1. Supersymmetric (SUSY) theories

①

SUSY : bose $\longleftrightarrow$ fermi

$(\mathrm{SUSY})^2 \sim$ (bosonic sym.)

1-1. 0d SUSY

$$Z = \int \underbrace{dx}_{\text{bos.}}\ \underbrace{d\psi_1 d\psi_2}_{\text{fermi}}\ e^{-S[x,\psi_1,\psi_2]}$$

一般的な作用：

$$S[x,\psi_1,\psi_2] = V(x) - M(x)\psi_1\psi_2$$

特に

$$V(x) = \frac{1}{2}(h'(x))^2, \quad M(x) = h''(x)$$

のとき，

$$h'(x) - h''(x)\psi_1\psi_2 = 0$$

$$h'\psi_1 = 0, \quad h''\psi_2 = 0$$

(※ 2d $N=(1,1)$ の dim. red.)

②

$$S[X,\psi_1,\psi_2]=\frac{1}{2}(h'(X))^2-h''(X)\psi_1\psi_2$$

次の変換を考える:

Grassmann odd parameter

$$\begin{cases}\delta_\varepsilon X=\varepsilon^1\psi_1+\varepsilon^2\psi_2\\ \delta_\varepsilon\psi_1=\varepsilon^2 h'(X)\\ \delta_\varepsilon\psi_2=-\varepsilon^1 h'(X)\end{cases}$$

"SUSY変換"

$$\begin{aligned}\delta_\varepsilon S&=h'h''(\varepsilon^1\psi_1+\varepsilon^2\psi_2)\\&\quad-h''(\varepsilon^2 h'\psi_2-\underbrace{\psi_1\varepsilon^1}_{=-\varepsilon^1\psi_1}h')\\&=0\end{aligned}$$

∴ S は SUSY 不変

代数が閉じているか？

$$\delta_\varepsilon \delta_{\hat{\varepsilon}} X = \delta_\varepsilon (\hat{\varepsilon}^1 \psi_1 + \hat{\varepsilon}^2 \psi_2)$$
$$= h' (\hat{\varepsilon}^1 \varepsilon^2 - \hat{\varepsilon}^2 \varepsilon^1)$$

$$\delta_{\hat{\varepsilon}} \delta_\varepsilon X = h' (\varepsilon^1 \hat{\varepsilon}^2 - \varepsilon^2 \hat{\varepsilon}^1)$$

$$\therefore \quad [\delta_\varepsilon , \delta_{\hat{\varepsilon}}] X = 0$$

同様に，

$$[\delta_\varepsilon , \delta_{\hat{\varepsilon}}] \psi_1 = h'' (\hat{\varepsilon}^2 \varepsilon^1 - \varepsilon^2 \hat{\varepsilon}^1) \psi_1$$
$$= 0 \quad (\text{up to e.o.m. } h'' \psi_1 = 0$$

$$[\delta_\varepsilon , \delta_{\hat{\varepsilon}}] \psi_2 = -h'' (\hat{\varepsilon}^1 \varepsilon^2 - \varepsilon^1 \hat{\varepsilon}^2) \psi_2$$
$$= 0$$

"on-shell SUSY"

Off-shell version

$$S[B, X, \psi_1, \psi_2] = \frac{1}{2}B^2 + iBh'(X) - h''(X)\psi_1\psi_2$$

$$\left(\begin{array}{l} \text{Bについて積分すると} \\ \qquad S \xrightarrow{B=-ih'} \frac{1}{2}(h')^2 - h''\psi_1\psi_2 \end{array} \right)$$

SUSY trans. :

$$\delta_\xi X = \varepsilon^1 \psi_1 + \varepsilon^2 \psi_2$$

$$\delta_\xi \psi_1 = i\varepsilon^2 B$$

$$\delta_\xi \psi_2 = -i\varepsilon^1 B$$

$$\delta_\xi B = 0$$

$$\delta_\xi S = iBh''(\varepsilon^1\psi_1 + \varepsilon^2\psi_2) - ih''(\varepsilon^2 B\psi_2 - \psi_1\varepsilon^1 B)$$

$$= 0$$

$$[\delta_\xi, \delta_{\bar\xi}](X, \psi_1, \psi_2, B) = 0 \quad \text{(w/o e.o.m.)}$$

1-2. SUSY QM

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}(h'(x))^2 + \frac{i}{2}(\bar{\psi}\dot{\psi} - \dot{\bar{\psi}}\psi) - h''(x)\bar{\psi}\psi$$

$$(\bar{\psi} = \psi^\dagger)$$

SUSY trans.:

$$\delta x = \varepsilon\bar{\psi} - \bar{\varepsilon}\psi$$

$$\delta\psi = \varepsilon(i\dot{x} + h'(x))$$

$$\delta\bar{\psi} = \bar{\varepsilon}(-i\dot{x} + h'(x))$$

Algebra:

$$[\delta_1, \delta_2]\, x = 2i(\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1)\,\dot{x}$$

$$[\delta_1, \delta_2]\, \psi \simeq 2i(\varepsilon_1\bar{\varepsilon}_2 - \varepsilon_2\bar{\varepsilon}_1)\,\dot{\psi}$$

時間並進

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保存charge:

$$Q = \bar{\psi}(i\dot{x} + h'), \quad \bar{Q} = \psi(-i\dot{x} + h')$$

$(\bar{Q} = Q^\dagger)$

実際,

$$\dot{Q} = \dot{\bar{\psi}}(i\dot{x} + h') + \bar{\psi}(i\ddot{x} + h''\dot{x})$$

$$\left(i\dot{\bar{\psi}} + h''\bar{\psi} = 0, \quad \ddot{x} + h'h'' + h'''\bar{\psi}\psi = 0\right)$$

$$= ih''\bar{\psi}(i\dot{x} + h') + \bar{\psi}(-ih'h'' + h''\dot{x})$$

$$= 0$$

Hamilton形式：

$$H = \frac{1}{2}p^2 + \frac{1}{2}(h'(x))^2 + \frac{1}{2}h''(x)(\bar{\psi}\psi - \psi\bar{\psi})$$

$$[x, p] = i, \quad \{\psi, \bar{\psi}\} = 1$$

$$Q = \bar{\psi}(ip + h'(x)), \quad \bar{Q} = \psi(-ip + h'(x))$$

$$[Q, H] = -ip\bar{\psi}h'' - \bar{\psi}h'h'' + h''\bar{\psi}(ip + h') = 0$$

$$[\bar{Q}, H] = 0$$

$$Q\bar{Q} = \bar{\psi}\psi(p^2 + i[p, h'] + (h')^2)$$

$$= \bar{\psi}\psi(p^2 + h'' + (h')^2)$$

$$\bar{Q}Q = \psi\bar{\psi}(p^2 - h'' + (h')^2)$$

$$\{Q, \bar{Q}\} = p^2 + (h')^2 + h''(\bar{\psi}\psi - \psi\bar{\psi})$$

$$= 2H$$

1-3. 4d SUSY theories (on $\mathbb{R}^4$)

Notation

$$\gamma_{m=1,2,3} = \begin{pmatrix} 0 & \sigma_m \\ \sigma_m & 0 \end{pmatrix}, \quad \gamma_4 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\gamma_5 \equiv -\gamma_{1234} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\chi\psi = \chi^a \psi_a = (\varepsilon^{ab}\chi_b)\psi_a$$

$$\left(\begin{array}{l} \varepsilon^{12} = 1 = -\varepsilon^{21} = \varepsilon_{12}, \\ \chi\psi = \psi\chi \end{array} \right)$$

Weyl spinor:

$$\psi \equiv \begin{pmatrix} \psi \\ 0 \end{pmatrix} \qquad \bar{\psi} \equiv \begin{pmatrix} 0 \\ \bar{\psi} \end{pmatrix}$$

$$\gamma_5 \psi = +\psi, \quad \gamma_5 \bar{\psi} = -\bar{\psi}$$

※ $\mathbb{R}^4$ で ψ と $\bar{\psi}$ は独立

$\mathcal{N}=1$ vector multiplet

$$(A_\mu, \lambda, \bar{\lambda}, D)$$

$$\mathcal{L}_{YM} = \frac{1}{g^2} \mathrm{tr}\left[\frac{1}{4} (F_{\mu\nu})^2 - (\bar{\lambda} \gamma^\mu D_\mu \lambda) \underbrace{- \frac{1}{2} D^2}_{\text{積分経路?}} \right]$$

SUSY trans. :

$$\delta A_\mu = i(\varepsilon \gamma_\mu \lambda) - i(\bar{\varepsilon} \gamma_\mu \lambda)$$

$$\delta \lambda = \frac{i}{2} \gamma^{\mu\nu} \varepsilon F_{\mu\nu} + D \varepsilon$$

$$\delta \bar{\lambda} = -\frac{i}{2} \gamma^{\mu\nu} \bar{\varepsilon} F_{\mu\nu} + D \bar{\varepsilon}$$

$$\delta D = -(\varepsilon \gamma^\mu D_\mu \bar{\lambda}) - (\bar{\varepsilon} \gamma^\mu D_\mu \lambda)$$

(Θ-term と FI-term は省略)

$\mathcal{N}=1$ chiral multiplet

$$L_{kin} = D_\mu\bar{\phi} D^\mu\phi - (\bar{\psi}\gamma^\mu D_\mu\psi) + \bar{\phi}D\phi - \bar{F}F$$
$$-\sqrt{2}\,\bar{\phi}\lambda\psi - \sqrt{2}\,\bar{\psi}\bar{\lambda}\phi$$

SUSY trans.:

$$\delta\phi = \sqrt{2}\,\varepsilon\psi$$
$$\delta\psi = -\sqrt{2}\,\gamma^\mu\bar{\varepsilon}D_\mu\phi + \sqrt{2}\,\varepsilon F$$
$$\delta F = -\sqrt{2}\,\bar{\varepsilon}\gamma^\mu D_\mu\psi - 2(\bar{\varepsilon}\bar{\lambda})\phi$$
$$\delta\bar{\phi} = \sqrt{2}\,\bar{\varepsilon}\psi$$
$$\delta\bar{\psi} = -\sqrt{2}\,\gamma^\mu\varepsilon D_\mu\bar{\phi} + \sqrt{2}\,\bar{\varepsilon}\bar{F}$$
$$\delta\bar{F} = -\sqrt{2}\,\varepsilon\gamma^\mu D_\mu\bar{\psi} - 2\bar{\phi}\varepsilon\lambda$$

$$L_{pot} = \frac{\partial W}{\partial\phi_i}F^i - \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(\psi^i\psi^j) + h.c.$$

(ここでは global sym. の charge 決めるのくらいにしか使わない)

$U(1)_R$ 対称性　$q_R(\theta)=1$

$$\Phi \to e^{iq_R}\Phi$$

$$q_R(A_\mu, \lambda, \bar{\lambda}, D) = (0, +1, -1, 0)$$

$$q_R(\varphi, \psi, F) = (q, q-1, q-2)$$

$$q_R(\bar{\varphi}, \bar{\psi}, \bar{F}) = (-\underbrace{q}, -(q-1), -(q-2))$$

さらにWなどから制限

SUSY algebra

$$[\delta, \delta']\varphi = 2(\underbrace{v^\mu \partial_\mu \varphi}_{\text{並進}} + i\underbrace{(-v^\mu A_\mu)}_{\text{gauge trans}}\varphi)$$

$$v^\mu = \bar{\varepsilon}\gamma^\mu \varepsilon' + \varepsilon \gamma^\mu \bar{\varepsilon}'$$

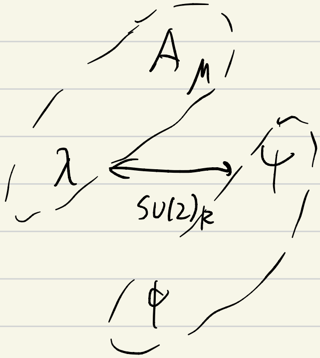
$$[Q, Q^\dagger] \sim \sigma^\mu P_\mu$$

$\left(r_{\mathcal{N}=1} = \frac{2}{3}\left(2R_{SU(2)} - r_{U(1)_r}\right)\right)$

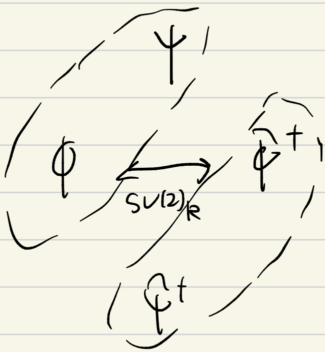
4d $\mathcal{N}=2$

$(W = \Phi q \tilde{q})$

$\mathcal{N}=2$ vector $=$ $\mathcal{N}=1$ vector $+$ adj. chiral $\left(q_R = \frac{2}{3}\right)$



$\mathcal{N}=2$ hyper $=$ conjugate pair of chirals $\left(q_R = \frac{2}{3}\right)$



4d $\mathcal{N}=3$

Lagrangianがある時は常に 4d $\mathcal{N}=4$

4d $\mathcal{N}=4$ (SYM)

$\mathcal{N}=4$ vector = $\mathcal{N}=2$ vector + adj. hyper

= $\mathcal{N}=1$ vector + 3 adj. chirals $(q_R = \frac{2}{3})$

$\to$ 6 scalars

$SO(6)_R \simeq SU(4)_R$