3. BH entropy from 4d SUSY Cardy formula ⑤

3-1. Cardy formula

Focus on $SU(N)$ $\mathcal{N}=4$ SYM :

$$I = \mathrm{Tr}_{BPS}\left[(-1)^F p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} v_1^{q_1} v_2^{q_2}\right]$$

$$r = \frac{2}{3}(Q_1+Q_2+Q_3),\quad q_{1,2} = Q_{1,2}-Q_3$$

	$U(1)_R$	$U(1)_1$	$U(1)_2$
Φ_1	$\frac{2}{3}$	1	0
Φ_2	$\frac{2}{3}$	0	1
Φ_3	$\frac{2}{3}$	-1	-1

- $\mathrm{Tr}\, R = 0$

- $V_2(a) = -\sum_{i\neq j}\left[\kappa(a_{ij}+m_1)+\kappa(a_{ij}+m_2)+\kappa(a_{ij}-m_1-m_2)\right]$

- $V_1(a) = \frac{1}{3}\sum_{i\neq j}\left[3\theta(a_{ij})-\theta(a_{ij}+m_1)-\theta(a_{ij}+m_2)-\theta(a_{ij}-m_1-m_2)\right]$

$$-\frac{N-1}{3}\left[\theta(m_1)+\theta(m_2)+\theta(-m_1-m_2)\right]$$

w/ $a_{ij} = a_i - a_j,\ \sum_{j=1}^{N} a_j = 0$

Let's consider the regime

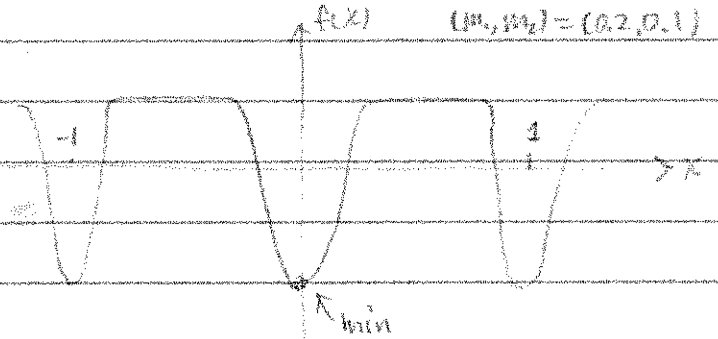
$$\mathrm{Re}\left(\frac{z}{i\sigma}\right) < 0$$

Integral is dominated by config. minimizing $V_2(a)$:

$$V_2(a) = \sum_{i<j} f(a_{ij}) + \frac{N-1}{2} f(0)$$

$$\begin{aligned} f(a_{ij}) = & \kappa(a_{ij} - \{m_1\}) - \kappa(a_{ij} + \{m_1\}) \\ & + \kappa(a_{ij} - \{m_2\}) - \kappa(a_{ij} + \{m_2\}) \\ & + \kappa(a_{ij} + \{m_1\} + \{m_2\}) - \kappa(a_{ij} - \{m_1\} - \{m_2\}) \end{aligned}$$

sufficient to show we can simultaneously minimize all $f(a_{ij})$



We can show $f(x)\big|_{min} - f(0) = 12\{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1)$

$V_2(a)$ is minimized by $a_{ij} = 0$, $\sum_j a_j = 0 \to a_j = 0$

$$\therefore\ V_2(a)\big|_{min} = V_2(0) = 6(N^2-1)\{m_1\}\{m_2\}(\{m_1\} + \{m_2\} - 1)$$

$O(\beta^{-1})$:

$$V_1(a)\big|_{a_j=0} = \frac{2(N^2-1)}{3}\left[\{m_1\}^2 + \{m_2\}^2 + \{m_1\}\{m_2\} - \{m_1\} - \{m_2\}\right]$$

$$\log I \underset{\beta\to 0}{\simeq} \frac{i\pi(N^2-1)}{\tau\sigma}\Big[\{m_1\}\{m_2\}(\{m_1\}+\{m_2\}-1)$$

$$+\frac{\tau+\sigma}{3}\left(\{m_1\}^2+\{m_2\}^2+\{m_1\}\{m_2\}-\{m_1\}-\{m_2\}\right)\Big]$$

move to $SO(6)_R$ language:

$$m_{1,2} = \Delta_{1,2} - \frac{\tau+\sigma}{3}$$

$$I = \mathrm{Tr}_{BPS}\left[(-1)^F p^{J_1+Q_3} q^{J_2+Q_3} e^{2\pi i\Delta_1(Q_1-Q_3)} e^{2\pi i\Delta_2(Q_2-Q_3)}\right]$$

$$= \mathrm{Tr}_{BPS}\left[p^{J_1} q^{J_2} \prod_{a=1}^{3} e^{2\pi i\Delta_a Q_a}\right]$$

$$\text{w/ } \Delta_1+\Delta_2+\Delta_3-\tau-\sigma-1\in 2\mathbb{Z}$$

$$\left((-1)^F = e^{2\pi i Q_3}\right)$$

$$\therefore \log I \underset{\beta\to 0}{\simeq} \frac{i\pi(N^2-1)\{\Delta_1\}\{\Delta_2\}(\{\Delta_1\}+\{\Delta_2\}-1-\sigma-\tau)}{\tau\sigma}$$

$$= -\frac{i\pi(N^2-1)\{\Delta_1\}\{\Delta_2\}\{\Delta_3\}}{\tau\sigma}$$

3-2. Comparison w/ BH entropy

$$e^{S_{CFT}(Q,J)} = \oint \frac{dp}{p^{J_1+1}} \frac{dq}{q^{J_2+1}} \left(\prod_{a=1}^{3} \frac{d e^{2\pi i \Delta_a}}{e^{2\pi i (Q_a+1)\Delta_a}} \right) I \Bigg|_{\sum_a \Delta_a - \tau - \sigma - 1 \in 2\mathbb{Z}}$$

For $|Q|, |J| \gg 1$, S_{CFT} is given by extremization:

$$S_{CFT}(Q,J) = S - 2\pi i \left(\sum_{a=1}^{3} X_a Q_a + \sum_{I=1}^{2} W_I J_I \right) - 2\pi i \Lambda \left(\sum_a X_a - \sum_I W_I - n \right) \Bigg|_{X_a, W_I}$$

$$\left(\begin{array}{l} S = \log I = -2\pi i \nu \dfrac{X_1 X_2 X_3}{W_1 W_2}, \\ \nu = \dfrac{N^2-1}{2}, \; W_1 = \sigma, \; W_2 = \tau, \; X_a = \{\Delta_a\}, \; n = 1 \end{array} \right)$$

Conditions:

$$\frac{\partial S}{\partial X_a} = 2\pi i (Q_a + \Lambda), \quad \frac{\partial S}{\partial W_I} = 2\pi i (J_I - \Lambda)$$

Using $S = \sum_{a=1}^{3} X_a \dfrac{\partial S}{\partial X_a} + \sum_{I=1}^{2} W_I \dfrac{\partial S}{\partial W_I}$

$$= 2\pi i \left(\sum_{a=1}^{3} X_a Q_a + \sum_{I=1}^{2} W_I J_I \right) + 2\pi i \Lambda \left(\sum_a X_a - \sum_I W_I \right),$$

$$S_{CFT} = 2\pi i \Lambda$$

w/ $0 = (Q_1+\Lambda)(Q_2+\Lambda)(Q_3+\Lambda) + \nu(J_1-\Lambda)(J_2-\Lambda)$

$$= \Lambda^3 + P_2\Lambda^2 + P_1\Lambda + P_0$$

$$\begin{pmatrix} P_0 = Q_1Q_2Q_3 + \nu J_1J_2 \\ P_1 = Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \nu(J_1+J_2) \\ P_2 = Q_1+Q_2+Q_3+\nu J_1J_2 \end{pmatrix}$$

$P_{0,1,2} \in \mathbb{R} \to$ 3 real sols. or 1 real and 2 complex conj. sols

3 real sols. $\downarrow$ $S \in i\mathbb{R}$

1 real and 2 complex conj. sols $\xrightarrow{\text{impose } S \in \mathbb{R}_{\geq 0}}$ 1 real and 2 imaginary conj.

Assume $0 = (\Lambda - i\alpha)(\Lambda + i\alpha)(\Lambda - \beta) = \Lambda^3 - \beta\Lambda^2 + \alpha^2\Lambda - \beta\alpha^2$

satisfied by $P_0 = P_1P_2$

$$\therefore \Lambda = -P_2, \pm i\sqrt{P_1}$$

Taking $\Lambda = -i P_1$,

$$S_{CFT} = 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2-1}{2}(J_1+J_2)}$$

for $^\forall N$

$$S_{BH} = 2\pi\sqrt{Q_1Q_2 + Q_2Q_3 + Q_3Q_1 - \frac{N^2}{2}(J_1+J_2)}$$

Comments:

1. The BH sol. satisfies $P_0 = P_1 P_2$ [Cabo-Bizet-Cassani-Martelli-Murthy] '18

2. We computed S_{CFT} for $\forall N$

Quantum BH entropy?

Using $c = \frac{N^2-1}{4}$,

$$S_{CFT} = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - 2c(J_1 + J_2)}$$

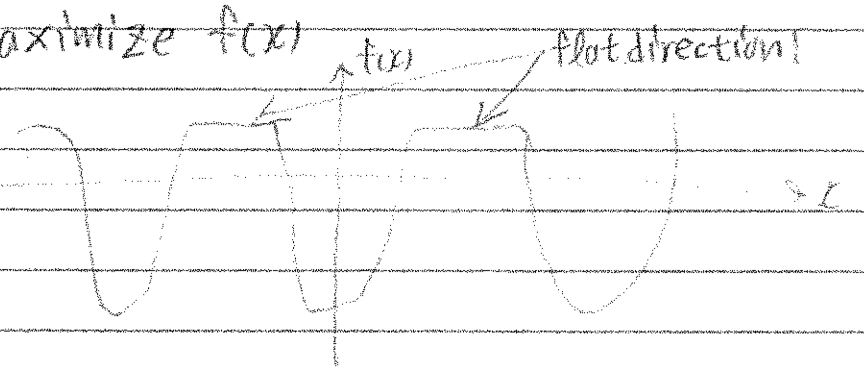
If we modify AdS/CFT dictionary as

$$\frac{\pi}{2 G_N g^3} = 4c,$$

$S_{CFT} = S_{BH}$ for $\forall N$ non-renormalization?

3. We have taken $\mathrm{Re}\left(\frac{z}{\tau\sigma}\right) < 0$ → what if $\mathrm{Re}\left(\frac{z}{\tau\sigma}\right) > 0$?

→ maximize $f(x)$



4. Generalization to gauge group G:

$$S_{CFT} = 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - \frac{\dim G}{2}(J_1 + J_2)}$$

$$= 2\pi \sqrt{Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_1 - 2c(J_1 + J_2)}$$

$$\left(c = \frac{\dim G}{4}\right)$$

5. Bethe ansatz approach [Benini-Milan '18]

$$N \to \infty, \quad \tau = \sigma \quad \text{(w/o Cardy limit)}$$

solution: $u_{ij} = \frac{\tau}{N}(i-j)$ w/ $\sum_j u_j = 0$

$$\ln I \underset{N\to\infty}{\simeq} \frac{i\pi N^2 \{\Delta_1\}\{\Delta_2\}(\{\Delta_1\}+\{\Delta_2\}-1-2\tau)}{\tau^2}$$

$$S_{\text{CFT}} = 2\pi\sqrt{Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}\times 2J} \qquad (J_1=J_2=J)$$

6. Matrix model [Cabo-Bizet-Murthy '19]

$$N \to \infty, \quad \tau = \sigma$$

$\exists\,\infty$ complex saddles $\ni$ BH saddle

dominant for small-τ

5. Bethe ansatz approach [Benini-Milan '18]

$N \to \infty$, $\tau = \sigma$ (w/o Cardy limit)

solution: $u_{ij} = \frac{\tau}{N}(i-j)$ w/ $\sum_j u_j = 0$

$$\ln I \underset{N\to\infty}{\sim} \frac{i\pi N^2 \{\Delta_1\}\{\Delta_2\}(\{\Delta_1\}+\{\Delta_2\}-1-2\tau)}{\tau^2}$$

$$S_{BH} = 2\pi\sqrt{Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2}{2}\times 2J} \quad (J_1=J_2=J)$$

6. Matrix model [Cabo-Bizet-Murthy '19]

$N \to \infty$, $\tau = \sigma$

$\exists\,\infty$ complex saddles $\ni$ BH saddle

dominant for small-τ