Complex Functions Examples c-2

Analytic Functions

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Introduction

This is the second book containing examples from the *Theory of Complex Functions*. The first topic will be examples of the necessary general *topological concepts*. Then follow some examples of *complex functions, complex limits* and *complex line integrals*. Finally, we reach the subject itself, namely the *analytic functions* in general. The more specific properties of these analytic functions will be given in the books to follow.

Even if I have tried to be careful about this text, it is impossible to avoid errors, in particular in the first edition. It is my hope that the reader will show some understanding of my situation.

Leif Mejlbro
30th May 2008

1 Some necessary theoretical results

This chapter must not be considered as a replacement of the usual textbooks, concerning the theory necessary for the examples. We shall always assume that all the fundamental definitions of continuity etc. are well-known. Furthermore, we are also missing some theoretical results. The focus here is solely on the most important theorems for this book. We start by quoting the three main theorems for the continuous functions:

Theorem 1.1 *If $f : \Omega \rightarrow \mathbb{C}$ is continuous and the domain $A \subseteq \Omega$ is compact,* (i.e. closed and bounded)*, then the range $f(A)$ is also compact.*

Theorem 1.2 *If $f : \Omega \rightarrow \mathbb{C}$ is continuous and the domain $A \subseteq \Omega$ is connected, then the range $f(A)$ is also connected.*

Theorem 1.3 *Any continuous map $f : \Omega \rightarrow \mathbb{C}$ is* uniformly continuous *on every compact subset $A \subseteq \Omega$.*

We see that the compact sets, i.e. the bounded and closed sets, are playing a central role in connection with continuous functions. This is why we have given them the name *compact sets*.

The complex plane $\mathbb{C}$ is in a natural correspondence with the real plane $\mathbb{R} \times \mathbb{R}$, by writing

$$z = x + iy \in \mathbb{C}, \qquad \text{corresponding to} \qquad (x, y) \in \mathbb{R} \times \mathbb{R}.$$

Then a complex function $f(z)$ can also be written

$$f(z) = u(x, y) + i\, v(x, y),$$

where

$$u(x, y) = \text{ Re } f(z) \qquad \text{and} \qquad v(x, y) = \text{ Im } f(z)$$

are the real part and the imaginary part respectively of the complex function $f(z)$ in the complex variable $z = x + iy \in \mathbb{C}$.

In the same way we consider a plane curve C as both lying in $\mathbb{C}$ and in $\mathbb{R} \times \mathbb{R}$. Since we formally have by a splitting into the real part and the imaginary part

$$f(z)\, dz = \{u(x, y) + i\, v(x, y)\}\{dx + i\, dy\} = \{u\, dx - v\, dy\} + i\{u\, dy + v\, dx\},$$

we *define* the *complex line integral* along C by

$$\int_C f(z)\, dz := \int_C \{u\, dx - v\, dy\} + i \int_C \{u\, dy + v\, dx\},$$

and then the complex line integral is reduced to a complex sum of of two ordinary real line integrals.

Definition 1.1 *Assume that Ω is an open non-empty subset of $\mathbb{C}$, and let $f : \Omega \to \mathbb{C}$ be a complex function. If the limit*

$$\lim_{\substack{z \in \Omega \\ z \to z_0}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists for some given $z_0 \in \Omega$, then we say that f is differentiable at z_0, and we use all the usual notations of the derivative from the real analysis like e.g. $f'(z_0)$.
If $f : \Omega \to \mathbb{C}$ is differentiable at every $z \in \Omega$, and the derivative $f'(z)$ is continuous in Ω, then we call f an analytical function.

Then we have the following theorem:

Theorem 1.4 *Assume that $f(z) = u(x, y) + i\, v(x, y)$ is defined in an open set Ω, and assume furthermore that both $u(x, y)$ and $v(x, y)$ are continuously differentiable with respect to both x and y. Then the complex function $f(z)$ is an analytic function, if and only if the pair $u(x, y)$ and $v(x, y)$ fulfil the* Cauchy-Riemann equations *in Ω:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \textit{and} \qquad \frac{\partial u}{\partial u} = -\frac{\partial v}{\partial x}.$$

If we instead use polar coordinates,

$$x = r \cdot \cos\theta, \qquad y = r \cdot \sin\theta,$$

in our description of a complex function, i.e.

$$f(z) = u(r, \theta) + i\, v(r, \theta),$$

then the same theorem still holds if and only if the *Cauchy-Riemann equations in polar coordinates* are satisfied,

$$\frac{\partial u}{\partial r} = \frac{1}{r}\,\frac{\partial v}{\partial \theta}, \qquad \frac{1}{r}\,\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}.$$

One of the main theorems of the Theory of Complex Functions is

Theorem 1.5 CAUCHY'S INTEGRAL THEOREM. *Assume that the function $f(z)$ is analytic in a* simply connected *domain* Ω (this means roughly speaking that the domain does not contain "holes"), *then the value of the line integral*

$$\int_{z_0}^{z} f(z)\,dz$$

is independent of the choice of the continuous and piecewise differentiable curve C in Ω from the fixed point $z_0 \in \Omega$ to the fixed point $z \in \Omega$.
In in particular the curve is closed*, then*

$$\oint_C f(z)\,dz = 0.$$

The next important result, which is given here, is also due to Cauchy:

Theorem 1.6 CAUCHY'S INTEGRAL FORMULA. *Assume that $f(z)$ is analytic in an open domain Ω. Assume that C is composed of simple and closed piecewise differentiable curves in Ω, run through in such a way that all points inside C* (this means to the left of C seen in the direction of the movement) *belong to Ω.*
Let z_0 be any point inside C in the sense above. Then

$$f\left(z_0\right) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz.$$

We also mention

Theorem 1.7 THE MEAN VALUE THEOREM. *The value of an analytic function $f(z)$ at a point z_0 is equal to the mean value of the function over any circle of centrum z_0 and radius r, assuming that the closed disc $B\left[z_0, r\right]$ of centrum z_0 and radius r is contained in Ω. We have for such $r > 0$,*

$$f\left(z_0\right) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + r\,e^{i\theta}\right)d\theta.$$

Finally, we mention

Theorem 1.8 CAUCHY'S INEQUALITIES. *Assume that $f(z)$ is analytic in a domain which contains the closed disc*

$$B\left[z_0, r\right] = \left\{z \in C \mid \left|z - z_0\right| \le r\right\},$$

and let M_r denote the maximum of $|f(z)|$ on the circle $\left|z-z_0\right| = r$. Then

$$\left|f^{(n)}\left(z_0\right)\right| \le M_r \cdot \frac{n!}{r^n} \qquad \text{for every } n \in \mathbb{N}_0.$$

2 Topological concepts

Example 2.1 *Let $\Omega = \{1, 2, 3, 4\}$. Find the smallest system of open sets in Ω, such that*

$$\{1\}, \qquad \{2, 4\}, \qquad \{1, 2, 3\}$$

are all open sets.

We shall find the open system, which is generated by

$$\{1\}, \qquad \{2, 4\}, \qquad \{1, 2, 3\}.$$

First of all, both $\emptyset$ and Ω must belong to the system.
Then all intersections must also be contained in the system, thus

$$\{1\} \cap \{2, 4\} = \emptyset, \quad \{1\} \cap \{1, 2, 3\} = \{1\}, \quad \{2, 4\} \cap \{1, 2, 3\} = \{2\}.$$

By this process we conclude that $\{2\}$ must also be open.
Finally, all unions of sets from the system must again be open. This gives

$$\begin{array}{ll} \{1\} \cup \{2\} = \{1, 2\}, & \{1\} \cup \{1, 2, 3\} = \{1, 2, 3\}, \\ \{1\} \cup \{2, 4\} = \{1, 2, 4\}, & \{2\} \cup \{2, 4\} = \{2, 4\}, \\ \{2\} \cup \{1, 2, 3\} = \{1, 2, 3\}, & \{2, 4\} \cup \{1, 2, 3\} = \{1, 2, 3, 4\} = \Omega. \end{array}$$

We have now exhausted all possibilities, so the system of open sets must consist of the sets

$$\begin{array}{llll} \emptyset, & \{1\}, & \{2\}, & \{1, 2\}, \\ \{2, 4\}, & \{1, 2, 3\}, & \{1, 2, 4\}, & \Omega = \{1, 2, 3, 4\}. \end{array}$$

Example 2.2 *Let $f : \mathbb{R} \mapsto \mathbb{R}$ be defined by*

$$f(x) = \frac{1}{2 + |x|}.$$

Prove that f is a contraction, and find the corresponding fixpoint.

We shall prove that there exists a constant $C < 1$, such that

$$|f(x) - f(y)| \leq C\,|x - y|.$$

By a small computation and an estimate,

$$|f(x) - f(y)| = \left| \frac{1}{2 + |x|} - \frac{1}{2 + |y|} \right| = \left| \frac{2 + |y| - 2 - |x|}{(2 + |x|)(2 + |y|)} \right| = \left| \frac{|y| - |x|}{(2 + |x|)(2 + |y|)} \right| \leq \frac{1}{4}\,|x - y|,$$

proving that the map is a contraction.

Now, $f(x) > 0$ for every $x \in \mathbb{R}$, so a fixpoint must necessarily be positive, thus $|x| = x$. Then we shall solve the equation

$$\frac{1}{2 + x} = x, \qquad x > 0.$$

This is equivalent to

$$x^2 + 2x = 1, \qquad x > 0,$$

hence

$$(x+1)^2 = x^2 + 2x + 1 = 2, \qquad x > 0,$$

and thus

$$x = \pm 2 - 1, \qquad x > 0.$$

It follows that the fixpoint is

$$x = \sqrt{2} - 1.$$

Example 2.3 *Let $f : \mathbb{R}\times\mathbb{R} \mapsto \mathbb{R}$ be given by*

$$f(x,y) = \left(x^2+1\right)\sinh y + x + \frac{y^2}{2}.$$

Prove that the equation $f(x,y)=0$ globally *determines y as a function of x.*
HINT: *Prove for every fixed x that $f(x,y)$ is a continuous and strictly increasing function of y, which takes on the value 0.*
Then find by implicit differentiation the approximating polynomial of at most second degree for y as a function of x from the point $(x_0,y_0)=(0,0)$.

We see that $f \in C^\infty(\mathbb{R}\times\mathbb{R})$ and

$$f_y'(x,y) = \left(x^2+1\right)\cosh y + y \ge \cosh y + y > 0,$$

so $f(x,y)$ is for every fixed x strictly increasing in. Furthermore, we clearly have

$$f(x,y)\to -\infty \text{ for } y \to -\infty \quad \text{og} \qquad f(x,y) \to +\infty \text{ for } y \to +\infty$$

for every fixed x. By the continuity there exists for every $x \in \mathbb{R}$ precisely one $y \in \mathbb{R}$, such that $f(x,y)=0$. This is another way of saying that y is determined as a function of x.

Now let $y=y(x)$ be given by the construction above. Then $y(0)=0$, because $(0,0)$ clearly satisfies the equation, and because the solution is unique. Then we get by implicit differentiation,

$$\left\{\left(x^2+1\right)\cosh y + y\right\}\frac{dy}{dx} + 2x\,\sinh y + 1 = 0.$$

If we here put $(x,y)=(0,0)$ and solve with respect to the derivative, we get

$$\left.\frac{dy}{dx}\right|_0 = -\frac{1}{1} = -1.$$

In general we get by another implicit differentiation,

$$\left\{\left(x^2+1\right)\cosh y\right\}\frac{d^2y}{dx^2} + \left\{\left(x^2+1\right)\sinh y + 1\right\}\left(\frac{dy}{dx}\right)^2 + 4x\,\cosh y\,\frac{dy}{dx} + 2\,\sinh y = 0.$$

If we here put $\left(x,y,\dfrac{dy}{dx}\right) = (0,0,-1)$, then we get by solving with respect to the second derivative,

$$\left.\frac{d^2y}{dx^2}\right|_{x=0} + 1\cdot 1 + 0 + 0 = 0, \qquad \text{hence} \qquad \left.\frac{d^2y}{dx^2}\right|_{x=0} = -1.$$

The approximating polynomial of at most second degree form the point of expansion $x=0$ is given by

$$P(x) = -x - \frac{1}{2}\,x^2.$$

3 Complex Functions

Example 3.1 *Let $w = u + iv$. Find in the following examples $u(x,y)$ and $v(x,y)$ as real functions in two real variables:*

$$(a)\ \ w = z^3, \quad (b)\ \ w = z + \frac{1}{z}, \quad (c)\ \ w = \frac{z}{1+z}, \quad (d)\ \ w = z\,e^z.$$

(a) If $z \in \mathbb{C}$, then

$$w = u + iv = z^3 = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3,$$

hence by a separation into real and imaginary parts

$$u(x,y) = x^3 - 3xy^2 \qquad \text{and} \qquad v(x,y) = 3xy^2 - y^3,$$

(b) If $z \in \mathbb{C} \setminus \{0\}$, then

$$w = u + iv = x + \frac{1}{z} = z + \frac{\overline{z}}{z \cdot \overline{z}} = x + iy + \frac{x - iy}{x^2 + y^2},$$

hence by separating into real and imaginary parts,

$$u(x,y) = x + \frac{x}{x^2+y^2}, \qquad v(x,y) = y - \frac{y}{x^2+y^2}.$$

The function

$$w = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

is also called *Joukowski's function*.

(c) If $z \in \mathbb{C} \setminus \{-1\}$, then

$$w = u + iv = \frac{z}{1+z} = \frac{1+z-1}{1+z} = 1 - \frac{1}{1+x+iy} = 1 - \frac{x+1-iy}{(x+1)^2+y^2},$$

hence by separating into real and imaginary parts,

$$u(x,y) = 1 - \frac{x+1}{(x+1)^2+y^2}, \qquad v(x,y) = \frac{y}{(x+1)^2+y^2}.$$

The function $w = \dfrac{z}{z+1}$ is an example of an *homography*, also called a *fractional linear transformation*.

(d) If $z \in \mathbb{C}$, then

$$\begin{aligned} w &= u + iv = z\,e^z = (x+iy)e^x(\cos y + i\,\sin y) \\ &= x\,e^x\cos y - y\,e^x\sin y + i\left\{x\,e^x\sin y + y\,e^x\cos y\right\}, \end{aligned}$$

hence by separation into real and imaginary parts,

$$u(x,y) = x\,e^x\cos y - y\,e^x\sin y, \qquad v(x,y) = x\,e^x\sin y + y\,e^x\cos y.$$

Example 3.2 *Find* $f(z+1)$, $f\left(\dfrac{1}{z}\right)$, *and* $f(f(z))$ *for*

$$(a)\;\; f(z)=z+1, \quad (b)\;\; f(z)=z^2, \quad (c)\;\; f(z)=\frac{1}{z}, \quad (d)\;\; f(z)=\frac{1+z}{1-z}.$$

(a) If $f(z)=z+1$, then

$$f(z+1)=z+2,$$

$$f\left(\frac{1}{z}\right)=\frac{1}{z}+1=\frac{z+1}{z}, \qquad \text{for } z\neq 0,$$

$$f(f(z))=f(z+1)=z+2.$$

(b) If $f(z)=z^2$, then

$$f(z+1)=(z+1)^2=z^2+2z+1,$$

$$f\left(\frac{1}{z}\right)=\frac{1}{z^2}, \qquad \text{for } z\neq 0,$$

$$f(f(z))=f\left(z^2\right)=z^4.$$

(c) If $f(z)=\dfrac{1}{z}$, then

$$f(z+1)=\frac{1}{z+1}, \qquad \text{for } z\neq -1,$$

$$f\left(\frac{1}{z}\right)=z, \qquad \text{for } z\neq 0,$$

$$f(f(z))=f\left(\frac{1}{z}\right)=z, \qquad \text{for } z\neq 0.$$

(d) If $f(z)=\dfrac{1+z}{1-z}$, then

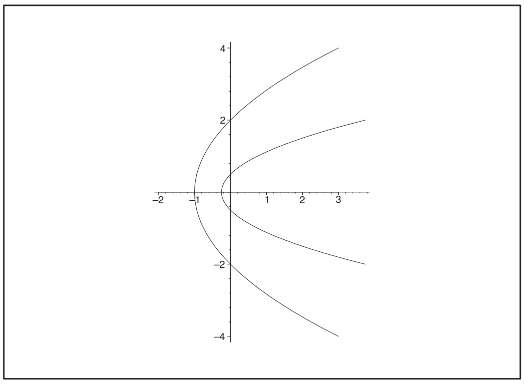
$$f(z+1)=\frac{1+(z+1)}{1-(z+1)}=\frac{2+z}{-z}=-\frac{z+2}{z}, \qquad \text{for } z\neq 0,$$

$$f\left(\frac{1}{z}\right)=\frac{1+\frac{1}{z}}{1-\frac{1}{z}}=\frac{z+1}{z-1}, \qquad \text{for } z\neq 0 \text{ og } z\neq 1,$$

$$f(f(z))=f\left(\frac{1+z}{1-z}\right)=\frac{1+\frac{1+z}{1-z}}{1-\dfrac{1+z}{1-z}}=\frac{1-z+1+z}{1-z-1-z}=\frac{2}{-2z}=-\frac{1}{z},$$

for $z\neq 0$ and $z\neq 1$.

Example 3.3 *Prove that the function $w = z^2$ maps the lines $y = c$, $c \in \mathbb{R}_+$, into parabolas in the w-plane.*
What is the image of the line $y = 0$?



If we put $z = t + ic$, $t \in \mathbb{R}$, then

$$w = z^2 = (t + ic)^2 = t^2 - c^2 + 2ict = u + iv,$$

hence by separation into real and imaginary part,

$$u = t^2 \qquad \text{og} \qquad v = 2ct.$$

If $y = 0$, thus $c = 0$, we get $u = t^2$ and $v = 0$, so the image of $y = 0$ is $\mathbb{R}_+ \cup \{0\}$ "run through twice".
If $c > 0$, it follows by eliminating t that we have the following equation of a parabola,

$$u = \frac{v^2}{4c^2} - c^2.$$

Example 3.4 *Find the image of*

$$\Omega = \{z \in \mathbb{C} \mid |Re(z)| < 1\}$$

by the following maps

$$(a)\ \ w = 2z + i, \qquad (b)\ \ w = (1+i)z + 1, \qquad (c)\ \ w = \frac{1}{z},$$

$$(d)\ \ w = 2z^2, \qquad (e)\ \ w = \frac{z+1}{z-1}.$$

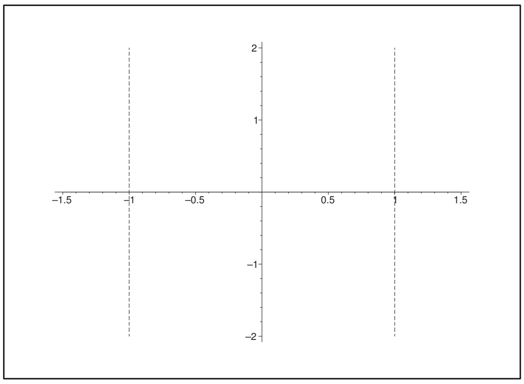


Figure 1: The domain Ω is the open parallel strip between the lines $x = -1$ and $x = 1$.

(a) Since the map $w = 2z + i$ is continuous, and Ω is connected, the range is by one of the main theorems also connected.

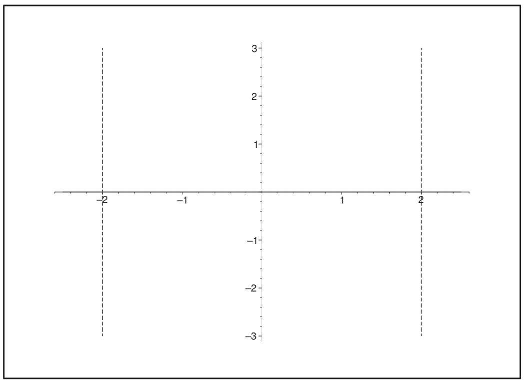


Figure 2: (a) The image of Ω is the open domain between the two vertical lines.

The map can be extended continuously to the boundary, and the map is an open map. It therefore suffices to find the images of the boundary curves $z = \pm 1 + iy$.
Since $w = \pm 2 + i(2y+1)$, we conclude that the range is

$$\{w \in \mathbb{C} \mid |\text{Re}(w)| < 2\}.$$

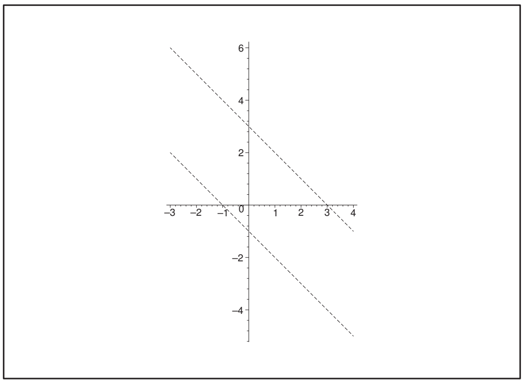


Figure 3: (b) The image of Ω is the open domain between the two oblique lines.

(b) The map $w = (1+i)z + 1$ is continuous and open. Therefore, it suffices as in (a) to find the images of the boundary curves.
If we put $z = -1 + iy$, $y \in \mathbb{R}$, then

$$w = (1+i)(-1+iy) + 1 = -1 - i + y(-1+i) + 1 = -i + y(i-1).$$

This is a parametric description of a line through the points $-i$ and -1 (put $y = 0$ and $y = 1$).
If we instead put $z = 1 + iy$, then

$$w = (1+i)(1+iy) + 1 = 2 + i + (-1+i)y,$$

which is the parametric description of a line. By putting $y = 0$, or $y = 1$, we get the points $2 + i$ and $1 + 2i$, and it is easy to sketch the domain.

(c) Consider the map

$$w = \frac{1}{z} = \frac{x - iy}{x^2 + y^2}, \qquad \text{for } z \neq 0.$$

The line $z = -1 + iy$ is mapped into the curve of the parametric description

$$u = -\frac{1}{1+y^2}, \qquad v = -\frac{y}{1+y^2}, \qquad y \in \mathbb{R},$$

hence $y = \dfrac{v}{u}$, since $u \neq 0$. Then

$$u + \frac{1}{1 + \left(\dfrac{v}{u}\right)^2} = 0,$$

or by some reformulation,

$$0 = u + \frac{u^2}{u^2+v^2} = \frac{u}{u^2+v^2}\left(u^2 + v^2 + u\right) = \frac{u}{u^2+v^2}\left\{\left(u + \frac{1}{2}\right)^2 + v^2 - \left(\frac{1}{2}\right)^2\right\}, \quad u \neq 0,$$

which we also write as

$$\left(u+\frac{1}{2}\right)^2+v^2=\left(\frac{1}{2}\right)^2, \qquad u \neq 0.$$

The image curve is therefore a circle (with the exception of one point) of centrum $-\frac{1}{2}$ and radius $\frac{1}{2}$, and where $u \neq 0$.

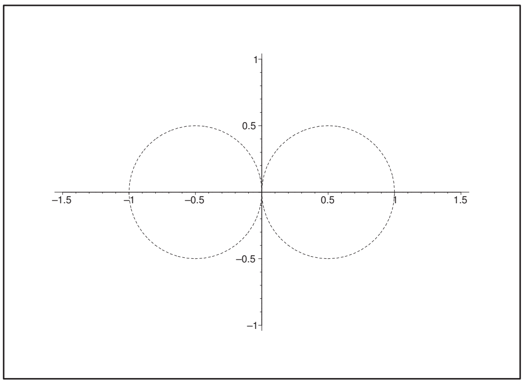


Figure 4: (c) The domain Ω is mapped into the open domain outside the two discs.

Analogously, the line $z = iy$ is mapped into the curve of the parametric description

$$u=\frac{1}{1+y^2}, \qquad v=-\frac{y}{1+y^2}, \qquad y \in \mathbb{R},$$

hence $y=-\frac{v}{u}$, $u \neq 0$, and thus

$$u-\frac{1}{1+\frac{v^2}{u^2}}=0, \qquad u \neq 0,$$

which we write as

$$\left(u-\frac{1}{2}\right)^2+v^2=\left(\frac{1}{2}\right)^2, \qquad u \neq 0.$$

Hence the image curve is a part of a circle of centrum $\frac{1}{2}$ and radius $\frac{1}{2}$, and where $u \neq 0$.
The map is open, so we get by a continuity argument (use e.g. that $z = i$ is mapped into $w = -i$) it follows that the range is the open domain outside the two discs.
NOTICE that the point $z = 0$ must be removed from the domain $|\mathrm{Re}(z)| < 1$, because the map is not defined at that point.

(d) The map $w = 2z^2$ is continuous. If we put $z = -1 + iy$, then

$$w=2(-1+iy)^2=2\left(1-y^2\right)-4iy,$$

hence by separation into real and imaginary part,

$$u=2\left(1-y^2\right) \qquad \text{and} \qquad v=-4y.$$

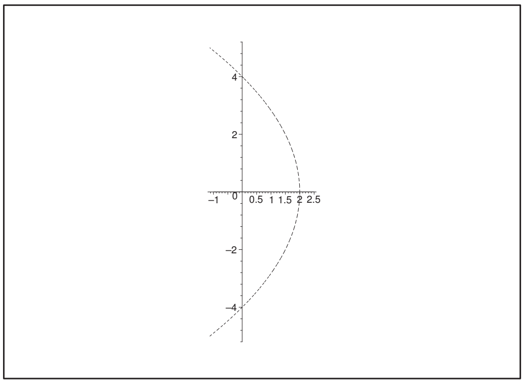


Figure 5: (d) The domain Ω is mapped onto the open interior of the parabola.

When y is eliminated, i.e. when we put $y = -\dfrac{v}{4}$, then

$$u = 2 - \frac{v^2}{8},$$

which is the equation of a parabola in the (u, v)-plane.

We conclude by the symmetry about $(0,0)$ that the image of the strips

$$-1 < \mathrm{Re}(z) \le 0 \qquad \text{and} \qquad 0 \le \mathrm{Re}(z) < 1$$

are identical.
Finally, if we put $z = iy$ (in Ω), then $w = -2y^2$, $y \in \mathbb{R}$, which is a parametric description of the negative real axis, run through twice.
By a continuity argument (e.g. by putting $z = \frac{1}{2}$, which is mapped into $w = \frac{1}{2}$), we conclude that the image is the interior of the parabola.

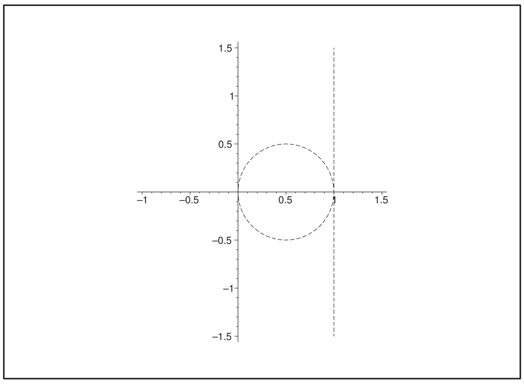


Figure 6: (e) The domain Ω is mapped into that part of the open left hand half plane determined by the line $x = 1$, also lying outside the disc.

(e) Finally, we consider the transformation

$$w = \frac{z+1}{z-1}, \qquad z \neq 1.$$

If we put $z = -1 + iy$, then

$$w = u + iv = \frac{iy}{-2+iy} = \frac{iy}{4+y^2}\,(-2-iy) = \frac{1}{4+y^2}\left(y^2 - 2iy\right).$$

We get by a separation into real and imaginary parts,

$$u = \frac{y^2}{4+y^2} \quad \text{og} \quad v = -\frac{2y}{4+y^2}, \qquad y \in \mathbb{R}.$$

If $y = 0$, we get $u = v = 0$, corresponding to the point $w = 0$.

If $y \neq 0$, then $v \neq 0$, and $\dfrac{u}{v} = -\dfrac{y}{2}$, thus $y = -2\,\dfrac{u}{v}$. Then by insertion,

$$\begin{aligned} 0 &= v + \frac{2y}{4+y^2} = v + \frac{-4\,\dfrac{u}{v}}{4 + 4\,\dfrac{u^2}{v^2}} = v - \frac{\dfrac{u}{v}}{1 + \dfrac{u^2}{v^2}} \\ &= v - \frac{uv}{u^2+v^2} = \frac{v}{u^2+v^2}\left(u^2+v^2-u\right), \qquad v \neq 0, \end{aligned}$$

and we conclude that

$$\left(u-\frac{1}{2}\right)^2+v^2=\left(\frac{1}{2}\right)^2, \qquad v \neq 0,$$

which describes a part of a circle of radius $\frac{1}{2}$ and centrum $\left(\frac{1}{2},0\right)$ and $v \neq 0$.
Notice that $(0,0)$ lies on the closure of this circle.

Finally, put $z = 1 + iy$. If $y \neq 0$, then

$$w=\frac{2+iy}{iy}=1-i\,\frac{2}{y},$$

which apart from the point $(u,v) = (1,0)$ corresponds to the line $u = 1$.

Summing up, the range is that domain which "lies between" the circle and the vertical straight line $u = 1$, thus the range is a part of the open left half plane given by $u = 1$, and which also lies outside the closed disc of centrum $\frac{1}{2}$ and radius $\frac{1}{2}$.

Example 3.5 *Find the image of*

$$\Omega = \{z \in \mathbb{C} \mid 1 < Im(z) < 2\}$$

by the following maps:

$$(a)\;\; w = 2z + i, \qquad (b)\;\; w = (1+i)z + 1, \qquad (c)\;\; w = \frac{1}{z},$$

$$(d)\;\; w = 2z^2, \qquad (e)\;\; w = \frac{z+1}{z-1}.$$

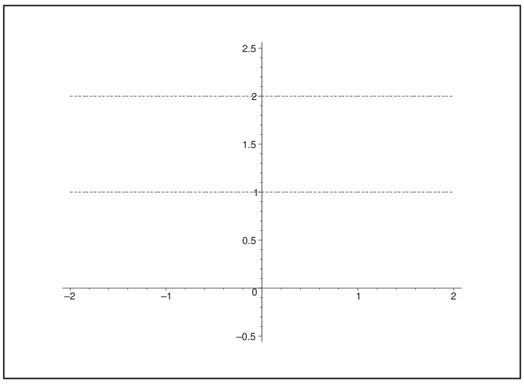


Figure 7: The domain Ω is the open parallel strip between the two horizontal lines.

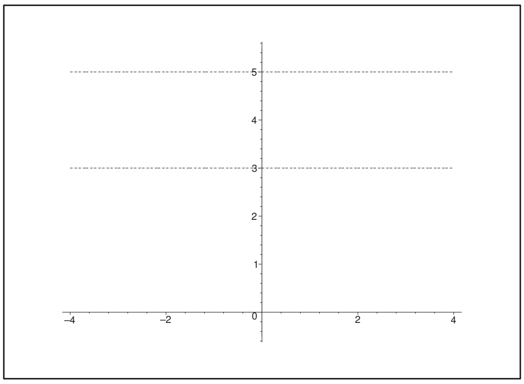


Figure 8: (a) The image of Ω is the open parallel strip between the two horizontal lines.

(a) If $\mathrm{Im}(z) \in]1, 2[$, then

$$\mathrm{Im}(w) = 2\,\mathrm{Im}(z) + 1 \in]3, 5[,$$

hence the strip $1 < \mathrm{Im}(z) < 2$ is mapped onto the strip $3 < \mathrm{Im}(w) < 5$.

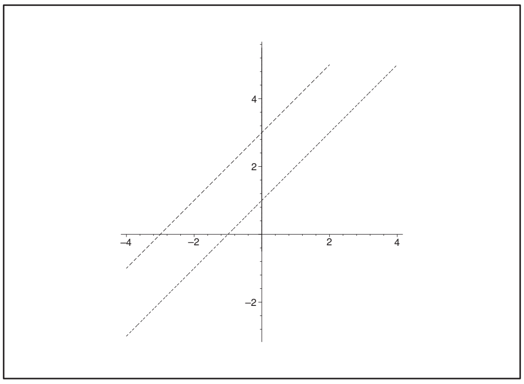


Figure 9: (b) The domain Ω is mapped into the open oblique parallel strip.

(b) The strip $1 < \mathrm{Im}(z) < 2$ has the boundary curves $y = 1$ and $y = 2$. If we put $z = x + i$, then

$$w = (1+i)(x+i)+1 = (1+i)x+i-1+1 = i+(1+i)x, \qquad x \in \mathbb{R}.$$

If we put $z = x + 2i$, then

$$w = (1+i)(x+2i)+1 = (1+i)x+2i-2+1 = (1+i)x-1+2i, \qquad x \in \mathbb{R}.$$

The range is the domain between these two parallel lines.

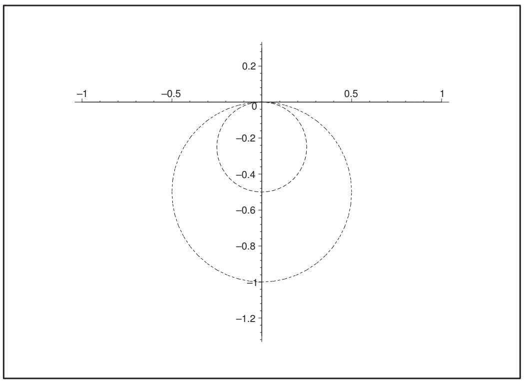


Figure 10: (c) The domain Ω is mapped into the open half moon between the two circles.

(c) Consider the map

$$w = \frac{1}{z} = \frac{x-iy}{x^2+y^2}, \qquad (x,y) \neq (0,0).$$

The curve $x = x + i$ is mapped into

$$w = \frac{x-i}{x^2+1},$$

hence by separation into real and imaginary part,

$$u = \frac{x}{x^2+1} \quad \text{and} \quad v = -\frac{1}{x^2+1}, \qquad v<0.$$

This gives fås $x = -\dfrac{u}{v}$, hence

$$v + \frac{1}{x^2+1} = v + \frac{v^2}{u^2+v^2} = 0, \qquad v<0.$$

This equation is then in the usual way written in the form

$$u^2 + \left(v+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2, \qquad v<0,$$

which is the equation of (a part of) a circle of centrum $-\dfrac{i}{2}$ and radius $\dfrac{1}{2}$, and where $v<0$.

The curve $z = x+2i$ is mapped into

$$w = u+iv = \frac{x-2i}{x^2+4},$$

hence by separation into real and imaginary part,

$$u = \frac{x}{x^2+4}, \qquad v = \frac{2}{x^2+4}.$$

When we eliminate $x = -2\,\dfrac{u}{v}$, $v<0$, we get

$$0 = v + \frac{2}{4+\left(2\cdot\dfrac{u}{v}\right)^2} = v + \frac{2v^2}{4v^2+4u^2} = \frac{v}{4\left(v^2+u^2\right)}\left(4v^2+4u^2+2v\right), \qquad v<0.$$

Hence

$$u^2+v^2+\frac{1}{2}\,v+\frac{1}{16} = u^2+\left(v+\frac{1}{4}\right)^2 = \left(\frac{1}{4}\right)^2, \qquad v<0,$$

which is the equation of (a part of) a circle with $w = -\dfrac{i}{4}$ as its centrum and radius $\dfrac{1}{4}$, and where $v<0$.
It follows by a trivial estimate that the range is bounded, hence the range must by the open half moon shaped domain between the two circles.

(d) We consider the map $w = 2z^2$. The strip $1 < \operatorname{Im}(z) < 2$ has the boundary curves $y=1$ and $y=2$. If we put $z = x+i$, $x\in\mathbb{R}$, then

$$w = 2(x+i)^2 = 2\left(x^2-1\right)+4ix = u+iv,$$

hence by separation into real and imaginary part,

$$u = 2x^2-2 \qquad \text{and} \qquad v = 4x.$$

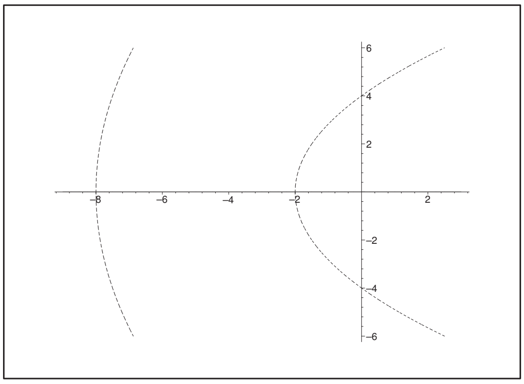


Figure 11: (d) The image of Ω is the open domain between the two arcs of parabolas.

When we eliminate the parameter x, we get the equation of the parabola

$$u = \frac{v^2}{8} - 2.$$

If instead we put $x = x + 2i$, then

$$w = 2(x+2i)^2 = 2\left(x^2 - 4\right) + 8ix = u + iv,$$

hence by separation into real and imaginary part,

$$u = 2x^2 - 8 \qquad \text{og} \qquad v = 8x.$$

When we eliminate the parameter x, we get the equation of the parabola

$$u = \frac{v^2}{32} - 8.$$

A continuity argument then shows that the image is the open domain between the two parabolas.

(e) Finally, consider the transformation

$$w = \frac{z+1}{z-1}, \qquad z \neq 1.$$

The boundary curve $z = x + i$ is mapped into

$$w = \frac{x+1+i}{x-1+i} = \frac{(x+1+i)(x-1-i)}{(x-1)^2+1} = \frac{x^2-2i}{(x-1)^2+1} = u + iv.$$

By separation of the real and the imaginary part we get

$$u = \frac{x^2}{(x-1)^2+1} \geq 0 \qquad \text{and} \qquad v = -\frac{2}{(x-1)^2+1} < 0,$$

and

$$\frac{u}{v} = -\frac{x^2}{2}, \qquad \text{thus} \qquad x = \pm\sqrt{-2\,\frac{u}{v}}.$$

On the other hand, if follows from the expression of v that

$$(x-1)^2 = -1 - \frac{2}{v}.$$

If we here put $x = \pm\sqrt{-2\,\frac{u}{v}}$, then

$$-2\,\frac{u}{v} \mp 2\sqrt{-2\,\frac{u}{v}} + 1 = -\frac{2}{v} - 1,$$

which is reduced to

$$\pm 2\sqrt{-2\,\frac{u}{v}} = \frac{2}{v} - 2\,\frac{u}{v} + 2 = 2 \cdot \frac{1-u+v}{v},$$

thus to

$$\pm\sqrt{-2\,\frac{u}{v}} = \frac{1+v-u}{v}.$$

Then by a squaring,

$$-2\,\frac{u}{v} = \frac{1}{v^2}\left(1 + v^2 + u^2 + 2v - 2u - 2uv\right),$$

thus

$$-2uv = u^2 + v^2 - 2u + 2v + 1 - 2uv,$$

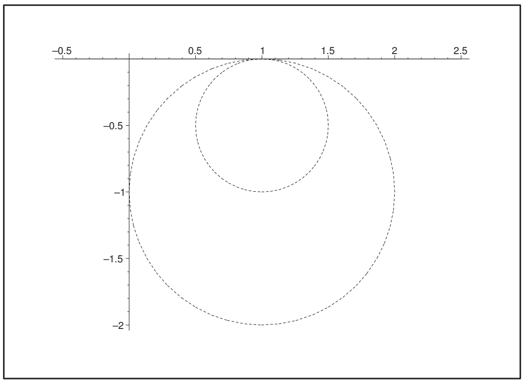


Figure 12: (e) The image of Ω is the open half moon shaped domain between the two circles.

which we write as

$$(u-1)^2+(v+1)^2=1, \qquad v<0,$$

describing (a part of) a circle of radius 1 and centrum $(1,-1)$, and $v<0$.
The curve $z=x+2i$ is mapped into

$$w=\frac{x+1+2i}{x-1+2i}=\frac{(x+1+2i)(x-1-2i)}{(x-1)^2+4}=\frac{x^2+3-4i}{(x-1)^2+4}=u+iv, \qquad v<0,$$

hence by separation into real and imaginary part,

$$u=\frac{x^2+3}{(x-1)^2+4}>0, \qquad v=-\frac{4}{(x-1)^2+4}<0.$$

Then

$$\frac{u}{v}=-\frac{1}{4}\left(x^2+3\right), \qquad \text{i.e.} \qquad x^2=-4\,\frac{u}{v}-3,$$

hence

$$x=\pm\sqrt{-4\,\frac{u}{c}-3},$$

and $(x-1)^2=-\dfrac{4}{v}-4$, so

$$-\frac{4}{v}-4=(x-1)^2=\left(\pm\sqrt{-4\,\frac{u}{v}-3}-1\right)^2=-4\,\frac{u}{v}-3\mp 2\sqrt{-4\,\frac{u}{v}-3}+1,$$

which is reduced to

$$\pm 2\sqrt{-4\,\frac{u}{v}-3}=\frac{4}{v}+4-4\,\frac{u}{v}-2=2\left(\frac{2+v-2u}{v}\right).$$

First we remove the common factor 2. Then we square once more,

$$-4\,\frac{u}{v}-3=\left(\frac{2+v-2u}{v}\right)^2=\frac{1}{v^2}\left\{4u^2-4uv-8u+v^2+4v+4\right\},$$

hence

$$-4uv-3v^2=4u^2-4uv+v^2-8u+4v+4,$$

which again is rewritten as

$$\begin{aligned}0 &= 4u^2+4v^2-8u+4v+4=4\left\{u^2-2u+1+v^2+v+\frac{1}{4}-\frac{1}{4}\right\}\\ &= 4\left\{(u-1)^2+\left(v+\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2\right\}, \qquad v<0,\end{aligned}$$

and we end up with the equation of (a part of) a circle,

$$(u-1)^2+\left(v+\frac{1}{2}\right)^2=\left(\frac{1}{2}\right)^2, \qquad v<0,$$

of centrum $\left(1,-\frac{1}{2}\right)$ and radius $\frac{1}{2}$, and where $v<0$.

Since the range is bounded, it must lie between the two circles.

Remark 3.1 We ought to have checked all our results. However, we shall later in another book in this Complex series obtain the same results in a much easier way by using the theory of conformal mapping, so we shall not bother with these tests. $\Diamond$

Example 3.6 *Find the image of the unit disc*

$$\Omega=\{z\in\mathbb{C}\mid |z|<1\}$$

by the following maps:

$$(a)\ \ w=2z+i, \qquad (b)\ \ w=(1+i)z+1, \qquad (c)\ \ w=\frac{1}{z},$$

$$(d)\ \ w=2z^2, \qquad (e)\ \ w=\frac{z+1}{z-1}.$$

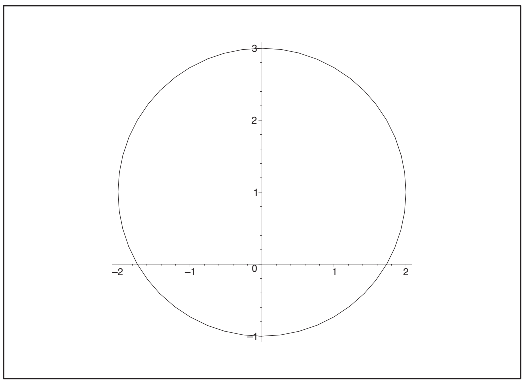


Figure 13: (a) The image of the open disc of centrum i and radius 2.

(a) When we solve the equation with respect to z we get

$$|z|=\frac{1}{2}\,|w-i|<1, \qquad \text{hence} \qquad |w-i|<2.$$

The image is the open disc of centrum $i\sim(0,1)$ and radius 2.

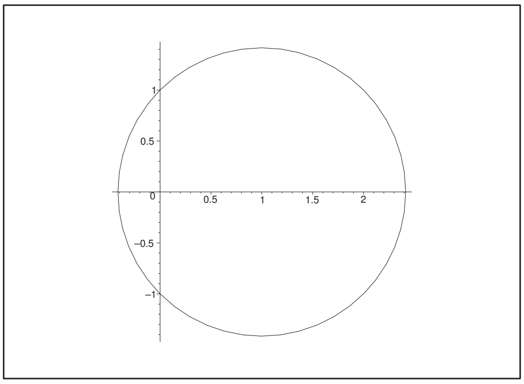


Figure 14: (b) The image of the open disc of centrum 1 and radius $\sqrt{2}$.

(b) By solving the equation with respect to z we get

$$z = \frac{w-1}{1+i},$$

thus

$$|z| = \frac{1}{\sqrt{2}}\,|w-1| < 1, \qquad \text{hence} \qquad |w-1| < \sqrt{2}.$$

The range is the open disc of centrum $w = 1$ and radius $\sqrt{2}$.

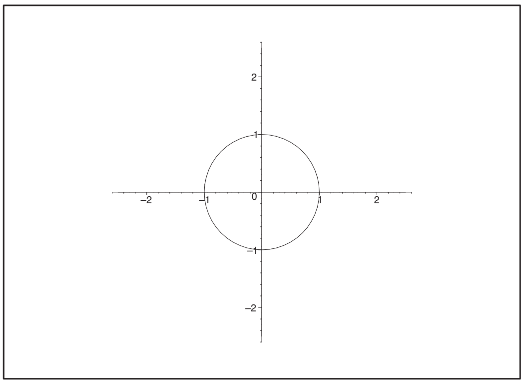


Figure 15: (c) The image is the open complementary set of a disc $w| > 1$.

(c) Here, $|z| = \dfrac{1}{|w|} < 1$, so $|w| > 1$ and $z \neq 0$. The image is the open complementary set of the disc of centrum 0 and radius 1.

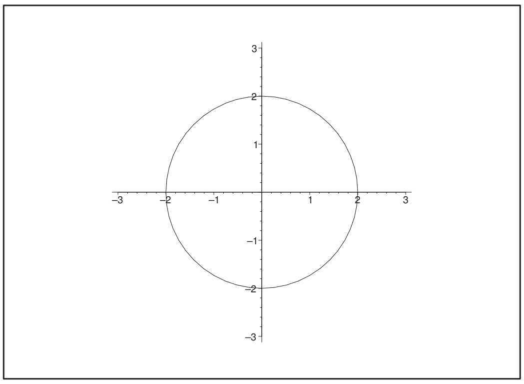


Figure 16: (d) The image is the open disc of centrum 0 and radius 2.

(d) It follows that $|w| = \left|2z^2\right| < 2$, hence the range is the open disc of centrum $w = 0$ and radius 2.

(e) We shall find the image of the boundary curve of the parametric description

$$z = e^{i\theta}, \qquad \theta \in]0, 2\pi[.$$

We get by insertion,

$$w = \frac{e^{i\theta} + 1}{e^{i\theta} - 1} = \frac{e^{i\,\frac{\theta}{2}} + e^{-i\,\frac{\theta}{2}}}{e^{i\,\frac{i\theta}{2}} - e^{-i\,\frac{\theta}{2}}} = \frac{2\,\cos\frac{\theta}{2}}{2i\,\sin\frac{\theta}{2}} = -i\,\cos\frac{\theta}{2}.$$

We conclude that the image of the boundary curve is the imaginary axis. Since $z = 0$ is mapped into $w = -1$, we conclude by the continuity that the range is the left hand half plane.

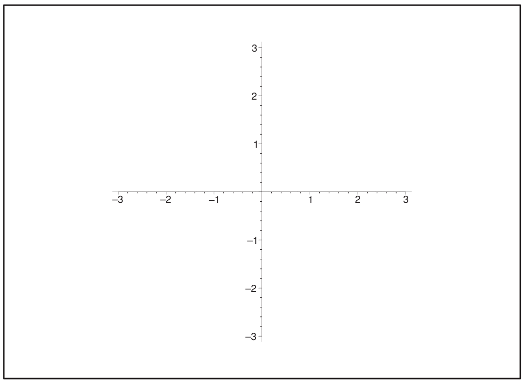


Figure 17: (e) The image of Ω is the open left hand half plane.

Example 3.7 *Find the equations of the curves in the (x, y)-plane, which by*

$$z = w + e^w$$

are mapped into $u =$ constant *and* $v =$ constant, *respectively.*
What is corresponding to the straight lines $v = 0$ and $v = \pi$?

If we put $z = x + iy$ and $w = u + iv$, then

$$z = x + iy = w + e^w = u + iv + e^{u+iv} = u + iv + e^u \cos v + i\, e^u \sin v.$$

By separation into real and imaginary part we get

$$x = u + e^u \cos v, \qquad y = v + e^u \sin v.$$

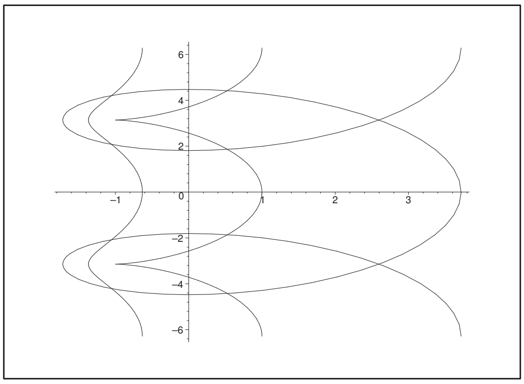


Figure 18: The images of the curves $u = k$ for $k = -1$, 0 and 1.

If $u = k$, and $v \in \mathbb{R}$ is considered as a parameter, we get the parametric description

$$x = k + e^k \cos v, \qquad y = v + e^k \sin v, \qquad v \in \mathbb{R},$$

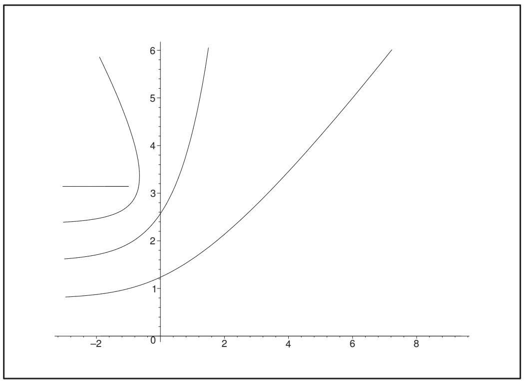


Figure 19: The images of the curves $v = k$ for $k = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$ and π.

which cannot be further reduced.
If $v = k$, and $u \in \mathbb{R}$ is considered as a parameter, then

$$x = u + e^u \cos k, \qquad y = k + e^u \sin k, \qquad u \in \mathbb{R}.$$

If $\sin k \neq 0$, i.e. $v = k \neq p\pi$, $p \in \mathbb{Z}$, then

$$e^u = \frac{y-k}{\sin k}, \quad \text{dvs.} \quad u = \ln\left(\frac{y-k}{\sin k}\right), \quad \text{forudsat, at } \frac{y-k}{\sin k} > 0.$$

Hence we obtain the explicit expression of the curve,

$$x = \ln\left(\frac{y-k}{\sin k}\right) + \frac{y-k}{\sin k} \cdot \cos k = \ln\left(\frac{y-k}{\sin k}\right) + (y-k)\cot k \qquad \text{for } \frac{y-k}{\sin k} > 0.$$

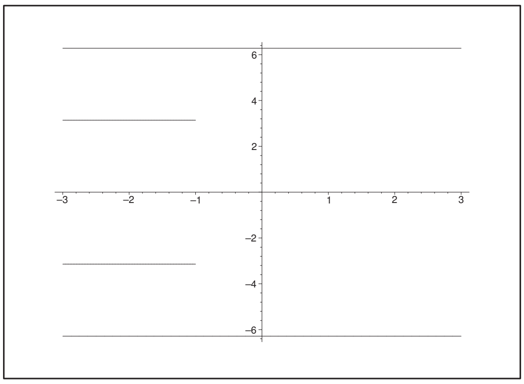


Figure 20: The images of the curves $v = k$ for $k = -2\pi$, $-\pi$, 0, π and 2π.

If $v = 2p\pi$, $p \in \mathbb{Z}$, then

$$x = u + e^u \qquad \text{and} \qquad y = 2p\pi, \qquad u \in \mathbb{R}.$$

Now, $x = e^u + u$ runs through all of $\mathbb{R}$, when u runs through $\mathbb{R}$, so the curve is the horizontal line $y = 2p\pi$. This is in particular true $p = 0$, so in this case the curve is the whole of the x-axis.

If $v = (2p+1)\pi$, $p \in \mathbb{Z}$, then

$$x = u - e^u \qquad \text{and} \qquad y = (2p+1)\pi, \qquad u \in \mathbb{R}.$$

Since

$$\frac{dx}{du} = 1 - e^u,$$

we conclude that the function $x(u)$ has a *maximum* for $u = 0$, corresponding to $x = -1$, and since $x(u) \to -\infty$ for $u \to +\infty$ and for $u \to -\infty$, we conclude that the half lines

$$x \leq -1, \qquad y = (2p+1)\pi, \qquad\qquad p \in \mathbb{Z},$$

are run through twice. This is in particular true for $p = 0$.

Example 3.8 *Sketch the curves*

$$u(x,y) = \text{ constant} \quad \textit{and} \quad v(x,y) = \text{ constant}$$

in the z-plane for the following functions:

$$(a)\;\; f(z) = \frac{1}{z}, \qquad (b)\;\; f(z) = z, \qquad (c)\;\; f(z) = (1-2i)z.$$

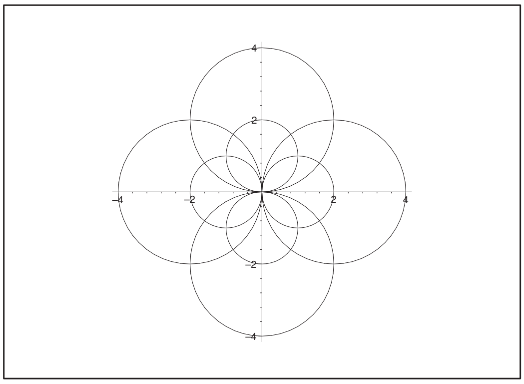


Figure 21: (a) The level curves of the dipole.

(a) We must clearly assume that $z \neq 0$. Once this is done we get

$$f(z) = \frac{1}{z} = \frac{1}{z} \cdot \frac{\overline{z}}{\overline{z}} = \frac{x-iy}{x^2+y^2}.$$

Then we separate the real and the imaginary parts,

$$u(x,y) = \frac{x}{x^2+y^2} \qquad \text{and} \qquad v(x,y) = -\frac{y}{x^2+y^2}.$$

The curves $u(x,y)=0$ are the two half lines given by

$$x = 0, \qquad y \neq 0,$$

thus the positive and the negative imaginary axis.

If $c \neq 0$, then $u(x,y) = \dfrac{1}{2c}$ is equivalent to $2cx = x^2+y^2$, thus

$$(x-c)^2 + y^2 = c^2, \qquad (x,y) \neq (0,0),$$

and the level curve is a circle of centrum $(c,0)$ and radius $|c|$, with the exception of the singular point $(0,0)$.

Analogously, $v(x,y)=0$ is described by the two half lines given by

$$x \neq 0, \qquad y = 0,$$

thus the positive and the negative part of the real axis.

If $c \neq 0$, then $v(x,y) = \dfrac{1}{2c}$ is equivalent to (notice the change of sign)

$$x^2 + (y+c)^2 = c^2, \qquad (x,y) \neq (0,0),$$

which describes a circle of centrum $(0,-c)$ and radius $|c|$ with the exception of the singular point $(0,0)$.

Remark 3.2 These level curves are the model of the field around a *dipole*. $\Diamond$

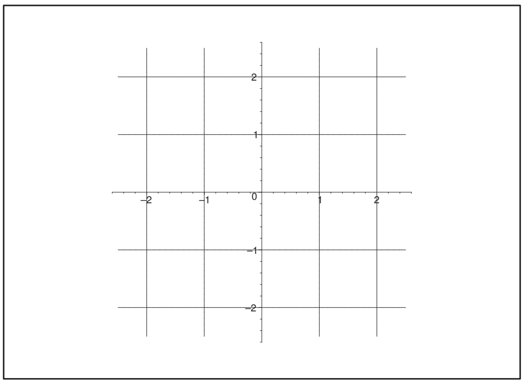


Figure 22: The level curves of (b).

(b) Here, $u(x,y) = x$ and $v(x,y) = y$, thus the level curves are the usual axis parallel lines.

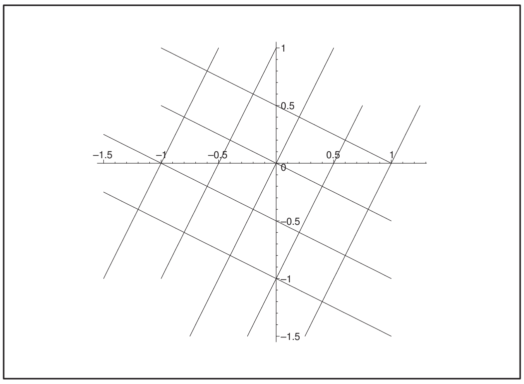


Figure 23: The level curves of (c).

(c) If

$$f(z) = (1-2i)z = (1-2i)(x+iy) = x+2y+i(-2x+y)$$

then it follows by a separation into real and imaginary parts,

$$u(x,y) = x + 2y \qquad \text{and} \qquad v(x,y) = -2x + y.$$

The level curves are the straight lines

$$u(x,y) = x + 2y = c, \qquad v(x,y) = -2x + c, \qquad c \in \mathbb{R}.$$

Remark 3.3 We see in all three cases that apart from the singular point $z = 0$ in (a), every curve from one system of curves is always perpendicular on any curve from the other system of curves. $\Diamond$.

Example 3.9 *Sketch the curves* $u(x,y) = \text{constant}$ *and* $v(x,y) = \text{constant}$ *in the* z*-plane for the following functions,*

$$(a)\ \ f(z) = z^2, \qquad (b)\ \ f(z) = z + z^2, \qquad (c)\ \ f(z) = \frac{z+i}{z-i}.$$

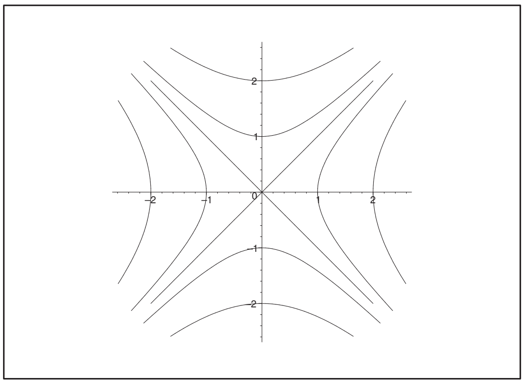


Figure 24: The level curves of (a).

(a) We get by a separation into real and imaginary part,

$$u(x,y) = x^2 - y^2 \qquad \text{og} \qquad v(x,y) = 2xy.$$

The level curves $u = k$ form a family of hyperbolas and the straight lines $y = x$ and $y = -x$.
The level curves $v = k$ are also a family of hyperbolas with the axes added.
We see that apart from in the singular point $(0,0)$, every curve from one system of curves is always orthogonal to any curve from the other system of curves.

(b) We first compute

$$u + iv = z + z^2 = x^2 - y^2 + x + i(2xy + y).$$

We get by a separation into real and imaginary part,

$$u(x,y) = x^2 + x - y^2 = \left(x + \frac{1}{2}\right)^2 - y^2 - \frac{1}{4}$$

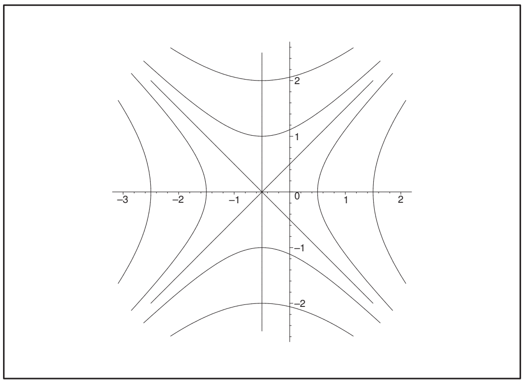


Figure 25: The level curves of (b).

and

$$v = 2xy + y = 2y\left(x + \frac{1}{2}\right),$$

thus the curves are the same as in (a), only the centrum has been translated to $\left(-\frac{1}{2}, 0\right)$.

(c) If $z \neq i$, then

$$f(z) = \frac{z+i}{z-i} = 1 + \frac{2i}{z-i} = 1 + \frac{2i(x+i\{1-y\})}{x^2+(y-1)^2} = 1 + \frac{2(y-1)+2ix}{x^2+(y-1)^2}.$$

Then by separation into the real and the imaginary parts,

$$u(x,y) = 1 + \frac{2(y-1)}{x^2+(y-1)^2} \qquad \text{and} \qquad v(x,y) = \frac{2x}{x^2+(y-1)^2}.$$

It follows that the curve $u(x,y) = 1$ is the line $y = 1$, with the exception of the point $(0,1)$, in which the denominator is always 0.

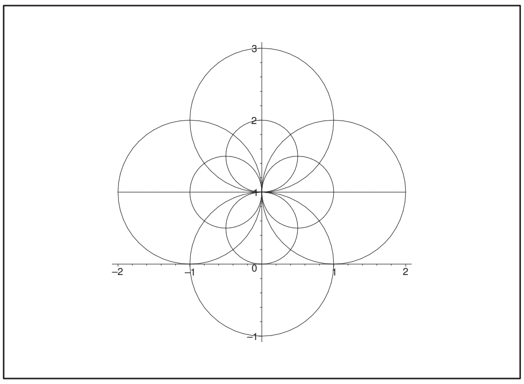


Figure 26: The level curves of (c), i.e. a field around a dipole at the point $(0,1)$.

We get for the level curve $u(x,y) = 1 + k$, $k \neq 0$, that

$$k = \frac{2(y-1)}{x^2+(y-1)^2}, \qquad (x,y) \neq (0,1),$$

thus

$$x^2 + (y-1)^2 - \frac{2}{k}(y-1) + \frac{1}{k^2} = \frac{1}{k^2}, \qquad (x,y) \neq (0,1),$$

which we also write

$$x^2 + \left(y - 1 - \frac{1}{k}\right)^2 = \frac{1}{k^2}, \qquad (x,y) \neq (0,1).$$

This is the equation of a circle of centrum $\left(0, \dfrac{k+1}{k}\right)$ and radius $\dfrac{1}{|k|}$, with the exception of the singular point $(0,1)$.

The case $v(x,y) = k$ is treated analogously.
If $v = 0$, then $x = 0$ (i.e. the y-axis), with the exception of the singular point $(0,1)$.
If $v = k \neq 0$, then we get instead,

$$x^2 + (y-1)^2 = 2 \cdot \frac{1}{k}\, x, \qquad (x,y) \neq (0,1),$$

which is written as the equation of (a part of) a circle,

$$\left(x-\frac{1}{k}\right)^2+(y-1)^2=\frac{1}{k^2}, \qquad (x,y)\neq(0,1),$$

of centrum $\left(\frac{1}{k},1\right)$ and radius $\frac{1}{|k|}$.

Remark 3.4 The example corresponds to a *dipole* at $z=i$. $\Diamond$

4 Limits

Example 4.1 *Check if the following limits exist:*

$$(a)\quad \lim_{z\to 0}\frac{\overline{z}}{z}, \qquad (b)\quad \lim_{z\to -1}\frac{z^4-2z^2+1}{z+1}.$$

(a) Put $z = x+iy \neq 0$. Then

$$\frac{\overline{z}}{z} = \frac{x-iy}{x+iy}.$$

If in particular we choose z real, $z = x$, we get

$$\frac{\overline{z}}{z} = 1, \qquad \text{hence} \qquad \lim_{x\to 0}\frac{\overline{x}}{x} = 1.$$

If we instead choose z imaginary, $z = iy$, then

$$\frac{\overline{z}}{z} = \frac{-iy}{iy} = -1, \qquad \text{hvoraf} \qquad \lim_{y\to 0}\frac{\overline{iy}}{iy} = -1.$$

Since we do *not* get the *same* limit value by the two different limits towards the same point, the limit value does *not* exist for $z \to 0$, by the definition.

(b) Since

$$z^4 - 2z^2 + 1 = \left(z^2-1\right)^2 = (z+1)^2(z-1)^2,$$

it follows for $z \neq -1$ that

$$\frac{z^2-2z^2+1}{z+1} = (z+1)(z-1)^2 \to 0\cdot(-1)^2 = 0 \qquad \text{for } z \to -1.$$

Hence the limit value exists ant

$$\lim_{z\to -1}\frac{z^4-2z^2+1}{z+1} = 0.$$

Example 4.2 *Check if any of the following functions (defined for $z \neq 0$), can be extended continuously to $z = 0$:*

$$(a)\quad \frac{Re(z)}{z}, \qquad (b)\quad \frac{z}{|z|}, \qquad (c)\quad \frac{Re\left(z^2\right)}{|z|^2}, \qquad (d)\quad \frac{z\,Re(z)}{|z|}.$$

(a) Since e.g.

$$\frac{\mathrm{Re}(z)}{z} = \frac{x}{x+iy} = \begin{cases} 1 & \text{for } y = 0 \text{ and } x \neq 0, \\ 0 & \text{for } x = 0 \text{ and } y \neq 0, \end{cases}$$

we cannot extend this function continuously to $z = 0$.

(b) Since e.g.

$$\frac{z}{|z|} = \begin{cases} 1 & \text{for } y=0 \text{ and } x>0, \\ -1 & \text{for } y=0 \text{ and } x<0, \end{cases}$$

we cannot extend this function continuously to $z=0$.

(c) Since e.g.

$$\frac{\operatorname{Re}\left(z^2\right)}{|z|^2} = \frac{x^2-y^2}{x^2+y^2} = \begin{cases} 1 & \text{for } y=0 \text{ and } x\neq 0, \\ -1 & \text{for } x=0 \text{ and } y\neq 0, \end{cases}$$

we cannot extend this function continuously to $z=0$.

(d) Since

$$\left|\frac{\operatorname{Re}(z)}{|z|}\right| = \frac{|x|}{\sqrt{x^2+y^2}} \le 1 \qquad \text{for every } z\neq 0,$$

we get

$$\left|\frac{z\operatorname{Re}(z)}{|z|} - 0\right| = \left|\frac{z\operatorname{Re}(z)}{|z|}\right| \le |z|\cdot 1 \to 0 \qquad \text{for } z\to 0,$$

and we conclude that this function can be extended continuously to $z=0$ with the value $f(0)=0$.

Example 4.3 *Check if the following limit values exist. In the case of existence, find the limit value:*

$$(a)\ \lim_{n\to+\infty}\frac{n!i^n}{n^n}, \qquad (b)\ \lim_{n\to+\infty} i^n, \qquad (c)\ \lim_{n\to+\infty} n\left(\frac{1+i}{n}\right)^2.$$

(a) We shall prove that the limit value is 0, thus

$$|a_n - 0| < \varepsilon \qquad \text{for every } n \ge N(\varepsilon).$$

This follows easily from the following trivial estimate

$$|a_n-0| = |a_n| = \left|\frac{n!\,i^n}{n^n}\right| = \frac{n!}{n^n} = \frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdots\frac{n-1}{n}\cdot\frac{n}{n} \le \frac{1}{n},$$

for $n\to+\infty$. This proves that the limit value exists and that

$$\lim_{n\to+\infty}\frac{n!i^n}{n^n} = 0.$$

(b) Since the sequence repeats cyclical the values i, -1, $-i$ and 2, the limit value does not exist.

(c) We shall prove that the limit value also in this case is 0, thus we shall prove that

$$|a_n - 0| = |a_n| \to 0 \qquad \text{for } n \to +\infty.$$

This follows from

$$|a_n| = \left| n \left(\frac{1+i}{n} \right)^n \right| = n \frac{(\sqrt{2})^n}{n^n} = \sqrt{2} \cdot \left(\frac{\sqrt{2}}{n} \right)^{n-1}.$$

If $n \geq 2$, er $\sqrt{2} < n$, then

$$\left(\frac{\sqrt{2}}{n} \right)^{n-1} \leq \left(\frac{\sqrt{2}}{n} \right)^1 = \frac{\sqrt{2}}{n}.$$

Hence, for $n \geq 2$,

$$|a_n - 0| = \sqrt{2} \cdot \left(\frac{\sqrt{2}}{n} \right)^{n-1} \leq \sqrt{2} \cdot \frac{\sqrt{2}}{n} = \frac{2}{n} \to 0 \qquad \text{for } n \to +\infty.$$

We conclude that the limit value exists and that

$$\lim_{n \to +\infty} n \left(\frac{1+i}{n} \right)^n = 0.$$

Example 4.4 *Check if* $\lim_{n \to +\infty} z_n$ *exists for any of the following sequences* (z_n)*, and in case of existence, find the limit value.*

$$(a)\;\; z_n = \frac{in}{n^2+i}, \quad (b)\;\; z_n = \exp\left(\frac{i\pi n}{4} \right), \quad (c)\;\; z_n = \left(\frac{1+i}{4} \right)^n, \quad (d)\;\; z_n = \exp\left(i \left\{ n^2 + n \right\} \pi \right).$$

We first note that z_n is defined for every $n \in \mathbb{N}$ in all four cases.

(a) The sequence $z_n = \dfrac{in}{n^2+i}$ converges towards the limit value 0. In fact,

$$0 \leq |z_n - 0| = |z_n| = \frac{n}{\sqrt{n^4+1}} \leq \frac{n}{\sqrt{n^4}} = \frac{n}{n^2} = \frac{1}{n} \to 0 \qquad \text{for } n \to +\infty.$$

(b) Since this sequence repeats cyclically the numbers

$$\frac{1+i}{\sqrt{2}}, \quad i, \quad \frac{-1+i}{\sqrt{2}}, \quad -1, \quad \frac{-1-i}{\sqrt{2}}, \quad -i, \quad \frac{1-i}{\sqrt{2}}, \quad 1,$$

it is divergent.

(c) The sequence $z_n = \left(\dfrac{1+i}{4} \right)^n$ converges towards the limit value 0. In fact,

$$0 \leq |z_n - 0| = |z_n| = \left(\frac{\sqrt{2}}{4} \right)^n \to 0 \qquad \text{for } n \to +\infty.$$

(d) Since $n^2+n=n(n+1)$ in an *even* number for every $n\in\mathbb{N}$, we see that

$$z_n=\exp\left(i\left\{n^2+n\right\}\pi\right)=\exp(2ip\pi)=1 \qquad \text{for every } n\in\mathbb{N},$$

where $2p=n(n+1)$.
The constant sequence is of course convergent with the limit value 1.

Example 4.5 *Check if* $\lim_{n\to+\infty} z_n$ *exists for any of the following sequences* (z_n)*, and find the limit value if it exists:*

$$(a)\quad \left(1+\frac{1}{n}\right)^{n^2}, \qquad (b)\quad \left(1+\frac{i}{n}\right)^{n^2}, \qquad (c)\quad \left|\left(1+\frac{i}{n}\right)^{n^2}\right|.$$

(a) It is well-known that

$$\left(1+\frac{1}{n}\right)^n \geq 1+\binom{n}{1}\frac{1}{n}=2,$$

so

$$\left(1+\frac{1}{n}\right)^{n^2}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^n \geq 2^n,$$

and we conclude that the sequence is divergent,

$$\left(1+\frac{1}{n}\right)^{n^2} \to +\infty \qquad \text{for } n \to +\infty.$$

(b) Since

$$\left|1+\frac{i}{n}\right|^{n^2} \geq 1 \qquad \text{for every } n,$$

any possible limit value cannot be 0. Since

$$\text{Arg}\left(1+\frac{i}{n}\right) = \text{ Arctan } \frac{1}{n},$$

we get

$$\arg\left(1+\frac{i}{n}\right)^{n^2} = \left\{n^2\text{Arctan } \frac{1}{n} + 2p\pi \mid p \in \mathbb{Z}\right\}.$$

Here,

$$n^2\text{Arctan } \frac{1}{n} = n^2\left\{\frac{1}{n} - \frac{1}{3}\cdot\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right\} = n - \frac{1}{3n} + o\left(\frac{1}{n}\right),$$

and it is almost obvious that the sequence with the elements $n^2\text{Arctan } \dfrac{1}{n}$ (mod 2π) does not have a limit value. Thus, $\text{Arg}\left(1+\dfrac{i}{n}\right)^{n^2}$ does not converge, and since the limit value cannot be zero for (z_n), we conclude that (z_n) is not convergent. (The angle is "turning" all the time).

c) Since

$$\left|\left(1+\frac{i}{n}\right)^{n^2}\right| = \left|1+\frac{i}{n}\right|^{n^2} = \left(\sqrt{1+\frac{1}{n^2}}\right)^{n^2} = \left(1+\frac{1}{n^2}\right)^{\frac{1}{2}n^2} = \left\{\left(1+\frac{1}{n^2}\right)^{n^2}\right\}^{\frac{1}{2}},$$

and

$$\left(1+\frac{1}{m}\right)^m \to e \qquad \text{for } m \to +\infty,$$

we conclude that

$$\lim_{n\to+\infty}\left|\left(1+\frac{i}{n}\right)^{n^2}\right| = \sqrt{e}.$$

Addition to (b) If we instead consider

$$z_n = \left(1+\frac{2\pi i}{n}\right)^{n^2}, \qquad n \in \mathbb{N},$$

then the sequence becomes *convergent*. In fact, since $n^2 \in \mathbb{N}$, we get

$$\begin{aligned}
|z_n| &= \left|1+\frac{2\pi i}{n}\right|^{n^2} = \left(\sqrt{1+\frac{4\pi^2}{n^2}}\right)^{n^2} = \left(1+\frac{4\pi^2}{n^2}\right)^{\frac{1}{2}\,n^2} \\
&= \left\{\left(1+\frac{1}{\frac{n^2}{4\pi^2}}\right)^{\frac{n^2}{4\pi^2}}\right\}^{2\pi^2} \to \exp\left(2\pi^2\right) \qquad \text{for } n \to +\infty,
\end{aligned}$$

and if we put $z_n = |z_n|\,\exp(i\theta_n)$, then

$$\begin{aligned}
\theta_n &= n^2 \text{Arg}\left(1+\frac{2\pi i}{n}\right) = n^2 \text{Arctan}\left(\frac{2\pi}{n}\right) \\
&= n^2\left\{\frac{2\pi}{n} - \frac{1}{3}\cdot\frac{8\pi^4}{n^3} + o\left(\frac{1}{n^3}\right)\right\} = 2\pi n - \frac{8\pi^2}{3}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right),
\end{aligned}$$

hence

$$\begin{aligned}
\exp(i\theta_n) &= \cos\theta_n + i\,\sin\theta_n = \cos\left(2\pi n - \frac{8\pi^2}{3}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right)\right) + i\,\sin\left(\pi n - \frac{8\pi^2}{3}\cdot\frac{1}{n} + o\left(\frac{1}{n}\right)\right) \\
&\to 1 \qquad \text{for } n \to +\infty.
\end{aligned}$$

Then by some rules of calculation for sequences,

$$|z_n|\,\exp(i\theta_n) \to \exp\left(2\pi^2\right) \qquad \text{for } n \to +\infty,$$

thus

$$\lim_{n\to+\infty}\left(1+\frac{2\pi i}{n}\right)^{n^2} = \exp\left(2\pi^2\right).$$

5 Line integrals

Example 5.1 *Let C be a simple, closed curve surrounding a domain Ω in the (x, y)-plane of the area S. Prove that*

$$(a)\quad \oint_C x\,dz = i\,S, \qquad (b)\quad \oint_C y\,dz = -S, \qquad (c)\quad \oint_C \overline{z}\,dz = 2i\,S.$$

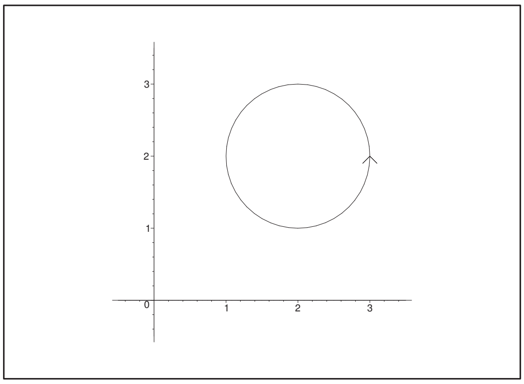


Figure 27: Example of a curve C, which surrounds a (bounded) domain Ω.

First it follows by a consideration of the figure that

$$\oint_C x\,dy = -\oint_C y\,dx = S.$$

It follows from

$$\oint_C x\,dx = \oint_C y\,dy = 0,$$

that

(a)

$$\oint_C x\,dz = \oint_C x\,dx + i\oint_C x\,dy = i\,S,$$

(b)

$$\oint_C y\,dz = \oint_C y\,dx + i\oint_C y\,dy = -S,$$

(c)

$$\begin{aligned}\oint_C \overline{z}\,dz &= \oint_C (x - iy)\,(dx + i\,dy) = \oint_C x\,dx - i\oint_C y\,dx + i\oint_C x\,dy + \oint_C y\,dy \\ &= 0 + i\,S + i\,S + 0 = 2i\,S.\end{aligned}$$

Example 5.2 *Let C denote the circle $|z| = 1$. Find*

$$(a)\ \oint_C \frac{dz}{z}, \quad (b)\ \oint_C \frac{dz}{|z|}, \quad (c)\ \oint_C \frac{dz}{z^2}, \quad (d)\ \oint_C \frac{dz}{|z^2|}.$$

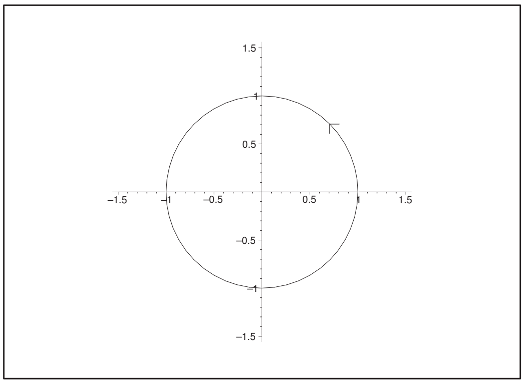


Figure 28: The circle C with direction of circulation.

We shall everywhere use the parametric description

$$z = e^{it}, \qquad t \in [0, 2\pi],$$

of the curve C. Then

$$dz = i\,e^{it}\,dt,$$

and we get:

(a)

$$\oint_C \frac{dz}{z} = \int_0^{2\pi} \frac{i\,e^{it}}{e^{it}}\,dt = i\int_0^{2\pi} dt = 2\pi i,$$

(b)

$$\oint_C \frac{dz}{|z|} = \int_0^{2\pi} \frac{i\,e^{it}}{|e^{it}|}\,dt = \int_0^{2\pi} i\,e^{it}\,dt = \left[e^{it}\right]_0^{2\pi} = 1-1=0,$$

(c)

$$\oint_C \frac{dz}{z^2} = \int_0^{2\pi} \frac{i\,e^{it}}{e^{2it}}\,dt = \int_0^{2\pi} i\,e^{-it}\,dt = \left[-e^{-it}\right]_0^{2\pi} = -1-(-1)=0,$$

(d)

$$\oint_C \frac{dz}{|z^2|} = \int_0^{2\pi} \frac{i\,e^{it}}{|e^{2it}|}\,dt = \int_0^{2\pi} i\,e^{it}\,dt = \left[e^{it}\right]_0^{2\pi} = 1-1=0.$$

Example 5.3 *Find the value of the complex line integral $\int_C |z|\,dz$, when the curve C is*

(a) *the line segment from $-i$ to i,*

(b) *the left half of the unit circle run through from $-i$ to i,*

(c) *the right half of the unit circle run through from $-i$ to i.*

(a) The parametric description is here

$$z = i\,y, \qquad y \in [-1,1],$$

so by insertion,

$$\int_C |z|\,dz = \int_{-1}^{1} |y|\cdot i\,dy = 2i\int_0^1 y\,dy = 2i\left[\frac{y^2}{2}\right]_0^1 = i.$$

(b) The left half of the unit circle from $-i$ to i has e.g. the parametric description

$$z = e^{-\theta}, \qquad \theta \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right],$$

hence

$$\int_C |z|\,dz = \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \left|e^{-i\theta}\right|(-i)e^{-i\theta}\,d\theta = \left[e^{-\theta}\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} = i-(-i) = 2i.$$

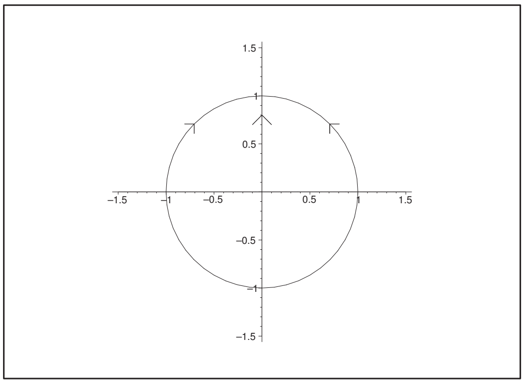


Figure 29: The three curves C from $-i$ to i with their directions.

(c) The right half of the unit circle from $-i$ to i has e.g. the parametric description

$$z = e^{i\theta}, \qquad \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right],$$

hence by insertion,

$$\int_C |z|\, dz = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} i\, e^{i\theta}\, d\theta = \left[e^{i\theta}\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = i - (-i) = 2i.$$

Remark 5.1 Clearly, the value of the line integral depends in this case of the path of integration. $\Diamond$

Example 5.4 *Compute* $\int_C \frac{1}{z}\, dz$*, where* C *denotes the curve with the parametric description*

$$z(t) = 2\,\cos t + 2i\,\sin t, \qquad t \in \left[0, \frac{\pi}{2}\right].$$

By insertion of the parametric description

$$z(t) = 2\,\cos t + i\,2\,\sin t = 2\,e^{it}, \qquad t \in \left[0, \frac{\pi}{2}\right],$$

we get

$$\int_C \frac{1}{z}\, dz = \int_0^{\frac{\pi}{2}} \frac{1}{2\,e^{it}} \cdot 2i\,e^{it}\, dt = i \int_0^{\frac{\pi}{2}} dt = \frac{i\pi}{2}.$$

Example 5.5 *Compute the line integral* $\int_C \left(x^2 + i\,y^3\right)\, dz$*, where* C *is the straight line segment from* $z = 1$ *to* $z = i$*.*

A parametric description of C is given by

$$x(t) = 1 - t, \qquad y(t) = t, \qquad t \in [0, 1].$$

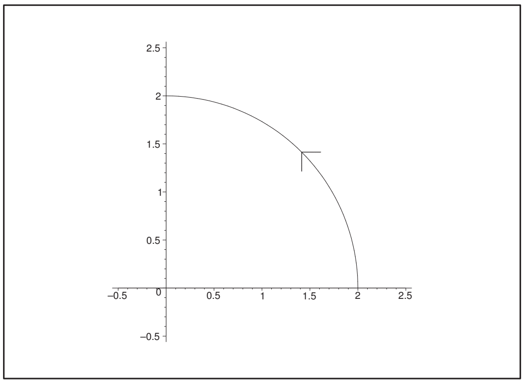


Figure 30: the path of integration C.

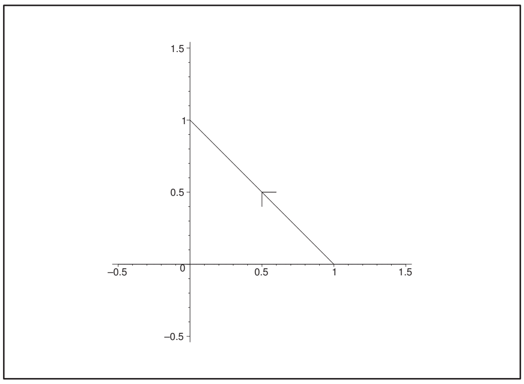


Figure 31: The path of integration C with its direction.

Hence, by insertion

$$\begin{aligned}\int_C \left(x^2+i\,y^3\right)\,dz &= \int_0^1 \left\{x(t)^2+i\,y(t=^3\right\}\cdot\left\{x'(t)+i\,y'(t)\right\}\,dt \\ &= \int_0^1 \left\{(1-t)^2+it^3\right\}\cdot(-1+i)\,dt \\ &= \int_0^1 \left\{-(1-t)^2-t^3\right\}\,dt+i\int_0^1 \left\{(1-t)^2-t^3\right\}\,dt \\ &= \left[-\frac{(t-1)^3}{3}-\frac{t^4}{4}\right]_0^1+i\left[\frac{(t-1)^3}{3}-\frac{t^4}{4}\right]_0^1 \\ &= \left(-\frac{1}{4}-\frac{1}{3}\right)+i\left(-\frac{1}{4}+\frac{1}{3}\right)=-\frac{7}{12}+\frac{1}{12}\,i.\end{aligned}$$

ALTERNATIVELY one may apply the following variant:

$$\begin{aligned}
\int_C \left(x^2 + i\,y^3\right)\,dz &= \int_0^1 \left\{(1-t)^2 + i\,t^3\right\} \cdot (-1+i)\,dt \\
&= (-1+i)\int_0^1 \left\{(t-1)^2 + i\,t^3\right\}\,dt \\
&= (-1+i)\left[\frac{(t-1)^3}{3} + i\,\frac{t^4}{4}\right]_0^1 = (-1+i)\left(\frac{1}{3} + \frac{1}{4}\,i\right) \\
&= -\frac{1}{3} - \frac{1}{4} + i\left(-\frac{1}{4} + \frac{1}{3}\right) = -\frac{7}{12} + \frac{1}{12}\,i.
\end{aligned}$$

Remark 5.2 This is not a so-called exact differential form, so the value of the line integral depends of the path of integration. $\Diamond$

Example 5.6 *Compute the line integral* $\int_C \left\{\left(x^2-y^2\right) dy - 2xy\, dy\right\}$*, where* C *denotes the straight line segment from* $1+i$ *to* $3+2i$*.*

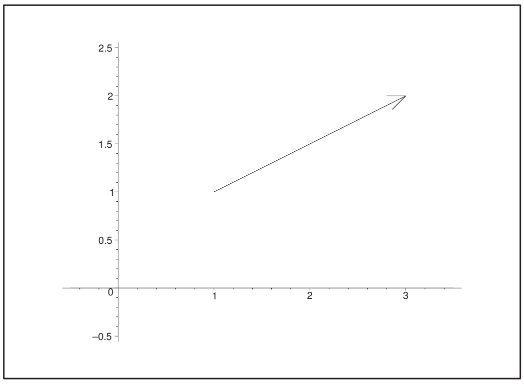


Figure 32: The line segment C from $1+i$ to $3+2i$.

Here we have at least three different variants:

1) *The "standard" method.* The curve C has the parametric description

$$z(t) = 1+i+(2+i)t, \qquad t \in [0,1],$$

hence by separation into real and imaginary parts,

$$x(t) = 1+2t \quad \text{and} \quad y(t) = 1+t \qquad \text{for } t \in [0,1].$$

Hence by insertion,

$$\begin{aligned}\int_C \left\{\left(x^2-y^2\right) dx - 2xy\, dy\right\} &= \int_0^1 \left\{\left((1+2t)^2-(1+t)^2\right)\cdot 2\, dt - 2(1+2t)(1+t)dt\right\}\\ &= \int_0^1 \left\{2(2+3t)t - 2\left(1+3t+2t^2\right)\right\} dt = 2\int_0^1 \left\{2t+3t^2-1-3t-2t^2\right\}\, dt\\ &= 2\int_0^1 \left(t^2-t-1\right) dt = 2\left(\frac{1}{3}-\frac{1}{2}-1\right) = \frac{2}{3}-3 = \frac{7}{3}.\end{aligned}$$

2) *Exact differential form.* The differential form under the integral is a (real) exact differential form, which can be seen by

$$\frac{\partial}{\partial y}\left(x^2-y^2\right) = -2y = \frac{\partial}{\partial x}(-2xy),$$

and the form is defined in the simply connected domain $\mathbb{R}^2$. Hence, the value can be found by some clever manipulation of the integrand,

$$\begin{aligned}
&\int_C \left\{\left(x^2-y^2\right)dx-2xy\,dy\right\}\\
&\qquad=\int_C\left\{x^2dx-\left(y^2dx+x\,d\left(y^2\right)\right)\right\}=\int_C d\left\{\frac{1}{3}\,x^3-xy^2\right\}=\left[\frac{1}{3}\,x^3-xy^2\right]_{(x,y)=(1,1)}^{(3,2)}\\
&\qquad=\left(\frac{1}{3}\cdot 3^3-3\cdot 2^2\right)-\left(\frac{1}{3}-1\right)=9-12-\frac{1}{3}+1=-2-\frac{1}{3}=-\frac{7}{3}.
\end{aligned}$$

3) *Complex Functions.* An even easier variant is to notice the connection with the Theory of Complex Functions, because

$$\begin{aligned}
&\int_C\left\{\left(x^2-y^2\right)dx-2xy\,dy\right\}=\operatorname{Re}\int_{1+i}^{3+2i}z^2dz=\operatorname{Re}\left\{\frac{1}{3}\,(3+2i)^3-\frac{1}{3}\,(1+i)^3\right\}\\
&\qquad=\frac{1}{3}\operatorname{Re}\{27+54i-36-8i-(1+3i-3-i)\}=\frac{1}{3}\,(27-36-1+3)=-\frac{7}{3}.
\end{aligned}$$

Example 5.7 *Sketch the curve C of the parametric description $z=1+it$, $t\in[0,1]$, and indicate its direction.*
Then compute

$$(a)\quad\int_C 4z^3dz,\qquad (b)\quad\int_C \overline{z}\,dx,\qquad (c)\quad\int_C\frac{1}{z}\,dz.$$

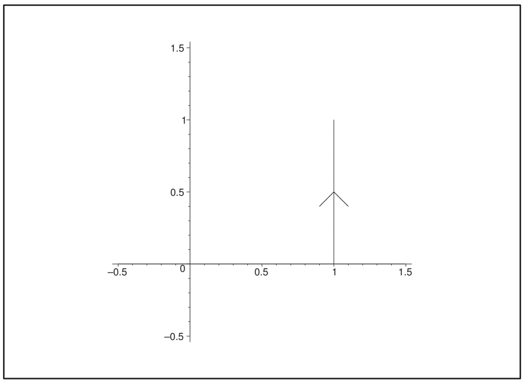


Figure 33: The curve C with its direction.

(a) We get by the Theory of Complex Functions that

$$\int_C 4z^3dz=\left[z^4\right]_1^{1+i}=(1+i)^4-1^4=-4-1=-5.$$

ALTERNATIVELY it follows by a computation of the corresponding line integral,

$$\begin{aligned}\int_C 4z^3 dz &= \int_0^1 4(1+it)^3 i\,dt = 4i\int_0^1 \left(1+3it-3t^2-i\,t^3\right) dt = 4i\left[t+\frac{3i}{2}\,t^2-t^3-\frac{i}{4}\,t^4\right]_0^1 \\ &= 4i\left\{1+\frac{3}{2}\,i-1-\frac{1}{4}\,i\right\} = i(6i-i) = -5.\end{aligned}$$

(b) Here we cannot apply the Theory of Complex Functions. Instead we insert the parametric description of C to get

$$\int_C \overline{z}\,dz = \int_0^1 (1-i\,t)i\,dt = \int_0^1 (i+t)\,dt = \left[it+\frac{t^2}{2}\right]_0^1 = \frac{1}{2}+i.$$

(c) Using the Theory of Complex Functions we get

$$\int_C \frac{1}{z}\,dz = [\mathrm{Log}\;z]_1^{1+i} = \;\mathrm{Log}(1+i) - \;\mathrm{Log}\;1 = \;\mathrm{Log}(1+i) = \frac{1}{2}\,\ln 2 + i\,\frac{\pi}{4}.$$

ALTERNATIVELY we get by the parametric description that

$$\begin{aligned}\int_C \frac{1}{z}\,dz &= \int_0^1 \frac{i}{1+it}\,dt = \int_0^1 \frac{i(1-it)}{1+t^2}\,dt = i\int_0^1 \frac{1}{1+t^2}\,dt + \int_0^1 \frac{t}{1+t^2}\,dt \\ &= i\,[\mathrm{Arctan}\;t]_0^1 + \frac{1}{2}\,\left[\ln\left(1+t^2\right)\right]_0^1 = \frac{1}{2}\,\ln 2 + i\,\frac{\pi}{4}.\end{aligned}$$

Example 5.8 *Sketch the curve C of the parametric description $z = e^{-i\pi t}$, $t \in [0,1]$, and indicate its orientation. Then compute*

$$(a) \quad \int_C 4z^3 dz, \qquad (b) \quad \int_C \overline{z}\, dz, \qquad (c) \quad \int_C \frac{1}{c}\, dz.$$

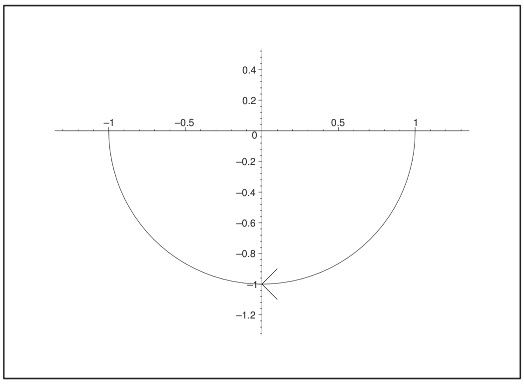


Figure 34: The curve C with its orientation.

(a) By using the Theory of Complex Functions we get

$$\int_C 4z^3\, dz = \left[z^4\right]_1^{-1} = (-1)^4 - 1^4 = 0.$$

Alternatively we apply the parametric description

$$\int_C 4z^3 dz = \int_0^1 4\, e^{-3i\pi t} \cdot (-i\pi) e^{-i\pi t}\, dt = \int_0^1 (-4i\pi) e^{-4i\pi t}\, dt = \left[e^{-4i\pi t}\right]_0^1 = 1 - 1 = 0.$$

(b) By insertion of the parametric description we get

$$\int_C \overline{z}\, dz = \int_0^1 e^{+i\pi t} \cdot (-i\pi) e^{-\pi t}\, dt = -i\pi.$$

(c) By insertion of the parametric description we get

$$\int_C \frac{1}{z}\, dz = \int_0^1 \frac{1}{e^{-i\pi t}}\, (-i\pi) e^{-i\pi t}\, dt = -i\pi.$$

Example 5.9 *Sketch the curve C of the parametric description $z = 3\,e^{2\pi i t}$, $t \in [0,1]$, and indicate its orientation. Then compute*

$$(a)\quad \int_C 4z^3 dz, \qquad (b)\quad \int_C \overline{z}\,dz, \qquad (c)\quad \int_C \frac{1}{z}\,dz.$$

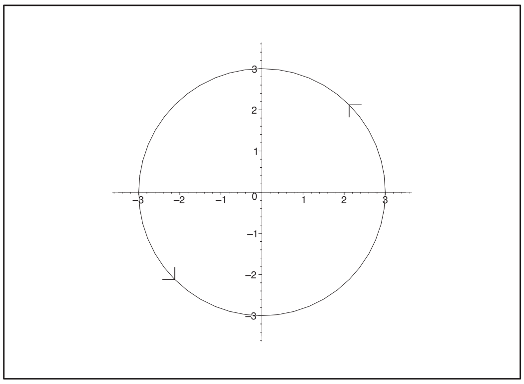


Figure 35: The curve C and its orientation.

(a) By using the Theory of Complex Functions we get

$$\int_C 4z^3 dz = \left[z^4\right]_3^3 = 0.$$

Alternatively it follows by using the parametric description,

$$\int_C 4z^3 dz = \int_0^1 4 \cdot 3^3 \cdot e^{6i\pi t} \cdot 3 \cdot 2i\pi \cdot e^{2i\pi t}\,dt = 3^4 \int_0^1 8i\pi \cdot e^{8i\pi t}\,dt = \left[81 \cdot e^{8i\pi t}\right]_0^1 = 0.$$

(b) We get by insertion of the parametric description that

$$\int_C \overline{z}\,dz = \int_0^1 3\,e^{-2i\pi t} \cdot 3 \cdot 2i\pi \cdot e^{2i\pi t}\,dt = 18\pi i.$$

(c) We get by insertion of the parametric description that

$$\int_C \frac{1}{z}\,dz = \int_0^1 \frac{1}{3\,e^{2i\pi t}} \cdot 3 \cdot 2i\pi\,e^{2i\pi t}\,dt = 2i\pi.$$

Example 5.10 *Sketch the curve C of the parametric description $z = e^{4i\pi t}$, $t \in [0,1]$, and indicate its orientation. Then compute*

$$(a)\quad \int_C 4z^3\,dz, \qquad (b)\quad \int_C \overline{z}\,dz, \qquad (c)\quad \int_C \frac{1}{z}\,dz.$$

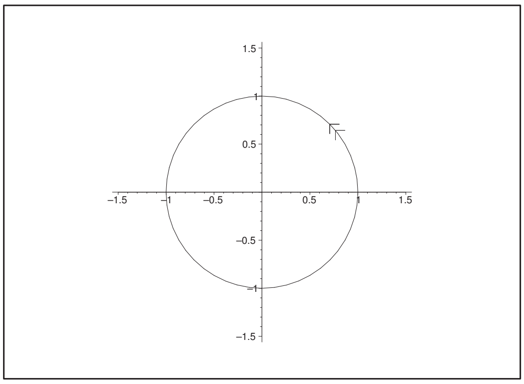


Figure 36: The curve C with its (double) orientation.

The curve C is the unit circle circulated twice in the positive direction.

(a) By using the Theory of Complex Functions we get

$$\int_C 4z^3 dz = \left[z^4\right]_1^1 = 0.$$

Alternatively it follows by insertion of the parametric description,

$$\int_C 4z^3 dz = \int_0^1 4\,e^{12\pi it}\cdot 4\pi i\,e^{4\pi it}dt = \int_0^1 16\pi i\,e^{16\pi it}dt = \left[e^{16\pi it}\right]_0^1 = 0.$$

(b) By insertion of the parametric description we get

$$\int_C \overline{z}\,dz = \int_0^1 e^{-4\pi it}\cdot 4\pi i\,e^{4\pi it}dt = 4\pi i.$$

(c) We get by insertion of the parametric description

$$\int_C \frac{1}{z}\,dz = \int_0^1 \frac{1}{e^{4\pi i}}\cdot 4\pi i\,e^{4\pi it}dt = 4\pi i.$$

Alternatively we notice that $\frac{1}{z} = \overline{z}$ for $|z| = 1$, because then $z \cdot \overline{z} = 1$. The result must be the same as in (c).

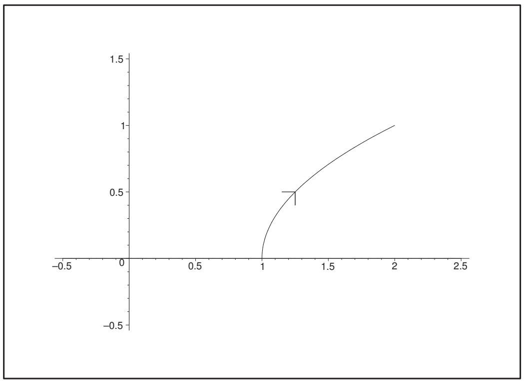


Figure 37: The curve C and its orientation.

Example 5.11 *Sketch the curve C of parametric description $z = 1 + it + t^2$, $t \in [0, 1]$, and indicate its orientation. Then compute*

$$(a)\quad \int_C 4z^3\,dz, \qquad (b)\quad \int_C \overline{z}\,dz, \qquad (c)\quad \int_C \frac{1}{z}\,dz.$$

The curve is part of a parabolic arc of vertex 1.

(a) We get by the Theory of Complex Functions that

$$\int_C 4z^3 dz = \left[z^4\right]_1^{2+i} = (2+i)^4 - 1 = (3+4i)^2 - 1 = -7 + 24i - 1 = -8 + 24i.$$

Alternatively (and not so smart) we insert the parametric description of the curve. Then we have the following computation,

$$\begin{aligned}
\int_C 4z^3 dz &= \int_0^1 4\left(1^+t^2+it\right)^3 (i+2t)dt \\
&= \int_0^1 4\left\{\left(1+t^2\right)^3+3it\left(1+t^2\right)^2-3t^2\left(1+t^2\right)-it^3\right\}(2t+i)dt \\
&= 4\int_0^1 \left\{\left(t^6+3t^4+3t^2+1-3t^4-3t^2\right)+i\left(3t^5+6t^3+3t-t^3\right)\right\}(2t+i)dt \\
&= 4\int_0^1 \left\{\left(2t^7+2t-3t^5-5t^3-3t\right)+i\left(t^6+1+6t^6+10t^4+6t^2\right)\right\}dt \\
&= 4\left[\left(\frac{1}{4}\,t^8-\frac{1}{2}\,t^6-\frac{5}{4}\,t^4-\frac{1}{2}\,t^2\right)+i\left(t^7+2t^5+2t^3+t\right)\right]_0^1 \\
&= 1-2-5-2+4i(1+2+2+1) = -8+24i.
\end{aligned}$$

(b) When we insert the parametric description we get

$$\begin{aligned}
\int_C \overline{z}\,dz &= \int_0^1 \left(1+t^2-it\right)(2t+i)\,dt = \int_0^1 \left\{\left(2t^3+2t+t\right)+i\left(t^2+1-2t^2\right)\right\}\,dt \\
&= \left[\frac{1}{2}\,t^4+\frac{3}{2}\,t^2+i\left(-\frac{1}{3}\,t^3+t\right)\right]_0^1 = 2+\frac{2}{3}\,i.
\end{aligned}$$

(c) Since Log z is an integral of $\dfrac{1}{z}$ in the right half plane, we get

$$\int_C \frac{1}{z}\,dz = [\text{Log } z]_1^{2+i} = \text{Log}(2+i) = \frac{1}{2}\ln 5 + i\,\text{Arctan}\,\frac{1}{2}.$$

ALTERNATIVELY (and less elegant) we insert the parametric description of the curve. Then

$$\begin{aligned}
\int_C \frac{1}{z}\,dz &= \int_0^1 \frac{2t+i}{t^2+1+it}\,dt = \int_0^1 \frac{t^2+1-it}{\left(t^2+1\right)^2+t^2}\cdot(2t+i)\,dt = \int_0^1 \frac{2t^3+3t+i\left(-t^2+1\right)}{t^4+3t^2+1}\,dt \\
&= \frac{1}{2}\int_0^1 \frac{4t^3+6t}{t^4+3t^2+1}\,dt + i\int_0^1 \frac{-t^2+1}{\left(t^2+\frac{3+\sqrt{5}}{2}\right)\left(t^2+\frac{3-\sqrt{5}}{2}\right)}\,dt \\
&= \frac{1}{2}\left[\ln\left(t^4+3t^2+1\right)\right]_0^1 - i\int_0^1 \left\{\frac{\frac{1+\sqrt{5}}{2}}{t^2+\frac{3+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2}}{t^2+\frac{3-\sqrt{5}}{2}}\right\}dt \\
&= \frac{1}{2}\ln 5 - i\left\{\frac{1+\sqrt{5}}{2}\cdot\sqrt{\frac{2}{3+\sqrt{5}}}\,\text{Arctan}\,\sqrt{\frac{2}{3+\sqrt{5}}} + \frac{1-\sqrt{5}}{2}\,\text{Arctan}\,\sqrt{\frac{2}{3-\sqrt{5}}}\right\}.
\end{aligned}$$

We stop the computations at this point because they should only serve as an illustration of the fact that in the Theory of Complex Functions it is worth always to look for alternatives which might be easier to apply. It is of course possible to reduce the computations above to the result

$$\frac{1}{2}\ln 5 + i\,\text{Arctan}\,\frac{1}{2},$$

by starting the reduction by

$$\sqrt{\frac{2}{3+\sqrt{5}}} = \sqrt{\frac{4}{6+2\sqrt{5}}} = \sqrt{\left(\frac{2}{\sqrt{5}+1}\right)^2} = \frac{2}{\sqrt{5}+1},$$

and analogously.

Example 5.12 *Compute $\int_C (z+1)dz$, where C is that part of the parabola of the equation $y = x^2$, which starts at $z = 0$ and ends at $z = 1 + i$.*

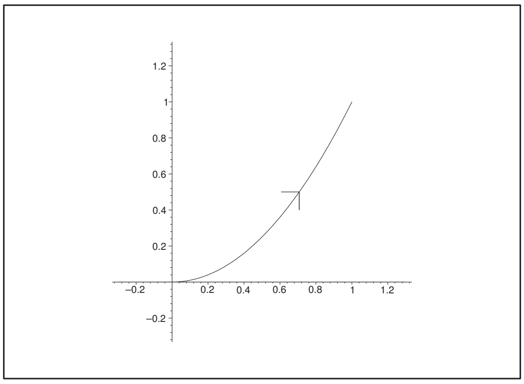


Figure 38: The curve C and its orientation.

If we use the parametric description

$$x = t, \qquad y = t^2, \qquad t \in [0,1],$$

it follows by insertion and computation that

$$\begin{aligned}\int_C (z+1)\,dz &= \int_0^1 \left(t + it^2 + 1\right)(1+2it)\,dt = \int_0^1 \left\{\left(1+t-2t^3\right) + i\left(t^2+2t+2t^2\right)\right\}\,dt \\ &= \left[t + \frac{t^2}{2} - \frac{t^4}{2} + i\left(t^3+t^2\right)\right]_0^1 = 1 + \frac{1}{2} - \frac{1}{2} + i(1+1) = 1+2i.\end{aligned}$$

Alternatively and a lot smarter we immediately see that $\dfrac{z^2}{2} + z$ is an integral. (Check this, i.e. differentiate!) Then

$$\int_C (z+1)\,dz = \left[\frac{z^2}{2} + z\right]_0^{1+i} = \frac{2i}{2} + 1 + i = 1 + 2i.$$

Example 5.13 *Find the value of the complex line integral*

$$\int_C z\,\exp\left(z^2\right)\,dz,$$

when C is

(a) *the straight line segment from $z = i$ to $z = -i+2$,*

(b) *the arc from $z = 0$ to $z = 1+i$ of the parabola of the equation $y = x^2$.*

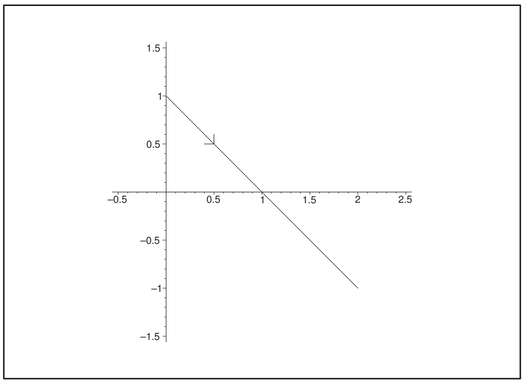


Figure 39: The curve of (a) and its orientation.

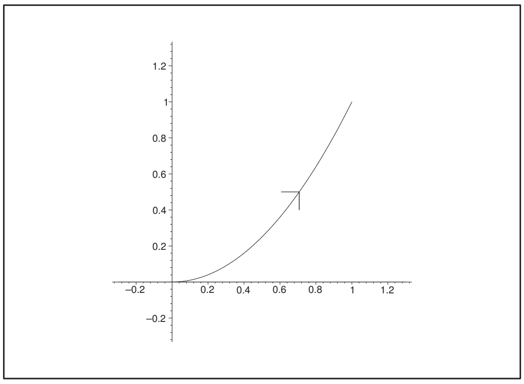


Figure 40: The curve of (b) and its orientation.

A simple check shows that $\frac{1}{2}\,\exp\left(z^2\right)$ is an integral of $z\,\exp\left(z^2\right)$. Hence

(a)

$$\begin{aligned}\int_C z\,\exp\left(z^2\right)\,dz &= \left[\frac{1}{2}\,\exp\left(z^2\right)\right]_i^{-i+2} = \frac{1}{2}\left\{\exp\left((-i+2)^2\right) - \exp(-1)\right\}\\ &= \frac{1}{2}\left\{e^3\cos 4 - \frac{1}{e} - i\,e^3\sin 4\right\}.\end{aligned}$$

(b) Analogously,

$$\int_C z\,\exp\left(z^2\right)\,dz = \left[\frac{1}{2}\,\exp\left(z^2\right)\right]_0^{1+i} = \frac{1}{2}\,\left\{e^{2i}-1\right\} = \frac{1}{2}\,\{\cos 2 - 1 + i\,\sin 2\}.$$

Example 5.14 *Let C be the curve of the parametric description*

$$x(t)=\cos t, \qquad y(t)=\sin t, \qquad t\in[0,\pi].$$

Prove that

$$\left|\int_C \frac{e^z}{z}\,dz\right| \le \pi\,e.$$

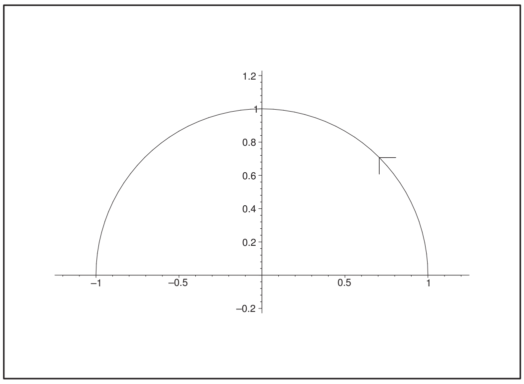


Figure 41: The curve C and its orientation.

The curve C is just a half circle in the upper half plan of radius 1 and centrum 0. It is well-known that its length is $L=\pi$.

If $z\in C$, then $|z|=\sqrt{x^2+y^2}=1$. In particular, $x\le 1$ on C, so we obtain the estimate

$$\left|\frac{e^z}{z}\right| = \frac{1}{|z|}\cdot\left|e^x\cdot e^{iy}\right| = \frac{1}{1}\cdot e^x\left|e^{iy}\right| = e^x \le e = M.$$

Then we have the estimate

$$\left|\int_C f(z)\,dz\right| = \left|\int_C \frac{e^z}{z}\,dz\right| \le M\cdot L = \pi\,e.$$

Remark 5.3 It can be proved that the exact value can be written

$$\int_C \frac{e^z}{z}\,dz = -\sum_{n=0}^{+\infty}\frac{2}{(2n+1)\cdot(2n+1)!} + i\pi.$$

However, this cannot be proved at the mathematical level of this book. $\Diamond$

Example 5.15 *It is possible the compute the integrals*

$$I_1 = \int_1^{+\infty} \frac{\cos x}{x + \frac{1}{x}}\, dx \qquad og \qquad I_2 = \int_1^{+\infty} \frac{sin x}{x + \frac{1}{x}}\, dx$$

numerically, though this is not an easy task, because the integrands decrease very slowly with increasing x. Apply the Theory of Complex Function, such that I_1 and I_2 are rewritten as integrals with fast decreasing integrands, such that it is easy to perform a numerical computation.

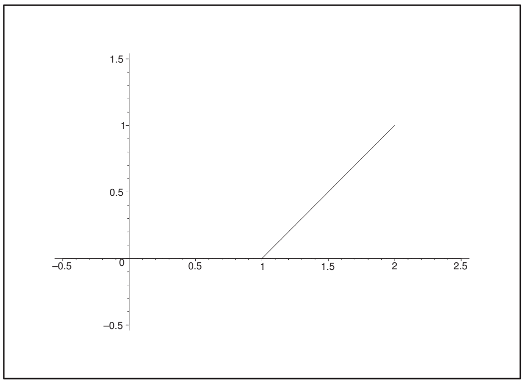


Figure 42: The path of integration C from 1 to $A + iB$.

Consider the complex line integral

$$I(A,B) = \int_C \frac{e^{iz}}{z + \frac{1}{z}}\, dz,$$

where C denotes the straight line segment from $z = 1$ to $z = A + iB$, where $A \geq 1$ and $B \geq 0$. In particular, we define

$$I(\infty, 0) = \lim_{A \to +\infty} I(A, 0) \qquad and \qquad I(1, \infty) = \lim_{B \to +\infty} I(1, B).$$

(a) *Prove that*

$$I(\infty, 0) = I_1 + i\, I_2.$$

(b) *Explain why*

$$I(\infty, 0) = I(1, \infty).$$

HINT: *Apply that* Jordan's lemma *for a half circle C_R^+ in the upper half plane also holds for a part of a half plane in the upper half plane.*

(c) *Use a real variable of integration t, where $z = \zeta(t) = 1 + i\, t$, in order to prove that one can write $I(1, \infty)$ in the form*

$$I(1, \infty) = e^i \int_0^{+\infty} e^{-t} \left\{ F_1(t) + i\, F_2(t) \right\} dt,$$

and find the real functions $F_1(t)$ and $F_2(t)$.

(a) If $B = 0$, then

$$I(A,0) = \int_C \frac{e^{iz}}{z + \frac{1}{z}}\, dz = \int_1^A \frac{e^{ix}}{x + \frac{1}{x}}\, dx = \int_0^A \frac{\cos x}{x + \frac{1}{x}}\, dx + i \int_1^A \frac{\sin x}{x + \frac{1}{x}}\, dx.$$

It follows that the limit $A \to +\infty$ gives (conditionally) convergent integrals and that

$$I(\infty,0) = \int_0^{+\infty} \frac{\cos x}{x + \frac{1}{x}}\, dx + i \int_0^{+\infty} \frac{\sin x}{x + \frac{1}{x}}\, dx = I_1 + i\, I_2.$$

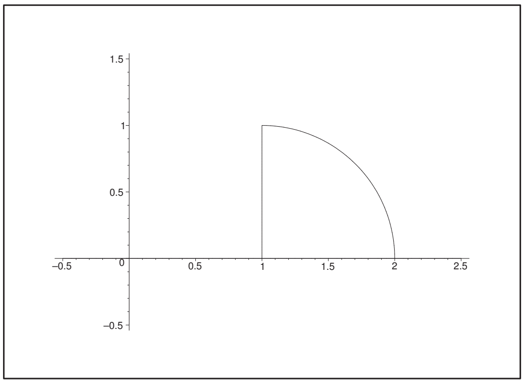


Figure 43: The curve C_1 is composed of the quarter circle of radius A and centrum $(1,0)$ and the corresponding line segments to centrum.

(b) Let C_1 be the closed curve on the figure. Since the integrand is analytic inside and on C_1, we have

$$\oint_{C_1} \frac{e^{iz}}{z+\frac{1}{z}}\,dz = 0.$$

Along the quarter circle we use the parametric description

$$\Gamma: \quad \zeta(t) = 1 + A\cdot e^{it}, \qquad t \in \left[0\,,\,\frac{\pi}{2}\right],$$

hence we get the estimate

$$\begin{aligned}\left|\int_\Gamma \frac{e^{iz}}{z+\frac{1}{z}}\,dz\right| &\le \int_0^{\frac{\pi}{2}} \frac{\left|\exp\left(i\left(1+A\,e^{it}\right)\right)\right|}{A-\frac{1}{A}}\cdot A\,dt \le \frac{A^2}{A^2-1}\int_0^{\frac{\pi}{2}} \exp(-A\,\sin t)\,dt \\ &\le \frac{A^2}{A^2-1}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{2}{\pi}\,A\cdot t\right)\,dt \le \frac{A^2}{A^2-1}\cdot\frac{\pi}{2}\cdot\frac{1}{A}\left\{1-e^{-A}\right\} \to 0 \text{ for } A \to +\infty,\end{aligned}$$

where we have used that

$$\frac{2}{\pi} \le \sin t \le t \qquad \text{for } t \in \left[0\,,\,\frac{\pi}{2}\right],$$

which follows easily by considering the graph of $\sin t$ in the given interval. This proves that

$$-\sin t \le -\frac{2}{\pi}\,t \qquad \text{for } t \in \left[0\,,\,\frac{\pi}{2}\right],$$

which was used in the estimates above.

Since

$$0 = \oint_{C_1} \frac{e^{iz}}{z+\frac{1}{z}}\,dz = I(A+1,0) + \int_\Gamma \frac{e^{iz}}{z+\frac{1}{z}}\,dz - I(1,A),$$

it follows by a rearrangement that

$$I(A+1,0) = I(1,A) - \int_\Gamma \frac{e^{iz}}{z+\dfrac{1}{z}}\,dz,$$

and hence by taking the limit,

$$I(\infty,0) = I(1,\infty).$$

(c) Finally, we apply the parametric description

$$z = \zeta(t) = 1 + i\,t, \qquad t \in [0,+\infty[,$$

in our computation of $I(1,\infty)$, thus

$$\begin{aligned} I(1,\infty) &= \int_0^\infty \frac{\exp(i(1+it))}{1+it+\dfrac{1}{1+it}}\,i\,dt = \int_0^\infty \frac{(1+it)\exp(-t+i)}{(1+it)^2+1}\,i\,dt \\ &= i\cdot e^i \int_0^\infty \frac{(1+it)\cdot e^{-t}}{2-t^2+2it}\,dt, \end{aligned}$$

where

$$\begin{aligned} \tilde{F}_1(t) + i\,\tilde{F}_2(t) &= i\cdot e^i \cdot \frac{1+it}{2-t^2+2it}\cdot\frac{2-t^2-2it}{2-t^2-2it} \\ &= (-\sin 1 + i\cdot\cos 1)\frac{2-t^2+2t^2+i\left\{2t-t^3-2t\right\}}{\left(2-t^2\right)^2+4t^2} \\ &= (-\sin 1 + i\,\cos 1)\cdot\frac{t^2+2-i\,t^3}{4+t^4} \\ &= \frac{1}{4+t^4}\left\{-\left(t^2+2\right)\sin 1 + t^3\cos 1 + i\left(\left(t^2+2\right)\cos 1 + t^3\sin 1\right)\right\}. \end{aligned}$$

By separation into real and imaginary part we therefore obtain

$$\tilde{F}_1(t) = \frac{t^3\cos 1 - \left(t^2+2\right)\sin 1}{t^4+4} \quad \text{og} \quad \tilde{F}_2(t) = \frac{t^3\sin 1 + \left(t^2+1\right)\cos 1}{t^4+4},$$

hence

$$I_1 = \int_1^{+\infty} \frac{\cos x}{x+\dfrac{1}{x}}\,dx = \int_0^{+\infty} \frac{t^3\cos 1 - \left(t^2+2\right)\sin 1}{t^4+4}\,e^{-t}\,dt,$$

and

$$I_2 = \int_1^{+\infty} \frac{\sin x}{x+\dfrac{1}{x}}\,dx = \int_0^{+\infty} \frac{t^3\sin 1 + \left(t^2+2\right)\cos 1}{t^4+4}\,e^{-t}\,dt.$$

Now we shall find and use

$$
\begin{aligned}
F_1(t) + i\,F_2(t) &= i\,\frac{1+i\,t}{2-t^2+2it}\cdot\frac{2-t^2-2it}{2-t^2-2it} = \frac{i}{t^4+4}\left\{2-t^2+2t^2+\left(2t-t^3-2t\right)i\right\} \\
&= i\cdot\frac{t^2+2-it^3}{t^4+4} = \frac{t^3}{t^4+4} + i\,\frac{t^2+2}{t^4+4},
\end{aligned}
$$

so we finally get

$$F_1(t) = \frac{t^3}{t^4+4} \qquad \text{and} \qquad F_2(t) = \frac{t^2+2}{t^4+4}.$$

6 Differentiable and analytic functions; Cauchy-Riemann's equations

Example 6.1 *Given the function*

$$f(x+iy) = \left(x^2+2y\right) + i\left(x^2+y^2\right).$$

Find the points z_0 in which $f'(z_0)$ exists.

Clearly, $f \in C^\infty\left(\mathbb{R}^2\right)$. Then by partial differentiation,

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2y, \quad \text{thus} \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ for } x = y,$$

and

$$\frac{\partial u}{\partial y} = 2, \quad \frac{\partial v}{\partial x} = 2x, \qquad \text{thus} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \text{ for } x = -1.$$

It follows that Cauchy-Riemann's equations are only fulfilled for $x = y = -1$, hence the complex derivative $f'(z_0)$ does only exist at the point $z_0 = -1-i$, where

$$f'(-1-i) = -2-2i.$$

Example 6.2 *Prove that the function*

$$2x^2+3y^2+4xy+5x+2y+3 + i\left(6x^2+2y^2+3xy+2x+7y+1\right)$$

is not analytic in any domain of the complex plane.

The function is clearly of class $C^\infty\left(\mathbb{R}^2\right)$. We shall therefore only check Cauchy-Riemann's equations. We get

$$\frac{\partial u}{\partial x} = 4x+4y+5, \qquad \frac{\partial v}{\partial y} = 4y+3x+7,$$

$$\frac{\partial u}{\partial y} = 6y+4x+2, \qquad \frac{\partial v}{\partial x} = 12x+3y+2.$$

The it follows from Cauchy-Riemann's equations that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \text{thus} \qquad 4x+4y+5 = 3x+4y+7,$$

hence $x = 2$, and

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}, \qquad \text{thus} \qquad 4x+6y+2 = -12x-3y-2.$$

We conclude from the latter equation that

$$16x+9y+4 = 0.$$

If we here put $x = 2$, we get $y = -4$, hence Cauchy-Riemann's equations are only fulfilled at the point $z = 2 - 4i$.

Now, a point does not contain an open domain, so we conclude that the function is not analytic.

Example 6.3 *Prove that none of the following functions is analytic at any point:*

$$(a)\ \ f(z) = xy + iy, \qquad (b)\ \ f(z) = e^y(\cos x + i\,\sin x).$$

All the given functions are of class $C^{\infty}\left(\mathbb{R}^2\right)$, so we shall only prove that Cauchy-Riemann's equations are not fulfilled in any open domain.

(a) It follows from

$$u(x,y) = xy \qquad \text{and} \qquad v(x,y) = y,$$

that

$$\frac{\partial u}{\partial x} = y, \quad \frac{\partial u}{\partial y} = x \qquad \text{og} \qquad \frac{\partial v}{\partial x} = 0, \quad \frac{\partial v}{\partial y} = 1,$$

and we conclude that Cauchy-Riemann's equations are only satisfied at the *point* $(0,1)$, and a point cannot contain any domain.

(b) Since

$$u(x,y) = e^y \cos x \quad \text{and} \quad v(x,y) = e^y \sin x,$$

it follows by differentiation that

$$\frac{\partial u}{\partial x} = -e^y \sin x, \qquad \frac{\partial u}{\partial y} = e^y \cos x,$$

$$\frac{\partial v}{\partial x} = e^y \cos x, \qquad \frac{\partial v}{\partial y} = e^y \sin x.$$

We see that Cauchy-Riemann's equations are only fulfilled when both $\sin x = 0$ and $\cos x = 0$ at the same time, and this is not possible, because

$$\cos^2 x + \sin^2 x = 1.$$

Example 6.4 *Given $\varphi(x,y) = x^3 y$. Is it possible to find a function $\psi(x,y)$, such that*

$$f(z) = \varphi(x,y) + i\,\psi(x,y)$$

becomes analytic?

According to Cauchy-Riemann's equations ψ must satisfy

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \varphi}{\partial y} = -x^3, \qquad \frac{\partial \psi}{\partial y} = \frac{\partial \varphi}{\partial x} = 3x^2 y.$$

It follows from the former equation that

$$\psi(x,y) = -\frac{1}{4}\,x^4 + C_1(y),$$

and then from the latter equation,

$$\psi(x,y) = \frac{3}{2}\,x^2y^2 + C_2(x).$$

It follows that these two expressions can never be identical, no matter how $C_1(y)$ and $C_2(x)$ are chosen. In fact, the variables occur in different terms in the former expression of ψ, while such a separation of the variables is impossible in the second expression.

Hence one cannot find such a function ψ.

ALTERNATIVELY (and better) we see that $\varphi(x,y) = x^3y$ is not harmonic:

$$\triangle\varphi = 6xy + 0 = 6xy \neq 0 \qquad \text{for } xy \neq 0,$$

and since $\triangle\varphi = 0$ is a necessary condition for φ being the real part of some analytic function, there does not exist such a ψ.

Example 6.5 *Prove that Cauchy-Riemann's equations are fulfilled for the function $f(z) = \sqrt{|xy|}$, $z = x + iy$, at the point $z = 0$, and that the derivative nevertheless does not exist.*

Since $f(z)$ is real, we have

$$u(x,y) = \sqrt{|xy|} \qquad \text{and} \qquad v(x,y) = 0.$$

Since $u(x,0) = 0$, we have

$$\frac{\partial u}{\partial x}\,(0,0) = 0 = \frac{\partial v}{\partial y}\,(0,0),$$

and since $u(0,y) = 0$, we also have

$$\frac{\partial u}{\partial y}\,(0,0) = -\frac{\partial v}{\partial x}\,(0,0) = 0,$$

and we have proved that Cauchy-Riemann's equations hold at the point $z \sim (0,0)$.

Let us approach $(0,0)$ along the curve of the parametric description

$$x(t) = t, \qquad y(t) = t, \qquad t \in \mathbb{R}_+.$$

Then

$$\varphi(t) = f(x(t),y(t)) = \sqrt{t^2} = t, \qquad t \in \mathbb{R}_+,$$

so $\varphi'(t) = 1 \neq 0$, and the derivative does not exist.

Example 6.6 *Prove that the function*

$$f(z)=\begin{cases}\exp\left(-z^{-4}\right) & \textit{for } z\in\mathbb{C}\setminus\{0\},\\[2mm] 0 & \textit{for } z=0,\end{cases}$$

satisfies Cauchy-Riemann's equations in all of $\mathbb{C}$*, and yet it is not differentiable at 0.*

Since $f(z)$ for $z\neq 0$ is the composition of analytic functions, it follows that f analytic for $z\neq 0$. In particular, $f(z)$ fulfils Cauchy-Riemann's equations for $z\neq 0$.

On the other hand, the function is not continuous at $z=0$, thus it cannot be analytic at $z=0$ either. In fact, if we choose the curve of the parametric description

$$z(t)=t\cdot\left(\frac{1+i}{\sqrt{2}}\right),\qquad t>0,$$

then

$$f(z(t)) = \exp\left(-\frac{1}{t^4\left(\frac{1+i}{\sqrt{2}}\right)^4}\right) = \exp\left(\frac{1}{t^4}\right) \to +\infty \quad \text{for } t \to 0+.$$

Now, $f(z) = u(x,y) + i\,v(x,y)$, so

$$f(x + i \cdot 0) = u(x,0) + i\,v(x,0) = \exp\left(-\frac{1}{x^4}\right) + i \cdot 0,$$

thus

$$u(x,0) = \begin{cases} \exp\left(-\frac{1}{x^4}\right) & \text{for } x \neq 0, \\ 0 & \text{for } x = 0, \end{cases} \quad \text{and} \quad v(x,0) = 0.$$

Analogously,

$$f(0 + iy) = u(0,y) + i \cdot v(0,y) = \exp\left(-\frac{1}{y^4}\right) + i \cdot 0,$$

thus

$$u(0,y) = \begin{cases} \exp\left(-\frac{1}{y^4}\right) & \text{for } y \neq 0, \\ 0 & \text{for } y = 0, \end{cases} \quad \text{and} \quad v(0,y) = 0.$$

Clearly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0 \qquad \text{for } z = 0.$$

Furthermore,

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x - 0} = \lim_{x \to 0} \frac{\exp\left(-\frac{1}{x^4}\right)}{x} = \lim_{xto0} \frac{\frac{1}{x}}{\exp\left(\frac{1}{x^4}\right)} = \lim_{|t| \to +\infty} \frac{t}{\exp(t^4)} = 0,$$

and analogously,

$$\frac{\partial u}{\partial y}(0,0) = 0.$$

It follows by inspection that Cauchy-Riemann's equations are also fulfilled at $(0,0)$, thus they hold all over $\mathbb{C}$.

The lesson is that Cauchy-Riemann's equations *alone* without any assumption of continuity are *not* sufficient for analyticity.

Example 6.7 *Prove by means of Cauchy-Riemanns equations that the function*

$$f(z) = (-e^x \sin y + 3) + i\,(e^x \cos y + 5)$$

is analytic everywhere in $\mathbb{C}$.

The functions

$$u(x,y) = -e^x \sin y + 3 \qquad \text{and} \qquad v(x,y) = e^x \cos y + 5$$

are both of class $C^\infty\left(\mathbb{R}^2\right)$, so it suffices to prove that Cauchy-Riemann's equations are fulfilled everywhere in $\mathbb{C}$. We get by differentiation

$$\frac{\partial u}{\partial x} = -e^x \sin y, \qquad \frac{\partial v}{\partial y} = -e^x \sin y, \qquad \text{dvs.} \qquad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -e^x \cos y, \qquad \frac{\partial v}{\partial x} = e^x \cos x, \qquad \text{dvs.} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Since Cauchy-Riemann's equations hold everywhere in $\mathbb{C}$, we conclude that $f(z)$ is analytic in all of $\mathbb{C}$.

Remark 6.1 Note that it is easy to find $f(z)$ by the following manipulations,

$$\begin{aligned} f(z) &= \{-e^x \sin y + 3\} + i\,\{e^x \cos y + 5\} = i^2 e^x \sin y + i\, e^x \cos y + (3+5i) \\ &= i\,\{e^x \cos y + i\, e^x \sin y\} + (3+5i) = i\, e^z + (3+5i). \qquad \Diamond \end{aligned}$$

Example 6.8 *Check the different concepts of logarithm in the Theory of Complex Functions.*

In the *Theory of Complex Functions* one use in particular the following three different forms of the logarithm:

1) The real natural logarithm $\ln : \mathbb{R}_+ \to \mathbb{R}$.

2) *The principal branch of the logarithm* $\text{Log} : \mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$, where $\Omega = \mathbb{C} \setminus (\mathbb{R}_- \cup \{0\})$ is an open sliced domain of the complex plane along the negative real axis, such that always $\text{Arg}\, z \in \,]-\pi, \pi[$ for $z \in \Omega$. In this case,

$$\text{Log}\, z := \ln|z| + i\,\text{Arg}\, z = \ln r + i\theta, \qquad \text{where } z = r\,e^{i\theta} \in \Omega, \text{ and } \theta \in \,]-\pi, \pi[.$$

By using Cauchy-Riemann's equations in polar coordinates it is easy to prove that $\text{Log}\, z$ is analytic in Ω with the derivative

$$f'(z) = e^{-i\theta}\left\{\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r} = \frac{1}{r}\,e^{-i\theta}\right\} = \frac{1}{r\,e^{i\theta}} = \frac{1}{z},$$

and it follows that $\text{Log}\, z$ defined on Ω is an extension of $\ln x$ defined on $\mathbb{R}_+$.

We also note that

$$\exp \circ \text{Log}\,(z) = \exp(\ln r + i\theta) = e^{\ln r}\, e^{i\theta} = r\,e^{i\theta} = z,$$

hence in Ω the functions exp and Log are inverse to each other.

3) Finally, we also have the *multiple-valued function* $\log z$, defined on $\mathbb{C} \setminus \{0\}$ by

$$\log z = \begin{cases} \operatorname{Log} z + 2ip\pi, & p \in \mathbb{Z}, \qquad \text{for } z \in \Omega, \\ \ln |z| + i\{2p+1\}\pi, & p \in \mathbb{Z}, \qquad \text{for } z \in \mathbb{R}_-. \end{cases}$$

This function is not uniquely determined, so we shall not call it an analytic function, even though it has many properties in common with the analytic functions. Notice that none of these logarithms is defined for $z = 0$.

Example 6.9 *Check the following functions, if they fulfil the Cauchy-Riemann equations in their domains:*

(a)

$$f(z) = x^2 - y^2 - 2ixy.$$

(b)

$$x^3 - 3y^2x + 2x + i\left(3x^2y - y^3 + 2y\right).$$

(c)

$$f(z) = \frac{1}{2}\ln\left(x^2 - y^2\right) + i \operatorname{Arctan} \frac{y}{x}.$$

(d)

$$f(z) = \frac{1}{2}\ln\left(x^2 + y^2\right) + i \operatorname{Arccot} \frac{x}{y}.$$

(a) The domain is $\mathbb{C}$, and by separating the real and the imaginary part we get

$$u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v = -2xy,$$

hence

$$\frac{\partial u}{\partial x} = 2x \qquad \text{and} \qquad \frac{\partial v}{\partial y} = 2x \neq \frac{\partial u}{\partial x} \quad \text{for } x \neq 0.$$

We conclude that Cauchy-Riemann's equations are not satisfied in any open domain.

Remark 6.2 If one also consider the second one of Cauchy-Riemann's equation, it can be proved that they are only fulfilled at $(0,0)$. $\Diamond$

Remark 6.3 Note that $f(z) = \overline{z}^2$. $\Diamond$

(b) The domain is $\mathbb{C}$, and we get by separation of the real and the imaginary part that

$$u(x,y) = x^3 - 3y^2x + 2x \qquad \text{and} \qquad v(x,y) = 3x^2y - y^3 + 2y,$$

thus

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2 = \frac{\partial v}{\partial y} \qquad \text{og} \qquad \frac{\partial u}{\partial y} = -6yx = -\frac{\partial v}{\partial x}.$$

We have proved that Cauchy-Riemann's equations are satisfied in all of $\mathbb{C}$.

Remark 6.4 In this case it follows that

$$\begin{aligned} f(z) &= x^3 - 3y^2x + 2x + i\left(3x^2y - y^3 + 2y\right) \\ &= x^3 + 3x^{\cdot} iy + 3x(iy)^2 + (iy)^3 + 2(x+iy) \\ &= z^3 + 2z. \qquad \diamondsuit \end{aligned}$$

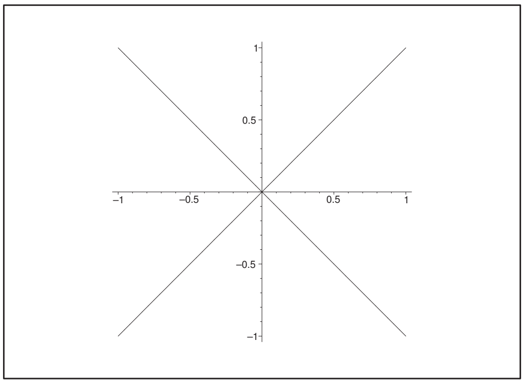


Figure 44: The domain in (c) is the union of the two open angular sets which contain the positive and the negative real half axis, resp..

(c) Here,

$$u(x,y) = \frac{1}{2}\ln\left(x^2 - y^2\right) \qquad \text{and} \qquad v(x,y) = \text{ Arctan } \frac{y}{x}$$

are defined and of class C^∞ in the open set given by $|x| > |y|$. We have assumed that $x \neq 0$, so

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{2x}{x^2 - y^2} = \frac{x}{x^2 - y^2},$$

and

$$\frac{\partial v}{\partial y} = \frac{\frac{1}{x}}{1 + \left(\frac{y}{x}\right)^2} = \frac{x}{x^2 + y^2} \neq \frac{\partial u}{\partial x}, \qquad \text{når } y \neq 0.$$

It follows that Cauchy-Riemman's equations are not satisfied in any open set, and the function cannot be analytic anywhere.

(d) Here,

$$u(x,y) = \frac{1}{2}\ln\left(x^2 + y^2\right) \qquad \text{and} \qquad v(x,y) = \text{ Arccot } \frac{x}{y}$$

are both defined and C^∞, when $y \neq 0$. Assuming this we get

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \qquad \frac{\partial v}{\partial y} = -\frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = \frac{x}{x^2 + y^2} = \frac{\partial u}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}, \qquad \frac{\partial v}{\partial x} = -\frac{1}{1 + \left(\frac{x}{y}\right)^2} \cdot \frac{1}{y} = -\frac{y}{x^2 + y^2} = -\frac{\partial u}{\partial y},$$

proving that Cauchy-Riemann's equations are fulfilled for $y \neq 0$.

Remark 6.5 This is an important example, because it can be proved that

$$f(z)=\begin{cases}\text{Log } z & \text{for } y>0,\\[2mm] \text{Log } z+i\pi & \text{for } y<0,\end{cases}$$ ◊

Example 6.10 *Check if the following functions are analytic in any domain of the plane,*

(a)

$$f(z)=x^2+y^2+2ixy.$$

(b)

$$f(z)=2x-3y+i(3x+2y).$$

(c)

$$f(z)=\frac{x+iy}{x^2+y^2}.$$

(d)

$$f(z)=\left|x^2-y^2\right|+2i\,|xy|.$$

(a) The function

$$f(z)=x^2+y^2+2ixy$$

is not analytic anywhere. It is seen by a separation of the real and the imaginary part that

$$u(x,y)=x^2+y^2 \qquad \text{and} \qquad v(x,y)=2xy$$

are both of class $C^\infty\left(\mathbb{R}^2\right)$. Then by a differentiation,

$$\frac{\partial u}{\partial x}=2x=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial y}{\partial y}=2y=\frac{\partial v}{\partial x}.$$

We see that one of the Cauchy-Riemann equations is satisfied, but the other

$$\frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x},$$

is only fulfilled for $y=0$, i.e. on the real axis, and the real axis does not contain any open domain of the plane.

(b) It is seen by INSPECTION that

$$f(z)=2x-3y+i(3x+2y)=2(x+iy)+3i(x+iy)=(2+3i)z,$$

which of course is analytic everywhere in $\mathbb{C}$.

ALTERNATIVELY,

$$u(x,y) = 2x - 3y \qquad \text{and} \qquad v(x,y) = 3x + 2y$$

are both of class $C^\infty\left(\mathbb{R}^2\right)$, and

$$\frac{\partial u}{\partial x} = 2 = \frac{\partial v}{\partial y} \qquad \text{og} \qquad \frac{\partial u}{\partial y} = -3 = -\frac{\partial v}{\partial x},$$

proving that Cauchy-Riemann's equations are fulfilled, and $f(z)$ is analytic everywhere in $\mathbb{C}$.

(c) An INSPECTION shows that if $z \neq 0$ (i.e. when $f(z)$ is defined), then

$$f(z) = \frac{x+iy}{x^2+y^2} = \frac{z}{z \cdot \overline{z}} = \frac{1}{\overline{z}}.$$

INDIRECT PROOF. *Assume* that $f(z) = \dfrac{1}{\overline{z}}$ were analytic for $z \neq 0$. Then according to the rules of computation, $\dfrac{1}{f(z)} = \overline{z}$ must be analytic in the same domain.
However, *since* $\dfrac{1}{f(z)} = \overline{z}$ is *not* analytic anywhere, $f(z)$ *cannot* be analytic.

ALTERNATIVELY,

$$u(x,y)=\frac{x}{x^2+y^2} \qquad \text{and} \qquad v(x,y)=\frac{y}{x^2+y^2}$$

are both of class C^∞ for $(x,y)\neq(0,0)$. Then

$$\frac{\partial u}{\partial x}=\frac{x^2+y^2-2x^2}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial y}=\frac{x^2+y^2-2y^2}{(x^2+y^2)^2}=\frac{x^2-y^2}{(x^2+y^2)^2},$$

$$\frac{\partial u}{\partial y}=-\frac{2xy}{(x^2+y^2)^2}, \qquad \frac{\partial v}{\partial x}=-\frac{2xy}{(x^2+y^2)^2}.$$

This proves that we obtain the "Cauchy-Riemann equations with the wrong signs", i.e.

$$\frac{\partial u}{\partial x}=-\frac{\partial v}{\partial y} \qquad \text{og} \qquad \frac{\partial u}{\partial y}=+\frac{\partial v}{\partial x}.$$

If therefore Cauchy-Riemann's equations are satisfied, then

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}=0 \qquad \text{og} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}=0.$$

Since $z\in\mathbb{C}\setminus\{0\}$, this is never the case.

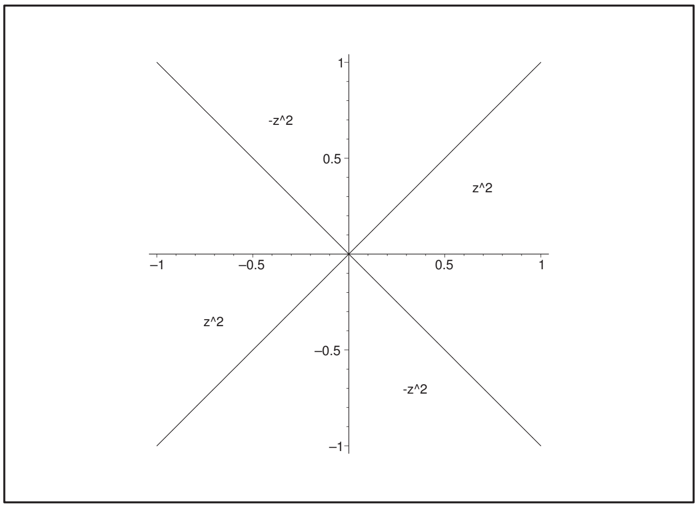


Figure 45: The four angular spaces, in which $f(z)$ is an analytic function, are indicated by the reduced expressions.

(d) Since

$$z^2=x^2-y^2+2ixy,$$

the analyticity here depends on the signs of x^2-y^2 and $2xy$.

If $x^2 - y^2 > 0$ and $2xy > 0$ (two angular spaces), then clearly

$$f(z) = \left|x^2 - y^2\right| + 2i|xy| = x^2 - y^2 + 2ixy = z^2$$

is analytic, cf. the figure.

If $x^2 - y^2 < 0$ and $2xy$ (again two angular spaces), then

$$f(z) = \left|x^2 - y^2\right| + 2i|xy| = -\left\{x^2 - y^2 + 2ixy\right\} = -z^2$$

is analytic, and $f(z)$ is (at least) analytic in the four marked angular spaces on the figure.

If $x^2 - y^2 < 0$ and $2xy > 0$ (again two angular spaces), then

$$f(z) = \left|x^2 - y^2\right| + 2i|xy| = -\left\{x^2 - y^2 - 2ixy\right\} = -\overline{z}^2.$$

If $x^2 - y^2 > 0$ and $2xy < 0$ (the two remaining angular spaces), then

$$f(z) = \left|x^2 - y^2\right| + 2i|xy| = x^2 - y^2 - 2ixy = \overline{z}^2.$$

The function $f(z)$ is not analytic in any of these latter cases. We have e.g. for $\overline{z}^2$ that

$$u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v(x,y) = -2xy,$$

hence

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = -2x,$$

$$\frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial v}{\partial x} = -2y.$$

It follows that Cauchy-Riemann's equations are only satisfied at $(0,0)$.

Example 6.11 *Prove that the following functions are analytic:*

(a)

$$f(z) = x^2 - y^2 - 2xy + i\left(x^2 - y^2 + 2xy\right), \qquad \text{for } z \in \mathbb{C}.$$

(b)

$$f(z) = \frac{x^3 + xy^2 + x + i\left(x^2 y + y^3 - y\right)}{x^2 + y^2}, \qquad \text{for } z \in \mathbb{C} \setminus \{0\}.$$

(c)

$$f(z) = \sin x \cdot \cosh y + i\,\cos x \cdot \sinh y, \qquad \text{for } z \in \mathbb{C}.$$

As usual it is worth while first to make an inspection. This will give us some variants in each part of the example.

(a) 1) INSPECTION. All terms contain the variables (x, y) of degree 2. Therefore, it is quite reasonable first to consider

$$z^2 = (x+iy)^2 ? x^2 - y^2 + 2ixy.$$

When we try to rewrite the expression of $f(z)$ by means of expressions which contain z^2, we see that

$$\begin{aligned} f(z) &= \left\{x^2-y^2\right\} - 2xy + i\left(x^2-y^2+\{2xy\}\right) \\ &= \left\{x^2-y^2+2ixy\right\} + i\left\{x^2-y^2+2ixy\right\} \\ &= (1+i)z^2. \end{aligned}$$

Since $f(z) = (1+i)z^2$ is a polynomial, we conclude that $f(z)$ is analytic.

2) CAUCHY-RIEMANN'S EQUATIONS. First note that

$$u(x,y) = x^2 - y^2 - 2xy \qquad \text{and} \qquad v(x,y) = x^2 - y^2 + 2xy$$

are both of class $C^\infty\left(\mathbb{R}^2\right)$. We shall therefore only prove that Cauchy-Riemann's equations are fulfilled. We find

$$u(x,y) = x^2-y^2-2xy, \qquad v(x,y) = x^2-y^2+2xy,$$

$$\frac{\partial u}{\partial x} = 2x - 2y, \qquad \frac{\partial v}{\partial y} = -2y + 2x,$$

$$\frac{\partial u}{\partial y} = -2y - 2x, \qquad \frac{\partial v}{\partial x} = 2x + 2y.$$

This implies that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x},$$

hence Cauchy-Riemann's are satisfied everywhere, and we conclude that $f(z)$ is analytic.

In order to prove that $f(z) = (1+i)z^2$, one must either go through the argument of (1) once more, or otherwise differentiate twice, in which case we get $f''(z) = 2(1+i)$. Then we get the result by two successive integrations.

(b) 1) INSPECTION. The degree of the numerator is 3, and the denominator has only degree 2. It is reasonable first to try a reduction. Here we get without any problems

$$x^3 + xy^2 = x\left(x^2+y^2\right) \qquad \text{and} \qquad x^2y + y^3 = y\left(x^2+y^2\right).$$

Then by insertion and reduction,

$$(1)\ \ f(z) = \frac{x^3+xy^2+x+i\left(x^2y+y^3-y\right)}{x^2+y^2} x + \frac{x}{x^2+y^2} + i\left\{y - \frac{y}{x^2+y^2}\right\}.$$

Since $x^2+y^2 = |z|^2 = z\cdot\overline{z}$, it follows from (1) that

$$f(z) = x + iy + \frac{x-iy}{x^2+y^2} = z + \frac{\overline{z}}{z\cdot\overline{z}} = z + \frac{1}{z},$$

which of course is analytic for $z \neq 0$.

2) CAUCHY-RIEMANN'S EQUATIONS. We shall here immediately use the reduction (1), which we here assume. If $z \neq 0$, then

$$u(x,y) = x + \frac{x}{x^2+y^2} \qquad \text{and} \qquad v(x,y) = y - \frac{y}{x^2+y^2},$$

which of course are both of class $C^\infty\left(\mathbb{R}^2 \setminus \{(0,0)\}\right)$. Then

$$\frac{\partial u}{\partial x} = 1 + \frac{1}{x^2+y^2} - \frac{2x^2}{\left(x^2+y^2\right)^2} = 1 + \frac{y^2-x^2}{\left(x^2+y^2\right)^2},$$

$$\frac{\partial v}{\partial y} = 1 - \frac{1}{x^2+y^2} + \frac{2y^2}{\left(x^2+y^2\right)^2} = 1 + \frac{y^2-x^2}{\left(x^2+y^2\right)^2} = \frac{\partial u}{\partial x},$$

and

$$\frac{\partial u}{\partial y} = -\frac{2xy}{\left(x^2+y^2\right)^2}, \qquad \frac{\partial v}{\partial x} = +\frac{2xy}{\left(x^2+y^2\right)^2} = -\frac{\partial u}{\partial y}.$$

Thus it follows that the Cauchy-Riemann equations are fulfilled, so $f(z)$ is analytic everywhere in its domain.

3) CAUCHY-RIEMANN'S EQUATIONS WITHOUT THE REDUCTION (1). This is the difficult variant, in which one checks if the original expressions

$$u(x,y) = \frac{x^3+yx+x}{x^2+y^2} \qquad \text{and} \qquad v(x,y) = \frac{x^2y+y^3-y}{x^2+y^2}$$

satisfy Cauchy-Riemann's equations.

By a mechanical computation we get

$$
\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\left(3x^2+y^2+1\right)\left(x^3+xy+x\right)\cdot 2x}{\left(x^2+y^2\right)^2} \\
&= \frac{3x^4+x^2y^2+x^2+3x^2y^2+y^4+y^2-2x^4-2x^2y^2-2x^2}{\left(x^2+y^2\right)^2} \\
&= \frac{x^2+2x^2y^2+y^4-x^2+y^2}{\left(x^2+y^2\right)^2} \qquad \left(=1+\frac{y^2-x^2}{\left(x^2+y^2\right)^2}\right), \\
\frac{\partial v}{\partial y} &= \frac{\left(x^3+3y^2-1\right)\left(x^2+y^2\right)-\left(x^2y+y^3-y\right)\cdot 2y}{\left(x^2+y^2\right)^2} \\
&= \frac{x^4+3x^2y^2-x^2+x^2y^2+3y^4-y^2-2x^2y^2-2y^4+2y^2}{\left(x^2+y^2\right)^2} \\
&= \frac{x^4+2x^2y^2+y^4-x^2+y^2}{\left(x^2+y^2\right)^2} \qquad \left(=1+\frac{y^2-x^2}{\left(x^2+y^2\right)^2}\right), \\
\frac{\partial u}{\partial y} &= \frac{2xy\left(x^2+y^2\right)-2y\left(x^3+xy^2+x\right)}{\left(x^2+y^2\right)^2} \\
&= \frac{2xy}{\left(x^2+y^2\right)^2}\left(x^2+y^2-x^2-y^2.1\right) = -\frac{2xy}{\left(x^2+y^2\right)^2}, \\
\frac{\partial vv}{\partial x} &= \frac{2xy\left(x^2+y^2\right)-2x\left(x^2y+y^3-y\right)}{\left(x^2+y^2\right)^2} \\
&= \frac{2xy}{\left(x^2+y^2\right)^2}\left(x^2+y^2-x^2-y^2+1\right) = +\frac{2xy}{\left(x^2+y^2\right)^2},
\end{aligned}
$$

and it follows that Cauchy-Riemann's equations

$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y} \qquad \text{og} \qquad \frac{\partial u}{\partial y}=-\frac{\partial v}{\partial x}$$

are fulfilled, hence $f(z)$ is analytic for $z \neq 0$.

(c) It will later be shown that $f(z)$ is the definition of the complex function $\sin z$, $z \in \mathbb{C}$. Since this function has not yet been defined, we must instead use the Cauchy-Riemann equations.
Clearly, the functions

$$u(x,y)=\sin x\cdot\cosh y \qquad \text{and} \qquad v(x,y)=\cos x\cdot\sinh y$$

are both of class $C^\infty\left(\mathbb{R}^2\right)$. Furthermore,

$$\frac{\partial u}{\partial x}=\cos x\cdot\cosh y=\frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y}=\sin x\cdot\sinh y=-\frac{\partial v}{\partial x},$$

hence Cauchy-Riemann's equations are satisfied, thus

$$f(z)=\sin x\cdot\cosh y+i\,\cos x\cdot\sinh y \qquad \{=\sin z=\sin(x+iy)\}$$

is analytic in all of $\mathbb{C}$.

Remark 6.6 We shall here show by applying Euler's formulæ that this definition of $\sin z$ is quite reasonable. In fact,

$$\begin{aligned}
f(z) &= \sin x \cdot \cosh y + i \cos x \cdot \sinh y \\
&= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \cdot \frac{1}{2}\left(e^{y} + e^{-y}\right) + i \cdot \frac{1}{2}\left(e^{ix} + e^{-ix}\right) \cdot \frac{1}{2}\left(e^{y} - e^{-y}\right) \\
&= \frac{1}{2i} \cdot \frac{1}{2}\left\{e^{ix}e^{y} + e^{ix}e^{-y} - e^{-ix}e^{y} - e^{ix}e^{y} + e^{ix}e^{-y} - e^{-ix}e^{y} + e^{-ix}e^{-y}\right\} \\
&= \frac{1}{2i} \cdot \frac{1}{2}\left\{2\,e^{ix}e^{-y} - 2\,e^{-ix}e^{y}\right\} = \frac{1}{2i}\left(e^{ix-y} - e^{-ix+y}\right) \\
&= \frac{1}{2i}\left\{e^{i(x+iy)} - e^{-i(x+iy)}\right\} = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right). \qquad \Diamond
\end{aligned}$$

Example 6.12 *Find some real constants a, b, c, d, such that the following functions become analytic:*

(a)

$$f(z) = x + a\,y + i(bx + cy).$$

(b)

$$f(x) = x^2 + a\,xy + b\,y^2 + i\left(c\,x^2 + d\,xy + y^2\right).$$

(c)

$$f(z) = \cos x \cdot \cosh y + a \cos x \cdot \sinh y i\,(b \sin x \cdot \sinh y + \sin x \cdot \cosh y)\,.$$

We first note that all the functions are of class $C^{\infty}\left(\mathbb{R}^2\right)$. Therefore, we shall only check if Cauchy-Riemann's equations are fulfilled.

(a) Since

$$u(x,y) = x + a\,y \qquad \text{and} \qquad v(x,y) = b\,x + c\,y,$$

it follows from Cauchy-Riemann's equations that

$$\frac{\partial u}{\partial x} = 1 = \frac{\partial v}{\partial y} = c \qquad \text{and} \qquad \frac{\partial u}{\partial y} = a = -\frac{\partial v}{\partial x} = -b.$$

Thus, the function is analytic, if and only if $c = 1$ and $a = -b$. In that case we also have

$$f(z) = x - b\,y + i(b\,x + y) = (1 + ib)(x + iy) = (1 + ib)z.$$

(b) Since

$$u(x,y) = x^2 + a\,xy + b\,y^2 \qquad \text{and} \qquad v(x,y) = c\,x^2 + d\,xy + y^2,$$

it follows from Cauchy-Riemann's equations that

$$\begin{aligned}
\frac{\partial u}{\partial x} &= 2x + a\,y = \frac{\partial v}{\partial y} = d \cdot x + 2y, \\
\frac{\partial u}{\partial y} &= a\,x + 2b\,y = -\frac{\partial v}{\partial x} = -2c\,x = -d \cdot y.
\end{aligned}$$

These equations must hold for all x and y, hence we conclude that

$$d = 2, \qquad a = 2, \qquad a = -2c \quad \text{and} \quad 2b = -d,$$

hence

$$a = 2, \qquad b = -\frac{1}{2}\,d = -1, \qquad c = -\frac{1}{2}\,a = -1, \qquad d = 2,$$

and we conclude that the only possible function is

$$\begin{aligned} f(z) &= x^2 + 2xy - y^2 + i\left\{-x^2 + 2xy + y^2\right\} = \left\{x^2 - y^2 + 2ixy\right\} - i\left\{x^2 - y^2 + 2ixy\right\} \\ &= (1-i)z^2. \end{aligned}$$

(c) Since

$$\begin{aligned} u(x,y) &= \cos c \cdot \cosh y + a \cdot \cos x \cdot \sinh y, \\ v(x,y) &= b \cdot \sin x \cdot \sinh y + \sin x \cdot \cosh y, \end{aligned}$$

we get

$$\begin{aligned} \frac{\partial u}{\partial x} &= -\sin x \cdot \cosh y - a \cdot \sin x \cdot \sinh y, \\ \frac{\partial v}{\partial y} &= b \cdot \sin x \cdot \cosh y + \sin x \cdot \sinh y, \\ \frac{\partial u}{\partial y} &= \cosh x \cdot \sinh y + a \cdot \cos x \cdot \cosh y, \\ \frac{\partial v}{\partial x} &= b \cdot \cos x \cdot \sinh y + \cos x \cdot \cosh y. \end{aligned}$$

It follows that Cauchy-Riemann's equations are satisfied, if

$$b = -1, \qquad -a = 1, \qquad b = -1 \quad \text{and} \quad a = -1,$$

thus for $a = b = -1$. For these values we obtain the analytic function

$$\begin{aligned} f(z) &= \cos x \cdot \cosh y - \cos x \cdot \sinh y + i\{-\sin x \cdot \sinh y + \sin x \cdot \cosh y\} \\ &= \cos x \cdot \{\cosh y - \sinh y\} + i\,\sin x \cdot \{\cosh y - \sinh y\} \\ &= \{\cos x + i\,\sin x\}e^{-y} = e^{ix-y} = e^{iz}. \end{aligned}$$

Example 6.13 *Find all analytic functions in $\mathbb{C}$ of the form*

$$f(z) = f(x+iy) = \varphi(x) + i\,\psi(y),$$

where φ and ψ are C^1-functions in one real variable.

We first assume that φ and ψ are real functions. A necessary and sufficient condition for $f(z)$ being analytic is that φ and ψ are both of class C^1, and that Cauchy-Riemann's equations are fulfilled, i.e.

$$\frac{\partial \varphi}{\partial x}(x) = \varphi'(x) = \frac{\partial \psi}{\partial y}(y) = \psi'(y),$$

because it is trivial that

$$\frac{\partial \varphi}{\partial y}(x) = -\frac{\partial \psi}{\partial x}(y) = 0$$

The equation $\varphi'(x) = \psi'(y)$ is only satisfied, if the common value is a *real* constant a. In fact, since $\varphi'(x)$ is independent of y, we conclude that $\psi'(y) = \varphi'(x)$ is a constant. Hence

$$\varphi'(x) = a \qquad \text{and} \qquad \psi'(y) = a, \qquad a \in \mathbb{R},$$

thus

$$\varphi(x) = a\,x + c_1 \qquad \text{and} \qquad \psi(y) = a\,y + c_2,$$

and

$$f(z) = \varphi(x) + i\,\psi(y) = a \cdot (x+iy) + c_1 + i\,c_2 = a\,z + c,$$

where $a \in \mathbb{R}$ and $c \in \mathbb{C}$. Finally,

$$f(z) = a\,z + c, \qquad a \in \mathbb{R} \text{ and } c \in \mathbb{C},$$

is clearly analytic.

Then assume that $\varphi(x)$ and $\psi(y)$ in the real variables have complex values. Then by separation into real and imaginary parts,

$$\varphi(x) = \varphi_1(x) + i\,\varphi_2(x) \qquad \text{and} \qquad \psi(y) = \psi_1(y) + i\,\psi_2(y),$$

hence

$$\begin{aligned} f(z) &= \varphi(x) + i\,\psi(y) \\ &= \{\varphi_1(x) - \psi_2(y)\} + i\,\{\varphi_2(x) + \psi_1(y)\}, \end{aligned}$$

and by a separation into real and imaginary parts,

$$u(x,y) = \varphi_1(x) - \psi_2(y), \qquad v(x,y) = \varphi_2(x) + \psi_1(y).$$

It follows from Cauchy-Riemann's equations that we get the conditions

$$\begin{aligned} \frac{\partial u}{\partial x} &= \varphi_1'(x) = \frac{\partial v}{\partial y} = \psi_1'(y) \\ \frac{\partial u}{\partial y} &= -\psi_2'(y) = -\frac{\partial v}{\partial x} = -\varphi_2'(x), \end{aligned}$$

and we conclude as above that there exist *real* constants a_1 and a_2, such that

$$\varphi_1'(x) = \psi_1'(y) = a_1 \qquad \text{and} \qquad \varphi_2'(x) = \psi_2'(y) = a_2,$$

thus

$$\varphi_1(x) = a_1 x + b_{11}, \qquad \psi_1(y) = a_1 y + b_{12},$$

$$\varphi_2(x) = a_2 x + b_{21}, \qquad \psi_2(y) = a_2 y + b_{22},$$

and by insertion,

$$\begin{aligned} f(z) &= \varphi(x) + i\,\psi(y) = (a_1 + i\,a_2)\,x + (b_{11} + i\,b_{21}) + i\,(a_1 + i\,a_2)\,y + i\,(b_{12} + i\,b_{22}) \\ &= a\,z + c, \qquad a,\, c \in \mathbb{C}, \end{aligned}$$

where

$$a = a_1 + i\,a_2 \qquad \text{og} \qquad c = b_{11} - b_{22} + i\,(b_{21} + b_{21}).$$

Example 6.14 *Prove that a shorthand of Cauchy-Riemann's equations is*

$$\frac{\partial f}{dx} = \frac{1}{i}\,\frac{\partial f}{\partial y}.$$

If we put $f = u + i\,v$, then it follows from CAUCHY-RIEMANN'S EQUATIONS that

$$\frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i\,\frac{\partial v}{\partial y}\right) = \frac{1}{i}\,\frac{\partial f}{\partial y}.$$

Example 6.15 Define

$$\frac{\partial f}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right) \qquad \left[= \frac{1}{2}\left(\frac{\partial f}{\partial x} - \frac{1}{i}\,\frac{\partial f}{\partial y}\right)\right].$$

Prove that CAUCHY-RIEMANN'S EQUATIONS *are equivalent to*

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Remark 6.7 One often says due to this result that the analytic functions only depend on z and not on $\overline{z}$. This claim can only be taken as a mnemonic rule, because $\dfrac{\partial f}{\partial \overline{z}}$ is *not* the derivative of f with respect to $\overline{z}$. It is only a shorthand for

$$\frac{1}{2}\left(\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y}\right). \qquad \Diamond$$

According to Example 6.14, the CAUCHY-RIEMANN EQUATIONS are equivalent to

$$\frac{\partial f}{\partial x} = \frac{1}{i}\,\frac{\partial f}{\partial y},$$

thus

$$0 = \frac{\partial f}{\partial x} - \frac{1}{i}\,\frac{\partial f}{\partial y} = \frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y} = 2\,\frac{\partial f}{\partial \overline{z}},$$

and it follows that Cauchy-Riemann's equations are equivalent to

$$\frac{\partial f}{\partial \overline{z}} = 0.$$

Example 6.16 *Assume that f is analytic in a domain Ω and that*

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \textit{in } \Omega.$$

Prove that f' is constant in Ω.

It follows from CAUCHY-RIEMANN'S EQUATIONS that

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} = 2\,\frac{\partial u}{\partial x},$$

hence

$$u(x,y) = u(y).$$

Analogously,

$$0 = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} + \frac{\partial v}{\partial y} = 2\,\frac{\partial v}{\partial y},$$

hence

$$v(x,y) = v(x).$$

Then it follows from the second of Cauchy-Riemann's equations that

$$\frac{\partial u}{\partial y} = u'(y) = -\frac{\partial v}{\partial x} = -v'(x),$$

independently of both x and y, hence the common value must be a real constant $-c$. Hereby we obtain the derivative

$$f'(z) = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x} = 0 + i(+c) = i\,c, \qquad c \in \mathbb{R}.$$

Example 6.17 *Find the domains of analyticity for each of the following functions, and then find their derivatives:*

$$(a)\;\left(z - \frac{1}{z}\right)^3, \qquad (b)\;\frac{z^2-4}{z^3-3z-2}, \qquad (c)\;\frac{z+i}{z-2}, \qquad (d)\;\frac{i}{z^3}.$$

Remark 6.8 Always start by inspecting the expression, if anything could be reduced. $\Diamond$

(a) The function

$$f(z) = \left(z - \frac{1}{z}\right)^3$$

is analytic in $\mathbb{C} \setminus \{0\}$. Then we get by the rules of computation that the derivative is

$$\begin{aligned} f'(z) &= 3\left(z-\frac{1}{z}\right)^2 \cdot \left(1+\frac{1}{z^2}\right) = 3\left(\frac{z^2-1}{z}\right)^2 \cdot \frac{z^1+1}{z^2} = \frac{3\left(z^4-1\right)\left(z^2-1\right)}{z^4} \\ &= \frac{3}{z^4}\left\{z^6-z^4-z^2+1\right\} = 3z^2 - 3 - \frac{3}{z^2} + \frac{3}{z^4}. \end{aligned}$$

(b) Here it pays off first to decompose. Since $z^3 - 3z - 2$ has the root $z = 2$, we easily get the expansion

$$z^3 - 3z - 2 = (z-2)(z+1)^2.$$

Since

$$z^2 - 4 = (z-2)(z+2),$$

we have the reduction

$$f(z) = \frac{z^2-4}{z^3-3z-2} = \frac{z+2}{(z+1)^2} = \frac{1}{(z+1)^2} + \frac{1}{z+1}.$$

Hence the function $f(z)$ is analytic in $\mathbb{C} \setminus \{2, -1\}$ with a removable singularity at $z = 2$. This means that the domain of analyticity can be extended to $\mathbb{C} \setminus \{-1\}$. In this extended domain the derivative is given by

$$f'(z) = -\frac{1}{(z+1)^2} - \frac{2}{(z+1)^3} = -\frac{z+3}{(z+1)^3}.$$

(c) We write the function in the following way,

$$f(z) = \frac{z+i}{z-2} = 1 + \frac{2+i}{z-2},$$

(note that we again start with a decomposition). This is clearly analytic in $\mathbb{C} \setminus \{2\}$. Since we have decomposed the function, it is very easy to compute the derivative,

$$f'(z) = -\frac{2+i}{(z-2)^2}.$$

Remark 6.9 If we did not decompose before the differentiation, the computations would have been somewhat larger, to put it mildly. $\Diamond$

(d) The function $f(z) = \dfrac{i}{z^3}$ is analytic in $\mathbb{C} \setminus \{0\}$ and its derivative is

$$f'(z) = -\frac{3i}{z^4}.$$

Example 6.18 *Find the derivatives of the following analytic function in their domains:*

$$(a)\;\; f(z) = z^5 - 3z^2 - 1, \qquad (b)\;\; f(z) = \frac{z}{1-z},$$

$$(c)\;\; f(z) = (1-z)^4\left(z^2+1\right)^3, \qquad (d)\;\; f(z) = \left(\frac{z-1}{z+1}\right)^4.$$

(a) The function is defined everywhere in $\mathbb{C}$, and it follows from the elementary rules of computation that

$$f'(z) = 5z^4 - 6z.$$

(b) Here it is better to start with a decomposition,

$$f(z) = \frac{z}{1-z} = -1 + \frac{1}{1-z},$$

which is defined in $\mathbb{C} \setminus \{1\}$, so the derivative is

$$f'(z) = \frac{1}{(1-z)^2}, \qquad z \in \mathbb{C} \setminus \{1\}.$$

(c) The function is defined in all of $\mathbb{C}$, and by the elementary rules of computation we get the derivative

$$\begin{aligned} f'(z) &= 4(z-1)^3\left(z^2+1\right)^3 + 6z(z-1)^4\left(z^2+1\right)^2 \\ &= 2(z-1)^3\left(z^2+1\right)^2\left\{2z^2+2+3z(z-1)\right\} \\ &= 2(z-1)^3\left(z^2+1\right)^2\left\{5z^2-3z+2\right\}. \end{aligned}$$

(d) The function $f(z) = \left(\frac{z-1}{z+1}\right)^4$ is defined in $\mathbb{C} \setminus \{-1\}$. Before the differentiation we note that

$$\frac{z-1}{z+1} = 1 - \frac{2}{z+1},$$

hence by differentiation of a composed function,

$$f'(z) = 4\left(\frac{z-1}{z+1}\right)^3 \cdot \frac{2}{(z+1)^2} = 8 \cdot \frac{(z-1)^3}{(z+1)^5}, \qquad z \in \mathbb{C} \setminus \{-1\}.$$

Example 6.19 **(a)** *Check where in the complex plane the functions u and v, given by*

$$u + iv = f(z) = (\overline{z})^2,$$

satisfy Cauchy-Riemann's equations.

(b) *Is $f(z)$ analytic at $z = 0$?*

(a) Since

$$u + iv = (\overline{z})^2 = (x - iy)^2 = x^2 - y^2 - 2ixy,$$

we get by a separation into the real and the imaginary parts that

$$u(x,y) = x^2 - y^2 \qquad \text{and} \qquad v(x,y) = -2xy.$$

These functions are both of class $C^\infty\left(\mathbb{R}^2\right)$. Then

$$\frac{\partial u}{\partial x} = 2x, \qquad \frac{\partial v}{\partial y} = -2x,$$

$$\frac{\partial u}{\partial y} = -2y, \qquad \frac{\partial v}{\partial x} = -2y.$$

It follows that Cauchy-Riemann's equations are fulfilled,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{for } x = 0,$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \qquad \text{for } y = 0.$$

This proves that Cauchy-Riemann's equations are only satisfied for $z = 0$.

(b) Since the domain of analyticity is always an open domain contained in the set where Cauchy-Riemann's equations are fulfilled, and since the set $\{(0,0)\} \sim \{0\}$ does not contain any open domain, the function $f(z)$ is not analytic at $z = 0$.

Remark 6.10 The difficult thing here is that the function $f(z)$ actually is *complex differentiable* at 0. In fact, if $z \neq 0$ we get for the difference quotient that

$$\left|\frac{f(z)-f(0)}{z-0}\right| = \left|\frac{(\overline{z})^2}{z}\right| = \frac{|z|^2}{|z|} = |z| \to 0 \qquad \text{for } z \to 0,$$

and it follows that

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0} = 0.$$

However, this does not assure that $f(z)$ is analytic at $z=0$. ♢

Example 6.20 *We shall check the map $w = z + z^2$, where $z = x+iy$ and $w = u+iv$.*

(a) *Prove that the straight lines $x = x_0$, where x_0 is a real constant $\neq -\frac{1}{2}$, is mapped into the parabolas*

$$(2)\ \ u = -\frac{v^2}{(1+2x_0)^2} + x_0\,(x_0+1)\,.$$

(b) *Which curves are the images of $y = y_0$, where y_0 is a real constant $\neq 0$?*

(c) *Find the image of $x_0 = -\frac{1}{2}$, and the image of $y_0 = 0$.*

(d) *Find a straight line $x = x_1$, where x_1 is a real constant $\neq x_0$, with the property that it is mapped into (2).*

(e) *What are the image curves of the circles $\left|z+\frac{1}{2}\right| = r_0$, where r_0 is a positive constant?*

(a) Since

$$w = f(z) = z + z^2 = x + iy + (x+iy)^2 = x + x^2 - y^2 + i(y + 2xy),$$

it follows by a separation of the real and the imaginary part that

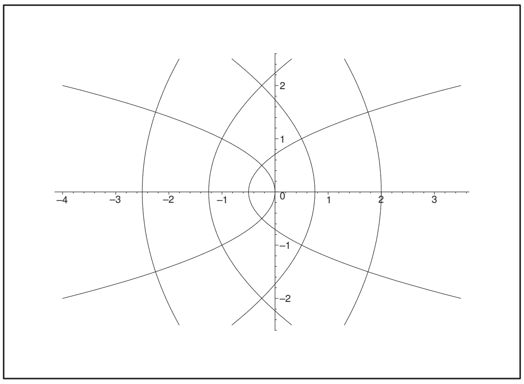
$$(3)\ \ u(x,y) = x + x^2 - y \qquad \text{and} \qquad v(x,y) = y(1+2x).$$

If therefore $z(t) = x_0 + i\,t$, $t \in \mathbb{R}$, then it follows from (3) that

$$u(t) = x_0\,(x_0+1) - t^2 \qquad \text{and} \qquad v(t) = (1+2x_0)\,t, \qquad t \in \mathbb{R},$$

and the task has now been reduced to eliminating the parameter t.
If $x_0 \neq -\frac{1}{2}$, then $t = \frac{v}{1+2x_0}$, hence by putting this into the expression of u gives (2), thus

$$u = -\frac{v^2}{(+2x_0)^2} + x_0\,(x_0+1)\,, \qquad x_0 \in \mathbb{R} \setminus \left\{-\frac{1}{2}\right\},$$



(c') We note that if $x_0 = -\dfrac{1}{2}$, then

$$u = -\frac{1}{2}\left(-\frac{1}{2}+1\right) - t^2 = -\frac{1}{4} - t^2 \qquad \text{and} \qquad v = 0, \qquad t \in \mathbb{R},$$

and the image is $\left]-\infty, -\dfrac{1}{4}\right]$ run through twice, and we have answered the first half of **(c)**.

(b) If instead $z(t) = t + iy_0$, $t \in \mathbb{R}$, $y_0 \neq 0$, then it follows from (3) that

$$u(t) = t + t^2 - y_0^2, \qquad v(t) = y_0(1+2t).$$

When we t eliminate from the last equation, we get

$$t = \frac{1}{2}\left(\frac{v}{y_0} - 1\right),$$

hence by insertion into the first equation,

$$u = \frac{1}{2}\left(\frac{v}{y_0} - 1\right) + \frac{1}{4}\left(\frac{v}{y_0} - 1\right)^2 - y_0^2 = \frac{v}{2y_0} - \frac{1}{2} + \frac{v^2}{4y_0^2} - \frac{1}{2}\,\frac{v}{y_0} + \frac{1}{4} - y_0^2 = \left(\frac{v}{2y_0}\right)^2 - y_0^2 - \frac{1}{4},$$

which again is a system of parabolas.

(c") If $y_0 = 0$, then

$$u(t) = t + t^2 = \left(t + \frac{1}{2}\right)^2 - \frac{1}{4} \qquad \text{and} \qquad v = 0, \qquad t \in \mathbb{R},$$

thus the interval $\left[-\frac{1}{4}, +\infty\right[$ is run through twice, and we have answered the latter part of **(c)**.

(d) If $x = x_1 \neq x_0$ and $x_0 \neq -\frac{1}{2}$ shall be mapped into (2), then the parabolas must have the same vertex, i.e.

$$x_0^2 + x_0 = \left(x_0 + \frac{1}{2}\right)^2 - \frac{1}{4} = \left(x_1 + \frac{1}{2}\right)^2 - \frac{1}{4} = x_1^2 + x_1.$$

Hence

$$0 = \left(x_0 + \frac{1}{2}\right)^2 - \left(x_1 + \frac{1}{2}\right)^2 = (x_0 + x_1 + 1)\cdot(x_0 - x_1)\,.$$

It follows from $x_1 \neq x_0$ that $x_1 = -1 - x_0$. Using this value we get

$$(1 + 2x_1)^2 = (1 - 2 - 2x_0)^2 = (1 + 2x_0)^2\,,$$

and we conclude that the two parabolas are identical.

(e) Finally, we shall find the images of the circles

$$\left|z + \frac{1}{2}\right| = r_0 > 0.$$

It follows from

$$w = z + z^2 = \left(z + \frac{1}{2}\right)^2 - \frac{1}{4},$$

that

$$\left(z + \frac{1}{2}\right)^2 = w + \frac{1}{4},$$

hence

$$\left|w+\frac{1}{4}\right| = \left|z+\frac{1}{2}\right|^2 = r_0^2,$$

and the image curve is the circle of centrum $-\dfrac{1}{4}$ and radius r_0^2 run through twice.

7 The polar Cauchy-Riemann's equations

.

Example 7.1 *Prove that the complex function $f(z)$, which is given in polar coordinates (r,θ) by*

$$f(z)=\sqrt{r}\cdot\cos\frac{\theta}{2}+i\sqrt{r}\cdot\sin\frac{\theta}{2}, \qquad z=r\,e^{i\theta},$$

is analytic in the sliced domain

$$\Omega=\left\{z=r\,e^{i\theta}\mid r\in\mathbb{R}_+,\,\theta\in]-\pi,\pi[\right\},$$

where we have removed the closed real negative axis. Then find its derivative.

Given that 0 does not belong to the domain Ω, and that

$$u(r,\theta)=\sqrt{r}\cdot\cos\frac{\theta}{2} \qquad \text{and} \qquad v(r,\theta)=\sqrt{r}\cdot\sin\frac{\theta}{2},$$

are of class C^∞ in this domain, we shall only check that the Cauchy-Riemann equations in polar coordinates are satisfied. We get

$$\frac{\partial u}{\partial r}=\frac{1}{2}\,\frac{1}{\sqrt{r}}\cdot\cos\frac{\theta}{2}, \qquad \frac{1}{r}\,\frac{\partial v}{\partial\theta}=\frac{1}{2}\,\frac{1}{\sqrt{r}}\cdot\cos\frac{\theta}{2},$$

$$\frac{1}{r}\,\frac{\partial u}{\partial\theta}=-\frac{1}{2}\,\frac{1}{\sqrt{r}}\cdot\sin\frac{\theta}{2}, \qquad \frac{\partial v}{\partial r}=\frac{1}{2}\,\frac{1}{\sqrt{r}}\cdot\sin\frac{\theta}{2},$$

and it follows that the polar Cauchy-Riemann equations are fulfilled, hence $f(z)$ is an analytic function.

The derivative is given by

$$\begin{aligned}f'(z) &= e^{-i\theta}\left(\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}\right)=e^{-i\theta}\left(\frac{1}{2\sqrt{r}}\,\cos\frac{\theta}{2}+i\,\frac{1}{2\sqrt{r}}\,\sin\frac{\theta}{2}\right)\\ &= \frac{1}{2}\,\frac{1}{\sqrt{r}}\,e^{-i\theta}\exp\left(i\,\frac{\theta}{2}\right)=\frac{1}{2}\,\frac{1}{\sqrt{r}}\cdot\frac{1}{\exp\left(i\,\dfrac{\theta}{2}\right)}=\frac{1}{2}\cdot\frac{1}{\sqrt{r}\,\cos\dfrac{\theta}{2}+i\sqrt{r}\,\sin\dfrac{\theta}{2}}=\frac{1}{2}\,\frac{1}{f(z)}.\end{aligned}$$

Remark 7.1 This is in fact the polar definition of the analytic function $f(z)=\sqrt{z}$ with the slit along the negative real axis. $\Diamond$

Example 7.2 *Prove that the complex function $f(z)$, which is given in polar coordinates (r, θ) by*

$$f(z) = e^{r\,\cos\theta}\cos(r\cdot\sin\theta) + i\,e^{r\,\cos\theta}\sin(r\cdot\sin\theta)$$

is analytic in the domain

$$\Omega = \left\{ z = r\,e^{i\theta} \mid r \in \mathbb{R}_+,\, \theta \in]-\pi,\pi[\right\},$$

and find its derivative.

It follows immediately from $x = r\cos\theta$ and $y = r\cdot\sin\theta$ that

$$f(z) = e^x\cos y + i\,e^x\sin y = e^x(\cos y + i\,\sin y) = e^x e^{iy} = e^z,$$

thus $f(z) = e^z$ is analytic with the derivative

$$f'(z) = e^z = f(z).$$

Alternatively we prove that Cauchy-Riemann's equations in polar coordinates are satisfied. First note that 0 does not belong to the domain. Then by a separation of the real and the imaginary parts,

$$u(r,\theta) = e^{r\,\cos\theta}\cos(r\,\sin\theta), \qquad v(r,\theta) = e^{r\,\cos\theta}\sin(r\,\sin\theta),$$

where both functions are of class C^∞. Then by a differentiation,

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \cos\theta \cdot e^{r\,\cos\theta} \cdot \cos(r\,\sin\theta) - \sin\theta \cdot e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta),\\
\frac{1}{r}\,\frac{\partial v}{\partial \theta} &= \frac{1}{r} \cdot r\,(-\sin\theta)e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta) + \frac{1}{r} \cdot r\,\cos\theta \cdot e^{r\,\cos\theta} \cdot \cos(r\,\sin\theta) = \frac{\partial u}{\partial r},\\
\frac{1}{r}\,\frac{\partial u}{\partial r} &= \frac{1}{r}\,(-r\,\sin\theta) \cdot e^{r\,\cos\theta} \cdot \cos(r\,\sin\theta) + \frac{1}{r}\,e^{r\,\cos\theta} \cdot r\,\cos\theta \cdot (-\sin(r\,\sin\theta)),\\
\frac{\partial v}{\partial r} &= \cos\theta \cdot e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta) + e^{r\,\cos\theta} \cdot \sin\theta \cdot \cos(r\,\sin\theta) = -\frac{1}{r}\,\frac{\partial u}{\partial \theta},
\end{aligned}$$

and we have proved that $f(z)$ is analytic. The derivative is given by

$$\begin{aligned}
f'(z) &= e^{-i\theta}\left\{\frac{\partial u}{\partial r} + i\,\frac{\partial v}{\partial r}\right\}\\
&= e^{-i\theta}\left\{\cos\theta \cdot e^{r\,\cos\theta} \cdot \cos(r\,\sin\theta) - \sin\theta \cdot e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta)\right.\\
&\qquad \left. +i\,\cos\theta \cdot e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta) + i\,e^{r\,\cos\theta} \cdot \sin\theta\ \cdot \cos(r\,\sin\theta)\right\}\\
&= e^{-\theta}\left\{(\cos\theta + i\,\sin\theta)e^{r\,\cos\theta} \cdot \cos(r\,\sin\theta) + i(\cos\theta + i\,\sin\theta)e^{r\,\cos\theta} \cdot \sin(r\,\sin\theta)\right\}\\
&= e^{-i\theta} \cdot e^{i\theta}\left\{e^{r\,\cos\theta}\cos(r\,\sin\theta) + i\,e^{r\,\cos\theta}\sin(r\,\sin\theta)\right\}\\
&= f(z).
\end{aligned}$$

Example 7.3 *Prove that the complex function $f(z)$, given in polar coordinates (r,θ) by*

$$f(z) = r\,\cos\theta \cdot \ln r - r\theta\,\sin\theta + i\{r\,\sin\sin\theta \cdot \ln r + r\theta \cdot \cos\theta\},$$

is analytic in the domain

$$\Omega = \left\{z = r\,e^{i\theta} \mid r \in \mathbb{R}_+,\, \theta \in \,]-\pi,\pi[\right\},$$

and find its derivative.

The domain Ω does not contain 0, and both functions

$$\begin{aligned}
u(r,\theta) &= r\,\cos\theta \cdot \ln r - r\,\theta \cdot \sin\theta,\\
v(r,\theta) &= r\,\sin\theta \cdot \ln r + r\,\theta \cdot \cos\theta,
\end{aligned}$$

are of class $C^\infty(\Omega)$. Hence, we shall only check the Cauchy-Riemann equations in polar coordinates.

We have

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \cos\theta \cdot \ln r + \cos\theta - \theta \cdot \sin\theta,\\
\frac{1}{r}\,\frac{\partial v}{\partial \theta} &= \cos\theta \cdot \ln r + \cos\theta - \theta \cdot \sin\theta = \frac{\partial u}{\partial r},\\
\frac{1}{r}\,\frac{\partial u}{\partial \theta} &= -\sin\theta \cdot \ln r - \sin\theta - \theta \cdot \cos\theta,\\
\frac{\partial v}{\partial r} &= \sin\theta \cdot \ln r + \sin\theta + \theta \cdot \cos\theta = -\frac{1}{r}\,\frac{\partial u}{\partial \theta},
\end{aligned}$$

so the polar Cauchy-Riemann equations are fulfilled and hence $f(z)$ is analytic in Ω.

Finally, the derivative is given by

$$\begin{aligned} f'(z) &= e^{-i\theta}\left\{\frac{\partial u}{\partial r}+i\,\frac{\partial v}{\partial r}\right\} \\ &= e^{-i\theta}\,\{\cos\theta\cdot\ln r+\cos\theta-\theta\cdot\sin\theta+i\,(\sin\theta\cdot\ln r+\sin\theta+\theta\cdot\cos\theta)\} \\ &= e^{-i\theta}\{\ln r(\cos\theta+i\,\sin\theta)+(\cos\theta+i\,\theta)+i\,\theta(\cos\theta+i\,\sin\theta)\} \\ &= (\ln r+1+i\theta)e^{-i\theta}(\cos\theta+i\,\sin\theta)=1+\text{ Log } z \qquad \text{for } z\in\Omega. \end{aligned}$$

Remark 7.2 It is also here possible to find the exact expression of $f(z)$ in Ω as a function in z. In fact, if $z\in\Omega$, then

$$\begin{aligned} f(z) &= r\,\cos\theta\cdot\ln r-r\,\theta\,\sin\theta+i\,\{r\,\sin\theta\cdot\ln r+r\,\theta\,\cos\theta\} \\ &= r\,\ln r\cdot(\cos\theta+i\,\sin\theta)+i\,r\,\theta\,(\cos\theta+i\,\sin\theta) \\ &= r\,\ln r\cdot e^{i\theta}+i\,r\,\theta\,e^{i\theta}=r\,e^{i\theta}(\ln r+i\theta) \\ &= z\,\text{Log}\,z. \qquad \diamond \end{aligned}$$

Example 7.4 *Sketch the curves* $u(x,y)=\text{constant}$ *and* $v(x,y)=\text{constant}$ *in the domains of the following functions:*

$$(a)\;\; f(z)=z^3, \qquad (b)\;\; f(z)=e^z, \qquad (x)\;\; f(z)=\text{ Log } z.$$

(a) We have in *rectangular* coordinates

$$f(z)=(x+iy)^3=x^3-3xy^2+i\left\{3x^2y-y^3\right\},$$

thus

$$(4)\;\; u(x,y)=x^3-3sy^2, \qquad v(x,y)=3x^2y-y^3,$$

If we instead use *polar* coordinate, then

$$f(z)=r^3e^{3i\theta}=r^3\cos3\theta+i\,r^3\sin3\theta,$$

hence

$$u(r,\theta)=r^3\cos3\theta, \qquad v(r,\theta)=r^3\sin3\theta.$$

Empirically the treatment of systems of curves of e.g. the form

$$u(x,y)=c,$$

is often causing some difficulties. Hence we shall go through this example in more detail than usually. We shall restrict ourselves to consider only the rectangular version (4).

First we consider the curves $u(x,y)=c$. If we insert a point (x,y), we only obtain one value of the constant c. Hence we conclude that curves belonging to different values of c do *not* intersect.

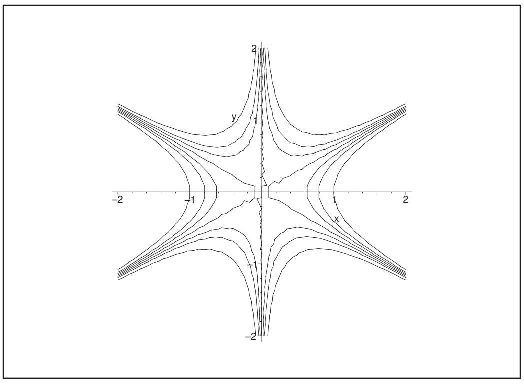


Figure 46: (a) The system of curves $u(x,y)=c$.

Let $c=0$. Then the equation is written as follows:

$$0 = x^3 - 3xy^2 = -3x\left(y^2 - \frac{1}{3}\,x^2\right) = -3x\left(y - \frac{1}{\sqrt{3}}\,x\right)\left(y + \frac{1}{\sqrt{3}}\,x\right).$$

Hence, the solution set for $c=0$ is the union of the three straight lines, given by the equations

$$x = 0, \qquad y = \frac{1}{\sqrt{3}}\,x \quad \text{and} \quad y = -\frac{1}{\sqrt{3}}\,x,$$

which intersect at $(0,0)$. Any other curve of the type $u(x,y)=c$, $c \neq 0$, is now confined to one of the six open angular spaces. Furthermore, the lines corresponding to $c=0$ must be *asymptotes* for any other curve $u(x,y)=c$, where $c \neq 0$.

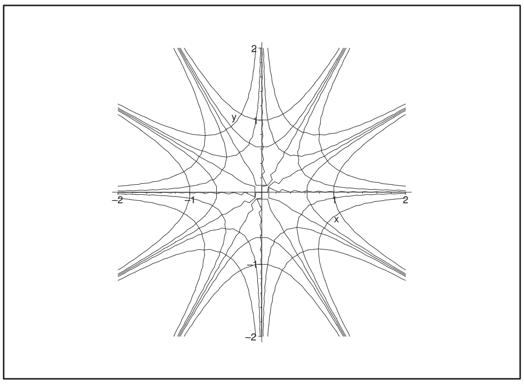


Figure 47: (a) The two curve systems $u(x,y)=c_1$ and $v(x,y)=c_2$ are orthogonal in $\mathbb{C} \setminus (0,0)$.

For the time being we shall neglect the vertical line $x=0$, which requires another though analogous treatment. Hence, we shall consider the lines

$$y = \pm\frac{1}{\sqrt{3}}\,x$$

and the corresponding angular spaces. Let (x_0, y_0) be a point on the claimed asymptote, and let (x, y) be a point on the curve. Then

$$x_0^3 - 3x_0y_0^2 = 0 \qquad \text{and} \qquad x^3 - 3xy^2 = c.$$

If we here put $x_0 = x$, it follows that the distance from the curve to one of the lines is smaller than or equal to either

$$|y - y_0| \qquad \text{or} \qquad |y - (-y_0)| = |y + y_0|\,.$$

Therefore, we shall only prove that one of these two distances tends towards zero, when $x = x_0 \to \pm\infty$. When we subtract the two equations, we get

$$3x_0\left(y^2 - y_0^2\right) = c,$$

thus

$$\left|y^2 - y_0^2\right| = |y - y_0| \cdot |y + y_0| = \left|\frac{c}{3x_0}\right|.$$

Since the right hand side tends towards zero for $x_0 \to \pm\infty$, at least one of the two factors $|y - y_0|$ and $|y + y_0|$ will tend towards zero by the same limit, and the claim is proved.

Then we threat the v-curves, either in the same way, or by using that the v-curves are orthogonal to the u-curves. A third method is simply by the symmetry to interchange x and y.

(b) We have in *rectangular* coordinates

$$f(z) = e^x \cos y + i\, e^x \sin y,$$

hence by separating the real and the imaginary part,

$$u(x, y) = e^x \cos y, \qquad v(x, y) = e^x \sin y.$$

In this case it is very difficult to handle the *polar* description, so we shall only consider the rectangular version.

We first investigate the u-curves

$$e^x \cos y = c, \qquad c \in \mathbb{R}.$$

If $c = 0$, then $\cos y = 0$, and we get the horizontal lines

$$y = \frac{\pi}{2} + p\pi, \qquad p \in \mathbb{Z}.$$

The curves corresponding to $c \neq 0$ must now be restricted to the horizontal "strips", defined by the curves corresponding to $c = 0$. If $c \neq 0$, it follows from the equation

$$e^x \cos y = c,$$

that $\cos y$ and c must have the same sign, thus

$$y \in \left] -\frac{\pi}{2} + 2p\pi\, ,\, \frac{\pi}{2} + 2p\pi \right[, \qquad p \in \mathbb{Z}, \text{ if } c > 0,$$

and

$$y \in \left] \frac{\pi}{2} + 2p\pi\, ,\, \frac{3\pi}{2} + 2p\pi \right[, \qquad p \in \mathbb{Z}, \text{ if } c < 0.$$

One may say that every second of the horizontal strips belong to positive c, and every second to negative c.

The point is that the equation can now be written, such that *x is expressed as a function of y*. This is due to the fact that Arccos is a somewhat difficult function, while cos is not. If c and $\cos y$ have the same sign, then

$$x = \ln\left(\frac{c}{\cos y}\right).$$

The minimum of x is obtained when $|\cos y| = 1$, corresponding to $y = q\pi$, $q \in \mathbb{Z}$, and $x = \ln|c|$. We note that this minimum is negative, if $|c| < 1$. If y approaches one of the boundary "curves", $\pm\frac{\pi}{2} + q\pi$, $q \in \mathbb{Z}$, then the denominator tends towards 0, thus $x \to +\infty$. Hence, the horizontal lines

$$y = \pm\frac{\pi}{2} + q\pi, \qquad q \in \mathbb{Z},$$

corresponding to the system of curves for $c = 0$, become the asymptotes. It follows from

$$x = \ln|c| - \ln|\cos y|,$$

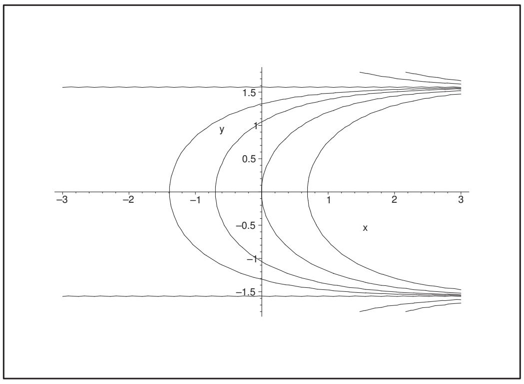


Figure 48: (b) The curves $u(x, y) = c$.

that all curves in one "strip" is obtained from $x = -\ln|\cos y|$ by a translation of the amount $\ln|c|$.

Finally, the v-curves are either obtained in the same way by noticing that

$$\sin y = \cos\left(\frac{\pi}{2} - y\right) = \cos\left(y - \frac{\pi}{2}\right),$$

so the v-curves are obtained from the u-curves by a translation of the amount $i\,\frac{\pi}{2}$. Alternatively, one may again apply the orthogonality of the u-curves and the v-curves.

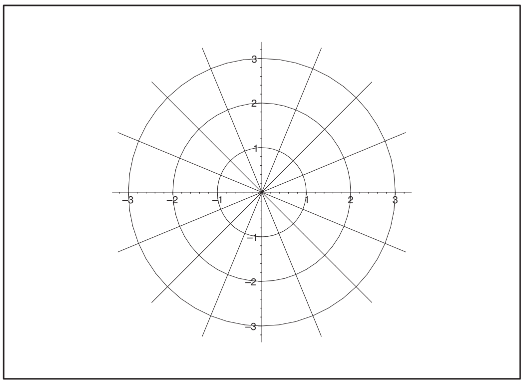


Figure 49: (c) The orthogonal curve systems $u(x, y) = c_1$ and $v(x, y) = c_2$.

(c) In this case it is most convenient to apply *polar* coordinates,

$$f(z) = \operatorname{Log} z = \ln r + i\,\theta, \qquad r > 0, \quad \theta \in]-\pi, \pi],$$

thus

$$u(r, \theta) = \ln r \qquad \text{and} \qquad v(r, \theta) = \theta.$$

The curves $u(r,\theta) = c$ are the circles $r = e^c$, and the curves $v(r,\theta) = c$ are the half lines $\theta = c \in \,]-\pi, \pi]$, where we usually include $\theta = \pi$, which gives a discontinuity in the function $\operatorname{Arg} z$.

The point $z = 0$ is a singular point.

We interpret this situation as the model of a point *source* at $z = 0$, where the v-curves denote the streamlines and the u-curves denote the equipotential curves.

Example 7.5 . Flow around a corner. *Given a real constant* $c > \dfrac{1}{2}$. *If* $z \in \mathbb{C} \setminus \{0\}$, *we define* $\operatorname{Log}^\star z$ *by*

$$\operatorname{Log}^\star z = \ln |z| + i\,\theta,$$

where θ *is the argument of* z, $z = r\,e^{i\theta}$, *for which* $\theta \in [0, 2\pi[$.
Consider the complex potential

$$F(z) = z^c := e^{c \operatorname{Log}^\star z}.$$

Write $z = r\,e^{i\theta}$. *Prove that the half lines* $\theta = 0$ *and* $\theta = \dfrac{\pi}{c}$ *are streamlines.*
(Hence, the pattern of the flow does not change, if we put barriers along these half lines).
Sketch the streamlines in the sector $0 \le \theta \le \dfrac{\pi}{c}$ *in the following three cases,*

$$(a)\;\; c = 4, \qquad (b)\;\; c = 1, \qquad (c)\;\; c = \frac{2}{3}.$$

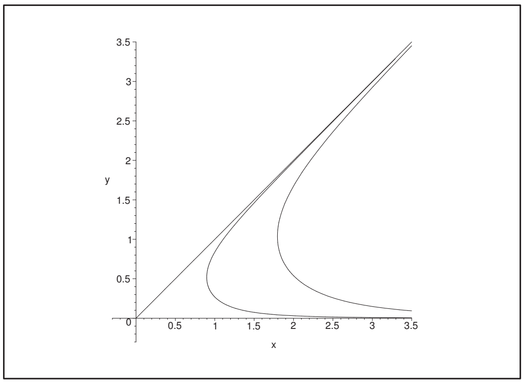


Figure 50: (a) The streamlines for $c = 4$.

First compute

$$F(z) = z^c := e^{c \operatorname{Log}^\star z} = e^{c\,\ln r + i\,c\,\theta} = r^c \cdot \{\cos(c\,\theta) + i\,\sin(c\,\theta)\},$$

which shows that the streamlines are given by

$$\psi(x,y) = \operatorname{Im}(F(z)) = r^c \cdot \sin(c\,\theta) = k,$$

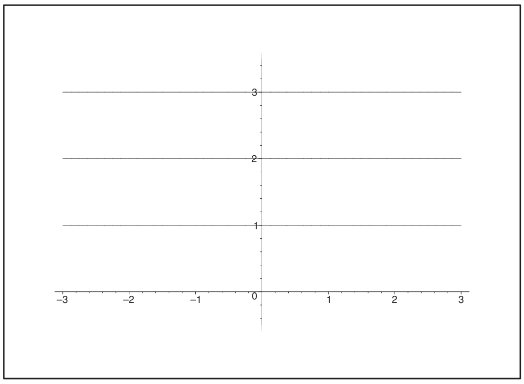


Figure 51: (b) The streamlines for $c = 1$.

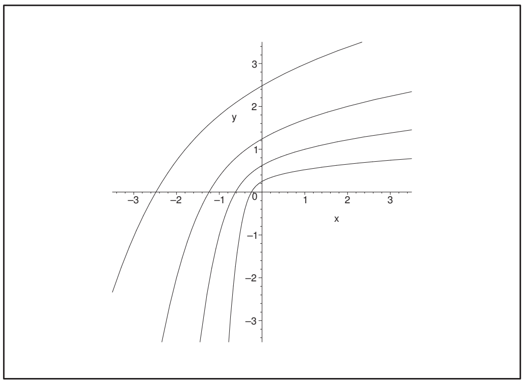


Figure 52: (c) The streamlines for $c = \dfrac{2}{3}$.

where

$$r^c := e^{c\,\ln r}.$$

If $\theta = 0$, then

$$\psi(x,y) = r^c \sin 0 = 0,$$

and if $\theta = \dfrac{\pi}{c} < 2\pi$, then

$$\psi(x,y) = r^c \sin\left(c \cdot \frac{\pi}{c}\right) = 0,$$

which proves that these half lines are streamlines, and the first claim is proved.

(a) If $c = 4$, then $\theta \in \left[0, \dfrac{\pi}{4}\right]$, so

$$r^4 \sin 4\theta = k \geq 0,$$

which corresponds to a flow in an angular sector, in which the angle is $\dfrac{\pi}{4}$.

(b) If $c = 1$, then $\theta \in [0, \pi]$, and we get

$$r \cdot \sin\theta = y = k \geq 0,$$

corresponding to a parallel flow.

(c) If $c = \frac{2}{3}$, then the requirement is that $\theta \in \left[0, \frac{3\pi}{2}\right]$, and the corresponding equation becomes

$$r^{\frac{2}{3}} \sin\left(\frac{2}{3}\theta\right) = k \geq 0,$$

which we interpret as the model of a flow around a rectangular corner.

Example 7.6 *Let* $f(z) = u + iv = z + \text{Log } z$ *be defined in the domain*

$$\Omega = \{z \in \mathbb{C} \mid Im(z) > 0\}.$$

Sketch the curves $u(x, y) = \text{constant}$ *and* $v(x, y) = \text{constant}$ *in* Ω.

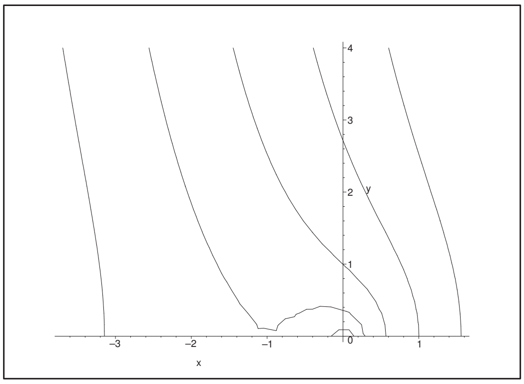


Figure 53: The curves $u(x, y) = \text{constant}$ in Ω of Example 7.6.

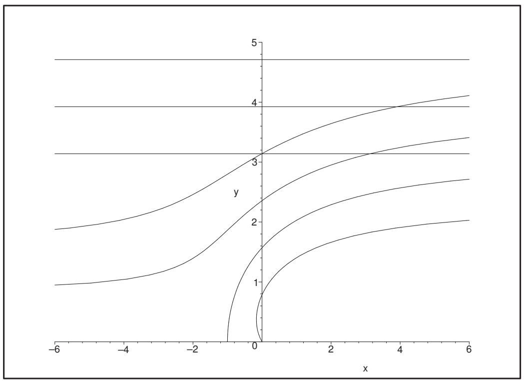


Figure 54: The curves $v(x, y) = \text{constant}$ in Ω of Example 7.6.

Since

$$z + \text{ Log } z = x + \frac{1}{2} \ln\left(x^2 + y^2\right) + i\left\{y + \text{ Arccot}\left(\frac{x}{y}\right)\right\}$$

for $y > 0$, we get by separation of the real and the imaginary part that

$$u(x, y) = x + \frac{1}{2} \ln\left(x^2 + y^2\right)$$

and

$$v(x, y) = y + \text{ Arccot}\left(\frac{x}{y}\right).$$

If we put $u(x,y) = c$, then we get by solving with respect to y that

$$y^2 = e^{2c-2x} - x^2 = C\, e^{-2x} - x^2, \qquad y > 0,$$

where we have put $C = e^{2c} > 0$.
Analogously, from $v(x,y) = k$ we get by solving with respect to x that

$$x = y \cdot \cot(k - y).$$

Example 7.7 *Let*

$$F(z) = c\,\mathrm{Log}(z-a) + c\,\mathrm{Log}(z+a),$$

where a, $c \in \mathbb{R}_+$ are given constants. We consider F in its domain Ω as a complex potential. Find the streamlines and the equipotential curves of F in Ω.
This example can be interpreted as the model of two sources *of the same strength at the points $z = a$ and $z = -a$.*

Obviously, we may choose $c = 1$ and $a = 1$, so we only consider

$$F(z) = \mathrm{Log}(z-1) + \mathrm{Log}(z+1), \qquad z \in \mathbb{C} \setminus (]-\infty, 1]),$$

where we must be aware of the branch cuts of the two principal logarithms. A separation into real and imaginary part gives

$$u(x,y) = \ln|z-1| + \ln|z+1| = \ln\left|z^2-1\right|,$$

and

$$v(x,y) = \mathrm{Arg}(z-1) + \mathrm{Arg}(z+1).$$

The curves

$$u(x,y) = \ln\left|z^2-1\right| = k, \qquad \text{thus} \qquad |z-1| \cdot |z+1| = c,$$

are the so-called *Cassini's rings*. These are the *equipotential curves*. I have not been able to let MAPLE give some reasonable sketches, so they are here omitted. On the other hand, it is easy to sketch the streamlines by using MAPLE, and then we may use that the equipotential curves are orthogonal on this system of curves.

It follows from

$$v(0,y) = \mathrm{Arg}(-1+iy) + \mathrm{Arg}(1+iy) = \pm\pi,$$

that the positive and the negative y-axis each form a streamline.
Since

$$\cot(u+v) = \frac{\cot u \cdot \cot v - 1}{\cos u + \cot v},$$

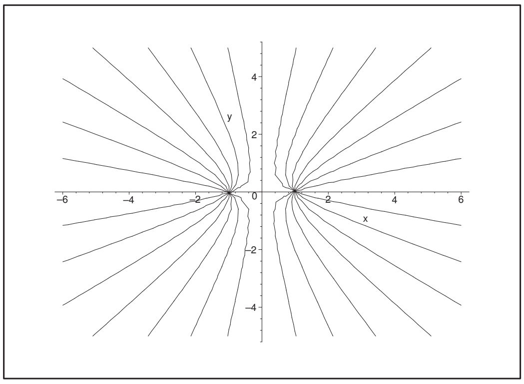


Figure 55: The streamlines $v(x,y) = c$.

and since for $x \neq 0$ and $y > 0$,

$$v(x,y) = \operatorname{Arccot}\left(\frac{x-1}{y}\right) + \operatorname{Arccot}\left(\frac{x+1}{y}\right) = c \in]0,\pi[\cup]\pi,2\pi[,$$

it follows for $y > 0$ that

$$\frac{\dfrac{x-1}{y}\cdot\dfrac{x+1}{y}-1}{\dfrac{x-1}{y}+\dfrac{x+1}{y}} = \frac{\dfrac{x^2-1}{y^2}-1}{\dfrac{2x}{y}} = \frac{x^2-y^2-1}{2xy} = \cot c,$$

which we also write

$$x^2 - y^2 - 2xy\,\cot c = 1.$$

This expression is extended by the obvious symmetry to the lower half plane. Hence the curve system becomes a system of hyperbolic arcs with the y-axis as one of their asymptotes, and where they all pass through either $(1,0)$ or $(-1,0)$. We shall of course add the lines $x = -1$ and $x = 1$ to this system.

8 Cauchy's Integral Theorem

Example 8.1 *Integrate e^z along a plane and closed curve C, which is composed of the interval $C_1 : [-1,1]$ on the real axis and the half circle C_2 of the parametric description $z(t) = e^{it}$, $t \in [0, \pi]$. Then find the value of $\int_{C_2} e^z \, dz$, and apply the result to prove that*

$$\int_0^\pi e^{\cos t} \cdot \sin(t + \sin t) \, dt = 2 \sinh 1.$$

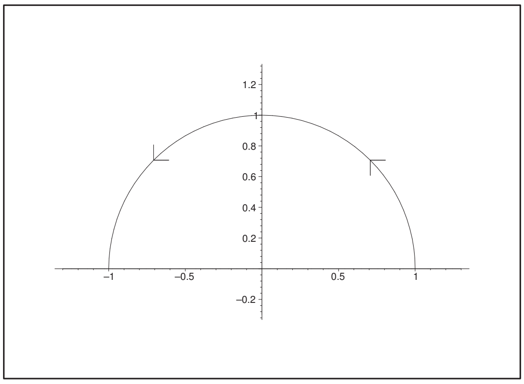


Figure 56: The closed curve C with its orientation.

Since C is a simple, closed curve in $\mathbb{C}$ and e^z is analytic in all of $\mathbb{C}$, it follows from *Cauchy's integral theorem* that

$$\oint_C e^z \, dz = \int_{C_1} e^z \, dz + \int_{C_2} e^z \, dz = 0.$$

Hence,

$$\begin{aligned}
\int_{C_2} e^z \, dz &= \int_0^\pi e^{\cos t + i \sin t} \, i \, e^{it} \, dt = i \int_0^\pi e^{\cos t} e^{i(t+\sin t)} \, dt \\
&= i \int_0^\pi e^{\cos t} \cdot \{\cos(t + \sin t) + i \sin(t + \sin t)\} \, dt \\
&= -\int_0^\pi e^{\cos t} \cdot \sin(t + \sin t) \, dt + i \int_0^\pi e^{\cos t} \cdot \cos(t + \sin t) \, dt \\
&= -\int_{C_1} e^z \, dz = -\int_{-1}^1 e^x \, dx = -\left[e^x\right]_{-1}^1 = -2 \sinh 1 + i \cdot 0.
\end{aligned}$$

Finally, by a separation into real and imaginary parts,

$$\int_0^\pi e^{\cos t} \cdot \sin(t + \sin t) \, dt = 2 \sinh 1,$$

and

$$\int_0^\pi e^{\cos t} \cdot \cos(t + \sin t) \, dt = 0.$$

Example 8.2 *Compute* $\int_1^{1+i} \left(2+3z^2+4z^3\right)\,dz$.

It follows by inspection that

$$F(z) = 2z + z^3 + z^4$$

is a primitive of $f(z) = 2+3z^2+4z^3$, thus

$$\begin{aligned}
\int_1^{1+i} \left(2+3z^2+4z^3\right)\,dz &= F(1+i) - F(1) \\
&= 2(1+i) + (1+i)^3 + (1+i)^4 - 2 - 1 - 1 \\
&= 2 + 2i + 2i(1+i) + (2i)^2 - 4 \\
&= 2 + 2i + 2i - 2 - 4 - 4 \\
&= -8 + 4i.
\end{aligned}$$

9 Cauchy's Integral Formula

Example 9.1 *Compute the values of the line integrals*

$$(a)\quad \oint_{|z|=1} \frac{z+2}{z(4-z)}\,dz, \qquad (b)\quad \oint_{|z|=2} \frac{1}{z(z-1)}\,dz, \qquad (c)\quad \oint_{|z|=2} \frac{z^3+3z^2-4}{z^2(z-1)}\,dz.$$

The method here is that first we decompose and then deform each integral into the line integral along some circle.

(a) It follows by a decomposition that

$$\frac{z+2}{z(4-z)} = -\frac{z+2}{z(z-4)} = \frac{1}{2}\cdot\frac{1}{z} - \frac{3}{2}\cdot\frac{1}{z-4}.$$

The function $\dfrac{1}{z-4}$ is analytic inside $|z|=1$, hence

$$\oint_{|z|=1} \frac{1}{z-4}\,dz = 0$$

by *Cauchy's integral theorem.*
Now,

$$\oint_{|z|=1} \frac{1}{z}\,dz = 2\pi\, i,$$

so

$$\oint_{|z|=1} \frac{z+2}{z(4-z)}\,dz = \frac{1}{2}\oint_{|z|=1} \frac{1}{z}\,dz - \frac{3}{2}\oint_{|z|=1} \frac{1}{z-4}\,dz = \frac{1}{2}\cdot 2\pi\, i + 0 = \pi\, i.$$

(b) A decomposition gives

$$\frac{1}{z(z-1)} = -\frac{1}{z} + \frac{1}{z-1}.$$

When we "reverse" the path of integration (indicated by a $\star$) and then deform it into some circle we get

$$\begin{aligned}
\oint^{\star}_{|z|=2} \frac{1}{z(z-1)}\,dz &= \oint^{\star}_{|z|=2} \left(-\frac{1}{z}\right) dz + \oint^{\star}_{|z|=2} \frac{1}{z-1}\,dz \\
&= +\oint_{|z|=1} \frac{dz}{z} - \oint_{|z-1|=1} \frac{dz}{z-1} = 2\pi\, i - 2\pi\, i = 0,
\end{aligned}$$

hence

$$\oint_{|z|=2} \frac{1}{z(z-1)}\,dz = -\oint^{\star}_{|z|=2} \frac{1}{z(z-1)}\,dz = 0.$$

(c) Since the numerator and the denominator have the same degree, and since $z = 1$ is a root of both the denominator and the numerator, we must be careful here, when we decompose. We get

$$\frac{z^3 + 3z^2 - 4}{z^2(z-1)} = \frac{\left(z^2 + 4z + 4\right)(z-1)}{z^2(z-1)} = \frac{z^2 + 4z + 4}{z^2} = 1 + \frac{4}{z} + \frac{4}{z^2}.$$

Here $f_1(z) = 1$ is analytic inside $|z| = 2$, so

$$\oint_{|z|=2} 1\, dz = 0.$$

Furthermore, $f_2(z) = -\dfrac{4}{z}$ is differentiable in $\Omega \setminus \{0\}$ with the derivative

$$f_2'(z) = \frac{4}{z^2},$$

so $\dfrac{4}{z^2}$ has the primitive $-\dfrac{4}{z}$, and we conclude that

$$\oint_{|z|=2} \frac{4}{z^2}\, dz = 0.$$

Hence,

$$\oint_{|z|=2} \frac{z^3 + 3z^2 - 4}{z^2(z-1)}\, dz = \oint_{|z|=2} 1\, dz + 4 \oint_{|z|=2} \frac{1}{z}\, dz + \oint_{|z|=2} \frac{4}{z^2}\, dz = 0 + 4 \cdot 2\pi\, i + 0 = 8\pi\, i.$$

Example 9.2 *Assume that $f : [0, +\infty[\to \mathbb{C}$ is continuous and that $f(t) = 0$ for $t > R$. Prove that the function $\mathcal{L}\{f\}$, given by*

$$\mathcal{L}\{f\}(z) = \int_0^{+\infty} f(t)\, e^{-zt}\, dt$$

is analytic in all of $\mathbb{C}$.
Assume that $f : [0, +\infty[\to \mathbb{C}$ is continuous and that there exist constants A, $B > 0$, such that

$$|f(t)| \le A\, e^{Bt} \qquad \text{for every } t \in [0, +\infty[.$$

Prove that one can find a $\sigma \in \mathbb{R}$, such that the function

$$\mathcal{L}\{f\}(z) = \int_0^{+\infty} f(t)\, e^{-zt}\, dt$$

is analytic in the half plane $Re(z) > \sigma$.
We call the analytic function $\mathcal{L}\{f\}(z)$ the Laplace-transformed *of f. The smallest real number σ_0, for which $\mathcal{L}\{f\}(z)$ is analytic in the half plane $Re(z) > \sigma_0$, is called the* abscissa of convergence.

When $f : [0, +\infty[\to \mathbb{C}$ is continuous, and $f(t) = 0$ for $t > R$, then the support of f is compact, so $|f(t)|$, which is also continuous, must have a maximum. In particular, f is bounded,

$$|f(t)| \le M \qquad \text{for every } t \ge 0,$$

and we get for every fixed $z \in \mathbb{C}$ that

$$\left| \int_0^{+\infty} f(t)\, e^{-zt}\, dt \right| \le M \int_0^R \left| e^{-(x+iy)t} \right| \, dt = M \int_0^R e^{-xt}\, dt < +\infty.$$

We conclude that $\mathcal{L}\{f\}(z)$ is defined for every $z \in \mathbb{C}$.

Since the integrand $f(t)\, e^{-zt}$ is continuous in t and since the derivative with respect to the parameter z is continuous and absolutely integrable, it follows that $\mathcal{L}\{f\}(z)$ is continuously differentiable, and its derivative is

$$\frac{d}{dz} \mathcal{L}\{f\}(z) = \int_0^{+\infty} \frac{\partial}{\partial z} \left\{ f(t)\, e^{-zt} \right\} \, dt = - \int_0^{+\infty} t\, f(t)\, e^{-zt}\, dt,$$

which proves that $\mathcal{L}\{f\}(z)$ is analytic in all of $\mathbb{C}$.

Then we assume that

$$|f(t)| \le A\, e^{Bt} \qquad \text{for every } t \in [0, +\infty[.$$

If $\sigma > B$, then we get for $\mathrm{Re}(z) \ge \sigma$ that

$$\left| \int_0^{+\infty} f(t)\, e^{-zt}\, dt \right| \le \int_0^{+\infty} A\, e^{Bt}\, e^{-t\,\mathrm{Re}(x)}\, dt \le A \int_0^{+\infty} e^{(\sigma - B)t}\, dt < +\infty,$$

so the Laplace-transform exists for every z, for which $\mathrm{Re}(z) \ge \sigma$.

When we differentiate the integrand with respect to the parameter z, then

$$\frac{\partial}{\partial z} \left\{ f(t)\, e^{-zt} \right\} = -t\, f(t)\, e^{-zt}.$$

If $\mathrm{Re}(z) \ge \sigma > b$, then we have the estimate

$$\left| \frac{\partial}{\partial z} \left\{ f(t)\, e^{-zt} \right\} \right| \le A\, t\, e^{Bt}\, e^{-\sigma t} = A\, t\, e^{-(\sigma - B)t}, \qquad t \ge 0.$$

It follows from the magnitudes that $\dfrac{\partial}{\partial z} \{f(t)\, e^{-zt}\}$ has an integrable majoring function, thus we conclude that $\mathcal{L}\{f\}$ is complex differentiable (and even continuously differentiable) for $\mathrm{Re}(z) > \sigma$. This proves that $\mathcal{L}\{f\}(z)$ is analytic in the half plane $\mathrm{Re}(z) > \sigma$.

Example 9.3 *Prove that*

$$\sum_{n=0}^{+\infty} \left\{ \exp\left(\frac{z^n}{n!} \right) - 1 \right\}$$

defines an analytic function $f : \mathbb{C} \to \mathbb{C}$.

If we put

$$f_n(z) = \exp\left(\frac{z^n}{n!} \right) - 1,$$

then $f_n(z)$ is analytic in all of $\mathbb{C}$.

If $|z| \leq R$, then

$$|f_n(z)| \leq \exp\left(\frac{R^n}{n!}\right) - 1,$$

so if we can prove that the series

$$\sum_{n=0}^{+\infty} \left\{\exp\left(\frac{R^n}{n!}\right) - 1\right\}$$

is convergent for every $R > 0$, then it follows that

$$f(z) = \sum_{n=0}^{+\infty} f_n(z)$$

is *uniformly convergent* on every compact subset of $\mathbb{C}$, and the claim is proved.

Then we have for every fixed R,

$$\frac{R^n}{n!} \to 0 \qquad \text{for } n \to +\infty.$$

Thus, we can find an N, such that for every $n \geq N$,

$$\exp\left(\frac{R^n}{n!}\right) \leq 1 + 2\,\frac{R^n}{n!}.$$

Then

$$\begin{aligned} 0 &< \sum_{n=0}^{+\infty}\left\{\exp\left(\frac{R^n}{n!}\right)-1\right\} = \sum_{n=0}^{N-1}\left\{\exp\left(\frac{R^n}{n!}\right)-1\right\} + \sum_{n=N}^{+\infty}\left\{\exp\left(\frac{R^n}{n!}\right)-1\right\} \\ &\leq \sum_{n=0}^{N-1}\left\{\exp\left(\frac{R^n}{n!}\right)-1\right\} + 2\sum_{n=N}^{+\infty}\frac{R^n}{n!} \leq \sum_{n=0}^{N-1}\left\{\exp\left(\frac{R^n}{n!}\right)-1\right\} + 2\,e^R < +\infty, \end{aligned}$$

and the claim is proved.

Remark 9.1 It follows that the derivative is given by

$$f'(z) = \sum_{n=1}^{+\infty}\frac{z^{n-1}}{(n-1)!}\,\exp\left(\frac{z^n}{n!}\right).$$

In this case it is easy to prove the uniform convergence over compact subsets. In fact, if $|z| \leq R$, then we can find an N, such that

$$\left|\exp\left(\frac{z^n}{n!}\right)\right| \leq M \qquad \text{for } |z| \leq R \text{ and } n \geq N,$$

thus

$$\left|\sum_{n=N+1}^{+\infty}\frac{z^{n-1}}{(n-1)!}\,\exp\left(\frac{z^n}{n!}\right)\right| \leq M\sum_{n=N+1}^{+\infty}\frac{R^n}{n!} \leq M\,e^R < +\infty. \qquad \Diamond$$

Example 9.4 *Assume that f is analytic in an open domain Ω. Define $\Omega^\star$ by*

$$\Omega^\star = \{\overline{z} \mid z \in \Omega\},$$

and a function $f^\star$ on $\Omega^\star$ by

$$f^\star(z) = \overline{f(\overline{z})}.$$

Prove that $f^\star$ is analytic in $\Omega^\star$.

Obviously, $f^\star$ is continuous and of class $C^\infty(\Omega^\star)$. Write $f = u + iv$, i.e.

$$f(z) = u(x,y) + i\,v(x,y).$$

Then

$$\begin{aligned} f^\star(z) &= u^\star(x,y) + i\,v^\star(x,y) = \overline{f(\overline{z})} \\ &= \overline{f(x-iy)} = u(x,-y) - i\,v(x,-y). \end{aligned}$$

If $(x,y) \in \Omega^\star$, then $(x,-y) \in \Omega$, and we obtain by partial differentiation,

$$\begin{aligned}
\frac{\partial u^\star}{\partial x}(x,y) &= \frac{\partial u}{\partial x}(x,-y),\\
\frac{\partial v^\star}{\partial y}(x,y) &= -\frac{\partial}{\partial y}\{v(x,-y)\} = +\frac{\partial v}{\partial y}(x,-y),\\
\frac{\partial u^\star}{\partial y}(x,y) &= \frac{\partial}{\partial y}\{u(x,-y\} = -\frac{\partial u}{\partial y}(x,-y),\\
\frac{\partial v^\star}{\partial x}(x,y) &= -\frac{\partial}{\partial x}\, v(x,-y) = -\frac{\partial v}{\partial x}(x,-y),
\end{aligned}$$

hence

$$\begin{aligned}
\frac{\partial u^\star}{\partial x}(x,y) &= \frac{\partial u}{\partial x}(x,-y) = \frac{\partial v}{\partial y}(x,-y) = \frac{\partial v^\star}{\partial y}(x,y),\\
\frac{\partial u^\star}{\partial y}(x,y) &= -\frac{\partial u}{\partial y}(x,-y) = \frac{\partial v}{\partial x}(x,-y) = -\frac{\partial v^\star}{\partial x}(x,y),
\end{aligned}$$

proving that $f^\star$ satisfies Cauchy-Riemann's equations everywhere in $\Omega^\star$, hence $f^\star$ is analytic in $\Omega^\star$.

Example 9.5 *Find by an application of Cauchy's integral formula*

$$(a)\ \frac{1}{2\pi i}\oint_{|z-2|=1}\frac{e^z}{z-2}\,dz, \qquad (b)\ \frac{1}{2\pi i}\oint_{|z|=1}\frac{z^2+4}{z}\,dz, \qquad (c)\ \oint_{|z|=4}\frac{\sin z}{z}\,dz.$$

(a) If we put $f(z) = e^z$, then $f(z)$ is analytic in $\mathbb{C}$. Since $z_0 = 2$ lies inside the circle $|z-2| = 1$, it follows from Cauchy's integral formula that

$$\frac{1}{2\pi i}\oint_{|z-2|=1}\frac{e^z}{z-2}\,dz = f(2) = e^2.$$

(b) If we put $f(z) = z^2+4$, then $f(z)$ is analytic in $\mathbb{C}$. Since $z_0 = 0$ lies inside the circle $|z| = 1$, it follows from Cauchy's integral formula that

$$\frac{1}{2\pi i}\oint_{|z|=1}\frac{z^2+4}{z}\,dz = f(0) = 4.$$

(c) If we put $f(z) = \sin z$, then $f(z)$ is analytic in $\mathbb{C}$. Since $z_0 = 0$ lies inside $|z| = 4$, it follows from Cauchy's integral formula that

$$\frac{1}{2\pi i}\oint_{|z|=4}\frac{\sin z}{z}\,dz = f(0) = \sin 0 = 0.$$

Example 9.6 *Find by applying Cauchy's integral formula*

$$(a)\ \oint_{|z|=4}\left(\frac{1}{z+1}+\frac{2}{z-3}\right)dz, \qquad (b)\ \oint_{|z|=2}\frac{1}{z^2-1}\,dz.$$

(a) If we put $f(z)=1$ and $g(z)=2$, then

$$\oint_{|z|=4}\left(\frac{1}{z+1}+\frac{2}{z-3}\right)dz=\oint_{|z|=4}\frac{f(z)}{z-(-1)}\,dz+\oint_{|z|=4}\frac{g(z)}{z-3}\,dz=2\pi\,i\{f(-1)+g(3)\}=6\pi\,i.$$

We have used that the curve $|z|=4$ is simple and closed and that the points -1 and 3 lie inside this curve.

(b) We get by a decomposition,

$$\frac{1}{z^2-1}=\frac{\frac{1}{2}}{z-1}-\frac{\frac{1}{2}}{z+1}.$$

Since 1 and -1 lie inside the simple, closed curve $|z|=2$, it follows as above that

$$\oint_{|z|=2}\frac{1}{z^2-1}\,dz=\frac{1}{2}\oint_{|z|=2}\frac{1}{z-1}\,dz-\frac{1}{2}\oint_{|z|=2}\frac{1}{z+1}\,dz=\frac{1}{2}\cdot 2\pi\,i-\frac{1}{2}\cdot 2\pi\,i=0.$$

Example 9.7 *Prove by means of Cauchy's integral formula that for every $k\in\mathbb{R}$,*

$$\oint_{|z|=1}\frac{e^{kz}}{z}\,dz=2\pi\,i.$$

Apply this result together with the parametric description $z=e^{i\theta}$, $\theta\in[-\pi,\pi[$, to prove that for every $k\in\mathbb{R}$,

$$\int_0^{\pi}e^{k\,\cos\theta}\cos(k\,\sin\theta)\,d\theta=\pi.$$

If we put $f(z)=e^{kz}$, then $f(z)$ is analytic in $\mathbb{C}$, hence by Cauchy's integral formula,

$$\oint_{|z|=1}\frac{e^{kz}}{z}\,dz=2\pi\,i\,f(0)=2\pi\,i.$$

Then put $z=e^{i\theta}$ to get

$$\begin{aligned}2\pi\,i &= \oint_{|z|=1}\frac{e^{kz}}{z}\,dz=\int_{-\pi}^{\pi}\frac{e^{k\,\cos\theta+i\,k\,\sin\theta}}{e^{i\theta}}\,i\,e^{i\theta}\,d\theta=i\int_{-\pi}^{\pi}e^{k\,\cos\theta}\{\cos(k\cdot\sin\theta)+i\,\sin(k\,\sin\theta)\}\,d\theta\\ &= i\int_{-\pi}^{\pi}e^{k\,\cos\theta}\cos(k\,\sin\theta)\,d\theta=2i\int_0^{\pi}e^{k\,\cos\theta}\cos(k\,\sin\theta)\,d\theta,\end{aligned}$$

because $e^{k\,\cos\theta}\sin(k\,\sin\theta)$ is odd and $e^{k\,\cos\theta}\cos(k\,\sin\theta)$ is even in θ. Finally, we get

$$\int_0^{\pi}e^{k\,\cos\theta}\cos(k\,\sin\theta)\,d\theta=\pi.$$

Example 9.8 *Assume that $f(z)$ is analytic in $\mathbb{C}$, and that there exist constants M, $R \in \mathbb{R}_+$ and $m \in \mathbb{N}_0$, such that*

$$|f(z)| \leq M \cdot |z|^m \qquad \text{for } |z| > R.$$

Prove that $f(z)$ is a polynomial of at most degree m.
HINT: *Apply Cauchy's inequality with $n = m+1$ to prove that*

$$\left|f^{(m+1)}(z_0)\right| \leq \frac{M(m+1)!\,(r+|z_0|)^m}{r^{m+1}}$$

for $z_0 \in \mathbb{C}$ and r sufficiently large. Then conclude that $f^{(m+1)}(z_0) = 0$ for every $z_0 \in \mathbb{C}$.

It follows from Cauchy's inequality for $n = m+1$ that

$$\left|f^{(m+1)}(z_0)\right| \leq \frac{M'(m+1)!}{r^{n+1}},$$

where M' is the maximum of $|f(z)|$ on the circle $|z - z_0| = r$, and where $r > R$.

It follows from the assumption that M' can be estimated by

$$M' \leq M \cdot |z|^m \leq M\,(r+|z_0|)^m\,,$$

hence by insertion,

$$\left|f^{(m+1)}(z_0)\right| \leq \frac{M \cdot (m+1)!\,(r+|z_0|)^m}{r^{m+1}} = \frac{1}{r} \cdot M \cdot (m+1)! \left(1 + \frac{|z_0|}{r}\right)^m.$$

This inequality holds for every $r > r_0$ and for every fixed z_0. We therefore conclude that

$$f^{(m+1)}(z_0) = 0,$$

and thus $f(z)$ is a polynomial of at most degree m.

Example 9.9 *Let $f(t)$ be continuous on $\mathbb{R}$. Prove that*

$$\int_0^1 \frac{f(t)}{1-zt}\,dt$$

is a function of z, which at least is analytic for $|z| < 1$.

Assume that $|z| \le R < 1$ and $t \in [0,1]$. Then we have the uniformly convergent series expansion

$$\frac{1}{1-zt} = \sum_{n=0}^{+\infty} z^n t^n,$$

thus by insertion,

$$\int_0^1 \frac{f(t)}{1-zt}\,dt = \sum_{n=0}^{\infty} \left(\int_0^1 t^n f(t)\,dt \right) z^n.$$

Here we get the estimate

$$\left| \int_0^1 t^n f(t)\,dt \right| \le M \int_0^1 t^n\,dt = \frac{M}{n+1},$$

and we conclude that the radius of convergence of the series is ≥ 1, hence the function

$$\int_0^1 \frac{f(t)}{1-zt}\,dt$$

is analytic in the domain given by $|z| < 1$, and possibly in a larger domain, depending on the structure of $f(t)$. One should e.g. check for a possible extension, if $f(1) = 0$, and $f'(1)$ exists.

Example 9.10 *Assume that $f(z)$ is analytic for $|z| \le 1$. Prove that*

$$\frac{1}{\pi} \int\int_{|z|\le 1} f(x+iy)\,dx\,dy = f(0).$$

HINT: *Express the integral in polar coordinates and apply Cauchy's integral formula.*

If we put

$$x = r \cdot \cos\theta \qquad \text{and} \qquad y = r \cdot \sin\theta,$$

then

$$\begin{aligned} \frac{1}{\pi} \int\int_{|z|\le 1} f(x+iy)\,dx\,dy &= \frac{1}{\pi} \int_0^{2\pi} \left\{ \int_0^1 f\left(r\,e^{i\theta}\right) r\,dr \right\} d\theta = \frac{1}{\pi} \int_0^1 r \left\{ \int_0^{2\pi} f\left(r\,e^{i\theta}\right)\,d\theta \right\} dr \\ &= \frac{1}{\pi} \int_0^1 r \cdot 2\pi\, f(0)\,dr = f(0) \int_0^1 2r\,dr = f(0), \end{aligned}$$

because if follows for every $r \in]0,1]$ by Cauchy's integral formula that

$$\int_0^{2\pi} f\left(r\, e^{i\theta}\right)\, d\theta = 2\pi\, f(0).$$

10 Simple applications in Hydrodynamics

Example 10.1 *A two-dimensional stationary from of a non-compressible ideal fluid is characterized by the complex potential*

(5) $F(z) = \dfrac{-1+i}{z}$.

(a) *Using polar coordinates (r,θ), where $z = r\,e^{i\theta}$, find an equation of the equipotential curves and an equation of the streamlines.*

(b) *Then find the complex field of velocity $\overline{F'(z)}$, the x and y parts of the real field of velocity $\mathbf{V}$, and the speed $|\mathbf{V}|$ of the field corresponding to (5), where everything should be expressed as functions in (r,θ).*

The function $F_1/\zeta) = -\dfrac{A}{\zeta}$, where $\zeta = \xi + i\,\eta$ and $A > 0$ is a constant, models a field of a dipole *where the dipole is lying in $\zeta = 0$ of* strength A *and of orientation in the positive ξ-direction. Find a transformation of the form $z = \zeta\,e^{i\alpha}$ (where α is a real constant), such that $F(z) = F_1(\zeta)$. Apply this result to show that the field corresponding to (5) is a field of a dipole. Find the position and strength of the dipole, and show its orientation on a sketch.*

(a) Since

$$\begin{aligned} F(z) &= \frac{-1+i}{z} = \frac{-1+i}{\sqrt{2}} \cdot \frac{\sqrt{2}}{r\,e^{i\theta}} = \frac{\sqrt{2}}{r} \cdot \exp\left(\frac{3\pi i}{4}\right) \cdot e^{-i\theta} \\ &= \frac{\sqrt{2}}{r}\exp\left(\left(\frac{3\pi}{4}-\theta\right)\right) = -\frac{\sqrt{2}}{r}\exp\left(-i\left(\frac{\pi}{4}+\theta\right)\right), \end{aligned}$$

we find

$$\varphi(r,\theta) = \frac{\sqrt{2}}{r}\cos\left(\frac{3\pi}{4}-\theta\right) \qquad \text{and} \qquad \psi(r,\theta) = \frac{\sqrt{2}}{r}\sin\left(\frac{3\pi}{4}-\theta\right),$$

so the equation of the *equipotential curves* becomes

(6) $r = \dfrac{\sqrt{2}}{k}\cos\left(\dfrac{3\pi}{4}-\theta\right) \qquad$ for $k \neq 0$,

and $\dfrac{3\pi}{4} - \theta = \dfrac{\pi}{2} + p\pi$, thus $\theta = \dfrac{\pi}{4}$ or $\theta = \dfrac{5\pi}{4}$ for $k = 0$.

In (6) the parameter either runs through

$$\theta \in \left]-\frac{3\pi}{4}, \frac{\pi}{4}\right[\qquad \text{or} \qquad \theta \in \left]\frac{\pi}{4}, \frac{5\pi}{4}\right[,$$

depending on whether $k > 0$ or $k < 0$, because we shall have $r > 0$

Analogously we get for the *streamlines* in polar coordinates,

$$r = \frac{\sqrt{2}}{k}\sin\left(\frac{3\pi}{4}-\theta\right) \qquad \text{for } k \neq 0,$$

and

$$\theta = -\frac{\pi}{4} \qquad \text{and} \qquad \theta = \frac{3\pi}{4} \qquad \text{for } k = 0.$$

In this connection it is not convenient to apply the polar coordinates, because we in rectangular coordinates obtain

$$F(z) = \frac{-1+i}{z} = (-1+i)\left(\frac{x}{x^2+y^2} - i\,\frac{y}{x^2+y^2}\right) = \frac{1}{x^2+y^2}\,\{-x+y+i(x+y)\},$$

and we get the following equation of the *equipotential curves*,

$$\varphi(x,y) = \frac{-x+y}{x^2+y^2} = k, \qquad k \in \mathbb{R},$$

and the equation of the *streamlines*,

$$\psi(x,y) = \frac{x+y}{x^2+y^2} = k, \qquad k \in \mathbb{R}.$$

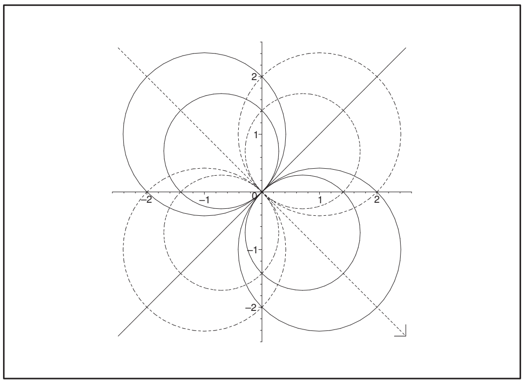


Figure 57: The equipotential curves are full-drawn lines, while the streamlines are dotted. Furthermore, the orientation is indicated on the straight streamline, and the the orientation in general follows by continuity.

If $k = 0$, then we get the equipotential curves

$$y = x, \qquad \text{for } (x,y) \neq (0,0),$$

i.e. two half lines, and the streamlines

$$y = -x, \qquad \text{for } (x,y) \neq (0,0),$$

i.e. again two half lines.
If $k \neq 0$, then we get for the *equipotential curves* that

$$\begin{aligned} 0 &= x^2 + y^2 + \frac{1}{k}\,x - \frac{1}{k}\,y + \left(\frac{1}{2k}\right)^2 + \left(\frac{1}{2k}\right)^2 - 2\cdot\left(\frac{1}{2k}\right)^2 \\ &= \left(x + \frac{1}{2k}\right)^2 + \left(y - \frac{1}{2k}\right)^2 - \left(\frac{1}{\sqrt{2}\,k}\right)^2, \end{aligned}$$

i.e. a circle with one point removed,

$$\left(x + \frac{1}{2k}\right)^2 + \left(y - \frac{1}{2k}\right)^2 = \left(\frac{1}{\sqrt{2}\,k}\right)^2, \qquad (x,y) \neq (0,0),$$

of centrum $\left(-\frac{1}{2k}, \frac{1}{2k}\right)$ and radius $\frac{1}{\sqrt{2}\,k}$.
For the *streamlines* we get the corresponding circle with one point removed,

$$\left(x - \frac{1}{2k}\right)^2 + \left(y - \frac{1}{2k}\right)^2 = \left(\frac{1}{\sqrt{2}\,k}\right)^2, \qquad (x,y) \neq (0,0),$$

of centrum $\left(\frac{1}{2k}, \frac{1}{2k}\right)$ and radius $\frac{1}{\sqrt{2}\,k}$.

(b) Then by a differentiation,

$$F'(z) = -\frac{-1+i}{z^2} = \frac{1-i}{r^2\,e^{2i\theta}},$$

and thus

$$\overline{F'(z)} = \frac{1+i}{r^2\, e^{-2i\theta}} = \frac{\sqrt{2}}{r^2}\,\exp\left(i\left(\frac{\pi}{4}+2\theta\right)\right).$$

We therefore conclude that

$$\mathbf{V} = \left(\frac{\sqrt{2}}{r^2}\,\cos\left(\frac{\pi}{4}+2\theta\right)\,,\,\frac{\sqrt{2}}{r^2}\,\sin\left(\frac{\pi}{4}+2\theta\right)\right), \qquad r>0,$$

and then the orientation of the streamlines (the dotted lines) is fixed. Finally,

$$|\mathbf{V}| = |\overline{F'(z)}| = \frac{\sqrt{2}}{r^2}, \qquad r>0.$$

(c) It follows from

$$\begin{aligned} F(z) &= -\frac{\sqrt{2}}{r}\,\exp\left(-i\left(\frac{\pi}{4}+\theta\right)\right) = -\frac{\sqrt{2}}{r\,\exp\left(i\left(\frac{\pi}{4}+\theta\right)\right)} \\ &= -\frac{\sqrt{2}}{z\,\exp\left(i\,\frac{\pi}{4}\right)} = -\frac{A}{\zeta} = F_1/\zeta), \end{aligned}$$

that the strength is $A=\sqrt{2}$. Furthermore, since $\zeta = z\,\exp\left(i\,\frac{\pi}{4}\right)$, the inverse is given by

$$z = \zeta\,\exp\left(-i\,\frac{\pi}{4}\right),$$

which is the wanted transformation.

Since the orientation corresponds to the positive ξ-axis, it follows that the orientation of the given dipole is $\theta = -\frac{\pi}{4}$.