Recent progress on five dimentional QFT

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1. Introduction

CFT is important!!

- string theory (2d)
- critical phenomena
- holography
- Start & goal of RG flows (for most of theories)

We are string theorists!

much harder to find non-free CFT in $d \geq 5$

- no SCFT in $d \geq 7$ (from superconformal algebra)
- no known CFT in $d \geq 7$
- $\exists$ many SCFTs in $d = 5, 6$
- $\exists$ very few known non-SUSY CFT in $d = 5$ even at conjectural level!!

↑

theme of this lecture!

Difficulty in finding 5d CFT

$\because$ most Lagrangian theories are perturbatively non-renormalizable

(flow to free th. in IR)

<u>scalar</u>

$$S = \int d^5x \left[\frac{1}{2} (\partial_\mu \phi)^2 - V(\phi) \right]$$

$$[\partial_\mu \phi]^2 = [m]^5 \rightarrow [\phi] = [m]^{\frac{3}{2}}$$

$$\Rightarrow \underbrace{[\phi^2] = [m]^3}_{\checkmark}, \quad \underbrace{[\phi^3] = [m]^{\frac{9}{2}}}_{\checkmark}, \quad \left. \begin{array}{l} [\phi^4] = [m]^6, \cdots \\ \text{non-renormalible} \end{array} \right.$$

<u>fermion</u>

$$S = \int d^5x \left[\bar{\Psi} i \not\partial \Psi + \cdots \right]$$

$$[\bar{\Psi} \not\partial \Psi] = [m]^5 \rightarrow [\Psi] = [m]^2$$

$$[\bar{\Psi}\Psi] = [m]^4 \checkmark \quad \left\{ \begin{array}{l} [(\bar{\Psi}\Psi)^2] = [m]^8 \\ [\phi \bar{\Psi}\Psi] = [m]^{\frac{11}{2}} \\ \vdots \end{array} \right.$$

Gauge field

- Yang-Mills term

$$S_{YM} \propto \frac{1}{g^2} \int d^5x \; \mathrm{tr}(f_{\mu\nu})^2$$

$$f_{\mu\nu} \equiv \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$$

$$[a_\mu] = [m]^1, \quad [(f_{\mu\nu})^2] = [m]^4$$

$$\rightarrow \quad [g] = [m]^{-1/2} \quad \text{non-renormalizable!}$$

- Chern-Simons term

$$S_{CS} \propto ik \int a \wedge f \wedge f \qquad (G = U(1))$$

$$\left(\propto ik \int d^5x \; \varepsilon^{\mu\nu\rho\sigma\tau} a_\mu f_{\nu\rho} f_{\sigma\tau} \right)$$

$$[k] = [m]^0 \quad \text{marginal!}$$

gauge theories w/ matters are
pert. non-renormalizable!

Comments

- "ϕ^3 theory" may have nontrivial IR f.p.
  (although potential is unstable)

- $\exists$ conjecture: [Fei-Giombi-Klebanov '14]
  5d $O(N)$ model has IR f.p. $(N \gg 1)$
  $$\left(\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^i)^2 + \frac{1}{2}(\partial_\mu \sigma)^2 + \frac{g_1}{2}\sigma \phi^i \phi^i + \frac{g_2}{6}\sigma^3 \right)$$
  [bootstrap: Nakayama-Ohtsuki '14]

- There might be UV finite case
  because of cancellation of diagrams (by huge symmetry)
  But, even maximal SYM in large-N limit has
  ∞ at 6-loop
  [Bern-Carrasco-Dixon-Douglas-Hippel-Johansson '12]

Basically,

5d gauge theory makes sense

only as effective theory

Goal:

show evidence for $\exists$ 5d non-SUSY CFT
which is UV completion of $SU(2)$ YM
[Genolini - MH - Kim - Tong - Vafa '20]

2. Preliminary : 5d gauge theory

2-0. differential form

rank-p tensor :

$$T^{(p)} = T^{(p)}_{\mu_1 \cdots \mu_p} dx^{\mu_1} \otimes dx^{\mu_2} \otimes \cdots \otimes dx^{\mu_p}$$

wedge product :

anti-symmetric !

$$dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

$$= \sum_{\sigma \in \text{permutation}} \text{sgn}(\sigma)\, dx^{\mu_{\sigma(1)}} \otimes dx^{\mu_{\sigma(2)}} \otimes \cdots \otimes dx^{\mu_{\sigma(p)}}$$

(ex.) $dx^\mu \wedge dx^\nu = dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu$)

rank-p anti-sym. tensor : "p-form"

$$A^{(p)} = \frac{1}{p!} A^{(p)}_{\mu_1 \cdots \mu_p} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_p}$$

$$(A^{(p)} \wedge B^{(q)})_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\, q!} A^{(p)}_{[\mu_1 \cdots \mu_p} B^{(q)}_{\mu_{p+1} \cdots \mu_{p+q}]}$$

$$A^{(p)} \wedge B^{(q)} = (-1)^{pq} B^{(q)} \wedge A^{(p)}$$

$$\left(A^{(p)} \wedge A^{(p)} = 0 \quad \text{for odd } p \right)$$

exterior derivative :

$$[dA^{(p)}]_{\mu_1 \cdots \mu_{p+1}} = (p+1)\, \partial_{[\mu_1} A^{(p)}_{\mu_2 \cdots \mu_{p+1}]}$$

$$\left(\text{ex. } \begin{aligned} dA^{(1)} &= \frac{1}{2!} [dA^{(1)}]_{\mu\nu}\, dx^\mu \wedge dx^\nu \\ &= \partial_\mu A^{(1)}_\nu\, dx^\mu \wedge dx^\nu \\ &= (\partial_\mu A^{(1)}_\nu - \partial_\nu A^{(1)}_\mu)\, dx^\mu \otimes dx^\nu \end{aligned} \right)$$

$$d^2 = 0$$

$$d(A^{(p)} \wedge B^{(q)}) = dA^{(p)} \wedge B^{(q)} + (-1)^p A^{(p)} \wedge dB^{(q)}$$

Integrations :

$$\int_{M_d} d^dx\, A^{(d)}_{01\cdots d\text{-}1} \equiv \int_{M_d} A^{(d)}$$ coordinate inv.

→ convenient to guess topological inv.

$$\int_{M_d} dA^{(d-1)} = \int_{\partial M_d} A^{(d-1)}$$ Stokes's theorem

Hodge star :

$$*\left(dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\right)$$

$$=\frac{\sqrt{|g|}}{(d-p)!}\,\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_d}\,dx^{\nu_{p+1}}\wedge\cdots\wedge dx^{\nu_d}$$

requires metric!

$$*A^{(p)}=\frac{\sqrt{|g|}}{p!\,(d-p)!}\,A^{(p)}_{\mu_1\cdots\mu_p}\,\varepsilon^{\mu_1\cdots\mu_p}{}_{\nu_{p+1}\cdots\nu_d}\,dx^{\nu_{p+1}}\wedge\cdots\wedge dx^{\nu_d}$$

$$**=(-1)^{p(d-p)+1}$$

ex. In $d=4$, for $F^{(2)}=da$

$$*F^{(2)}=\frac{\sqrt{|g|}}{4}\,F^{(2)}_{\mu\nu}\,\varepsilon^{\mu\nu}{}_{\rho\sigma}\,dx^\rho\wedge dx^\sigma$$

$$\left(*F^{(2)}\right)_{0i}\propto\varepsilon_{ijk}F^{(2)\,jk}$$

$$\varepsilon^{ijk}\left(*F^{(2)}\right)_{jk}\propto F^{(2)\,0i}$$

more precisely

$$\mathbb{E}\overset{*}{\longleftrightarrow}\mathbb{B}\qquad *F=(\mathbb{B},-\mathbb{E})$$

2-1. U(1) gauge theory

kinetic term

$$S_{kin} = -\frac{1}{4g^2}\int_{M_5} d^5x \sqrt{|g|}\; f_{\mu\nu}^2$$

$$(f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu)$$

gauge trans. : $\delta a_\mu = \partial_\mu \lambda(x)$

Using differential form,

$$f = da, \qquad \delta a = d\lambda$$

$$*f = \frac{1}{2!\,3!} f_{\mu\nu}\, \varepsilon^{\mu\nu}{}_{\rho\sigma\tau}\, dx^\rho \wedge dx^\sigma \wedge dx^\tau$$

$$S_{kin} = -\frac{1}{2g^2}\int_{M_5} f \wedge * f$$

Chern-Simons (CS) term:

$$S_{CS} = \frac{k}{24\pi^2}\int_{M_5} a \wedge f \wedge f \quad (k: \text{CS level})$$

"small" gauge trans.:

$$\delta S_{CS} = \frac{k}{24\pi^2}\int_{M_5} d\lambda \wedge f \wedge f = \frac{k}{24\pi^2}\int_{M_5} d(\lambda \wedge f \wedge f)$$

$$= 0 \quad (\text{w/o bdy.})$$

not invariant under "large" gauge trans.

$$\left(\begin{array}{l} \text{ex. for } M_5 = S^1_\beta \times M_4 \\ \qquad A_0 \longrightarrow A_0 + \frac{2\pi}{\beta} \\ \qquad S_{CS} \longrightarrow S_{CS} + \frac{k}{4\pi}\int_{M_4} F \wedge F \end{array}\right)$$

for spin M_4 $\quad \frac{1}{4\pi}\int_{M_4} F \wedge F = 2\pi\mathbb{Z}$

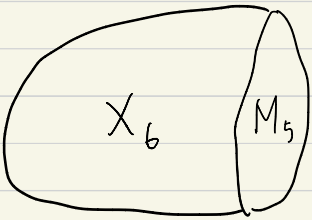
But, it is enough to take k s.t.

$$S_{CS} \rightarrow S_{CS} + 2\pi\mathbb{Z}$$

$$\left(\because \; e^{iS_{CS}} \rightarrow e^{iS_{CS}}\right)$$

Condition for k ?

Extends the space to 6d



$\partial X_6 = M_5$

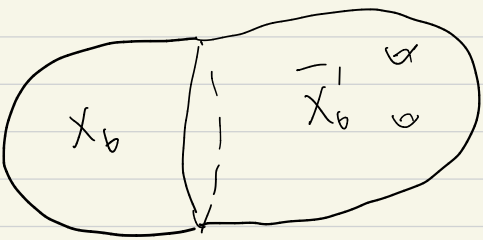
6d θ term

$$S_{cs}[a;X_6] = \frac{k}{24\pi^2}\int_{X_6} d(a\wedge f\wedge f) = \frac{k}{24\pi^2}\int_{X_6} f\wedge f\wedge f$$

gauge inv. but may depend on choice of X_6 ?

→ compare w/ another choice X_6' ($\partial X_6' = M_5$)

$$S_{cs}[a;X_6] - S_{cs}[a;X_6'] = \underbrace{\frac{k}{24\pi^2}\int_{Y_6} f\wedge f\wedge f}_{=\ 2\pi\mathbb{Z}\ ?}$$



Y_6

[cf. Witten '96]

It is known

$$\frac{1}{4\pi^2}\int_{Y_6} f\wedge f\wedge f = \begin{cases} 2\pi\mathbb{Z} & \text{generic closed oriented } Y_6 \\ 12\pi\mathbb{Z} & \text{"usually" (spin, } p_1=0) \end{cases}$$

For $k\in\mathbb{Z}$, gauge inv.

mixed CS term

When we have multiple $U(1)$ sym. : $U(1)_1\times U(1)_2\times\cdots$ gauge & global

$$\sum_{a,b,c}\frac{k_{abc}}{24\pi^2}\int_{M_5} A^a\wedge F^b\wedge F^c$$

(take $k_{abc}\in\mathbb{Z}$)

Symmetries

- Charge Conjugation

$$C: a \to -a \,, \quad S_{CS} \to -S_{CS} \qquad \cancel{C}$$

- Parity

textbook one is not appropriate for odd d

($\because \det(-1) = 1$)

Take $P: \ x^\mu \to \begin{cases} x^\mu & \mu \neq 1 \\ -x^\mu & \mu = 1 \end{cases}$

$$a^\mu(x) \to \begin{cases} a^\mu(Px) & \mu \neq 1 \\ -a^\mu(Px) & \mu = 1 \end{cases}$$

$$S_{CS} \to -S_{CS} \qquad \cancel{P}$$

- Time reversal

$$T: \ x^0 \to -x^0$$

$$a^0(x) \to a^0(Tx), \quad a^i(x) \to -a^i(Tx)$$

$$S_{CS} \to S_{CS}$$

(cf. In 3d, $C \ \cancel{P} \ \cancel{T}$)

- topological sym.

$$J^{top} = \frac{1}{8\pi^2} * (f \wedge f)$$

$$\partial_\mu (J^{top})^\mu = 0 \quad (d * J^{top} = 0)$$

$$Q^{top} \equiv \int_{\Sigma_4} * J^{top} = \frac{1}{8\pi^2} \int_{\Sigma_4} f \wedge f$$

"instanton sym.", "$U(1)_I$ sym."

We can have mixed CS terms between $U(1)_{gauge}$ & $U(1)_I$

$$\frac{k_{ggI}}{8\pi^2} \int A_I \wedge f \wedge f$$

$$\frac{k_{gII}}{8\pi^2} \int a \wedge F_I \wedge F_I$$

magnetic dual

$$a^{(1)} \xrightarrow{d} f^{(2)} \xrightarrow{*} \hat{f}^{(d-2)} \xleftarrow{d} b^{(d-3)}$$

$$\begin{cases} b^{(0)} & d=3 \text{ dual photon} \\ b^{(1)} & d=4 \text{ monopole (particle)} \\ b^{(2)} & d=5 \text{ monopole (string)} \end{cases}$$

brief look:

Lagrange multiplier

$$S \sim \frac{1}{2g^2} f \wedge * f + b^{(2)} \wedge df$$

e.o.m. for $b^{(2)}$: $df = 0$ $\rightarrow f = da$

$$\rightarrow S \sim \frac{1}{2g^2} da \wedge * da$$

e.o.m. for f : $\frac{1}{g^2} * f - db^{(2)} = 0$

$$\rightarrow S \sim \frac{g^2}{2} db^{(2)} \wedge * db^{(2)}$$

kinetic term of $b^{(2)}$

w/ $g^2 \rightarrow \sim \frac{1}{g^2}$

2-2. SU(2) gauge theory

$$a = a_\mu^p \tau^p dx^\mu \qquad (p = 1, 2, 3)$$

$$\tau^p = \frac{\sigma^p}{2}, \quad [\tau^p, \tau^q] = i\varepsilon^{pqr}\tau^r, \quad \mathrm{tr}(\tau^p\tau^q) = \frac{\delta^{pq}}{2}$$

kinetic term

$$S_{kin} = \frac{1}{2g^2}\int_{M_5} \sqrt{|g|}\, \mathrm{tr}(f_{\mu\nu})^2 = \frac{1}{2g^2}\int_{M_5} \mathrm{tr}(f \wedge * f)$$

$$f = da - i \underbrace{a \wedge a}_{\neq 0 \text{ since it's matrix!}}$$

gauge trans. :

$$a \longrightarrow \Omega a \Omega^{-1} + i \Omega d\Omega^{-1} \qquad \Omega(x) \in su(2)$$

topological sym.

$$J^{top} = \frac{1}{8\pi^2} * \mathrm{tr}(f \wedge f)$$

$$Q^{top} = \frac{1}{8\pi^2}\int_{\Sigma_4} \mathrm{tr}(f \wedge f) \quad \leftarrow \text{instanton \#}$$

CS term (diagonal)

For generic gauge group,

$$S_{CS} = \frac{k}{24\pi^2} \int \mathrm{Tr}\left(a\wedge f\wedge f + \frac{i}{2} a\wedge a\wedge a\wedge f - \frac{1}{10} a\wedge a\wedge a\wedge a\wedge a \right)$$

but this is 0 for SU(2) case

$$\left(\because \ d^{pqr} = \frac{1}{2} \mathrm{tr}\left(t^p \{ t^q, t^r \}\right) = 0 \right)$$

$\Longrightarrow$ no diagonal SU(2) CS term

$U(1)_I - SU(2)_{gauge}$ mixed CS-term

$$S_{CS} = \frac{k_{Igg}}{8\pi^2} \int A_I \wedge \mathrm{tr}(f\wedge f)$$

no $U(1)^2 - SU(2)$ CS-term

$$\because \ \mathrm{tr} f = 0$$

discrete θ term [Douglas-Katz-Vafa '96]

$$\pi_4(SU(2)) = \mathbb{Z}_2 \qquad \left(\begin{array}{l} \text{cf. } \pi_4(SU(N\geq 3)) = 0 \\ \pi_4(USp(N)) = \mathbb{Z}_2 \end{array} \right)$$

We have two types of contribution:

$$\underset{\substack{\uparrow \\ \text{trivial } \pi_4}}{Z_0}\ , \quad \underset{\substack{\uparrow \\ \text{non-trivial } \pi_4}}{Z_1}$$

and can choose relative weight between them:

$$Z = Z_0 \underset{\substack{\uparrow \\ \text{"}\theta=0\text{"}}}{+} Z_1$$

no known local functional description

or

$$Z = Z_0 \underset{\substack{\uparrow \\ \text{called "discrete } \theta\text{-term with } \theta=\pi\text{"}}}{-} Z_1$$

(analogy

$$\pi_3(SU(N)) = \mathbb{Z}$$

in 4d SU(N) gauge theory

$$Z = \sum_{\substack{n \\ \uparrow \\ \text{inst. \#}}} e^{i\theta n} Z_n$$

)

追記

discrete θ-term is absorbed into sign of fermion mass

therefore, when all the fermions are massless,

physics is independent of θ

(cf. in 4d,

θ-term is absorbed into phase of fermion mass)