Application of Quantum Computation to High Energy Physics
– Quantum Field Theory –

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Intensive lectures @RESCEU（day 2）

Plan of the intensive lectures

Day 1 (If 2nd lecture of each day ends early, then we start hands-on early)

- Lecture 1: introduction, basics of quantum computation
- Lecture 2: quantum simulation of spin system
- Hands-on 1: Basics on IBM's qiskit

Day 2

- Lecture 3: quantum field theory (QFT)
- Lecture 4: QFT on quantum computer
- Hands-on 2: Time evolution of spin system

Day 3

- Lecture 5: quantum error correction
- Lecture 6: some advanced topics, future prospects
- Hands-on 3: Constructing ground state of spin system

Purpose of lecture 3

I’d like to express…

- What is quantum field theory?
- How is this interesting & nontrivial?
- conventional numerical approach
- when the conventional approach doesn’t work

Plan

1. Introduction to Quantum field theory

2. (1+1)d scalar field theory

3. Gauge theory

4. Lattice field theory

5. Summary

One particle QM (operator formalism)

Hamiltonian:

$$H(x,p) = \frac{1}{2m}p^2 + V(x)$$

Schrodinger equation:

$$\hat{H}(\hat{x},\hat{p})\,|\psi\rangle = E\,|\psi\rangle, \qquad [\hat{x},\hat{p}] = i$$

$$\hat{H}(\hat{x},\hat{p})\,\psi(x) = E\,\psi(x), \qquad \hat{p} = -i\frac{d}{dx} \qquad (\psi(x) \coloneqq \langle x|\psi\rangle)$$

Expectation value:

$$\langle\psi|\,\hat{\mathcal{O}}(\hat{x},\hat{p})\,|\psi\rangle$$

One particle QM (path integral formalism)

Hamiltonian:

$$H(x,p) = \frac{1}{2m}p^2 + V(x)$$

$$\Downarrow \qquad \dot{x} := \frac{\partial H}{\partial p} = \frac{p}{m}$$

Lagrangian:

One particle QM (path integral formalism)

Hamiltonian:

$$H(x,p) = \frac{1}{2m}p^2 + V(x)$$

$$\Downarrow \qquad \dot{x} \coloneqq \frac{\partial H}{\partial p} = \frac{p}{m}$$

Lagrangian:

$$L(x,\dot{x}) \coloneqq \dot{x}p - H = \frac{m}{2}\dot{x}^2 - V(x)$$

Path integral:

One particle QM (path integral formalism)

Hamiltonian:

$$H(x,p) = \frac{1}{2m}p^2 + V(x)$$

$$\dot{x} := \frac{\partial H}{\partial p} = \frac{p}{m}$$

Lagrangian:

$$L(x,\dot{x}) := \dot{x}p - H = \frac{m}{2}\dot{x}^2 - V(x)$$

Path integral:

$$S[x] := \int dt\, L(x,\dot{x})$$

$$\int_{x(0)=x_i}^{x(T)=x_f} Dx\, e^{iS[x]} = \langle x_f | e^{-iHT} | x_i \rangle$$

integral over $x(t)$ for all t

∞-dimensional!

Imaginary (Euclid) time & temperature

$$t = i\tau, \qquad x(\tau+\beta) = x(\tau)$$

Partition function:

$$Z(\beta) := \int Dx\, e^{-S[x]} = \mathrm{Tr}[e^{-\beta H}]$$ (β: inverse temperature)

$$S[x] := \int d\tau\, L(x), \quad L(x) = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)$$

Expectation value:

Imaginary (Euclid) time & temperature

$$t = i\tau, \qquad x(\tau + \beta) = x(\tau)$$

Partition function:

$$Z(\beta) := \int Dx\, e^{-S[x]} = \mathrm{Tr}\left[e^{-\beta H}\right]$$ (β: inverse temperature)

$$S[x] \coloneqq \int d\tau\, L(x), \quad L(x) = \frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)$$

Expectation value:

$$\langle \mathcal{O}(x) \rangle = \frac{\int Dx\, \mathcal{O}(x) e^{-S[x]}}{\int Dx\, e^{-S[x]}} = \frac{\mathrm{Tr}\left[\mathcal{O}\, e^{-\beta H}\right]}{\mathrm{Tr}\left[e^{-\beta H}\right]}$$

$$\xRightarrow{\beta \to \infty} \langle \mathrm{vac} |\, \mathcal{O}\, | \mathrm{vac} \rangle$$

(0+1)d real scalar field theory

Just $x \to \phi$ (& $m = 1$) in one particle QM

Lagrangian:

$$L(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

(0+1)d real scalar field theory

Just $x \to \phi$ (& $m = 1$) in one particle QM

Lagrangian:

$$L(\phi, \dot{\phi}) = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\Big\Downarrow$$

conjugate momentum:

$$\Pi := \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

Hamiltonian:

(0+1)d real scalar field theory

Just $x \to \phi$ (& $m = 1$) in one particle QM

Lagrangian:

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conjugate momentum:

$$\Pi := \frac{\partial L}{\partial \dot{\phi}} = \dot{\phi}$$

Hamiltonian:

$$H(\phi, \Pi) := \dot{\phi}\Pi - L = \frac{1}{2}\Pi^2 + V(\phi) \qquad [\phi, \Pi] = i$$

Schrodinger eq. :

$$H(\phi, \Pi)\,\Psi(\phi) = E\,\Psi(\phi), \qquad \Pi = -i\frac{d}{d\phi}$$

(0+1)d multi scalar field theory (N scalars)

Lagrangian:

$$L(\phi_i, \dot{\phi}_i) = \frac{1}{2}\sum_{i=1}^{N} \dot{\phi}_i^2 \; - V(\phi_i)$$

(0+1)d multi scalar field theory (N scalars)

Lagrangian:

$$L(\phi_i, \dot{\phi}_i) = \frac{1}{2}\sum_{i=1}^{N} \dot{\phi}_i^2 \; - V(\phi_i)$$

$\Downarrow$

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$$\Pi_{\mathrm{i}} := \frac{\partial L}{\partial \dot{\phi}_i} = \dot{\phi}_i$$

Hamiltonian:

(0+1)d multi scalar field theory (N scalars)

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$$L(\phi_i, \dot{\phi}_i) = \frac{1}{2}\sum_{i=1}^{N} \dot{\phi}_i^2 \;\; - V(\phi_i)$$

$$\Downarrow$$

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$$\Pi_\mathrm{i} := \frac{\partial L}{\partial \dot{\phi}_i} = \dot{\phi}_i$$

Hamiltonian:

$$H(\phi_i, \Pi_\mathrm{i}) := \sum_{i=1}^{N} \dot{\phi}_\mathrm{i} \Pi_i - L = \frac{1}{2}\sum_{i=1}^{N} \Pi_i^2 \;\; + V(\phi_i) \qquad [\phi_i, \Pi_\mathrm{j}] = i\,\delta_{ij}$$

Schrodinger eq. :

$$H(\phi_i, \Pi_\mathrm{i})\,\Psi(\phi_i) = E\,\Psi(\phi_i), \qquad \Pi_\mathrm{j} = -i\frac{\partial}{\partial \phi_j}$$

From QM to (relativistic) QFT

(0+1) dim.

(d+1) dim.

Spacetime:

$$\boldsymbol{R}^{\mathbf{0,1}}\text{: } ds^2 = dt^2$$

Isometry:

translation

"Field":

$$\phi(t)$$

Lagrangian:

$$L = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

From QM to (relativistic) QFT

(0+1) dim.

Spacetime:

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"Field":

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Lagrangian:

$$L = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

(d+1) dim.

$$\boldsymbol{R}^{\boldsymbol{d},\mathbf{1}}:\ ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

$$\left(\begin{array}{l} x^\mu = (t, \boldsymbol{x}) \\ \eta_{\mu\nu} = \mathrm{diag}(+1, -1, \cdots, -1) \end{array}\right)$$

From QM to (relativistic) QFT

(0+1) dim.

Spacetime:

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translation, $SO(d,1)$ rotation

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(0+1) dim.

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translation, $SO(d,1)$ rotation

$$\phi(t, \boldsymbol{x})$$

From QM to (relativistic) QFT

(0+1) dim.

Spacetime:

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Isometry:

translation

"Field":

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Lagrangian:

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(d+1) dim.

$$\boldsymbol{R}^{\boldsymbol{d},\mathbf{1}}:\ ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu$$

$$\left(\begin{array}{l} x^\mu = (t, \boldsymbol{x}) \\ \eta_{\mu\nu} = \mathrm{diag}(+1, -1, \cdots, -1) \end{array}\right)$$

translation, $SO(d,1)$ rotation

$$\phi(t, \boldsymbol{x})$$

$$\left\{\begin{array}{l} L = \int d^d\boldsymbol{x}\, \mathcal{L}[\phi(t, \boldsymbol{x})] \\ \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi) \end{array}\right.$$

Classical scalar field theory (Lagrange)

Action:

$$S[\phi] = \int d^{d+1}x\, \mathcal{L}[\phi(x)]\,, \qquad \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Variation:

Classical scalar field theory (Lagrange)

Action:

$$S[\phi] = \int d^{d+1}x\, \mathcal{L}[\phi(x)]\,, \qquad \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Variation:

(under appropriate b.c.)

$$\delta S[\phi] = \int d^{d+1}x\, \delta\mathcal{L} = \int d^{d+1}x \left[\eta^{\mu\nu}(\partial_\mu\delta\phi)(\partial_\nu\phi) - \frac{\partial V}{\partial\phi}\delta\phi\right]$$

$$= \int d^{d+1}x \left[-\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{\partial V}{\partial\phi}\right]\delta\phi$$

Equation of motion:

Classical scalar field theory (Lagrange)

Action:

$$S[\phi] = \int d^{d+1}x \, \mathcal{L}[\phi(x)] \,, \qquad \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

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$$\delta S[\phi] = \int d^{d+1}x \, \delta\mathcal{L} = \int d^{d+1}x \left[\eta^{\mu\nu}(\partial_\mu\delta\phi)(\partial_\nu\phi) - \frac{\partial V}{\partial\phi}\delta\phi\right]$$

$$= \int d^{d+1}x \left[-\eta^{\mu\nu}\partial_\mu\partial_\nu\phi - \frac{\partial V}{\partial\phi}\right]\delta\phi$$

Equation of motion:

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\partial V}{\partial\phi} = 0$$

(Cf. Klein-Gordon eq.

$$(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0$$
)

Classical scalar field theory (Hamilton)

Lagrangian (density):

$$\mathcal{L}[\phi(x)] = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Conjugate momentum:

$$\Pi(x) \coloneqq \frac{\partial\mathcal{L}[\phi(x)]}{\partial\dot{\phi}(x)} = \dot{\phi}(x)$$

Hamiltonian (density):

Classical scalar field theory (Hamilton)

Lagrangian (density):

$$\mathcal{L}[\phi(x)] = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Conjugate momentum:

$$\Pi(x) \coloneqq \frac{\partial\mathcal{L}[\phi(x)]}{\partial\dot{\phi}(x)} = \dot{\phi}(x)$$

Hamiltonian (density):

$$\partial_i\phi = (\boldsymbol{\nabla}\phi)_i$$

$$H := \int d^d\boldsymbol{x}\ \mathcal{H}[\phi(x), \Pi(x)]$$

$$\mathcal{H}(x) := \Pi(x)\dot{\phi}(x)\ -\ \mathcal{L}[\phi(x)] = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

Classical scalar field theory (Hamilton, cont'd)

Hamiltonian:

$$H = \int d^d\boldsymbol{x}\ \mathcal{H}(\boldsymbol{x}), \qquad \mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

Poisson bracket:

$$\frac{\delta f(\boldsymbol{x})}{\delta f(\boldsymbol{y})} \coloneqq \delta^{(d)}(\boldsymbol{x}-\boldsymbol{y})$$

$$\{F,G\}_P \coloneqq \int d^d x \left[\frac{\delta F}{\delta\phi(\boldsymbol{x})}\frac{\delta G}{\delta\Pi(\boldsymbol{x})} - \frac{\delta G}{\delta\phi(\boldsymbol{x})}\frac{\delta F}{\delta\Pi(\boldsymbol{x})}\right]$$

$$\Longrightarrow \{\phi(\boldsymbol{x}),\Pi(\boldsymbol{y})\}_P = \delta^{(d)}(\boldsymbol{x}-\boldsymbol{y})$$

Classical scalar field theory (Hamilton, cont'd)

Hamiltonian:

$$H = \int d^d\boldsymbol{x}\ \mathcal{H}(\boldsymbol{x}), \qquad \mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

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$$\Longrightarrow \quad \{\phi(\boldsymbol{x}),\Pi(\boldsymbol{y})\}_P = \delta^{(d)}(\boldsymbol{x}-\boldsymbol{y})$$

Equation of motion:

$$\left(\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\partial V}{\partial\phi} = 0\right)$$

$$\dot{\phi}(\boldsymbol{x}) = \{\phi(\boldsymbol{x}),H\}_P = \Pi(\mathbf{x})$$

$$\dot{\Pi}(\boldsymbol{x}) = \{\Pi(\boldsymbol{x}),H\}_P = \partial_i\partial_i\phi - \frac{\partial V}{\partial\phi}$$

Scalar field theory (path integral formalism)

Action:

$$S[\phi] = \int d^{d+1}x\, \mathcal{L}[\phi(x)]\,, \qquad \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Partition function:

$$\int_{\phi(0,\boldsymbol{x})=\phi_i}^{\phi(T,\boldsymbol{x})=\phi_f} D\phi\; e^{iS[\phi]} = \langle\phi_f|e^{-iHT}|\phi_i\rangle$$

integral over field for all x

∞*-dim!*

Scalar field theory (path integral formalism)

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Partition function:

$$\int_{\phi(0,\boldsymbol{x})=\phi_i}^{\phi(T,\boldsymbol{x})=\phi_f} D\phi \; e^{iS[\phi]} = \langle \phi_f | e^{-iHT} | \phi_i \rangle$$

integral over field for all x

∞-*dim!*

Expectation value of operator:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi \; \mathcal{O}(\phi) e^{iS[\phi]}}{\int D\phi \; e^{iS[\phi]}}$$

Scalar field theory (operator formalism)

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi)$$

$$\Downarrow \qquad \Pi(\mathbf{x}) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$$

Hamiltonian:

$$\mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi)$$

Scalar field theory (operator formalism)

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Hamiltonian:

$$\mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi)$$

Commutation relation:

$$[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = i\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

Scalar field theory (operator formalism)

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

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Commutation relation:

$$[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = i\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

"Schrodinger eq." :

$$H[\phi, \Pi]\, \Psi(\phi) = E\, \Psi(\phi), \qquad \Pi = -i\frac{\delta}{\delta\phi(\boldsymbol{x})}$$

Euclid spacetime (imaginary time)

$$t = i\tau, \qquad \phi(\tau+\beta,\boldsymbol{x}) = \phi(\tau,\boldsymbol{x})$$

Partition function:

$\left(\boldsymbol{R}^{d+1}\colon\ ds^2 = \delta_{\mu\nu}dx^\mu dx^\nu\right)$

$$Z(\beta) := \int D\phi\; e^{-S[\phi]} = \mathrm{Tr}[e^{-\beta H}]$$

$$S[\phi] \coloneqq \int d^{d+1}x\, L(\phi), \qquad \mathcal{L} = \frac{1}{2}\delta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + V(\phi)$$

Euclid spacetime (imaginary time)

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$$S[\phi] := \int d^{d+1}x\, L(\phi), \qquad \mathcal{L} = \frac{1}{2}\delta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) + V(\phi)$$

Expectation value:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi\, \mathcal{O}(x) e^{-S[\phi]}}{\int D\phi\, e^{-S[\phi]}} = \frac{\mathrm{Tr}\left[\mathcal{O}\, e^{-\beta H}\right]}{\mathrm{Tr}[e^{-\beta H}]}$$

$$\xrightarrow{\beta \to \infty} \langle \mathrm{vac} |\, \mathcal{O}\, | \mathrm{vac} \rangle$$

Comment: scalar on curved spacetime

Flat spacetime:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \qquad \eta_{\mu\nu} = \mathrm{diag}(+1, -1, \cdots, -1)$$

$$S[\phi] = \int d^{d+1}x \left[\frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right]$$

Curved spacetime:

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu$$

$$S[\phi] = \int d^{d+1}x \sqrt{-\det g}\left[\frac{1}{2}g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)\right]$$

Short summary

Action:

$$S[\phi] = \int d^{d+1}x \, \mathcal{L}[\phi(x)] \, , \qquad \mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Path integral:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi \, \mathcal{O}(\phi) e^{iS[\phi]}}{\int D\phi \, e^{iS[\phi]}}$$

Hamiltonian:

$$\mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

Commutation relation:

$$[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = i\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

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(1+1)d free massive scalar

Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_x \phi)^2 - \frac{m^2}{2}\phi^2 \qquad (m\text{: mass})$$

$$\Pi(x) = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \partial_t \phi$$

Hamiltonian:

$$H = \int dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{2}\phi^2\right]$$

$$[\phi(x), \Pi(y)] = i\delta(x-y)$$

Let’s solve this theory!

(1+1)d free massive scalar (cont'd)

$$H = \int dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{2}\phi^2\right], \quad [\phi(x), \Pi(y)] = i\delta(x-y)$$

Fourier expansion:

$$\begin{cases} \phi(x) = \int dp\, \phi_p e^{ipx}, \\ \Pi(x) = \int dp\, \Pi_p e^{ipx} \end{cases} \qquad [\phi_p, \Pi_q] = i\delta(p+q)$$

$$\Downarrow$$

(1+1)d free massive scalar (cont'd)

$$H = \int dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x \phi)^2 + \frac{m^2}{2}\phi^2\right], \quad [\phi(x), \Pi(y)] = i\delta(x-y)$$

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$$\Downarrow$$

$$H = \int dp \left[\frac{1}{2}\Pi_p \Pi_{-p} + \frac{p^2+m^2}{2}\phi_p \phi_{-p}\right]$$

∞ many harmonic oscillators!

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\frac{1}{2}\Pi_p\Pi_{-p} + \frac{p^2+m^2}{2}\phi_p\phi_{-p}\right], \qquad [\phi_p, \Pi_q] = i\delta(p+q)$$

Creation/annihilation op. :

$$\left(\omega_p^2 = p^2 + m^2\right)$$

$$\begin{cases} a_p = \sqrt{\dfrac{\omega_p}{2}}\phi_p + \dfrac{i}{\sqrt{2\omega_p}}\Pi_p\,, \\ a_p^\dagger = \sqrt{\dfrac{\omega_p}{2}}\phi_{-p} - \dfrac{i}{\sqrt{2\omega_p}}\Pi_{-p} \end{cases}$$

$$\left[a_p, a_q^\dagger\right] = \delta(p-q)$$

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\frac{1}{2}\Pi_p\Pi_{-p} + \frac{p^2+m^2}{2}\phi_p\phi_{-p}\right], \qquad [\phi_p, \Pi_q] = i\delta(p+q)$$

Creation/annihilation op. :

$$(\omega_p^2 = p^2 + m^2)$$

$$\begin{cases} a_p = \sqrt{\frac{\omega_p}{2}}\phi_p + \frac{i}{\sqrt{2\omega_p}}\Pi_p\,, \\ a_p^\dagger = \sqrt{\frac{\omega_p}{2}}\phi_{-p} - \frac{i}{\sqrt{2\omega_p}}\Pi_{-p} \end{cases}$$

$$\left[a_p, a_q^\dagger\right] = \delta(p-q)$$

$$\Longrightarrow \quad H = \int dp \left[\omega_p \left(\underbrace{a_p^\dagger a_p}_{\text{number operator!}} + \frac{1}{2}\right)\right]$$

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\omega_p a_p^\dagger a_p\right], \qquad \left[a_p, a_q^\dagger\right] = \delta(p-q)$$

Fock vacuum:

$$a_p|\mathrm{vac}\rangle = 0, \qquad H|\mathrm{vac}\rangle = 0$$

Energy eigenstates:

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\omega_p a_p^\dagger a_p\right], \qquad \left[a_p, a_q^\dagger\right] = \delta(p-q)$$

Fock vacuum:

$$a_p|\text{vac}\rangle = 0, \qquad H|\text{vac}\rangle = 0$$

Energy eigenstates: $\left(\omega_p^2 = p^2 + m^2\right)$

$$a_p^\dagger\,|\text{vac}\rangle \qquad E = \omega_p$$

one particle w/ mass m
& momentum p

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\omega_p a_p^\dagger a_p\right], \qquad \left[a_p, a_q^\dagger\right] = \delta(p-q)$$

Fock vacuum:

$$a_p|\text{vac}\rangle = 0, \qquad H|\text{vac}\rangle = 0$$

Energy eigenstates:

$(\omega_p^2 = p^2 + m^2)$

$a_p^\dagger\,|\text{vac}\rangle$ $\quad E = \omega_p$ $\quad$ one particle w/ mass m & momentum p

$a_{p_1}^\dagger a_{p_2}^\dagger|\text{vac}\rangle$ $\quad E = \omega_{p_1} + \omega_{p_2}$ $\quad$ 2 particles

(1+1)d free massive scalar (cont'd)

$$H = \int dp \left[\omega_p a_p^\dagger a_p\right], \qquad \left[a_p, a_q^\dagger\right] = \delta(p-q)$$

Fock vacuum:

$$a_p|\text{vac}\rangle = 0, \qquad H|\text{vac}\rangle = 0$$

Energy eigenstates:

$$\left(\omega_p^2 = p^2 + m^2\right)$$

$a_p^\dagger|\text{vac}\rangle$ $\qquad E = \omega_p$ $\qquad$ one particle w/ mass m & momentum p

$a_{p_1}^\dagger a_{p_2}^\dagger|\text{vac}\rangle$ $\qquad E = \omega_{p_1} + \omega_{p_2}$ $\qquad$ 2 particles

$a_{p_1}^\dagger a_{p_2}^\dagger a_{p_3}^\dagger|\text{vac}\rangle$ $\qquad E = \omega_{p_1} + \omega_{p_2} + \omega_{p_3}$ $\qquad$ 3 particles

$\vdots \qquad\qquad \vdots$

Free massive scalar on S^1_L

$$H = \int_0^L dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2\right]$$

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Fourier expansion:

discrete!

$$\phi(x) = \sum_{n\in\mathbf{Z}} \phi_n e^{\frac{2\pi i n}{L}x}, \quad \Pi(x) = \sum_{n\in\mathbf{Z}} \Pi_n e^{\frac{2\pi i n}{L}x}, \quad [\phi_m, \Pi_n] = i\delta_{m+n,0}$$

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Creation/annihilation op. :

$$a_n = \sqrt{\frac{\omega_n}{2}}\phi_n + \frac{i}{\sqrt{2\omega_n}}\Pi_n , \quad \omega_n^2 = \left(\frac{2\pi n}{L}\right)^2 + m^2 \quad \left[a_m, a_n^\dagger\right] = \delta_{mn}$$

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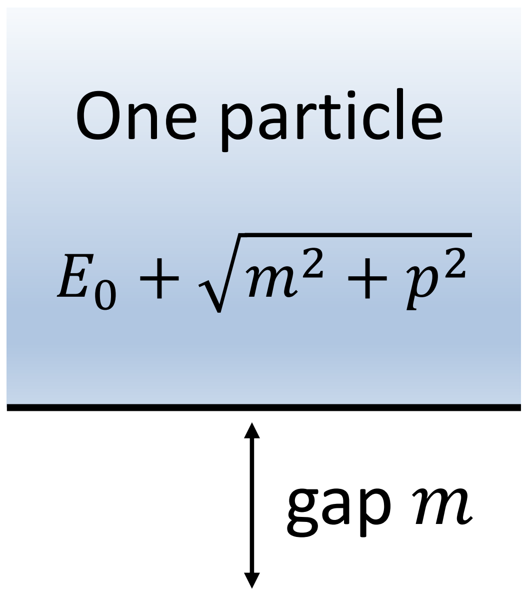
$$\Longrightarrow \quad H = \sum_{n\in\mathbf{Z}} \omega_n \left(a_n^\dagger a_n + \frac{1}{2}\right)$$

eigenstates
= Fock states

Energy spectrum of free massive scalar

∞ volume



unique gapped vacuum

finite volume

Energy spectrum of free massive scalar

∞ volume

One particle

$$E_0 + \sqrt{m^2 + p^2}$$

gap m

unique gapped vacuum

finite volume

One particle

$$E_0 + \sqrt{m^2 + \left(\frac{2\pi n}{L}\right)^2}$$

gap m

unique gapped vacuum

Interacting case (ϕ^4 theory)

Hamiltonian:

$$H = H_0 + H_{\text{int}}$$

$$H_0 = \int dx \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_x\phi)^2 + \frac{m^2}{2}\phi^2\right], \quad H_{\text{int}} = \lambda \int dx\, \phi^4(x)$$

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Fourier expansion:

$$\left(\phi(x) = \int dp\, \phi_p e^{ipx}\right)$$

$$H_0 = \int dp \left[\frac{1}{2}\Pi_p \Pi_{-p} + \frac{p^2 + m^2}{2}\phi_p \phi_{-p}\right]$$

$$H_{\text{int}} = \int dx \int dp_1 dp_2 dp_3 dp_4\, e^{i(p_1+p_2+p_3+p_4)x} \phi_{p_1}\phi_{p_2}\phi_{p_3}\, \phi_{p_4}$$

$$= \int dp_1 dp_2 dp_3 dp_4\, \delta(p_1 + p_2 + p_3 + p_4) \phi_{p_1}\phi_{p_2}\phi_{p_3}\, \phi_{p_4}$$

Interacting case (ϕ^4 theory, cont'd)

Consider time evolution of single particle state:

$$e^{-iHt}\left(a_p^\dagger\,|\mathrm{vac}_0\rangle\right) \qquad\qquad \left(a_p|\mathrm{vac}_0\rangle = 0\right)$$

For $t \ll 1$, up to $\mathcal{O}\left(t^2\right)$,

$$(1 - iHt)\,a_p^\dagger\,|\mathrm{vac}_0\rangle = (\mathrm{const.}\ -iH_{\mathrm{int}}t)\,a_p^\dagger\,|\mathrm{vac}_0\rangle$$

Interacting case (ϕ^4 theory, cont'd)

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In particular,

$$\phi_p = \frac{1}{\sqrt{2\omega_p}}\left(a_p + a_{-p}^\dagger\right)$$

$$H_{\text{int}}a_p^\dagger\,|\text{vac}_0\rangle = \int dp_1 dp_2 dp_3 dp_4\,\delta\left(\sum_{i=1}^{4} p_i\right)\phi_{p_1}\phi_{p_2}\phi_{p_3}\phi_{p_4}a_p^\dagger\,|\text{vac}_0\rangle$$

includes 1 particle, 3 particles, 5 particles states

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includes 1 particle, 3 particles, 5 particles states

The number of particles can change dynamically in QFT

Does perturbation theory work?

$$Z = \int D\phi \; e^{-S_0[\phi]-\lambda S_{\text{int}}[\phi]} \; = \int D\phi \; e^{-S_0[\phi]} \sum_n \frac{(-\lambda \, S_{\text{int}}[\phi])^n}{n!}$$

$$\sim \sum_n \frac{(-\lambda)^n}{n!} \int D\phi \; e^{-S_0[\phi]} (S_{\text{int}}[\phi])^n$$

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perturbative series in QFT is non-convergent

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However, it is known typically

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Ex.) ▪QM w/ quartic potential

$$\left[-\frac{d^2}{dx^2} + \lambda x^4\right] \psi(x) = E\,\psi(x)$$

▪"0d" theory

$$Z = \int_{-\infty}^{\infty} dx \; e^{-x^2 - g x^4}$$

Best way by Naïve sum = Truncation

N-th order approximation of a function $P(g)$:

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell$$

"error" of the approximation:

$$\delta_N(g) \equiv P_{N+1}(g) - P_N(g) = c_{N+1} g^{N+1}$$

Optimized order N_*: (given g)

$$\frac{\partial}{\partial N}\delta_N(g)\Big|_{N=N_*} = 0 \quad \xrightarrow{N \gg 1} \quad \frac{\partial}{\partial N}(\log c_N + N \log g)\Big|_{N=N_*} = 0$$

Best way by Naïve sum = Truncation (Cont'd)

$$P_N(g) \equiv \sum_{\ell=0}^{N} c_\ell \, g^\ell \quad \xrightarrow{\text{optimize}} \quad \frac{\partial}{\partial N}\left(\log c_N + N \log g\right)_N \Big|_{N=N_*} = 0$$

In QFT, typically

$$c_\ell \sim \ell! \, A^\ell \quad (\ell \gg 1)$$

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Then,

$$0 = \frac{\partial}{\partial N}\left(N\log N - N + N\log(\mathrm{Ag})\right)\Big|_{N=N_*} \quad \Rightarrow \quad N_* = \frac{1}{Ag}$$

Best way by Naïve sum = Truncation (Cont'd)

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In QFT, typically

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Then,

$$0 = \frac{\partial}{\partial N}\left(N \log N - N + N \log(Ag)\right)\Big|_{N=N_*} \Longrightarrow N_* = \frac{1}{Ag}$$

Error of the truncation:

$$\delta_{N_*}(g) = c_{N_*+1} g^{N_*+1} \sim e^{-N_*} = \underbrace{e^{-\frac{1}{Ag}}}_{\text{Non-perturbative effect}}$$

Thus, we should be careful unless coupling is small…

d.o.f. not naively seen -soliton-

Take $V(\phi) = \frac{g^2}{2}\left(v^2 - \phi^2\right)^2$ & find nontrivial classical sol.

Energy for static config. :

$$E = H = \int_{-\infty}^{\infty} dx \left[\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 + \frac{g^2}{2}\left(v^2 - \phi^2\right)^2\right]$$

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$$= \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\frac{d\phi}{dx} \mp g(v^2 - \phi^2)\right)^2 \pm g \int_{-\infty}^{\infty} dx \,(v^2 - \phi^2)\frac{d\phi}{dx}$$

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$$\geq \pm g\left[v^2\phi - \frac{1}{3}\phi^3\right]_{\phi(-\infty)}^{\phi(\infty)}$$

"Bogomolny bound"

d.o.f. not naively seen -soliton- (cont'd)

$$E = \frac{1}{2}\int_{-\infty}^{\infty} dx \left(\frac{d\phi}{dx} \mp g(v^2 - \phi^2)\right)^2 \pm g\left[v^2\phi - \frac{1}{3}\phi^3\right]_{\phi(-\infty)}^{\phi(\infty)}$$

The bound is saturated when $\frac{d\phi}{dx} \mp g\left(v^2 - \phi^2\right) = 0$

d.o.f. not naively seen -soliton- (cont'd)

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trivial

$$\phi(x) = \pm v, \qquad E = 0$$

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"kink"

$$\phi(x) = v\tanh(gvx - \mathrm{const.}), \qquad E = \frac{4}{3}gv^3$$

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3. $\phi(\pm\infty) = \mp v$

"anti-kink"

$$\phi(x) = -v\tanh(gvx - \mathrm{const.}), \qquad E = \frac{4}{3}gv^3$$

~particles?

Typical problems in QFT

- compute observables $\rightarrow$ compare w/ experiments
- determine vacuum structure/effective theory
  - —— unique or degenerate vacua?
  - —— gapped or gapless?
  - —— what symmetry preserved/broken?
  - —— topological phase?
- determine phase structures as changing parameters

(e.g. temperature, chemical potential, coupling)

Short summary

- Energy gap $\sim$ mass of lightest particle
- QFT describes states w/ any # of particles
- The number of particles can change dynamically
- perturbative series in QFT is typically non-convergent
- actual energy spectrum may be quite different from naïve guess from Lagrangian/Hamiltonian
- QFT is complicated…
  Analytically well-controlled cases are rare

Plan

1. Introduction to Quantum field theory
2. (1+1)d scalar field theory
3. Gauge theory
4. Lattice field theory
5. Summary

Classical Maxwell theory

Maxwell equations:

$$\partial_i E^i = 0, \qquad \partial_t E^i - \epsilon^{ijk}\partial_j B_k = 0$$

$$\partial_i B^i = 0, \qquad \partial_t B^i + \epsilon^{ijk}\partial_j E_k = 0$$

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Vector potential:

$$E^i = -\partial^i \varphi - \partial_t A^i, \qquad B^i = \epsilon^{ijk}\partial_j A_k$$

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Vector potential:

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$$\Downarrow$$

$$A^\mu = (\varphi, A^1, A^2, A^3)$$

$$E^i = F^{0i}, B^i = \epsilon^{ijk} F_{jk}$$

Relativistic form:

Classical Maxwell theory

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$$E^i = F^{0i}, B^i = \epsilon^{ijk} F_{jk}$$

Relativistic form:

$$\partial_\nu F^{\mu\nu} = 0, \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Lagrange formalism for Maxwell theory

Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Euler-Lagrange eq.:

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(coming from definition of $F_{\mu\nu}$)

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Gauge symmetry:

$$A_\mu \to A_\mu + \partial_\mu f(x)$$

Hamilton formalism for Maxwell theory

Lagrangian:

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Conjugate momenta:

$$\Pi^\mu := \frac{\partial \mathcal{L}}{\partial \dot{A}_\mu} =$$

Hamilton formalism for Maxwell theory

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This is constrained system! (related to gauge sym.)

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Lagrange multiplier

$$\mathcal{H} = \Pi^\mu \dot{A}_\mu - \mathcal{L} + \lambda \Pi^0$$

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$$= \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 + E^i \partial_i A_0 + \lambda \Pi^0$$

Hamilton formalism for Maxwell theory (cont'd)

Hamiltonian:

$$\mathcal{H} = \frac{1}{2}\boldsymbol{E}^2 + \frac{1}{2}\boldsymbol{B}^2 + E^i \partial_i A_0 + \lambda \Pi^0$$

$$\{A_i, E_j\}_P = \delta_{ij}$$

Consistency w/ constraint:

$$0 \simeq \dot{\Pi}^0 = \{\Pi^0, H\}_P = \partial_i E^i$$

new constraint

||

Gauss's law

$$\{\partial_i E^i, H\}_P = 0$$

no more constraint

Coupling to matters

Complex scalar:

$$\mathcal{L} = (\partial_\mu \bar{\phi})(\partial^\mu \phi) + V\left(\bar{\phi}\phi\right)$$

$U(1)$ global sym. : $\phi(x) \rightarrow e^{i\theta}\phi(x)$

Coupling to matters

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$U(1)$ global sym. : $\phi(x) \rightarrow e^{i\theta}\phi(x)$

Promotion to local: $\theta \rightarrow \theta(x)$

$$\partial_\mu \phi(x) \rightarrow \mathrm{e}^{i\theta}\left(\partial_\mu \phi + i(\partial_\mu \theta)\phi\right) \neq \mathrm{e}^{i\theta}\partial_\mu \phi$$

Coupling to matters

Complex scalar:

$$\mathcal{L} = (\partial_\mu \bar{\phi})(\partial^\mu \phi) + V\left(\bar{\phi}\phi\right)$$

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Introduction of gauge field:

$$\partial_\mu \phi \to D_\mu \phi \coloneqq \left(\partial_\mu - iA_\mu(x)\right)\phi \quad \text{w/ } A_\mu(x) \to A_\mu(x) + \partial_\mu \theta(x)$$

"covariant derivative" *gauge transformation!*

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Gauge invariant Lagrangian:

$$\mathcal{L} = (D_\mu \bar{\phi})(D^\mu \phi) + V\left(\bar{\phi}\phi\right)$$

Topological terms

For each dimension, $^{\exists}$particular gauge invariant term

Theta term (even dim. spacetime)

$$\frac{\theta}{4\pi}\int d^2x\ \epsilon^{\mu\nu}F_{\mu\nu}, \qquad \frac{\theta}{8\pi^2}\int d^4x\ \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \qquad \cdots$$

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Chern-Simons term (odd dim. spacetime)

$$k\int dx\, A, \quad \frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}, \quad \frac{k}{24\pi^2}\int d^3x\, \epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau}, \quad \cdots$$

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These are known to be

topological, parity odd & imaginary in Euclid spacetime

Ex. (1+1)d Maxwell theory w/ θ

Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu}$$

Hamiltonian on S^1_{L} :

(in temporal gauge $A_0 = 0$)

$$H = \frac{1}{2}\int_{S^1_L} dx \left(\Pi(\mathrm{x}) - \frac{\theta}{2\pi}\right)^2$$

$$\Pi = \frac{1}{\mathrm{g}^2}\dot{A} + \frac{\theta}{2\pi}$$

$\Big\downarrow$ Gauss law: $\partial_x \Pi(x) = 0$

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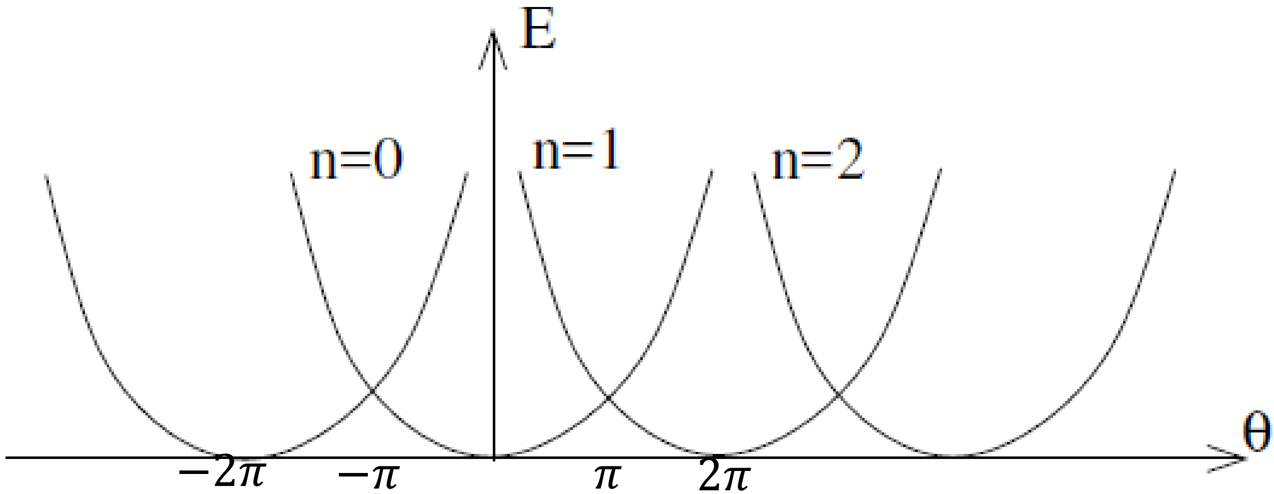
Energy eigenstates:

$$\Pi|n\rangle = n|n\rangle \ \ (n \in \boldsymbol{Z})$$

Vacuum structure of the (1+1)d Maxwell theory

Ground state energy: $E_0 = \frac{L}{2} \min_{n \in \boldsymbol{Z}} \left(n - \frac{\theta}{2\pi} \right)^2$

$H = \frac{L}{2} \left(\Pi - \frac{\theta}{2\pi} \right)^2$



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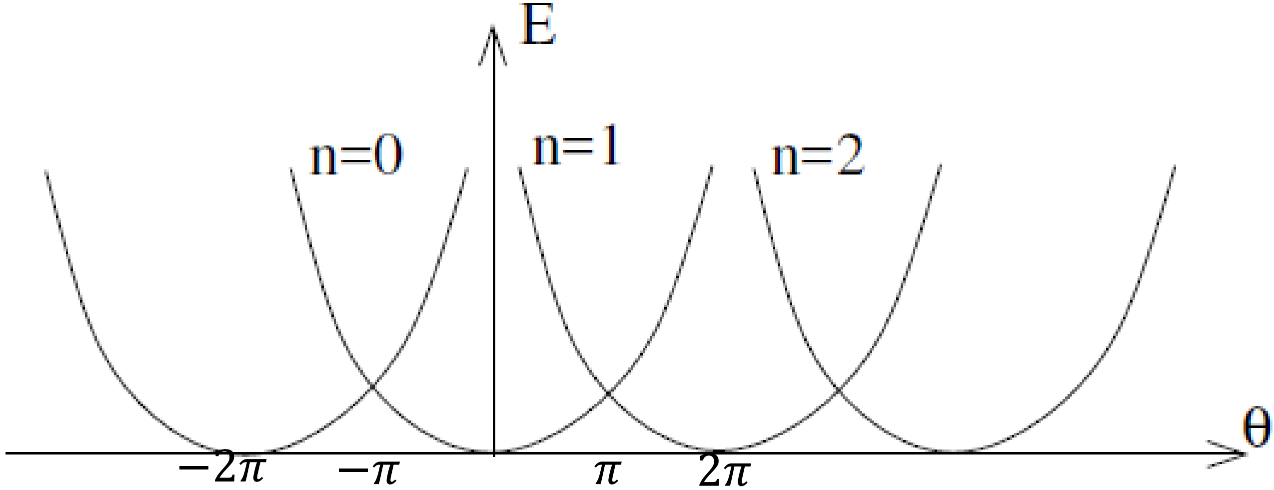

$^{\exists}$ two vacua at $\theta = \pi$!!

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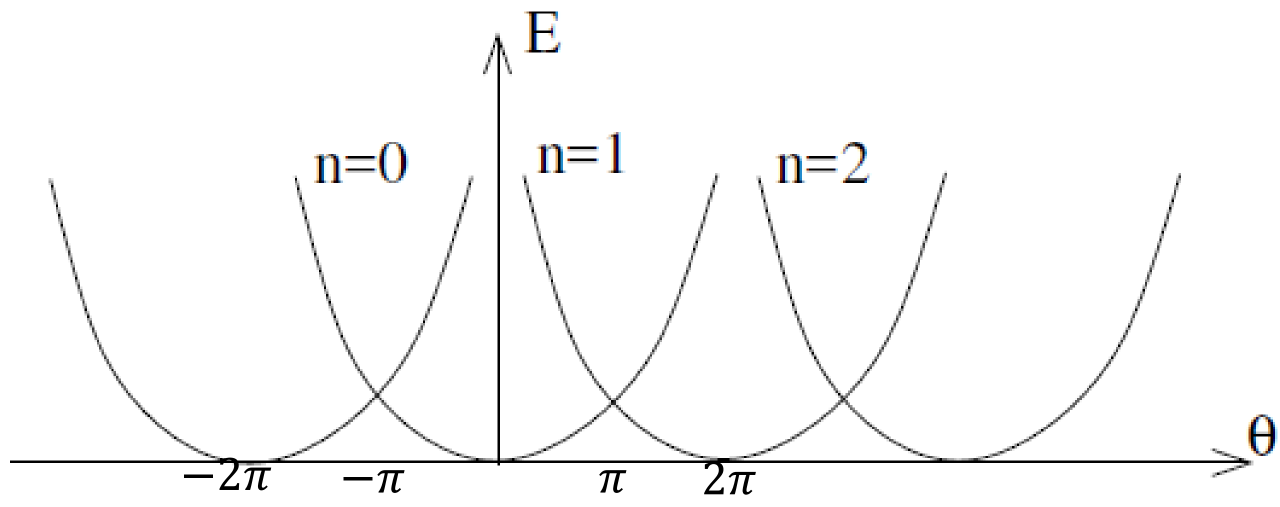

∃ two vacua at $\theta = \pi$!!

$$\Pi = \frac{1}{g^2} \dot{A} + \frac{\theta}{2\pi}$$

$$\langle F_{01} \rangle = \begin{cases} -\frac{g^2}{2} \text{ as } \theta \to \pi - 0 \\ +\frac{g^2}{2} \text{ as } \theta \to \pi + 0 \end{cases}$$

Parity (& charge conj.) is spontaneously broken!

Non-abelian gauge theory (Yang-Mills theory)

Gauge field:

$$A_\mu(x) = A_\mu^a(x) T^a \qquad (T^a\text{: generator of gauge group } G)$$

Gauge trans. :

$$A_\mu \to \Omega(x) A_\mu \Omega^{-1}(\mathrm{x}) + \mathrm{i}\Omega(x)\partial_\mu \Omega^{-1}(\mathrm{x}) \qquad \Omega(\mathrm{x}) \in G$$

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Field strength :

$$F_{\mu\nu} \coloneqq \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \qquad F_{\mu\nu} \to \Omega(x) F_{\mu\nu} \Omega^{-1}(\mathrm{x})$$

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Lagrangian :

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu})$$

Yang–Mills existence and mass gap

From Wikipedia, the free encyclopedia

The **Yang–Mills existence and mass gap problem** is an unsolved problem in mathematical physics and mathematics, and one of the seven Millennium Prize Problems defined by the Clay Mathematics Institute, which has offered a prize of US$1,000,000 for its solution.

Millennium Prize Problems
Birch and Swinnerton-Dyer conjecture
Hodge conjecture
Navier–Stokes existence and smoothness
P versus NP problem
Poincaré conjecture (solved)
Riemann hypothesis
Yang–Mills existence and mass gap
V · T · E

The problem is phrased as follows:[1]

> *Yang–Mills Existence and Mass Gap.* Prove that for any compact simple gauge group G, a non-trivial quantum Yang–Mills theory exists on $\mathbb{R}^4$ and has a mass gap $\Delta > 0$. Existence includes establishing axiomatic properties at least as strong as those cited in Streater & Wightman (1964), Osterwalder & Schrader (1973) and Osterwalder & Schrader (1975).

In this statement, a quantum Yang–Mills theory is a non-abelian quantum field theory similar to that underlying the Standard Model of particle physics; $\mathbb{R}^4$ is Euclidean 4-space; the mass gap Δ is the mass of the least massive particle predicted by the theory.

Therefore, the winner must prove that:

- Yang–Mills theory exists and satisfies the standard of rigor that characterizes contemporary mathematical physics, in particular constructive quantum field theory,[2][3] and
- The mass of all particles of the force field predicted by the theory are strictly positive.

For example, in the case of G=SU(3)—the strong nuclear interaction—the winner must prove that glueballs have a lower mass bound, and thus cannot be arbitrarily light.

The general problem of determining the presence of a spectral gap in a system is known to be undecidable.[4][5]

Background [edit]

> [...] one does not yet have a mathematically complete example of a quantum gauge theory in four-dimensional space-time, nor even a precise definition of quantum gauge theory in four dimensions. Will this change in the 21st century? We hope so!
>
> — From the Clay Institute's official problem description by Arthur Jaffe and Edward Witten.

Well known gauge theories in high energy physics

- Quantum Electrodynamics (QED)

$U(1)$ gauge field + charged fermion

photon *electron*

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$U(1)$ gauge field + charged fermion

photon electron

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$SU(3)$ gauge field + fundamental fermions

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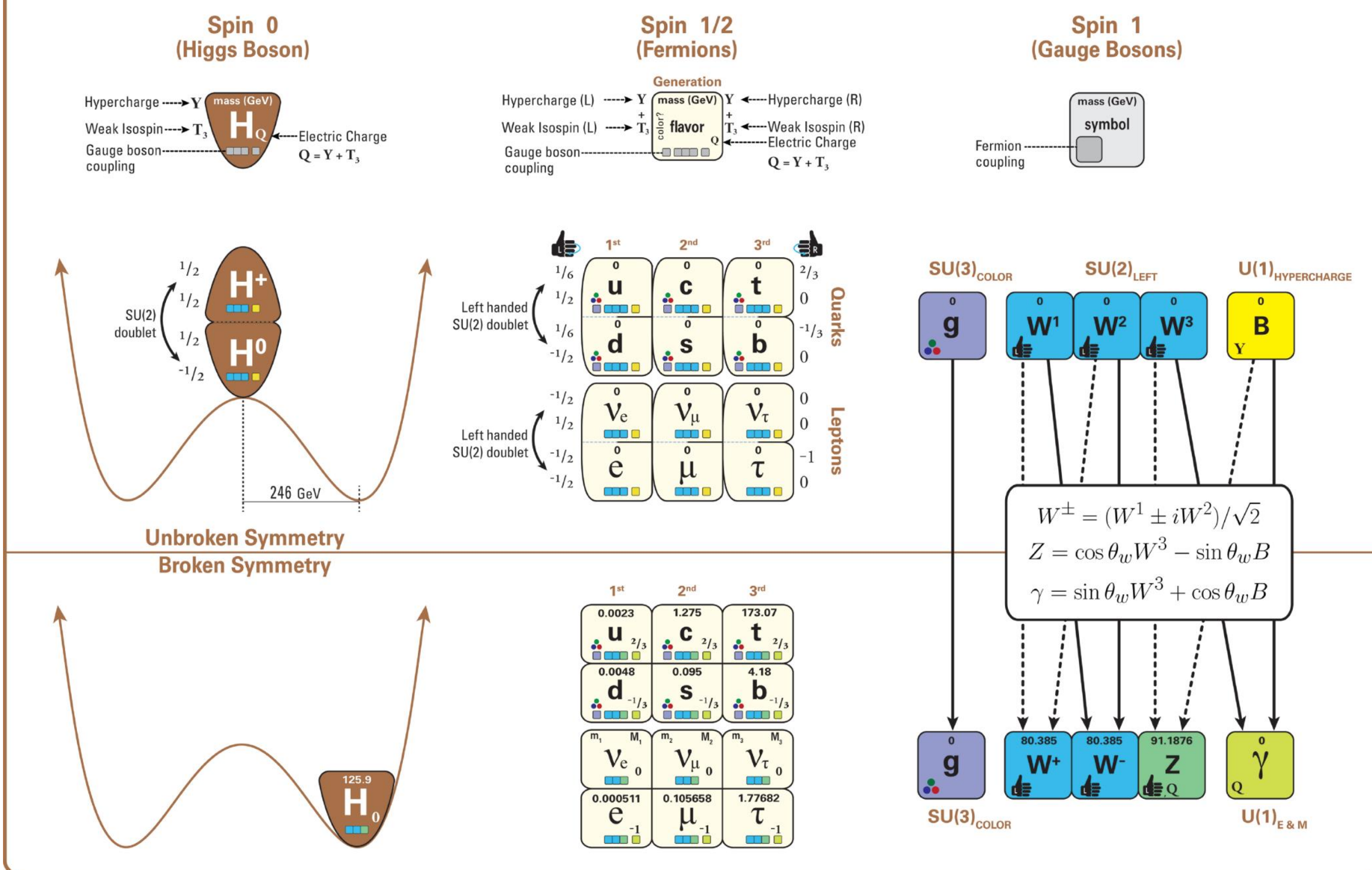
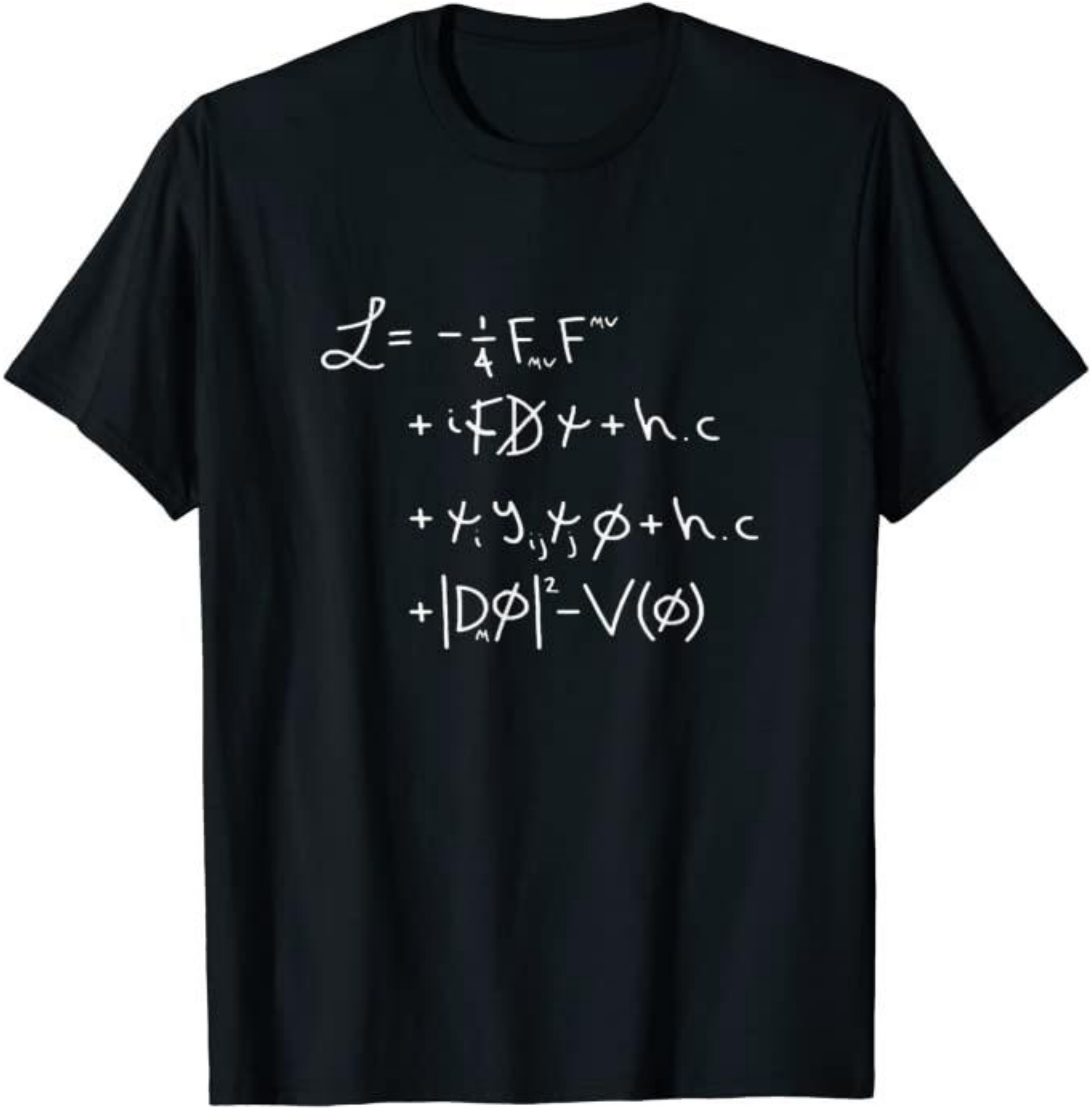
- Standard model

$$G = SU(3) \times SU(2) \times U(1)$$

+complicated combination of matters

The Standard Model of Particle Physics [wikipedia]




$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$
$+ i \bar{\psi} \not{D} \psi + h.c$
$+ \psi_i y_{ij} \psi_j \phi + h.c$
$+ |D_\mu \phi|^2 - V(\phi)$

Short summary

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{tr}(F_{\mu\nu}F^{\mu\nu}) \qquad F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$$

- gauge theory is constrained system

- topological terms:

$$\frac{\theta}{4\pi}\int d^2x\, \epsilon^{\mu\nu}F_{\mu\nu}, \quad \frac{k}{4\pi}\int d^3x\, \epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho}, \quad \frac{\theta}{8\pi^2}\int d^4x\, \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \quad \cdots$$

parity odd, imaginary in Euclid spacetime

- non-abelian gauge theory is difficult to solve…

but you also want $1,000,000 (?)

Plan

1. Introduction to Quantum field theory
2. (1+1)d scalar field theory
3. Gauge theory
4. Lattice field theory
5. Summary

Scalar field theory (continuum)

Action (Euclidean):

$$S[\phi] = \int d^d x \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + V(\phi)\right]$$

Vacuum expectation value can be given by path integral:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi \; \mathcal{O}(\phi) e^{-S[\phi]}}{\int D\phi \; e^{-S[\phi]}}$$

But it is ∞-dimensional & can't be simulated practically

coming from the fact that spacetime has ∞-many points!

Scalar field theory on lattice

Discretize the spacetime by lattice:

$\phi(x)$ $\qquad$ $\phi(x_n \equiv an)$

a: lattice spacing, $n \equiv (n_1, \cdots n_d)$

The simplest lattice action:

$$\int d^d x \to a^d \sum_n , \qquad \partial_\mu \phi(x) \to \Delta_\mu \phi(x_n) \equiv \frac{\phi(x_n + a e_\mu) - \phi(x_n)}{a}$$

$$S[\phi] = \int d^d x \left[\frac{1}{2} (\partial_\mu \phi)^2 + V(\phi) \right] \Longrightarrow a^d \sum_n \left[\frac{1}{2} \sum_\mu (\Delta_\mu \phi)^2 + V(\phi) \right]$$

Gauge invariant interaction on lattice

Let's consider complex scaler field theory

Continuum

$$\int d^d x \left[(\partial_\mu \bar{\phi})(\partial_\mu \phi) + V\left(\bar{\phi}\phi\right)\right]$$

$U(1)$ global sym. : $\phi(x) \to e^{i\theta}\phi(x)$

Promotion to gauge: $\theta \to \theta(x)$

But,

$$\partial_\mu \phi(x) \to \mathrm{e}^{i\theta}\left(\partial_\mu \phi + i(\partial_\mu \theta)\phi\right) \neq \mathrm{e}^{i\theta}\partial_\mu \phi$$

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w/ $A_\mu(x) \to A_\mu(x) + \partial_\mu \theta(x)$

Lattice

$$a^d \sum_n \left[\sum_\mu (\Delta_\mu \bar{\phi})(\Delta_\mu \phi) + V\left(\bar{\phi}\phi\right)\right]$$

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But,

$$\Delta_\mu \phi(x) \to \frac{e^{i\theta(x_n + ae_\mu)}\phi(x_n + ae_\mu) - e^{i\theta(x_n)}\phi(x_n)}{a}$$

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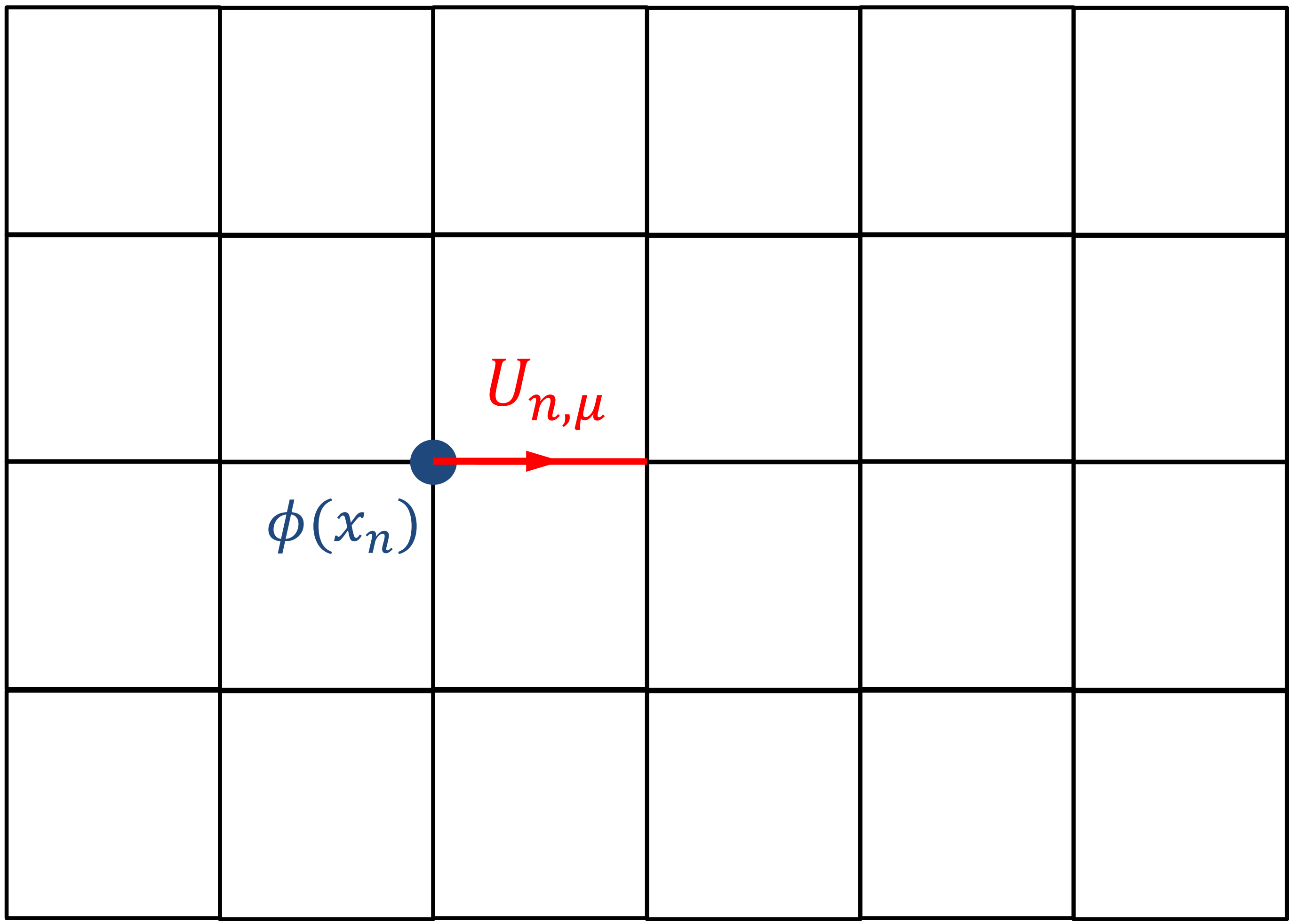
Introduction of "gauge field":

$$\Delta_\mu \phi(x) \to \frac{\phi\left(x_n + ae_\mu\right) - U_{n,\mu}\phi(x_n)}{a}$$

w/ $U_{n,\mu} \to e^{i\theta(x_n + ae_\mu)} U_{n,\mu} e^{-i\theta(x_n)}$

living on link between x_n & $x_n + ae_\mu$



$$\phi(x_n) \to e^{i\theta(x_n)}\phi(x_n), \quad U_{n,\mu} \to e^{i\theta(x_n+ae_\mu)}U_{n,\mu}e^{-i\theta(x_n)}$$

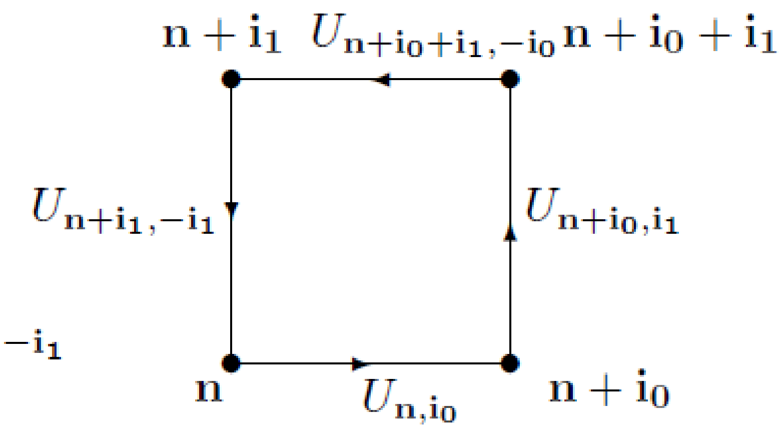
$$\Delta_\mu \phi(x) \to \frac{\phi\left(x_n + ae_\mu\right) - U_{n,\mu}\phi(x_n)}{a}$$

Lattice gauge theory ($G = SU(N)$)

Action:

$$S(U) = \sum_P \frac{1}{g^2} Tr\left(\prod_P U + H.c.\right)$$

$$\prod_P U = U_{\mathrm{n},\mathrm{i}_0} U_{\mathrm{n}+\mathrm{i}_0,\mathrm{i}_1} U_{\mathrm{n}+\mathrm{i}_0+\mathrm{i}_1,-\mathrm{i}_0} U_{\mathrm{n}+\mathrm{i}_1,-\mathrm{i}_1}$$



Gauge trans.:

$$U_{\boldsymbol{n},\boldsymbol{i}} \to V_{\boldsymbol{n}+\boldsymbol{i}} U_{\boldsymbol{n},\boldsymbol{i}} V_{\boldsymbol{n}}^{\dagger} \qquad V_{\boldsymbol{n}} \in SU(N)$$

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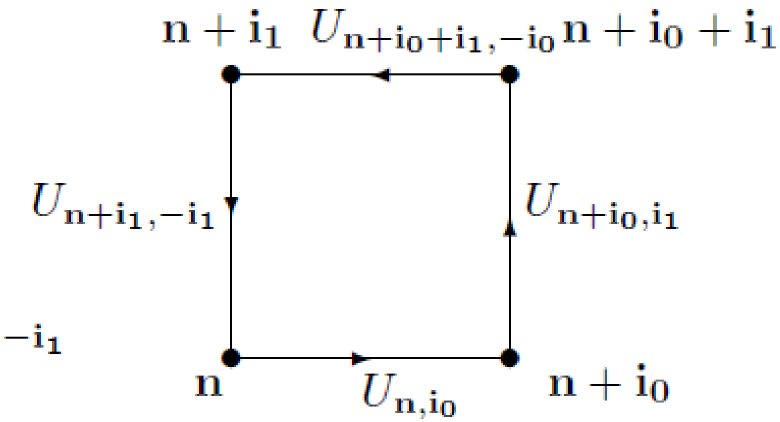

Gauge trans.:

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"Path integral" :

$$Z := \int [DU]\, e^{-S[U]} \qquad DU \equiv \prod_{\mathrm{n},\mathrm{i}} dU_{\mathrm{n},\mathrm{i}}$$

Haar measure

$$\langle \mathcal{O}(U) \rangle := \frac{1}{Z} \int [DU] \mathcal{O}(U)\, e^{-S[U]}$$

Conventional approach to simulate QFT

① Discretize Euclidean spacetime by lattice:

& make path integral finite dimensional:

$$\int D\phi\ \mathcal{O}(\phi)e^{-S[\phi]} \quad \Longrightarrow \quad \int d\phi\ \mathcal{O}(\phi)e^{-S(\phi)}$$

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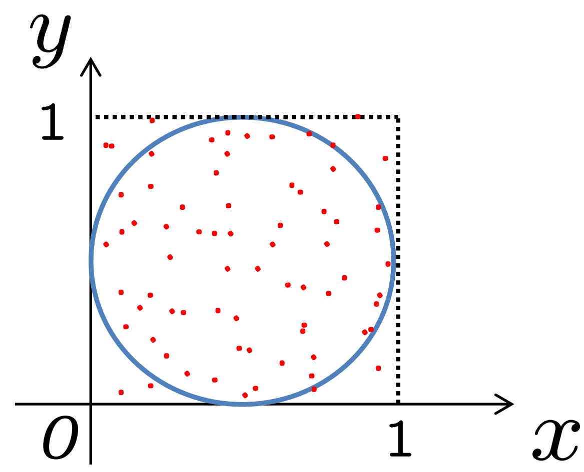
② Numerically Evaluate it by Markov chain Monte Carlo method regarding the Boltzmann factor as a probability:

$$\langle \mathcal{O}(\phi) \rangle \simeq \frac{1}{\sharp(\text{samples})} \sum_{i \in \text{samples}} \mathcal{O}(\phi_i)$$

“Direct” Monte Carlo method

Ex.) The area of the circle with the radius 1/2



① Distribute random numbers many times

$$x \in [0,1),\ \ y \in [0,1)$$

② Count the number of points which satisfy

$$\left(x-\frac{1}{2}\right)^2+\left(y-\frac{1}{2}\right)^2 \leq \frac{1}{4}$$

③ Estimate the ratio

$$\frac{(\text{Number of points inside the circle})}{(\text{Numer of points for distribution})} \simeq (\text{Area})$$

Markov chain Monte Carlo method

Consider a Markov process w/ transition probability $P^{(a)}$:

$$x^{(0)} \to x^{(1)} \to \cdots \to \cdots \to x^{(M-1)} \to x^{(M)} \to \cdots$$

$P^{(1)}(x^{(0)}, x^{(1)})$ $P^{(2)}(x^{(1)}, x^{(2)})$ $P^{(M)}(x^{(M-1)}, x^{(M)})$

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Under some conditions,

transition prob. converges to an equilibrium prob.

$$\lim_{M\to\infty} P^{(M)}(x^{(M-1)}, x^{(M)}) = P_{eq}(x^{(M)})$$ *thermalization*

Markov chain Monte Carlo method

Consider a Markov process w/ transition probability $P^{(a)}$:

$$x^{(0)} \underset{P^{(1)}(x^{(0)},x^{(1)})}{\rightarrow} x^{(1)} \underset{P^{(2)}(x^{(1)},x^{(2)})}{\rightarrow} \cdots \rightarrow \cdots \rightarrow x^{(M-1)} \underset{P^{(M)}(x^{(M-1)},x^{(M)})}{\rightarrow} x^{(M)} \rightarrow \cdots$$

Under some conditions,

transition prob. converges to an equilibrium prob.

$$\lim_{M\to\infty} P^{(M)}(x^{(M-1)}, x^{(M)}) = P_{eq}(x^{(M)}) \quad \textit{thermalization}$$

We can compute exp. values by an algorithm to generate

$$P_{eq}(x) \propto e^{-S(x)}$$

Ex.) Gaussian ensemble by heat bath algorithm

$$\langle O(x,y)\rangle = \frac{\int dxdy\ O(x,y)P(x,y)}{\int dxdy\ P(x,y)} \qquad P(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

①

②

Ex.) Gaussian ensemble by heat bath algorithm

$$\langle O(x,y)\rangle = \frac{\int dxdy\ O(x,y)P(x,y)}{\int dxdy\ P(x,y)} \qquad P(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

① Generate random configurations with Gaussian weight many times

$$\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy = \frac{1}{2\pi}re^{-\frac{r^2}{2}}drd\theta = d\xi d\eta \qquad (\eta, \xi \in [0,1))$$

$$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \qquad \begin{pmatrix} \theta = 2\pi\eta \\ r = \sqrt{-2\log\xi} \end{pmatrix}$$

The uniform random numbers generate the Markov chain:

$$(x^{(0)}, y^{(0)}) \to \underset{P(x^{(1)},y^{(1)})}{(x^{(1)}, y^{(1)})} \to \cdots\cdots \to \underset{P(x^{(M)},y^{(M)})}{(x^{(M)}, y^{(M)})}$$

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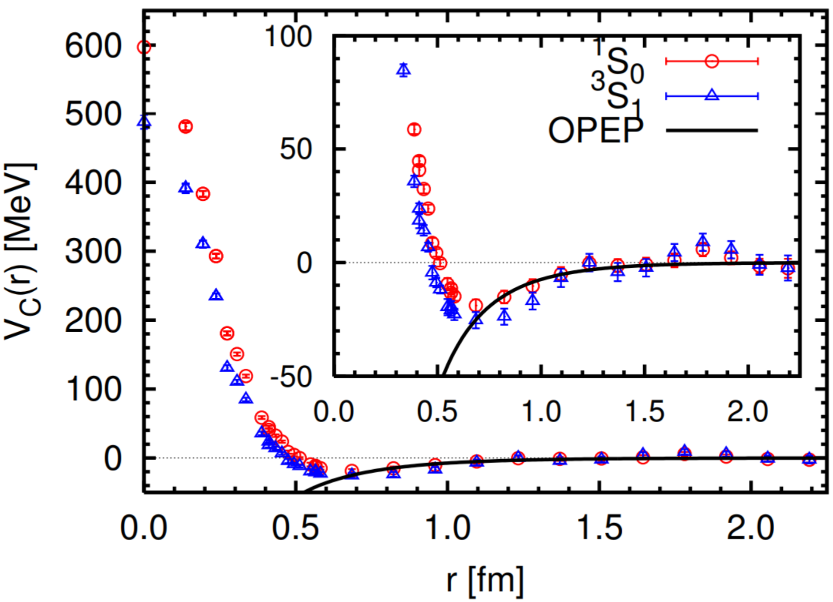
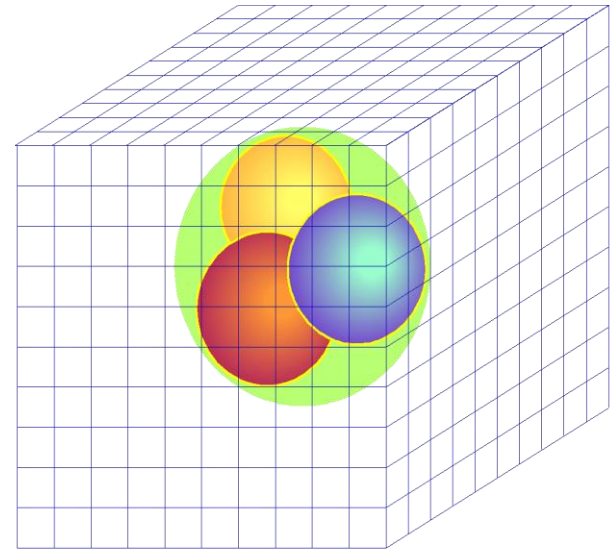
② Measure observable and take its average:

$$\frac{1}{M}\sum_{a=1}^{M} O(x^{(a)},y^{(a)}) \simeq \langle O(x,y)\rangle$$

Success of lattice QCD (e.g. nuclear force)





Nuclear Force from Lattice QCD

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Nucleon-nucleon (NN) potential is studied by lattice QCD simulations in the quenched approximation, using the plaquette gauge action and the Wilson quark action on a 32^4 ($\simeq$ (4.4 fm)^4) lattice. A NN potential $V_{\rm NN}(r)$ is defined from the equal-time Bethe-Salpeter amplitude with a local interpolating operator for the nucleon. By studying the NN interaction in the ^{1}S$_0$ and ^{3}S$_1$ channels, we show that the central part of $V_{\rm NN}(r)$ has a strong repulsive core of a few hundred MeV at short distances ($r \lesssim 0.5$ fm) surrounded by an attractive well at medium and long distances. These features are consistent with the known phenomenological features of the nuclear force.

Sign problem in Monte Carlo simulation

Markov Chain Monte Carlo:

$$\int d\phi \ \mathcal{O}(\phi)\underbrace{e^{-S(\phi)}}_{\text{probability}}$$

can't directly apply when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$

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Naïve way to avoid = reweighting:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi\ \mathcal{O}(\phi) e^{-S[\phi]}}{\int D\phi\ e^{-S[\phi]}}$$

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$$= \frac{\langle \mathcal{O}(\phi) \cdot \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}{\langle \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}$$

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$$= \frac{\langle \mathcal{O}(\phi)\cdot\mathrm{phase}(e^{-S})\rangle_{\mathrm{no-phase}}}{\langle \mathrm{phase}(e^{-S})\rangle_{\mathrm{no-phase}}}$$

For highly oscillating integral, $\sim \frac{0}{0}$ $\Rightarrow$ needs huge statistics

"sign problem"

Sign problem in Monte Carlo simulation (cont'd)

Markov Chain Monte Carlo:

$$\int d\phi\ \mathcal{O}(\phi)\underbrace{e^{-S(\phi)}}_{\text{probability}}$$

problematic when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$ & is highly oscillating

Examples w/ sign problem:

- topological term —— complex action
- chemical potential —— indefinite sign of fermion determinant
- real time —— " $e^{iS(\phi)}$ " *much worse*

In operator formalism suitable for quantum simulation,

sign problem is absent from the beginning

(∃ various approaches within framework of path integral formalism but I'll skip it)

Summary

Summary

- Quantum field theory is interesting
  but typically difficult to solve analytically

- To do numerical analysis, we have to regularize
  QFT in some way

- Conventional approach is to cut spacetime
  by lattice & numerically evaluate path integral

- Monte Carlo method suffers from sign problem
  in some important situations

  (e.g. systems w/ real time, chemical potential, topological term)

Here is the end of lecture 3 !

Appendix

Ordinary continuous symmetry (modern way)

Conserved current (1-form): [Gaiotto-Kapustin-Seiberg-Willett '04]

$$d * J^{(1)} = 0$$

Conserved charge:

$$Q(M_{d-1}) = \oint_{M_{d-1}} * J^{(1)}$$

codimension 1 closed manifold

Trans. generated by Q ($U(1)$ case):

$$U = \exp(i\alpha Q)$$

α: closed 0-form i.e. constant

$$U\phi(x)U^{-1} = e^{iq\alpha}\phi(x)$$

acting on local op. (=0-dim. object)

q-form continuous symmetry

Conserved current ($q+1$-form): [Gaiotto-Kapustin-Seiberg-Willett '04]

$$d * J^{(q+1)} = 0$$

Conserved charge :

$$Q^{(q)}(M_{d-q-1}) = \oint_{M_{d-q-1}} * J^{(q+1)}$$

codimension $(q+1)$ closed manifold

Trans. generated by the charge:

$$U = \exp(iQ), \qquad Q = \oint_{M_{d-1}} \alpha^{(q)} \wedge * J^{(q+1)}$$

closed q-form

$$U \, V_{\bar{q}} \, U^{-1} = e^{i\bar{q}\alpha} \cdot V_{\bar{q}}$$

acting on q-dim. object

Ex. 4d Maxwell theory

Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F^{(2)} \wedge * F^{(2)}$$

Identities:

- E.o.m. $\quad d * F^{(2)} = 0$
- Bianchi id. $\quad dF^{(2)} = 0$

These are interpreted as the current conservations:

- $d * J_e^{(2)} = 0, \quad J_e^{(2)} = \frac{2}{g^2} F^{(2)}$ electric 1-form symmetry
- $d * J_m^{(2)} = 0, \quad J_m^{(2)} = \frac{1}{2\pi} * F^{(2)}$ magnetic 1-form symmetry

Electric $U(1)$ 1-form symmetry

Conserved charge:

$$J_e^{(2)} = \frac{2}{g^2} F^{(2)}, \qquad Q_e^{(1)}(M_2) = \oint_{M_2} * J_e^{(2)}$$

codimension 2 closed manifold

Trans. generated by the charge:

$$U = \exp(iQ_e), \qquad Q_e = \oint_{M_3} \alpha^{(1)} \wedge * J_e^{(2)}$$

closed 1-form

$$U\, V_{q_e} U^{-1} = e^{iq_e\alpha} \cdot V_{q_e}$$

acting on 1-dim. object i.e. line

In particular, it transforms the Wilson line as

$$W = e^{i\int A} \to \; e^{i\alpha} W$$

Magnetic $U(1)$ 1-form symmetry

Conserved charge:

$$J_m^{(2)} = \frac{1}{2\pi} * F^{(2)}, \quad Q_m^{(1)}(M_2) = \oint_{M_2} * J_m^{(2)}$$

codimension 2 closed manifold

Trans. generated by the charge:

$$U = \exp(iQ_m), \qquad Q_m = \oint_{M_3} \alpha^{(1)} \wedge * J_m^{(2)}$$

closed 1-form

$$U\, V_{q_m} U^{-1} = e^{i q_m \alpha} \cdot V_{q_m}$$

acting on 1-dim. object i.e. line

In particular, it transforms the ’t Hooft line as

$$H \to \ e^{i\alpha} H$$

Ex. 3d Maxwell theory

Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F^{(2)} \wedge * F^{(2)}$$

Identities:

- E.o.m. $\quad d * F^{(2)} = 0$
- Bianchi id. $\quad dF^{(2)} = 0$

These are interpreted as the current conservations:

- $d * J_e^{(2)} = 0, \quad J_e^{(2)} = \frac{2}{g^2} F^{(2)}$ electric 1-form symmetry
- $d * J_m^{(1)} = 0, \quad J_m^{(1)} = \frac{1}{2\pi} * F^{(2)}$ magnetic 0-form symmetry

Ex. 2d Maxwell theory

Lagrangian:

$$\mathcal{L} = -\frac{1}{4g^2} F^{(2)} \wedge * F^{(2)}$$

Identity:

E.o.m. $$d * F^{(2)} = 0$$

This is interpreted as the current conservation:

$$d * {J_e}^{(2)} = 0, \qquad J_e^{(2)} = \frac{2}{g^2} F^{(2)}$$

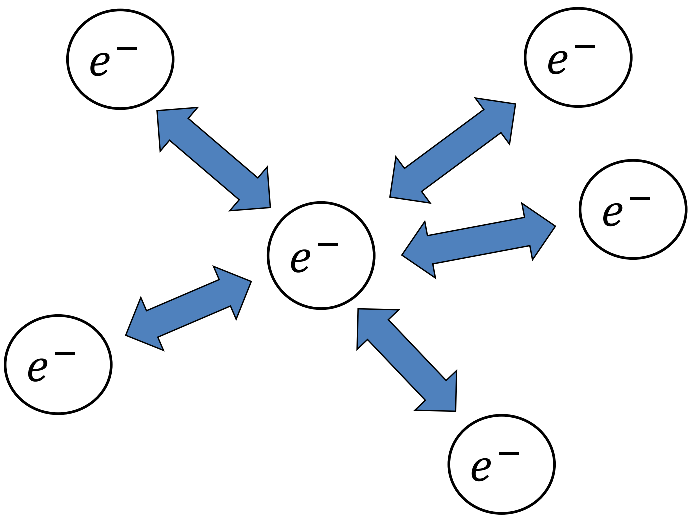
electric 1-form symmetry

No magnetic symmetry

Why perturbative series is not convergent

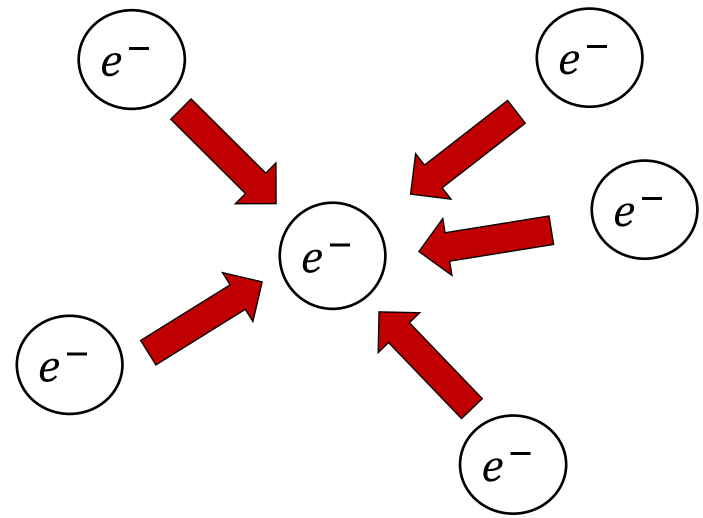
～Dyson's original argument (very rough)～ [Dyson '52]

World w/ $e^2 > 0$



repulsive

World w/ $e^2 < 0$



attractive, prefer to be dense

looks qualitatively different $\Longrightarrow$ non-analytic?

Why perturbative series is not convergent

～technical reasons～

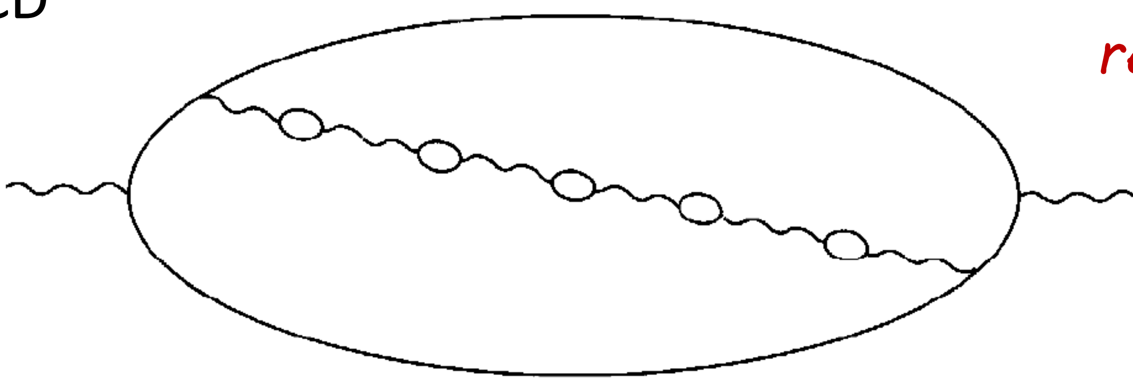
① (# of n-loop Feynmann diagrams) ～ n!

proliferation

② $^{\exists}$Feynmann diagrams contributing by ～n!

Ex.) QCD

renormalon



[Fig.20.3 in Weinberg's book,
cf. Takaura-san's lectures]

Hybrid Monte Carlo algorithm

[Duane-Kennedy-Pendleton-Roweth '87] [Cf. Rothe, Aoki's textbooks]

$$\langle O(x)\rangle = \frac{\int d^N x\ O(x)e^{-S(x)}}{\int d^N x\ e^{-S(x)}} = \frac{\int d^N x d^N p\ O(x) e^{-\sum_i \frac{p_i^2}{2} + S(x)}}{\int d^N x d^N p\ e^{-\sum_i \frac{p_i^2}{2} + S(x)}}$$

Regard as the "conjugate momentum"

① Take an initial condition freely

② Generate the momentum with Gaussian weight

③ Solve "Molecular dynamics" "Hamiltonian": $H = \sum_i \frac{p_i^2}{2} + S(x)$

$$\frac{dx_i(\tau)}{d\tau} = \frac{\partial H(x(\tau), p(\tau))}{\partial p_i(\tau)} = p_i(\tau), \quad \frac{dp_i(\tau)}{d\tau} = -\frac{\partial H(x(\tau), p(\tau))}{\partial x_i(\tau)} = -\frac{\partial S(x(\tau))}{\partial x_i(\tau)}$$

$$(\ x(0),\ p(0)\ ) = (x, p), \qquad (\ x(\tau_f),\ p(\tau_f)\ ) = (x', p')$$

④ Metropolis test $\Delta H = H(x', p') - H(x, p)$

If $\Delta H < 0$, $(x,p) \to (x', p')$ accepted

If $\Delta H > 0$,

$$\begin{cases} (x,p) \to (x',p') & \text{accepted with prob. } e^{-\Delta H} \\ (x,p) \to (x,p) & \text{rejected with prob. } 1 - e^{-\Delta H} \end{cases}$$