:

$$\mathcal{H} = \frac{1}{2} E_i^2 + \frac{1}{2} B_i^2$$

$$[A_i(\boldsymbol{x}), E_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

Gauss law:

$$\partial_i E^i = 0$$

Pure Maxwell theory (lattice)

Continuum:

$$\mathcal{H} = \frac{1}{2}E_i^2 + \frac{1}{2}B_i^2 \qquad \partial_i E^i = 0$$

$$\Downarrow$$

Lattice:

$$\mathcal{H} = \frac{a^d}{2}\sum_{\boldsymbol{n},i} L_{\boldsymbol{n},i}^2 + \mathrm{Re}\sum_{\text{plaquette}}\sum_{i<j}\prod_{P\in\text{plaquette}} U_P$$

$$[U_{\boldsymbol{m},i}, L_{\boldsymbol{n},j}] = i\delta_{ij}\delta_{\boldsymbol{m},\boldsymbol{n}}$$

Gauss law:

$$\sum_i (L_{\boldsymbol{n}+\boldsymbol{e}_i,i} - L_{\boldsymbol{n},i}) = 0$$

Ex. (1+1)d pure Maxwell theory w/ θ

Continuum:

$$\mathcal{L} = \frac{1}{2g^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01} \quad \xrightarrow{\Pi = \frac{1}{g^2}\dot{A} + \frac{\theta}{2\pi}} \quad \mathcal{H} = \frac{1}{2}\left(\Pi - \frac{\theta}{2\pi}\right)^2$$

Lattice:

$$H = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2$$

$$L_n \leftrightarrow -\frac{\Pi(x)}{g}$$

Gauss law:

$$L_{n+1} - L_n = 0$$

Ex. (1+1)d pure Maxwell theory w/ θ

Continuum:

$$\mathcal{L} = \frac{1}{2g^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01} \xrightarrow{\Pi = \frac{1}{g^2}\dot{A} + \frac{\theta}{2\pi}} \mathcal{H} = \frac{1}{2}\left(\Pi - \frac{\theta}{2\pi}\right)^2$$

Lattice:

$$H = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2 \qquad L_n \leftrightarrow -\frac{\Pi(x)}{g}$$

Gauss law:

$$L_{n+1} - L_n = 0$$

- open b.c.

$$L_n = L_{n-1} = L_{n-2} = \cdots = L_1 = (b.c.)$$

- p.b.c.

$$L_n = L_{n-1} = \cdots = L_1 = \cdots = L_{n+1} = L_n$$

one d.o.f. remains

Short summary

(repeated)

Hilbert space of QFT is typically ∞ dimensional

⟶ Make it finite dimensional!

- Fermion is easiest (up to doubling problem)

  — Putting on spatial lattice, Hilbert sp. is finite dimensional

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  (need truncation or additional regularization)

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Plan

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

Schwinger model w/ topological term

Continuum ①:

(will be used for the case w/ probes)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

Continuum ②:

(equivalent via “chiral anomaly”, used here)

[Fujikawa’79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

Schwinger model w/ topological term

Continuum ①: (will be used for the case w/ probes)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

Continuum ②: (equivalent via "chiral anomaly", used here) [Fujikawa'79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

Taking temporal gauge $A_0 = 0$, $(\,\Pi = \dot{A}^1)$

$$\hat{H} = \int dx\left[-i\bar{\psi}\gamma^1(\partial_1 + igA_1)\psi + m\bar{\psi}e^{i\theta\gamma^5}\psi + \frac{1}{2}\Pi^2\right]$$

Physical states are constrained by Gauss law:

$$0 = -\partial_1\Pi - g\bar{\psi}\gamma^0\psi$$

Sign problem in path integral formalism

In Minkowski space,

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \right] + \frac{g\theta}{4\pi} \int F \in \boldsymbol{R}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \mathcal{O}\ e^{iS}}{\int DAD\psi D\bar{\psi}\ e^{iS}}$$

highly oscillating

In Euclidean space,

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi \right] + i\frac{g\theta}{4\pi} \int F \in \boldsymbol{C}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \mathcal{O}\ e^{-S}}{\int DAD\psi D\bar{\psi}\ e^{-S}}$$

highly oscillating for non-small θ

Accessible region by analytic computation

- Massive limit:

The fermion can be integrated out

&

the theory becomes effectively pure Maxwell theory w/ θ

Accessible region by analytic computation

- Massive limit:

The fermion can be integrated out

&

the theory becomes effectively pure Maxwell theory w/ θ

- Bosonization:

[Coleman '76]

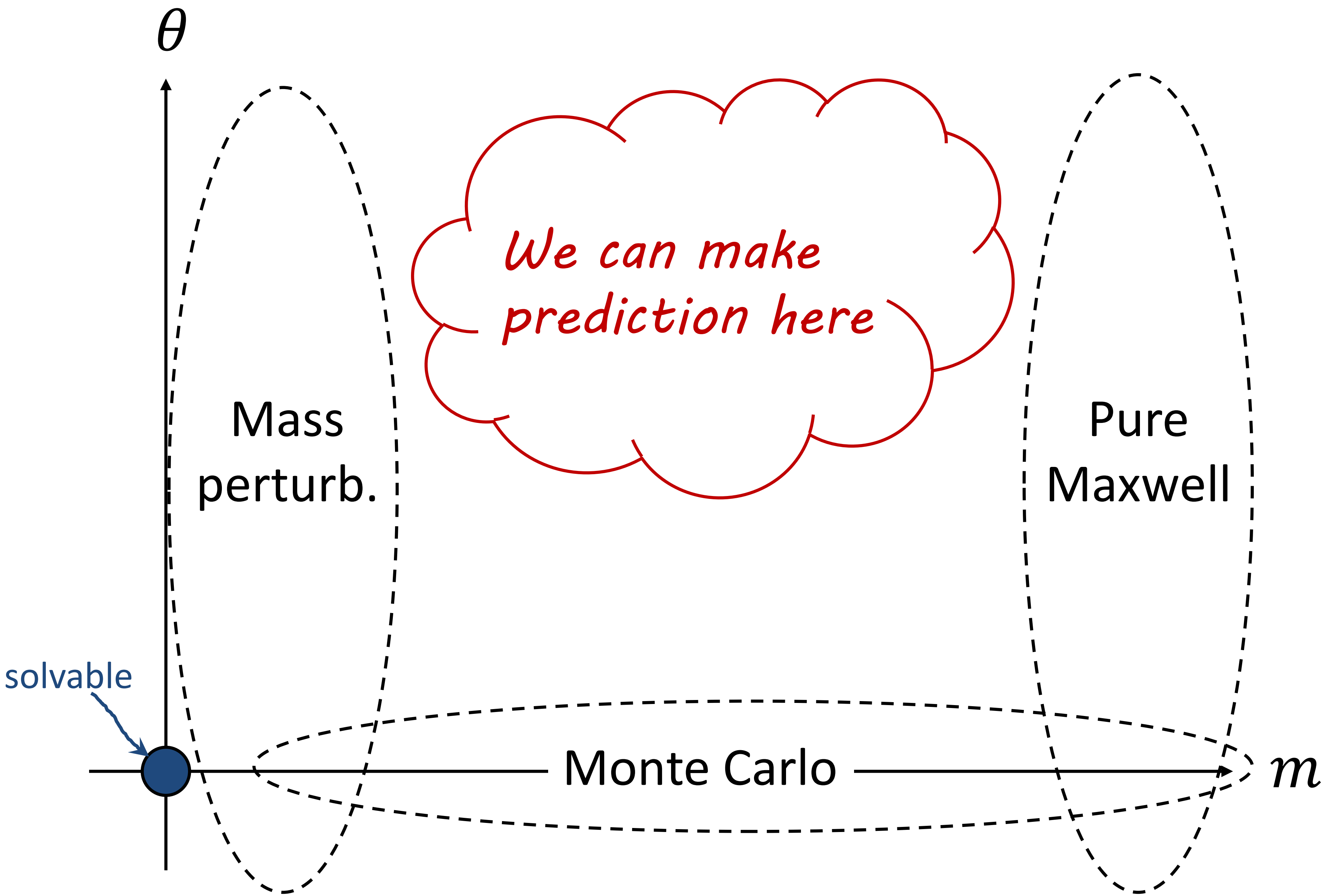
$$\mathcal{L} = \frac{1}{8\pi}\left(\partial_\mu \phi\right)^2 - \frac{g^2}{8\pi^2}\phi^2 + \frac{e^\gamma g}{2\pi^{3/2}} m \cos(\phi + \theta)$$

exactly solvable for $m = 0$

&

small m regime is approximated by perturbation

Map of accessibility/difficulty



Put the theory on lattice

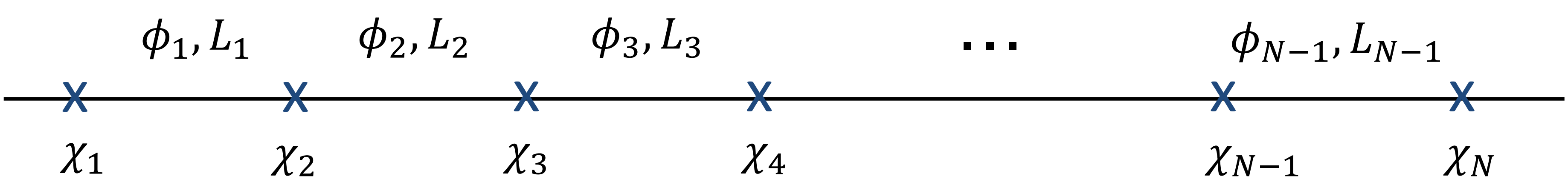
▪ **Fermion** (on site):

"Staggered fermion" [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{\underbrace{a}_{\text{lattice spacing}}{}^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \begin{array}{l} \rightarrow \text{odd site} \\ \rightarrow \text{even site} \end{array}$$

▪ **Gauge field** (on link):

$$\phi_n \leftrightarrow -agA^1(x), \qquad L_n \leftrightarrow -\frac{\Pi(x)}{g}$$



Lattice theory w/ staggered fermion

Hamiltonian:

$$\hat{H} = -i\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right)\left[\chi_n^\dagger e^{i\phi_n}\chi_n - \text{h.c.}\right]$$
$$+m\cos\theta\sum_{n=1}^{N}(-1)^n\chi_n^\dagger\chi_n + J\sum_{n=1}^{N-1}L_n^2 \qquad \left(w = \frac{1}{2a}, J = \frac{g^2 a}{2}\right)$$

Commutation relation:

$$\{\chi_n^\dagger, \chi_m\} = \delta_{mn},\ \ \{\chi_n, \chi_m\} = 0,\ \ [\phi_n, L_m] = i\delta_{mn}$$

Gauss law:

$$L_n - L_{n-1} = \chi_n^\dagger\chi_n - \frac{1-(-1)^n}{2}$$

Eliminate gauge d.o.f.

1. Take open b.c. & solve Gauss law:

$$L_n = \sum_{\ell=1}^{n-1} \left[\chi_\ell^\dagger \chi_\ell - \frac{1-(-1)^\ell}{2} \right] \qquad \text{(took } L_0 = 0\text{)}$$

2. Redefine fermion to absorb ϕ_n:

$$\chi_n \to \prod_{\ell<n} \left[e^{-i\phi_\ell} \right] \chi_n$$

Then,

$$\hat{H} = -i \sum_{n=1}^{N-1} \left(w - (-1)^n \frac{m}{2} \sin\theta \right) \left[\chi_n^\dagger \chi_{n+1} - \text{h.c.} \right] + m\cos\theta \sum_{n=1}^{N} (-1)^n \chi_n^\dagger \chi_n$$
$$+ J \sum_{n=1}^{N-1} \left[\sum_{\ell=1}^{n-1} \left(\chi_\ell^\dagger \chi_\ell - \frac{1-(-1)^\ell}{2} \right) \right]^2$$

This acts on finite dimensional Hilbert space

Going to spin system

$$\{\chi_n^\dagger, \chi_m\} = \delta_{mn}, \ \{\chi_n, \chi_m\} = 0$$

This is satisfied by the operator:

"Jordan-Wigner transformation"

[Jordan-Wigner'28]

$$\chi_n = \frac{X_n - \mathrm{i}Y_n}{2}\left(\prod_{i=1}^{n-1} -\mathrm{i}Z_i\right)$$

Now the system is purely a spin system:

$$\widehat{H} = H_{ZZ} + H_{\pm} + H_Z$$

$$\begin{cases} H_{ZZ} = \dfrac{J}{2}\displaystyle\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell, \\ H_{\pm} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n \frac{m}{2}\sin\theta\right)\left[X_n X_{n+1} + Y_n Y_{n+1}\right], \\ H_Z = \dfrac{m\cos\theta}{2}\displaystyle\sum_{n=1}^{N}(-1)^n Z_n - \frac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell=1}^{n} Z_\ell \end{cases}$$

Qubit description of the Schwinger model !!

Comments on choices of setup

There were many choices of setup to come here…

- Formulation of continuum theory?
- Type of lattice fermion?
- Boundary condition?
- Impose Gauss law?
- How to map fermion to spin system?
- Even N or odd N?

Choice of continuum theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

(used for the case w/ probes)

$$\Updownarrow$$ "chiral anomaly" [cf. Fujikawa'79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

(used for the case w/o probes)

- Equivalent for continuum theory w/o bdy.

—— (generically) inequivalent for theory on lattice or w/ bdy.

- The latter doesn't violate θ-periodicity even for open b.c.

Choice of boundary conditions

Gauss law: $L_n - L_{n-1} = q\left[\chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}\right]$

Open b.c.

- $L_n =$ (fermion op.)

$\longrightarrow \quad \dim(\mathcal{H}_{\text{phys}}) < \infty$

- θ-periodicity is lost
- momentum not conserved

Periodic b.c.

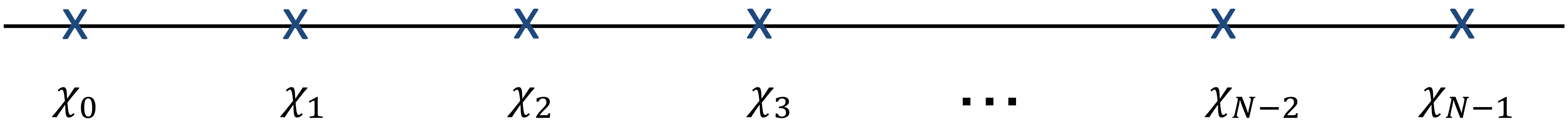
- one of L_n's remains

$\longrightarrow \quad \dim(\mathcal{H}_{\text{phys}}) = \infty$

additional truncation needed

- $^\exists \theta$-periodicity
- momentum conserved

Even N or odd N?



Staggered fermion: $\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$ $\begin{matrix} \to \text{odd site} \\ \to \text{even site} \end{matrix}$

- Usually even N is taken (p.b.c. allows only even N)
- Open b.c. allows both but parity is different: $\chi_n \to i(-1)^n \chi_{N-n-1}$

	n mod 2	$\bar{\psi}\psi \sim \sum_n (-1)^n \chi_n^\dagger \chi_n$	$\bar{\psi}\gamma^5\psi \sim \sum_n (-1)^n (\chi_n^\dagger \chi_{n+1} - \mathrm{h.c.})$
even N	changes	flipped	invariant
odd N	invariant	invariant	flipped

Odd N seems more like the continuum theory?

Plan

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

Time evolution operator

Suzuki-Trotter decomposition:

$$e^{-i\widehat{H}t} = \left(e^{-i\widehat{H}\frac{t}{M}}\right)^M$$

(M: large positive integer)

$$\simeq \left(e^{-iH_Z\frac{t}{M}}e^{-iH_{ZZ}\frac{t}{M}}e^{-iH_{XX}\frac{t}{M}}e^{-iH_{YY}\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\begin{cases} H_Z = \dfrac{m\cos\theta}{2}\displaystyle\sum_{n=1}^{N}(-1)^n Z_n - \dfrac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell=1}^{n} Z_\ell \\ H_{ZZ} = \dfrac{J}{2}\displaystyle\sum_{n=2}^{N-1}\sum_{1\leq k<\ell\leq n} Z_k Z_\ell, \\ H_{XX} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right)X_n X_{n+1} \\ H_{YY} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right)Y_n Y_{n+1} \end{cases}$$

Can we express it in terms of elementary gates?

Time evolution operator (cont'd)

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_Z \frac{t}{M}} e^{-iH_{ZZ}\frac{t}{M}} e^{-iH_{XX}\frac{t}{M}} e^{-iH_{YY}\frac{t}{M}}\right)^M$$

The 1st one is trivial:

$$e^{-icZ} = R_Z(2c)$$

The 2nd one appeared in Ising model:

$$e^{-icZ_1Z_2} = CXR_Z^{(2)}(2c)CX$$

The 3rd one (see next slide):

$$e^{-icX_1X_2} = CXR_X^{(1)}(2c)CX$$

The 4th one:

$$e^{-icY_1Y_2} = R_Z^{(1)}\left(-\frac{\pi}{2}\right) R_Z^{(2)}\left(-\frac{\pi}{2}\right) e^{-icX_1X_2} R_Z^{(2)}\left(\frac{\pi}{2}\right) R_Z^{(1)}\left(\frac{\pi}{2}\right)$$

Time evolution operator (Cont'd)

$$e^{-icX_1X_2} = CXR_X^{(1)}(2c)CX$$

Proof:

$$CXR_X^{(1)}(2c)CX|0\rangle \otimes |\psi\rangle$$

$$= CXR_X^{(1)}(2c)|0\rangle\otimes|\psi\rangle = CX\left[\cos c|0\rangle\otimes|\psi\rangle - i\sin c|1\rangle\otimes|\psi\rangle\right]$$

$$= \cos c|0\rangle\otimes|\psi\rangle - i\sin c|1\rangle\otimes X|\psi\rangle = \cos c|0\rangle\otimes|\psi\rangle - i\sin c\ X|0\rangle\otimes X|\psi\rangle$$

$$CXR_X^{(1)}(2c)CX|1\rangle \otimes |\psi\rangle$$

$$= CXR_X^{(1)}(2c)|1\rangle\otimes X|\psi\rangle = CX\left[\cos c|1\rangle\otimes X|\psi\rangle - i\sin c|0\rangle\otimes X|\psi\rangle\right]$$

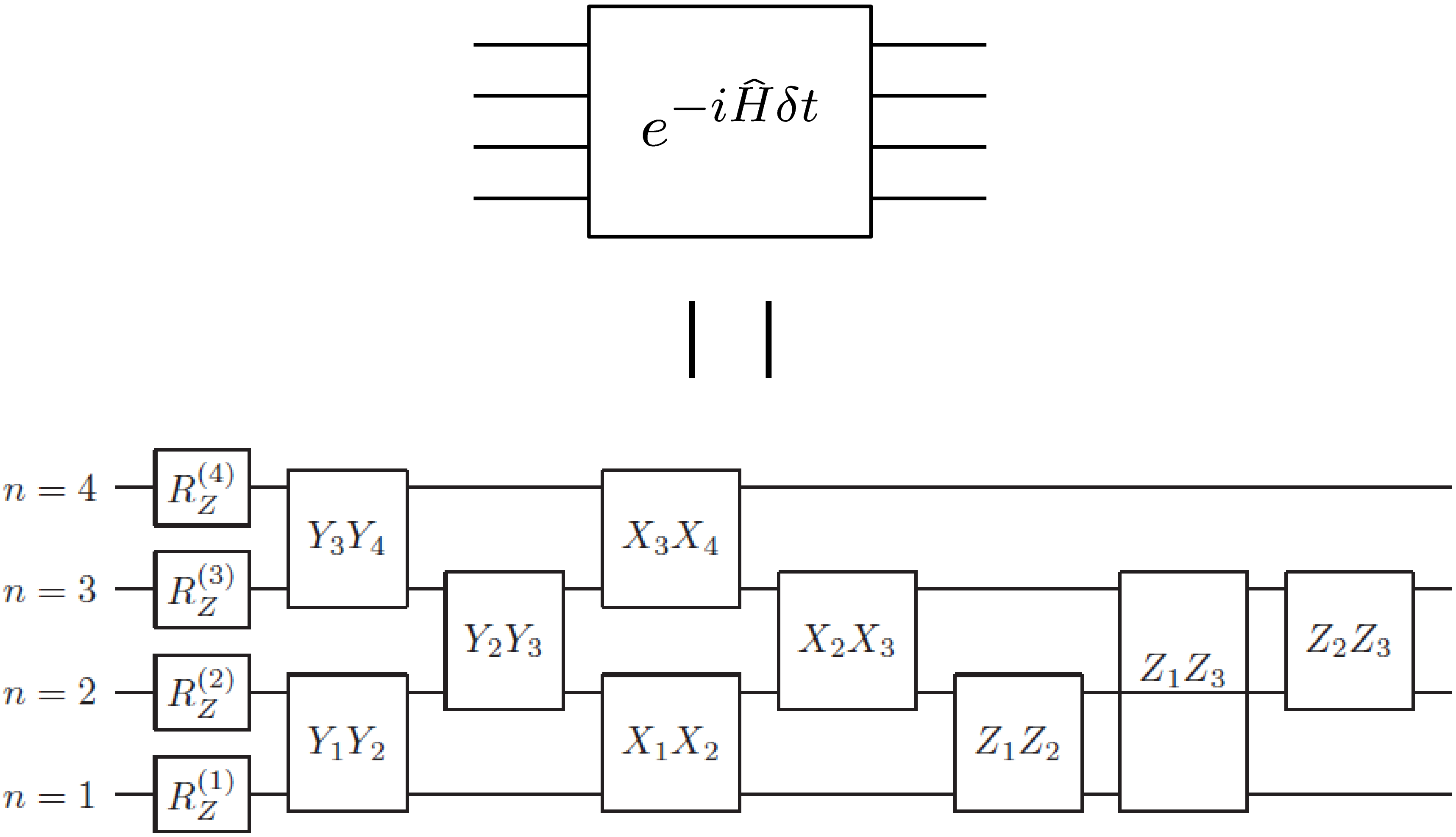
$$= \cos c|1\rangle\otimes|\psi\rangle - i\sin c|0\rangle\otimes X|\psi\rangle = \cos c|1\rangle\otimes|\psi\rangle - i\sin c\ X|1\rangle\otimes X|\psi\rangle$$

Thus,

$$CXR_X^{(1)}(2c)CX|\varphi\rangle\otimes|\psi\rangle = \cos c|\varphi\rangle\otimes|\psi\rangle - i\sin c\ X|\varphi\rangle\otimes X|\psi\rangle$$

$$= e^{-icX_1X_2}|\varphi\rangle \otimes |\psi\rangle$$

Quantum circuit for time evolution op. (N=4)



Improvement of Suzuki-Trotter decomposition

The leading order decomposition:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\delta t}e^{-iH_2\delta t} + \mathcal{O}(\delta t^2)$$

The 2nd order improvement:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\frac{\delta t}{2}}e^{-iH_2\delta t}e^{-iH_1\frac{\delta t}{2}} + \mathcal{O}(\delta t^3)$$

cf. Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}$$

This increases the number of gates at each time step
but we can take larger δt (smaller M) to achieve similar accuracy.
Totally we save the number of gates.

Survival probability of massive vacuum

[cf. Martinez etal. **Nature** 534 (2016) 516-519]

The ground state in the large mass limit is

$$(\text{mass term}) \propto m \sum_{n=1}^{N} (-1)^n Z_n$$

$$|\text{massive}\rangle = |0101\cdots 01\rangle$$

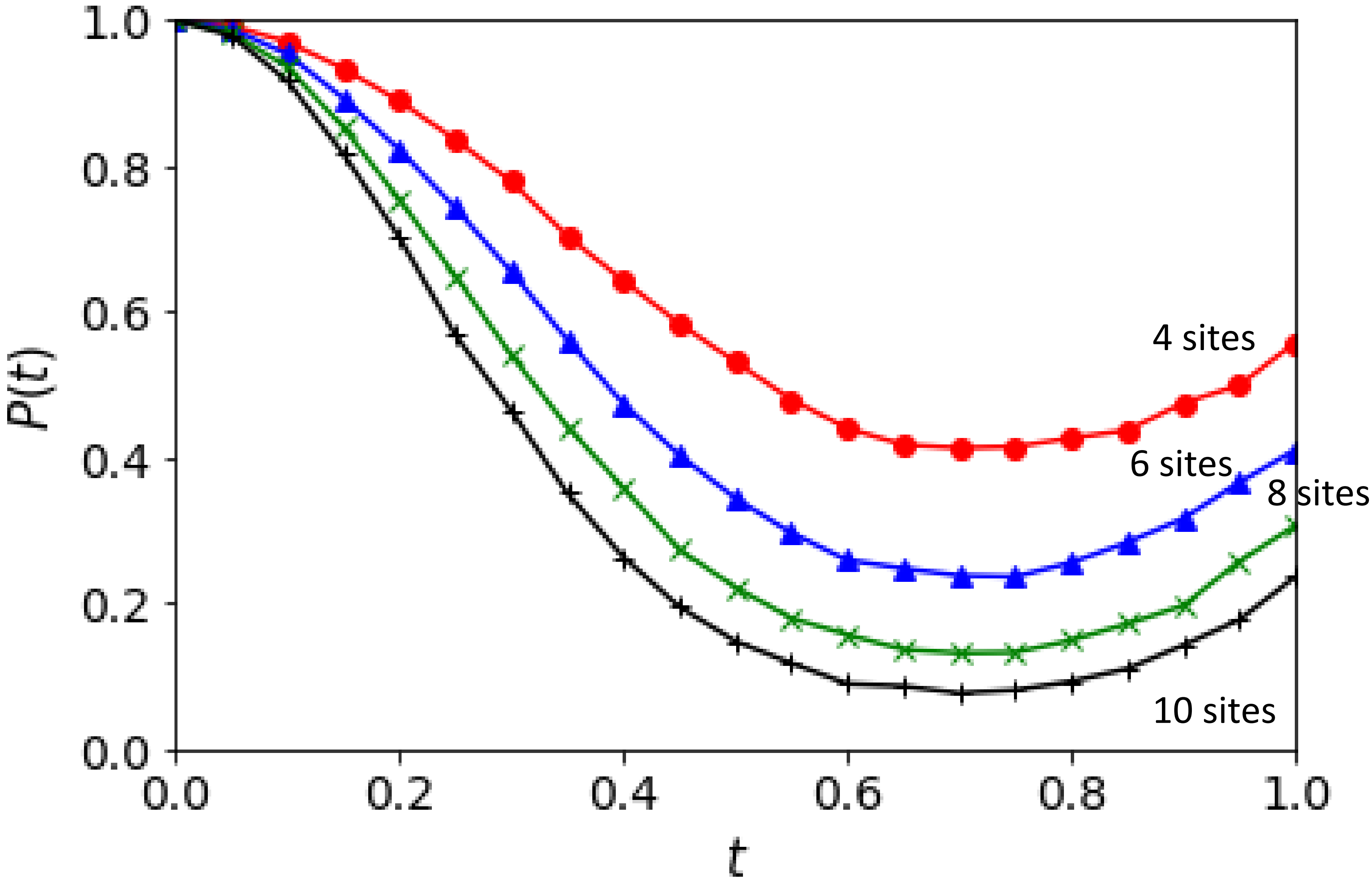
Survival probability:

$$P(t) = \left| \langle \text{massive}|\ e^{-i\hat{H}t}\ |\text{massive}\rangle \right|^2$$

$$= \left| \langle 00\cdots 0|\ X_N \cdots X_4 X_2 e^{-i\hat{H}t} X_2 X_4 \cdots X_N\ |00\cdots 0\rangle \right|^2$$

Result of simulator (10000 shots)

$J = 1,\ w = 1,\ m = 1,\ \theta = 0,\ \delta t = 0.01,$ 100 time steps



Tradeoff of symmetries in Suzuki-Trotter dec.

Suzuki-Trotter decomposition:

(more precisely, we generically use its improvement)

$$(M \in \boldsymbol{Z}, M \gg 1)$$

$$e^{-iHt} = \left(e^{-iH\frac{t}{M}}\right)^M \simeq \left(e^{-iH_1\frac{t}{M}} e^{-iH_2\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\Rightarrow \quad H_{\text{eff}} = \frac{1}{-it} \log\left(e^{-iH_1\frac{t}{M}} e^{-iH_2\frac{t}{M}}\right)^M$$

Symmetries may be broken by decomposition

Tradeoff of symmetries in Suzuki-Trotter dec.

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$$\Rightarrow \quad H_{\text{eff}} = \frac{1}{-it} \log\left(e^{-iH_1\frac{t}{M}} e^{-iH_2\frac{t}{M}}\right)^M$$

Symmetries may be broken by decomposition

Tradeoff:

- Parity friendly (& translation if p.b.c.)

$$H = H_{XX} + H_{YY} + H_{ZZ} + H_Z \qquad \cancel{U(1)}$$

- $U(1)$ friendly

$$H = H_{XX+YY}^{(\text{even})} + H_{XX+YY}^{(\text{odd})} + H_{ZZ} + H_Z \qquad \cancel{P}$$

Plan

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

VEV of mass operator (chiral condensation)

$$\langle \bar{\psi}(x)\psi(x)\rangle = \langle \mathrm{vac}|\bar{\psi}(x)\psi(x)|\mathrm{vac}\rangle$$

Instead of the local op., we analyze the average over the space:

$$\frac{1}{2Na}\langle \mathrm{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\mathrm{vac}\rangle$$

Once we get the vacuum, we can compute the VEV as

$$\frac{1}{2Na}\langle \mathrm{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\mathrm{vac}\rangle = \frac{1}{2Na}\sum_{n=1}^{N}(-1)^n \sum_{i_1\cdots i_N=0,1}\langle \mathrm{vac}|Z_n|i_1\cdots i_N\rangle\langle i_1\cdots i_N|\mathrm{vac}\rangle$$

$$= \frac{1}{2Na}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{n+i_n}\,|\langle i_1\cdots i_N|\mathrm{vac}\rangle|^2$$

Adiabatic state preparation of vacuum (repeated)

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\mathrm{vac}_0\rangle$ is known and unique

Step 2: Introduce adiabatic Hamiltonian $H_A(t)$ s.t.

$$\begin{cases} \cdot\ H_A(0) = H_0,\ H_A(T) = H_{\mathrm{target}} \\ \cdot\ \left|\frac{dH_A}{dt}\right| \ll 1 \ \text{for}\ T \gg 1 \end{cases}$$

Step 3: Use the adiabatic theorem

If $H_A(t)$ has a unique ground state w/ a finite gap for $\forall t$, then the ground state of H_{target} is obtained by

$$\boxed{|\mathrm{vac}\rangle = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\, H_A(t)\right)|\mathrm{vac}_0\rangle}$$

Features of adiabatic state preparation

$$(\text{"systematic error"}) \sim \frac{1}{T\,(\text{gap})^2}$$

Advantage:

- guaranteed to be correct for $T \gg 1$ & $\delta t \ll 1$ if $H_A(t)$ has a unique gapped vacuum
- can directly get excited states under some conditions

Disadvantage:

Features of adiabatic state preparation

$$
(\text{"systematic error"}) \sim \frac{1}{T\,(\text{gap})^2}
$$

😄 Advantage:

- guaranteed to be correct for $T \gg 1$ & $\delta t \ll 1$ if $H_A(t)$ has a unique gapped vacuum
- can directly get excited states under some conditions

☹️ Disadvantage:

- doesn't work for degenerate vacua
- costly — likely requires many gates

⇒ perhaps not so efficient in NISQ era

Adiabatic state preparation in the Schwinger model

$$|\mathrm{vac}> = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\ H_A(t)\right)|\mathrm{vac}_0>$$

$$\simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\mathrm{vac}_0>$$

$$\left(U(t) = e^{-iH_A(t)\delta t}\right)$$

Here we choose

$$\begin{cases} H_0 = H_{ZZ} + H_Z|_{m\to m_0, \theta\to 0} \quad \Rightarrow \quad |\mathrm{vac}_0\rangle = |0101\cdots 01\rangle \\ H_A(t) = \hat{H}\Big|_{w\to w(t), \theta\to\theta(t), m\to m(t)} \\ w(t) = \frac{t}{T}w,\ \theta(t) = \frac{t}{T}\theta,\ m(t) = \left(1-\frac{t}{T}\right)m_0 + \frac{t}{T}m \end{cases}$$

m_0 can be any positive number in principle
but it is practically chosen to have small systematic error

Massless case

For massless case,

θ is absorbed by chiral rotation ⇨ $\theta = 0$ w/o loss of generality

No sign problem

Nevertheless,

it's difficult in conventional approach because computation of fermion determinant becomes very costly

$^\exists$ Exact result:

[Hetrick-Hosotani '88]

$$\langle \bar{\psi}(x)\psi(x)\rangle = -\frac{e^{\gamma}}{2\pi^{3/2}}g \simeq -0.160g$$

Can we reproduce it?

Estimation of systematic errors

Approximation of vacuum: [Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]

$$|\text{vac}> \simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\text{vac}_0> \equiv |\text{vac}_A\rangle$$

Approximation of VEV:

$$\langle \mathcal{O} \rangle \equiv \langle \text{vac}|\mathcal{O}|\text{vac}\rangle \simeq \langle \text{vac}_A|\mathcal{O}|\text{vac}_A\rangle$$

Introduce the quantity

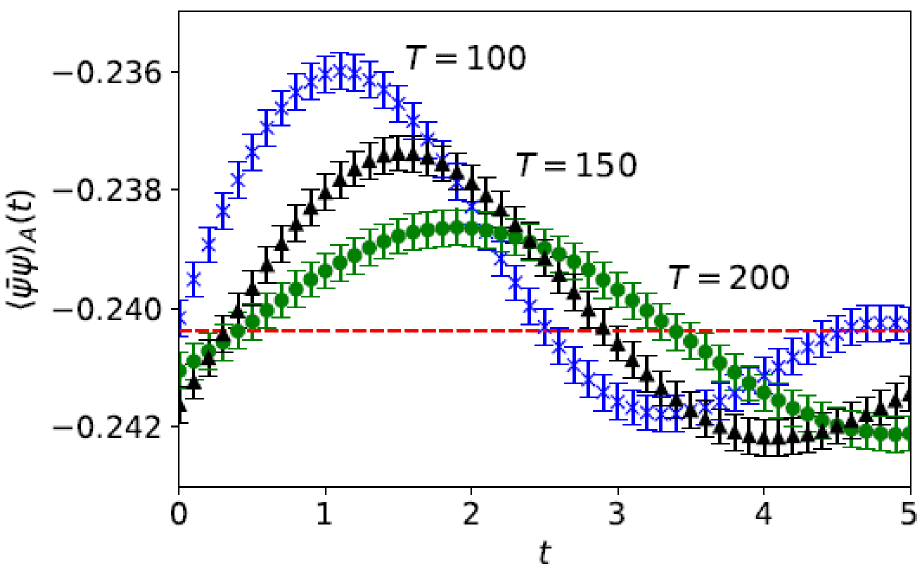
$$\langle \mathcal{O} \rangle_A(t) \equiv \langle \text{vac}_A|e^{i\hat{H}t}\mathcal{O}e^{-i\hat{H}t}|\text{vac}_A\rangle$$

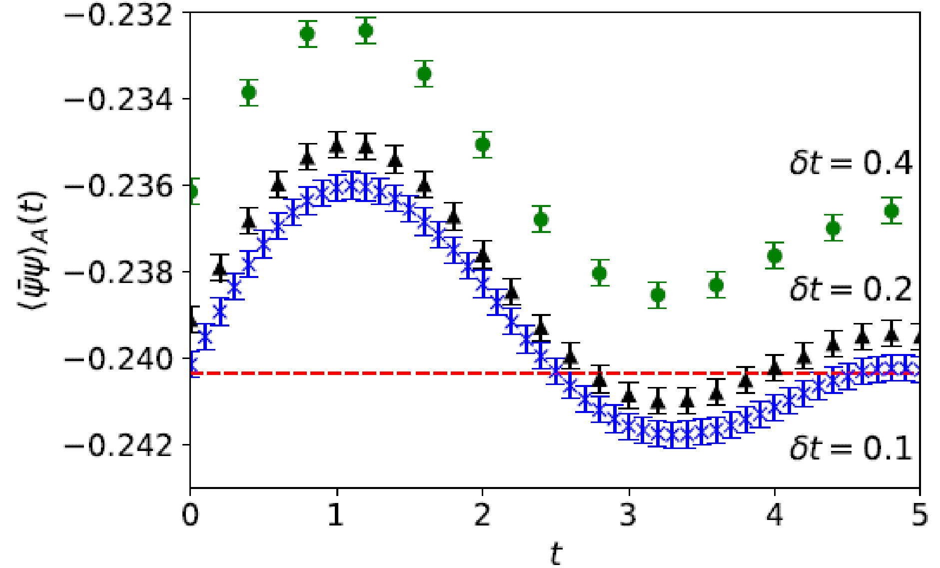
$$\begin{cases} \text{independent of t if } |\text{vac}_A\rangle = |\text{vac}\rangle \\ \text{dependent on t if } |\text{vac}_A\rangle \neq |\text{vac}\rangle \end{cases}$$

This quantity describes intrinsic ambiguities in prediction

⇒ Useful to estimate systematic errors

Estimation of systematic errors (Cont'd)





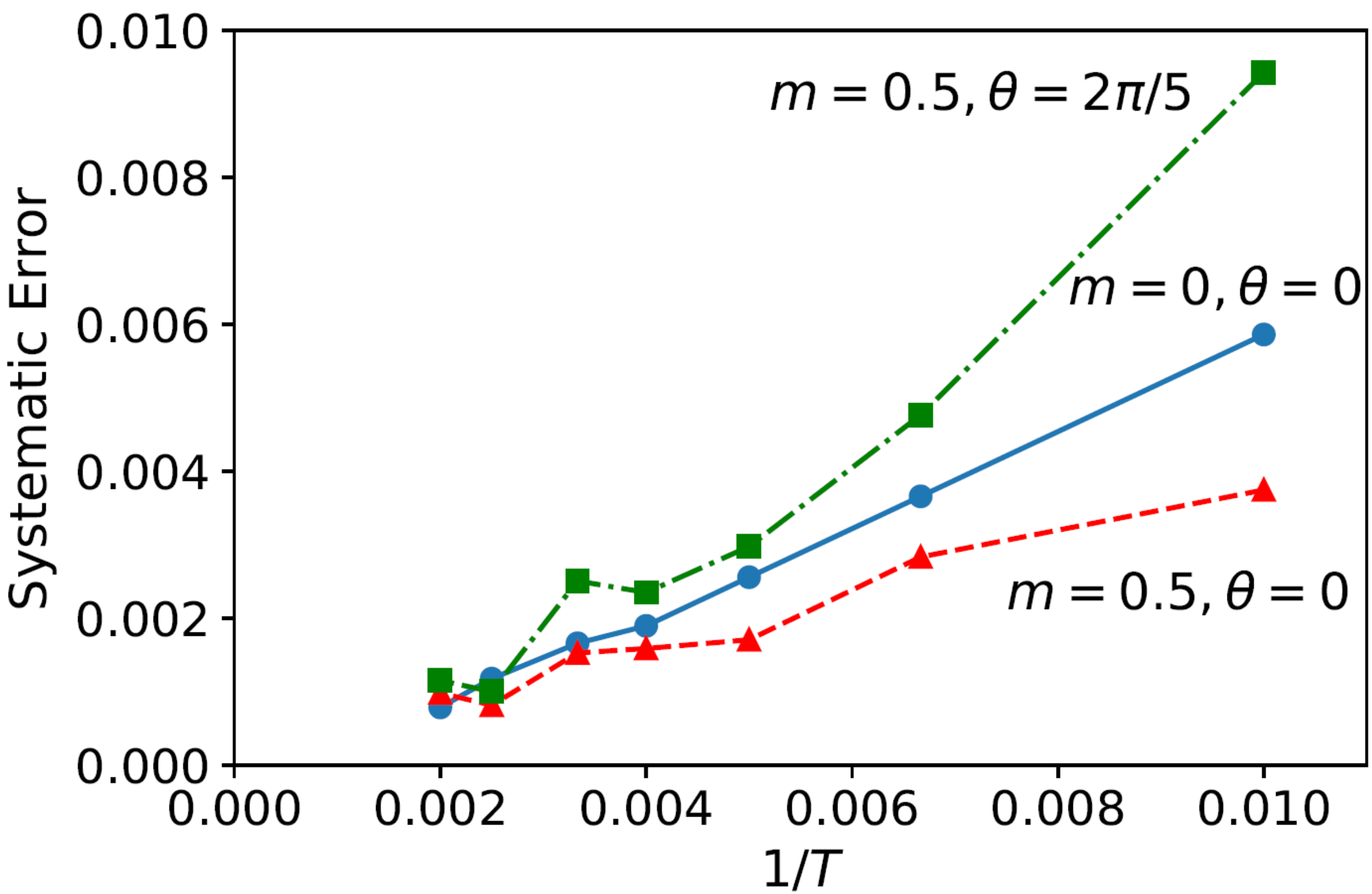
Oscillating around the correct value

Define central value & error as

$$\frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) + \min\langle\mathcal{O}\rangle_A(t)\right) \quad \& \quad \frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) - \min\langle\mathcal{O}\rangle_A(t)\right)$$

T-dependence of the systematic errors

Parameters: $g = 1, a = 1, N = 8, 10^6$ shots

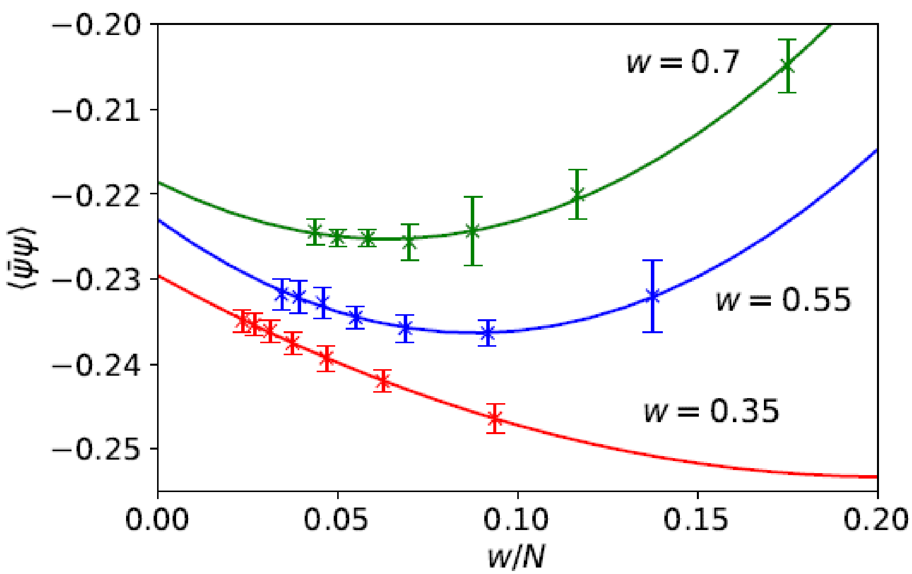


Thermodynamic & Continuum limit

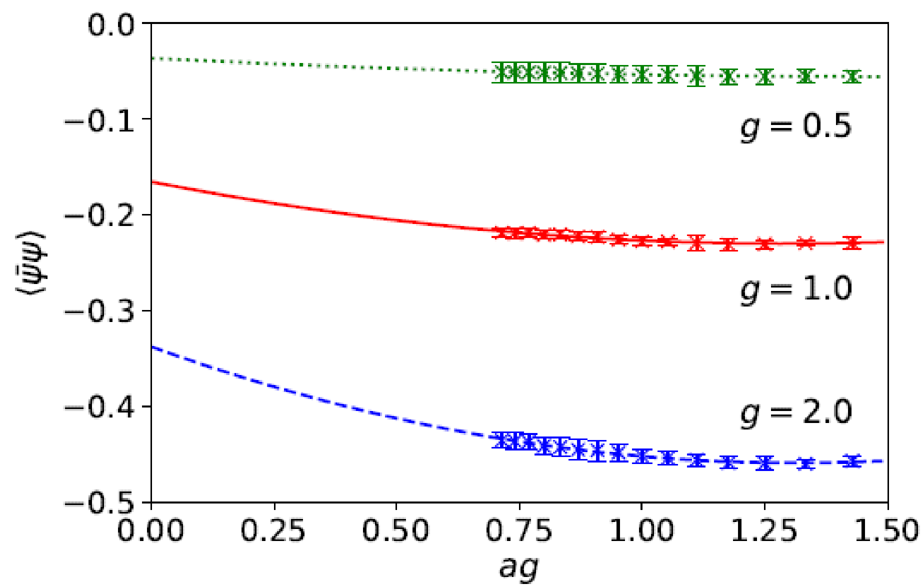
$$g = 1, m = 0, N_{\text{max}} = 16, T = 100, \delta t = 0.1, 1M \text{ shots}$$

#(measurements)

Thermodynamic limit: ($N \to \infty$, fixed a)



Continuum limit: ($a \to 0$ after $aN \to \infty$)

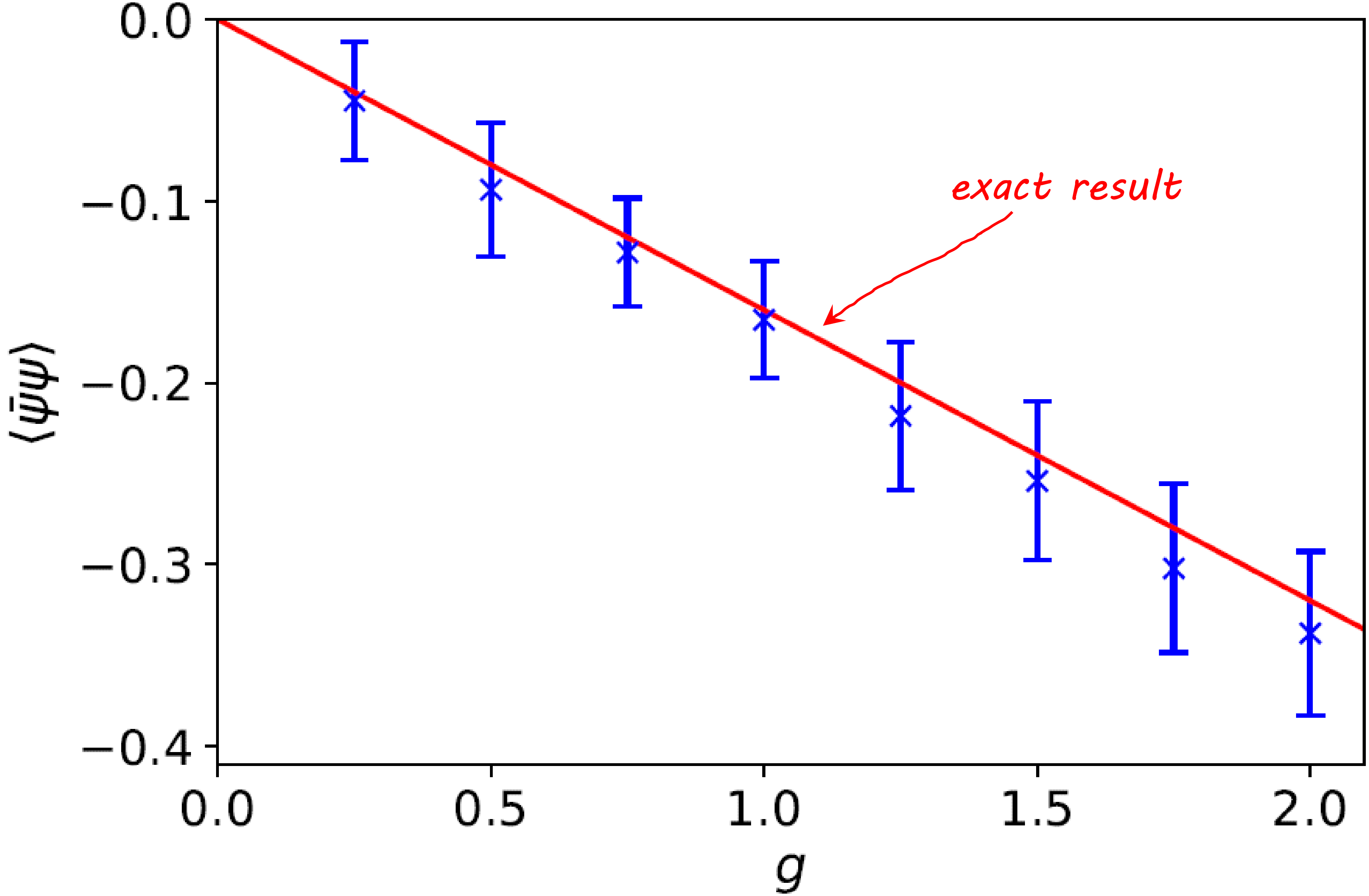


$$\left(w \coloneqq \frac{1}{2a}\right)$$

Result for massless case (after continuum limit)

$T = 100, \delta t = 0.1, N_{\mathrm{max}} = 16, 1M$ shots

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]



Massive case

Result of mass perturbation theory: [Adam '98]

$$\langle \bar{\psi}(x)\psi(x)\rangle \sim -0.160g + 0.322m\cos\theta$$

However,

$\exists$ Subtlety in comparison: this quantity is UV divergent

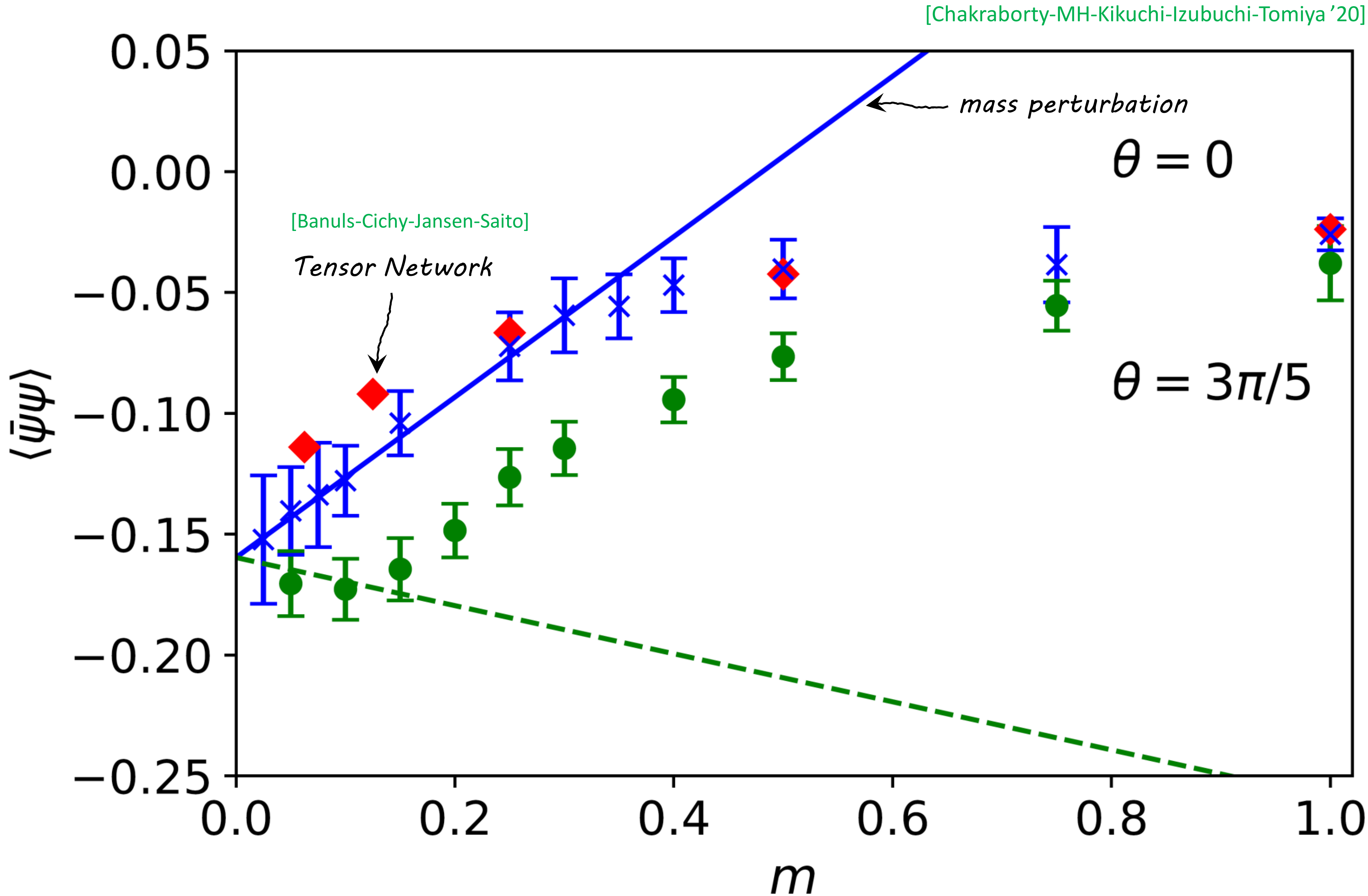
$(\sim m \log \Lambda)$

$\Rightarrow$ need a regularization!

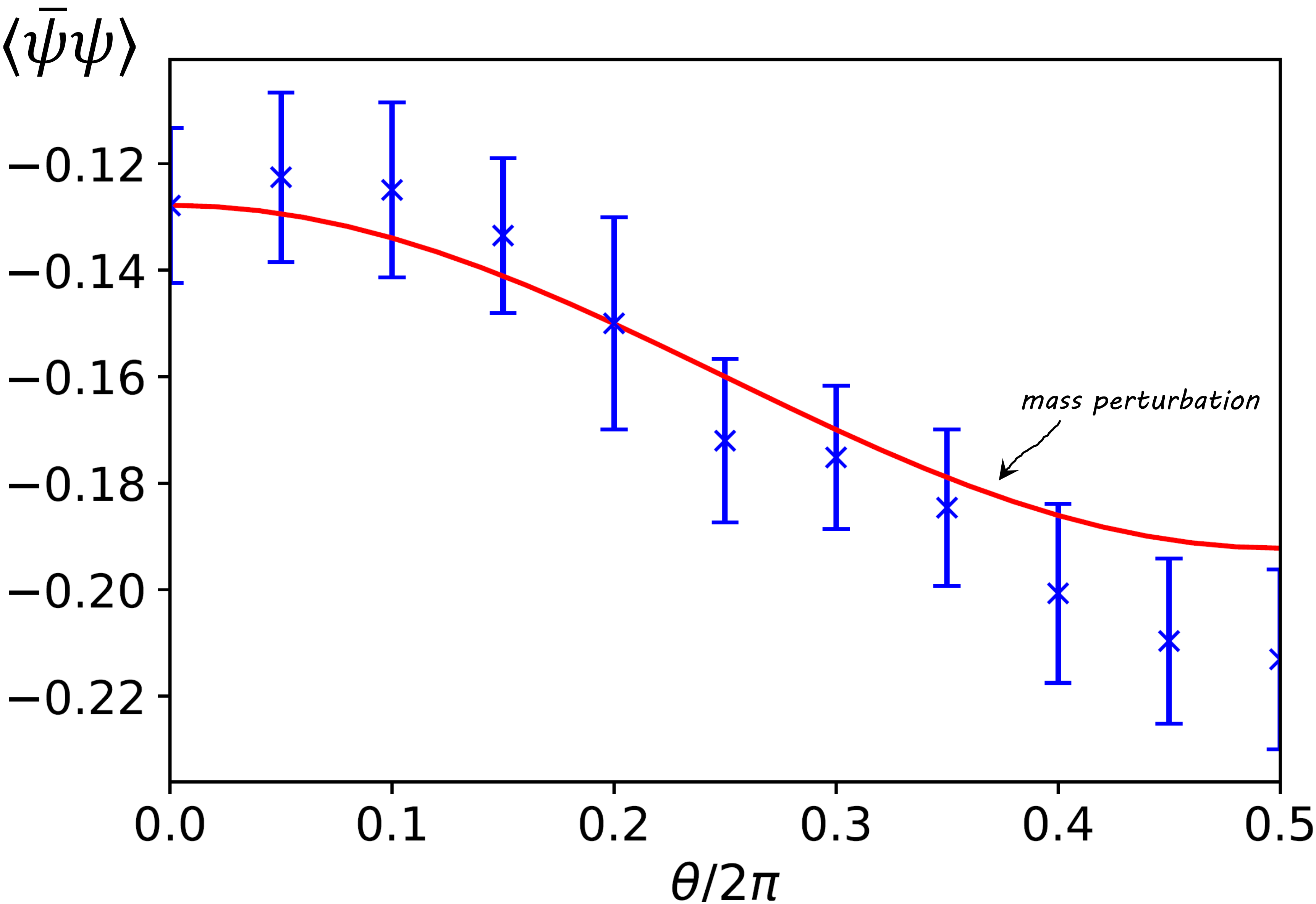
Here we subtract free theory result before taking continuum limit:

$$\lim_{a\to 0}\left[\langle \bar{\psi}\psi\rangle - \langle \bar{\psi}\psi\rangle_{\text{free}}\right]$$

Chiral condens. for massive case at g=1



θ dependence at $m = 0.1$ & $g = 1$



Summary

Summary

- Quantum computation is suitable for operator formalism which is free from sign problem
- QFT typically has ∞ dimensional Hilbert space and regularization is needed for simulation in operator formalism
- For QFT w/ physical bosonic d.o.f., extra truncation is needed even after putting it on lattice
- We've constructed the vacuum of Schwinger model w/ the topological term by adiabatic state preparation
- found agreement in the chiral condensate with the exact result for $m = 0$ & mass perturbation theory for small m

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

Here is the end of lecture 4!