GRAPHICAL SOLUTION OF DIFFICULT CROSSING PUZZLES

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1. Introduction. In this article we present a graphical method for solving "difficult crossing" puzzles such as the cannibals and missionaries puzzle or the puzzle of the jealous husbands. The method is extremely simple and makes the solution of many such puzzles easy and quick. It also makes the connection between the two aforementioned puzzles easily apparent.

The basic idea of the method is to regard the players in the melodrama as forming a "system" which can be in a number of "states." A representation of these states and the possible transitions between them is then given by a graph. That is, a graph in the sense of graph theory. See, for example, reference [3]. The idea of using a graph-theoretical approach to the analysis of games and puzzles is not new, of course—see [3]—and has in fact been applied specifically to difficult crossing puzzles by B. Schwartz, [4]. However, Schwartz placed his emphasis on solving the puzzles by matrix operations, and his paper does not as a matter of fact contain any graphs. Apparently the graphical solution to be presented here has not previously been published.

2. The state diagram. Let us first recall the statement of the cannibals and missionaries puzzle, as given by Schwartz.

"A group consisting of three cannibals and three missionaries seeks to cross a river. A boat is available which will hold up to two people. If the missionaries on either side of the river are outnumbered at any time by the cannibals on that side, even momentarily, the cannibals will do away with the unfortunate, outnumbered missionaries. What schedule of crossings can be devised to permit the entire party to cross safely?"

Following Schwartz, we let

$$m = \text{number of missionaries on the first bank,}$$

$$c = \text{number of cannibals on the first bank.}$$

Then the pair (c, m) denotes the *state* of the system at any time that the boat is not in midstream. It is not necessary also to give the number of missionaries and cannibals on the second bank since the total number of each must always be three (assuming that the number of missionaries does not suffer an unfortunate decline). Since $0 \leqq m \leqq 3$, $0 \leqq c \leqq 3$, there are sixteen possible pairs, but some of these must be excluded. For example, $m=1$, $c=3$ is excluded. So is $m=2$, $c=1$, since this corresponds to one missionary and two cannibals on the far bank of the river. In fact, the allowable states (c, m) must satisfy these restrictions:

$$\text{(a)}\ 0 \leqq c \leqq 3$$

$$\text{(b)}\ 0 \leqq m \leqq 3$$

$$\text{(c)}\ c = m \quad \text{or} \quad m = 0 \quad \text{or} \quad m = 3.$$

We now graph the allowable states (c, m) on a rectangular coordinate system, as in Fig. 1. There are ten states, lying in the shape of a letter Z. We call this figure the *state diagram* of the system.

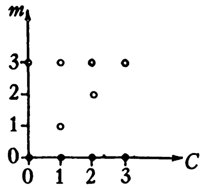

FIG. 1.

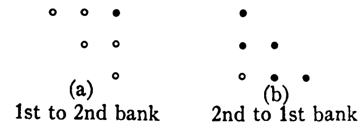

FIG. 2. Reachable points

3. Transitions between states. Transitions between states can be shown geometrically as directed lines or arcs joining the points of the state diagram. In this way, a *directed graph* is obtained, the points or *vertices* corresponding to states and the *edges* corresponding to crossings of the river. Which vertices will thus be joined by edges? We observe that a passage of the boat from the first to the second bank reduces the values of c and m since it removes people from the first bank. The total decrease in c and m is either one or two since the capacity of the boat is two. Thus the only points of the graph which are reachable from a given point on such a passage lie in a right triangle with the given point at the upper right vertex, as in Fig. 2a. Of course, some of these reachable points may not be allowable. Similarly passage of the boat from the second to the first bank of the river corresponds to increasing c or m. The points reachable from a given point in such a transition are shown in Fig. 2b.

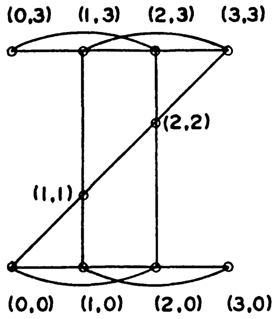

FIG. 3.

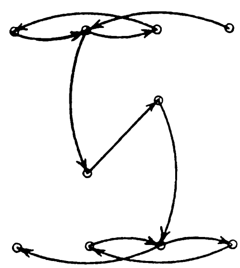

FIG. 4.

Keeping these rules in mind, we can construct the graph of the puzzle, shown in Fig. 3. Each edge can be traversed in either direction, corresponding to the fact that all ferryings are reversible (by turning the boat around and returning its passengers to their starting point). It should be kept in mind that passages from the first to the second bank must alternate with passages from the second to the first bank. This means that on the graph edges leading down or to the left (to reachable points as in Fig. 2a) must alternate with edges leading up or to the right (to reachable points as in Fig. 2b). Thus, (3, 3) to (2, 2) to (1, 1) to (0,0) is not a permissible solution of the puzzle.

A method for finding a solution to the puzzle can now be described in the following terms. Starting at the upper right corner of the graph, make a sequence of transitions to allowable points in the reachable triangles, moving alternately down or to the left and up or to the right, until the lower left corner is reached. After a little experimentation, the solution shown in Fig. 4 is obtained.

4. Existence, uniqueness, minimization. It is natural to ask whether the solution shown in Fig. 4 is the only solution to the puzzle. More generally, if one starts with some other numbers of cannibals and missionaries, is the puzzle solvable, and if so which solution (if there are several) requires the fewest crossings of the river?

The graphical method is of assistance in answering some of these questions. Let us classify the allowable states into three kinds:

(T) those along the top ($m=3$),

(B) those along the bottom ($m=0$),

(D) those along the diagonal ($0<m=c<3$).

If the boat holds only two people, it is impossible to go directly from a state of type T to one of type B. Hence any solution path must include a type D state. Moreover, if we move from (2, 3) or (3, 3) to (2, 2), then at the next step we must go to (2, 3) or (3, 3), and nothing is achieved. Therefore when the solution path leaves the T states, it must go to (1, 1). The only way to do this is from (1, 3). The only way to reach B states is then to go to (2, 2) and (2, 0). Thus, the solution path must contain the sequence

$$\cdots \to (1, 3) \to (1, 1) \to (2, 2) \to (2, 0) \to \cdots$$

Finally, (1, 3) can be reached from (3, 3) by

$$(3, 3) \to (1, 3) \to (2, 3) \to (0, 3) \to (1, 3)$$

or

$$(3, 3) \to (2, 2) \to (2, 3) \to (0, 3) \to (1, 3)$$

and similarly (0, 0) can be reached from (2,0) in two ways. Hence there are exactly four possible distinct solutions of the puzzle. All four require eleven crossings of the river (five and a half round trips).

In the same way, reasoning from the graphical representation makes it easy to answer questions of the existence and uniqueness of solutions and choice of the solution requiring a minimal number of crossings for certain more general puzzles. For example, four cannibals and four missionaries cannot be taken safely across a river with a boat holding only two people. To see this, we refer to the state diagram in Fig. 5. In order to leave the T states without immediately returning, one must go from (2, 4) to (2, 2). The next step must be to (3, 3), or back to (2, 4). From (3, 3), there is no way to get to the bottom states, nor to (1, 1), and the puzzle has no solution.

If the boat will hold three people instead of two, the reachable states are as indicated in Fig. 6. (We require that the boat not contain more cannibals than missionaries.) It is easy to see that up to five cannibals and five missionaries can now safely cross the river, but not six cannibals and six missionaries.

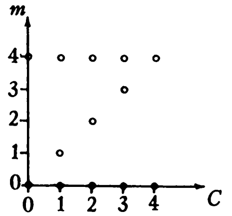

FIG. 5.

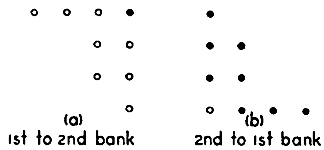

FIG. 6.

On the other hand, if the boat will hold four or more people, then any number of missionaries and cannibals can be transported across the river, for it is then possible to move down along the diagonal of the state diagram, as shown in Fig. 7 for the case of six cannibals, six missionaries, and a boat holding four. The diagonal solution is not always the one requiring the fewest crossings, however. For example, if there are six missionaries and six cannibals and the boat holds five, the solution which sticks to the diagonal is still the one in Fig. 7, but the solution in Fig. 8 requires only seven crossings. However, one can show that if the boat holds an even number, B, of people, and $4 \leqq B < M$ where M is the number of missionaries (or cannibals), then no path can reach (0, 0) in

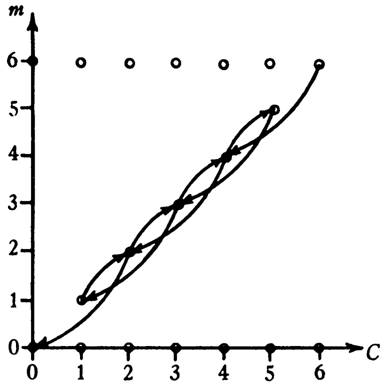

FIG. 7.

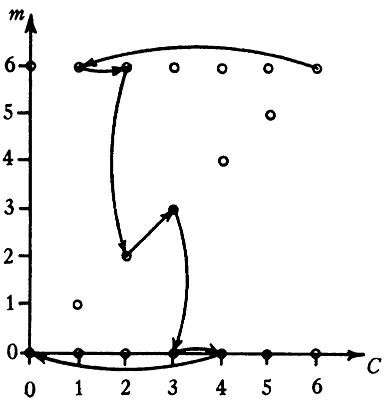

FIG. 8.

fewer steps than the one down the diagonal. To see this, we first note that the path must pass through a diagonal state and the best way to leave the top states (other than to follow the diagonal) is to send B cannibals across on the first step, return one cannibal, and then send $B-1$ missionaries, as shown in Fig. 9a. Thus a net of $B-1$ missionaries and $B-1$ cannibals is transported in three crossings. On the other hand, the same result can be achieved by following the diagonal as in Fig. 9b. Similarly one can show that no advantage can be gained by leaving the diagonal to go to the bottom states when this becomes possible. For if this is done, at most $2B-1$ persons can be transported in 3 steps versus $2B-2$ for the diagonal route, and since it is necessary to transport an even number of persons the difference cannot make it possible to reach

(0, 0) in fewer steps (see Fig. 10, where two possible cases are depicted). We leave it to the reader to analyze the case when B is odd.

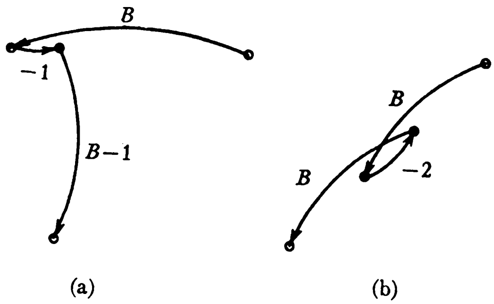

FIG. 9. (Numbers beside arcs indicate numbers of persons transported from 1st to 2nd bank.)

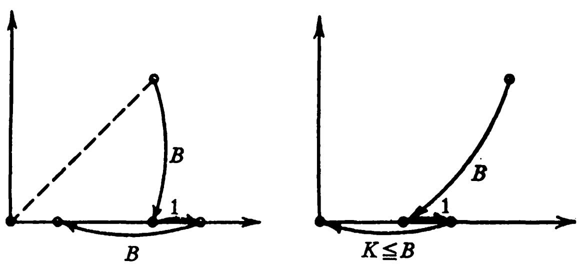

FIG. 10.

5. More general puzzles. The graphical technique makes it easy to analyze some simple variants of the basic puzzle. For example, it may be specified that the boat not only has a maximum capacity but also a minimum capacity (greater than one). In fact, arbitrary constraints can be imposed, for this simply means a change in the set of allowable states to be shown on the state diagram. This set of allowable states can even be a function of the number of crossings, the last state, or the sequence of states which has been used. In complicated cases of this sort, it may be necessary to use a computer to carry out the solution.

If the number of types of individuals is increased, the puzzle becomes more complicated. For example, Schwartz uses his analytic method to discuss a puzzle in which there are M missionaries, all of whom can row, R cannibals who can row, and C cannibals who cannot row. The state of the system is then described by an ordered triple of numbers (c, r, m). The graph of this puzzle can be shown on the plane in a convenient way by placing all points with the same value of m on a horizontal line. The complete graph for the case $M=3$, $R=1$, $C=2$ is shown in Fig. 11; from it the solution $(2, 1, 3)\to(1, 0, 3)\to(1, 1, 3)\to(0, 0, 3)\to(0, 1, 3)\to(0, 1, 1)\to(1, 1, 2)\to(1, 0, 1)\to(2, 0, 2)\to(2, 0, 0)\to(2, 1, 0)\to(1, 0, 0)\to(1, 1, 0)\to(0, 0, 0)$ can be obtained. In more general situations, with many classes of individuals, it may be necessary to use computer solutions based on the analytical method of Schwartz or the dynamic programming method of Bellman, [1].

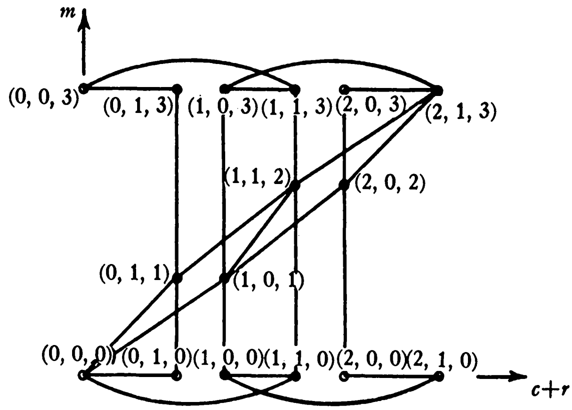

FIG. 11. State (c, r, m)

6. Puzzle of the jealous husbands. We shall close by demonstrating the relation of the puzzle of the jealous husbands to the preceding discussion. This puzzle may be stated as follows.

"Three jealous husbands and their wives must cross a river in a boat that holds only two persons. How can this be done so that a wife is never left in the company of either or both of the other women's husbands unless her own husband is present?"

This puzzle is an ancient one whose origin is apparently unknown. The puzzle was known to the famous Italian mathematician Tartaglia (born 1510, died 1557), who gave an incorrect solution of the puzzle with four families. (See Lucas, [2], for additional historical information.) The cannibals and missionaries puzzle is a later version.

In order to obtain a graphical interpretation of this new puzzle, we shall define the states to be ordered pairs (w, h), where h is the number of husbands and w is the number of wives on the first bank of the river. At first sight, this seems to be an unsatisfactory definition because seemingly different situations will be put together into a single state. For example, if we refer to the three husbands as A, B, and C and their respective wives as a, b, and c, then the situation in which the persons on the first bank are A and a and those situations in which they are A and b or A and c all have $h=w=1$. Thus, all are classed together as the state (1, 1), although the latter two are not allowable. However, since we intend to show only allowable states on our state diagram this is not really an objection. It will still be true that the state (1, 1) can be realized by having A and a, B and b, or C and c on the left bank, but we have no need to distinguish among these cases.

Now that the states of the system have been defined, we can draw the state diagram. The result is exactly the same as Fig. 1, except that the labels on the axes must be changed from c, m, to w, h. It is to be understood that the diagonal states are allowable only if the husbands and wives are matched. Thus (2, 2) can be achieved by A, B, a, b or A, C, a, c or B, C, b, c but not by A, B, a, c, etc. It is now clear that solutions of the cannibals and missionaries puzzle such as in Fig. 4 provide the only possible solutions of the husbands puzzle. To see that Fig. 4, for example, actually yields a solution of the husbands puzzle, it is only necessary to check that the transitions $(1, 3)\rightarrow(1, 1)\rightarrow(2, 2)$ can be made without violating the requirement that the husbands and wives be matched, as is indeed the case.

References

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Editorial Comment on r! Several readers have written in to point out that the result

$$r! = \sum_{i=0}^{r} (-1)^i \binom{r}{i} (n - i)^r$$

conjectured by Tepper [5] and proved by Long [3] was known as early as the time of L. Euler; see [1, p. 62]. Essentially a well-known result in the calculus of finite differences, it can readily be deduced from the fact that $\Delta^r f(x) = a_r r!$ for any polynomial $f(x)$ of degree r with leading coefficient a_r and that

$$\Delta^r f(x) = \sum_{i=0}^{r} (-1)^r \binom{r}{i} f(x + r - i).$$

One has only to consider the special case $f(x) = (x - r)^r$. See [4; p. 10 and problem 4, p. 19]. For two additional proofs, see also [2].

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