5. Computation of 4d SCI

5-1. Localization

$$Z = \int D\Phi \, e^{-S[\Phi]}$$

Q: fermionic off-shell sym.

$$QS[\Phi]=0\,, \quad Q^2 = h_B$$

generator of bosonic sym.

Q-exact 変形:

$$Z(t) \equiv \int D\Phi \, e^{-S[\Phi]-tQV[\Phi]}$$

$V[\Phi]$: fermionic functional s.t. $h_B V=0$

$$\frac{dZ(t)}{dt} = -\int D\Phi\,(QV)\,e^{-S-tQV}$$

$$= -\int D\Phi\,\underbrace{Q\left(V e^{-S-tQV}\right)}_{\text{配位空間での全微分}}$$

$$= 0 \qquad (\text{"bdy" がなければ})$$

よって，

$$Z = \lim_{t\to 0} Z(t) = Z(t)$$

$$= \lim_{t\to\infty} \int D\Phi\, e^{-S[\Phi]-tQV[\Phi]}$$

QV の saddle pt. に支配される！

$$Z = \sum_{\Phi_0 \in \text{saddles}} e^{-S[\Phi_0]} \underbrace{Z_{1\text{-loop}}(\Phi_0)}_{\text{QVの}\Phi_0\text{周りでの 1-loop det.}}$$

Q 不変な op. の期待値についても同様

5-2. 0d SUSY QFT

$$S(x, B, \psi_1, \psi_2) = \frac{1}{2}B^2 + iBh'(x) - h''(x)\psi_1\psi_2$$

$h(x)$: x の n 次多項式

SUSY trans.:

$$\begin{cases} \delta_1 x = \varepsilon^1 \psi_1 \\ \delta_1 B = 0 \\ \delta_1 \psi_1 = 0 \\ \delta_1 \psi_2 = -i\varepsilon^1 B \end{cases}$$

$$\delta_2 x = -\varepsilon^2 \psi_2$$
$$\delta_2 B = 0$$
$$\delta_2 \psi_1 = -i\varepsilon^2 B$$
$$\delta_2 \psi_2 = 0$$

この系の分配関数は簡単に計算可：

$$Z = \frac{1}{2\pi}\int dx\, dB\, d\psi_1 d\psi_2\, e^{-\frac{1}{2}B^2 + iBh' + h''\psi_1\psi_2}$$

$$= \frac{1}{\sqrt{2\pi}}\int dx\, d\psi_1 d\psi_2\, e^{-\frac{1}{2}(h')^2 + h''\psi_1\psi_2}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dx\, h'\, e^{-\frac{1}{2}(h')^2}$$

$$= \frac{1}{\sqrt{2\pi}}\int dy\, e^{-\frac{1}{2}y^2} \qquad (y = h'(x))$$

※ただし、$x \to y$ は 1 to 1 map とは限らない

ex.) $h(x) = \frac{c}{3}x^3$, $h'(x) = cx^2 \to x = \pm\sqrt{\frac{y}{c}}$

$\to Z = 0$

また、$x \to \pm\infty$ が $y \to \pm\infty$ か $\mp\infty$ かが
$h(x)$ の最高次の係数の符号で決まる

$$Z = \begin{cases} 0 & \text{for odd } n \\ \mathrm{sign}(\partial_x^n h) & \text{for even } n \end{cases}$$

Localizationによる計算

$$\delta_1\delta_2\left(\frac{1}{2}\psi_1\psi_2-h\right)=\delta_1\left(-\frac{i}{2}\varepsilon^1 B\psi_2+h'\varepsilon^2\psi_2\right)$$

$$=\frac{1}{2}B^2\varepsilon^1\varepsilon^2+h''\varepsilon^1\psi_1\varepsilon^2\psi_2-ih'B\varepsilon^2\varepsilon^1$$

$$=(\varepsilon^1\varepsilon^2)\,S$$

S 自体が Q-exact

$$\rightarrow\quad Z=\frac{1}{2\pi}\lim_{t\to\infty}\int dB\,dx\,d\psi_1\,d\psi_2\;e^{-tS[B,X,\psi_1,\psi_2]}$$

Sのsaddleに支配される!

$$S=\frac{1}{2}(B+ih')^2+\frac{1}{2}(h')^2-h''\psi_1\psi_2$$

$$\rightarrow\quad \underbrace{B=-ih',\quad h'=0}_{B=0,\ X=x_0},\quad \psi_1=\psi_2=0$$

$$\rightarrow\quad S[B=0,X=x_0,\psi_1=0,\psi_2=0]=0$$

1-loop det.

$$X = x_0 + \frac{1}{\sqrt{t}}\delta X,\quad B = \frac{1}{\sqrt{t}}\delta B,$$

$$\psi_1 = \frac{1}{\sqrt{t}}\delta\psi_1 \quad , \quad \psi_2 = \frac{1}{\sqrt{t}}\delta\psi_2$$ として展開:

$$tB \simeq \frac{1}{2}\left(B + i h''(x_0)\,\delta X\right)^2 + \frac{1}{2}(h''(x_0))^2\,\delta X^2$$
$$- \frac{1}{2}h''(x_0)\,\psi_1\psi_2$$

$$\therefore\ Z_{1\text{-loop}} = 2\pi\,\frac{h''(x_0)}{|h''(x_0)|}$$

よって,
$$Z = \sum_{x_0}\frac{h''(x_0)}{|h''(x_0)|}$$

$$= \begin{cases} 0 & \text{for odd } n \\ \mathrm{sign}(\partial_x^n h) & \text{for even } n \end{cases}$$

5-3. 4d SCI

$\varepsilon=\varepsilon_L$の場合にLocalizationを適用！

$$I=\mathrm{tr}\left[(-1)^F t^{R+2J_L} x^{2J_R} \prod_i e^{\mu_i F_i}\right]$$

$$=\mathrm{tr}\left[(-1)^F p^{j_L+j_R+\frac{R}{2}} q^{j_L-j_R+\frac{R}{2}} \prod_i e^{\mu_i F_i}\right]$$

実はこのとき，

S そのものが "Q-exact"

→ $QV=S$に選べば，

元々の理論でのsaddle pt. 解析が

exactな結果を与える

"weak coupling = strong coupling"

Saddle は A_μ 以外 trivial;

$\lambda = 0$, $D = 0$, $\phi = \bar{\phi} = 0$, $\psi = \bar{\psi} = 0$, $F = \bar{F} = 0$,

$\underline{F_{\mu\nu} = 0}$

$S^1 \times S^3$ 上の flat connection

$$\rightarrow \quad A = \underbrace{a}_{S^1\text{方向の holonomy}} dx^4$$

$Z_{1\text{-loop}}$ は saddle の周りの $S^1 \times S^3$ 上の mode 展開から求まる

一般の4d $\mathcal{N}=1$ Lagrangian 理論の場合の公式！ 9

(ゲージ群: G)

G の rank

$$I = \frac{(p;p)^{|G|}(q;q)^{|G|}}{|W(G)|}\int_{-\frac{1}{2}}^{\frac{1}{2}} d^{|G|}a \; Z_{1\text{-loop}}(a)$$

Weyl 群

$$(p;p) = \prod_{k=0}^{\infty}(1-p^{k+1})$$ Q-pochhammer

$$Z_{1\text{-loop}} = Z_{vec}\, Z_{chi}$$

$$Z_{vec} \equiv \frac{1}{\prod_{\alpha \in \text{roots}} \Gamma_e(e^{2\pi i \alpha \cdot a}; p, q)}$$

$$\Gamma_e(z; p, q) = \prod_{j,k \geq 0} \frac{1 - z^{-1} p^{j+1} q^{k+1}}{1 - z p^j q^k}$$

"elliptic gamma function"

$$Z_{chi} \equiv \prod_{I \in \text{chirals}} \prod_{\rho_I \in R_I} \Gamma_e\left((pq)^{\frac{R_I}{2}} e^{2\pi i \rho_I \cdot a}; p, q \right)$$

R-charge

I番目のmatterの表現

weight vector

Ex. 1 Massive theory w/ single vac.

$$W = m Q_a Q_b \to r_a + r_b = 2$$

$$\exists\, U(1)_f : \quad Q_a \to e^{i\theta} Q_a, \quad Q_b \to e^{-i\theta} Q_b$$

$$I = \Gamma\left((pq)^{\frac{r_a}{2}} u \right) \Gamma\left((pq)^{\frac{r_b}{2}} u^{-1} \right)$$

$$= \prod_{j,k=0}^{\infty} \frac{1 - (pq)^{1-\frac{r_a}{2}} u^{-1} p^j q^k}{1 - (pq)^{\frac{r_a}{2}} u\, p^j q^k}$$

$$\times \prod_{j,k=0}^{\infty} \frac{1 - (pq)^{1-\frac{2-r_a}{2}} u\, p^j q^k}{1 - (pq)^{\frac{2-r_a}{2}} u^{-1} p^j q^k}$$

$$= 1 \qquad \checkmark$$

Ex, 2 ~~SUSY~~

Polonyi model (F-term ~~SUSY~~)

$$W = \eta Q \to r = 2$$

$$I = \Gamma(pq; p, q)$$

$$= \prod_{j,k=0}^{\infty} \frac{\overbrace{1 - p^j q^k}^{(j,k)=0\text{のとき}0}}{1 - (pq)p^j q^k}$$

$$= 0 \quad ✓$$

Ex. 3 Runaway vac.

$$W = \eta \underbrace{Q}_{r=2} + \frac{\lambda}{2} Q^2 \underbrace{S}_{r=-2}$$

vac.: $\eta + \lambda QS = 0, \quad Q^2 = 0$

$\therefore \quad Q \to 0, \ S \to \infty \quad$ w/ $QS = -\frac{\eta}{\lambda}$

$$I = \underbrace{\Gamma(pq; p, q)}_{\downarrow \ 0} \ \underbrace{\Gamma((pq)^{-1}; p, q)}_{\downarrow \ \infty}$$

$$\left(\Gamma((pq)^{-1}; p, q) = \prod_{j,k=0}^{\infty} \frac{1 - p^{j+2} q^{k+2}}{1 - p^{j-1} q^{k-1}} \to \infty \right)$$

SCI is ill-defined!

signal of flat direction