Application of Quantum Computation to Quantum Field Theory
– Basics & Spin system –

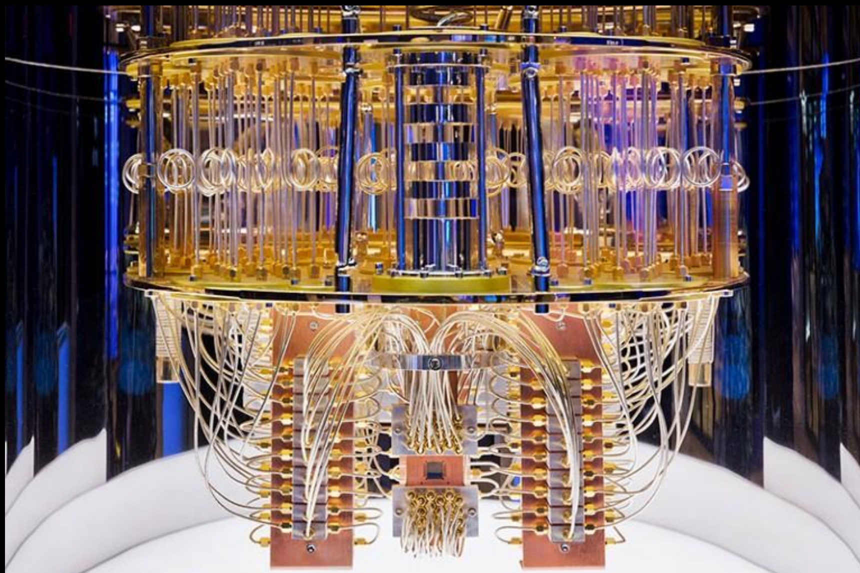
Masazumi Honda

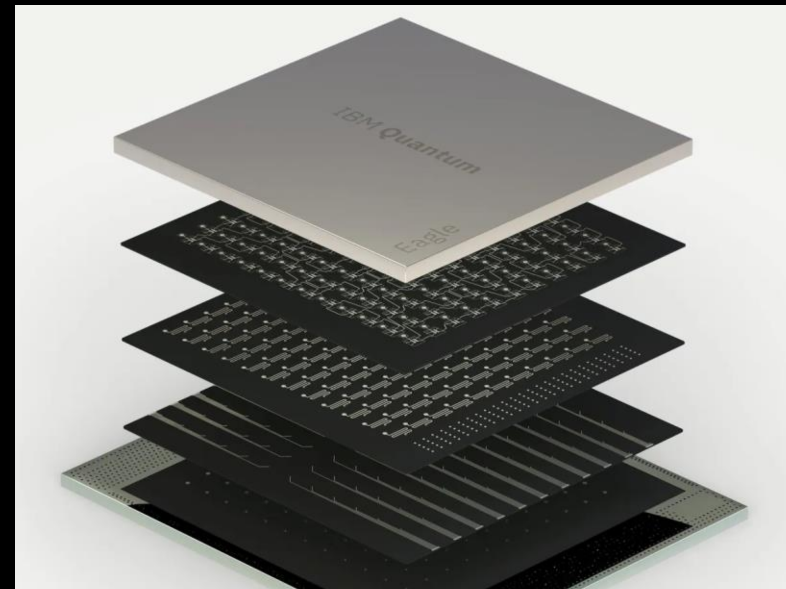
(本多正純)

6th Jul., 2023

Intensive lectures @Niigata U.（day 1）

Quantum computer sounds growing well…





Article

Evidence for the utility of quantum computing before fault tolerance

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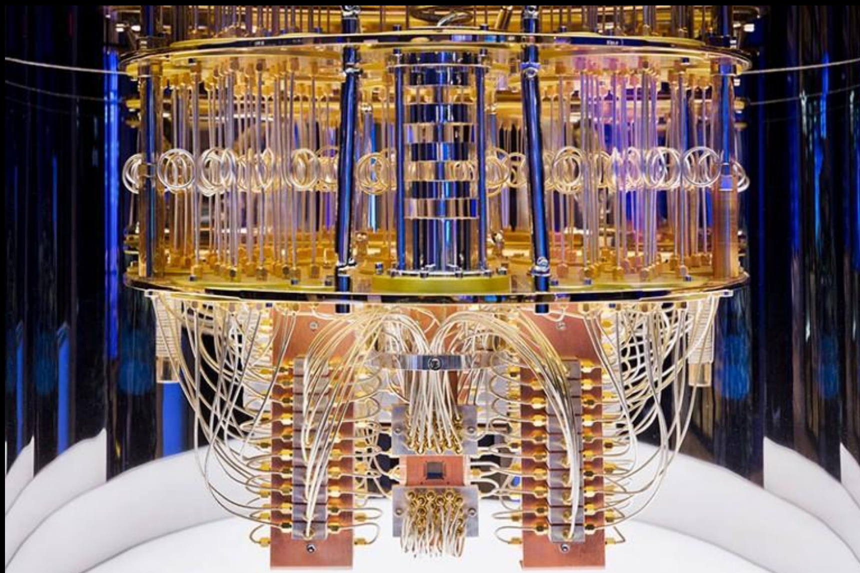
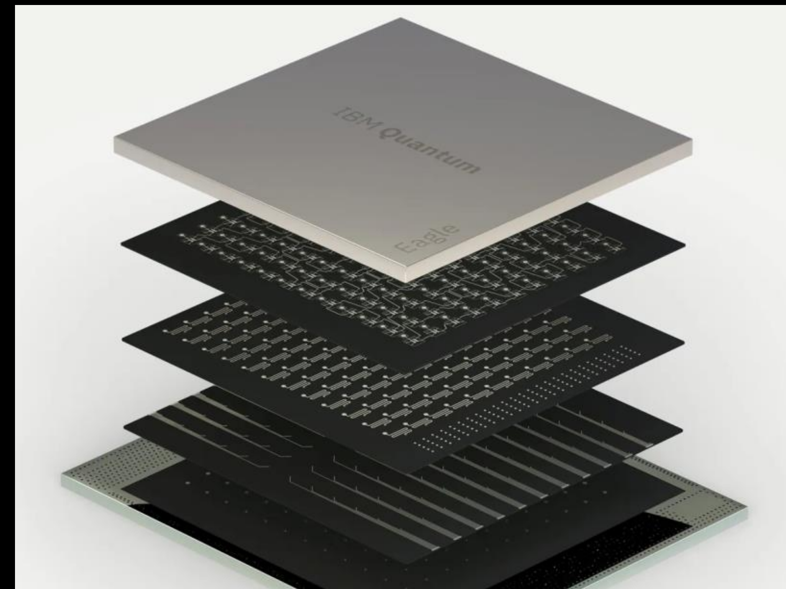
Quantum computing promises to offer substantial speed-ups over its classical

Quantum computer sounds growing well…



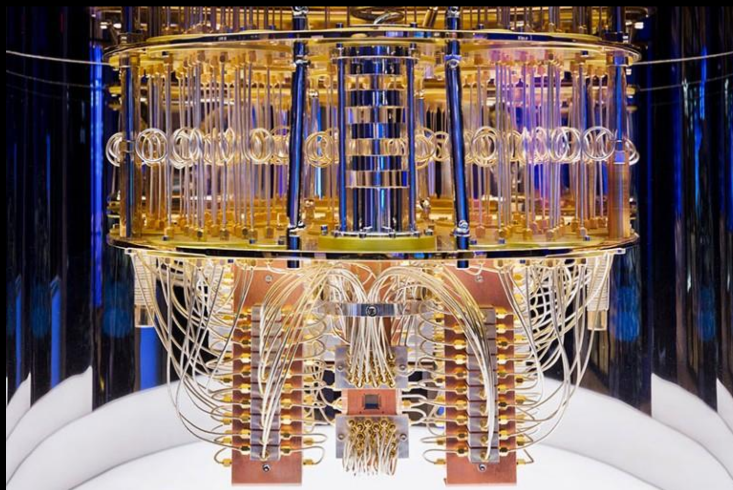


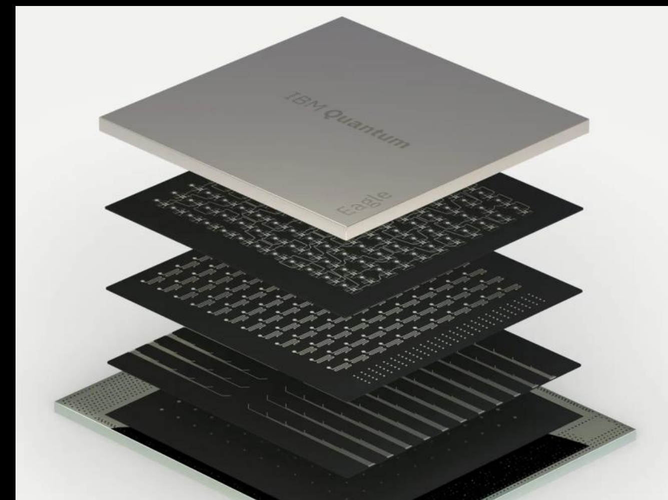
Article

Evidence for the utility of quantum computing before fault tolerance

How can we use it for us?

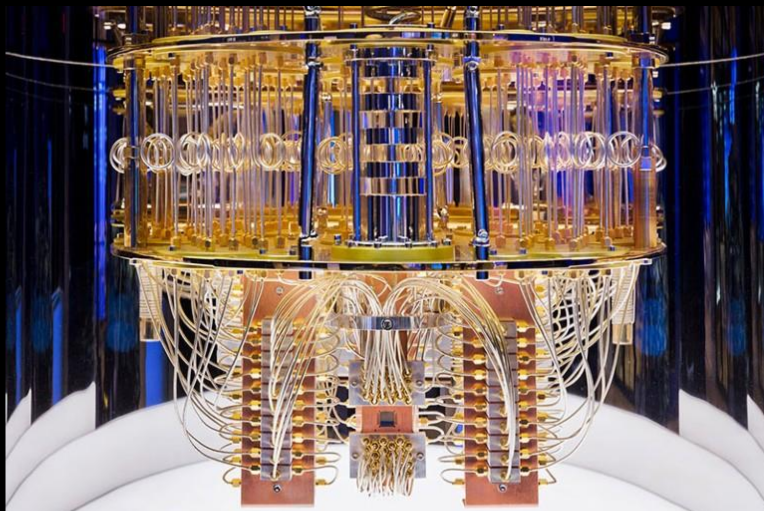
Applications mentioned in media ?

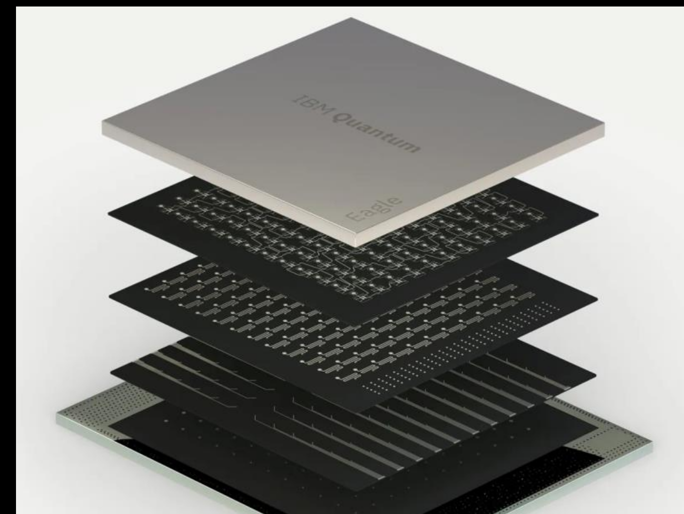


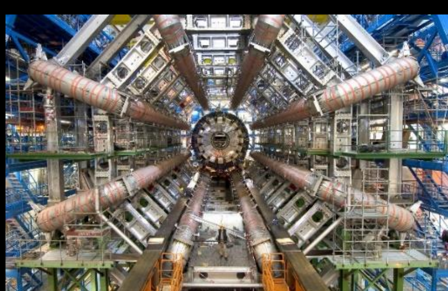


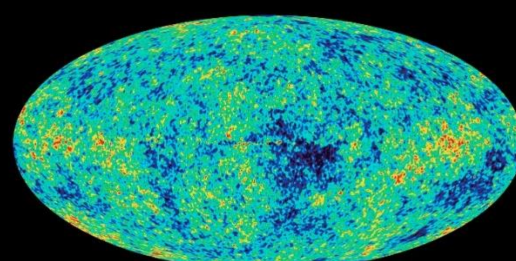
etc...

In my mind...









etc...

This lecture is on

Application of Quantum Computation to Quantum Field Theory (QFT)

- <u>Generic motivation:</u>

  simply would like to use powerful computers?

- <u>Specific motivation:</u>

This lecture is on

Application of Quantum Computation
to
Quantum Field Theory (QFT)

- Generic motivation:

simply would like to use powerful computers?

- Specific motivation:

Quantum computation is suitable for operator formalism

$\longrightarrow$ Liberation from infamous sign problem in Monte Carlo?

(next slide)

Sign problem in Monte Carlo simulation

Conventional approach to simulate QFT: (this point will be elaborated tomorrow)

① Discretize Euclidean spacetime by lattice:

& make path integral finite dimensional:

$$\int D\phi\ \mathcal{O}(\phi)e^{-S[\phi]} \Rightarrow \int d\phi\ \mathcal{O}(\phi)e^{-S(\phi)}$$

②

Sign problem in Monte Carlo simulation

Conventional approach to simulate QFT:

(this point will be elaborated tomorrow)

① Discretize Euclidean spacetime by lattice:

& make path integral finite dimensional:

$$\int D\phi\ \mathcal{O}(\phi) e^{-S[\phi]} \quad \Longrightarrow \quad \int d\phi\ \mathcal{O}(\phi) e^{-S(\phi)}$$

② Numerically Evaluate it by (Markov Chain) Monte Carlo method regarding the Boltzmann factor as a probability:

$$\langle \mathcal{O}(\phi) \rangle \simeq \frac{1}{\sharp(\text{samples})} \sum_{i \in \text{samples}} \mathcal{O}(\phi_i)$$

Sign problem in Monte Carlo simulation (Cont'd)

Markov Chain Monte Carlo:

$$\int d\phi\ \mathcal{O}(\phi)\underbrace{e^{-S(\phi)}}_{\text{probability}}$$

problematic when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$ & is highly oscillating

Examples w/ sign problem:

- topological term —— complex action
- chemical potential —— indefinite sign of fermion determinant
- real time —— " $e^{iS(\phi)}$ " much worse

Sign problem in Monte Carlo simulation (Cont'd)

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Examples w/ sign problem:

- topological term —— complex action
- chemical potential —— indefinite sign of fermion determinant
- real time —— " $e^{iS(\phi)}$ " *much worse*

In operator formalism,

sign problem is absent from the beginning

($^\exists$various approaches within framework of path integral formalism but I'll skip it)

Cost of operator formalism

We have to play with huge vector space

since QFT typically has $\underbrace{\infty\text{-dim.}}_{\text{regularization needed!}}$ Hilbert space

Technically, computers have to

memorize huge vector & multiply huge matrices

Cost of operator formalism

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Technically, computers have to

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Quantum computers do this job?

Should we care now as “users”?

Quantum computers don’t have sufficient powers yet.

Shouldn’t we start to care after quantum supremacy comes?

Should we care now as "users"?

Quantum computers don't have sufficient powers yet.

Shouldn't we start to care after quantum supremacy comes?

I personally think:

$^\exists$ Many things to do even now in various contexts

(numerical/analytic/purely algorithmic/lat/th/ph/astro/cosmo)

For instance,

- we haven't established
  - how to put QCD efficiently on quantum computers
  - how to efficiently pick up various real time physics

    (e.g. scattering/dynamical hadronization)

- $^\exists$ only few examples so far to take a serious continuum limit

[cf. Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

Plan of the lectures

(If 2nd lecture in each day ends early, then we start hands-on early)

Day 1

- Lecture 1: introduction, basics of quantum computation
- Lecture 2: Spin system on quantum computer (QC)
- Hands-on 1: Basics on IBM's qiskit

Day 2

- Lecture 3: Quantum field theory (QFT) on QC
- Lecture 4: QFT on QC, error correction & future prospects
- Hands-on 2: Time evolution of spin system

Plan of lecture 1

0. Introduction

1. Qubits and gates

2. Some demonstrations in IBM Q Experience

3. Quantum simulation of Spin system

4. Summary

Qubit = Quantum Bit

Qubit = Quantum system w/ 2 dim. Hilbert space

Basis:

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

"computational basis"

Generic state:

$$\alpha|0\rangle + \beta|1\rangle \quad \text{w/} \quad |\alpha|^2 + |\beta|^2 = 1$$

Ex.) Spin 1/2 system:

$$|0\rangle = |\uparrow\rangle, \qquad |1\rangle = |\downarrow\rangle$$

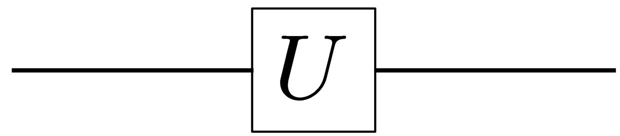
(We don't need to mind how it is realized as "users")

Single qubit operations

- **Acting unitary operator:** $|\psi\rangle \;\rightarrow\; U|\psi\rangle$ (multiplying 2x2 unitary matrix)

  In quantum circuit notation,

$|\psi\rangle$



$U|\psi\rangle$

$$= \alpha|0\rangle + \beta|1\rangle$$

$$= \alpha U|0\rangle + \beta U|1\rangle$$

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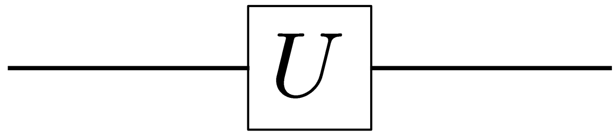
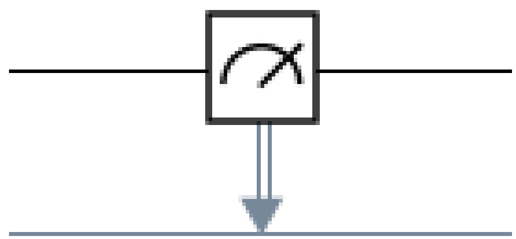


$$U|\psi\rangle = \alpha U|0\rangle + \beta U|1\rangle$$

- **Measurement:**

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$



c (classical number)

$$\begin{cases} c = 0 \text{ w/ probability } |\alpha|^2 \\ c = 1 \text{ w/ probability } |\beta|^2 \end{cases}$$

Single qubit gates used here

X, Y, Z gates : (just Pauli matrices)

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

X is "NOT": $X|0\rangle = |1\rangle, \ X|1\rangle = |0\rangle$

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$\underline{R_X, R_Y, R_Z \text{ gates :}}$

$$R_X(\theta) = e^{-\frac{i\theta}{2}X}, \qquad R_Y(\theta) = e^{-\frac{i\theta}{2}Y}, \qquad R_Z(\theta) = e^{-\frac{i\theta}{2}Z}$$

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$\underline{\text{Hadamard gate :}}$

$$H = \frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle, \qquad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle$$

Single qubit gates used here

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T gate :

$$T = e^{\frac{\pi i}{8}} R_Z\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

Multiple qubits

2 qubits – 4 dim. Hilbert space:

$$|\psi\rangle = \sum_{i,j=0,1} c_{ij}|ij\rangle, \qquad |ij\rangle \equiv |i\rangle \otimes |j\rangle$$

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \qquad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \qquad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \qquad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

N qubits – 2^N dim. Hilbert space:

$$|\psi\rangle = \sum_{i_1,\cdots i_N=0,1} c_{i_1\cdots i_N}|i_1\cdots i_N\rangle,$$

$$|i_1 i_2 \cdots i_N\rangle \equiv |i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_N\rangle$$

Two qubit gates used here

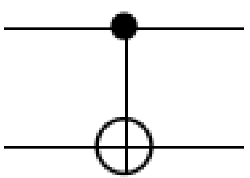
Controlled X (NOT) gate:

$$\begin{cases} CX|00\rangle = |00\rangle, & CX|01\rangle = |01\rangle, \\ CX|10\rangle = |11\rangle, & CX|11\rangle = |10\rangle \end{cases}$$

or equivalently

$$CX|0\rangle \otimes |\psi\rangle = |0\rangle \otimes |\psi\rangle, \qquad CX|1\rangle \otimes |\psi\rangle = |1\rangle \otimes X|\psi\rangle$$

$$CX = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} =$$



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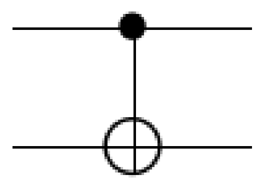
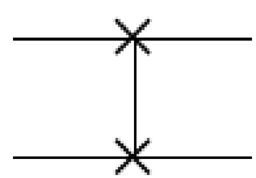


SWAP gate:

$$\mathsf{SWAP}|\psi\rangle \otimes |\phi\rangle = |\phi\rangle \otimes |\psi\rangle$$

$$\mathsf{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$



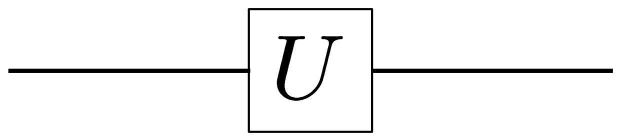
We'll see this is useful to compute Renyi entropy

Rule of the game

Do something interesting by a combination of

action of Unitary operators:

$|\psi\rangle$

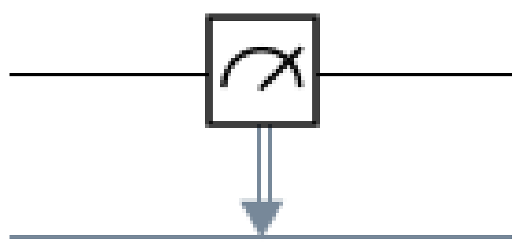


$U|\psi\rangle$

&

measurements:

$|\psi\rangle$



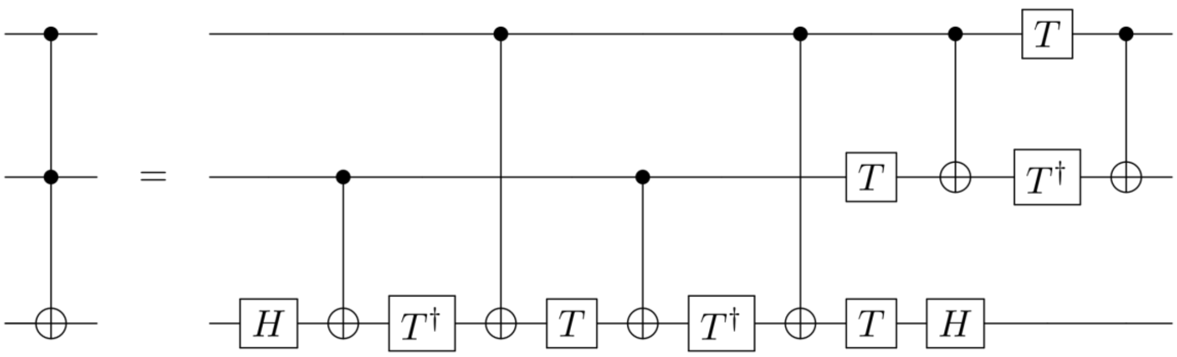
c (classical number)

Universality

- Any unitary gate is a combination of single qubit gates & CX ("Single qubit gates & CX are universal")

Ex.) Toffoli

(controlled-controlled-NOT)



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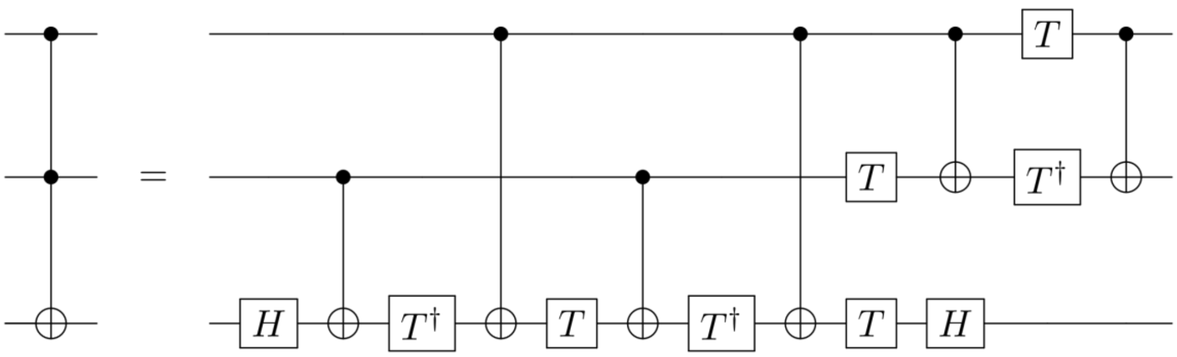


- Any single qubit gate is approximated by a combination of H & T in arbitrary precision (next slide)

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

- H, T & CX are universal

Approximation of single qubit gate by H & T

① Get a rotation with angle $2\pi \times (\text{irrational})$:

$$THTH = e^{\frac{i\pi}{4}} R_{\vec{n}}(\theta) \qquad \text{with} \;\; R_{\vec{n}}(\theta) \equiv e^{-\frac{i}{2}\vec{n}\cdot\vec{\sigma}}$$

where

$$\vec{n} = \frac{1}{\sqrt{1+\cos^2(\pi/8)}} \begin{pmatrix} cos(\pi/8) \\ \sin(\pi/8) \\ cos(\pi/8) \end{pmatrix} \quad \& \quad \underbrace{\cos(\theta/2)}_{2\pi \times (\text{irrational})!} \equiv \cos^2(\pi/8)$$

② Use Weyl's uniform distribution theorem:

$\frac{\theta}{2\pi}\boldsymbol{Z}$ is uniformly distributed mod 1 ⇒ approximate $R_{\vec{n}}(\alpha)$ for $\forall\alpha$

③ Construct rotation around another axis:

$$HR_{\vec{n}}(\alpha)H = R_{\vec{m}}(\alpha) \qquad \text{with} \quad \vec{m} = \frac{1}{\sqrt{1+\cos^2(\pi/8)}} \begin{pmatrix} cos(\pi/8) \\ -\sin(\pi/8) \\ cos(\pi/8) \end{pmatrix}$$

④ Approximate ∀single qubit gate: $R_{\vec{n}}(\alpha) R_{\vec{m}}(\beta)\, R_{\vec{n}}(\gamma)$

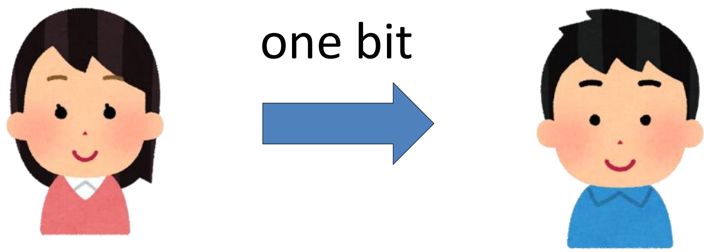
(To achieve accuracy ϵ, it requires $\mathcal{O}(\log^c(1/\epsilon)\,)$ gates w/ $c \sim 2$) [Solovay '95, Kitaev '97]

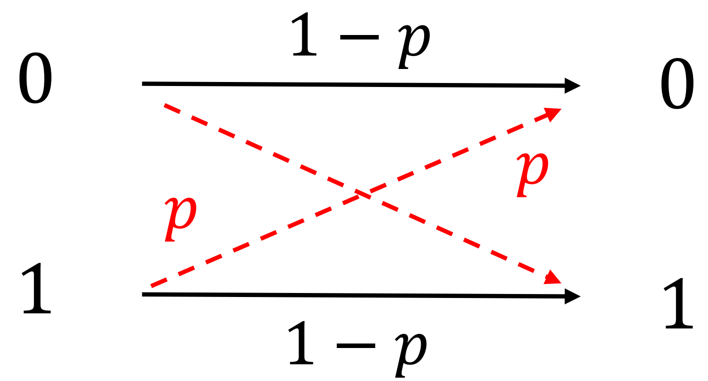
Errors in classical computers

Computer interacts w/ environment ⇒ error/noise

Errors in classical computers

Computer interacts w/ environment $\Rightarrow$ error/noise



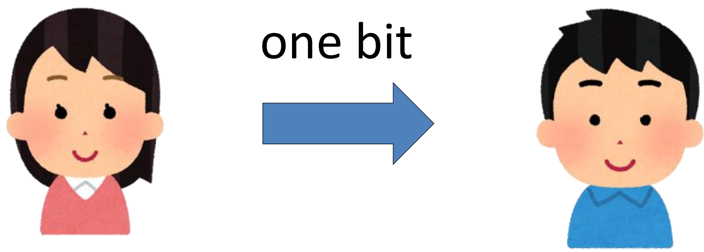


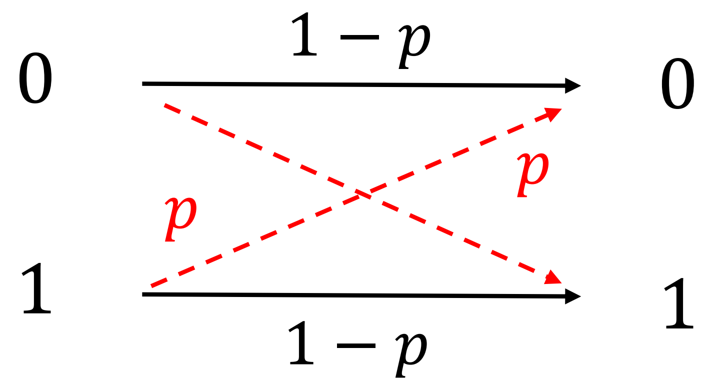
Suppose we send a bit but have “error” in probability p

A simple way to correct errors:

Errors in classical computers

Computer interacts w/ environment $\Rightarrow$ error/noise





Suppose we send a bit but have "error" in probability p

A simple way to correct errors:

① Duplicate the bit (encoding): $0 \to 000, \quad 1 \to 111$

② Error detection & correction by "majority voting":

$$001 \to 000, \quad 011 \to 111, \quad \text{etc...}$$

$\Rightarrow$ $P_{\text{failed}} = 3p^2(1-p) + p^3$ (improved if $p < 1/2$)

Errors in quantum computers

Computer interacts w/ environment ⇨ error/noise

(In addition to decoherence & measurement errors)

Unknown unitary operators are multiplied:

$$|\psi\rangle \xrightarrow{\text{error!}} U|\psi\rangle$$

not only bit flip!

We need to include "quantum error corrections"
but it seems to require a huge number of qubits

$\sim$ major obstruction of the development

(We will come back to this point on the 3rd day)

(Classical) simulator for Quantum computer

Quantum computation $\subset$ Linear algebra

The same algorithm can be implemented in classical computer but w/o speed-up (1 quantum step = many classical steps)

Simulator = Tool to simulate quantum computer by classical computer

- Doesn't have errors → ideal answers

  (More precisely, classical computer also has errors but its error correction is established)

- The same code can be run in quantum computer w/ speed-up

Useful to test algorithm & estimate computational resources

(~# of qubits, gates)

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Short summary

- Qubit = Quantum bit
- Important gates:

$$R_X(\theta) = e^{-\frac{i\theta}{2}X}, \quad R_Y(\theta) = e^{-\frac{i\theta}{2}Y}, \quad R_Z(\theta) = e^{-\frac{i\theta}{2}Z}$$

$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \equiv |+\rangle, \quad H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \equiv |-\rangle$$

$$CX|0\rangle \otimes |\psi\rangle = |0\rangle \otimes |\psi\rangle, \quad CX|1\rangle \otimes |\psi\rangle = |1\rangle \otimes X|\psi\rangle$$

- Do something interesting by a combination of acting unitary op. & measurement
- H, T & CX are universal

$$T = e^{\frac{\pi i}{8}} R_Z\left(\frac{\pi}{4}\right) = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

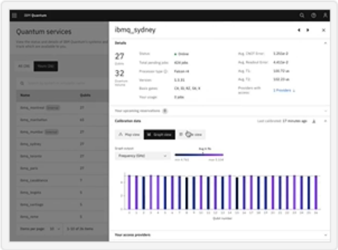
- Real quantum computer has errors
- Simulator = Tool to simulate quantum computer by classical computer

Some demonstrations in IBM Quantum

IBM Quantum

Real quantum computers. Right at your fingertips.

IBM offers cloud access to the most advanced quantum computers available.
Learn, develop, and run programs with our quantum applications and systems.



View quantum system details

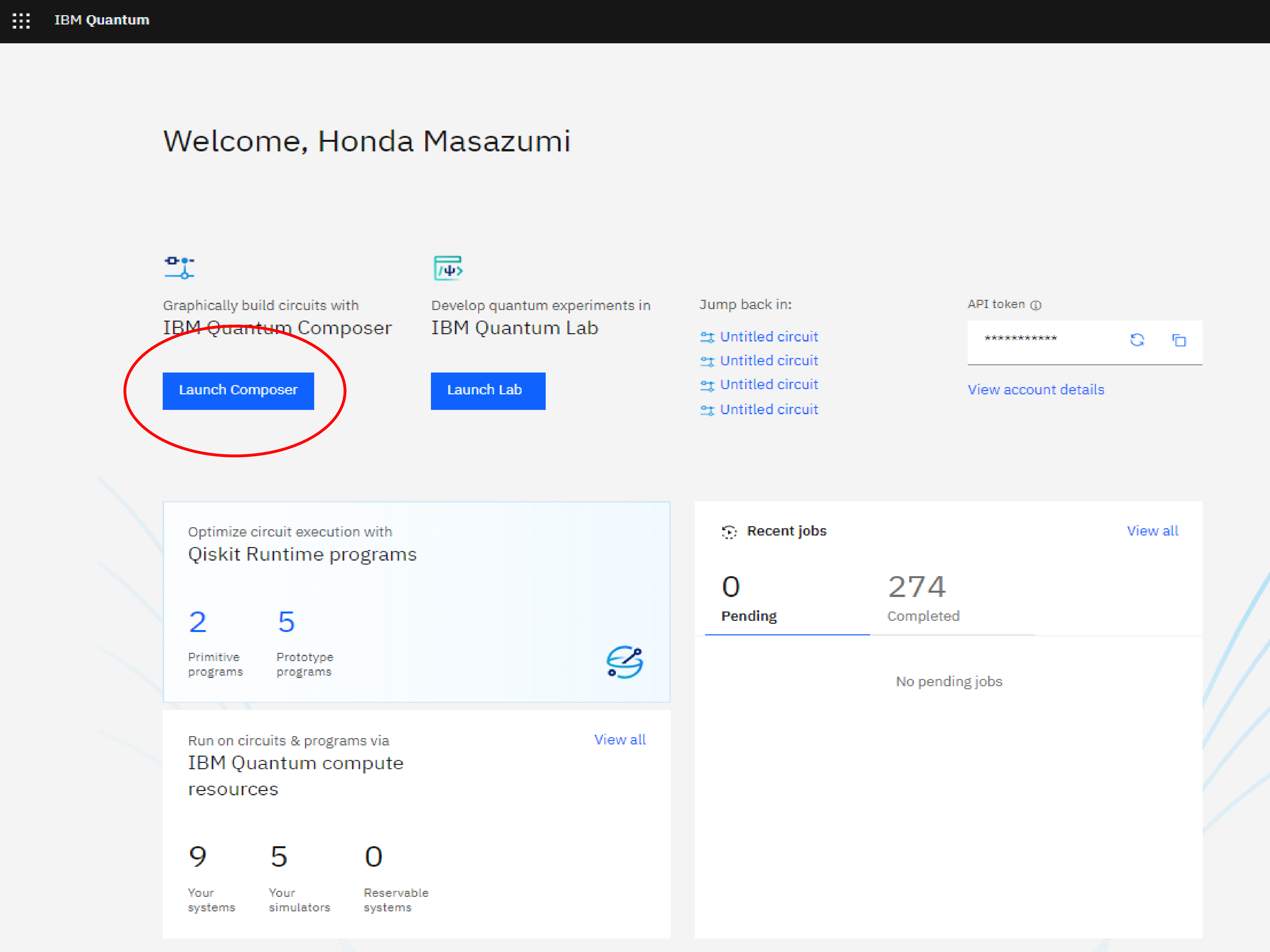
Check out the status, topology, calibration data, and access details of your IBM quantum systems.

Sign in to IBM Quantum

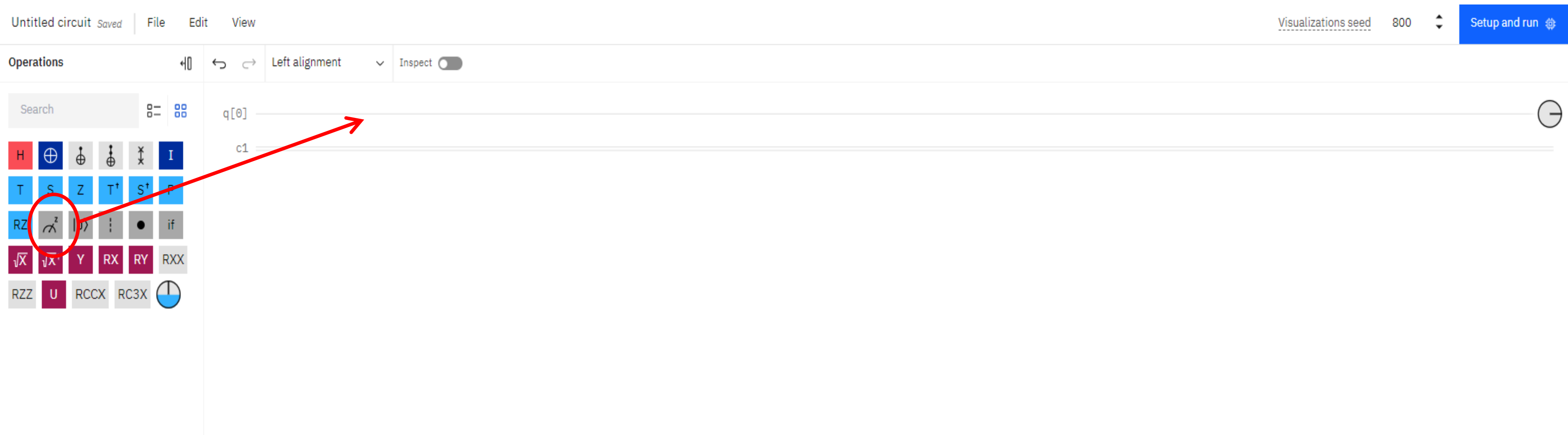
IBMid

New to IBM Quantum?
Create an IBMid account.

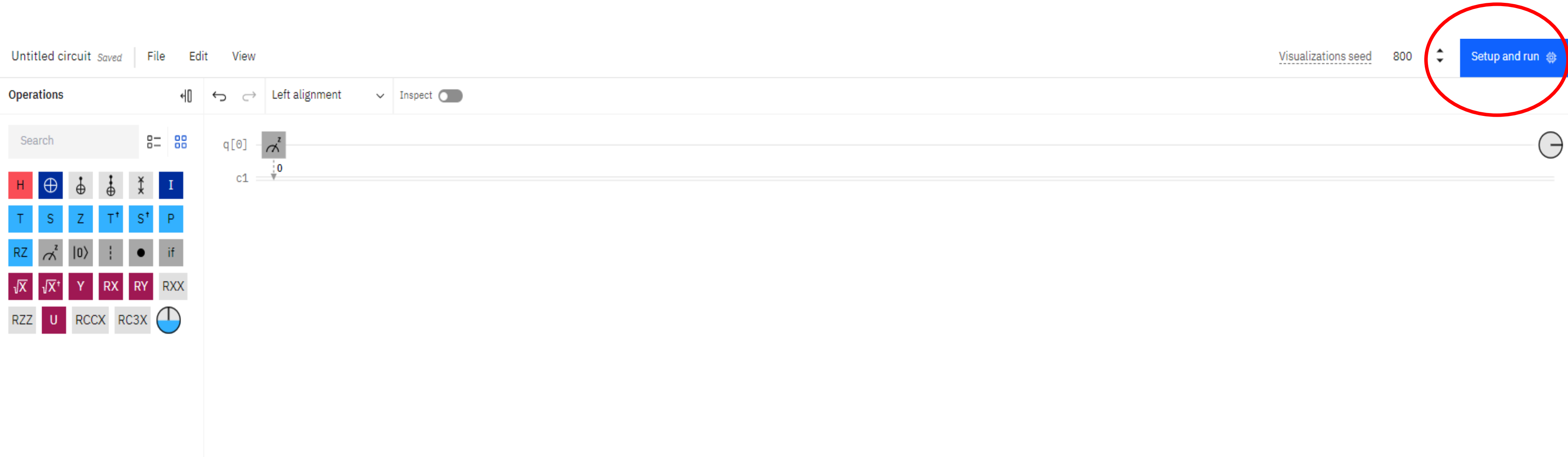
Having trouble signing in?


IBM Quantum
Welcome, Honda Masazumi
Graphically build circuits with
IBM Quantum Composer
Launch Composer
Develop quantum experiments in
IBM Quantum Lab
Launch Lab
Jump back in:
Untitled circuit
Untitled circuit
Untitled circuit
Untitled circuit
API token
****
View account details
Optimize circuit execution with
Qiskit Runtime programs
2
Primitive programs
5
Prototype programs
Recent jobs
View all
0
Pending
274
Completed
No pending jobs
Run on circuits & programs via
IBM Quantum compute resources
View all
9
Your systems
5
Your simulators
0
Reservable systems

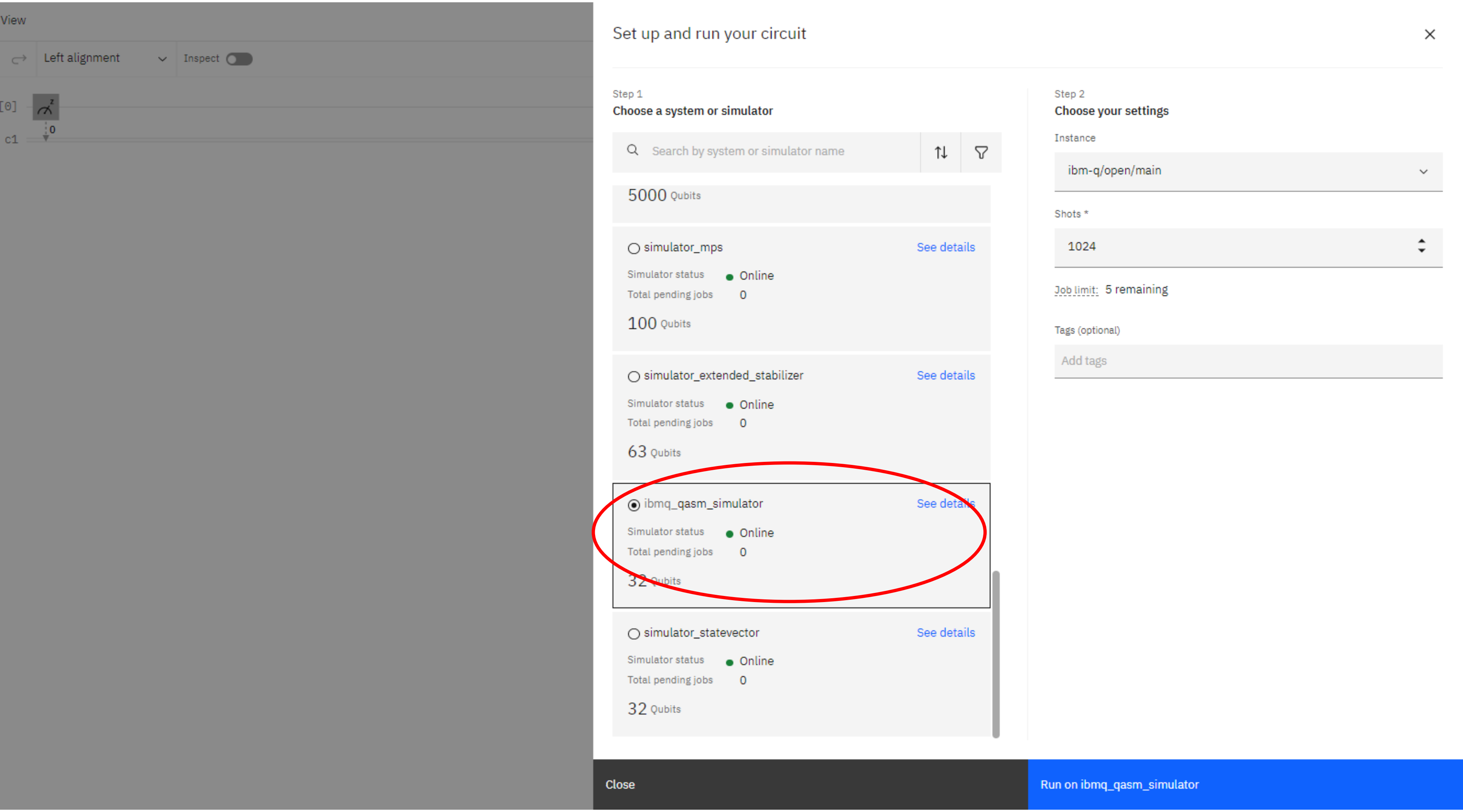
A trivial problem: measure $|0\rangle$



A trivial problem: measure $|0\rangle$ (Cont'd)



Measure 1024 times in simulator



Trivial result

Details

582ms
Total completion time

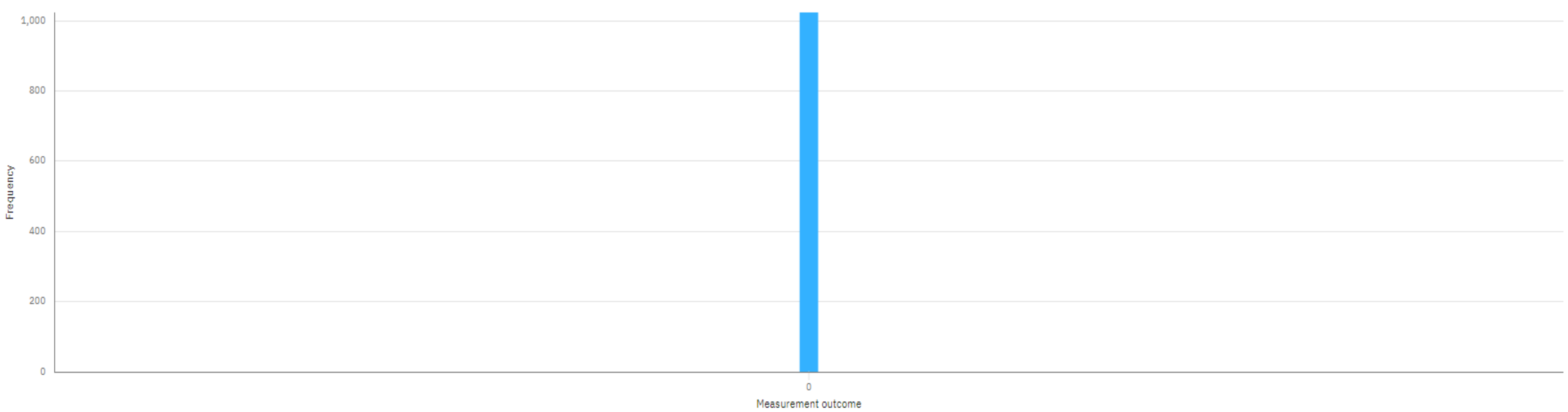
ibmq_qasm_simulator
Compute resource

Sent from: Untitled circuit
Status: Completed
Instance: ibm-q/open/main
Program: circuit-runner
# of shots: 1024
# of circuits: 1

Status Timeline

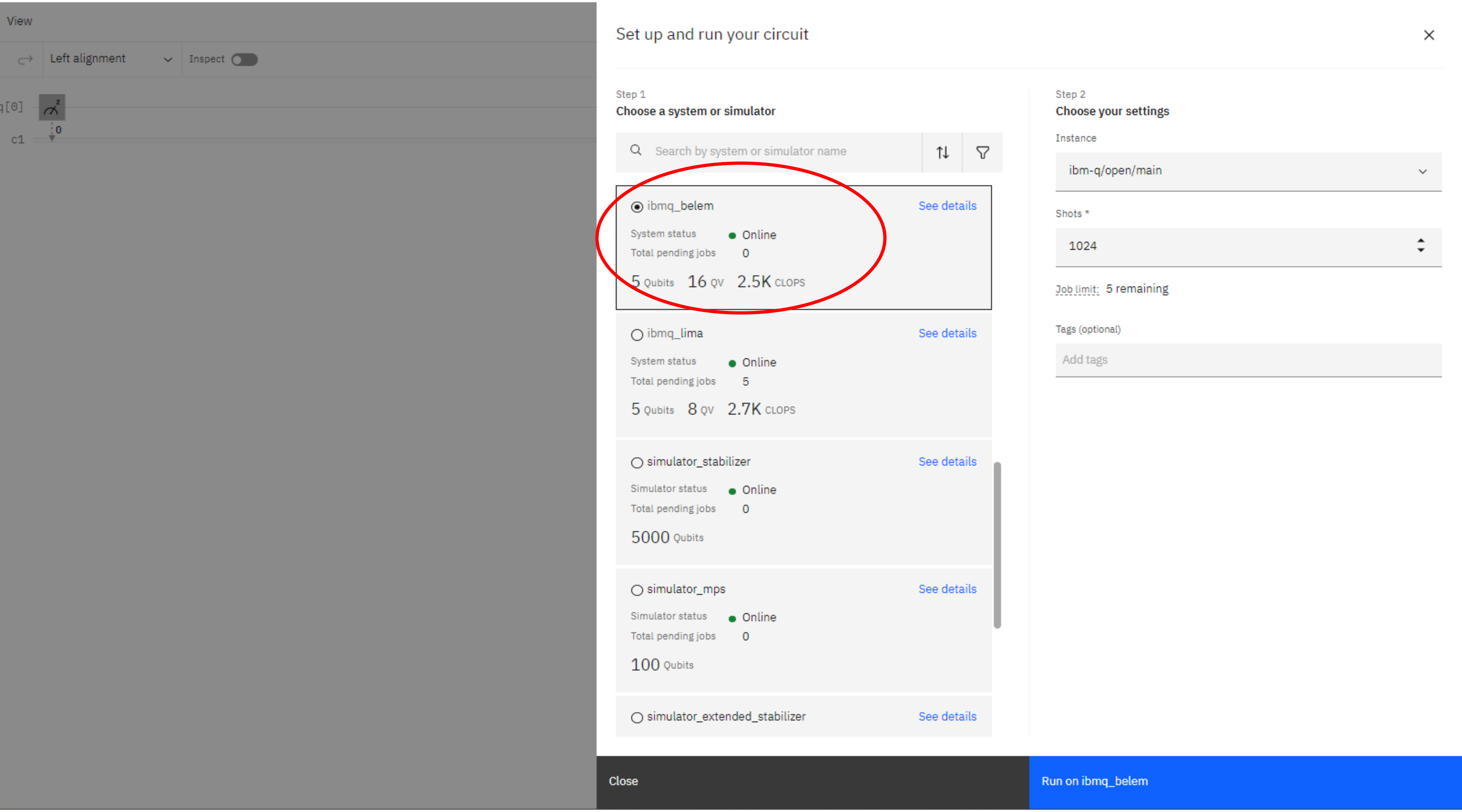
- Created: Apr 16, 2023 12:10 AM
- In queue: less than a second
- Running: Apr 16, 2023 12:10 AM
  time in classical and quantum computation 1s
- Completed: Apr 16, 2023 12:10 AM

Histogram



Of Course!

Measure 1024 times in quantum computer



Result of quantum computer?

Details

6.8s
Total completion time

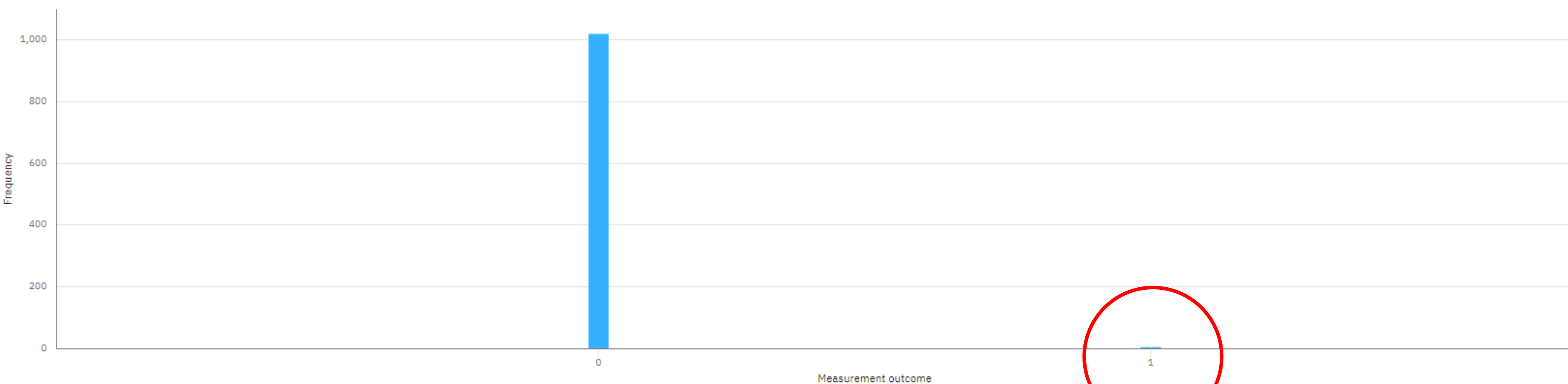
ibmq_belem
Compute resource

Sent from: Untitled circuit
Status: Completed
Instance: ibm-q/open/main
Program: circuit-runner
# of shots: 1024
# of circuits: 1

Status Timeline

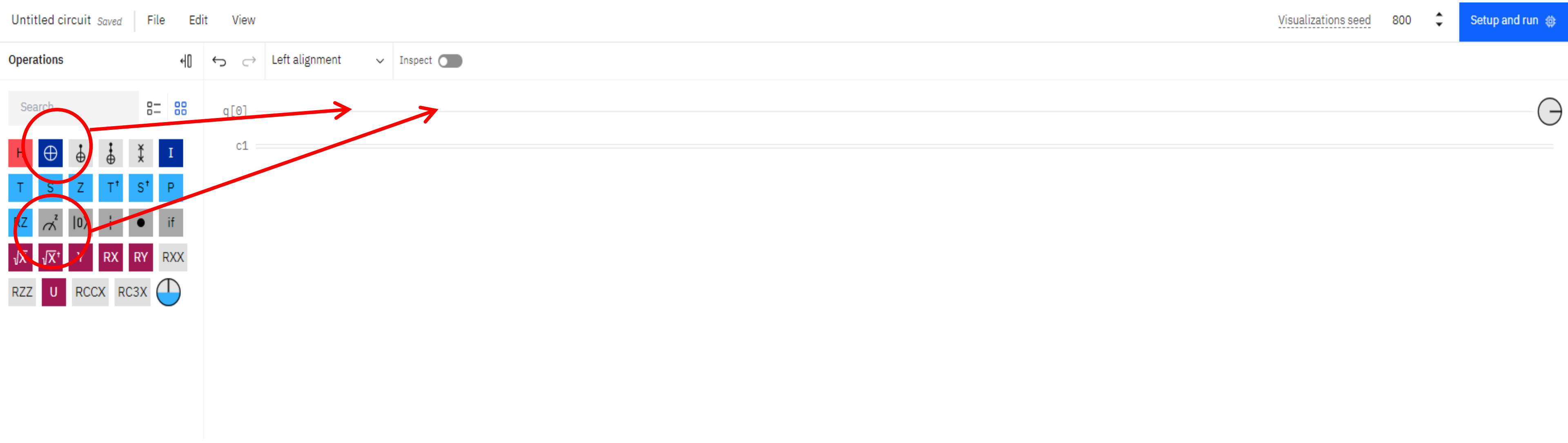
- Created: Apr 16, 2023 12:14 AM
- In queue: less than a second
- Running: Apr 16, 2023 12:14 AM
  time in classical and quantum computation 5s
- Completed: Apr 16, 2023 12:14 AM

Histogram



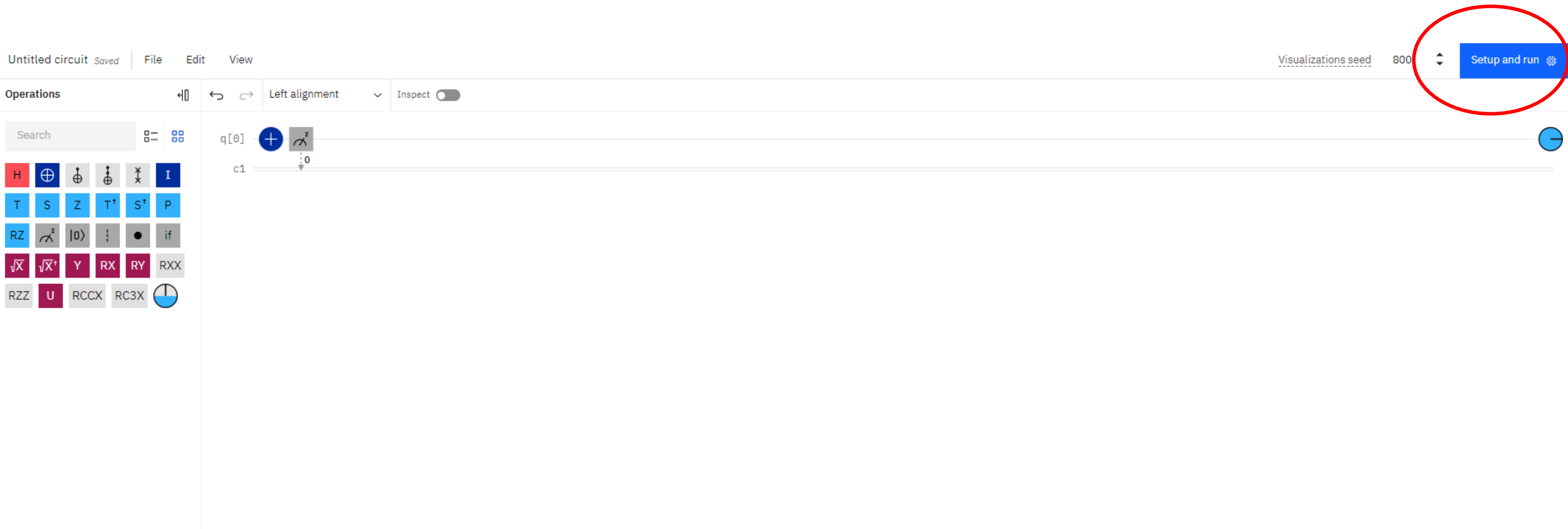
This is the error!

Another trivial problem: measure $|1\rangle$



$$X|0\rangle = |1\rangle$$

Another trivial problem: measure $|1\rangle$ (Cont'd)



Result of simulator (1024 shots)

Details

730ms
Total completion time

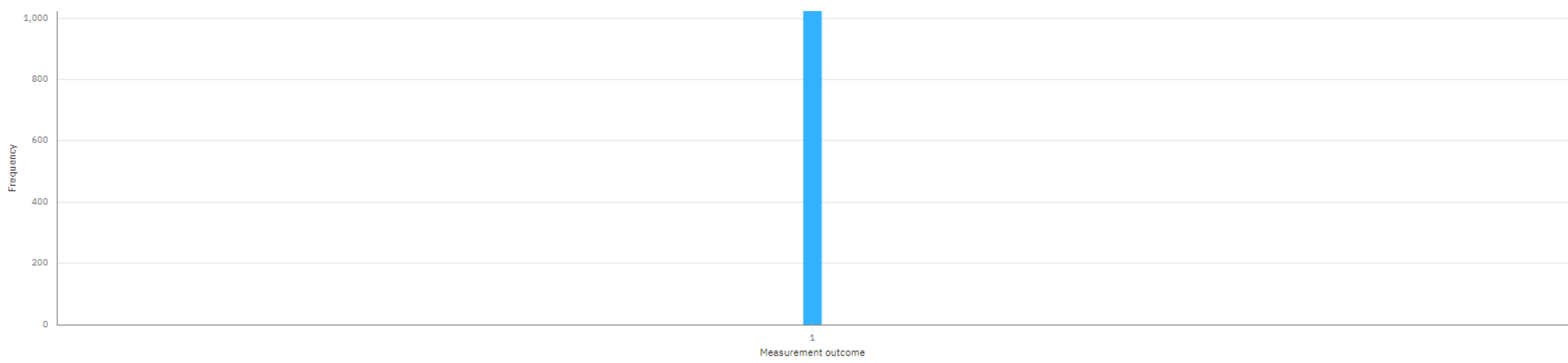
ibmq_qasm_simulator
Compute resource

Sent from: Untitled circuit
Status: Completed
Instance: ibm-q/open/main
Program: circuit-runner
# of shots: 1024
# of circuits: 1

Status Timeline

- Created: Apr 16, 2023 12:24 AM
- In queue: less than a second
- Running: Apr 16, 2023 12:24 AM
  time in classical and quantum computation 1s
- Completed: Apr 16, 2023 12:24 AM

Histogram



Result of quantum computer (1024shots)

Details

3.5s
Total completion time

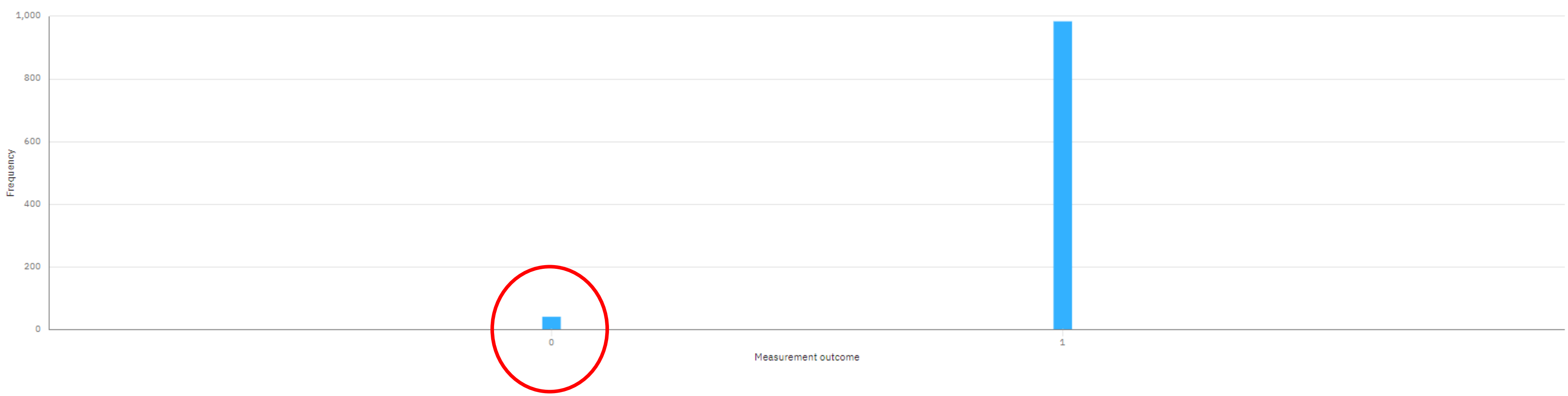
ibmq_belem
Compute resource

Sent from: Untitled circuit
Status: Completed
Instance: ibm-q/open/main
Program: circuit-runner
# of shots: 1024
# of circuits: 1

Status Timeline

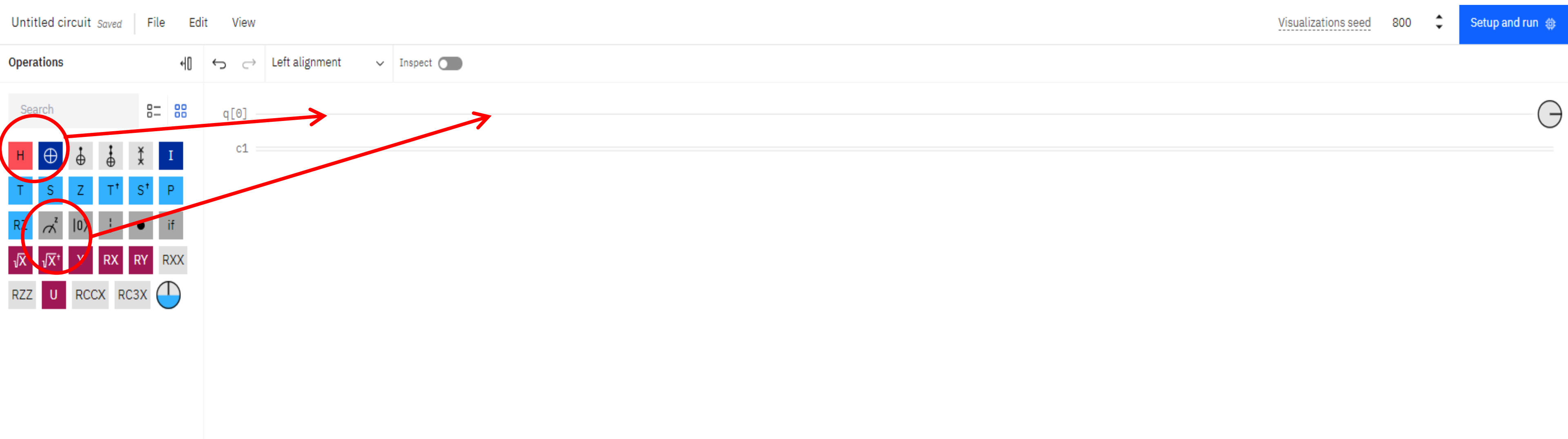
- Created: Apr 16, 2023 12:24 AM
- In queue: less than a second
- Running: Apr 16, 2023 12:24 AM
  time in classical and quantum computation 2s
- Completed: Apr 16, 2023 12:24 AM

Histogram



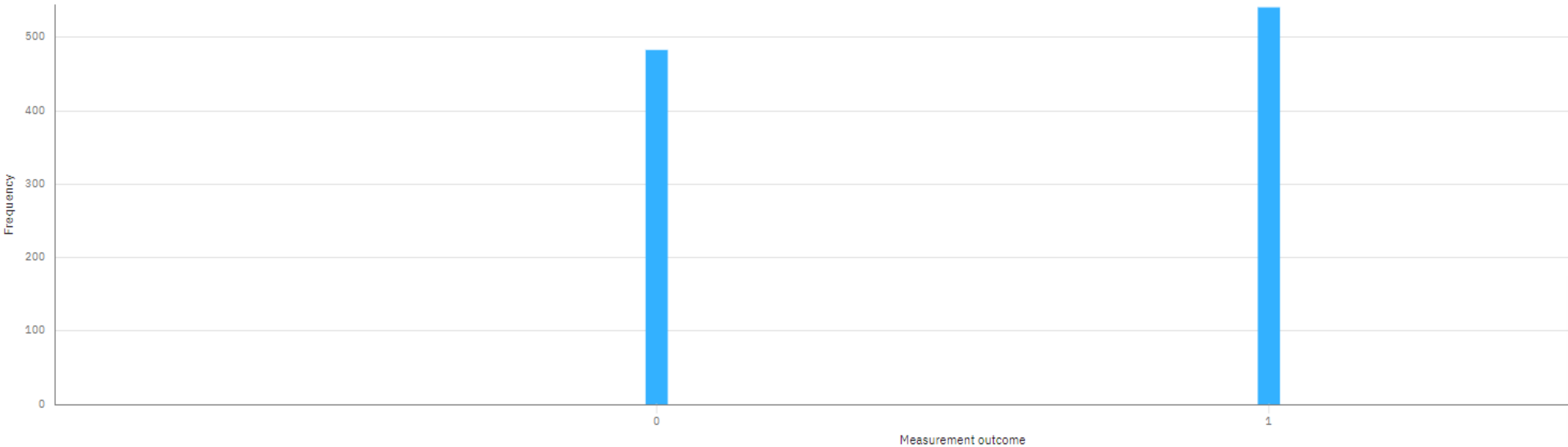
Error again

The simplest nontrivial problem: Hadamard gate



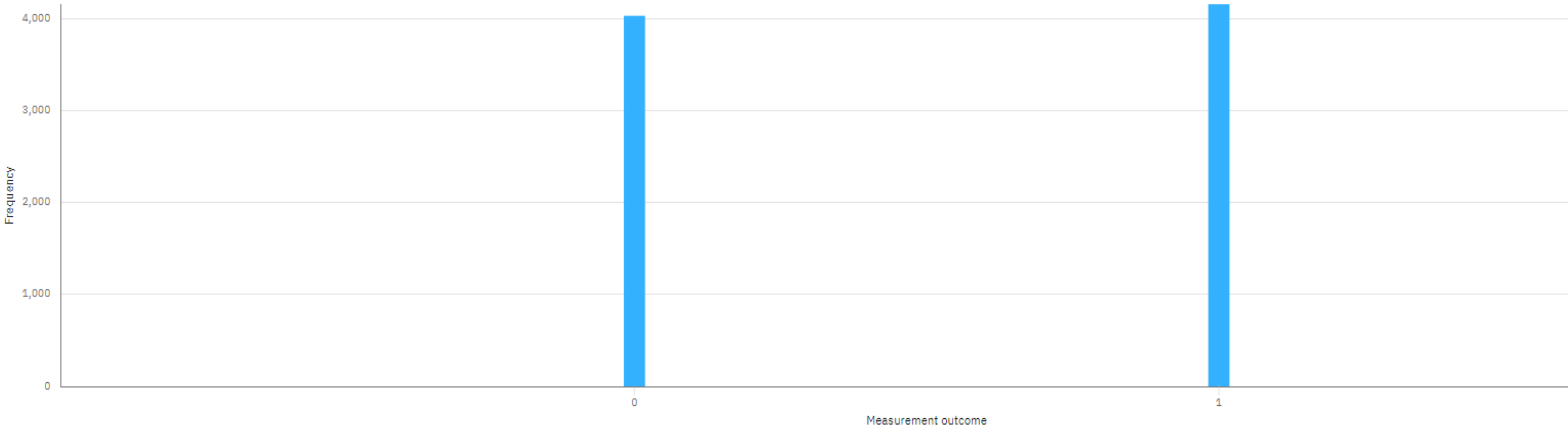
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

Result of simulator (1024 shots)



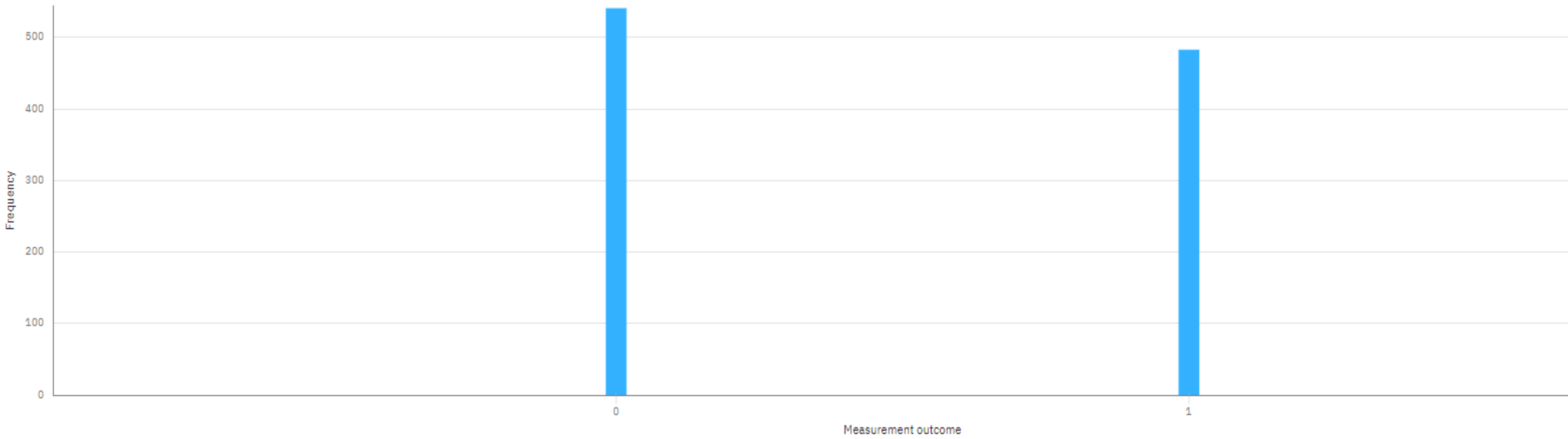
Not exactly 50:50 because of **statistical errors**

Result of simulator (8192 shots)



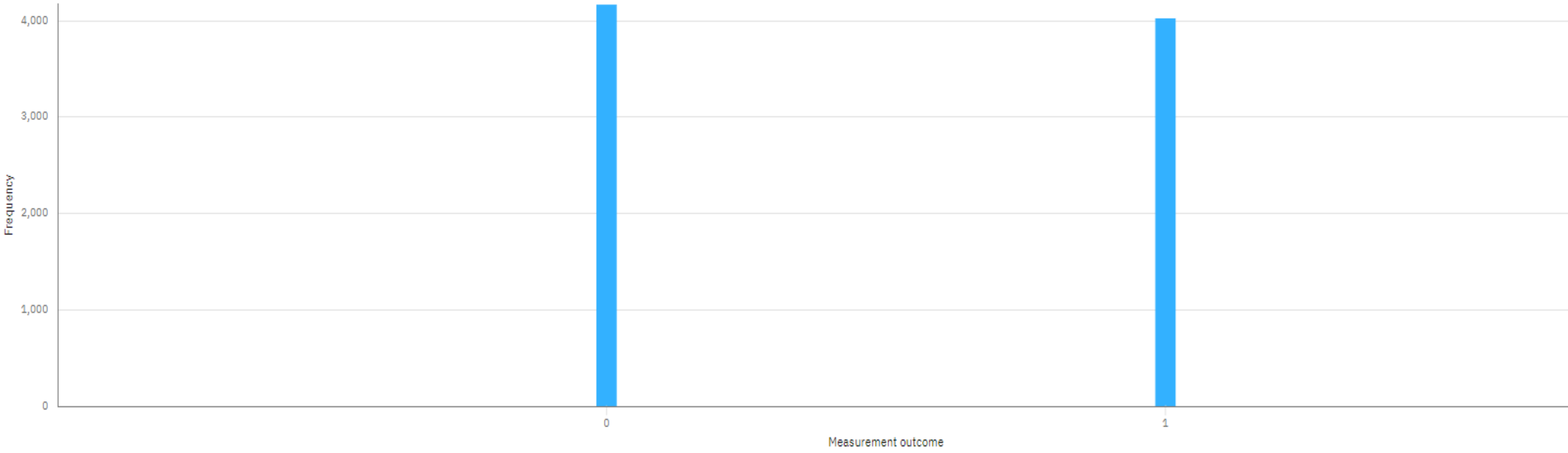
Improved!

Result of quantum computer (1024 shots)



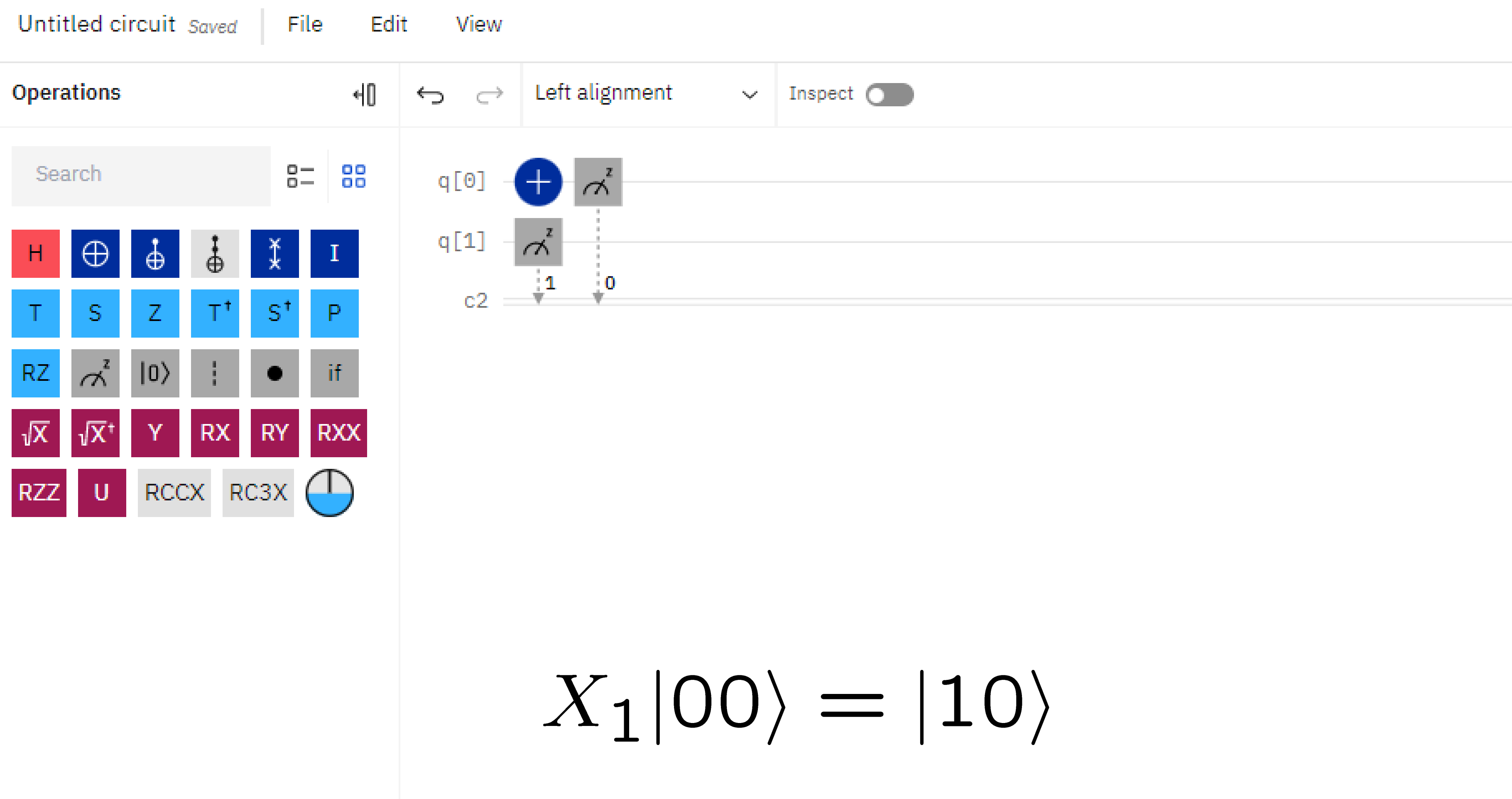
∃ Both errors & statistical errors

Result of quantum computer (8192 shots)



Statistical errors are improved

A trivial problem for 2 qubits



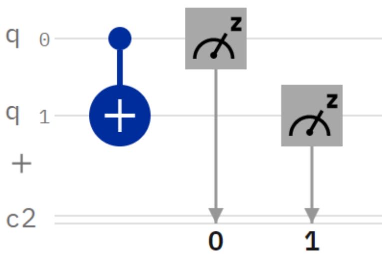
$$X_1|00\rangle = |10\rangle$$

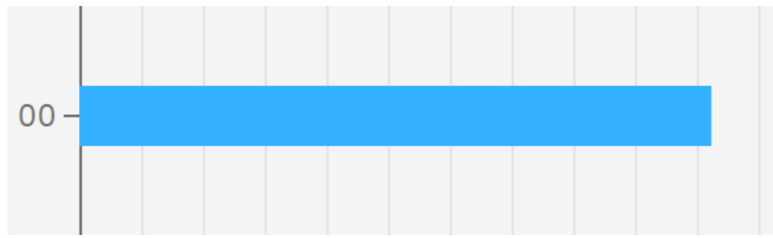
Result of simulator (1024 shots)

$$\boxed{X_1|00\rangle = |10\rangle}$$

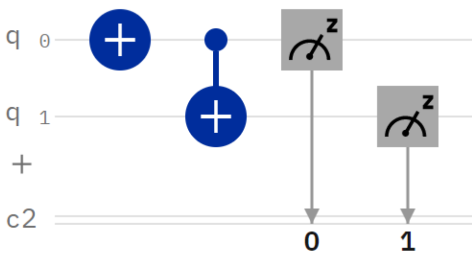
Note that notation is different from the ket notation

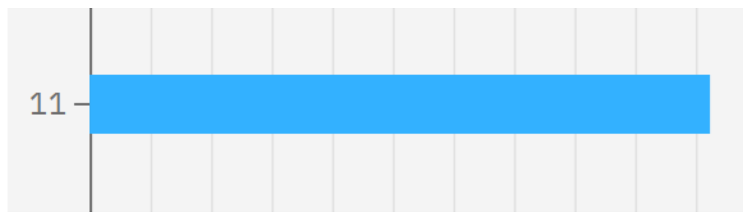
2 qubit operation by simulator



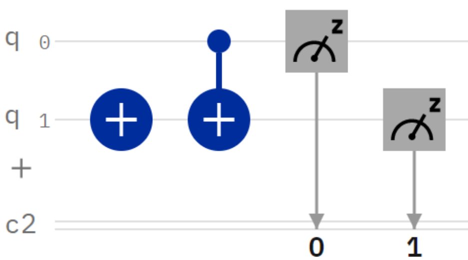


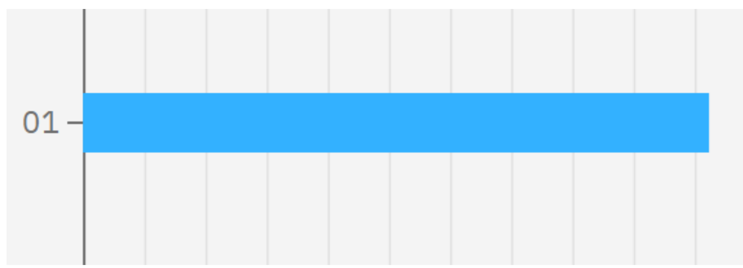
$$CX|00\rangle = |00\rangle$$



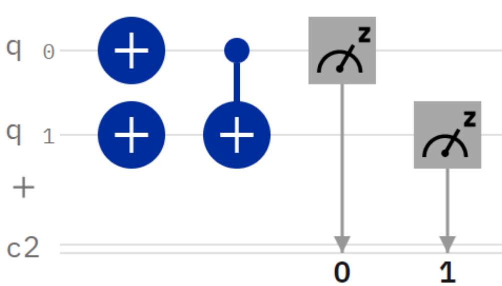


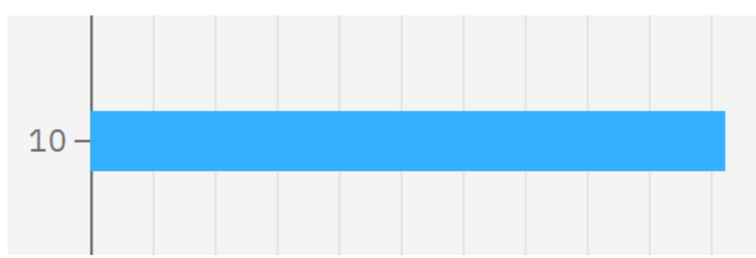
$$CX|10\rangle = |11\rangle$$





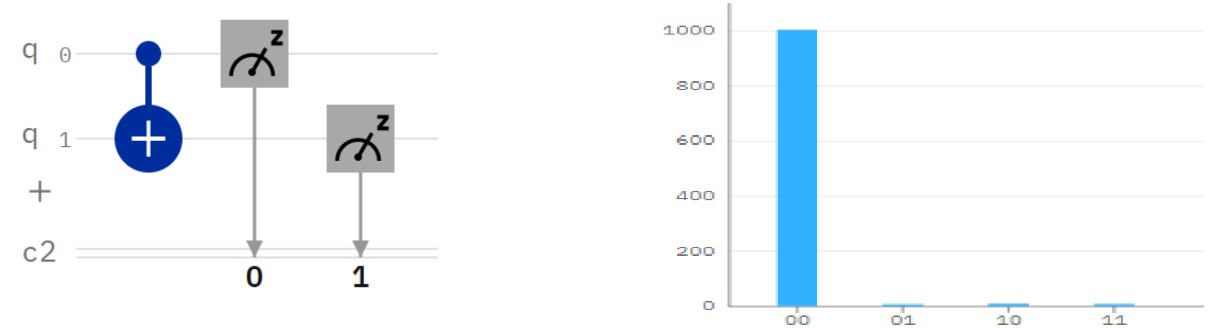
$$CX|01\rangle = |01\rangle$$



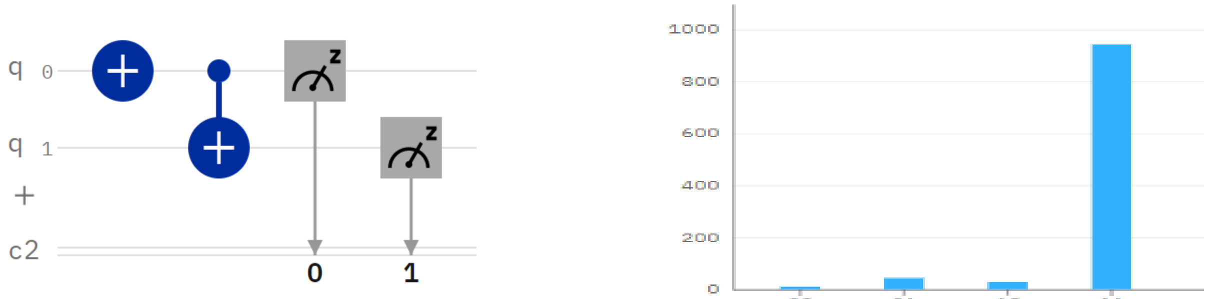


$$CX|11\rangle = |10\rangle$$

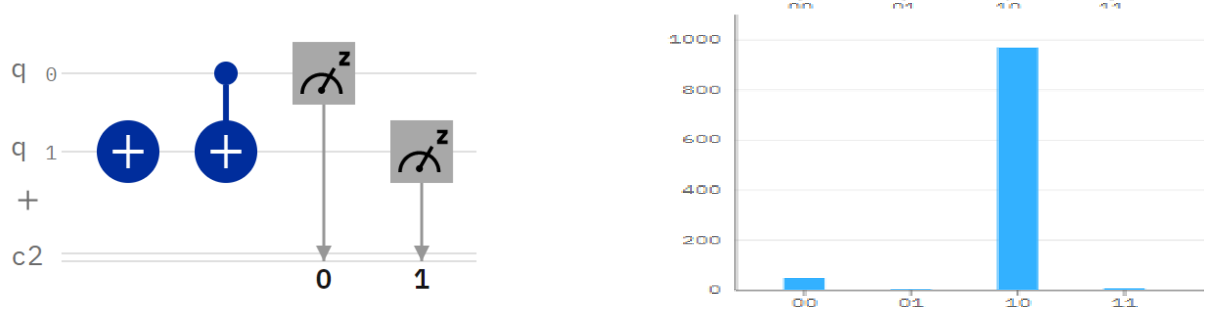
2 qubit operation by quantum computer (1024 shots)



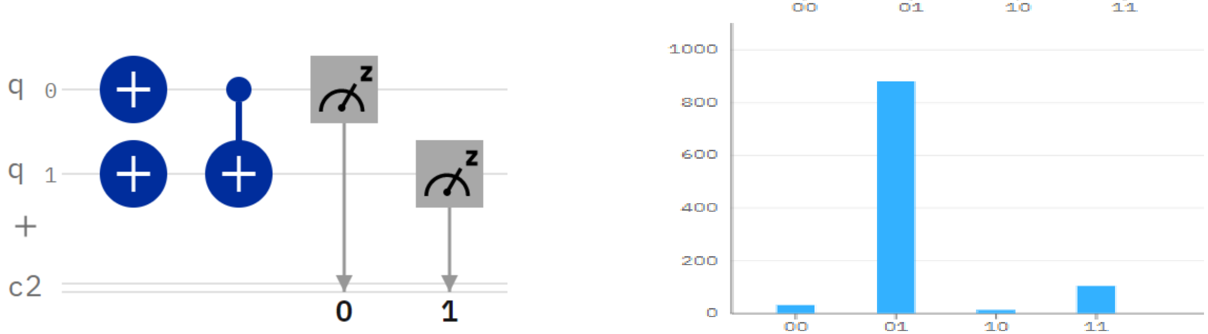
$$CX|00\rangle = |00\rangle$$



$$CX|10\rangle = |11\rangle$$



$$CX|01\rangle = |01\rangle$$



$$CX|11\rangle = |10\rangle$$

Tutorial 0: Play with IBMQ

(until the end of this class)

(Here is the expected end of the lecture part of the 1st class)

Plan of the lectures

(If 2nd lecture in each day ends early, then we start hands-on early)

Day 1

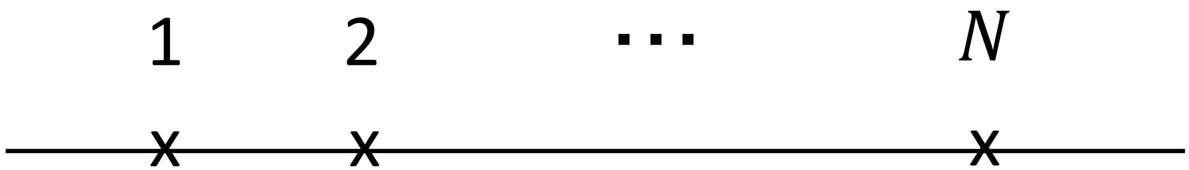
- Lecture 1: introduction, basics of quantum computation
- Lecture 2: Spin system on quantum computer (QC)
- Hands-on 1: Basics on IBM's qiskit

Day 2

- Lecture 3: Quantum field theory (QFT) on QC
- Lecture 4: QFT on QC, error correction & future prospects
- Hands-on 2: Time evolution of spin system

Quantum simulation of Spin system

The (1+1)d transverse Ising model



Hamiltonian (w/ open b.c.): (X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

$$\hat{H} = -J\sum_{n=1}^{N-1} Z_n Z_{n+1} - h\sum_{n=1}^{N} X_n - m\sum_{n=1}^{N} Z_n$$

For simplicity, take $N = 2$ & $m = 0$ for a while:

$$\boxed{\hat{H} = -JZ_1Z_2 - h(X_1 + X_2)}$$

Let’s construct the time evolution op. $e^{-i\hat{H}t}$

Warm up: 2-site transverse Ising model

$$\hat{H} = -JZ_1Z_2 - h(X_1 + X_2)$$

We are going to

- construct time evolution operator
- obtain vacuum state
- compute vacuum expectation values
- compute Renyi entropy

Time evolution operator

Time evolution of any state is studied by acting the operator

$$e^{-i\hat{H}t} = e^{-i(H_X + H_{ZZ})t}$$

where

$$H_X = -h(X_1 + X_2), \quad H_{ZZ} = -JZ_1Z_2$$

How do we express this in terms of elementary gates?

(such as $X, Y, Z, R_{X,Y,Z}, CX$ etc...)

Time evolution operator

Time evolution of any state is studied by acting the operator

$$e^{-i\hat{H}t} = e^{-i(H_X + H_{ZZ})t}$$

where

$$H_X = -h(X_1 + X_2), \quad H_{ZZ} = -JZ_1Z_2$$

How do we express this in terms of elementary gates?

(such as $X, Y, Z, R_{X,Y,Z}, CX$ etc...)

Step 1: Suzuki-Trotter decomposition:

($^\exists$ higher order improvements)

$$e^{-i\hat{H}t} = \left(e^{-i\hat{H}\frac{t}{M}}\right)^M$$

(M: large positive integer)

$$\simeq \left(e^{-iH_X\frac{t}{M}}e^{-iH_{ZZ}\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

Time evolution operator (Cont'd)

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_X \frac{t}{M}} e^{-iH_{ZZ}\frac{t}{M}}\right)^M$$

The 1st one is trivial:

acting on qubit 2 acting on qubit 1

$$e^{-iH_X \frac{t}{M}} = e^{-i\frac{ht}{M}X_2} e^{-i\frac{ht}{M}X_1} = R_X^{(2)}\left(\frac{2ht}{M}\right) R_X^{(1)}\left(\frac{2ht}{M}\right)$$

Time evolution operator (Cont'd)

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_X\frac{t}{M}}e^{-iH_{ZZ}\frac{t}{M}}\right)^M$$

The 1st one is trivial:

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The 2nd one is nontrivial:

$$e^{-iH_{ZZ}\frac{t}{M}} = e^{-i\frac{Jt}{M}Z_1Z_2} = \cos\frac{Jt}{M} - iZ_1Z_2\sin\frac{Jt}{M}$$

One can show (see next slide)

$$e^{-i\frac{Jt}{M}Z_1Z_2} = CX\,R_Z^{(2)}\left(\frac{2Jt}{M}\right)CX$$

Time evolution operator (Cont'd)

$$e^{-icZ_1Z_2} = CXR_Z^{(2)}(2c)CX$$

Proof:

$$CXR_Z^{(2)}(2c)CX|0\rangle \otimes |\psi\rangle$$

$$= CXR_Z^{(2)}(2c)|0\rangle \otimes |\psi\rangle = CX|0\rangle \otimes R_Z(2c)|\psi\rangle$$

$$= |0\rangle \otimes R_Z(2c)|\psi\rangle = \cos c|0\rangle \otimes |\psi\rangle - i\sin c\ Z|0\rangle \otimes Z|\psi\rangle$$

$$CXR_Z^{(2)}(2c)CX|1\rangle \otimes |\psi\rangle$$

$$= CXR_Z^{(2)}(2c)|1\rangle \otimes X|\psi\rangle = CX|1\rangle \otimes R_Z(2c)X|\psi\rangle = |1\rangle \otimes XR_Z(2c)X|\psi\rangle$$

$$= \cos c|1\rangle \otimes XX|\psi\rangle - i\sin c\ |1\rangle \otimes XZX|\psi\rangle$$

$$= \cos c|1\rangle \otimes |\psi\rangle - i\sin c\ Z|1\rangle \otimes Z|\psi\rangle$$

Thus,

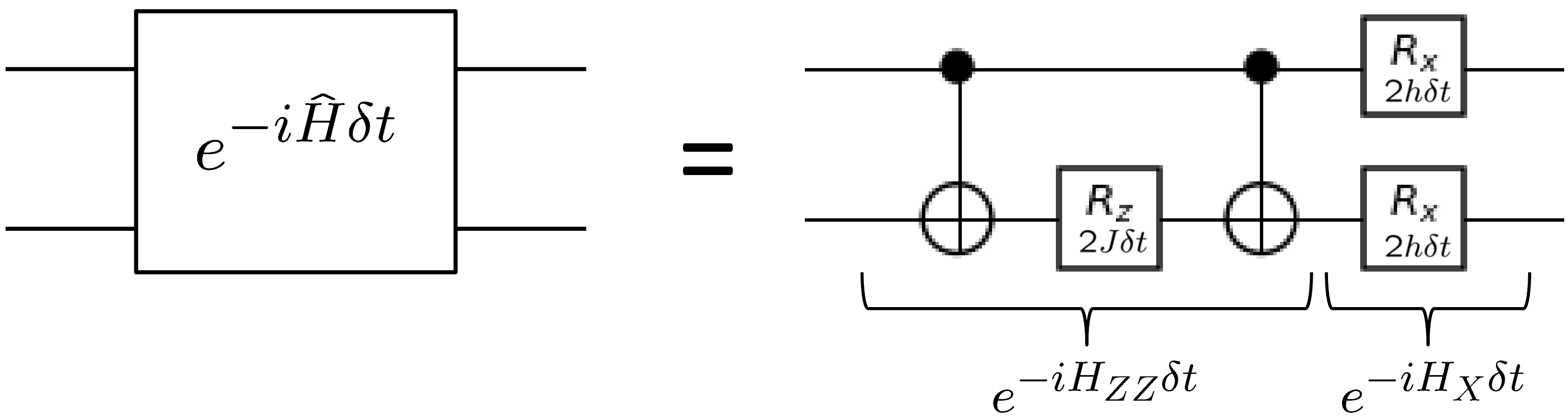
$$CXR_Z^{(2)}(2c)CX|\varphi\rangle \otimes |\psi\rangle = \cos c|\varphi\rangle \otimes |\psi\rangle - i\sin c\ Z|\varphi\rangle \otimes Z|\psi\rangle$$

$$= e^{-icZ_1Z_2}|\varphi\rangle \otimes |\psi\rangle$$

Quantum circuit for time evolution op.

$$H_X = -h(X_1 + X_2), \qquad H_{ZZ} = -JZ_1Z_2$$

$$\delta t = \frac{t}{M} \ll 1$$



$$+\mathcal{O}(\delta t)$$

Survival probability of free vacuum

For $J = 0$, the ground state is

$$\hat{H}|_{J=0} = -h(X_1 + X_2)$$

$$|++\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}} = H^{(2)}H^{(1)}|00\rangle$$

We can compute survival probability of the free vacuum:

$$P(t) = \left|\langle ++|e^{-i\hat{H}t}|++\rangle\right|^2$$

toy version of Schwinger effect

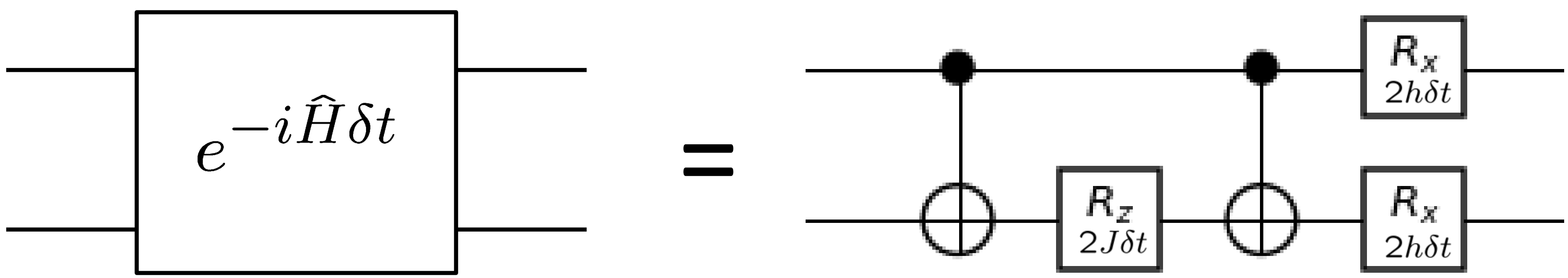
$$= \left|\langle 00|H^{(2)}H^{(1)}e^{-i\hat{H}t}H^{(2)}H^{(1)}|00\rangle\right|^2$$

Measure the probability having $|00\rangle$ inside the state

$$H^{(2)}H^{(1)}e^{-i\hat{H}t}H^{(2)}H^{(1)}|00\rangle$$

Demonstration for the survival probability

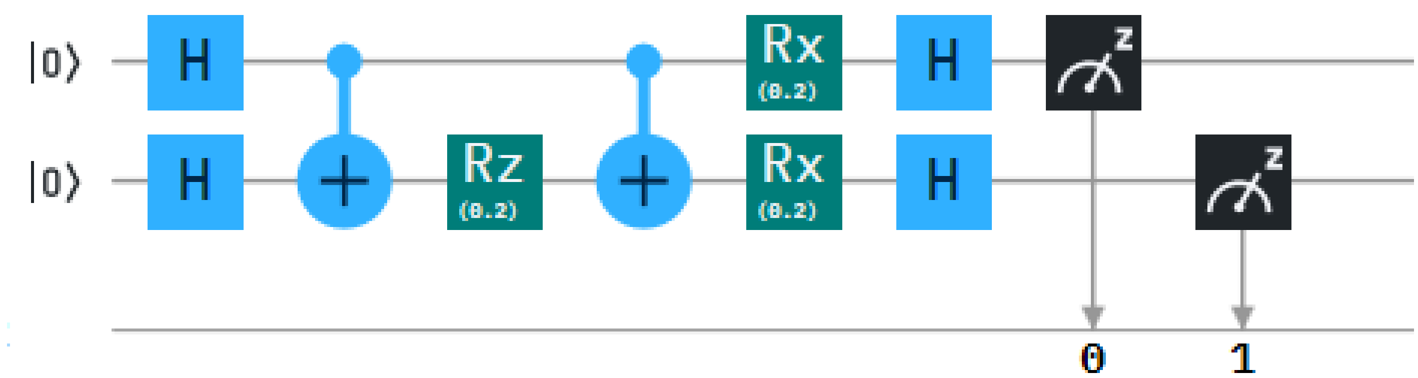
$$P(t) = \left|\langle ++|e^{-i\widehat{H}t}|++\rangle\right|^2 = \left|\langle 00|H^{(2)}H^{(1)}e^{-i\widehat{H}t}H^{(2)}H^{(1)}|00\rangle\right|^2$$



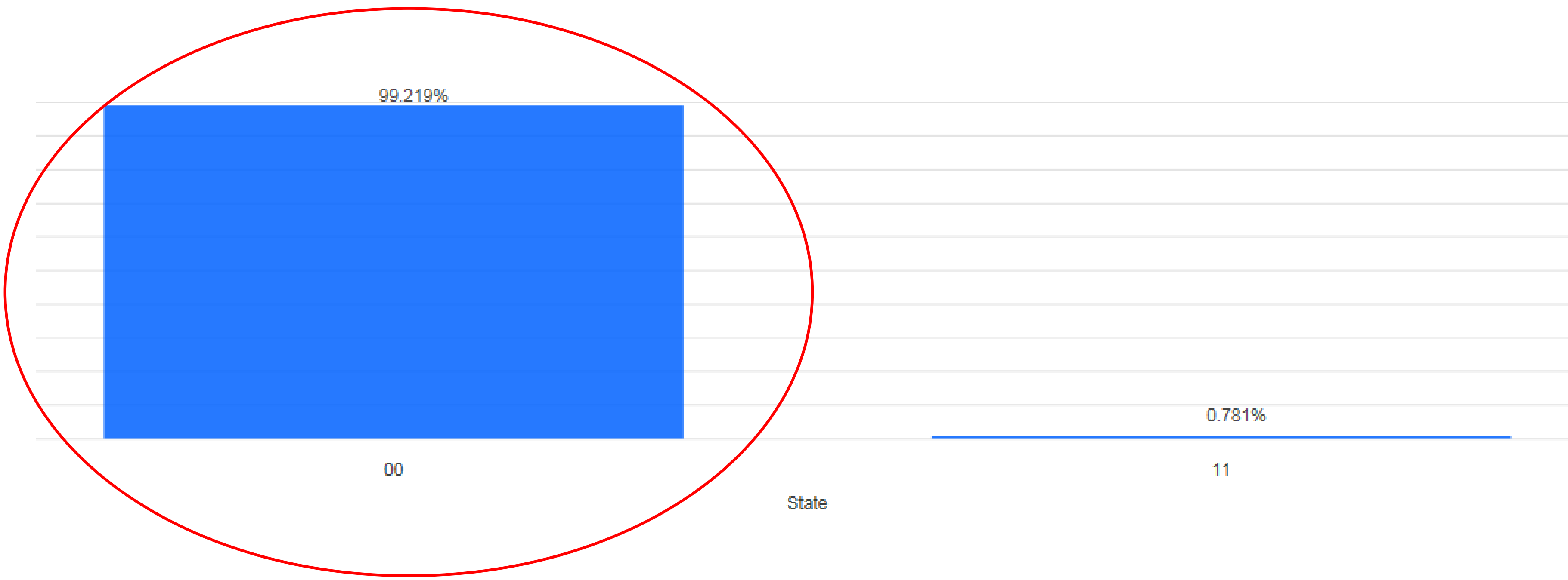
Let's compute it for $J = 1, h = 1, t = 0.1, M = 1$

$$\delta t = \frac{t}{M}$$

Demonstration for the survival probability (Cont'd)

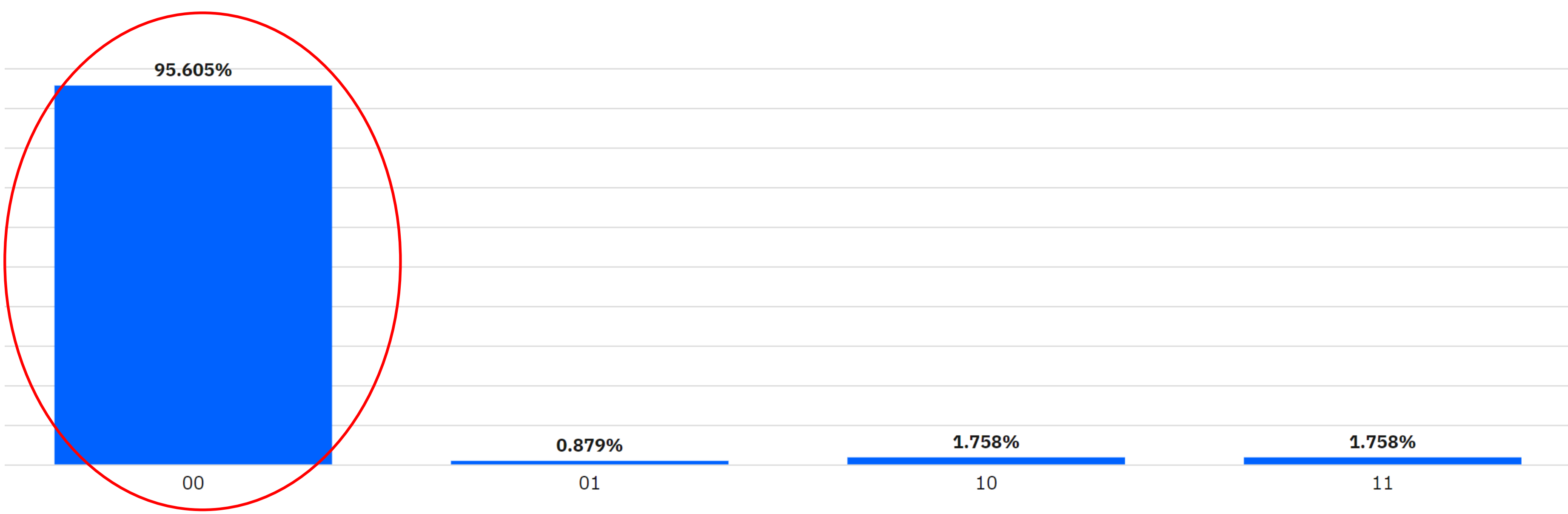


Result by simulator w/ 1024 shots:



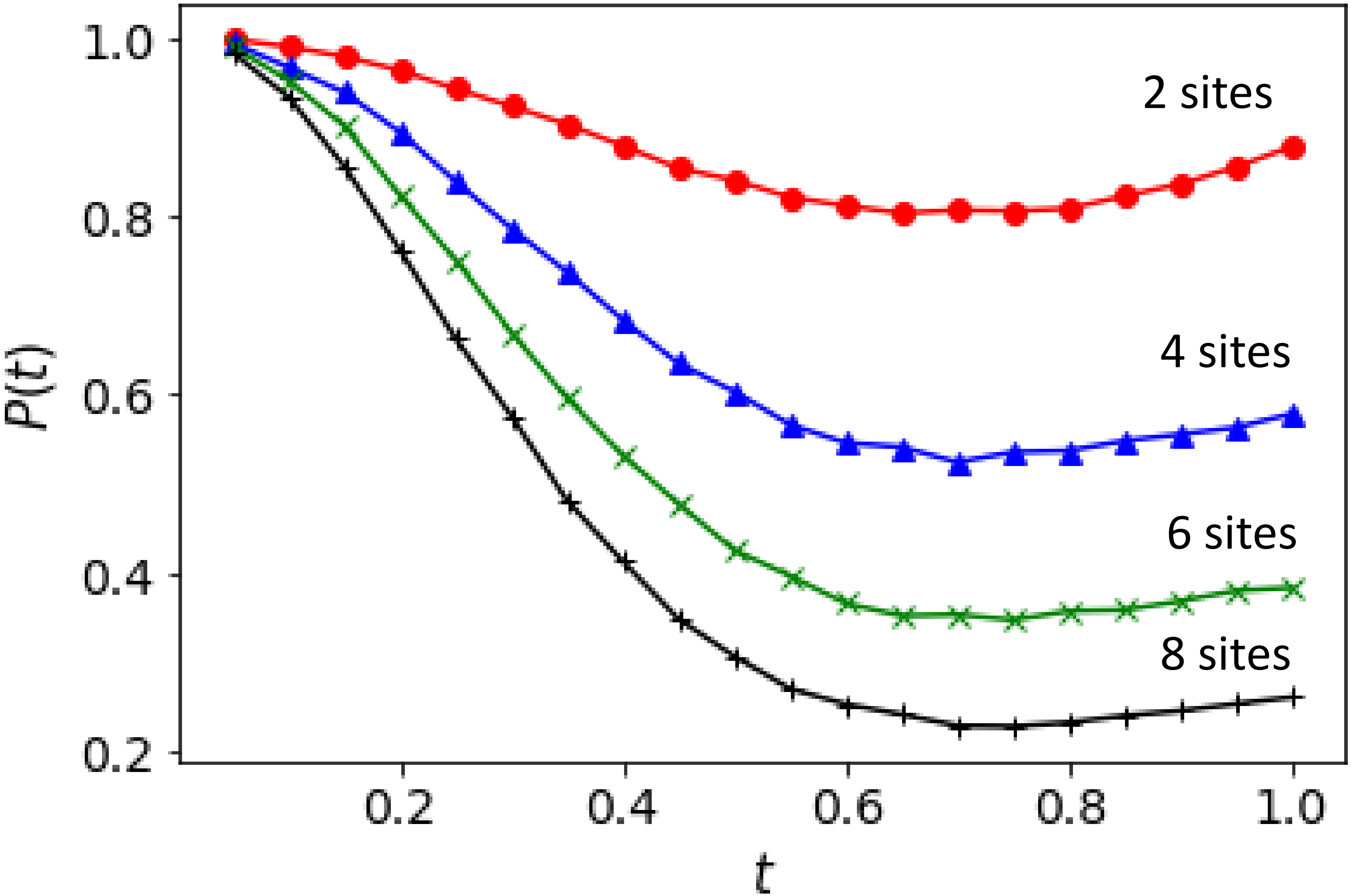
Result of simulator (1024 shots):

Result of quantum computer (1024 shots):



More serious computation

$J = 1, h = 1, t = 1, M = 100, 10000$ shots



Computational costs for large size system

$$P(t) = |\langle + \cdots + |\, e^{-i\hat{H}t}\, | + \cdots + \rangle|^2$$

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_X \frac{t}{M}} e^{-iH_{ZZ} \frac{t}{M}} \right)^M$$

Classical computer

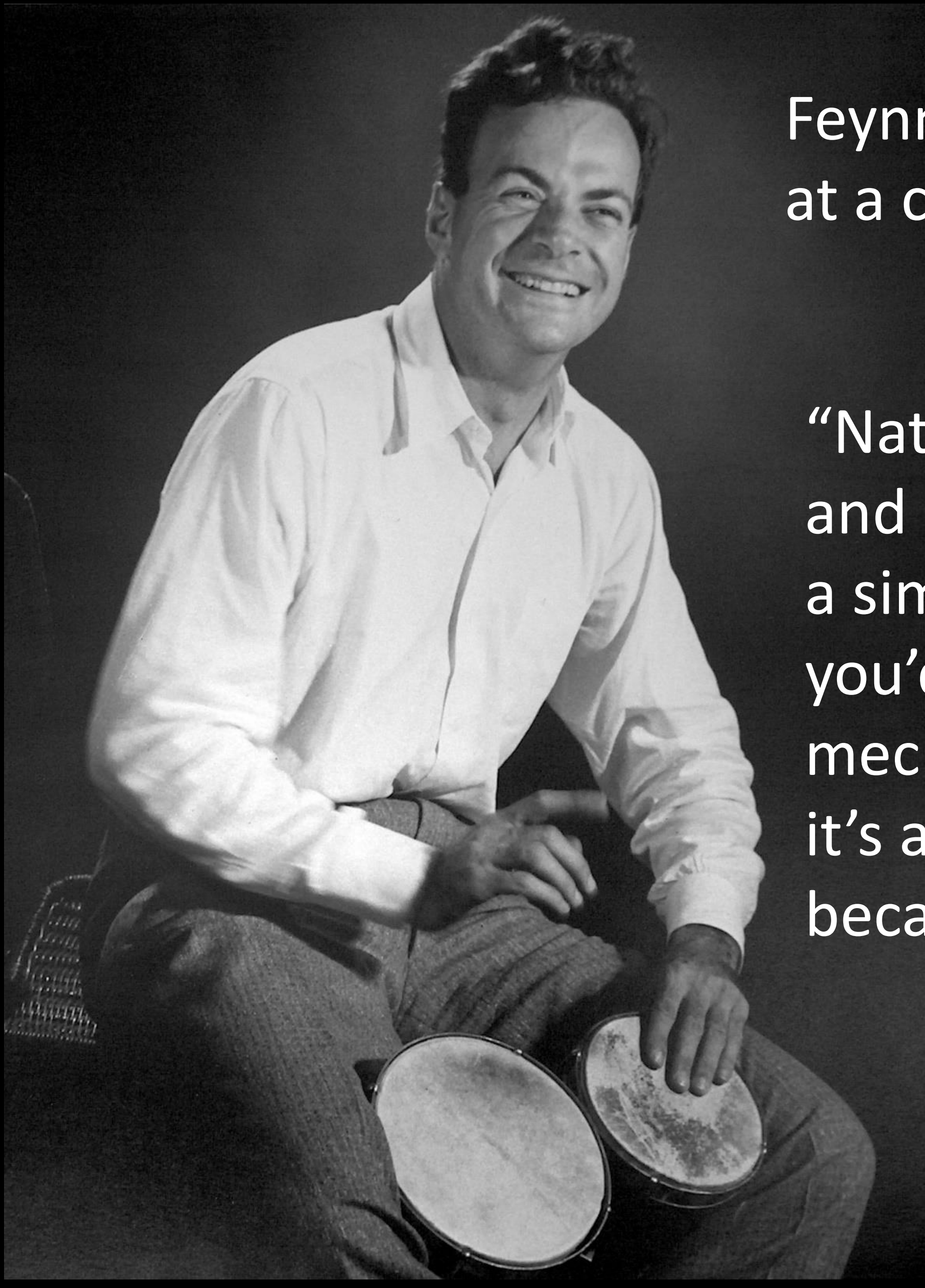
multiplications of matrices to vectors w/ sizes $= 2^N$

exponentially large steps

Quantum computer

- time evolution $= \mathcal{O}(NM)$ experimental operations
- taking the inner product is done by acting N gates & a measurement

polynomial steps



Feynman as a keynote speaker at a conference in MIT (1981):

"Nature isn't classical, dammit, and if you want to make a simulation of Nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem because it doesn't look so easy."

Constructing vacuum (ground state)

$^{\exists}$various quantum algorithms to construct vacuum:

- adiabatic state preparation
- algorithms based on variational method
- imaginary time evolution

etc...

Here, let's apply

adiabatic state preparation

Adiabatic state preparation of vacuum

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\mathrm{vac}_0\rangle$ is known and unique

Step 2:

Step 3:

Adiabatic state preparation of vacuum

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\text{vac}_0\rangle$ is known and unique

Step 2: Introduce adiabatic Hamiltonian $H_A(t)$ s.t.

$$\begin{cases} \bullet\ H_A(0) = H_0,\ H_A(T) = H_{\text{target}} \\ \bullet\ \left|\frac{dH_A}{dt}\right| \ll 1 \text{ for } T \gg 1 \end{cases}$$

Step 3:

Adiabatic state preparation of vacuum

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\mathrm{vac}_0\rangle$ is known and unique

Step 2: Introduce adiabatic Hamiltonian $H_A(t)$ s.t.

$$\begin{cases} \cdot\ H_A(0) = H_0,\ H_A(T) = H_{\mathrm{target}} \\ \cdot\ \left|\frac{dH_A}{dt}\right| \ll 1 \ \text{ for } \ T \gg 1 \end{cases}$$

Step 3: Use the adiabatic theorem

If $H_A(t)$ has a unique ground state w/ a finite gap for $\forall t$, then the ground state of H_{target} is obtained by

$$\boxed{|\mathrm{vac}\rangle = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\, H_A(t)\right) |\mathrm{vac}_0\rangle}$$

For transverse Ising model

$$\hat{H} = -J\sum_{n=1}^{N-1} Z_n Z_{n+1} - h\sum_{n=1}^{N} X_n - m\sum_{n=1}^{N} Z_n$$

Choose

$$\begin{cases} H_0 = -h\sum_{n=1}^{N} X_n \quad \Rightarrow \quad |\mathsf{vac}_0\rangle = |+\cdots+\rangle \\ H_A(t) = \left(1 - \frac{t}{T}\right) H_0 + \frac{t}{T}\hat{H} \end{cases}$$

Discretize the integral:

$$\mathcal{T}\exp\left(-i\int_0^T dt\ H_A(t)\right)|\mathsf{vac}_0> \simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\mathsf{vac}_0>$$

where

$$U(t) = e^{-iH_A(t)\delta t},\ \ \delta t = \frac{T}{M} \ll 1$$

Magnetization

Once we get the vacuum, we can compute VEV of operators:

$$\langle \text{vac}|\mathcal{O}|\text{vac}\rangle$$

It is easiest to compute magnetization:

$$\frac{1}{N}\langle \text{vac}|\sum_{n=1}^{N} Z_n|\text{vac}\rangle = \frac{1}{N}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}\langle \text{vac}|Z_n|i_1\cdots i_N\rangle\langle i_1\cdots i_N|\text{vac}\rangle$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{i_n}\,|\langle i_1\cdots i_N|\text{vac}\rangle|^2$$

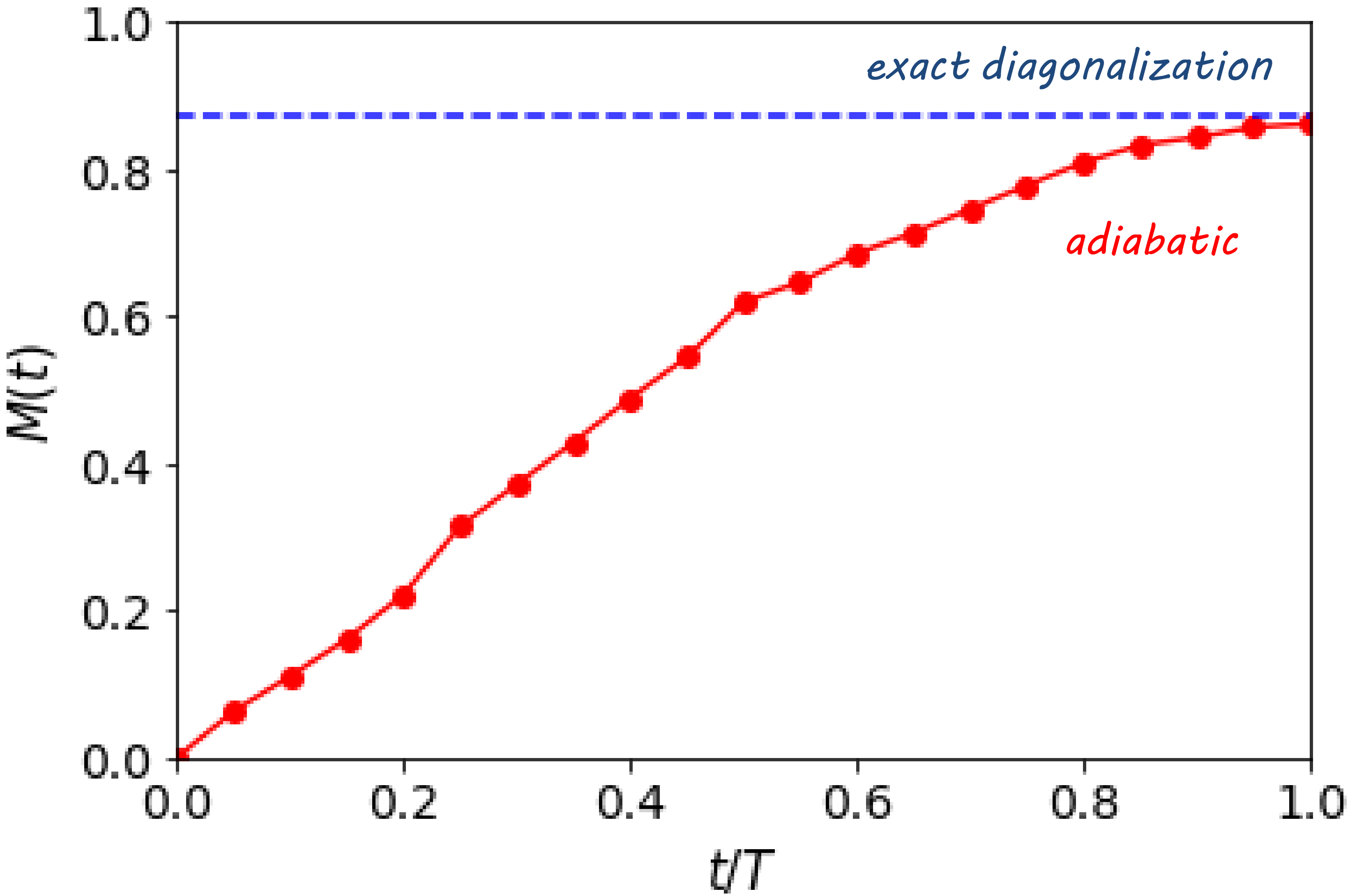
Transverse one is a bit more tricky:

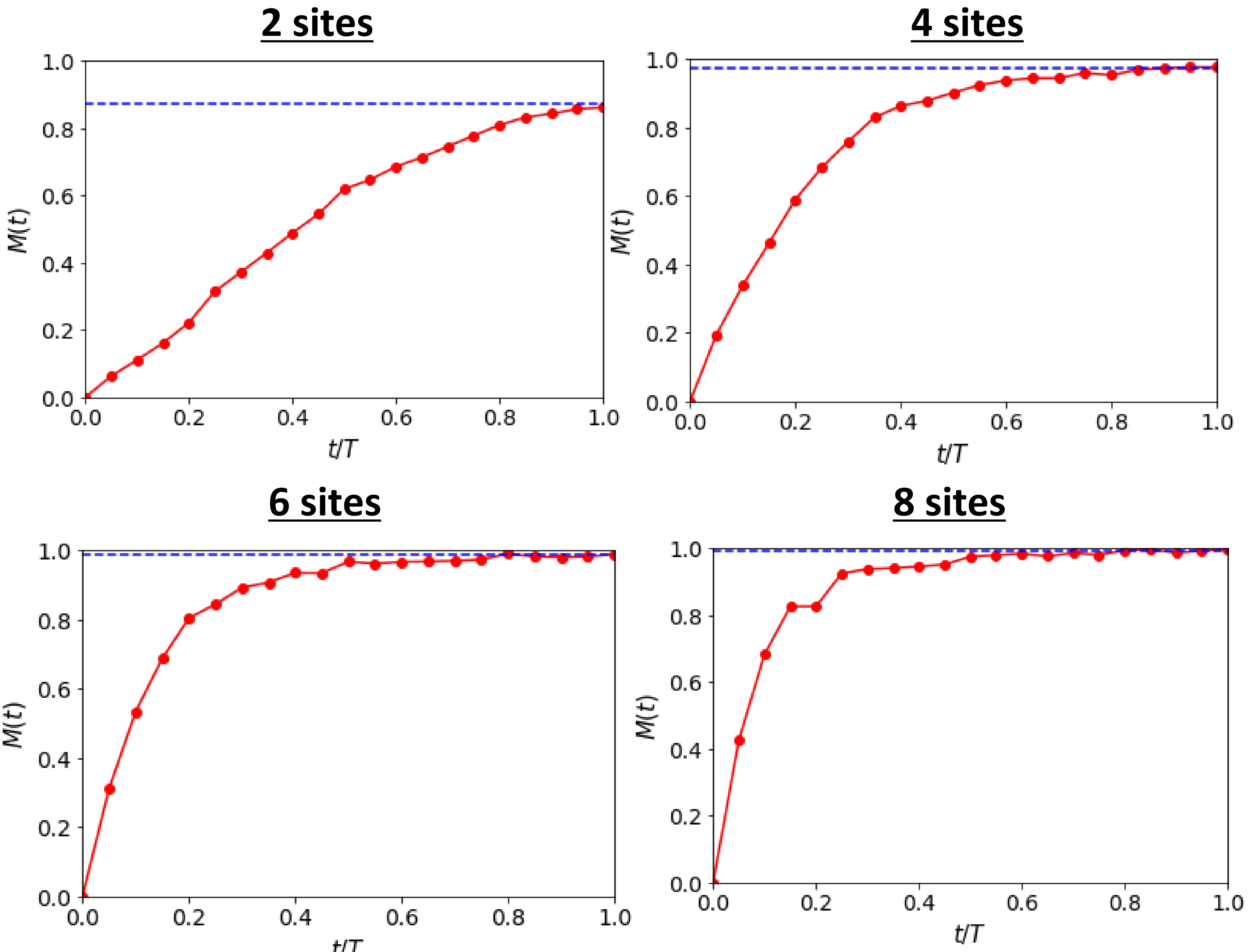
$$\frac{1}{N}\langle \text{vac}|\sum_{n=1}^{N} X_n|\text{vac}\rangle = \frac{1}{N}\langle \text{vac}|\sum_{n=1}^{N} H^{(n)}Z_nH^{(n)}|\text{vac}\rangle$$

$$= \frac{1}{N}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{i_n}\left|\langle i_1\cdots i_N|H^{(n)}|\text{vac}\rangle\right|^2$$

Result by simulator (10000 shots)

2 sites, $J = 1, h = 1, m = 1, T = 100, \delta t = 0.05,$ 2000 time steps




2 sites
4 sites
6 sites
8 sites
M(t)
t/T
0.0
0.2
0.4
0.6
0.8
1.0

Renyi entropy

Dividing total Hilbert space as

$$\mathcal{H}_{\text{tot}} = \mathcal{H}_A \otimes \mathcal{H}_B,$$

reduced density matrix is defined as

$$\rho_A = \text{tr}_{\mathcal{H}_B}(\rho_{\text{tot}})$$

Entanglement entropy:

$$S_A = -\text{tr}_{\mathcal{H}_A}(\rho_A \log \rho_A)$$

n-th Renyi entropy:

$$S_n = \frac{1}{1-n} \log \text{tr}_{\mathcal{H}_A}(\rho_A^n) \qquad \left(S_A = \lim_{n \to 1} S_n \right)$$

Quantum algorithm for 2nd Renyi entropy

Consider $(N_A + N_B)$-qubit system and the density matrix

$$\rho_{N_A+N_B} = |\Psi\rangle\langle\Psi|$$

Let's divide the system into two systems: $\mathcal{H}_{N_A+N_B} = \mathcal{H}_{N_A} \otimes \mathcal{H}_{N_B}$

& consider the 2nd Renyi entropy

$$S_2 = -\log \mathrm{tr}_{\mathcal{H}_{N_A}}\left(\rho_A^2\right), \quad \rho_A = \mathrm{tr}_{\mathcal{H}_{N_B}}\left(\rho_{N_A+N_B}\right)$$

Quantum algorithm for 2nd Renyi entropy

Consider $(N_A + N_B)$-qubit system and the density matrix

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Let's divide the system into two systems: $\mathcal{H}_{N_A+N_B} = \mathcal{H}_{N_A} \otimes \mathcal{H}_{N_B}$

& consider the 2nd Renyi entropy

$$S_2 = -\log \mathrm{tr}_{\mathcal{H}_{N_A}}\left(\rho_A^2\right), \quad \rho_A = \mathrm{tr}_{\mathcal{H}_{N_B}}\left(\rho_{N_A+N_B}\right)$$

One can show (next slide) [Hastings-Gonzalez-Kallin-Melko'10]

$$\boxed{\mathrm{tr}_{\mathcal{H}_{N_A}}\left(\rho_A^2\right) = \langle\Psi| \otimes \langle\Psi| \ \mathrm{SWAP}_A \ |\Psi\rangle \otimes |\Psi\rangle}$$

$$\mathrm{SWAP}_A : \text{Exchange of } A-\text{part in } |\Psi\rangle \otimes |\Psi\rangle$$

For $|\Psi\rangle = \sum_{i,j} c_{ij} |i_1 \cdots i_{N_A} j_1 \cdots j_{N_B}\rangle,$

$$\mathrm{SWAP}_A |\Psi\rangle \otimes |\Psi\rangle \equiv \sum_{i,j,i',j'} c_{ij} c_{i'j'} |i'_1 \cdots i'_{N_A} j_1 \cdots j_{N_B}\rangle \otimes |i_1 \cdots i_{N_A} j'_1 \cdots j'_{N_B}\rangle$$

Quantum algorithm for 2nd Renyi entropy (Cont'd)

$$\mathrm{tr}_{\mathcal{H}_{N_A}}\left(\rho_A^2\right) = \langle\Psi| \otimes \langle\Psi|\ \mathsf{SWAP}_A\ |\Psi\rangle \otimes |\Psi\rangle$$

Proof:

$$\langle\Psi| \otimes \langle\Psi|\ \mathsf{SWAP}_A\ |\Psi\rangle \otimes |\Psi\rangle$$

$$= \sum_{k,\ell,k',\ell'} \bar{c}_{k\ell}\bar{c}_{k'\ell'}\langle\{k'\}\{\ell'\}|\otimes\langle\{k\}\{\ell\}| \sum_{i,j,i',j'} c_{ij}c_{i'j'}|\{i'\}\{j\}\rangle\otimes|\{i\}\{j'\}\rangle$$

$$= \sum_{i,j,i',j'} c_{ij}\bar{c}_{i'j}c_{i'j'}\bar{c}_{ij'}$$

$$(\rho_A)_{ii'} = \sum_j \langle\{i\}\{j\}|\ \rho_{N_A+N_B}\ |\{i'\}\{j\}\rangle = \sum_j c_{ij}\bar{c}_{i'j}$$

$$= \sum_{i,i'} (\rho_A)_{ii'}(\rho_A)_{i'i} = \mathrm{tr}_{\mathcal{H}_{N_A}}\left(\rho_A^2\right)$$

Demonstration: 2nd Renyi entropy of Bell state

Bell state:

$$|B\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

Reduced density matrix:

$$\rho_{\mathrm{red}} = \mathrm{tr}_2 |B\rangle\langle B| = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|)$$

2nd Renyi entropy:

$$\mathrm{tr}\rho_{\mathrm{red}}^2 = \mathrm{tr}\begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} = \frac{1}{2}$$

$$S_2 = -\log \mathrm{tr}\rho_{\mathrm{red}}^2 = \log 2$$

Let’s reproduce it in IBM Quantum!

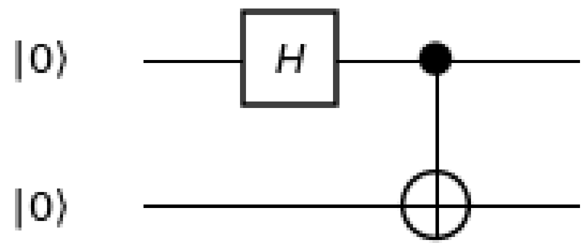
Demonstration: 2nd Renyi entropy of Bell state (Cont'd)

We know

$$|B\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$\mathrm{tr}\rho_{\mathrm{red}}^2 = \langle B| \otimes \langle B| \ \mathrm{SWAP}^{(1,3)} \ |B\rangle \otimes |B\rangle$$

The Bell state is written as



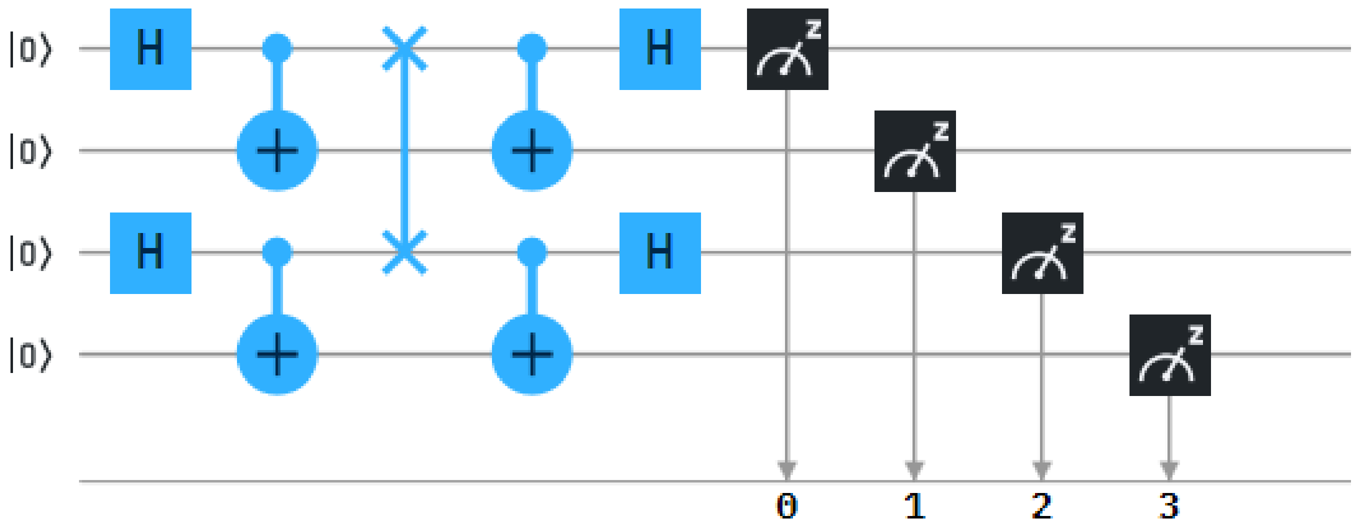
$$= |B\rangle$$

Therefore,

$$\mathrm{tr}\rho_{\mathrm{red}}^2 = \langle 0000| \ U^\dagger \mathrm{SWAP}^{(1,3)} U \ |0000\rangle \qquad (|B\rangle \otimes |B\rangle \equiv U|0000\rangle)$$

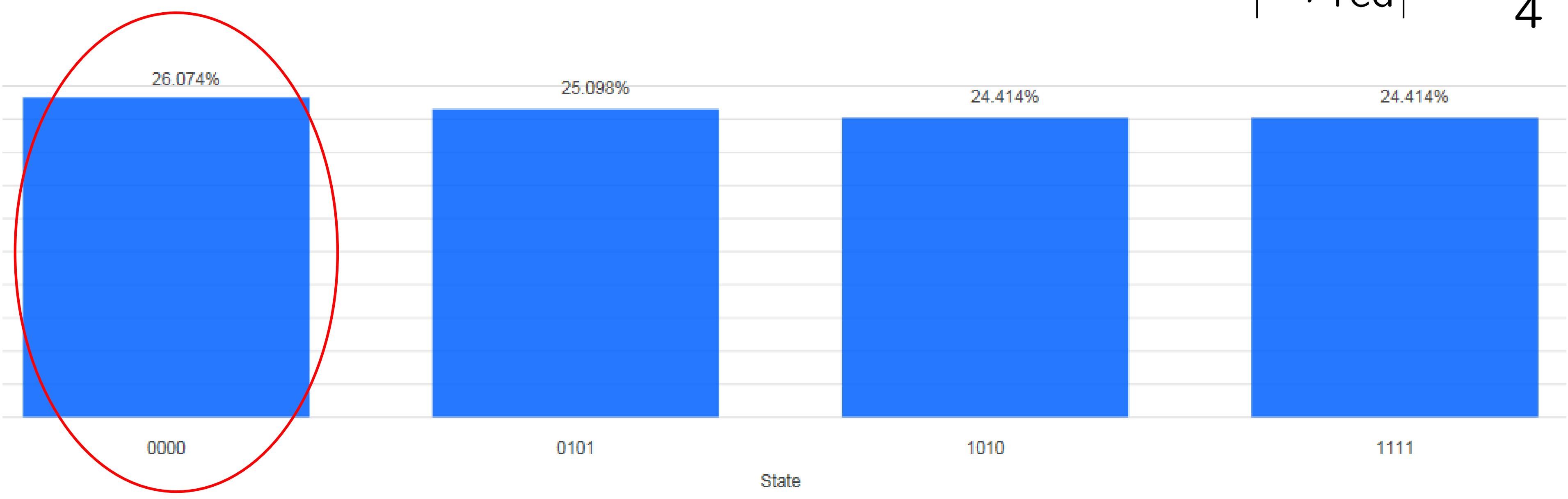
$$\left|\mathrm{tr}\rho_{\mathrm{red}}^2\right|^2 = \left|\langle 0000| \ U^\dagger \mathrm{SWAP}^{(1,3)} U \ |0000\rangle\right|^2$$

Demonstration: 2nd Renyi entropy of Bell state (Cont'd)

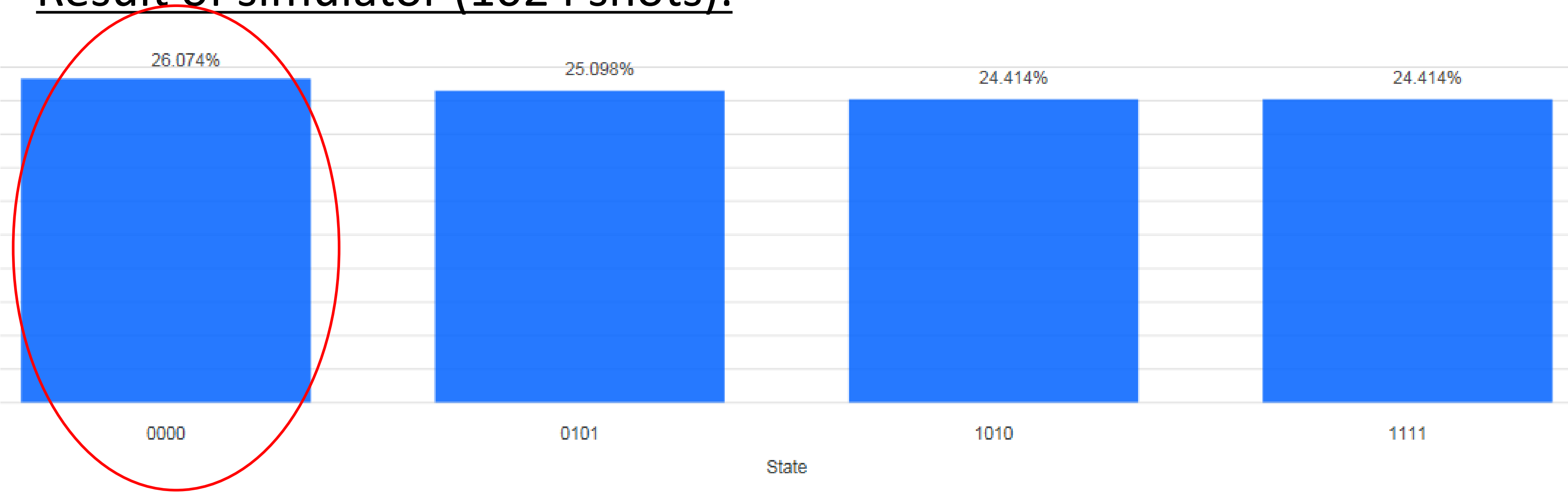


Result of simulator (1024 shots):

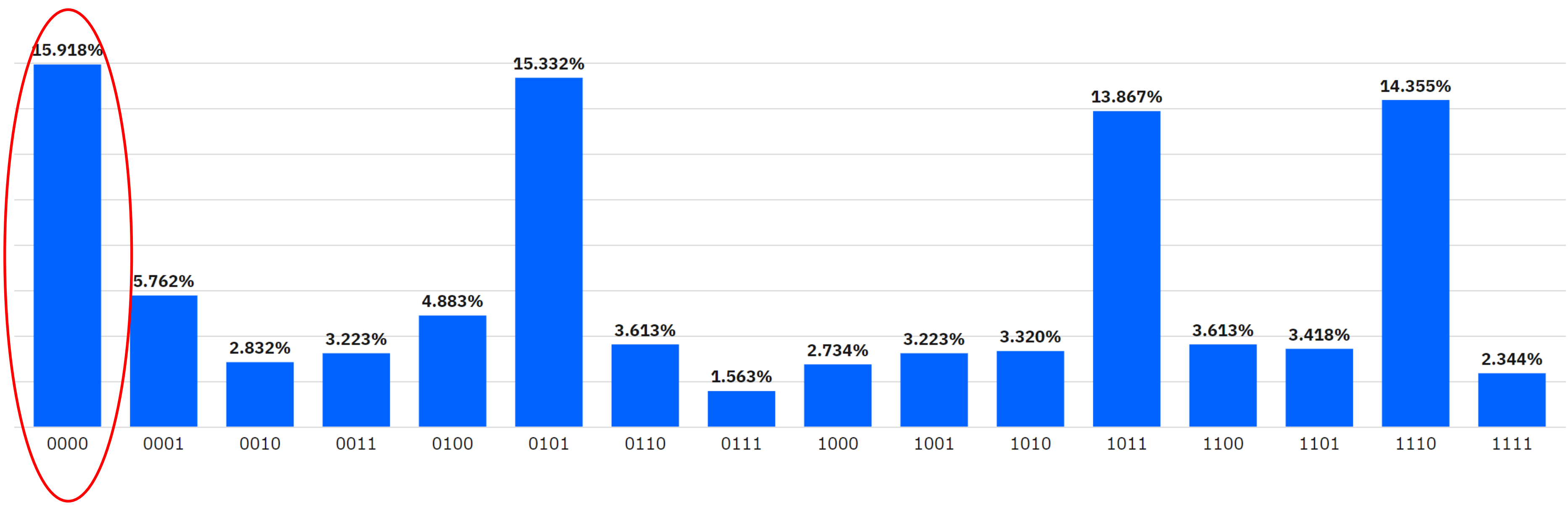
$$\left|\mathrm{tr}\rho_{\mathrm{red}}^2\right|^2 = \frac{1}{4}$$



Result of simulator (1024 shots):



Result of quantum computer (1024 shots):



More direct way?

We've directly computed

$$\left|\mathrm{tr}\rho_{\mathrm{red}}^2\right|^2 = \left|\langle 0000|\; U^\dagger \mathrm{SWAP}^{(1,3)} U \;|0000\rangle\right|^2$$

rather than itself:

$$\mathrm{tr}\rho_{\mathrm{red}}^2 = \langle 0000|\; U^\dagger \mathrm{SWAP}^{(1,3)} U \;|0000\rangle$$

Can we directly compute it?

—— Yes, there is a way to compute expectation value of unitary op. under any state: (next slide)

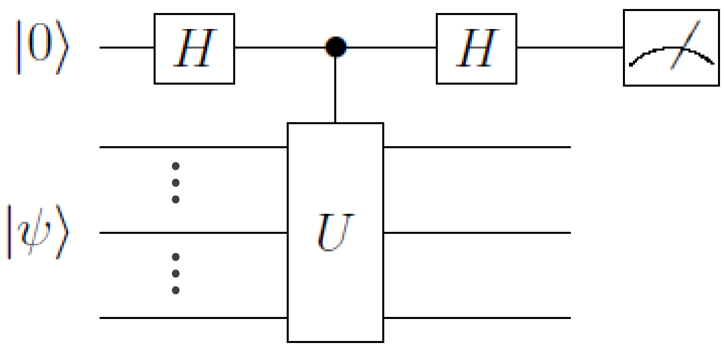
$$\langle\psi|\, \mathrm{U}\, |\psi\rangle$$

“Hadamard test”: standard way to compute $\langle\psi|\ \mathrm{U}\ |\psi\rangle$

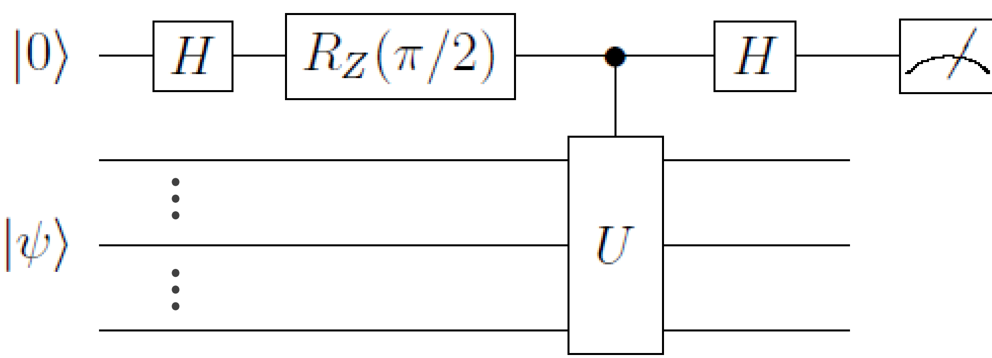
① Extend Hilbert space & consider the state

$$\underbrace{|0\rangle}_{\text{“ancillary qubit”}} \otimes |\psi\rangle$$

② We can compute $\langle\psi|\ U\ |\psi\rangle$ by using the 2 circuits: (next slide)



$\Rightarrow$ $\mathrm{Re}(\langle\psi|U|\psi\rangle)$

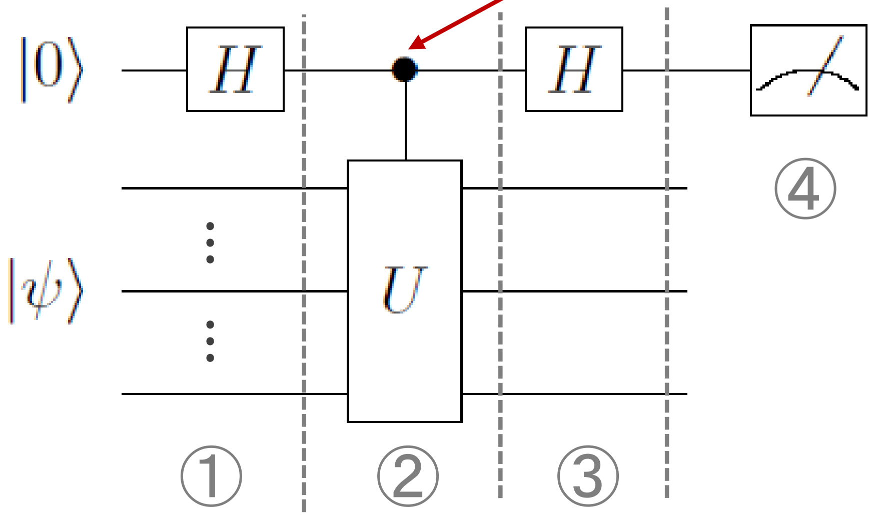


$\Rightarrow$ $\mathrm{Im}(\langle\psi|U|\psi\rangle)$

"Hadamard test": standard way to compute $\langle\psi|\,\mathrm{U}\,|\psi\rangle$ (Cont'd)

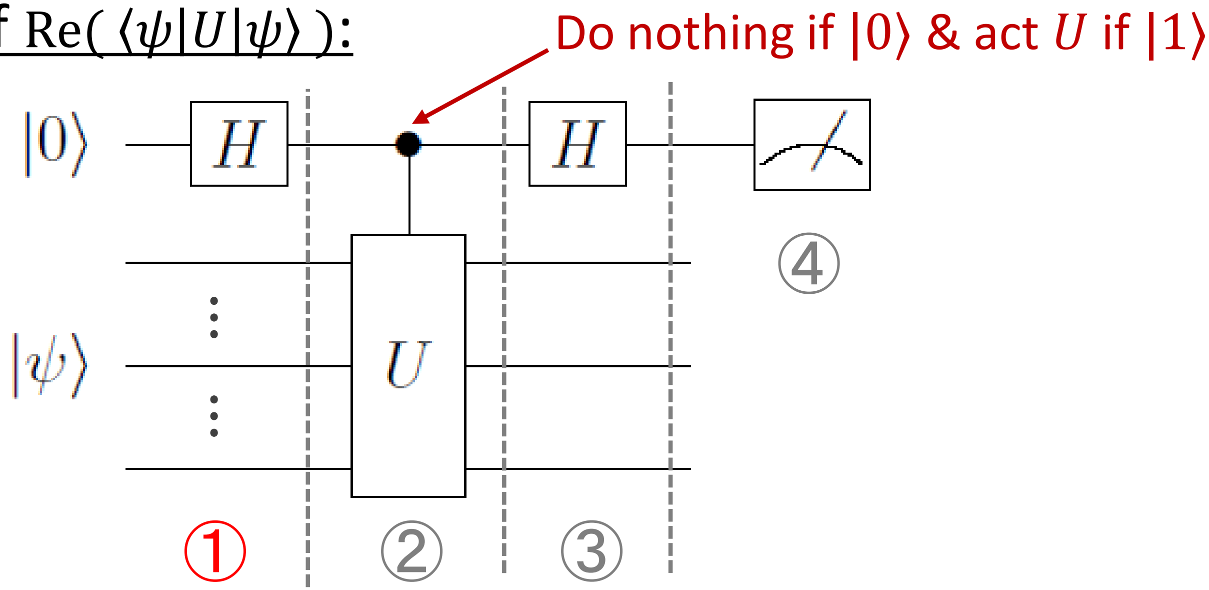
Computation of $\mathrm{Re}(\,\langle\psi|U|\psi\rangle\,)$:

Do nothing if $|0\rangle$ & act U if $|1\rangle$



"Hadamard test": standard way to compute $\langle\psi|\ U\ |\psi\rangle$ (Cont'd)

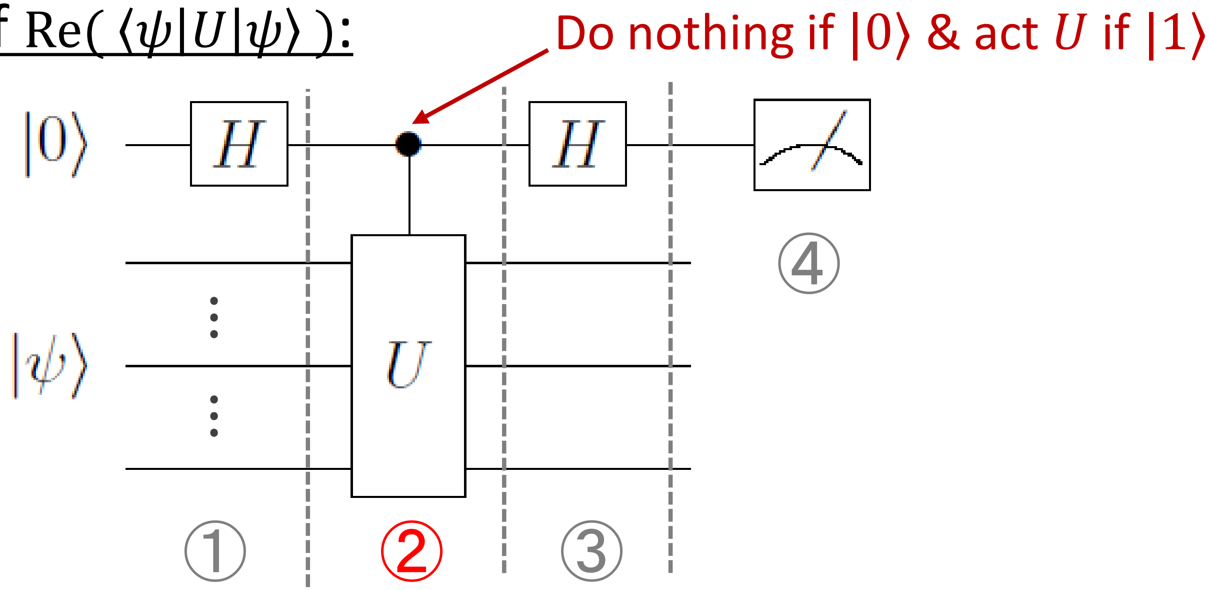
Computation of $\mathrm{Re}(\langle\psi|U|\psi\rangle)$:



① $H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |\psi\rangle$

"Hadamard test": standard way to compute $\langle\psi|\ U\ |\psi\rangle$ (Cont'd)

Computation of $\mathrm{Re}(\langle\psi|U|\psi\rangle)$:



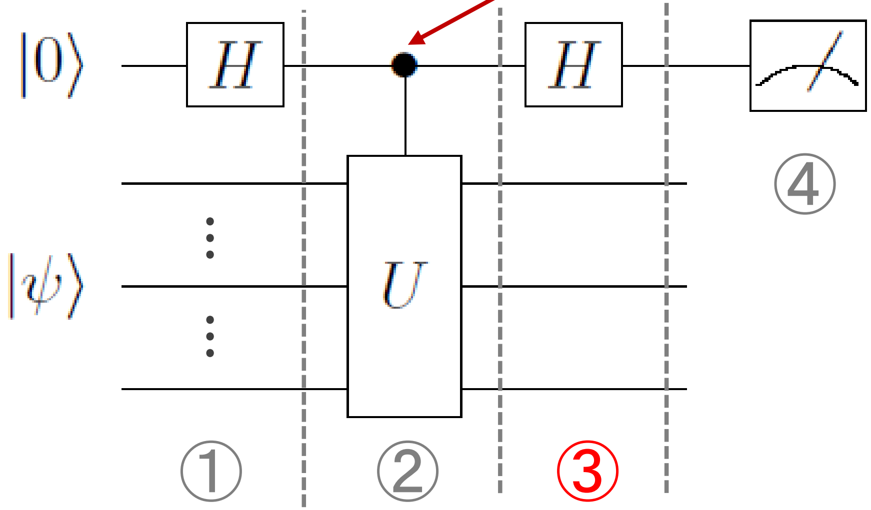
① $H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |\psi\rangle$

② $\frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes U|\psi\rangle$

“Hadamard test”: standard way to compute $\langle\psi|\ U\ |\psi\rangle$ (Cont’d)

Computation of $\mathrm{Re}(\langle\psi|U|\psi\rangle)$:

Do nothing if $|0\rangle$ & act U if $|1\rangle$



① $H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes |\psi\rangle$

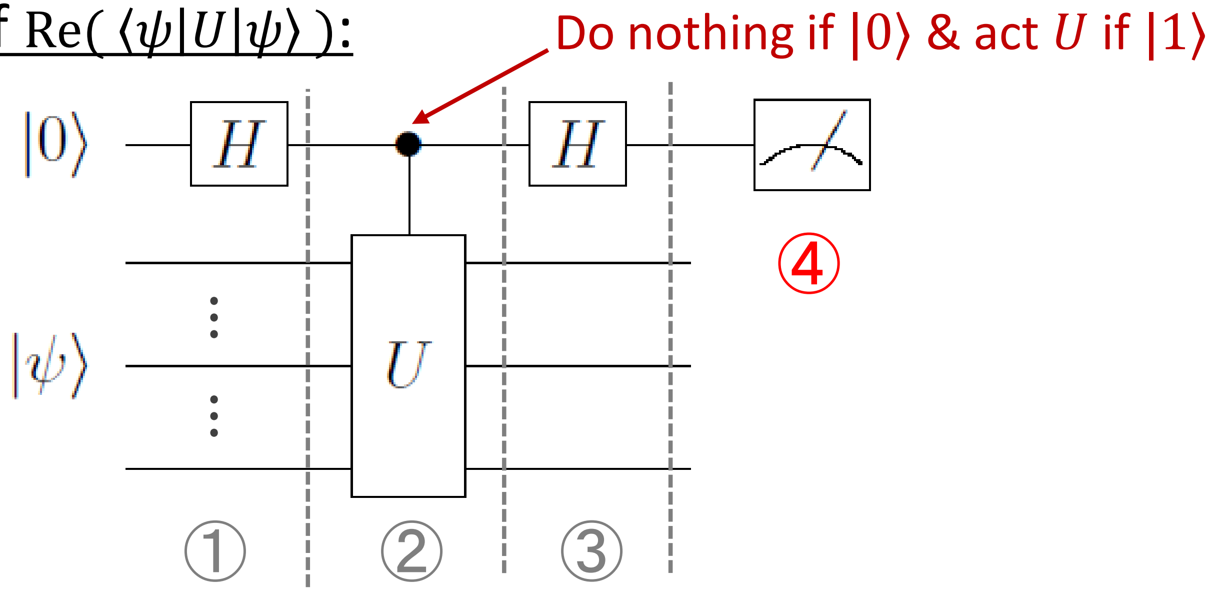
② $\frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle \otimes U|\psi\rangle$

③ $\frac{1}{2}(|0\rangle + |1\rangle) \otimes |\psi\rangle + \frac{1}{2}(|0\rangle - |1\rangle) \otimes U|\psi\rangle$

$= \frac{1}{2}|0\rangle \otimes (1 + U)|\psi\rangle + \frac{1}{2}|1\rangle \otimes (1 - U)|\psi\rangle$

"Hadamard test": standard way to compute $\langle\psi|\ U\ |\psi\rangle$ (Cont'd)

Computation of $\mathrm{Re}(\ \langle\psi|U|\psi\rangle\ )$:



① $H|0\rangle \otimes |\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle\ + \frac{1}{\sqrt{2}}|1\rangle \otimes |\psi\rangle$

② $\frac{1}{\sqrt{2}}|0\rangle \otimes |\psi\rangle\ + \frac{1}{\sqrt{2}}|1\rangle \otimes U|\psi\rangle$

③ $\frac{1}{2}(|0\rangle + |1\rangle) \otimes |\psi\rangle\ + \frac{1}{2}(|0\rangle - |1\rangle) \otimes U|\psi\rangle$

$= \frac{1}{2}|0\rangle \otimes (1 + U)|\psi\rangle\ + \frac{1}{2}|1\rangle \otimes (1 - U)|\psi\rangle$

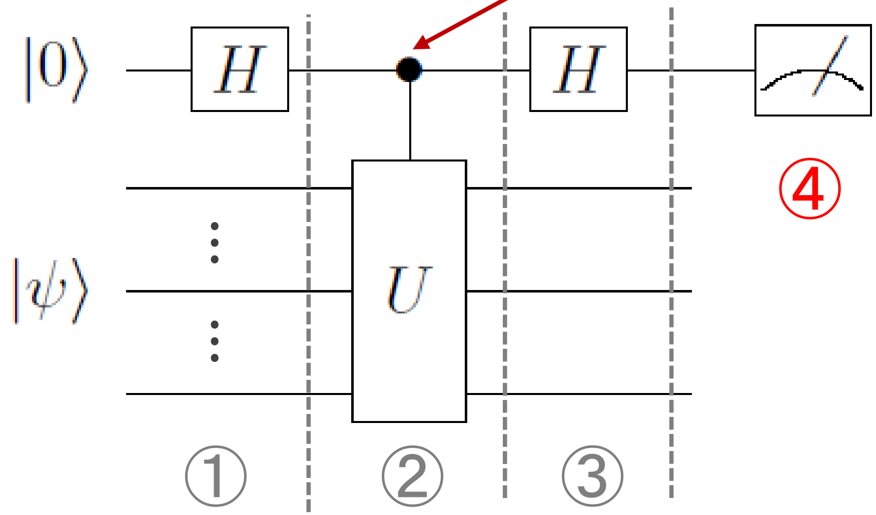
④ $P_0 = \frac{1}{4}|(1 + U)|\psi\rangle|^2\ = \frac{1}{2}(\ 1 + \mathrm{Re}\langle\psi|U|\psi\rangle\ )$

$P_1 = \frac{1}{4}|(1 - U)|\psi\rangle|^2\ = \frac{1}{2}(\ 1 - \mathrm{Re}\langle\psi|U|\psi\rangle\ )$

“Hadamard test”: standard way to compute $\langle\psi|\,U\,|\psi\rangle$ (Cont’d)

Computation of $\mathrm{Re}(\,\langle\psi|U|\psi\rangle\,)$:

Do nothing if $|0\rangle$ & act U if $|1\rangle$



① $H|0\rangle\otimes|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle\otimes|\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes|\psi\rangle$

② $\frac{1}{\sqrt{2}}|0\rangle\otimes|\psi\rangle + \frac{1}{\sqrt{2}}|1\rangle\otimes U|\psi\rangle$

③ $\frac{1}{2}(|0\rangle+|1\rangle)\otimes|\psi\rangle + \frac{1}{2}(|0\rangle-|1\rangle)\otimes U|\psi\rangle$

$= \frac{1}{2}|0\rangle\otimes(1+U)|\psi\rangle + \frac{1}{2}|1\rangle\otimes(1-U)|\psi\rangle$

④ $P_0 = \frac{1}{4}|(1+U)|\psi\rangle|^2 = \frac{1}{2}(\,1+\mathrm{Re}\langle\psi|U|\psi\rangle\,)$

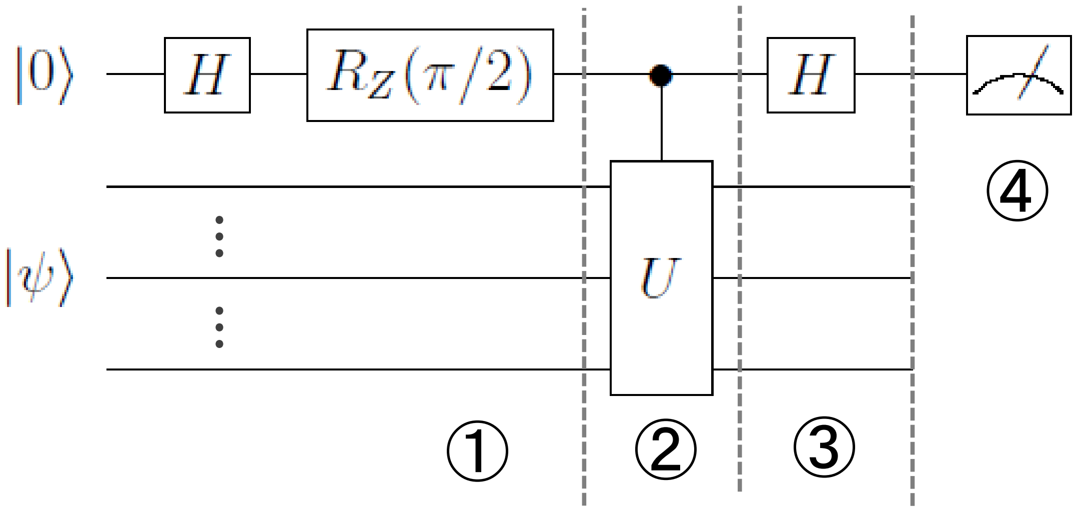
$P_1 = \frac{1}{4}|(1-U)|\psi\rangle|^2 = \frac{1}{2}(\,1-\mathrm{Re}\langle\psi|U|\psi\rangle\,)$

$\Rightarrow$ $\boxed{\mathrm{Re}\langle\psi|U|\psi\rangle = P_0 - P_1}$

"Hadamard test": standard way to compute $\langle\psi|\ U\ |\psi\rangle$ (Cont'd)

Computation of $\mathrm{Im}(\ \langle\psi|U|\psi\rangle\ )$:

$$\left(R_Z(\theta) = e^{-\frac{i\theta}{2}Z} \right)$$



① $\mathrm{R_Z}(\pi/2) H|0\rangle \otimes |\psi\rangle = \dfrac{e^{-\frac{\pi i}{4}}}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \dfrac{e^{+\frac{\pi i}{4}}}{\sqrt{2}}|1\rangle \otimes |\psi\rangle$

② $\dfrac{e^{-\frac{\pi i}{4}}}{\sqrt{2}}|0\rangle \otimes |\psi\rangle + \dfrac{e^{+\frac{\pi i}{4}}}{\sqrt{2}}|1\rangle \otimes U|\psi\rangle$

③ $\dfrac{e^{-\frac{\pi i}{4}}}{2}|0\rangle \otimes (1 + iU)|\psi\rangle + \dfrac{e^{-\frac{\pi i}{4}}}{2}|1\rangle \otimes (1 - iU)|\psi\rangle$

④ $P_0 = \frac{1}{4}|(1 + iU)|\psi\rangle|^2 = \frac{1}{2}(\ 1 - \mathrm{Im}\langle\psi|U|\psi\rangle\ )$

$P_1 = \frac{1}{4}|(1 - iU)|\psi\rangle|^2 = \frac{1}{2}(\ 1 + \mathrm{Im}\langle\psi|U|\psi\rangle\ )$

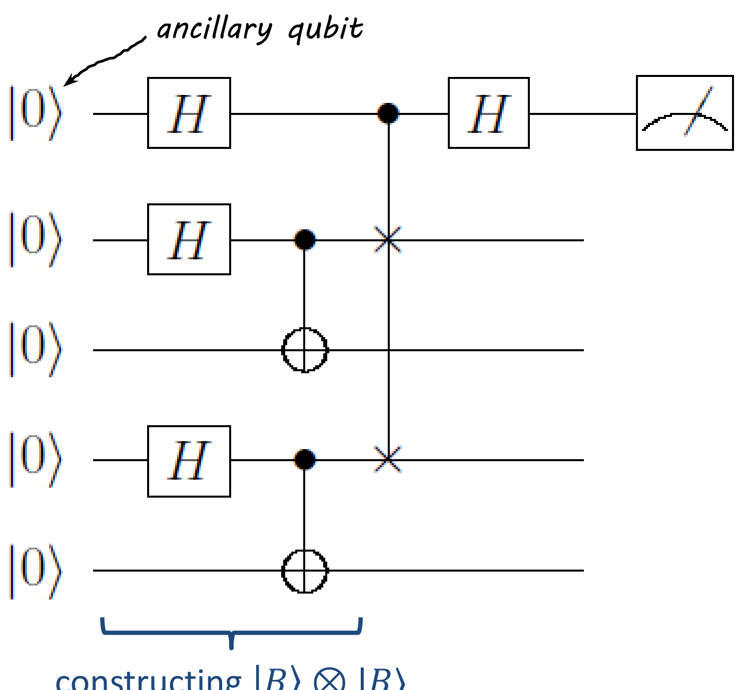
$$\Rightarrow \boxed{\mathrm{Im}\langle\psi|U|\psi\rangle = P_1 - P_0}$$

Coming back to the Renyi entropy of Bell state

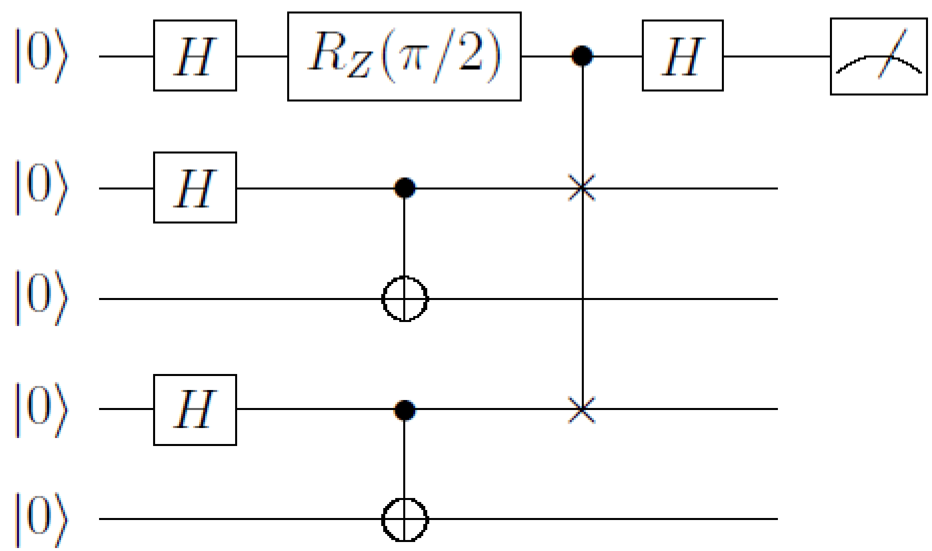
Taking $|\psi\rangle = |B\rangle \otimes |B\rangle$ & $U = \mathrm{SWAP}^{(1,3)}$, we can directly compute

$$\mathrm{tr}\rho_{\mathrm{red}}^2 = \langle B| \otimes \langle B|\ \mathsf{SWAP}^{(1,3)}\ |B\rangle \otimes |B\rangle$$

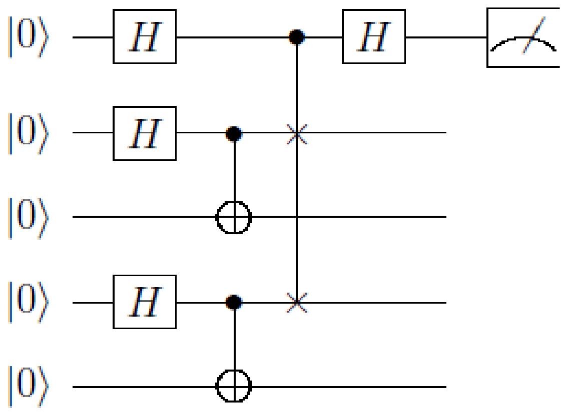
Real part:

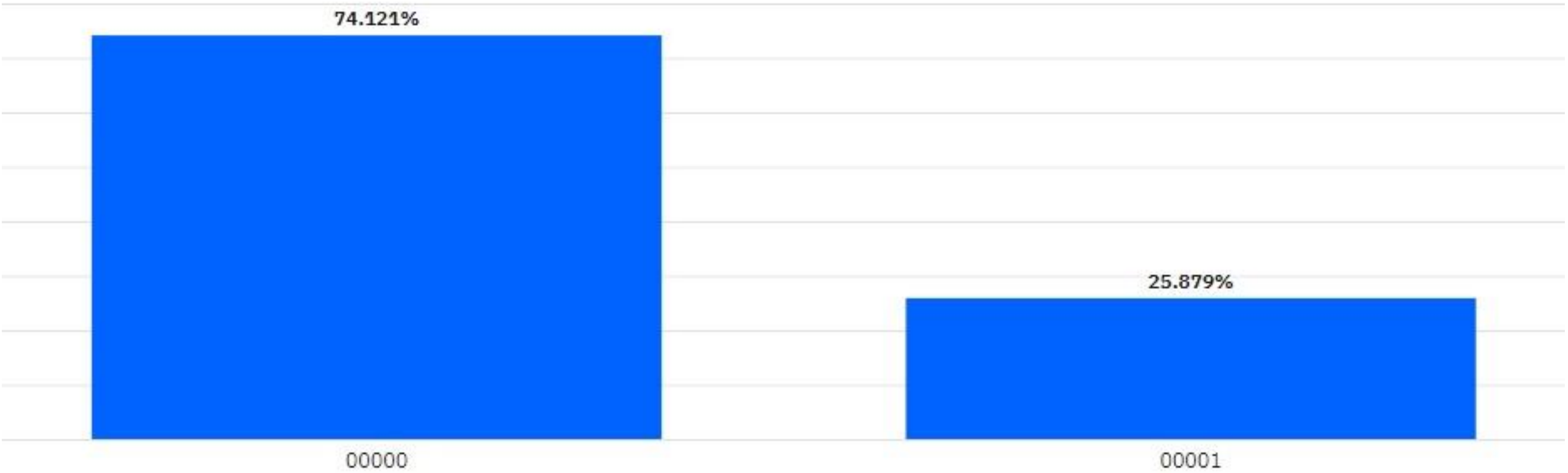


Imaginary part:



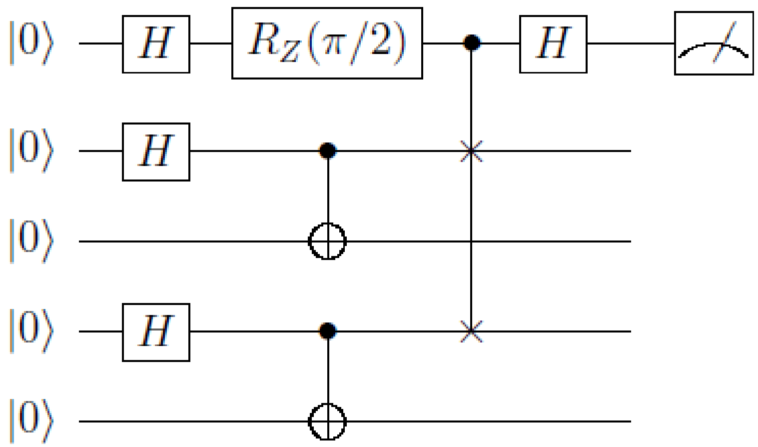
Result of simulator (real part, 1024 shots)

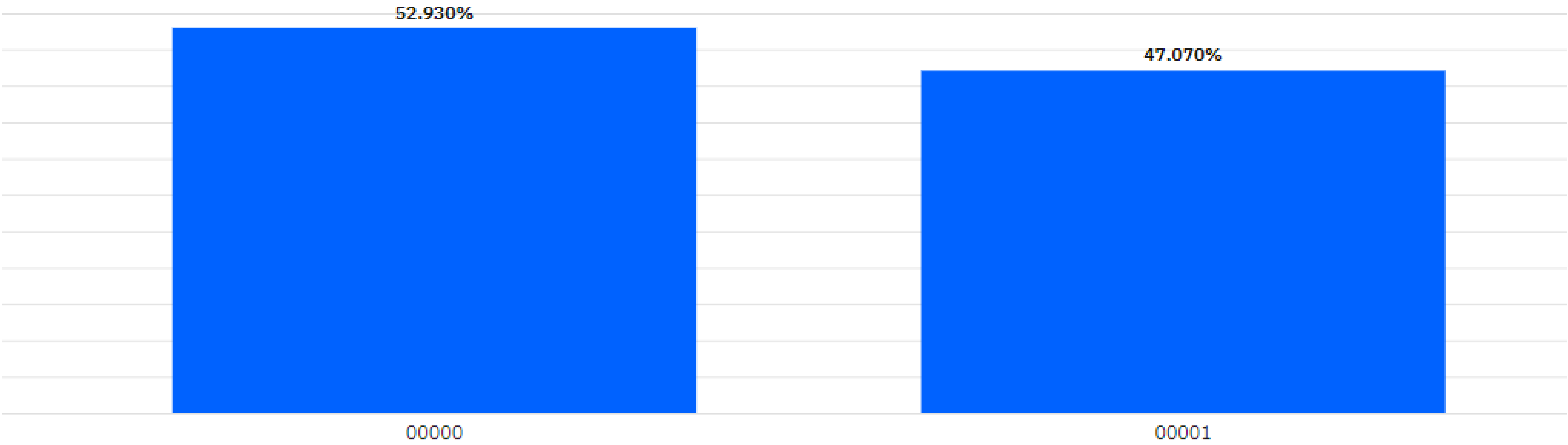




Expectation: $P_0 - P_1 = \mathrm{Re}\,\mathrm{tr}\rho_{\mathrm{red}}^2 = \frac{1}{2}$

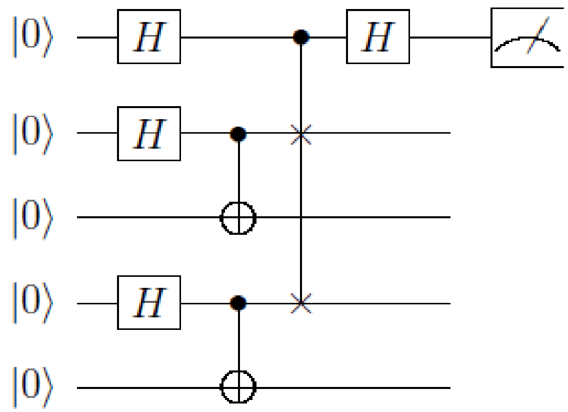
Result of simulator (imaginary part, 1024 shots)

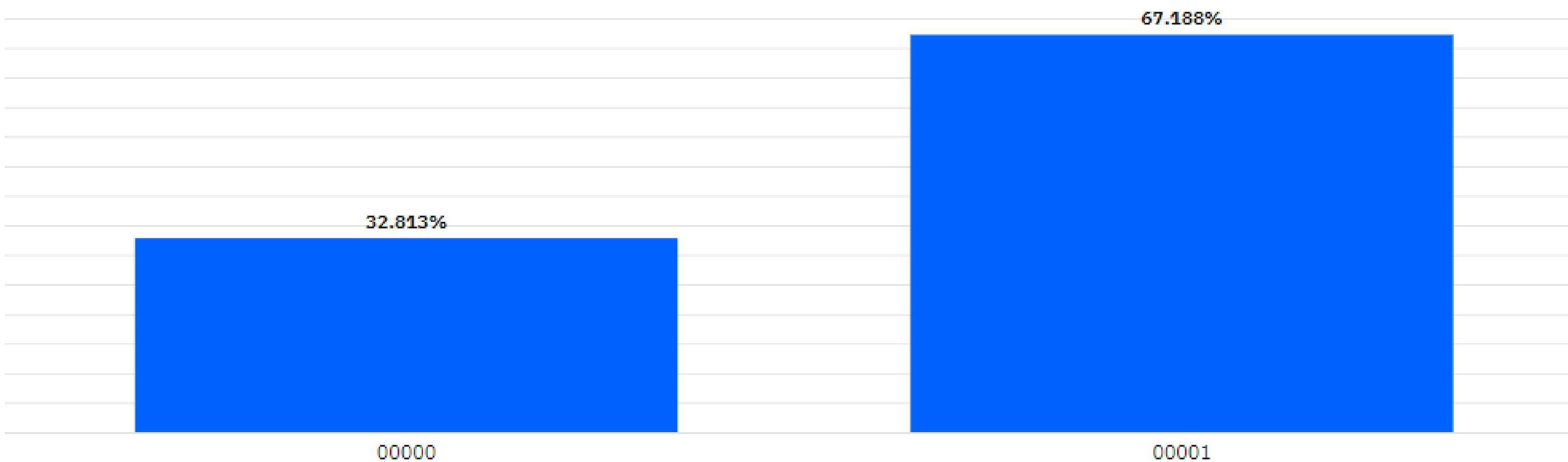




Expectation: $P_1 - P_0 = \mathrm{Im}\ \mathrm{tr}\rho_{\mathrm{red}}^2 = 0$

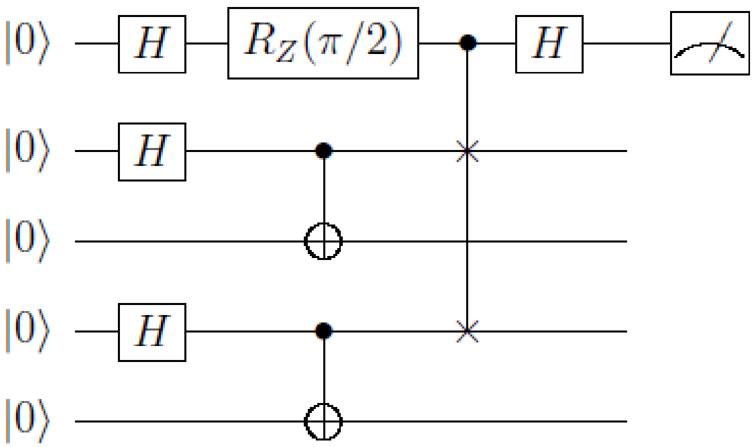
Result of quantum computer (real part, 1024 shots)

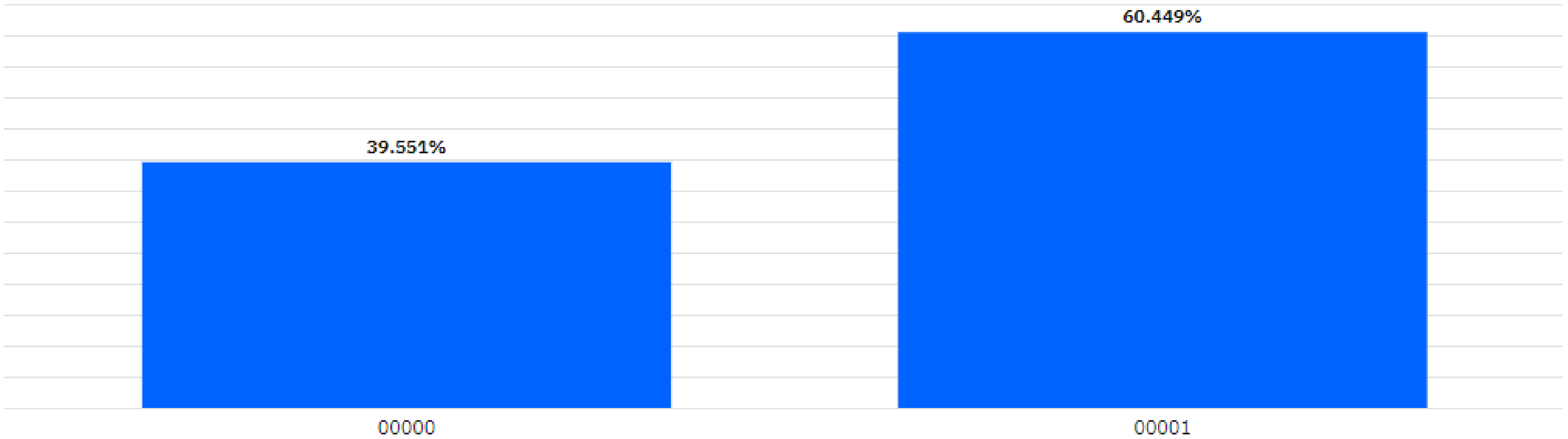




Expectation: $P_0 - P_1 = \mathrm{Re}\,\mathrm{tr}\rho_{\mathrm{red}}^2 = \frac{1}{2}$

Result of quantum computer (imaginary part, 1024 shots)





Expectation: $P_1 - P_0 = \mathrm{Im}\ \mathrm{tr}\rho_{\mathrm{red}}^2 = 0$

Summary of the lecture part of the 1st day

Summary

- Quantum computation is suitable for operator formalism which is free from sign problem

- Instead we have to deal with huge vector space. Quantum computers in future may do this job.

- ”Rule” of quantum computation
  = Do something interesting by a combination of
  acting unitary op. & measurement

- Real quantum computer has errors

- Quantum error correction is important

(Here is the expected end of the lecture 2)

Hands-on 1

Download a file from my github:

https://github.com/masazumihonda/lectures/tree/main/2023_Niigata_QC

Appendix

What if we replace T by something else?

$$T = e^{\frac{i\pi}{8}} R_Z(\pi/4) \quad \Rightarrow \quad T' \equiv R_Z(\phi)\ ??$$

We have the identity:

$$T'HT'H = R_{\vec{n}}(\theta)$$

where

$$\vec{n} = \frac{1}{\sqrt{1+\cos^2(\phi/2)}} \begin{pmatrix} cos(\phi/2) \\ \sin(\phi/2) \\ cos(\phi/2) \end{pmatrix} \quad \& \quad \cos(\theta/2) \equiv \cos^2(\phi/2)$$

We can approximate any single qubit gate

by combining H & T' if $\theta/2\pi$ is irrational