Application of Quantum Computation to Quantum Field Theory

– QFT on Quantum Computer –

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PRESTO SAKIGAKE

7th Jul., 2023

Intensive lectures @Niigata U.（day 2）

Plan of the lectures

(If 2nd lecture in each day ends early, then we start hands-on early)

Day 1

- Lecture 1: introduction, basics of quantum computation
- Lecture 2: Spin system on quantum computer (QC)
- Hands-on 1: Basics on IBM's qiskit, time evolution of Ising

Day 2

- Lecture 3: Quantum field theory (QFT) on QC
- Lecture 4: QFT on QC, error correction & future prospects
- Hands-on 2: vacuum of Ising, Renyi entropy

What is meant by

"Application of Quantum Computation to Quantum Field Theory" ??

In general, it is

to replace (a part of) computations by quantum algorithm

Therefore,

physical meaning of qubits in quantum computer depends on contexts

Here,

qubits = states in physical system

Plan of lecture 3

0. Conventional numerical approach to QFT

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

Scalar field theory (continuum)

Action (Euclidean):

$$S[\phi] = \int d^d x \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 + V(\phi)\right]$$

Vacuum expectation value can be given by path integral:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi \, \mathcal{O}(\phi) e^{-S[\phi]}}{\int D\phi \, e^{-S[\phi]}}$$

But it is ∞-dimensional & can't be simulated practically

coming from the fact that spacetime has ∞-many points!

Scalar field theory on lattice

Discretize the spacetime by lattice:

$$\phi(x) \qquad\qquad \phi(x_n \equiv an)$$

a: lattice spacing, $n \equiv (n_1, \cdots n_d)$

The simplest lattice action:

$$\int d^d x \to a^d \sum_n , \qquad \partial_\mu \phi(x) \to \Delta_\mu \phi(x_n) \equiv \frac{\phi(x_n + a e_\mu) - \phi(x_n)}{a}$$

$$S[\phi] = \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi\right)^2 + V(\phi) \right] \Rightarrow a^d \sum_n \left[\frac{1}{2} \sum_\mu \left(\Delta_\mu \phi\right)^2 + V(\phi) \right]$$

Gauge invariant interaction on lattice

Let's consider complex scaler field theory

Continuum

$$\int d^d x \left[(\partial_\mu \bar{\phi})(\partial_\mu \phi) + V\left(\bar{\phi}\phi\right)\right]$$

$U(1)$ global sym. : $\phi(x) \rightarrow e^{i\theta}\phi(x)$

Promotion to gauge: $\theta \rightarrow \theta(x)$

But,

$$\partial_\mu \phi(x) \rightarrow \mathrm{e}^{i\theta}\left(\partial_\mu \phi + i(\partial_\mu \theta)\phi\right) \neq \mathrm{e}^{i\theta}\partial_\mu \phi$$

Introduction of gauge field:

$$\partial_\mu \phi(x) \rightarrow \left(\partial_\mu - iA_\mu(x)\right)\phi(x)$$

w/ $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu \theta(x)$

Lattice

$$a^d \sum_n \left[\sum_\mu (\Delta_\mu \bar{\phi})(\Delta_\mu \phi) + V\left(\bar{\phi}\phi\right)\right]$$

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But,

$$\Delta_\mu \phi(x) \to \frac{e^{i\theta(x_n + ae_\mu)}\phi\left(x_n + ae_\mu\right) - e^{i\theta(x_n)}\phi(x_n)}{a}$$

Gauge invariant interaction on lattice

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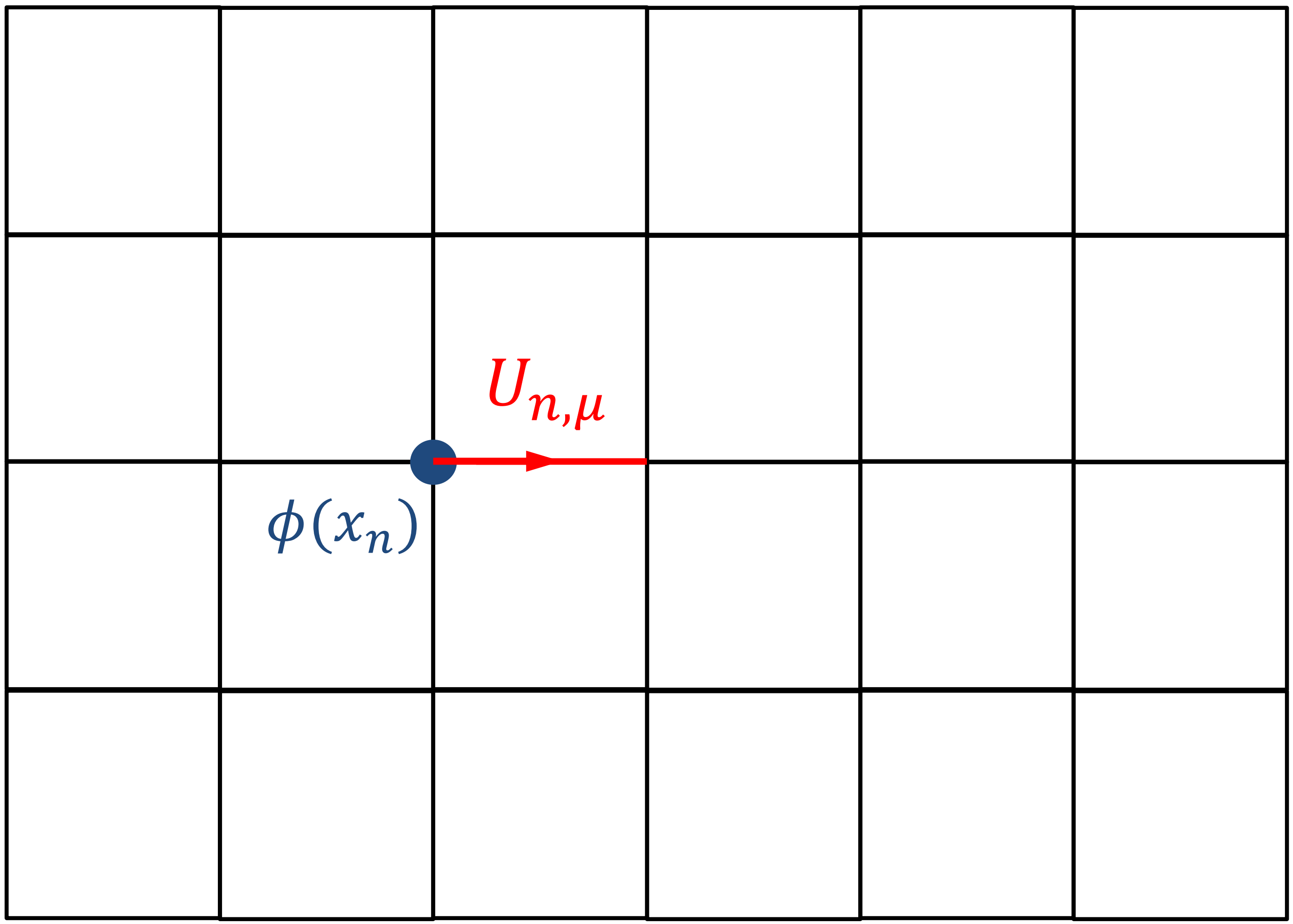
Introduction of "gauge field":

$$\Delta_\mu \phi(x) \to \frac{\phi(x_n + ae_\mu) - U_{n,\mu}\phi(x_n)}{a}$$

w/ $U_{n,\mu} \to e^{i\theta(x_n + ae_\mu)} U_{n,\mu} e^{-i\theta(x_n)}$

living on link between x_n & $x_n + ae_\mu$



$$\phi(x_n) \to e^{i\theta(x_n)}\phi(x_n), \quad U_{n,\mu} \to e^{i\theta(x_n+ae_\mu)}U_{n,\mu}e^{-i\theta(x_n)}$$

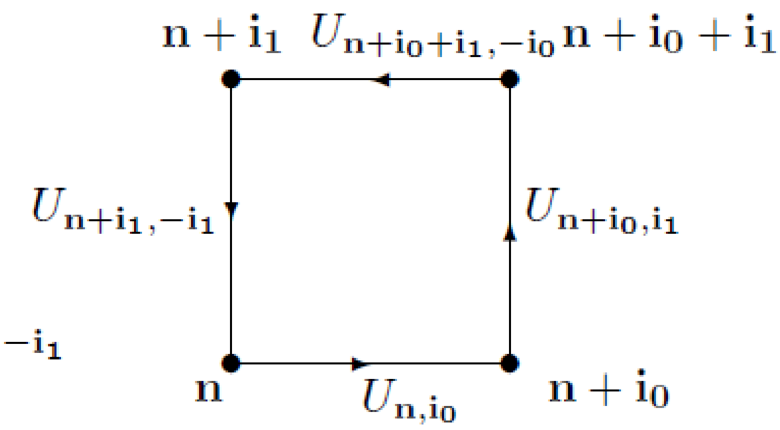
$$\Delta_\mu\phi(x) \to \frac{\phi\left(x_n + ae_\mu\right) - U_{n,\mu}\phi(x_n)}{a}$$

Lattice gauge theory ($G = SU(N)$)

Action:

$$S(U) = \sum_P \frac{1}{g^2} Tr \left(\prod_P U + H.c. \right)$$

$$\prod_P U = U_{\mathbf{n},\mathbf{i_0}} U_{\mathbf{n}+\mathbf{i_0},\mathbf{i_1}} U_{\mathbf{n}+\mathbf{i_0}+\mathbf{i_1},-\mathbf{i_0}} U_{\mathbf{n}+\mathbf{i_1},-\mathbf{i_1}}$$



Gauge trans.:

$$U_{\boldsymbol{n},\boldsymbol{i}} \to V_{\boldsymbol{n}+\boldsymbol{i}} U_{\boldsymbol{n},\boldsymbol{i}} V_{\boldsymbol{n}}^{\dagger} \qquad V_{\boldsymbol{n}} \in SU(N)$$

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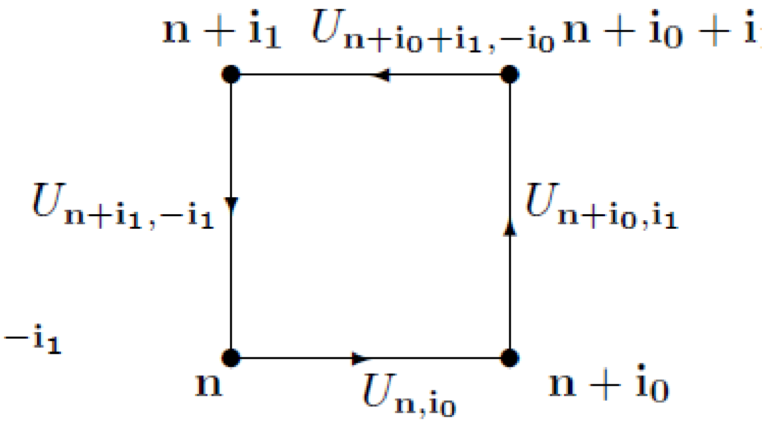
$$\prod_P U = U_{\mathrm{n},\mathrm{i}_0} U_{\mathrm{n}+\mathrm{i}_0,\mathrm{i}_1} U_{\mathrm{n}+\mathrm{i}_0+\mathrm{i}_1,-\mathrm{i}_0} U_{\mathrm{n}+\mathrm{i}_1,-\mathrm{i}_1}$$



Gauge trans.:

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"Path integral" :

$$Z := \int [DU]\, e^{-S[U]} \qquad DU \equiv \prod_{\mathrm{n},\mathrm{i}} dU_{\mathrm{n},\mathrm{i}} \qquad \textit{Haar measure}$$

$$\langle \mathcal{O}(U) \rangle := \frac{1}{Z} \int [DU] \mathcal{O}(U)\, e^{-S[U]}$$

Conventional approach to simulate QFT

① Discretize Euclidean spacetime by lattice:

& make path integral finite dimensional:

$$\int D\phi\ \mathcal{O}(\phi)e^{-S[\phi]} \Rightarrow \int d\phi\ \mathcal{O}(\phi)e^{-S(\phi)}$$

Conventional approach to simulate QFT

① Discretize Euclidean spacetime by lattice:

& make path integral finite dimensional:

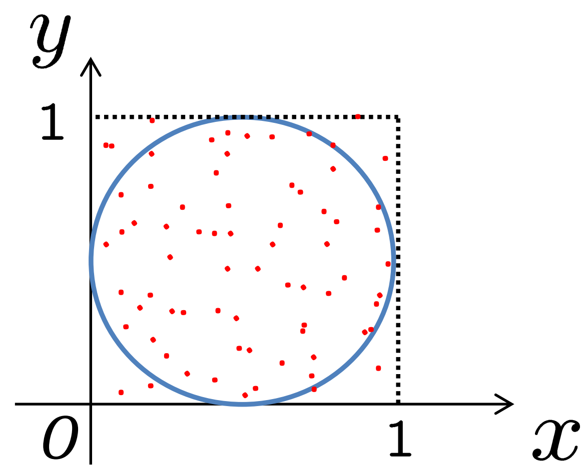
$$\int D\phi\ \mathcal{O}(\phi)e^{-S[\phi]} \quad \Longrightarrow \quad \int d\phi\ \mathcal{O}(\phi)e^{-S(\phi)}$$

② Numerically Evaluate it by Markov chain Monte Carlo method regarding the Boltzmann factor as a probability:

$$\langle \mathcal{O}(\phi) \rangle \simeq \frac{1}{\sharp(\text{samples})} \sum_{i \in \text{samples}} \mathcal{O}(\phi_i)$$

“Direct” Monte Carlo method

Ex.) The area of the circle with the radius 1/2



① Distribute random numbers many times

$$x \in [0, 1), \;\; y \in [0, 1)$$

② Count the number of points which satisfy

$$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 \leq \frac{1}{4}$$

③ Estimate the ratio

$$\frac{\text{(Number of points inside the circle)}}{\text{(Numer of points for distribution)}} \simeq \text{(Area)}$$

Markov chain Monte Carlo method

Consider a Markov process w/ transition probability $P^{(a)}$:

$$x^{(0)} \to x^{(1)} \to \cdots \to \cdots \to x^{(M-1)} \to x^{(M)} \to \cdots$$

$P^{(1)}(x^{(0)}, x^{(1)})$ $P^{(2)}(x^{(1)}, x^{(2)})$ $P^{(M)}(x^{(M-1)}, x^{(M)})$

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Under some conditions,

transition prob. converges to an equilibrium prob.

$$\lim_{M\to\infty} P^{(M)}(x^{(M-1)}, x^{(M)}) = P_{eq}(x^{(M)})$$ *thermalization*

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transition prob. converges to an equilibrium prob.

$$\lim_{M\to\infty} P^{(M)}(x^{(M-1)}, x^{(M)}) = P_{eq}(x^{(M)})$$ *thermalization*

We can compute exp. values by an algorithm to generate

$$P_{eq}(x) \propto e^{-S(x)}$$

Ex.) Gaussian ensemble by heat bath algorithm

$$\langle O(x,y)\rangle = \frac{\int dxdy\ O(x,y)P(x,y)}{\int dxdy\ P(x,y)} \qquad P(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$$

①

②

Ex.) Gaussian ensemble by heat bath algorithm

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① Generate random configurations with Gaussian weight many times

$$\frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}dxdy = \frac{1}{2\pi}re^{-\frac{r^2}{2}}drd\theta = d\xi d\eta \qquad (\eta, \xi \in [0,1))$$

$$\begin{pmatrix} x = r\cos\theta \\ y = r\sin\theta \end{pmatrix} \qquad \begin{pmatrix} \theta = 2\pi\eta \\ r = \sqrt{-2\log\xi} \end{pmatrix}$$

The uniform random numbers generate the Markov chain:

$$\underset{P(x^{(1)},y^{(1)})}{(x^{(0)},y^{(0)}) \to (x^{(1)},y^{(1)})} \to \cdots\cdots \to \underset{P(x^{(M)},y^{(M)})}{(x^{(M)},y^{(M)})}$$

②

Ex.) Gaussian ensemble by heat bath algorithm

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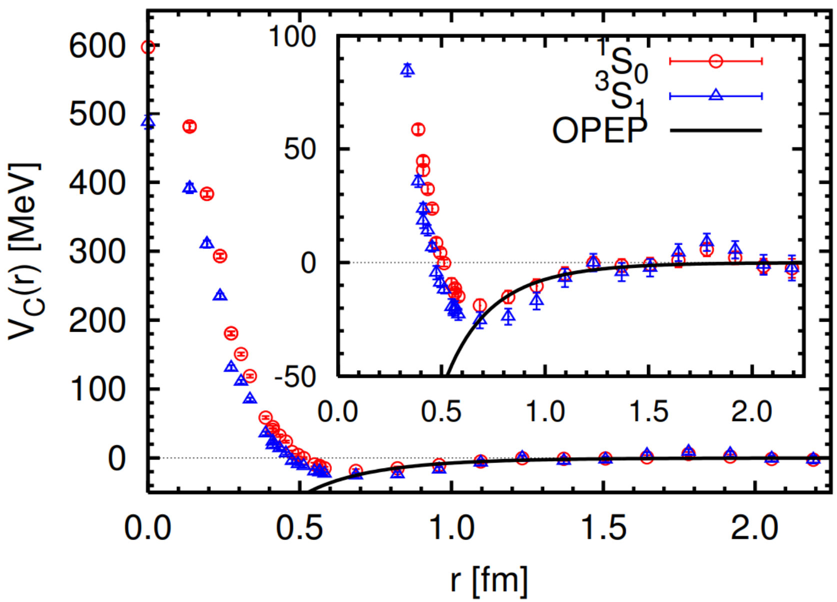
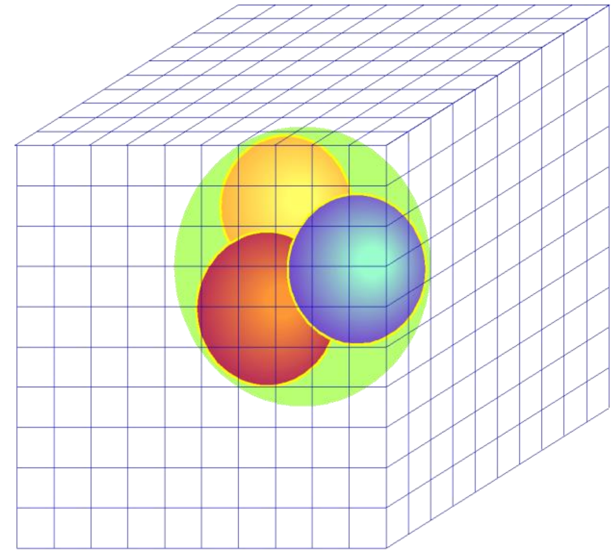
$$(x^{(0)},y^{(0)}) \to \underset{P(x^{(1)},y^{(1)})}{(x^{(1)},y^{(1)})} \to \cdots\cdots \to \underset{P(x^{(M)},y^{(M)})}{(x^{(M)},y^{(M)})}$$

② Measure observable and take its average:

$$\frac{1}{M}\sum_{a=1}^{M} O(x^{(a)},y^{(a)}) \simeq \langle O(x,y)\rangle$$

Success of lattice QCD (e.g. nuclear force)





Nuclear Force from Lattice QCD

N. Ishii[1,2], S. Aoki[3,4] and T. Hatsuda[2]

[1] *Center for Computational Sciences, University of Tsukuba, Tsukuba 305–8577, Ibaraki, JAPAN,*
[2] *Department of Physics, University of Tokyo, Tokyo 113–0033, JAPAN,*
[3] *Graduate School of Pure and Applied Sciences, University of Tsukuba, Tsukuba 305–8571, Ibaraki, JAPAN and*
[4] *RIKEN BNL Research Center, Brookhaven National Laboratory, Upton, New York 11973, USA*

Nucleon-nucleon (NN) potential is studied by lattice QCD simulations in the quenched approximation, using the plaquette gauge action and the Wilson quark action on a 32^4 ($\simeq (4.4\ \mathrm{fm})^4$) lattice. A NN potential $V_{\mathrm{NN}}(r)$ is defined from the equal-time Bethe-Salpeter amplitude with a local interpolating operator for the nucleon. By studying the NN interaction in the $^1\mathrm{S}_0$ and $^3\mathrm{S}_1$ channels, we show that the central part of $V_{\mathrm{NN}}(r)$ has a strong repulsive core of a few hundred MeV at short distances ($r \lesssim 0.5$ fm) surrounded by an attractive well at medium and long distances. These features are consistent with the known phenomenological features of the nuclear force.

Sign problem in Monte Carlo simulation

Markov Chain Monte Carlo:

$$\int d\phi \ \mathcal{O}(\phi)\underbrace{e^{-S(\phi)}}_{\text{probability}}$$

can't directly apply when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$

Sign problem in Monte Carlo simulation

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can't directly apply when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$

Naïve way to avoid = reweighting:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int D\phi \ \mathcal{O}(\phi) e^{-S[\phi]}}{\int D\phi \ e^{-S[\phi]}}$$

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$$= \frac{\langle \mathcal{O}(\phi) \cdot \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}{\langle \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}$$

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$$= \frac{\langle \mathcal{O}(\phi) \cdot \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}{\langle \mathrm{phase}(e^{-S}) \rangle_{\mathrm{no-phase}}}$$

For highly oscillating integral, $\sim \frac{0}{0}$ ⇒ needs huge statistics

"sign problem"

Sign problem in Monte Carlo simulation (cont'd)

Markov Chain Monte Carlo:

$$\int d\phi\ \mathcal{O}(\phi)\underbrace{e^{-S(\phi)}}_{\text{probability}}$$

problematic when Boltzmann factor isn't $\mathbf{R}_{\geqq 0}$ & is highly oscillating

Examples w/ sign problem:

- topological term —— complex action
- chemical potential —— indefinite sign of fermion determinant
- real time —— " $e^{iS(\phi)}$ " much worse

In operator formalism suitable for quantum simulation,

sign problem is absent from the beginning

(∃ various approaches within framework of path integral formalism but I'll skip it)

Plan of lecture 3

0. Conventional numerical approach to QFT

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

“Regularization” of Hilbert space

Hilbert space of QFT is typically ∞ dimensional

$\longrightarrow$ Make it finite dimensional!

- **Fermion** is easiest (up to doubling problem)

  — Putting on spatial lattice, Hilbert sp. is finite dimensional

- scalar

  — Hilbert sp. at each site is ∞ dimensional

  (need truncation or additional regularization)

- gauge field (w/ kinetic term)

  — no physical d.o.f. in 0+1D/1+1D (w/ open bdy. condition)

  — ∞ dimensional Hilbert sp. in higher dimensions

Citation history of “Hamiltonian Formulation of Wilson's Lattice Gauge Theories” by Kogut-Susskind

(totally 2177 at this moment)

(1+1)d free Dirac fermion (continuum)

Lagrangian:

$$\mathcal{L} = \int dx\left[i\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - m\bar{\psi}\psi\right] \qquad \{\gamma^{\mu},\gamma^{\nu}\} = 2\eta^{\mu\nu}$$

$$\Big\Downarrow \qquad \frac{\partial\mathcal{L}}{\partial(\partial_t\psi)} = \bar{\psi}$$

Hamiltonian:

$$H = \int dx\left[-i\bar{\psi}\gamma^{1}\partial_{1}\psi + m\bar{\psi}\psi\right]$$

$$\{\psi(x),\bar{\psi}(y)\} = \delta(x-y)$$

(1+1)d free Dirac fermion (lattice)

Continuum:

$$H = \int dx\left[-i\bar{\psi}\gamma^1\partial_1\psi + m\bar{\psi}\psi\right]$$

$$\psi(x) = \begin{pmatrix}\psi_u(x)\\ \psi_d(x)\end{pmatrix} \qquad \gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2$$

$$= \int dx\left[-i(\psi_u^\dagger\partial_1\psi_d + \psi_d^\dagger\partial_1\psi_u) + m(\psi_u^\dagger\psi_u - \psi_d^\dagger\psi_d)\right]$$

$$\Downarrow$$

Lattice (w/ N sites and spacing a):

"Staggered fermion" [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix}\psi_u\\ \psi_d\end{pmatrix}$$

$\psi_u \rightarrow$ odd site

$\psi_d \rightarrow$ even site

(1+1)d free Dirac fermion (lattice)

Continuum:

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$$\psi(x) = \begin{pmatrix}\psi_u(x)\\ \psi_d(x)\end{pmatrix} \qquad \gamma^0 = \sigma_3, \quad \gamma^1 = i\sigma_2$$

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$$\frac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix}\psi_u \\ \psi_d\end{pmatrix}\begin{matrix}\rightarrow \text{odd site}\\ \rightarrow \text{even site}\end{matrix}$$

$$H = -\frac{i}{2a}\sum_{n=1}^{N-1}\left(\chi_n^\dagger\chi_{n+1} - \chi_{n+1}^\dagger\chi_n\right) + m\sum_{n=1}^{N}(-1)^n\chi_n^\dagger\chi_n$$

$$\{\chi_m, \chi_n^\dagger\} = \delta_{\mathrm{mn}}, \ \{\chi_m, \chi_n\} = 0$$

Jordan-Wigner transformation

$$\{\chi_m, \chi_n^\dagger\} = \delta_{\mathrm{mn}}, \ \{\chi_m, \chi_n\} = 0$$

This is satisfied by the operator: [Jordan-Wigner'28]

$$\chi_n = \frac{X_n - \mathrm{i}Y_n}{2}\left(\prod_{i=1}^{n-1} -\mathrm{i}Z_i\right)$$

(X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

Jordan-Wigner transformation

$$\{\chi_m, \chi_n^\dagger\} = \delta_{\mathrm{mn}}, \ \{\chi_m, \chi_n\} = 0$$

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(X_n, Y_n, Z_n: $\sigma_{1,2,3}$ at site n)

Then the system is mapped to the spin system:

$$\boxed{\hat{H} = \frac{w}{2}\sum_{n=1}^{N-1}\left(X_n X_{n+1} + Y_n Y_{n+1}\right) + \frac{m}{2}\sum_{n=1}^{N}(-1)^n Z_n}$$

Now we can apply quantum algorithms to QFT!

Scalar field theory (continuum)

Lagrangian:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

$$\Big\Downarrow \qquad \Pi(\mathbf{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi)} = \partial_t\phi$$

Hamiltonian:

$$\mathcal{H}(\boldsymbol{x}) = \frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i\phi)^2 + V(\phi)$$

$$[\phi(\boldsymbol{x}), \Pi(\boldsymbol{y})] = i\delta^{(d)}(\boldsymbol{x}-\boldsymbol{y})$$

Scalar field theory (lattice)

Continuum Hamiltonian:

$$H = \int d^d \boldsymbol{x} \left[\frac{1}{2}\Pi^2 + \frac{1}{2}(\partial_i \phi)^2 + V(\phi) \right]$$

$$\int d^d x \to \ a^d \sum_n \ ,$$

$$\partial_\mu \phi(x) \to \Delta_\mu \phi(x_n) \equiv \frac{\phi(x_n + a e_\mu) - \phi(x_n)}{a}$$

Lattice Hamiltonian (simplest):

$$H = a^d \sum_n \left[\frac{1}{2}\Pi_{\mathrm{n}}^2 + \frac{1}{2}\sum_i (\Delta_i \phi_n)^2 + V(\phi_n) \right]$$

$$[\phi(\boldsymbol{x_m}), \Pi(\boldsymbol{x_n})] = i\delta_{\boldsymbol{m,n}}$$

technically the same as multi-particle QM

Regularization for single particle QM

$$\hat{H} = \frac{1}{2}\hat{p}^2 + \frac{\omega^2}{2}\hat{x}^2 + V_{\mathrm{int}}(\hat{x})$$

Most naïve approach = truncation in harmonic osc. basis:

$$\hat{a} = \sqrt{\frac{\omega}{2}}\,\hat{x} + \frac{i}{\sqrt{2\omega}}\,\hat{p} \;=\; \sum_{n=0}^{\infty} \sqrt{n+1}\,|n\rangle\langle n+1|$$

$$\xRightarrow[\textit{regularize!}]{} \quad \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

Regularization for single particle QM

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$$\xrightarrow{\text{regularize!}} \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

Then replace $\hat{p}$ & $\hat{x}$ by

$$\hat{x}\Big|_{\text{regularized}} \equiv \frac{1}{\sqrt{2\omega}}(\hat{a} + \hat{a}^\dagger)\Big|_{\text{regularized}}$$

$$\hat{p}\Big|_{\text{regularized}} \equiv \frac{1}{i}\sqrt{\frac{\omega}{2}}\,(\hat{a} - \hat{a}^\dagger)\Big|_{\text{regularized}}$$

Regularization for single particle QM (Cont'd)

$$\hat{a}\Big|_{\text{regularized}} = \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

We can rewrite the Fock basis in terms of qubits:

$$|n\rangle = |b_{K-1}\rangle|b_{K-2}\rangle \cdots |b_0\rangle \qquad K \equiv \log_2 \Lambda$$

$$n = \mathrm{b}_{\mathrm{K}-1}2^{\mathrm{K}-1} + \mathrm{b}_{\mathrm{K}-2}2^{\mathrm{K}-2} + \cdots + b_0 2^0$$ (binary representation)

Regularization for single particle QM (Cont'd)

$$\hat{a}\Big|_{\text{regularized}} = \sum_{n=0}^{\Lambda-2} \sqrt{n+1}\,|n\rangle\langle n+1|$$

We can rewrite the Fock basis in terms of qubits:

$$|n\rangle = |b_{K-1}\rangle|b_{K-2}\rangle \cdots |b_0\rangle \qquad K \equiv \log_2 \Lambda$$

$$n = \mathrm{b}_{\mathrm{K}-1}2^{\mathrm{K}-1} + \mathrm{b}_{\mathrm{K}-2}2^{\mathrm{K}-2} + \cdots + b_0 2^0$$ (binary representation)

Then,

$$|n\rangle\langle n+1| = \otimes_{\ell=0}^{K-1} \underbrace{(|b'_\ell\rangle\langle b_\ell|)}_{\text{either one of}}$$

$$\begin{pmatrix} |0\rangle\langle 0| = \dfrac{1_2 - \sigma_z}{2}, & |1\rangle\langle 1| = \dfrac{1_2 + \sigma_z}{2}, \\ |0\rangle\langle 1| = \dfrac{\sigma_x + i\sigma_y}{2}, & |1\rangle\langle 0| = \dfrac{\sigma_x - i\sigma_y}{2} \end{pmatrix}$$

Pure Maxwell theory (continuum)

Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \qquad (F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu)$$

temporal gauge $A_0 = 0$

$$E^i = \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = \dot{A}^i$$

Hamiltonian:

$$\mathcal{H} = \frac{1}{2} E_i^2 + \frac{1}{2} B_i^2$$

$$[A_i(\boldsymbol{x}), E_j(\boldsymbol{y})] = i\delta_{ij}\delta^{(d)}(\boldsymbol{x} - \boldsymbol{y})$$

Gauss law:

$$\partial_i E^i = 0$$

Pure Maxwell theory (lattice)

Continuum:

$$\mathcal{H} = \frac{1}{2} E_i^2 + \frac{1}{2} B_i^2 \qquad \partial_i E^i = 0$$

$$\Downarrow$$

Lattice:

$$\mathcal{H} = \frac{a^d}{2} \sum_{\boldsymbol{n},i} L_{\boldsymbol{n},i}^2 + \mathrm{Re} \sum_{\text{plaquette}} \sum_{i<j} \prod_{P \in \text{plaquette}} U_P$$

$$[U_{\boldsymbol{m},i}, L_{\boldsymbol{n},j}] = i \delta_{ij} \delta_{\boldsymbol{m},\boldsymbol{n}}$$

Gauss law:

$$\sum_i (L_{\boldsymbol{n}+\boldsymbol{e}_i,i} - L_{\boldsymbol{n},i}) = 0$$

Ex. (1+1)d pure Maxwell theory w/ θ

Continuum:

$$\mathcal{L} = \frac{1}{2g^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01}$$

$$\Pi = \frac{1}{g^2}\dot{A} + \frac{\theta}{2\pi}$$

$$\mathcal{H} = \frac{1}{2}\left(\Pi - \frac{\theta}{2\pi}\right)^2$$

Lattice:

$$H = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2$$

$$L_n \leftrightarrow -\frac{\Pi(x)}{g}$$

Gauss law:

$$L_{n+1} - L_n = 0$$

Ex. (1+1)d pure Maxwell theory w/ θ

Continuum:

$$\mathcal{L} = \frac{1}{2g^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01} \xrightarrow{\Pi = \frac{1}{g^2}\dot{A} + \frac{\theta}{2\pi}} \mathcal{H} = \frac{1}{2}\left(\Pi - \frac{\theta}{2\pi}\right)^2$$

Lattice:

$$H = \frac{g^2 a}{2} \sum_n \left(L_n + \frac{\theta}{2\pi}\right)^2 \qquad L_n \leftrightarrow -\frac{\Pi(x)}{g}$$

Gauss law:

$$L_{n+1} - L_n = 0$$

- open b.c.

$$L_n = L_{n-1} = L_{n-2} = \cdots = L_1 = (b.c.)$$

- p.b.c.

$$L_n = L_{n-1} = \cdots = L_1 = \cdots = L_{n+1} = L_n$$

one d.o.f. remains

Short summary

(repeated)

Hilbert space of QFT is typically ∞ dimensional

$\longrightarrow$ Make it finite dimensional!

- Fermion is easiest (up to doubling problem)
  — Putting on spatial lattice, Hilbert sp. is finite dimensional
- scalar
  — Hilbert sp. at each site is ∞ dimensional
  (need truncation or additional regularization)
- gauge field (w/ kinetic term)
  — no physical d.o.f. in 0+1D/1+1D (w/ open bdy. condition)
  — ∞ dimensional Hilbert sp. in higher dimensions

Plan of lecture 3

0. Conventional numerical approach to QFT

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

Schwinger model w/ topological term

Continuum ①: (will be used for the case w/ probes)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

Continuum ②: (equivalent via "chiral anomaly", used here) [Fujikawa'79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^{5}}\psi$$

Schwinger model w/ topological term

Continuum ①:

(will be used for the case w/ probes)

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

Continuum ②:

(equivalent via "chiral anomaly", used here)

[Fujikawa'79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

Taking temporal gauge $A_0 = 0$, $(\Pi = \dot{A}^1)$

$$\hat{H} = \int dx\left[-i\bar{\psi}\gamma^1(\partial_1 + igA_1)\psi + m\bar{\psi}e^{i\theta\gamma^5}\psi + \frac{1}{2}\Pi^2\right]$$

Physical states are constrained by Gauss law:

$$0 = -\partial_1\Pi - g\bar{\psi}\gamma^0\psi$$

Sign problem in path integral formalism

In Minkowski space,

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \right] + \frac{g\theta}{4\pi} \int F \in \boldsymbol{R}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \mathcal{O}\ e^{iS}}{\int DAD\psi D\bar{\psi}\ e^{iS}}$$

highly oscillating

In Euclidean space,

$$S = \int d^4x \left[-\frac{1}{4} F_{\mu\nu}^2 + \bar{\psi}\left(i\gamma^\mu D_\mu - m\right)\psi \right] + i\,\frac{g\theta}{4\pi} \int F \in \boldsymbol{C}$$

$$\langle \mathcal{O} \rangle = \frac{\int DAD\psi D\bar{\psi}\ \mathcal{O}\ e^{-S}}{\int DAD\psi D\bar{\psi}\ e^{-S}}$$

highly oscillating when θ isn't small

Accessible region by analytic computation

- Massive limit:

The fermion can be integrated out

&

the theory becomes effectively pure Maxwell theory w/ θ

Accessible region by analytic computation

- Massive limit:

The fermion can be integrated out

&

the theory becomes effectively pure Maxwell theory w/ θ

- Bosonization:

[Coleman '76]

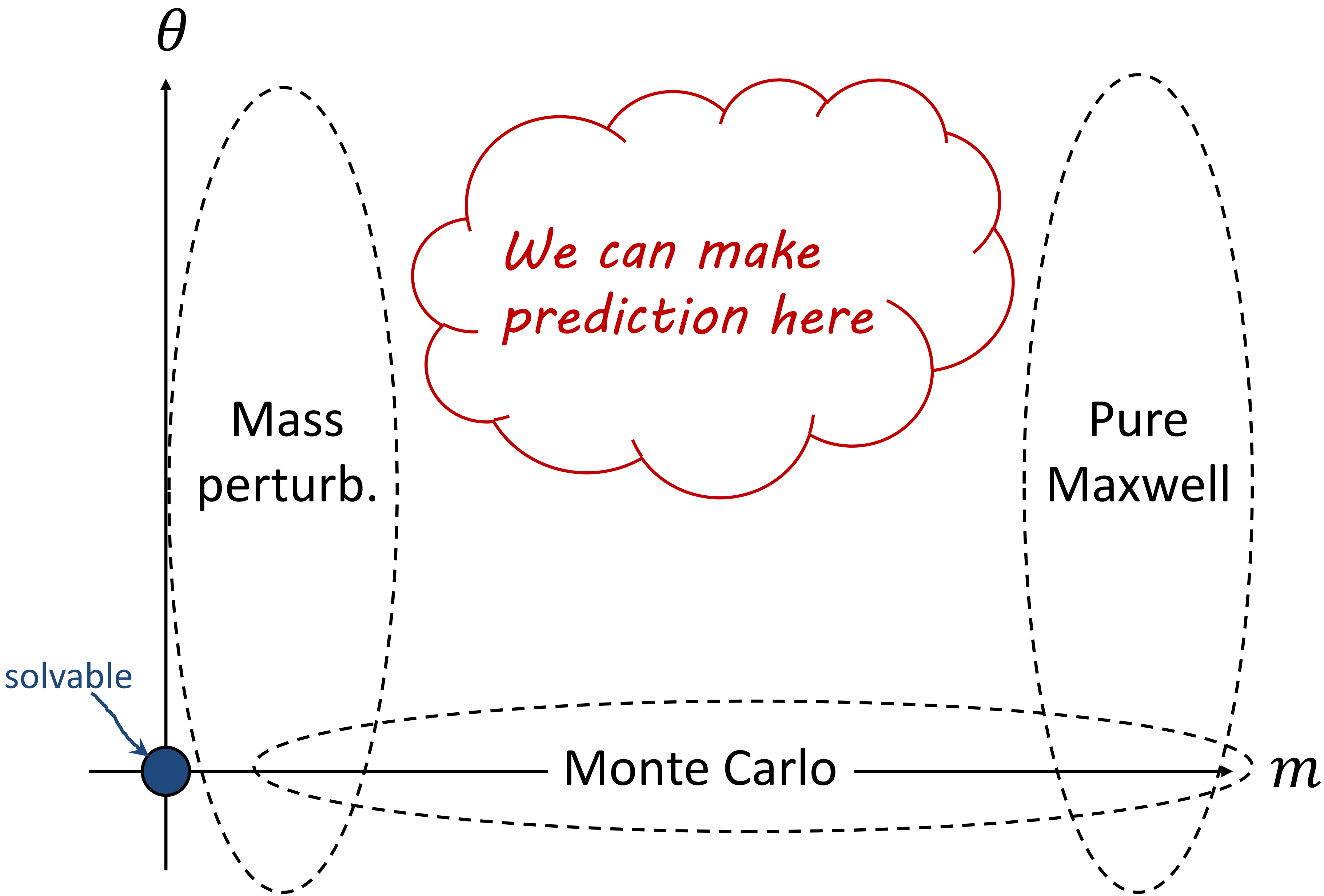
$$\mathcal{L} = \frac{1}{8\pi}\left(\partial_\mu \phi\right)^2 - \frac{g^2}{8\pi^2}\phi^2 + \frac{e^\gamma g}{2\pi^{3/2}} m \cos(\phi + \theta)$$

exactly solvable for $m = 0$

&

small m regime is approximated by perturbation

Map of accessibility/difficulty



Put the theory on lattice

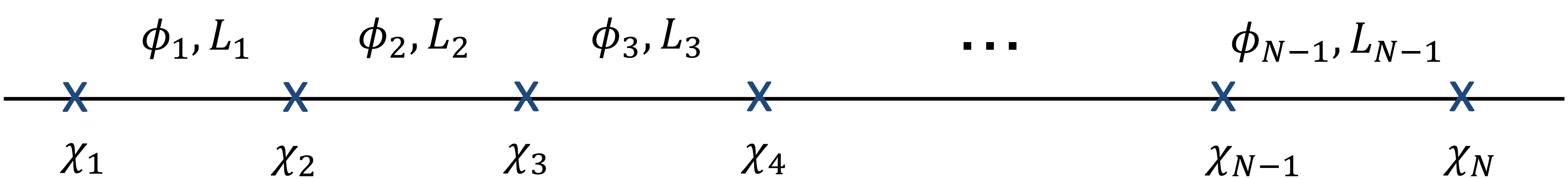
- **Fermion** (on site):

"Staggered fermion" [Susskind, Kogut-Susskind '75]

$$\frac{\chi_n}{\underbrace{a^{1/2}}_{\text{lattice spacing}}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix} \begin{array}{l} \rightarrow \text{odd site} \\ \rightarrow \text{even site} \end{array}$$

- **Gauge field** (on link):

$$\phi_n \leftrightarrow -agA^1(x), \qquad L_n \leftrightarrow -\frac{\Pi(x)}{g}$$



Lattice theory w/ staggered fermion

Hamiltonian:

$$\hat{H} = -i\sum_{n=1}^{N-1}\left(w - (-1)^n \frac{m}{2}\sin\theta\right)\left[\chi_n^\dagger e^{i\phi_n}\chi_n - \text{h.c.}\right]$$
$$+ m\cos\theta \sum_{n=1}^{N}(-1)^n \chi_n^\dagger \chi_n + J\sum_{n=1}^{N-1} L_n^2 \qquad \left(w = \frac{1}{2a}, J = \frac{g^2 a}{2}\right)$$

Commutation relation:

$$\{\chi_n^\dagger, \chi_m\} = \delta_{mn}, \;\; \{\chi_n, \chi_m\} = 0, \;\; [\phi_n, L_m] = i\delta_{mn}$$

Gauss law:

$$L_n - L_{n-1} = \chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}$$

Eliminate gauge d.o.f.

1. Take open b.c. & solve Gauss law:

$$L_n = \sum_{\ell=1}^{n-1}\left[\chi_\ell^\dagger \chi_\ell - \frac{1-(-1)^\ell}{2}\right] \qquad \text{(took } L_0 = 0\text{)}$$

2. Redefine fermion to absorb ϕ_n:

$$\chi_n \to \prod_{\ell<n}\left[e^{-i\phi_\ell}\right] \chi_n$$

Then,

$$\hat{H} = -i\sum_{n=1}^{N-1}\left(w - (-1)^n \frac{m}{2}\sin\theta\right)\left[\chi_n^\dagger \chi_{n+1} - \text{h.c.}\right] + m\cos\theta \sum_{n=1}^{N}(-1)^n \chi_n^\dagger \chi_n$$
$$+J\sum_{n=1}^{N-1}\left[\sum_{\ell=1}^{n-1}\left(\chi_\ell^\dagger \chi_\ell - \frac{1-(-1)^\ell}{2}\right)\right]^2$$

This acts on finite dimensional Hilbert space

Going to spin system

$$\{\chi_n^\dagger, \chi_m\} = \delta_{mn}, \ \ \{\chi_n, \chi_m\} = 0$$

This is satisfied by the operator:

"Jordan-Wigner transformation"

[Jordan-Wigner'28]

$$\chi_n = \frac{X_n - \mathrm{i}Y_n}{2}\left(\prod_{i=1}^{n-1} -\mathrm{i}Z_i\right)$$

Now the system is purely a spin system:

$$\widehat{H} = H_{ZZ} + H_{\pm} + H_Z$$

$$\begin{cases} H_{ZZ} = \dfrac{J}{2}\displaystyle\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell, \\ H_{\pm} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right)\left[X_n X_{n+1} + Y_n Y_{n+1}\right], \\ H_Z = \dfrac{m\cos\theta}{2}\displaystyle\sum_{n=1}^{N}(-1)^n Z_n - \frac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell=1}^{n} Z_\ell \end{cases}$$

Qubit description of the Schwinger model !!

Comments on choices of setup

There were many choices of setup to come here…

- Formulation of continuum theory?
- Type of lattice fermion?
- Boundary condition?
- Impose Gauss law?
- How to map fermion to spin system?
- Even N or odd N?

Choice of continuum theory

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{g\theta}{4\pi}\epsilon_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}\psi$$

(used for the case w/ probes)

$\Updownarrow$ "chiral anomaly" [cf. Fujikawa'79]

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\gamma^{\mu}(\partial_{\mu} + igA_{\mu})\psi - m\bar{\psi}e^{i\theta\gamma^5}\psi$$

(used for the case w/o probes)

- Equivalent for continuum theory w/o bdy.

—— (generically) inequivalent for theory on lattice or w/ bdy.

- The latter doesn't violate θ-periodicity even for open b.c.

Choice of boundary conditions

Gauss law: $L_n - L_{n-1} = q\left[\chi_n^\dagger \chi_n - \frac{1-(-1)^n}{2}\right]$

Open b.c.

- $L_n =$ (fermion op.)

$\longrightarrow \quad \dim(\mathcal{H}_{\text{phys}}) < \infty$

- θ-periodicity is lost
- momentum not conserved

Periodic b.c.

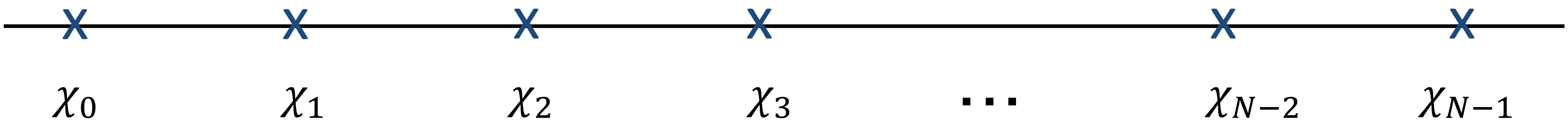
- one of L_n's remains

$\longrightarrow \quad \dim(\mathcal{H}_{\text{phys}}) = \infty$

additional truncation needed

- $^\exists\theta$-periodicity
- momentum conserved

Even N or odd N?



Staggered fermion: $\dfrac{\chi_n}{a^{1/2}} \longleftrightarrow \psi(x) = \begin{pmatrix} \psi_u \\ \psi_d \end{pmatrix}$ $\begin{array}{l} \longrightarrow \text{odd site} \\ \longrightarrow \text{even site} \end{array}$

- Usually even N is taken (p.b.c. allows only even N)
- Open b.c. allows both but parity is different: $\chi_n \to i(-1)^n \chi_{N-n-1}$

	n mod 2	$\bar{\psi}\psi \sim \sum_n (-1)^n \chi_n^\dagger \chi_n$	$\bar{\psi}\gamma^5\psi \sim \sum_n (-1)^n (\chi_n^\dagger \chi_{n+1} - \mathrm{h.c.})$
even N	changes	flipped	invariant
odd N	invariant	invariant	flipped

Odd N seems more like the continuum theory?

Plan of lecture 3

0. Conventional numerical approach to QFT

1. QFT as qubits (mapping to spin system)

2. Schwinger model as qubits

3. Time evolution operator

4. Simulation of Schwinger model

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

5. Summary

Time evolution operator

Suzuki-Trotter decomposition:

$$e^{-i\widehat{H}t} = \left(e^{-i\widehat{H}\frac{t}{M}}\right)^M$$

(M: large positive integer)

$$\simeq \left(e^{-iH_Z\frac{t}{M}}e^{-iH_{ZZ}\frac{t}{M}}e^{-iH_{XX}\frac{t}{M}}e^{-iH_{YY}\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\begin{cases} H_Z = \dfrac{m\cos\theta}{2}\displaystyle\sum_{n=1}^{N}(-1)^n Z_n - \dfrac{J}{2}\sum_{n=1}^{N-1}(n \bmod 2)\sum_{\ell=1}^{n} Z_\ell \\ H_{ZZ} = \dfrac{J}{2}\displaystyle\sum_{n=2}^{N-1}\sum_{1\le k<\ell\le n} Z_k Z_\ell, \\ H_{XX} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right) X_n X_{n+1} \\ H_{YY} = \dfrac{1}{2}\displaystyle\sum_{n=1}^{N-1}\left(w - (-1)^n\frac{m}{2}\sin\theta\right) Y_n Y_{n+1} \end{cases}$$

Can we express it in terms of elementary gates?

Time evolution operator (cont'd)

$$e^{-i\hat{H}t} \simeq \left(e^{-iH_Z \frac{t}{M}} e^{-iH_{ZZ} \frac{t}{M}} e^{-iH_{XX} \frac{t}{M}} e^{-iH_{YY} \frac{t}{M}} \right)^M$$

The 1st one is trivial:

$$e^{-icZ} = R_Z(2c)$$

The 2nd one appeared in Ising model:

$$e^{-icZ_1Z_2} = CXR_Z^{(2)}(2c)CX$$

The 3rd one (see next slide):

$$e^{-icX_1X_2} = CXR_X^{(1)}(2c)CX$$

The 4th one:

$$e^{-icY_1Y_2} = R_Z^{(1)}\left(-\frac{\pi}{2}\right) R_Z^{(2)}\left(-\frac{\pi}{2}\right) e^{-icX_1X_2} R_Z^{(2)}\left(\frac{\pi}{2}\right) R_Z^{(1)}\left(\frac{\pi}{2}\right)$$

Time evolution operator (Cont'd)

$$e^{-icX_1X_2} = CXR_X^{(1)}(2c)CX$$

Proof:

$$CXR_X^{(1)}(2c)CX|0\rangle \otimes |\psi\rangle$$

$$= CXR_X^{(1)}(2c)|0\rangle\otimes|\psi\rangle = CX\left[\cos c|0\rangle\otimes|\psi\rangle - i\sin c|1\rangle\otimes|\psi\rangle\right]$$

$$= \cos c|0\rangle\otimes|\psi\rangle - i\sin c|1\rangle\otimes X|\psi\rangle = \cos c|0\rangle\otimes|\psi\rangle - i\sin c\; X|0\rangle\otimes X|\psi\rangle$$

$$CXR_X^{(1)}(2c)CX|1\rangle \otimes |\psi\rangle$$

$$= CXR_X^{(1)}(2c)|1\rangle\otimes X|\psi\rangle = CX\left[\cos c|1\rangle\otimes X|\psi\rangle - i\sin c|0\rangle\otimes X|\psi\rangle\right]$$

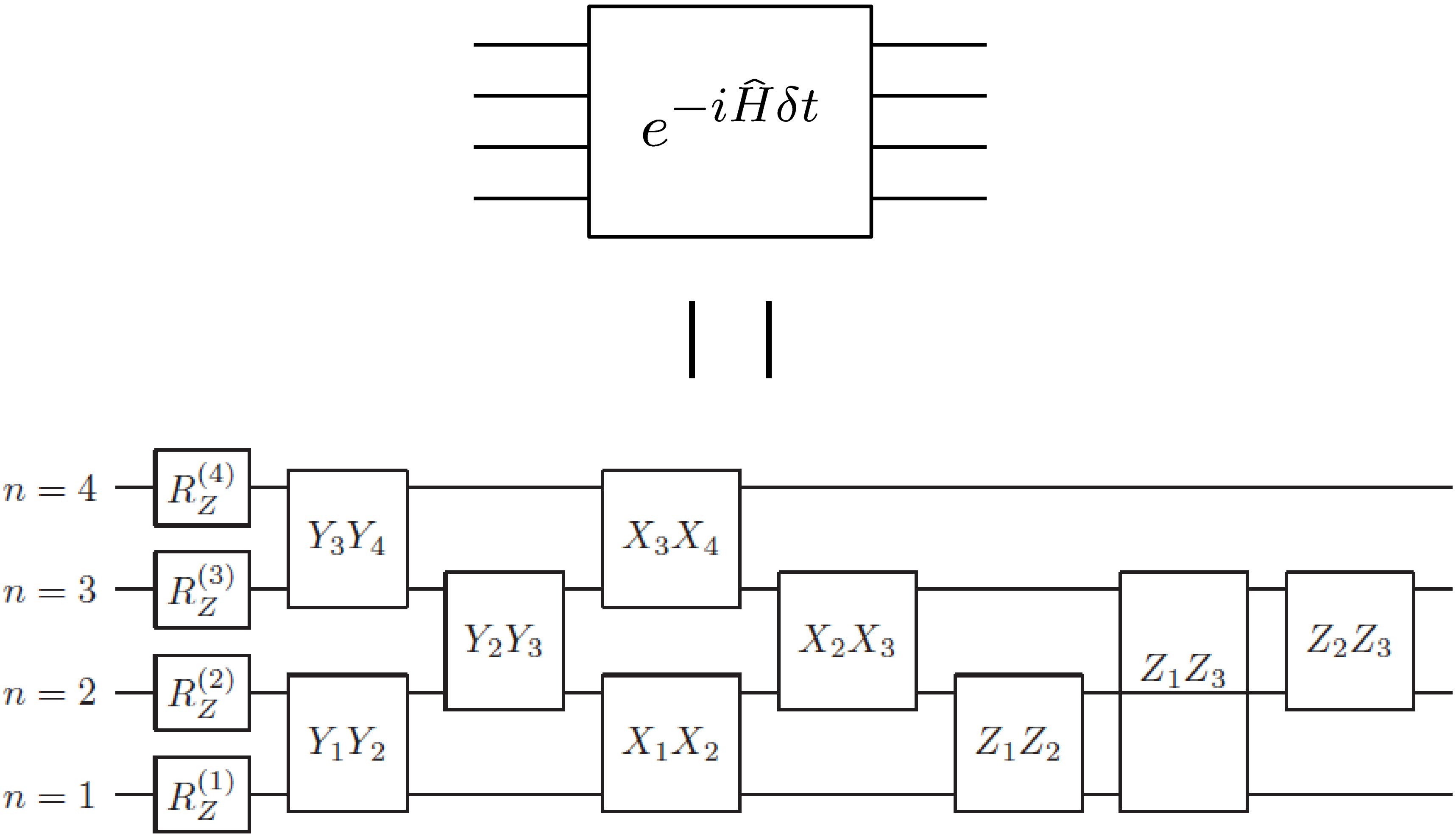
$$= \cos c|1\rangle\otimes|\psi\rangle - i\sin c|0\rangle\otimes X|\psi\rangle = \cos c|1\rangle\otimes|\psi\rangle - i\sin c\; X|1\rangle\otimes X|\psi\rangle$$

Thus,

$$CXR_X^{(1)}(2c)CX|\varphi\rangle\otimes|\psi\rangle = \cos c|\varphi\rangle\otimes|\psi\rangle - i\sin c\; X|\varphi\rangle\otimes X|\psi\rangle$$

$$= e^{-icX_1X_2}|\varphi\rangle \otimes |\psi\rangle$$

Quantum circuit for time evolution op. (N=4)



Improvement of Suzuki-Trotter decomposition

The leading order decomposition:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\delta t}e^{-iH_2\delta t} + \mathcal{O}(\delta t^2)$$

The 2nd order improvement:

$$e^{-i(H_1+H_2)\delta t} = e^{-iH_1\frac{\delta t}{2}}e^{-iH_2\delta t}e^{-iH_1\frac{\delta t}{2}} + \mathcal{O}(\delta t^3)$$

cf. Baker-Campbell-Hausdorff formula:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]+\cdots}$$

This increases the number of gates at each time step
but we can take larger δt (smaller M) to achieve similar accuracy.
Totally we save the number of gates.

Survival probability of massive vacuum

[cf. Martinez etal. **Nature** 534 (2016) 516-519]

The ground state in the large mass limit is

$$(\text{mass term}) \propto m \sum_{n=1}^{N} (-1)^n Z_n$$

$$|\text{massive}\rangle = |0101\cdots01\rangle$$

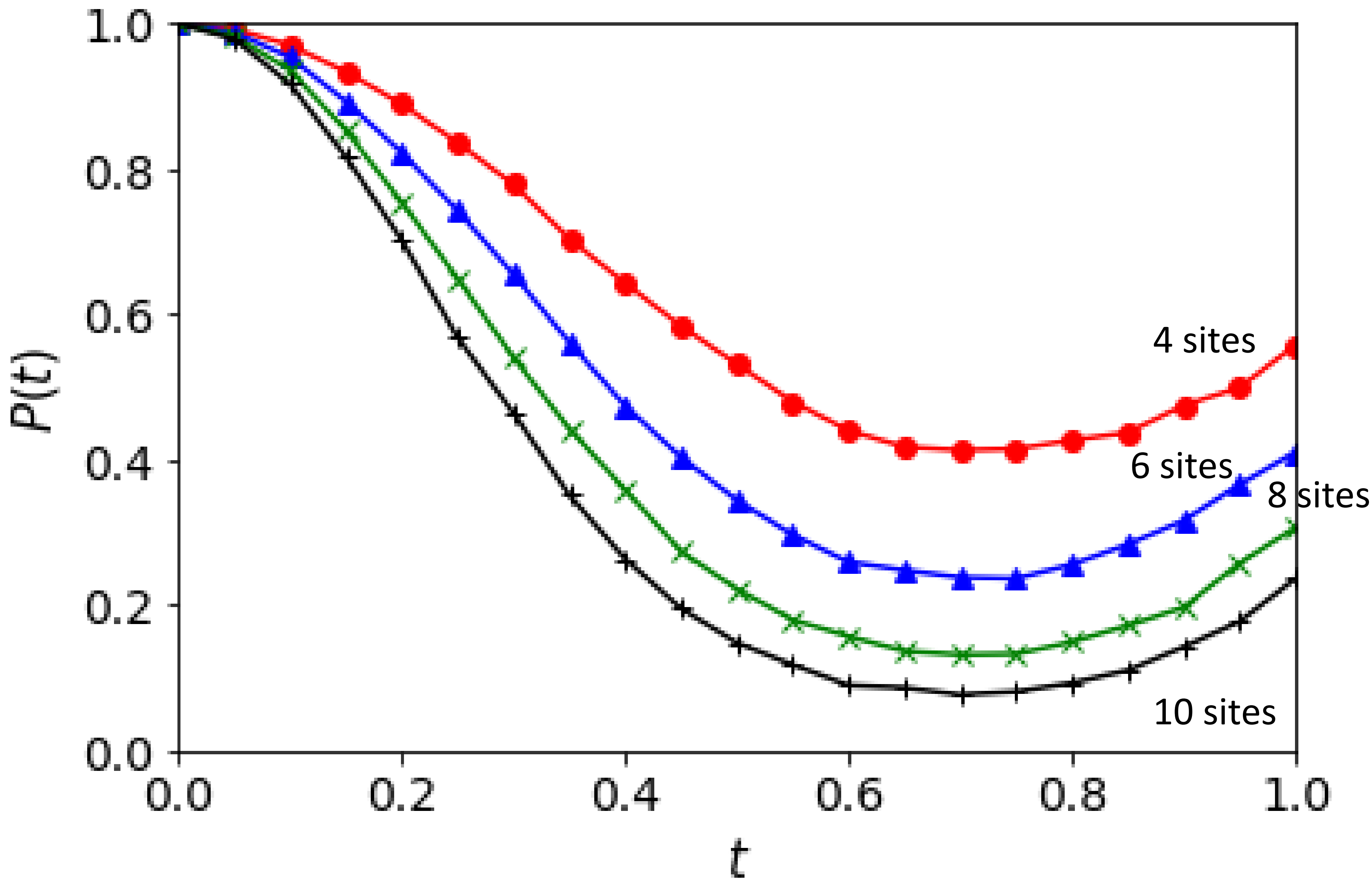
Survival probability:

$$P(t) = \left| \langle\text{massive}|\ e^{-i\hat{H}t}\ |\text{massive}\rangle \right|^2$$

$$= \left| \langle 00\cdots0|\ X_N \cdots X_4 X_2 e^{-i\hat{H}t} X_2 X_4 \cdots X_N\ |00\cdots0\rangle \right|^2$$

Result of simulator (10000 shots)

$J = 1,\ w = 1,\ m = 1,\ \theta = 0,\ \delta t = 0.01,$ 100 time steps



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VEV of mass operator (chiral condensation)

$$\langle\bar{\psi}(x)\psi(x)\rangle = \langle \text{vac}|\bar{\psi}(x)\psi(x)|\text{vac}\rangle$$

Instead of the local op., we analyze the average over the space:

$$\frac{1}{2Na}\langle\text{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\text{vac}\rangle$$

Once we get the vacuum, we can compute the VEV as

$$\begin{aligned}\frac{1}{2Na}\langle\text{vac}|\sum_{n=1}^{N}(-1)^n Z_n|\text{vac}\rangle &= \frac{1}{2Na}\sum_{n=1}^{N}(-1)^n \sum_{i_1\cdots i_N=0,1}\langle\text{vac}|Z_n|i_1\cdots i_N\rangle\langle i_1\cdots i_N|\text{vac}\rangle \\ &= \frac{1}{2Na}\sum_{n=1}^{N}\sum_{i_1\cdots i_N=0,1}(-1)^{n+i_n}\,|\langle i_1\cdots i_N|\text{vac}\rangle|^2\end{aligned}$$

Adiabatic state preparation of vacuum (repeated)

Step 1: Choose an initial Hamiltonian H_0 of a simple system whose ground state $|\mathrm{vac}_0\rangle$ is known and unique

Step 2: Introduce adiabatic Hamiltonian $H_A(t)$ s.t.

$$\begin{cases} \cdot\ H_A(0) = H_0,\ H_A(T) = H_{\mathrm{target}} \\ \cdot\ \left|\frac{dH_A}{dt}\right| \ll 1 \text{ for } T \gg 1 \end{cases}$$

Step 3: Use the adiabatic theorem

If $H_A(t)$ has a unique ground state w/ a finite gap for $\forall t$, then the ground state of H_{target} is obtained by

$$\boxed{|\mathrm{vac}\rangle = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\, H_A(t)\right) |\mathrm{vac}_0\rangle}$$

Adiabatic state preparation in the Schwinger model

$$|\text{vac}> = \lim_{T\to\infty} \mathcal{T} \exp\left(-i\int_0^T dt\ H_A(t)\right)|\text{vac}_0>$$

$$\simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\text{vac}_0>$$

$$\left(U(t) = e^{-iH_A(t)\delta t}\right)$$

Here we choose

$$\begin{cases} H_0 = H_{ZZ} + H_Z|_{m\to m_0,\theta\to 0} \quad \Rightarrow \quad |\text{vac}_0\rangle = |0101\cdots 01\rangle \\ H_A(t) = \widehat{H}\Big|_{w\to w(t),\theta\to\theta(t),m\to m(t)} \\ w(t) = \frac{t}{T}w,\ \theta(t) = \frac{t}{T}\theta,\ m(t) = \left(1-\frac{t}{T}\right)m_0 + \frac{t}{T}m \end{cases}$$

m_0 can be any positive number in principle
but it is practically chosen to have small systematic error

Massless case

For massless case,

θ is absorbed by chiral rotation $\Rightarrow$ $\theta = 0$ w/o loss of generality

No sign problem

Nevertheless,

it's difficult in conventional approach because computation of fermion determinant becomes very costly

$\exists$ Exact result:

[Hetrick-Hosotani '88]

$$\langle \bar{\psi}(x)\psi(x) \rangle = -\frac{e^{\gamma}}{2\pi^{3/2}} g \simeq -0.160 g$$

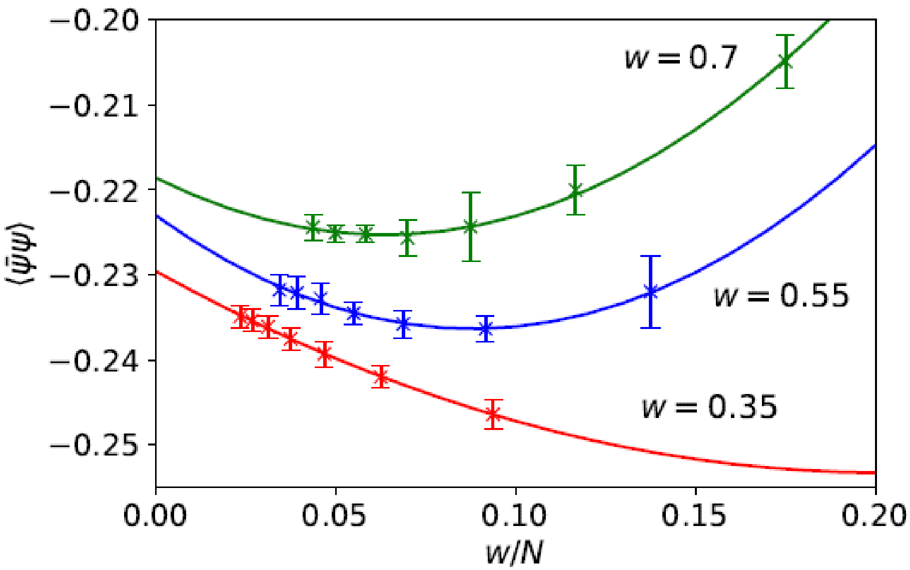
Can we reproduce it?

Thermodynamic & Continuum limit

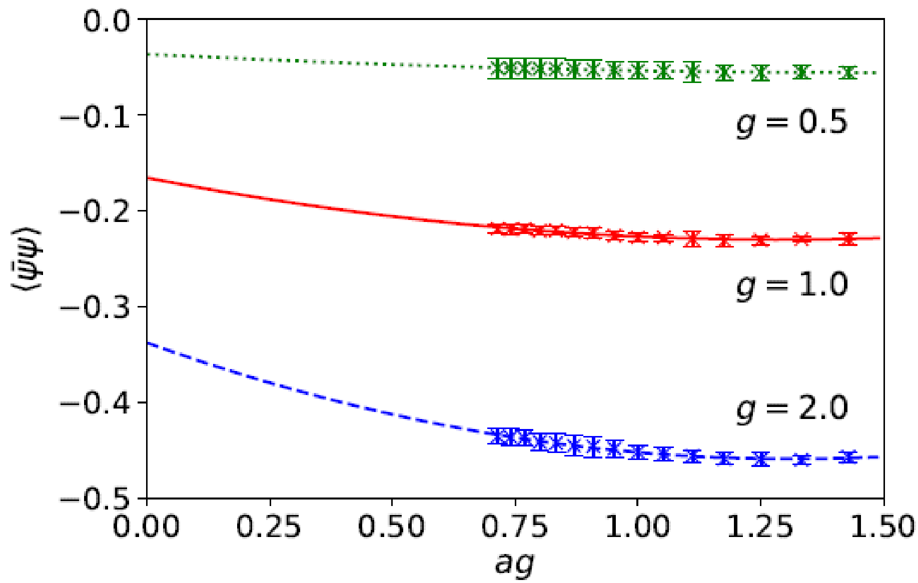
$g = 1, m = 0, N_{\text{max}} = 16, T = 100, \delta t = 0.1, 1M$ shots

#(measurements)

Thermodynamic limit: $(N \to \infty$, fixed $a)$



Continuum limit: $(a \to 0$ after $aN \to \infty)$

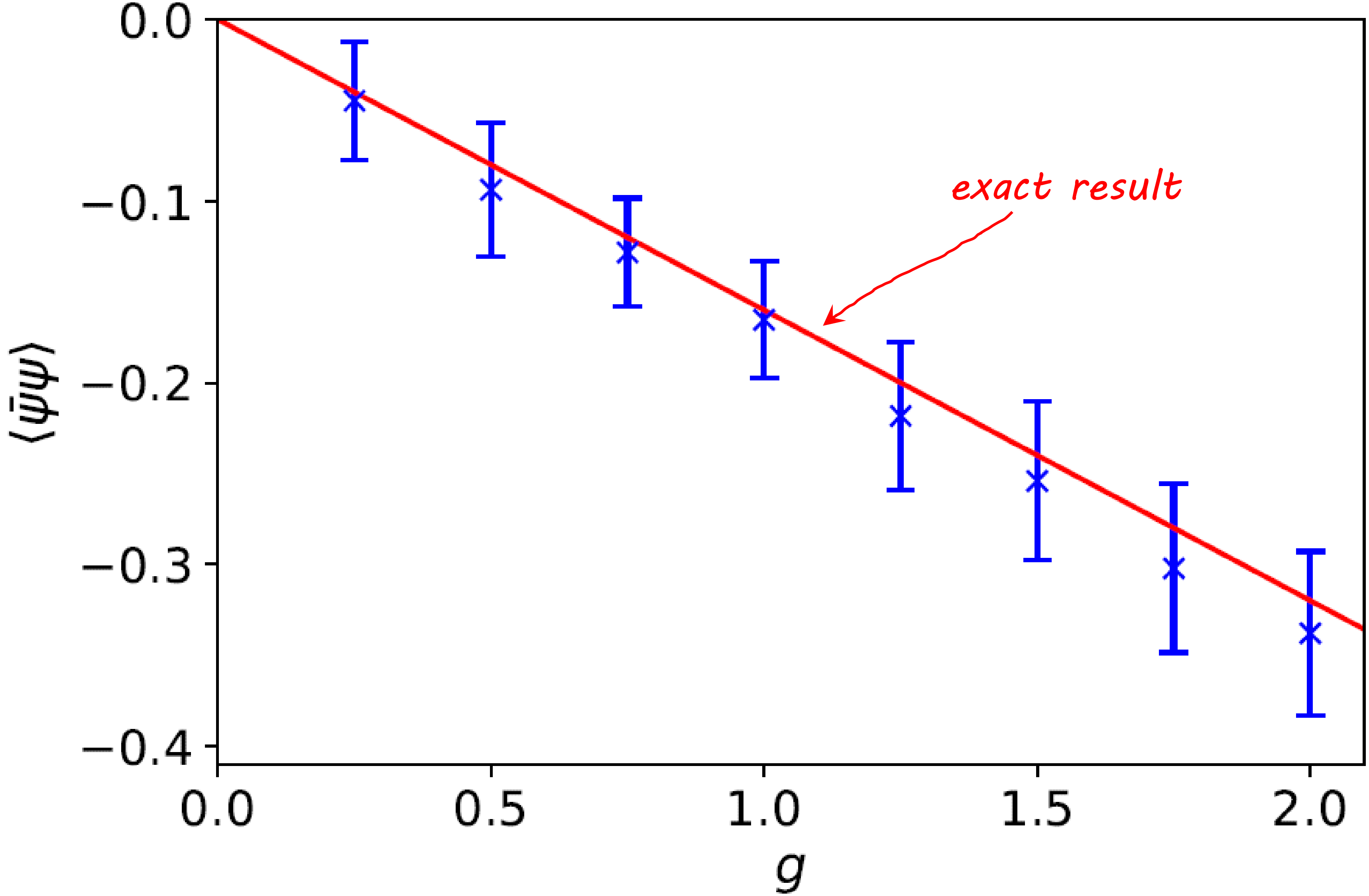


$$\left(w \coloneqq \frac{1}{2a}\right)$$

Result for massless case (after continuum limit)

$T = 100, \delta t = 0.1, N_{\max} = 16, 1M$ shots

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]



Massive case

Result of mass perturbation theory: [Adam '98]

$$\langle \bar{\psi}(x)\psi(x)\rangle \simeq -0.160g + 0.322m\cos\theta + \mathcal{O}(m^2)$$

However,

$\exists$ Subtlety in comparison: this quantity is UV divergent

$(\sim m \log \Lambda)$

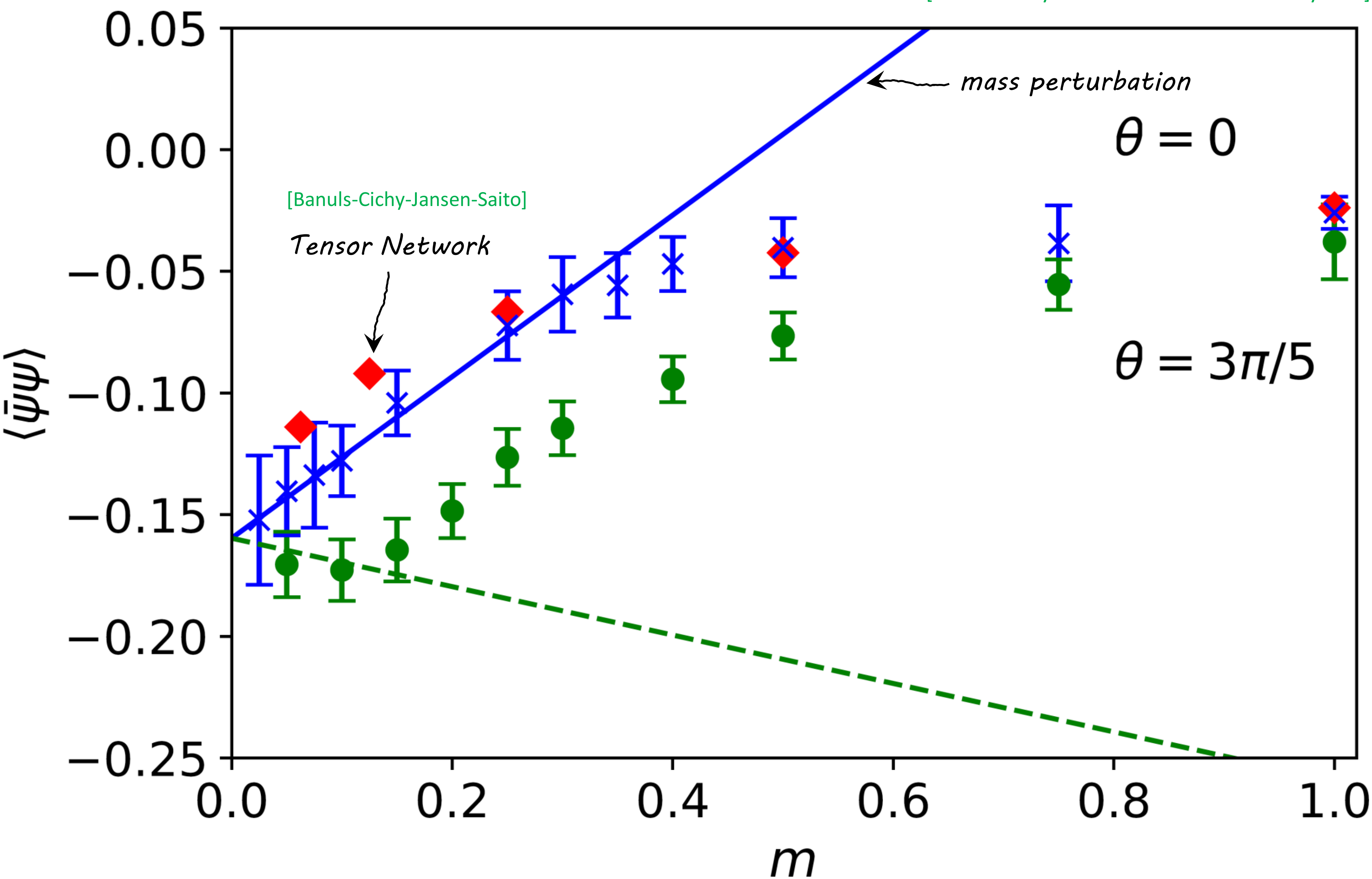
$\Rightarrow$ need a regularization!

Here we subtract free theory result before taking continuum limit:

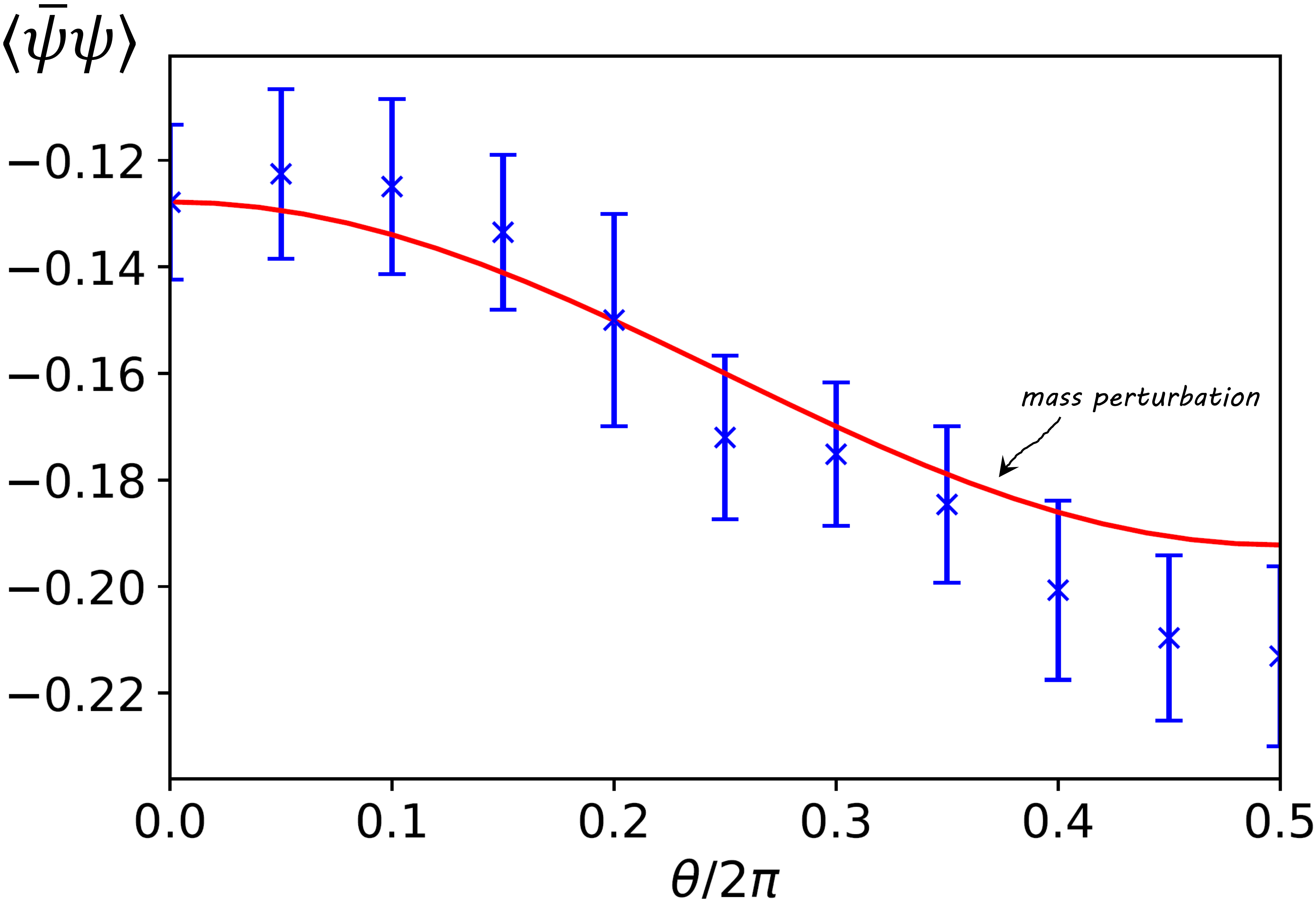
$$\lim_{a\to 0}\left[\langle\bar{\psi}\psi\rangle - \langle\bar{\psi}\psi\rangle_{\mathrm{free}}\right]$$

Chiral condens. for massive case at g=1

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]



θ dependence at $m = 0.1$ & $g = 1$



Summary

Summary

- Quantum computation is suitable for operator formalism which is free from sign problem
- QFT typically has ∞ dimensional Hilbert space and regularization is needed for simulation in operator formalism
- For QFT w/ physical bosonic d.o.f., extra truncation is needed even after putting it on lattice
- We've constructed the vacuum of Schwinger model w/ the topological term by adiabatic state preparation
- found agreement in the chiral condensate with the exact result for $m = 0$ & mass perturbation theory for small m

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya '20]

Here is the end of lecture 3!

Appendix

Estimation of systematic errors

Approximation of vacuum:

[Chakraborty-MH-Kikuchi-Izubuchi-Tomiya ’20]

$$|\mathrm{vac}> \simeq U(T)U(T-\delta t)\cdots U(2\delta t)U(\delta t)|\mathrm{vac}_0> \equiv |\mathrm{vac}_A\rangle$$

Approximation of VEV:

$$\langle\mathcal{O}\rangle \equiv \langle\mathrm{vac}|\mathcal{O}|\mathrm{vac}\rangle \simeq \langle\mathrm{vac}_A|\mathcal{O}|\mathrm{vac}_A\rangle$$

Introduce the quantity

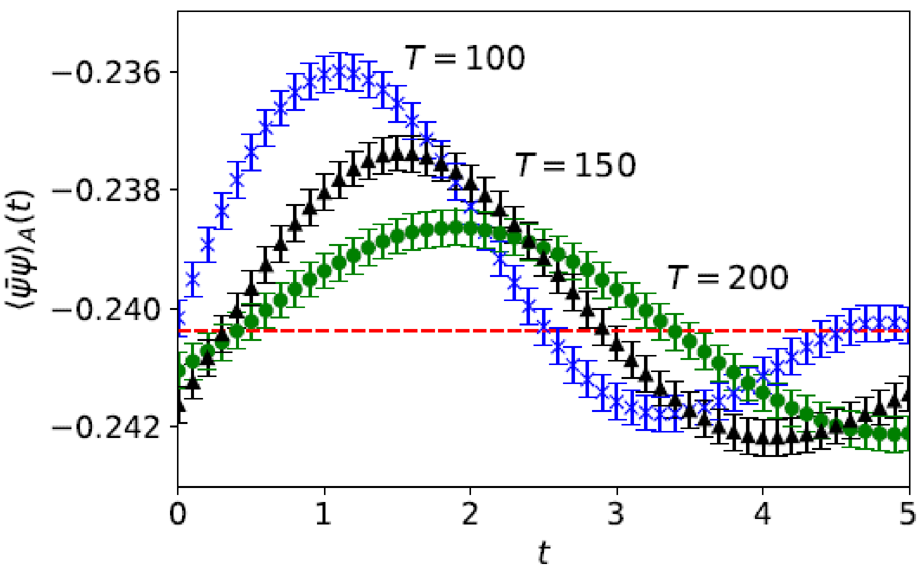
$$\langle\mathcal{O}\rangle_A(t) \equiv \langle\mathrm{vac}_A|e^{i\hat{H}t}\mathcal{O}e^{-i\hat{H}t}|\mathrm{vac}_A\rangle$$

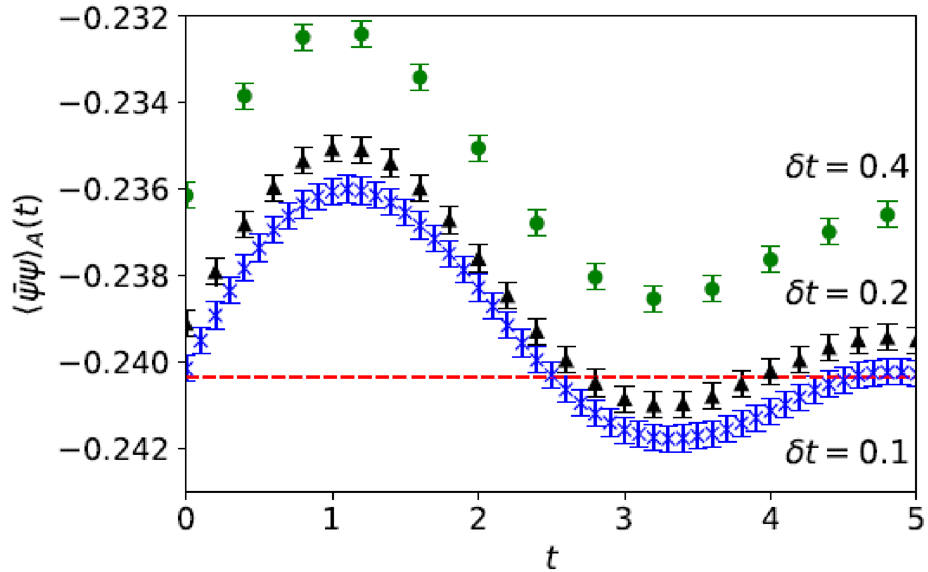
$$\begin{cases} \text{independent of t if } |\mathrm{vac}_A\rangle = |\mathrm{vac}\rangle \\ \text{dependent on t if } |\mathrm{vac}_A\rangle \neq |\mathrm{vac}\rangle \end{cases}$$

This quantity describes intrinsic ambiguities in prediction

➡ Useful to estimate systematic errors

Estimation of systematic errors (Cont'd)





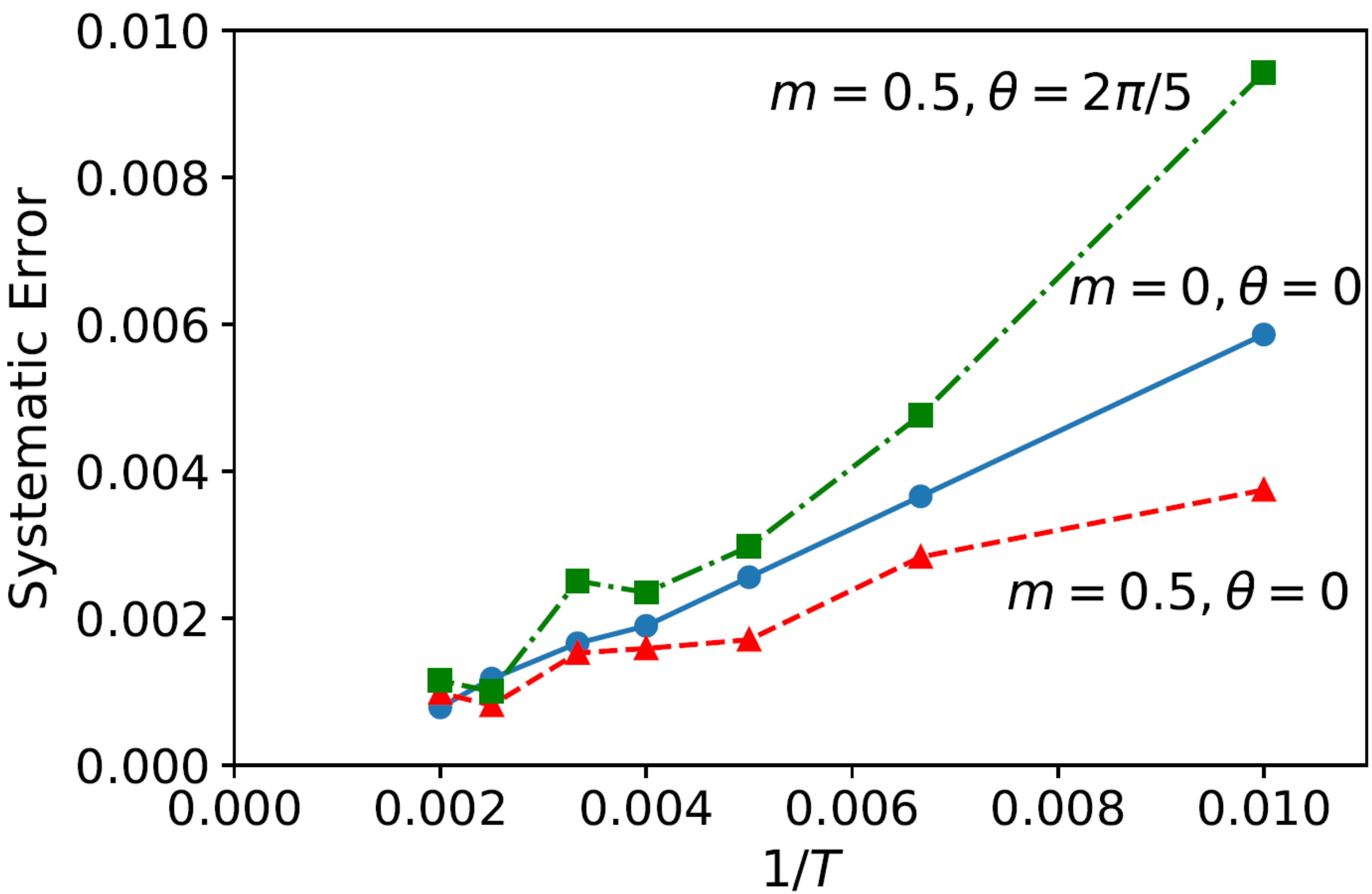
Oscillating around the correct value

Define central value & error as

$$\frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) + \min\langle\mathcal{O}\rangle_A(t)\right) \quad \& \quad \frac{1}{2}\left(\max\langle\mathcal{O}\rangle_A(t) - \min\langle\mathcal{O}\rangle_A(t)\right)$$

T-dependence of the systematic errors

Parameters: $g = 1, a = 1, N = 8, 10^6$ shots



Tradeoff of symmetries in Suzuki-Trotter dec.

Suzuki-Trotter decomposition:

(more precisely, we generically use its improvement)

$$(M \in \boldsymbol{Z}, M \gg 1)$$

$$e^{-iHt} = \left(e^{-iH\frac{t}{M}}\right)^M \simeq \left(e^{-iH_1\frac{t}{M}}e^{-iH_2\frac{t}{M}}\right)^M + \mathcal{O}(1/M)$$

$$\Rightarrow \quad H_{\mathrm{eff}} = \frac{1}{-it}\log\left(e^{-iH_1\frac{t}{M}}e^{-iH_2\frac{t}{M}}\right)^M$$

Symmetries may be broken by decomposition

Tradeoff:

- Parity friendly (& translation if p.b.c.)

$$H = H_{XX} + H_{YY} + H_{ZZ} + H_Z \qquad \cancel{U(1)}$$

- $U(1)$ friendly

$$H = H_{XX+YY}^{(\mathrm{even})} + H_{XX+YY}^{(\mathrm{odd})} + H_{ZZ} + H_Z \qquad \cancel{P}$$