SUSY index

有限温度分配関数 on $S^1_\beta \times M_{d-1}$;

$$Z(\beta, \mu) = \mathrm{Tr}\left[e^{-\beta H} \prod_i e^{\mu_i Q_i} \right]$$

boson : p.b.c , fermion : a.p.b.c. → ~~SUSY~~

強結合計算ムズかしい

SUSY index on $S^1_\beta \times M_{d-1}$;

$$I(\beta, \mu) = \mathrm{Tr}\left[(-1)^F e^{-\beta H} \prod_i e^{\mu_i Q_i} \right]$$

fermion : p.b.c. (または twisted)

SUSY 不変で強結合計算が(ほぼ)可

Witten index in SUSY QM

$$2H = \{Q, Q^\dagger\}$$

$$I(\beta) = \mathrm{tr}\left[(-1)^F e^{-\beta H}\right]$$

$$H|E\rangle = E|E\rangle$$

$$\begin{aligned} H(Q|E\rangle) &= \frac{1}{2}(QQ^\dagger + Q^\dagger Q)Q|E\rangle \\ &= \frac{1}{2}Q(2H - QQ^\dagger)|E\rangle \\ &= E(Q|E\rangle) \end{aligned}$$

$E \neq 0$のとき,

bosonic state と fermionic stateが"ペアリング"

$$\begin{aligned} \Rightarrow I(\beta) &= \sum_E e^{-\beta E}\left[(\#\text{ of }|bos\rangle\text{ w/ }E) - (\#\text{ of }|fer\rangle\text{ w/ }E)\right] \\ &= (\#\text{ of }|bos.\rangle\text{ w/ }E=0) - (\#\text{ of }|fer\rangle\text{ w/ }E=0) \\ &= \mathrm{tr}\left[(-1)^F\right] \end{aligned}$$

βに独立!

他の示し方!

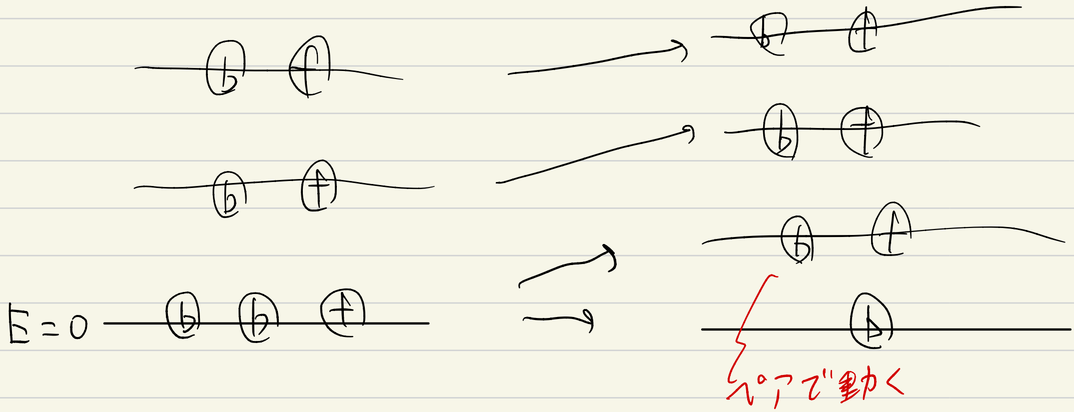
$$\frac{dI}{d\beta} = -\,\mathrm{tr}\left[(-1)^F H e^{-\beta H}\right]$$

$$= -\,\mathrm{tr}\left[(-1)^F (QQ^\dagger + Q^\dagger Q) e^{-\beta H}\right]$$

$$\left(\begin{aligned} \mathrm{tr}\left[(-1)^F QQ^\dagger e^{-\beta H}\right] &= -\,\mathrm{tr}\left[Q(-1)^F Q^\dagger e^{-\beta H}\right] \\ &= -\,\mathrm{tr}\left[(-1)^F Q^\dagger e^{-\beta H} Q\right] \\ &= -\,\mathrm{tr}\left[(-1)^F Q^\dagger Q e^{-\beta H}\right] \end{aligned}\right)$$

$$\rightarrow \frac{dI}{d\beta} = 0$$

I は理論の"連続変形"の下で不変



⇒ 弱結合の計算から強結合の結果が得られる!

ただし、"離散変形"の下では I はジャンプし得る

ex.) ポテンシャルの漸近形を変えた場合

よく、SUSY の SSB の判定に使われる

$I \neq 0$ なら SUSY は破れてない

$I = 0$ なら破れてるかも

例：1変数 SUSY QM

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$$

$$\delta x = \varepsilon\bar{\psi} - \bar{\varepsilon}\psi$$

$$\delta\psi = \varepsilon(i\dot{x} + h'(x))$$

$$\delta\bar{\psi} = \bar{\varepsilon}(-i\dot{x} + h'(x))$$

$$\downarrow$$

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi})$$

$$[x, p] = i, \quad \{\psi, \bar{\psi}\} = 1$$

$$Q = \bar{\psi}(ip + h'(x)), \quad \bar{Q} = \psi(-ip + h'(x))$$

SUSY ground state;

$$|\Psi\rangle \quad \text{w/} \quad Q|\Psi\rangle = 0 = \bar{Q}|\Psi\rangle$$

$$|\Psi\rangle \equiv |\phi_1\rangle_b \otimes |0\rangle_f + |\phi_2\rangle_b \otimes \bar{\psi}|0\rangle_f$$

$$(\psi|0\rangle_f = 0)$$

"波動関数"

$$\Psi(x) = \phi_1(x)|0\rangle_f + \phi_2(x)\bar{\psi}|0\rangle_f$$

$$Q|\Psi\rangle = 0 \to \left(\frac{d}{dx} + h'(x)\right)\phi_1(x)\,\bar{\psi}|0\rangle_f = 0$$

$$\bar{Q}|\Psi\rangle = 0 \to \left(-\frac{d}{dx} + h'(x)\right)\phi_2(x)\,\psi\bar{\psi}|0\rangle_f = 0$$

$$\therefore \quad \phi_1(x) = c_1 e^{-h(x)}, \quad \phi_2(x) = c_2 e^{+h(x)}$$

1

φ_2 : non-normalizable

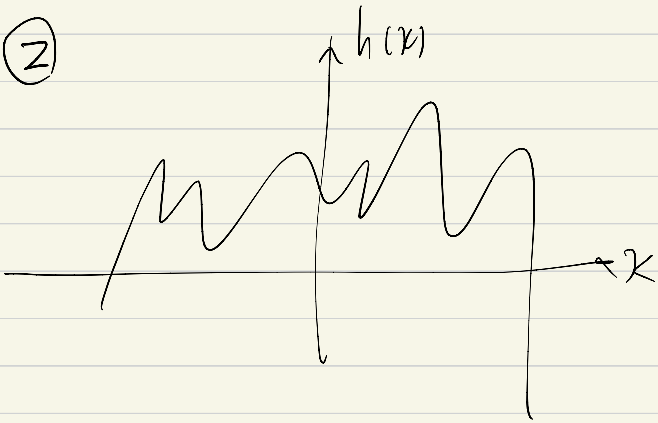
$(C_2 = 0)$

$\Psi(x) = \varphi_1(x)\,|0\rangle_f$

bosonic

$\rightarrow \quad I = 1$

2



φ_1 : non-normalizable

$\Psi(x) = \varphi_2(x)\,\bar{\psi}\,|0\rangle_f$

termionic

$\rightarrow \quad I = -1$

3

φ_1, φ_2 : non-normalizable

$\rightarrow$ no SUSY ground state

$\rightarrow \quad I = 0$

特にharmonic osc. の場合;

$$h(x) = \frac{w}{2}x^2$$

$$H = H_b + H_f$$

$$H_b \equiv \frac{1}{2}p^2 + \frac{w^2}{2}x^2 \quad , \quad H_f \equiv \frac{w}{2}[\bar{\psi}, \psi]$$

$$E_b = \left(n+\frac{1}{2}\right)|w| \qquad (n=0,1,2,\cdots)$$

$$E_f = \underbrace{-\frac{w}{2}}_{F=0}, \quad \underbrace{+\frac{w}{2}}_{F=1}$$

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \mathrm{Tr}_b(e^{-\beta H_b}) \cdot \mathrm{Tr}_f(e^{-\beta H_f})$$

$$= \left(\sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})|w|}\right)\left(e^{-\frac{\beta w}{2}} + e^{+\frac{\beta w}{2}}\right)$$

$$= \frac{e^{+\frac{\beta}{2}w} + e^{-\frac{\beta}{2}w}}{e^{\frac{\beta}{2}|w|} - e^{-\frac{\beta}{2}|w|}} = \coth\left(\frac{\beta}{2}|w|\right)$$

$$I = \mathrm{Tr}[(-1)^F e^{-\beta H}] = \frac{e^{+\frac{\beta w}{2}} - e^{-\frac{\beta w}{2}}}{e^{\frac{\beta}{2}|w|} - e^{-\frac{\beta}{2}|w|}} = \mathrm{sign}(w)$$

$\uparrow$

βに依らない!

経路積分との対応！

$$Z_{S^1_\beta} \equiv \int Dx D\psi D\bar{\psi}\, e^{-\oint dt\, L(t)}$$

$$S_{bos} = \oint_0^\beta dt \left(\frac{1}{2}\dot{x}^2 + \frac{w^2}{2} x^2 \right)$$

$$\left(x(t) = \frac{1}{\sqrt{\beta}} \sum_{n=-\infty}^{\infty} x_n e^{\frac{2\pi i}{\beta} n t} \quad (x_n^* = x_{-n}) \right)$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{2} \left(\left(\frac{2\pi n}{\beta}\right)^2 + w^2 \right) x_n x_{-n}$$

$$Z_{bos} = \int Dx\, e^{-S_{bos}} = \frac{\sqrt{2\pi}}{w} \prod_{n=1}^{\infty} \frac{2\pi}{\left(\frac{2\pi n}{\beta}\right)^2 + w^2}$$

$$= \frac{\sqrt{2\pi}}{|w|} \left(\prod_{n=1}^{\infty} \frac{\beta^2}{2\pi n^2} \right) \prod_{n=1}^{\infty} \frac{1}{1 + \left(\frac{\beta w}{2\pi n}\right)^2}$$

$$\left(\begin{array}{l} \prod_{n=1}^{\infty} \left(1 + \left(\frac{z}{\pi n}\right)^2 \right) = \frac{\sinh z}{z}, \quad \prod_{n=1}^{\infty} c = e^{\zeta(0) \ln c} = e^{-\frac{1}{2}\ln c} = \frac{1}{\sqrt{c}} \\ \prod_{n=1}^{\infty} \frac{1}{n^2} = e^{-2\sum_{n=1}^{\infty} \ln n} = e^{2\zeta'(0)} = e^{-\ln(2\pi)} = \frac{1}{2\pi} \end{array} \right)$$

$$= \frac{\sqrt{2\pi}}{w} \cdot \frac{\sqrt{2\pi}}{\beta} \cdot \frac{1}{2\pi} \; \frac{\frac{\beta |w|}{2}}{\sinh \frac{\beta |w|}{2}}$$

$$= \frac{1}{2\sinh \frac{\beta |w|}{2}} = \mathrm{Tr}\left(e^{-\beta H_b}\right) \checkmark$$

$$S_{fer} = \oint dt\ \bar{\Psi}(\partial_t + W)\Psi$$

$$= \sum_n \bar{\Psi}_{-n}\left(\frac{2\pi i n}{\beta} + W\right)\Psi_n$$

$$n = \begin{cases} \mathbb{Z} & \text{for p.b.c.} \\ \mathbb{Z}+\frac{1}{2} & \text{for a.p.b.c.} \end{cases}$$

$$Z^{pbc}_{fer} = W \prod_{n=1}^{\infty}\left(\left(\frac{2\pi n}{\beta}\right)^2 + W^2\right)$$

$$= \operatorname{sign}(W)\cdot 2\sinh\frac{\beta W}{2} = \mathrm{Tr}\left[(-1)^F e^{-\beta H_t}\right] \quad \checkmark$$

$$Z^{apbc}_{fer} = \prod_{n=0}^{\infty}\left(\left(\frac{2\pi(n+\frac{1}{2})}{\beta}\right)^2 + W^2\right)$$

$$= \prod_{n=0}^{\infty}\left(\frac{2\pi}{\beta}\right)^2\left(n+\frac{1}{2}\right)^2 \prod_{n=0}^{\infty}\left(1 + \left(\frac{\beta W}{2\pi}\right)^2\frac{1}{(n+\frac{1}{2})^2}\right)$$

$$\left(\begin{array}{l} \displaystyle\prod_{n=0}^{\infty}\left(1+\frac{z^2}{\pi^2(n+\frac{1}{2})^2}\right) = \cosh z \qquad \zeta(s,a) = \sum_{n=0}^{\infty}\frac{1}{(n+a)^s} \\ \displaystyle\prod_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^2 = e^{2\sum_n \ln(n+\frac{1}{2})} = e^{-2\zeta'(0,\frac{1}{2})} = e^{\ln 2} = 2 \end{array}\right.$$

$$= \left(\frac{2\pi}{\beta}\right)\times\left(\frac{\beta}{2\pi}\right)\times 2\times\cosh\frac{\beta W}{2}$$

$$= 2\cosh\frac{\beta W}{2} = \mathrm{Tr}\left[e^{-\beta H_A}\right] \quad \checkmark$$

$$Z^{apbc} = Z_{bos} Z_{ter}^{apbc} = \frac{2\cosh\frac{\beta w}{2}}{2\sinh\frac{\beta|w|}{2}} = \coth\left(\frac{\beta|w|}{2}\right)$$

$$= \mathrm{Tr}\left[e^{-\beta H} \right]$$

$$Z^{pbc} = Z_{bos} Z_{ter}^{pbc} = \frac{2\sinh\frac{\beta w}{2}}{2\sinh\frac{\beta|w|}{2}} = \mathrm{sign}(w)$$

$$= \mathrm{Tr}\left[(-1)^F e^{-\beta H} \right]$$

$$\mathrm{Det}(-\partial_t + w)$$

$$= \prod_{n=-\infty}^{\infty} \left(i \frac{2\pi(n+\frac{1}{2})}{\beta} + w \right)$$ $n=0$ はっこぬく

$$= \prod_{n=0}^{\infty} \left(i \frac{2\pi(n+\frac{1}{2})}{\beta} + w \right)$$

$$\prod_{m=1}^{\infty} \left(-i \frac{2\pi(m-\frac{1}{2})}{\beta} + w \right)$$

より一般的なindex

Q_i : superchargeと交換するop.

(global sym. のchargeの適当な重ね合せ)

$$\mathrm{Tr}\left[(-1)^F e^{-\beta H}\right] \longrightarrow \mathrm{Tr}\left[(-1)^F e^{-\beta H} \prod_i e^{\mu_i Q_i}\right]$$

- 同様にβや連続変形に依らない
- より詳細な情報を得ることができる

ex.)

- Q_i : flavor charge
- T^d 上の理論を考えて $Q_i = P_i$
  ($d=2$: elliptic genus)
- $S^1 \times M_{d-1}$ 上で, $Q_i =$ (generators of isometry + R)