3. 5d SU(2) SYM & E_1 SCFT

3-1. 5d $\mathcal{N}=1$ vector multiplet

on-shell d.o.f.

6d $\mathcal{N}=(1,0)$ $\quad (A_\mu, \lambda_m)$

dim. reduction $\downarrow$

?? (no 5d Majorana)

5d $\mathcal{N}=1$ $\quad (A_\mu, \phi, \lambda_m)$

$\downarrow$

2 Majorana's

4d $\mathcal{N}=2$ $\quad (A_\mu, \phi_1, \phi_2, \lambda_m)$

$SU(2)_R$ doublet $(m=1,2)$

Symplectic ($SU(2)$) Majorana:

$$\lambda^m = \varepsilon^{mn} (\psi^c)_n$$

charge conjugation

3-2. SUSY U(1) theory

$$V = (a_\mu, \phi, D^i, \lambda_m)$$

D^i: auxiliary field, $\underline{3}$ of $SU(2)_R$, $i = 1, 2, 3$

kinetic term

$$S_{SYM} = \frac{1}{2g^2}\int d^5x \left(-\frac{1}{2} f_{\mu\nu}^2 - (D_\mu\phi)^2 - i\bar{\lambda}\gamma^\mu D_\mu\lambda + (D^i)^2 + i\bar{\lambda}[\phi, \lambda]\right)$$

$$(\bar{\lambda} \equiv \lambda^\dagger \gamma^0)$$

SUSY trans.

$$\delta a_\mu = i\bar{\lambda}\gamma_\mu\varepsilon$$

$$\delta\phi = \bar{\lambda}\varepsilon$$

$$\delta\lambda^m = \frac{1}{2} f_{\mu\nu}\gamma^{\mu\nu}\varepsilon^m - iD_\mu\phi\gamma^\mu\varepsilon^m + iD^i(\sigma^i)^m{}_n\varepsilon^n$$

$$\delta D^i = D_\mu\bar{\lambda}\gamma^\mu\sigma^i\varepsilon - [\phi, \bar{\lambda}]\sigma^i\varepsilon$$

diagonal SUSY CS term

(mixed between $U(1)_I$) $V_I = (A_I, \phi_I, D^i_I, (\lambda_I)_m)$

$$S_{SCS} = \frac{\overset{3K_{Igg}}{k_g}}{24\pi^2} \int \left[\overset{A_I}{a} \wedge f \wedge f + \left(-3\overset{\bar\lambda_I}{\bar\lambda} \gamma^{\mu\nu} \lambda f_{\mu\nu} + 6 i \bar\lambda \sigma^i \overset{D^i_I}{D^i} \lambda \right) d^5x \right]$$

$$+ \frac{\overset{K_{Igg}}{k_g}}{2\pi^2} \underset{\phi_I}{\phi} \left[\underbrace{ -\frac{1}{2} f_{\mu\nu}^2 - (D_\mu \phi)^2 - i \bar\lambda \gamma^\mu D_\mu \lambda + (D^I)^2 + i \bar\lambda [\phi, \lambda] }_{\mathcal{L}_{SYM}!} \right]$$

$$+ \frac{k}{\pi^2} D^i_I \phi D^i$$

taking $\phi_I \sim \frac{1}{g^2}$ gives S_{SYM}

$\rightarrow$ S_{SYM} is a part of SUSY mixed CS

~~SUSY~~ for generic V_I

SUSY for $\phi_I = \text{const.}$, (others) $= 0$

Prepotential

$\mathcal{F}(V)$: analytic func., determines action etc...

For this case,

$$\mathcal{F}_{cl}(V, V_I) = \underbrace{\frac{k_{Igg}}{2} V_I V^2}_{S_{SYM}} + \underbrace{\frac{k_g}{6} V^3}_{S_{SCS}}$$

We often write only scalar part :

$$F_{cl}(\phi, \phi_I) = \frac{k_{Igg}}{2} \phi_I \phi^2 + \frac{k_g}{6} \phi^3$$

$\left(\text{usually } k_{Igg} \phi_I = \frac{\pi^2}{g^2} \right)$

$\left(\frac{k_{abc}}{6} V_a V_b V_c \longleftrightarrow \text{mixed CS w/ level } k_{abc} \right)$

3-3. SU(2) SYM

Take $U(1)_I - [SU(2)_g]^2$ SUSY CS term w/ level 2:
(discrete $\theta = 0$)

$$S = \int d^5x \left\{ \frac{2\phi_I}{2\pi^2} \mathrm{tr}\left[-\frac{1}{2} f_{\mu\nu}^2 - (D_\mu\phi)^2 - i\bar{\lambda}\gamma^\mu D_\mu \lambda + (D^i)^2 + i\bar{\lambda}[\phi, \lambda] \right] + \underset{2\times\frac{1}{\pi^2}}{\frac{2}{\pi^2}} D_I^i \,\mathrm{tr}(\phi D^i) \right\}$$

$$+ \underset{\frac{2}{8\pi^2}}{\frac{1}{4\pi^2}} \int \mathrm{tr}\left[A_I \wedge f \wedge f + (-3\bar{\lambda}_I \gamma^{\mu\nu} \lambda_I f_{\mu\nu} + 6i\bar{\lambda}\sigma^i D_I^i \lambda) d^5x \right]$$

$$F_{cl}(V, V_I) = V_I \,\mathrm{tr} V^2 \rightarrow \phi_I \,\mathrm{tr}\, \phi^2$$

SUSY is preserved for

$$\phi_I = \underset{\substack{\parallel \\ \frac{\pi}{g^2}}}{\text{const.}}, \quad (\text{others}) = 0$$

later turned on

3-4. Low energy effective theory

[Seiberg '96]

vacua

$\phi =$ const. gives classical vacua "Coulomb branch"

We can take

$$\langle \phi \rangle = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \qquad a \geq 0$$

$$SU(2)_g \to U(1)_g \qquad \text{for } a \neq 0$$

$\underbrace{\text{charged d.o.f.}}_{\text{2 components of the fields}}$ of $U(1)_g$ become massive

$\to$ Integrate out !!

1-loop result

$$F_{\text{1-loop}} = \underbrace{2\phi_I a^2}_{\text{classical}} + \underbrace{\frac{4}{3}a^3}_{\text{1-loop effect.}}$$

$[U(1)_g]^3$ CS term w/ level 8

interpretation?

1回目おわり

massive fermion & CS level shift

CS term gives the 3-pt coupling:

$$\frac{\delta}{\delta A^{\mu}(p_1)} \frac{\delta}{\delta A^{\nu}(p_2)} \frac{\delta}{\delta A^{\rho}(p_3)} \int DA\, e^{iS_{CS}}$$

$$\sim i k\, \varepsilon_{\mu\nu\rho\sigma\tau}\, p_1^{\sigma} p_2^{\tau}$$

can be shifted by matters via

$$\langle J_\mu J_\nu J_\rho \rangle$$

It's known that a fermion w/ charge q & mass m gives

$$\delta k = -\frac{q^3}{2}\,\mathrm{sign}(m)$$

$$\left(\text{cf. mixed ver:} \quad \delta k_{abc} = -\frac{q_a q_b q_c}{2}\,\mathrm{sign}(m) \right)$$

$$\left(\text{cf. 3d mixed CS level shift:} \quad \delta k_{ab} = -\frac{q_a q_b}{2}\,\mathrm{sign}(m) \right)$$

For the present case,

$$\mathrm{tr}\left(i\bar{\lambda}[\phi,\lambda]\right) \qquad \left(\phi=\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix},\ \lambda=\lambda^+\tau^- + \lambda^-\tau^+ + \lambda^3\sigma^3\right)$$

$$= 2ia\,\bar{\lambda}^-\lambda^+ - 2ia\,\bar{\lambda}^+\lambda^-$$

	$Q[U(1)_g]$	mass
λ^+	2	$-2a$
λ^-	-2	$2a$
λ^3	0	0

$$\therefore\ \delta k_g = -\frac{1}{2}\,2^3\,\mathrm{sign}(-2a) - \frac{1}{2}(-2)^3\,\mathrm{sign}(2a)$$

$$= 8\,\mathrm{sign}(a)$$

$$= 8 \qquad (\because a \geq 0)$$

SUSY $\rightarrow$ SUSY $[U(1)_g]^3$ CS w/ level 8

追記：Subtleties (cf. Closset – Del Zotto – Saxena '18)

(cf. 3d case
Closset – Dumitrescu – Festuccia – Komargodski – Seiberg '12
Seiberg – Senthil – Wang – Witten '16)

① having CS level k

$$\rightarrow \frac{ik}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma\kappa} p^\sigma q^\kappa \subset \langle J_\mu(p) J_\nu(q) J_\rho(-p-q) \rangle$$

opposite statement may or may not be true

(true for $k \in \mathbb{Z}$, but sometimes $k \notin \mathbb{Z}$
it is better to use η-invariant)

② single massless Dirac fermion coupled to A

classically, $\exists$ parity & gauge sym.

But, we don't have regularization to preserve both

If we choose to keep gauge sym. but give up parity,

$$\frac{i\,k_{\text{Dirac}}}{24\pi^2} \varepsilon_{\mu\nu\rho\sigma\kappa} p^\sigma q^\kappa \subset \langle J_\mu J_\nu J_\rho \rangle$$

[Alvarez-Gaume-Della Pietra -Moore '85]

$$k_{\mathrm{Dirac}} = -\frac{1}{2} \ \underbrace{\mathrm{mod}\ 1}$$ "parity anomaly"

fixing integer part

$\Leftrightarrow$ choice of reguralization scheme

(It is often said "level $-\frac{1}{2}$ CS term" but it is oversimplified)

③ single massive Dirac fermion

level shift formula:

$$\lim_{|m|\to\infty} k_{\mathrm{Dirac}}(m) - k_{\mathrm{Dirac}}(0) = -\frac{1}{2}\,\mathrm{sgn}(m)$$

If we choose a regularization scheme called "$U(1)_{-1/2}$ quantization" s.t. $k_{\mathrm{Dirac}}(0) = -\frac{1}{2}$,

$$\lim_{m\to+\infty} k_{\mathrm{Dirac}}(m) = 0$$

$$\lim_{m\to-\infty} k_{\mathrm{Dirac}}(m) = -1$$

Implications

$$F = 2\phi_I a^2 + \frac{4}{3} a^3$$

$$\frac{\partial F}{\partial a} = 4a(\phi_I + a) = T_{mono} = m_W m_I$$

$$\frac{\partial^2 F}{\partial a^2} = \underbrace{4\phi_I + 8a} = \frac{1}{g^2_{eff}} = 2(m_W + m_I)$$

- * $U(1)_g$ is mixed w/ $U(1)_I$
- * smooth func. of a & no Landau pole

→ can take $g \to \infty$?

SCFT?

Indeed string th.

- . gives the same g^2_{eff}
- . predicts $g = \infty$ has enhanced global sym.

$$U(1)_I \to E_1 \equiv SU(2)_I$$

"E_1 SCFT"

Central charge

$$\rightarrow \quad m_W = 2a, \quad m_I = 2(\phi_I + a)$$

Duality trans.:

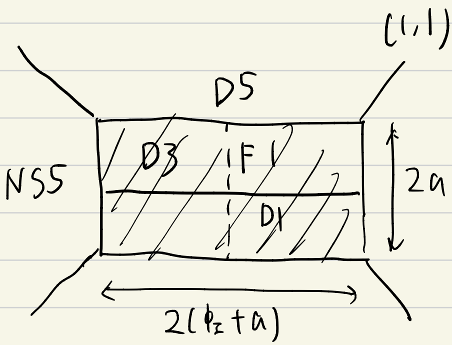
$$\phi_I \rightarrow -\phi_I, \quad a \rightarrow a + \phi_I$$

($g = \infty$ is fixed pt.)

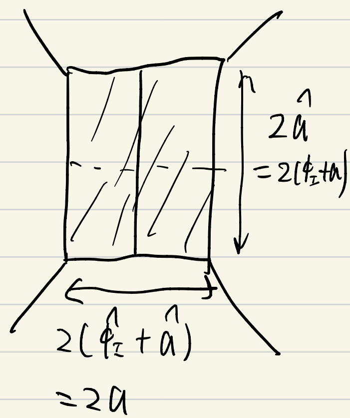
Brane interpretation:

x^5 ↑ → x^6

	0–4	5	6	7–9
D5	o		o	
NS5	o	o		



$\overset{S}{\longleftrightarrow}$



D1 : inst.

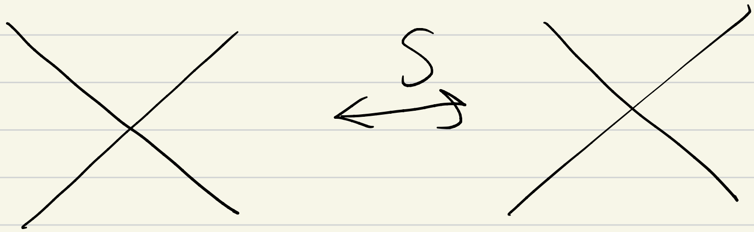
F1 : W-boson

D3 : mono. string

$$\frac{1}{g_{eff}^2} = (\text{perimeter})$$

追記

In SCFT point,



$\exists$ no dimensionful parameters

— consistent w/ scale invariance

cf. other theories

- U(1) w/ N_f electrons

$$\frac{1}{g_{\mathrm{eff}}^2} = 2\phi_I - N_f|\phi| \qquad \exists \text{ Landau pole!}$$

- SU(2) w/ N_f fundamentals

$$\frac{1}{g_{\mathrm{eff}}^2} = 4\phi_I + (8-N_f)\phi$$

no Landau pole for $N_f \leq 8$

$\longrightarrow$ can take $g=\infty$? "E_{N_f+1} SCFT"

$\exists$ various evidence after the paper!

(superconformal index, Nekrasov part, etc...)

3-5. SU(2) SYM from E_1 SCFT

Assume E_1 SCFT

$\exists$ $SU(2)_I$ current : J_μ^a $(a=1,2,3)$

$\xrightarrow{\text{SUSY}} (J_\mu^a, M^a, \mu^{ai}, \psi^a)$

$SU(2)_R$ $\underline{1}$ $\underline{1}$ $\underline{3}$ $\underline{2}$

coupled to $SU(2)_I$ b.g. vector multiplet

$((A_I)_\mu^a, \phi_I^a, D_I^{ai}, \lambda_I^a)$

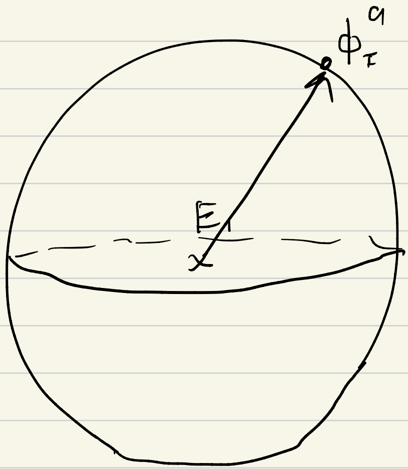
SUSY preserving relevant deformation :

$$\delta \mathcal{L} = \underbrace{\phi_I^a}_{\text{b.g.}} M^a \qquad SU(2)_I \to U(1)_I$$

$$\xrightarrow{\text{IR}} \text{“} \frac{1}{g^2} \mathrm{tr}\, f_{\mu\nu}^2 \text{”} \qquad \text{SYM}$$

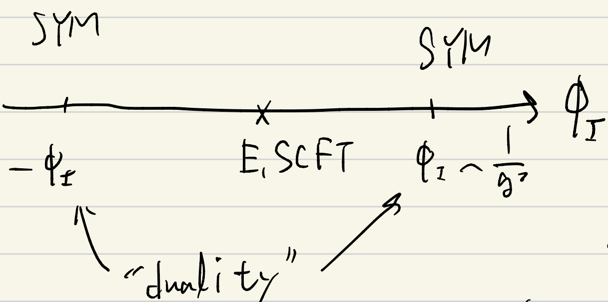
[cf. classification of deformations in SCFT : Cordova-Dumitrescu -Intriligator '16]

$\llcorner \phi_I^a$



pt's on the sphere describe the same physics

Take $\phi_I^a = \phi_I \hat{n}^a = (0, 0, \phi_I)$



$\mathbb{Z}_2^I : M^a \to -M^a$

J_μ^3 : $U(1)_I$ current

$J_\mu^1 \pm i J_\mu^2$: broken

(massive, charged under $U(1)_I$ $\to$ instanton)