LEIF MEJLBRO

REAL FUNCTIONS OF SEVERAL VARIABLES – DESCRIPTIO..
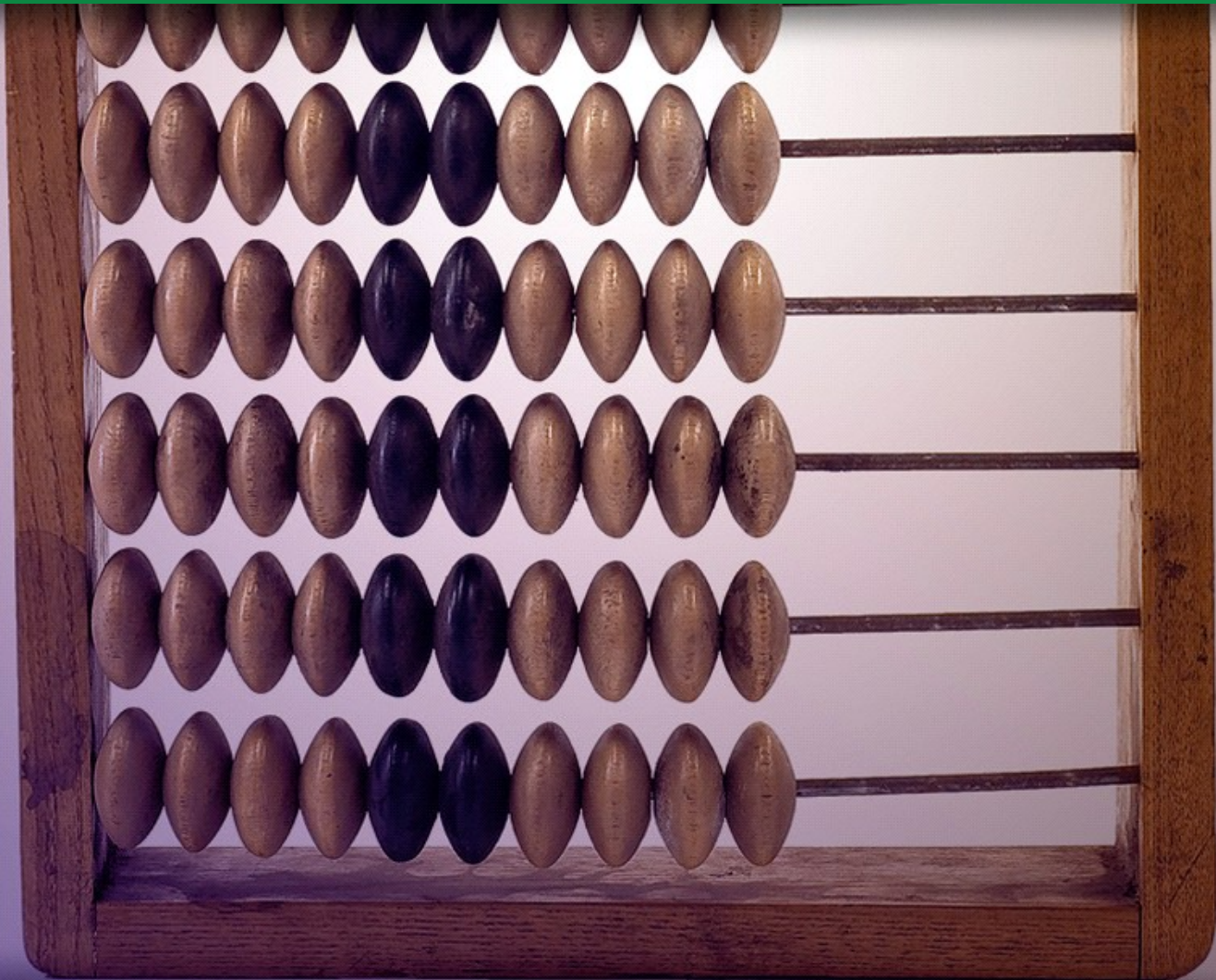
Leif Mejlbro

Real Functions of Several Variables

Examples of Description of Surfaces Partial Derivatives, Gradient, Directional Derivative and Taylor's Formula

Calculus 2c-2

Real Functions of Several Variables – Examples of Description of Surfaces Partial
Derivatives, Gradient, Directional Derivative and Taylor's Formula – Calculus 2c-2

Contents

Preface

In this volume I present some examples of *Description of Surfaces, Partial Derivatives, Gradient, Directional Derivatives* and *Taylor's Formula*, cf. also *Calculus 2b, Functions of Several Variables*. Since my aim also has been to demonstrate some solution strategy I have as far as possible structured the examples according to the following form

A *Awareness*, i.e. a short description of what is the problem.

D *Decision*, i.e. a reflection over what should be done with the problem.

I *Implementation*, i.e. where all the calculations are made.

C *Control*, i.e. a test of the result.

This is an ideal form of a general procedure of solution. It can be used in any situation and it is not linked to Mathematics alone. I learned it many years ago in the Theory of Telecommunication in a situation which did not contain Mathematics at all. The student is recommended to use it also in other disciplines.

One is used to from high school immediately to proceed to **I**. *Implementation*. However, examples and problems at university level are often so complicated that it in general will be a good investment also to spend some time on the first two points above in order to be absolutely certain of what to do in a particular case. Note that the first three points, **ADI**, can always be performed.

This is unfortunately not the case with **C** *Control*, because it from now on may be difficult, if possible, to check one's solution. It is only an extra securing whenever it is possible, but we cannot include it always in our solution form above.

I shall on purpose not use the logical signs. These should in general be avoided in Calculus as a shorthand, because they are often (too often, I would say) misused. Instead of $\wedge$ I shall either write "and", or a comma, and instead of $\vee$ I shall write "or". The arrows $\Rightarrow$ and $\Leftrightarrow$ are in particular misunderstood by the students, so they should be totally avoided. Instead, write in a plain language what you mean or want to do.

It is my hope that these examples, of which many are treated in more ways to show that the solutions procedures are not unique, may be of some inspiration for the students who have just started their studies at the universities.

Finally, even if I have tried to write as careful as possible, I doubt that all errors have been removed. I hope that the reader will forgive me the unavoidable errors.

Leif Mejlbro
17th October 2007

1   Description of surfaces

Example 1.1 *In the following there are given some surfaces in the form* $\mathbf{x} = \mathbf{r}(u, v)$, $(u, v) \in \mathbb{R}^2$. *Find in each of these cases an equation of the surface by eliminating the parameters* (u, v), *and then describe the type of the surface.*

1) $\mathbf{r}(u, v) = (u, u + 2v, v - u)$.

2) $\mathbf{r}(u, v) = (u, \sin v, 3 \cos v)$.

3) $\mathbf{r}(u, v) = (u \cos v, u \sin v, u^2 \sin 2v)$.

4) $\mathbf{r}(u, v) = (a(\cos v - u \sin v), b(\sin v + u \cos v), cu)$.

5) $\mathbf{r}(u, v) = (u \cos v, 2u \sin v, u^2)$.

6) $\mathbf{r}(u, v) = (u + v, u - v, 4v^2)$.

7) $\mathbf{r}(u, v) = (u + v, u^2 + v^2, u^3 + v^3)$.

A Description of surfaces.

D Eliminate (u, v) to obtain some known relationship between x, y, z.

I 1) Here

$$x = u, \qquad y = u + 2v, \qquad z = v - u,$$

hence

$$y - 2z = u + 2v - 2v + 2u = 3u = 3x,$$

or

$$3x - y + 2z = 0.$$

This is the equation of a plane through $(0, 0, 0)$ with the normal vector $(3, -1, 2)$.

2) Here

$$x = u, \qquad y = \sin v, \qquad z = 3 \cos v,$$

i.e.

$$y^2 + \left(\frac{z}{3}\right)^2 = 1, \qquad x = u, \quad u \in \mathbb{R}.$$

This is a cylinder surface with the X axis as its axis and the ellipse of centrum $(0, 0)$ and half axes 1 and 3 in the YZ plane as the generating curve.

3) It follows from

$$x = u \cos v, \qquad y = u \sin v, \qquad z = u^2 \sin 2v$$

that

$$2xy = 2u^2 \cos v \cdot \sin v = u^2 \sin 2v = z,$$

i.e.

$$z = 2xy,$$

which describes an hyperbolic paraboloid.

4) Here

$$\frac{x}{a} = \cos v - u \sin v, \quad \frac{y}{b} = \sin v + u \cos v, \quad \frac{z}{c} = u,$$

hence

$$\left(\frac{x}{a}\right)^2 = \cos^2 v - 2u \sin v \cdot \cos v + u^2 \sin^2 v,$$

$$\left(\frac{y}{b}\right)^2 = \sin^2 v + 2u \sin v \cdot \cos v + u^2 \cos^2 v,$$

and accordingly

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 + u^2 = 1 + \left(\frac{z}{c}\right)^2.$$

This is the equation of an hyperboloid with one net.

5) It follows from

$$x = u \cos v, \qquad \frac{y}{2} = u \sin v, \qquad z = u^2$$

that

$$x^2 + \left(\frac{y}{2}\right)^2 = u^2 = z,$$

which is the equation of an elliptic paraboloid.

6) It follows from

$$x = u + v, \qquad y = u - v, \qquad z = 4v^2$$

that $2v = x - y$, i.e.

$$z = 4v^2 = (x - y)^2.$$

This is the equation of a cylinder surface with the line $y = x$ as its axis and a parabola as its generating curve.

7) It follows from

$$x = u + v, \qquad y = u^2 + v^2, \qquad z = u^3 + v^3$$

that

$$2z = 2(u^3 + v^3) = (u + v)(2u^2 - 2uv + 2v^2) = x(2y - 2uv),$$

where

$$2uv = (u + v)^2 - (u^2 + v^2) = x^2 - y.$$

Then by insertion,

$$2z = x(3y - x^2).$$

This equation contains terms of first, second and third order.

Example 1.2 *Sketch the following cylinder surfaces.*

1) $x = \cos \varphi$, $y = \sin \varphi$, $\varphi \in \left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right]$, $z \in [1, 2\varphi]$.

2) $xy = 1$, $y \in \left[\dfrac{1}{2}, 2\right]$, $z \in [0, x]$.

3) $y = e^{-x}$, $z \in [y, 1]$.

4) $x = y^2$, $z \in [x, y]$.

A Cylinder surfaces.

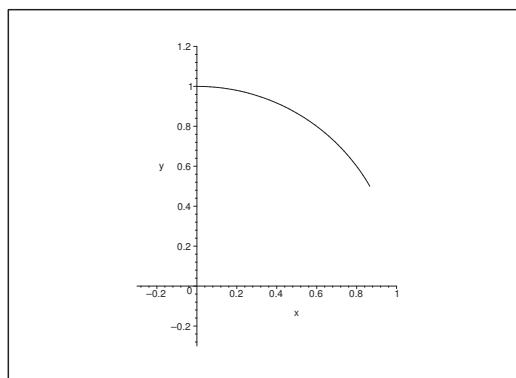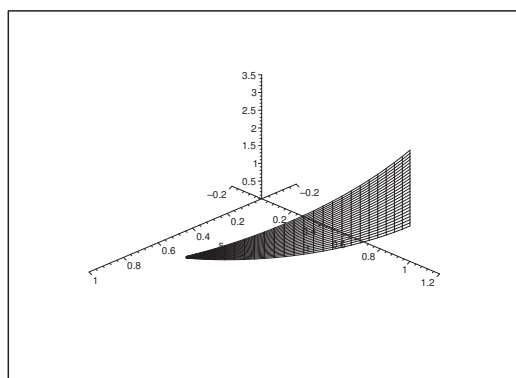**D** First sketch the projection onto the XY plane.

Figure 1: The projection onto the XY plane.

Figure 2: The cylinder surface of 1).
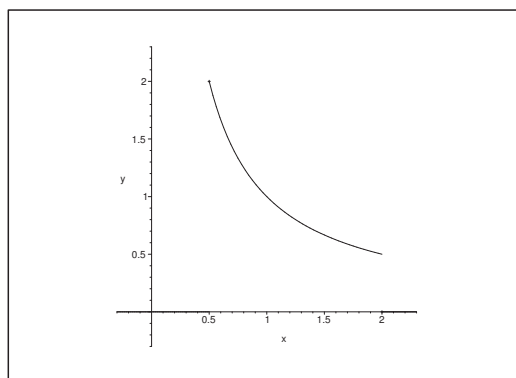
Figure 3: The projection onto the XY plane.

I 1) Here we get a circular arc in the XY- plane.

   2) The projection onto the XY plane is an arc of an hyperbola, lying in the first quadrant. Note that $x \in [\frac{1}{2}, 2]$.

Figure 4: The cylinder surface of 2).
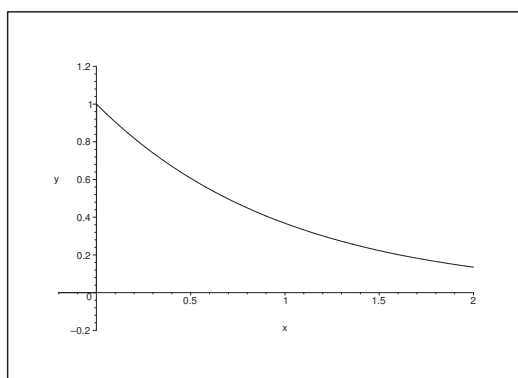
Figure 5: The projection onto the XY plane.
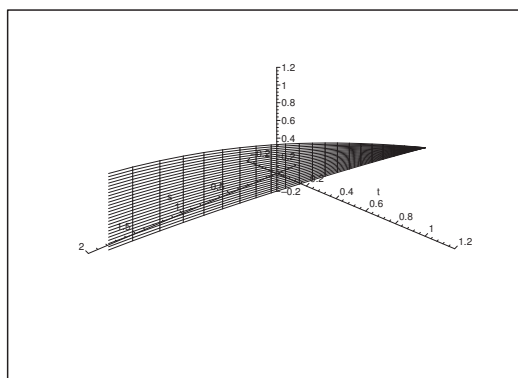
Figure 6: The cylinder surface of 3).

3) Since $y \leq 1$, we must have $x \geq 0$.

4) From $x = y^2 \leq z \leq y$ we get the condition $0 \leq y \leq 1$. On the figure the surface looks wrong. There may here be an error in the MAPLE programme, though I am not sure.
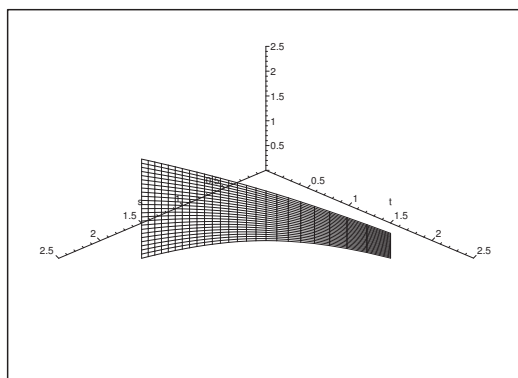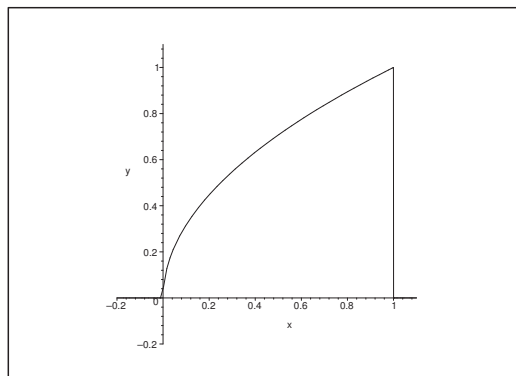
Figure 7: The projection onto the XY plane.
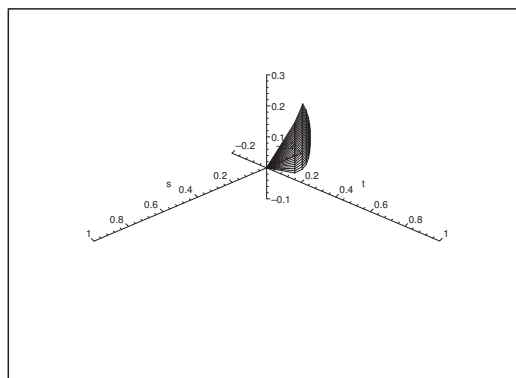
Figure 8: The cylinder surface of 4).

Example 1.3 *In the following there are given some equations of meridian curves. Set up in each case an equation of the corresponding surface of revolution $\mathcal{O}$ and find the name of $\mathcal{O}$.*

1) $z = \varrho$.

2) $\varrho = |z|$.

3) $\varrho = a$.

4) $z^2 + 2\varrho^2 = 2az$.

5) $z^2 - \varrho^2 = a^2$.

6) $\varrho^2 - z^2 = a^2$.

A Surfaces of revolution with a given meridian curve.

D First sketch the meridian curve in the PZ half plane.

I 1) This is a cone of vertex $(0, 0, 0)$.

   2) This is a double cone of vertex $(0, 0, 0)$.
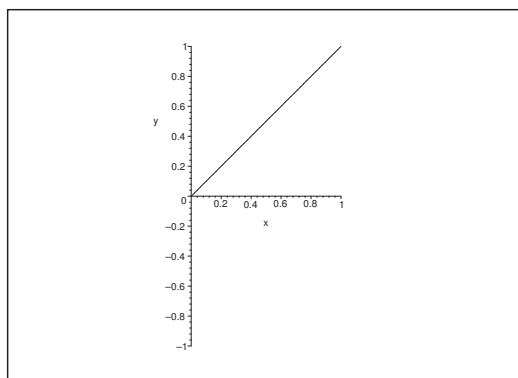
Figure 9: The meridian curve of 1).
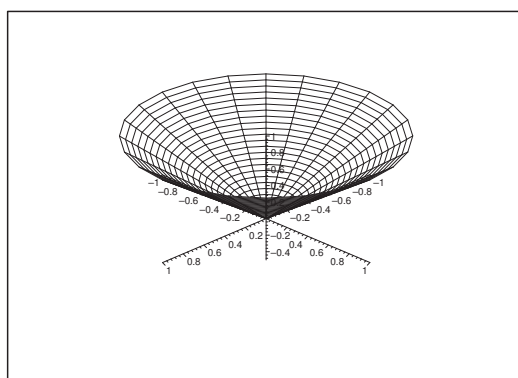
Figure 10: The surface of 1).
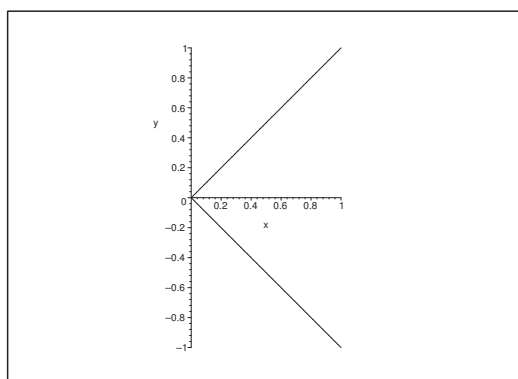
Figure 11: The meridian curve of 2).

3) This is clearly a cylinder.

4) It follows by a small rearrangement that the equation is equivalent to

$$(z - a)^2 + 2\varrho^2 = a^2,$$
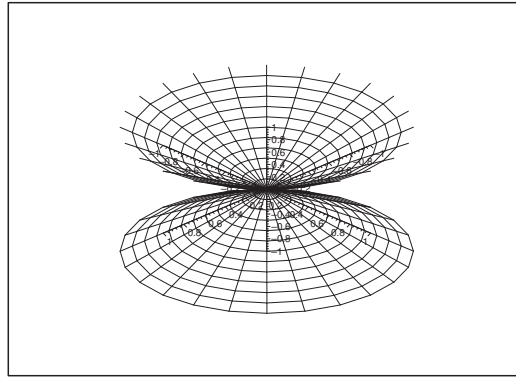
Figure 12: the surface of 2).
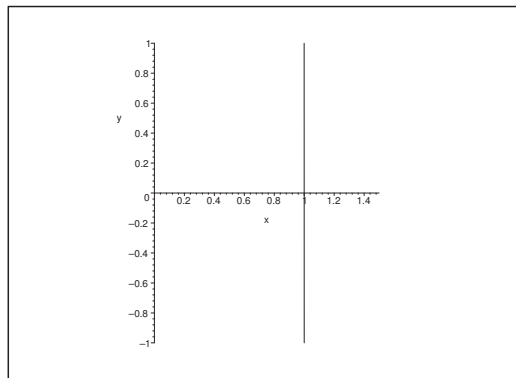
Figure 13: The meridian curve of 3).
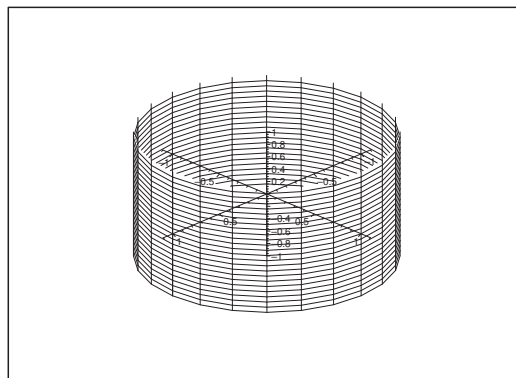
Figure 14: The surface of 3).

i.e. in the canonical form

$$\left(\frac{\varrho}{\frac{a}{\sqrt{2}}}\right)^2 + \left(\frac{z-a}{a}\right)^2 = 1, \qquad \varrho \geq 0.$$

The meridian curve is an half ellipse in the PZ half plane of centrum $(0, a)$ and half axes $\dfrac{a}{\sqrt{2}}$ and a.
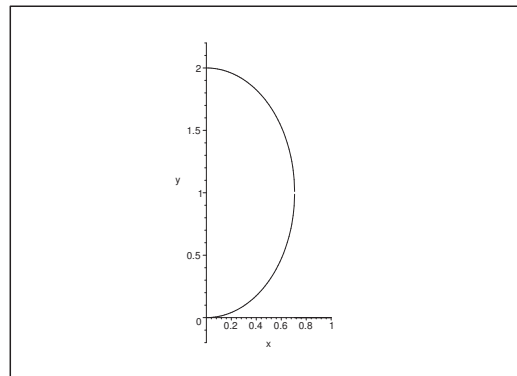
Figure 15: The meridian curve of 4).

The surface of revolution is the surface of an ellipsoid of centrum $(0, 0, a)$ and half axes $\dfrac{a}{\sqrt{2}}$, $\dfrac{a}{\sqrt{2}}$ and a. Notice that one of the top points lies at $(0, 0, 0)$. Also note that the scales are different on the axes on the figure.
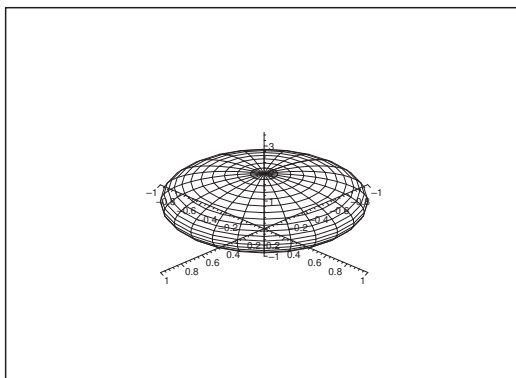
Figure 16: The surface of 4).

5) In this case the meridian curves consist of two halves of branches of an hyperbola.
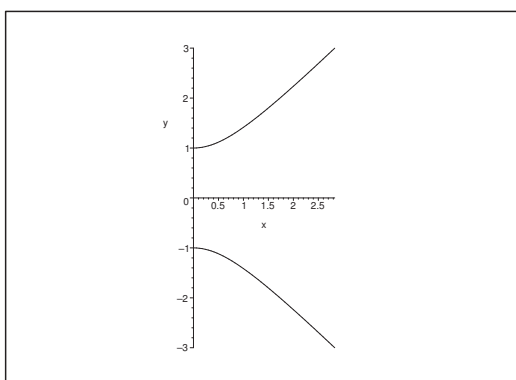
Figure 17: The meridian curves of 5).

By the revolution we get an hyperboloid with two nets. Only the upper net is sketched on the figure (and we use different scales on the axes). There is a similar surface in the lower half space.

6) The curve $\varrho^2 - z^2 = a^2$, $\varrho \geq 0$, is a branch of an hyperbola with its top point at $(a, 0)$ and its half axes a and a. The surface of revolution is an hyperboloid with one net and of centrum $(0, 0, 0)$ and with the Z axis as its axes of revolution and with the half axes a, a, a.
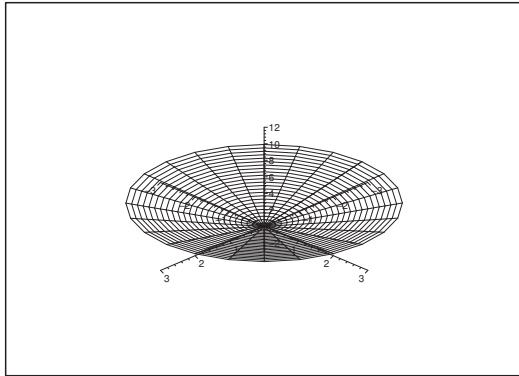
Figure 18: The upper surface of 5).
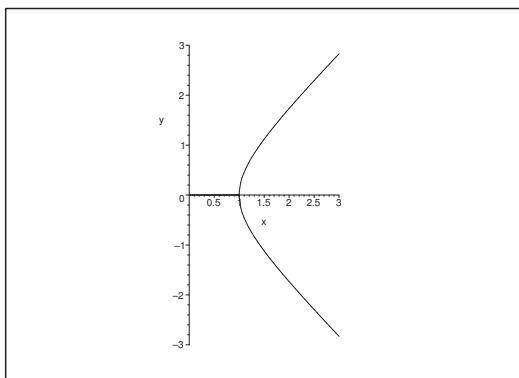
Figure 19: The meridian curve of 6).
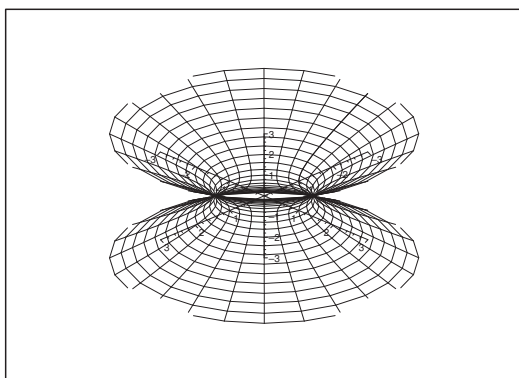
Figure 20: The surface of 6).

2 Gradient

Example 2.1 *Assume that the function $f : A \to \mathbb{R}$, $A \subseteq \mathbb{R}^k$, satisfies*

$$|f(\mathbf{x}) - f(\mathbf{u})| \le a\|\mathbf{x} - \mathbf{u}\|^{c+1}, \qquad \mathbf{x} \in K(\mathbf{u}; b),$$

where $\mathbf{u}$ is a fixed point in the open domain A of f and where b is so small that $K(\mathbf{u}; b) \subset A$. Prove that f is differentiable at the point $\mathbf{u}$ with the gradient $\mathbf{0}$.

A Differentiability; gradient.

D Analyze the definition of differentiability.

I If we put $\mathbf{x} = \mathbf{u} + \mathbf{h}$, then $\mathbf{h} = \mathbf{x} - \mathbf{u}$, and the assumption of the example can be written

$$|f(\mathbf{u} + \mathbf{h}) - f(\mathbf{u})| \le a\|\mathbf{h}\|^{1+c} = \mathbf{0} \cdot \mathbf{h} + \tilde{\varepsilon}(\mathbf{h}) \cdot \|\mathbf{h}\|,$$

where $\tilde{\varepsilon}(\mathbf{h}) = a\|\mathbf{h}\|^c \to 0$ for $\mathbf{h} \to \mathbf{0}$. This shows that there exist a function $\varepsilon(\mathbf{h})$ with $|\varepsilon(\mathbf{h})| \le \tilde{\varepsilon}(\mathbf{h})$, such that

$$f(\mathbf{u} + \mathbf{h}) - f(\mathbf{u}) = \mathbf{0} \cdot \mathbf{h} + \varepsilon(\mathbf{h}) \cdot \|\mathbf{h}\|.$$

According to the definition, f is differentiable at $\mathbf{u}$ and its gradient is

$$\bigtriangledown f(\mathbf{u}) = \mathbf{0}.$$

Example 2.2 *Let $P(x, y)$ be an* homogeneous *polynomial of degree n in two variables. Prove that*

$$xP'_x(x, y) + yP'_y(x, y) = n P(x, y).$$

Formulate and prove an analogous theorem for an homogeneous polynomial of degree n in k variables.

A Homogeneous polynomials.

D Split $P(x, y)$ into its parts and differentiate.

I A typical term in $P(x, y)$ is of the form

$$P_k(x, y) = a_k x^k y^{n-k},$$

from which we get

$$
\begin{aligned}
x\left(P_k\right)'_x + y\left(P_k\right)'_y &= a_k k x \cdot c^{k-1} y^{n-k} + a_k (n-k) s^k y \cdot y^{n-k-1} \\
&= a_k\, k\, x^k y^{n-k} + a_k (n-k) x^k y^{n-k} \\
&= n a_k x^k y^{n-k} = n\, P_k(x, y).
\end{aligned}
$$

Since differentiation and multiplication by x (or by y) are linear operations, it follows by adding all such terms that we have for any homogeneous polynomial $P(x, y)$ of degree n that

$$xP'_x(x, y) + yP'_y(x, y) = n P(x, y).$$

In general we have that if $P(x_1, \ldots, x_m)$ is an *homogeneous* polynomial of degree n in m variables, then

$$x_1 P'_{x_1}(\mathbf{x}) + \cdots + x_m P'_{x_m}(\mathbf{x}) = \sum_{j=1}^{m} x_j P'_{x_j}(\mathbf{x}) = n P(\mathbf{x}).$$

In fact, $P(\mathbf{x})$ is built up by linear combinations of terms of the form

$$Q(\mathbf{x}) = x_1^{k_1} x_1^{k_2} \cdots x_m^{k_m}, \quad k_1, \ldots, k_m \geq 0 \text{ og } k_1 + \cdots + k_m = n,$$

where

$$\sum_{j=1}^{m} x_j Q'_{x_j}(\mathbf{x}) = \sum_{j=1}^{m} k_j x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (k_1 + \cdots + k_m) Q(\mathbf{x}) = n\, Q(\mathbf{x}).$$

This holds for every term in any homogeneous polynomial $P(\mathbf{x})$, and then it follows by the linearity that it also holds for $P(\mathbf{x})$ itself.

Example 2.3 *Find in each of the following cases the gradient of the given function in two variables.*

1) $f(x, y) = \text{Arctan} \, \dfrac{x}{y}$, *for* $y \neq 0$.

2) $f(x, y) = \text{Arctan} \, \dfrac{y}{x}$, *for* $x \neq 0$.

3) $f(x, y) = \ln \dfrac{3 + xy}{4 + \sin y}$, *for* $(x, y) \in \mathbb{R}^2$, $3 + xy > 0$.

4) $f(x, y) = \ln \sqrt{x^2 + y^2}$, *for* $(x, y) \neq (0, 0)$.

A Gradients.

D Differentiate.

I 1) When $f(x, y) = \text{Arctan} \, \dfrac{x}{y}$, $y \neq 0$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \frac{1}{y} = \frac{y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{1}{1 + \left(\dfrac{x}{y}\right)^2} \cdot \left(-\frac{x}{y^2}\right) = -\frac{x}{x^2 + y^2},$$

hence

$$\nabla f(x, y) = \left(\frac{y}{x^2 + y^2}, -\frac{x}{x^2 + y^2}\right), \qquad y \neq 0.$$

2) REMARK. One might be misled to believe that this result can be derived from 1), but it turns up that it is not so. ◊

After the warning in the remark above we calculate as above for $f(x, y) = \text{Arctan} \, \dfrac{y}{x}$, $x \neq 0$, that

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{1}{1 + \left(\dfrac{y}{x}\right)^2} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2},$$

so

$$\nabla f(x, y) = \left(-\frac{y}{x^2 + y^2}, \frac{x}{x^2 + y^2}\right), \qquad x \neq 0.$$

3) When $xy > -3$, the function is defined an of class C^∞, so

$$\frac{\partial f}{\partial x} = \frac{y}{3 + xy}, \qquad \frac{\partial f}{\partial y} = \frac{x}{3 + xy} - \frac{\cos y}{4 + \sin y},$$

and

$$\nabla f(x, y) = \left(\frac{y}{3 + xy}, \frac{x}{3 + xy} - \frac{\cos y}{4 + \sin y}\right), \qquad \text{for } xy > -3.$$
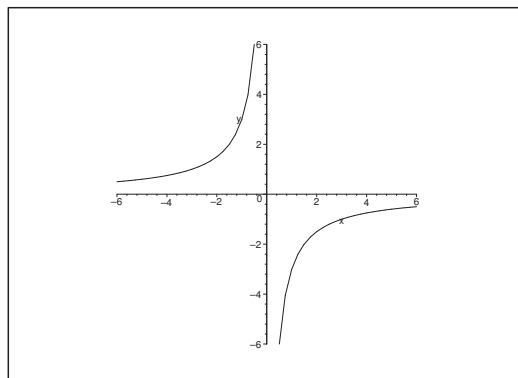
Figure 21: The domain of 3).

4) When $f(x, y) = \ln \sqrt{x^2 + y^2} = \frac{1}{2} \ln(x^2 + y^2)$, $(x, y) \neq (0, 0)$, we get

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2},$$

thus

$$\bigtriangledown f(x, y) = \left(\frac{x}{x^2 + y^2}, \frac{y}{x^2 + y^2} \right).$$

Example 2.4 *Find in each of the following cases the gradient of the given function in three variables.*

1) $f(x, y, z) = (x + y)(y + z)(z + x)$, *for* $(x, y, z) \in \mathbb{R}^3$.

2) $f(x, y, z) = x\, 3^{y+xz}$, *for* $(x, y, z) \in \mathbb{R}^3$.

3) $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, *for* $(x, y, z) \neq (0, 0, 0)$.

4) $f(x, y, z) = \exp\left(x^2 - y + z \right)$, *for* $(x, y, z) \in \mathbb{R}^3$.

5) $f(x, y, z) = x\, \tan(yz^2) + \cos(x^3 z)$, *for* $yz^2 \neq \left(p + \frac{1}{2} \right) \pi$, $p \in \mathbb{Z}$.

A Gradients.

D Differentiate.

I 1) It follows from $f(x, y, z) = (x + y)(y + z)(z + x)$ that

$$\frac{\partial f}{\partial x} = (x + z)(y + z) + (x + y)(y + z) = (2x + y + z)(y + z) = (x + y + z)^2 - x^2.$$

In this case it follows by the symmetry that we can simply interchange the letters in order to get

$$\frac{\partial f}{\partial y} = (x + y + z)^2 - y^2, \qquad \frac{\partial f}{\partial z} = (x + y + z)^2 - z^2,$$

hence

$$\nabla f = \left((x+y+z)^2 - x^2, (x+y+z)^2 - y^2, (x+y+z)^2 - z^2\right).$$

2) It follows from

$$f(x,y,z) = x\,3^{y+xz} = x\exp\{(y + xz)\ln 3\}$$

that
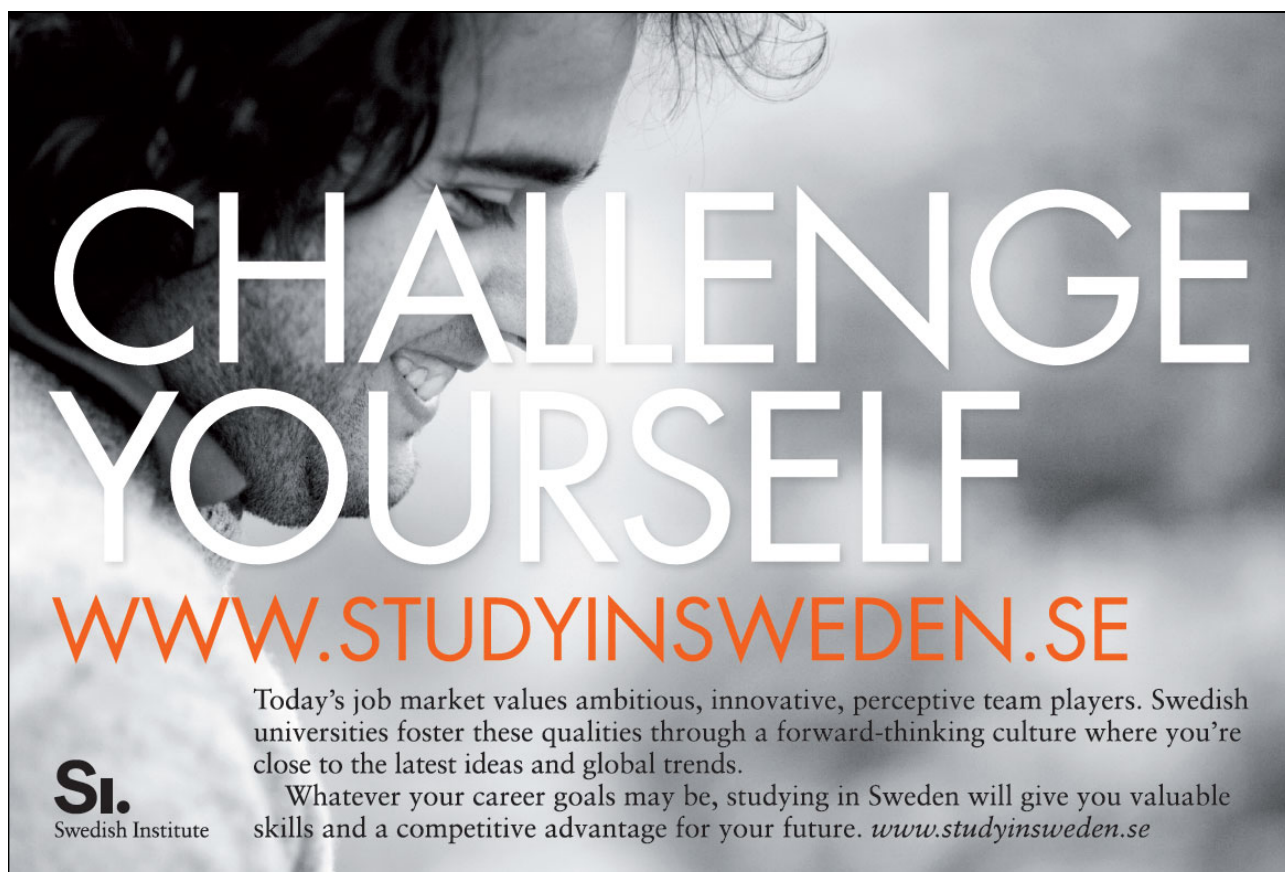
$$\frac{\partial f}{\partial x} = 3^{y+xz} + x\,3^{y+xz}z\,\ln 3 = e^{y+xz}(1 + xz\ln 3),$$

$$\frac{\partial f}{\partial y} = x\,e^{y+xz}\ln 3, \qquad \frac{\partial f}{\partial z} = x^2\ln 3 \cdot 3^{y+xz},$$

and accordingly,

$$\nabla f(x,y,z) = 3^{y+xz}\left(1 + xz\ln 3, x\ln 3, x^2\ln 3\right).$$

3) When

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}, \qquad (x, y, z) \neq (0, 0, 0),$$

we get

$$\frac{\partial f}{\partial x} = -\frac{1}{2} \cdot \frac{2x}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3} = -\frac{x}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3},$$

and by the symmetry, analogous expressions for $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$, so

$$\bigtriangledown f = -\frac{(x, y, z)}{\left(\sqrt{x^2 + y^2 + z^2}\right)^3}, \qquad (x, y, z) \neq (0, 0, 0).$$

REMARK. If we introduce the notation

$$\mathbf{r} = (x, y, z), \qquad r = \sqrt{x^2 + y^2 + z^2},$$

then this *important* result can be written in the short form

$$\bigtriangledown r = -\frac{\mathbf{r}}{r^3}. \qquad \Diamond$$

4) When $f(x, y, z) = \exp(x^2 - y + z)$, then

$$\frac{\partial f}{\partial x} = \exp(x^2 - y + z) \cdot 2x,$$

$$\frac{\partial f}{\partial y} = \exp(x^2 - y + z) \cdot (-1),$$

$$\frac{\partial f}{\partial z} = \exp(x^2 - y + z),$$

hence

$$\bigtriangledown f(x, y, z) = \exp(x^2 - y + z)(2x, -1, 1).$$

5) We see that the function

$$f(x, y, z) = x \tan(yz^2) + \cos(x^3 z)$$

is defined and of class C^∞, when $yz^2 \neq \dfrac{\pi}{2} + p\pi$, $p \in \mathbb{Z}$. Then by a differentiation

$$\frac{\partial f}{\partial x} = \tan(yz^2) - 3x^2 z \sin(x^3 z),$$

$$\frac{\partial f}{\partial y} = xz^2\{1 + \tan^2(yz^2)\} = \frac{xz^2}{\cos^2(yz^2)},$$

$$\frac{\partial f}{\partial z} = 2xyz\{1 + \tan^2(yz^2)\} - x^3 \sin(x^3 z) = \frac{2xyz}{\cos^2(yz^2)} - x^3 \sin(x^3 z),$$

and accordingly in the given domain,

$$\bigtriangledown f = \left(\tan(yz^2) - 3x^2 z \sin(x^3 z), \frac{xz^2}{\cos^2(yz^2)}, \frac{2xyz}{\cos^2(yz^2)} - x^3 \sin(x^3 x)\right).$$

Example 2.5 *In some of the cases where it is not possible to decide only by using the rules of calculation whether a given function of several variables is differentiable at some given point, one may try instead to use the definition directly in the following way.*
Use restrictions to see if the partial derivatives exist at the point. When this is the case, then insert the values into the definition of differentiability, in which the ε function occurs; then check if this ε function has the required property.

Use this procedure to prove the following claims:

- *In 1)–3) the function is not differentiable at $(0,0)$.*

- *In 4)–5) the function is differentiable at $(0,0)$ with the gradient zero.*

1) $f(x,y) = \sqrt{x^2 + y^2}$.

2) $f(x,y) = |x + y|$.

3)
$$f(x,y) = \begin{cases} \dfrac{x^3}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

4) $f(x,y) = \sqrt{x^4 + y^4}$.

5) $f(x,y) = |x^2 - y^2|$.

A Gradients by using the definition.

D Follow the given description.

I First note that if f is differentiable, then
$$f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) = \mathbf{h} \cdot \nabla f(\mathbf{x}) + \varepsilon(\mathbf{h})\|\mathbf{h}\|,$$
where $\varepsilon(\mathbf{h}) \to 0$ for $\mathbf{h} \to \mathbf{0}$.

1) Here,
$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}},$$
hence
$$\frac{\partial f}{\partial x}(x,0) \to \begin{cases} 1 & \text{for } x \to 0+, \\ -1 & \text{for } x \to 0-, \end{cases} \qquad \frac{\partial f}{\partial y}(0,y) \to \begin{cases} 1 & \text{for } y \to 0+, \\ -1 & \text{for } y \to 0-. \end{cases}$$
Then
$$\varepsilon(x,y) = \frac{1}{\sqrt{x^2 + y^2}} \left\{ f(x,y) - f(0,0) - x \frac{\text{``}\partial f\text{''}}{\partial x} - y \frac{\text{``}\partial f\text{''}}{\partial y} \right\}$$
$$= \begin{cases} 1 - \dfrac{x}{\sqrt{x^2 + y^2}} - \dfrac{y}{\sqrt{x^2 + y^2}} & \text{for } x > 0, y > 0, \\ 1 + \dfrac{x}{\sqrt{x^2 + y^2}} - \dfrac{y}{\sqrt{x^2 + y^2}} & \text{for } x < 0, y > 0, \\ 1 + \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{y}{\sqrt{x^2 + y^2}} & \text{for } x < 0, y < 0, \\ 1 - \dfrac{x}{\sqrt{x^2 + y^2}} + \dfrac{y}{\sqrt{x^2 + y^2}} & \text{for } x > 0, y < 0. \end{cases}$$

By using polar coordinates we see that these expressions do not tend to zero in the given domains, when $(x, y) \to (0, 0)$. The function is accordingly not differentiable at $(0, 0)$.

2) Here

$$\frac{\partial f}{\partial x}(x, 0) = \begin{cases} 1 & \text{for } x > 0, \\ -1 & \text{for } x < 0, \end{cases} \qquad \frac{\partial f}{\partial y}(0, y) = \begin{cases} 1 & \text{for } y > 0, \\ -1 & \text{for } y < 0, \end{cases}$$

so

$$\varepsilon(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \{|x + y| - |x| - |y|\},$$

which does not tend towards zero for $(x, y) \to (0, 0)$. [Try e.g. $y = -x$.]

3) Here,

$$\frac{\partial f}{\partial x}(x, 0) = 1 \qquad \text{og} \qquad \frac{\partial f}{\partial y}(0, y) = 0,$$

so

$$\varepsilon(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \left\{ \frac{x^3}{x^2 + y^2} - x \right\} = -\frac{xy^2}{(\sqrt{x^2 + y^2})^3} = -\cos\varphi \cdot \sin^2\varphi$$

in polar coordinates. This expression does not tend to 0 for $\varrho = \sqrt{x^2 + y^2} \to 0$.

4) Here

$$\frac{\partial f}{\partial x}(x, 0) = \frac{\partial}{\partial x}(x^2) = 2x \to 0 \qquad \text{for } x \to 0,$$

and analogously

$$\frac{\partial f}{\partial y}(0, y) = 2y \to 0 \qquad \text{for } y \to 0,$$

hence

$$\varepsilon(x, y) = \frac{1}{\sqrt{x^2 + y^2}} \left\{ \sqrt{x^4 + y^4} - 0 - 0 \right\} = \varrho\sqrt{\cos^4\varphi + \sin^4\varphi} \to 0$$

for $\varrho \to 0$.

Therefore, the function is differentiable at $\mathbf{0}$ and

$$\bigtriangledown f(\mathbf{0}) = \mathbf{0}.$$

5) Here $f(x, 0) = x^2$, so

$$\frac{\partial f}{\partial x}(x, 0) = 2x \to 0 \qquad \text{for } x \to 0,$$

and $f(0, y) = y^2$, and thus

$$\frac{\partial f}{\partial y}(0, y) = 2y \to 0 \qquad \text{for } y \to 0.$$

Then

$$\varepsilon(x, y) = \frac{|x^2 - y^2|}{\sqrt{x^2 + y^2}} = \frac{\varrho^2}{\varrho}|\cos^2\varphi - \sin^2\varphi| = \varphi|\cos 2\varphi| \to 0$$

for $\varrho \to 0$, and we conclude that the function is differentiable at $\mathbf{0}$ and

$$\bigtriangledown f(0) = \mathbf{0}.$$

Example 2.6 *Find in each of the following cases the gradient of the given function $f : \mathbb{R}^3 \to \mathbb{R}$. The vector $\mathbf{a}$ is constant.*

1) $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a}$.

2) $f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a})^2$.

3) $f(\mathbf{x}) = \|\mathbf{x} \times \mathbf{x}\|$.

4) $f(\mathbf{x}) = \mathbf{x} \times (\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a}$.

A Gradients.

D Calculate the expressions and then differentiate.

I 1) Since $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{a} = x_1 a_1 + x_2 a_2 + x_3 a_3$, it follows that

$$\nabla f(\mathbf{x}) = \mathbf{a}.$$

2) We get from $f(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{a})^2 = \{x_1 a_1 + x_2 a_2 + x_3 a_3\}^2$ that

$$\frac{\partial f}{\partial x_i} = 2a_i(x_1 a_1 + x_2 a_2 + x_3 a_3) = 2a_i\,(\mathbf{x} \cdot \mathbf{a}),$$

so we get as expected,

$$\bigtriangledown f(\mathbf{x}) = 2(\mathbf{x} \cdot \mathbf{a})\,\mathbf{a}.$$

3) Since $f(\mathbf{x}) = \|\mathbf{x} \times \mathbf{x}\| = 0$, we get $\bigtriangledown f(\mathbf{x}) = \mathbf{0}$.

4) First calculate

$$\mathbf{x} \times \mathbf{a} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = (x_2a_3 - x_3a_2, x_3a_1 - x_1a_3, x_1a_2 - x_2a_1)\,,$$

whence

$$
\begin{aligned}
\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ x_2a_3 - x_3a_2 & x_3a_1 - x_1a_3 & x_1a_2 - x_2a_1 \end{vmatrix} \\
&= (x_2(x_1a_2 - x_2a_1) - x_3(x_3a_1 - x_1a_3))\,\mathbf{e}_1 \\
&\quad + (x_3(x_2a_3 - x_3a_2) - x_1(x_1a_2 - x_2a_1))\,\mathbf{e}_2 \\
&\quad + (x_1(x_3a_1 - x_1a_3) - x_2(x_2a_3 - x_3a_2))\,\mathbf{e}_3.
\end{aligned}
$$

We conclude that

$$
\begin{aligned}
\mathbf{x} \times (\mathbf{x} \times \mathbf{a}) \cdot \mathbf{a} &= a_1(x_1x_2a_2 - x_2^2a_1 - x_3^2a_1 + x_1x_3a_3) \\
&\quad + a_2(x_2x_3a_3 - x_3^2a_3 - x_1^2a_2 + x_1x_2a_1) \\
&\quad + a_3(x_1x_3a_1 - x_1^2a_3 - x_2^2a_3 + x_2x_3a_2) \\
&= -x_1^2(a_2^2 + a_3^2) - x_2^2(a_1^2 + a_3^2) - x_3^2(a_1^2 + a_2^2) \\
&\quad + 2x_1x_2a_1a_2 + 2x_1x_3a_1a_3 + 2x_2x_3a_2a_3,
\end{aligned}
$$

which can be further reduced. This is, however, not necessary here, because we shall only need the derivatives in the following,

$$
\begin{aligned}
\frac{\partial f}{\partial x_1} &= -2x_1(a_1^2 + a_3^2) + 2x_2a_1a_2 + 2x_3a_1a_3 \\
&= -2x_1(a_1^2 + a_2^2 + a_3^2) + 2a_1(x_1a_1 + x_2a_2 + x_3a_3) \\
&= -2x_1\|\mathbf{a}\|^2 + 2a_1\mathbf{a} \cdot \mathbf{x}),
\end{aligned}
$$

$$\frac{\partial f}{\partial x_2} = -2x_2\|\mathbf{a}\|^2 + 2a_2(\mathbf{a} \cdot \mathbf{x}),$$

$$\frac{\partial f}{\partial x_3} = -2x_3\|\mathbf{a}\|^2 + 2a_2(\mathbf{a} \cdot \mathbf{x}).$$

These are the coordinates of $\bigtriangledown f$, so all things put together we finally get

$$\bigtriangledown f(\mathbf{x}) = -2(\mathbf{a} \cdot \mathbf{a})\,\mathbf{x} + 2(\mathbf{x} \cdot \mathbf{a})\,\mathbf{a}.$$

Example 2.7 *Let A denote the point set where we have removed the coordinate axes from the plane* $\mathbb{R}^2$, *i.e.*

$$A = \{(x,y) \mid xy \neq 0\}.$$

We define a function $f : A \to \mathbb{R}$ *by putting* $f(x,y)$ *equal to the number of the quadrant, which* (x,y) *belongs to. Find* $\bigtriangledown f$.

A Gradient.

D Use that f is constant on every connected component of A.

I The task is now trivial, because $f(x,y)$ is constant on each of the four open quadrants, where it is defined, hence $\bigtriangledown f = \mathbf{0}$.

3 The chain rule

Example 3.1 . *Use the chain rule to calculate the derivative of the function $F(u) = f(\mathbf{X}(u))$, i.e. without finding $F(u)$ explicitly, in the following cases:*

1) $f(x, y) = xy$, *where* $\mathbf{X}(u) = (e^u, \cos u)$, $u \in \mathbb{R}$.

2) $f(x, y) = e^{xy}$, *where* $\mathbf{X}(u) = (3u^2, u^3)$, $u \in \mathbb{R}$.

3) $f(x, y) = x^3 + y^3 - 3xy$, *where* $\mathbf{X}(u) = \left(u^2, \dfrac{3u}{1+u} \right)$, $u > -1$.

4) $f(x, y) = y^x$, *where* $\mathbf{X}(u) = (\sin u, u^3)$, $u > 0$.

5) $f(x, y) = y\, e^x$, *where* $\mathbf{X}(u) = (\text{Arctan}(1 + u), e^u)$, $u \in \mathbb{R}$.

6) $f(x, y) = y \sin x$, *where* $\mathbf{X}(u) = \left(-u, \sqrt{1 + u^2}\right)$, $u \in \mathbb{R}$.

A The chain rule.

D Start by formulating the general chain rule. No matter the formulation we shall nevertheless also calculate $F(u)$ and find the derivative in the usual way, so that it is possible to compare the two methods.

I The task is to insert (correctly) into the chain rule,

$$F'(u) = \frac{\partial f}{\partial x} \frac{dx}{du} + \frac{\partial f}{\partial y} \frac{dy}{du},$$

where x and y are the coordinates of $\mathbf{X} = (x, y)$.

1) When $f(x, y) = xy$ and $(x, y) = (e^u, \cos u)$, we get

$$F'(u) = y\, \frac{dx}{du} + x\, \frac{dy}{du} = \cos u \cdot e^u - e^u \sin u = e^u(\cos u - \sin u).$$

TEST. By insertion we also have

$$F(u) = e^u \cos u,$$

so

$$F'(u) = e^u(\cos u - \sin u).$$

We see that we get the same result, and in this case the application of the chain rule is not easier than the traditional method. ◊

2) When $f(x, y) = e^{xy}$ and $(x, y) = (3u^2, u^3)$, we get by the chain rule,

$$F'(u) = e^{xy}\, y\, \frac{dx}{dy} + e^{xy}\, x\, \frac{dy}{du} = e^{3u^5} \cdot u^3 \cdot 6u + e^{3u^5} \cdot 3u^2 \cdot 3u^2 = 15u^4 \exp(3u^5).$$

TEST. By insertion we get

$$F(u) = e^{xy} = \exp(3u^5),$$

so by a differentiation,

$$F'(u) = 15u^4 \exp(3u^5).$$

We see that the two results agree, and also that the direct method is easier to apply in this case than the chain rule. ◊

3) When $f(x, y) = x^3 + y^3 - 3xy$ and $(x, y) = \left(u^2, \dfrac{3u}{1 + u}\right)$, $u > -1$, we get

$$
\begin{aligned}
F'(u) &= (3x^2 - 3y)\frac{dx}{du} + (3y^2 - 3x)\frac{dy}{du} \\
&= 3\left(u^4 - \frac{3u}{1 + u}\right)2u + 3\left(\frac{9u^2}{(1 + u)^2} - u^2\right)\cdot\frac{3(1 + u) - 3u}{(1 + u)^2} \\
&= 6u^2\left(u^3 - \frac{3}{1 + u}\right) + 9\frac{u^2}{(1 + u)^2}\left\{\frac{9}{(1 + u)^2} - 1\right\} \\
&= \frac{81u^2}{(1 + u)^4} - \frac{9u^2}{(1 + u)^2} - \frac{18u^2}{1 + u} + 6u^5 \\
&= 6u^5 + \frac{9u^2}{(1 + u)^4}\left\{9 - (1 + u)^2 - 2(1 + u)^3\right\} \\
&= 6u^5 + \frac{9u^2}{(1 + u)^4}\{9 - 1 - 2u - u^2 - 2 - 6u - 6u^2 - 2u^3\} \\
&= 6u^5 - \frac{9u^2}{(1 + u)^4}\{2u^3 + 7u^2 + 8u - 6\}.
\end{aligned}
$$

TEST. By insertion we get

$$
F(u) = u^6 + \frac{27u^3}{(1 + u)^3} - \frac{9u^3}{1 + u},
$$

hence

$$
\begin{aligned}
F'(u) &= 6u^5 - \frac{27u^2}{1 + u} + \frac{9u^3}{(1 + u)^3} + \frac{81u^2}{(1 + u)^3} - \frac{81u^3}{(1 + u)^4} \\
&= 6u^5 + \frac{9u^2}{(1 + u)^4}\{-3(1 + u)^3 + u(1 + u)^2 + 9(1 + u) - 9u\} \\
&= 6u^5 - \frac{9u^5}{(1 + u)^4}\{3 + 9u + 9u^2 + 3u^3 - u - 2u^2 - u^3 - 9 - 9u + 9u\} \\
&= 6u^5 - \frac{9u^2}{(1 + u)^4}\{2u^3 + 7u^2 + 8u - 6\}.
\end{aligned}
$$

The two results agree. This time the two methods are more comparable in effort than in the previous ones. $\Diamond$

4) When $f(x, y) = y^x$ and $(x, y) = (\sin u, u^3)$, $u > 0$, we get

$$
\frac{\partial f}{\partial x} = \ln y \cdot y^x, \qquad \frac{\partial f}{\partial y} = x\, y^{x-1}, \qquad \frac{dx}{du} = \cos u, \qquad \frac{dy}{du} = 3u^2,
$$

so

$$
\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} \\
&= \{\ln y \cdot y^x\}\cos u + xy^{x-1}\cdot 3u^2 \\
&= \ln(u^3)\cdot u^{3\sin u}\cos u + \sin u \cdot u^{3(\sin u - 1)}\cdot 3u^2 \\
&= 3\ln u \cdot u^{3\sin u}\cos u + 3u^{3\sin u - 1}\sin u.
\end{aligned}
$$

TEST. We get by insertion

$$
F(u) = u^{3\sin u} = \exp(3\sin u \cdot \ln u), \qquad u > 0,
$$

hence

$$F'(u) = u^{3\sin u}\{3\ln u \cdot \cos u + 3\,\frac{1}{u}\,\sin u\} = 3\ln u \cdot u^{3\sin u}\cos u + 3u^{2\sin u - 1}\sin u.$$

The two results agree. ◊

5) When $f(x,y) = y\,e^x$ and $(x,y) = (\text{Arctan}(1+u), e^u)$, we get

$$\frac{\partial f}{\partial x} = y\,e^x, \quad \frac{\partial f}{\partial y} = e^x, \quad \frac{dx}{du} = \frac{1}{1+(1+u)^2}, \quad \frac{dy}{du} = e^u,$$

hence

$$
\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} \\[2mm]
&= e^u \cdot e^{\text{Arctan}(1+u)} \cdot \frac{1}{1+(1+u)^2} + e^{\text{Arctan}(1+u)} \cdot e^u \\[2mm]
&= \left\{1 + \frac{1}{1+(1+u)^2}\right\}\exp(u + \text{Arctan}(1+u)).
\end{aligned}
$$

TEST. An insertion gives

$$F(u) = e^u e^{\text{Arctan}(1+u)} = \exp\{u + \text{Arctan}(1+u)\},$$

so

$$F'(u) = \left\{1 + \frac{1}{1+(1+u)^2}\right\} \exp\{u + \text{Arctan}(1+u)\}.$$

The two results agree. ◊

6) When $f(x,y) = y \sin x$ and $(x,y) = (-u, \sqrt{1+u^2})$, we get

$$
\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x}\frac{dx}{du} + \frac{\partial f}{\partial y}\frac{dy}{du} \\
&= y \cos x \cdot (-1) + \sin x \cdot \frac{u}{\sqrt{1+u^2}} \\
&= -\sqrt{1+u^2} \cdot \cos(-u) + \sin(-u) \cdot \frac{u}{\sqrt{1+u^2}} \\
&= -\sqrt{1+u^2} \cdot \left\{\cos u + \frac{u \sin u}{1+u^2}\right\}.
\end{aligned}
$$

TEST. By insertion,

$$F(u) = -\sqrt{1+u^2} \cdot \sin u,$$

hence

$$
\begin{aligned}
F'(u) &= -\sqrt{1+u^2} \cdot \cos u - \frac{u}{\sqrt{1+u^2}} \cdot \sin u \\
&= -\sqrt{1+u^2} \cdot \left\{\cos u + \frac{u \sin u}{1+u^2}\right\}.
\end{aligned}
$$

The two results agree. ◊

REMARK. From a pedagogical point of view it is very inconvenient that the usual method is easier to apply in all cases than the chain rule. It is therefore here not very convincing that the chain rule is a practical device in some situations, where the usual calculation becomes messy. The reader is referred to **Example 3.6**, where the direct calculation is not possible, and yet the result can be obtained by using the chain rule instead. ◊

Example 3.2 *Calculate the partial derivatives of the function $F(u, v) = f(\mathbf{X}(u, v))$ by means of the chain rule, i.e. without finding $F(u, v)$ explicitly, in the following cases:*

1) $f(x, y) = x^2 y$, $\mathbf{X}(u, v) = (u + v, uv)$, *where* $(x, y) \in \mathbb{R}^2$.

2) $f(x, y) = \dfrac{x}{x + y}$, $\mathbf{X}(u, v) = (u^2 + v^2, 2uv)$, *where* $(u, v) \in \mathbb{R}_+^2$.

3) $f(x, y) = \operatorname{Arctan} \dfrac{y}{x}$, $\mathbf{X}(u, v) = (u^2 - uv + v^2, 2uv)$, *where* $(u, v) \neq (0, 0)$.

4) $f(x, y) = \operatorname{Arctan}(x + y^2)$, $\mathbf{X}(u, v) = (u, \exp(u \sin v))$, *where* $(u, v) \in \mathbb{R}^2$.

5) $f(x, y) = x \cos y$, $\mathbf{X}(u, v) = \sqrt{1 + u^2 + v^2} \cdot \operatorname{Arcsin} u$, *where* $|u| < 1$ *and* $v \in \mathbb{R}$.

6) $f(x, y) = x \sinh y$, $\mathbf{X}(u, v) = (u^3 v, \ln u + \ln v)$, *where* $(u, v) \in \mathbb{R}_+^2$.

A Partial derivatives of composite functions by the chain rule.

D Set up the chain rule. Then differentiate in each case and insert. In spite of the text we shall nevertheless check the result by using the traditional method in the test.

I The chain rule is written in two versions,

$$\frac{\partial F}{\partial u} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial F}{\partial v} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial v},$$

where one should be very careful to insert the right coordinates. Whenever f and x and y are present, we first calculate in the intermediate coordinates x and y, and then afterwards we put $x = x(u, v)$ and $y = y(u, v)$. Therefore, in the rough workings we obtain a mixed result in which both x and y as well as u and v occur. Then x and y are eliminated in the next step.

1) When $f(x, y) = x^2 y$ and $(x, y) = \mathbf{X}(u, v) = (u + v, uv)$, then

$$\frac{\partial f}{\partial x} = 2xy \qquad \text{og} \qquad \frac{\partial f}{\partial y} = x^2,$$

and

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = v, \qquad \frac{\partial x}{\partial v} = 1, \quad \frac{\partial y}{\partial v} = u,$$

so

$$\frac{\partial F}{\partial u} = 2xy \cdot 1 + x^2 \cdot v = 2(u + v)uv + (u + v)v = v(u + v)(3u + v),$$

and

$$\frac{\partial F}{\partial v} = 2xy \cdot 1 + x^2 \cdot u = 2(u + v)uv + (u + v)^2 u = u(u + v)(u + 3v).$$

TEST. We get by insertion

$$F(u, v) = (u + v)^2 uv,$$

thus

$$\frac{\partial F}{\partial u} = 2(u + v)v + (u + v)^2 v = v(u + v)(3u + v),$$

and

$$\frac{\partial F}{\partial v} = \frac{2u(v-u)}{(u+v)^3}.$$

The results agree. ◊

2) Consider

$$f(x,y) = \text{Arctan } \frac{y}{x}, \qquad \mathbf{X}(u,v) = (u^2 - uv + v^2, 2uv), \quad (u,v) \neq (0,0).$$

We first check that the composite function is defined (and of class C^∞, where it is defined). Here we shall just check that $x \neq 0$ for $(u,v) \neq (0,0)$. Now

$$x(u,v) = u^2 - uv + v^2 = \left(u - \frac{1}{2}v\right)^2 + \frac{3}{4} \neq 0 \quad \text{for } (u,v) \neq (0,0).$$

Therefore, $f(\mathbf{X}(u,v))$ is defined and of class C^∞ for $(u,v) \neq (0,0)$.

In the calculations we shall need $x^2 + y^2$ expressed by u and v. We see that

$$\begin{aligned}
x^2 + y^2 &= (u^2 - uv + v^2)^2 + 4u^2v^2 \\
&= u^4 + u^2v^2 + v^4 - 2u^3v + 2u^2v^2 - 2uv^3 + 4u^2v^2 \\
&= u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4.
\end{aligned}$$

REMARK. It is not worth trying the variant

$$x(u,v) = u^2 - uv + v^2 = \frac{u^3 + v^3}{u+v} \qquad \text{for } u \neq -v,$$

because the following expressions are very complicated. ◊

Then by a calculation,

$$\frac{\partial f}{\partial x} = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{x}{x^2 + y^2},$$

and

$$\frac{\partial x}{\partial u} = 2u - v, \quad \frac{\partial y}{\partial u} = 2v, \qquad \frac{\partial x}{\partial v} = 2v - u, \quad \frac{\partial y}{\partial v} = 2u,$$

so

$$\begin{aligned}
\frac{\partial F}{\partial u} &= -\frac{y}{x^2 + y^2} \cdot (2u - v) + \frac{x}{x^2 + y^2} \cdot 2v = \frac{-2uv(2u - v) + (u^2 - uv + v^2) \cdot 2v}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4} \\
&= \frac{2v(-2u^2 + uv + u^2 - uv + v^2)}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4} = \frac{2v(v^2 - u^2)}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4},
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial F}{\partial v} &= -\frac{y}{x^2 + y^2} \cdot (2v - u) + \frac{x}{x^2 + y^2} \cdot 2u = \frac{-2uv(2v - u) + (u^2 - uv + v^2)2u}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4} \\
&= \frac{2u(-2v^2 + uv + u^2 - uv + v^2)}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4} = \frac{2u(u^2 - v^2)}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4}.
\end{aligned}$$

Test. We get by insertion that

(1) $F(u,v) = \mathrm{Arctan}\left(\dfrac{2uv}{u^2 - uv + v^2}\right) = F(v,u),$

thus

$$
\begin{aligned}
\frac{\partial F}{\partial u} &= \frac{1}{1 + \left\{\dfrac{2uv}{u^2 - uv + v^2}\right\}^2} \cdot \left\{\frac{2v(u^2 - uv + v^2) - 2uv(2u - v)}{(u^2 - uv + v^2)^2}\right\} \\
&= \frac{2v(u^2 - uv + v^2 - 2u^2 + uv)}{(u^2 - uv + v^2)^2 + 4u^2v^2} = \frac{2v(v^2 - u^2)}{u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4}.
\end{aligned}
$$

Due to the symmetry of (1) we obtain $\dfrac{\partial F}{\partial v}$ by interchanging u and v.

The results agree. $\diamond$

REMARK. One may wonder why there is given no attempt to reduce the denominator $u^4 - 2u^3v + 7u^2v^2 - 2uv^3 + v^4$ as a product of factors $u - av$ of first degree. The reason is that a then must satisfy the equation

$$a^4 - 2a^3 + 7a^2 - 2a + 1 = 0,$$

of fourth degree and with $\pm$ integers as coefficients. It can be proved that the only *possible* rational roots must be of the form $a = \pm\dfrac{1}{1} = \pm 1$, and it is easily seen that none of these possibilities satisfies the equation. The problem is therefore to solve an equation of fourth degree without any rational solutions, and such a procedure is not commonly known in Calculus courses. $\Diamond$

3) When $f(x,y) = \operatorname{Arctan}(x + y^2)$ and $(x,y) = \mathbf{X}(u,v) = \left(u, e^{u\sin v}\right)$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (x + y^2)^2} \qquad \text{and} \qquad \frac{\partial f}{\partial y} = \frac{2y}{1 + (x + y^2)^2},$$

and

$$\frac{\partial x}{\partial u} = 1, \quad \frac{\partial y}{\partial u} = \sin v \cdot e^{u\sin v}, \qquad \frac{\partial x}{\partial v} = 0, \quad \frac{\partial y}{\partial v} = u\cos v \cdot e^{u\sin v},$$

hence

$$\begin{aligned}
\frac{\partial F}{\partial u} &= \frac{1}{1 + (x + y^2)^2} \cdot 1 + \frac{2y}{1 + (x + y^2)^2} \cdot \sin v \cdot e^{u\sin v} \\
&= \frac{1}{1 + (u + e^{2u\sin v})^2} \cdot (1 + 2\sin v \cdot e^{2u\sin v}),
\end{aligned}$$

and

$$\frac{\partial F}{\partial v} = \frac{1}{1 + (x + y^2)^2} \cdot 0 + \frac{2y}{1 + (x + y^2)^2} \cdot e^{u\sin v} \cdot u\cos v = \frac{2u\cos v \cdot e^{2u\sin v}}{1 + (u + e^{2u\sin v})^2}.$$

TEST. We get by insertion,

$$F(u,v) = \operatorname{Arctan}(u + e^{2u\sin v}),$$

hence

$$\frac{\partial F}{\partial u} = \frac{1 + 2\sin v e^{2u\sin v}}{1 + (u + e^{2u\sin v})^2} \quad \text{and} \quad \frac{\partial F}{\partial v} = \frac{2u\cos v \cdot e^{2u\sin v}}{1 + (u + e^{2u\sin v})^2}.$$

The results agree. $\Diamond$

4) When $f(x,y) = x\cos y$ and $(x,y) = \mathbf{X}(u,v) = (\sqrt{1 + u^2 + v^2}, \operatorname{Arcsin} u)$, it follows that the composite function is defined and of class C^∞ for $|u| < 1$ and $v \in \mathbb{R}$. Then,

$$\frac{\partial f}{\partial x} = \cos y \qquad \text{and} \qquad \frac{\partial f}{\partial y} = -x\sin y,$$

as well as

$$\frac{\partial x}{\partial u} = \frac{u}{\sqrt{1 + u^2 + v^2}}, \qquad \frac{\partial y}{\partial u} = \frac{1}{\sqrt{1 - u^2}},$$

$$\frac{\partial x}{\partial v} = \frac{v}{\sqrt{1+u^2+v^2}}, \qquad \frac{\partial y}{\partial v} = 0.$$

We get accordingly,

$$\frac{\partial F}{\partial u} = \frac{\cos(\text{Arcsin } u) \cdot u}{\sqrt{1+u^2+v^2}} - \frac{\sqrt{1+u^2+v^2} \cdot \sin(\text{Arcsin } u)}{\sqrt{1-u^2}} = \frac{u\sqrt{1-u^2}}{\sqrt{1+u^2+v^2}} - \frac{u\sqrt{1+u^2+v^2}}{\sqrt{1-u^2}}$$

$$= u \left\{ \sqrt{\frac{1-u^2}{1+u^2+v^2}} - \sqrt{\frac{1+u^2+v^2}{1-u^2}} \right\}$$

and

$$\frac{\partial F}{\partial v} = \frac{\cos(\text{Arcsin } u) \cdot v}{\sqrt{1+u^2+v^2}} + 0 = +v\sqrt{\frac{1-u^2}{1+u^2+v^2}}.$$

TEST. We get by insertion,

$$F(u,v) = \sqrt{1+u^2+v^2} \cdot \cos(\text{Arcsin } u) = +\sqrt{1+u^2+v^2} \cdot \sqrt{1-u^2},$$

hence

$$\frac{\partial F}{\partial u} = \frac{u\sqrt{1-u^2}}{\sqrt{1+u^2+v^2}} - \frac{u\sqrt{1+u^2+v^2}}{\sqrt{1-u^2}} = u \left\{ \sqrt{\frac{1-u^2}{1+u^2+v^2}} - \sqrt{\frac{1+u^2+v^2}{1-u^2}} \right\},$$

and

$$\frac{\partial F}{\partial v} = v\sqrt{\frac{1-u^2}{1+u^2+v^2}}.$$

The results agree. $\lozenge$

5) When $f(x,y) = x \sinh y$ and $(x,y) = \mathbf{X}(u,v) = u^3 v, \ln v + \ln v)$, $(u,v) \in \mathbb{R}_+^2$, then the composition of the functions is defined and of class C^∞. From

$$\frac{\partial f}{\partial x} = \sinh y, \qquad \frac{\partial f}{\partial y} = x \cosh y,$$

and

$$\frac{\partial x}{\partial u} = 3u^2 v, \quad \frac{\partial y}{\partial u} = \frac{1}{u}, \qquad \frac{\partial x}{\partial v} = u^3, \quad \frac{\partial y}{\partial v} = \frac{1}{v},$$

follows that

$$\frac{\partial F}{\partial u} = \sinh y \cdot 3u^2 v + x \cdot \cosh y \cdot \frac{1}{u},$$

and

$$\frac{\partial F}{\partial v} = \sinh y \cdot u^3 + x \cdot \cosh y \cdot \frac{1}{v}.$$

Since $y(u,v) = \ln u + \ln v$, we have

$$\sinh y = \frac{1}{2}\left(uv - \frac{1}{uv}\right) \quad \text{og} \quad \cosh y = \frac{1}{2}\left(uv + \frac{1}{uv}\right).$$

Then by insertion,

$$
\begin{aligned}
\frac{\partial F}{\partial u} &= 3u^2v \cdot \frac{1}{2}\left(uv - \frac{1}{uv}\right) + u^3v \cdot \frac{1}{2}\left(uv + \frac{1}{uv}\right) \cdot \frac{1}{u} \\
&= \frac{3}{2}\,u^3v^2 - \frac{3}{2}\,u + \frac{1}{2}\,u^3v^2 + \frac{1}{2}\,u \\
&= 2u^3v - u,
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial F}{\partial v} &= \frac{1}{2}\left(uv - \frac{1}{uv}\right) \cdot u^3 + u^3v \cdot 12\left(uv + \frac{1}{uv}\right) \cdot 1v \\
&= \frac{1}{2}\,u^4v - \frac{1}{2}\,\frac{u^2}{v} + \frac{1}{2}\,u^4v + \frac{1}{2}\,\frac{u^2}{v} \\
&= u^4v.
\end{aligned}
$$

TEST. We get by insertion

$$
F(u,v) = u^3v \cdot 12\left(uv - \frac{1}{uv}\right) = \frac{1}{2}\,u^4v^2 - \frac{1}{2}\,u^2,
$$

hence

$$
\frac{\partial F}{\partial u} = 2u^3v - u \quad \text{and} \quad \frac{\partial F}{\partial v} = u^4v.
$$

The results agree. $\lozenge$

REMARK. All these examples are very simple because they should train the reader to use a new method. Unfortunately, in all the chosen examples the usual method is easier to apply; but there exist examples, like e.g. **Example 3.6**, where the chain rule is the most efficient one. However, in the previous two examples, **Example 3.1** and **Example 3.2** we must admit that the chain rule is more difficult to apply. $\lozenge$

Example 3.3 *It can be proved that the differential equation*

$$
\frac{dw}{du} = w^2 + u^2, \qquad u \in \mathbb{R},
$$

among its solutions has

$$
w = X(u), \quad u \in \mathbb{R}, \qquad \text{where } X(0) = 1,
$$

and

$$
w = Y(u), \quad u \in \mathbb{R}, \qquad \text{where } Y(0) = 2.
$$

Let

$$
F(u) = f(X(u), Y(u)), \qquad f(x,y) = \ln(1 + xy^2).
$$

Find the derivative $F'(0)$.

A The chain rule.

D Since the functions $X(u)$ and $Y(u)$ cannot be found explicitly by elementary methods, we shall try the chain rule instead.

REMARK. The non-linear differential equation above is a so-called Ricatti equation. Such equations cannot be solved in general *unless one knows one solution*. It can be proved that the equation then can be completely solved. Therefore, one usually says that the Ricatti equation can only be solved by guessing. This is not true. There are some special cases, in which the Ricatti equation can be completely solved without knowing a solution in advance. The considered equation is actually of this type, but since its solution lies far beyond what can be mentioned here, we shall not solve it. ◊

I First note that for $xy^2 > -1$,

$$\frac{\partial f}{\partial x} = \frac{y^2}{1 + xy^2} \quad \text{and} \quad \frac{\partial f}{\partial y} = \frac{2xy}{1 + xy^2}.$$

Then

$$\frac{dX}{du} = X(u)^2 + u^2 \quad \text{and} \quad \frac{dY}{du} = Y(u)^2 + u^2,$$

so when we apply the chain rule we get

$$
\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x}\frac{dX}{du} + \frac{\partial f}{\partial y}\frac{dY}{du} = \frac{y^2}{1+xy^2}\left\{X(u)^2 + u^2\right\} + \frac{2xy}{1+xy^2}\left\{Y(u)^2 + u^2\right\} \\
&= \frac{Y(u)^2}{1 + X(u)Y(u)^2}\left\{X(u)^2 + u^2\right\} + \frac{2X(u)Y(u)}{1 + X(u)Y(u)^2}\left\{Y(u)^2 + u^2\right\}.
\end{aligned}
$$

Now $X(0) = 1$ and $Y(0) = 2$, so $X(u)Y(u)^2 > -1$ in an interval around $u = 0$, and $F'(0)$ is defined. We get the value by inserting the values of the calculations above.

$$F'(0) = \frac{4}{1 + 1 \cdot 4}\{1 + 0\} + \frac{2 \cdot 1 \cdot 2}{1 + 1 \cdot 4}\{4 + 0\} = \frac{4}{5} + \frac{4}{5} \cdot 4 = 4.$$

Example 3.4 *Let u and w denote two functions in two variables. We assume that they fulfil the differential equations*

$$a\,\frac{\partial w}{\partial t} = -\frac{\partial u}{\partial z}, \qquad b\,\frac{\partial u}{\partial t} = -\frac{\partial w}{\partial z}, \qquad (z,t) \in \mathbb{R}^2.$$

We also consider two C^1-functions F, $G : \mathbb{R} \to \mathbb{R}$, and we put

$$u(z,t) = F(x + ct) + G(z - ct),$$

$$w(z,t) = \gamma\{F(z + ct) - G(z - ct)\}.$$

Prove that one can choose the constants c and γ such that the differential equations are satisfied.

A System of partial differential equations.

D Insert the given functions and find c and γ.

I By partial differentiation we get

$$\frac{\partial w}{\partial t} = \gamma\{cF'(z + ct) + cG'(z - ct)\} = c\gamma\{F'(z + ct) + G'(z - ct)\},$$

$$\frac{\partial w}{\partial z} = \gamma\{F'(z + ct) - G'(z - ct)\},$$

and

$$\frac{\partial u}{\partial z} = F'(z + ct) + G'(x - ct),$$

$$\frac{\partial u}{\partial t} = c\,F'(z + ct) - c\,G'(z - ct) = c\{F'(z + ct) - G'(z - ct)\}.$$

It follows from the equation $a\,\dfrac{\partial w}{\partial t} = -\dfrac{\partial u}{\partial z}$ that

$$ac\gamma\{F'(z + ct) + G'(z - ct)\} = -\{F'(z + ct) + G'(z - ct)\}.$$

Since F and G are arbitrary, we get $ac\gamma = -1$.

Then it follows from the equation $b\,\dfrac{\partial u}{\partial t} = -\dfrac{\partial w}{\partial z}$ that

$$bc\{F'(z+ct) - G'(z-ct)\} = -\gamma\{F'(z+ct) - G'(z-ct)\}.$$

Since F and G are arbitrary, we get $bc = -\gamma$.

Then solve the system of two equations

$$ac\gamma = -1 \qquad \text{and} \qquad bc = -\gamma$$

in c and γ for given a, $b > 0$ by eliminating γ, i.e. $-abc^2 = -1$, and then accordingly

$$c = +\frac{1}{\sqrt{ab}}.,$$

where we have chosen the sign $+$, such that $c > 0$. If we instead choose the sign $-$, we interchange F and G.

By the choices above of c we get $\gamma = -bc = -\sqrt{\dfrac{b}{a}}$, thus

$$c = \frac{1}{\sqrt{ab}} \qquad \text{and} \qquad \gamma = -\sqrt{\frac{b}{a}}.$$

The system has the solutions

$$\begin{cases} u(z,t) = F\left(z + \dfrac{t}{\sqrt{ab}}\right) + G\left(z - \dfrac{t}{\sqrt{ab}}\right), \\ w(z,t) = -\sqrt{\dfrac{b}{a}}\left\{F\left(z + \dfrac{t}{\sqrt{ab}}\right) - G\left(z - \dfrac{t}{\sqrt{ab}}\right)\right\}. \end{cases}$$

These solutions are valid for any C^1-functions $F, G : \mathbb{R} \to \mathbb{R}$.

REMARK 1. If F and G are of class C^2, then the functions are solutions of the wave equation. $\Diamond$

REMARK 2. The reason why the example is placed here is that one latently applies the chain rule in a very simple version when we calculate the partial derivative. However, this cannot be clearly seen due to all the other messages in the example. $\Diamond$

Example 3.5 *Let $P_n(x, y, z)$ be an homogeneous polynomial of degree n in three variables. Consider P_n as a function of the spherical coordinates (r, θ, φ). Prove by using the result of* **Example 2.2** *that*

$$r \frac{\partial P_n}{\partial r} = n P_n.$$

A Homogeneous polynomial in $\mathbb{R}^3$.

D Apply **Example 2.2**.

I We have according to **Example 2.2**,

$$x \frac{\partial P_n}{\partial x} + y \frac{\partial P_n}{\partial y} + z \frac{\partial P_n}{\partial z} = n P_n.$$

Then notice that

$$r \frac{\partial x}{\partial r} = r \frac{\partial}{\partial r} \{r \sin\theta \cos\varphi\} = r \sin\theta \cos\varphi = x \quad \text{for } r > 0,$$

and analogously for the other rectangular coordinate functions, so

$$r \frac{\partial x}{\partial r} = x, \quad r \frac{\partial y}{\partial r} = y, \quad r \frac{\partial z}{\partial r} = z, \qquad \text{for } r > 0.$$

Then we get by the chain rule

$$r \frac{\partial P_n}{\partial r} = r \frac{\partial x}{\partial r} \frac{\partial P_n}{\partial x} + r \frac{\partial y}{\partial r} \frac{\partial P_n}{\partial y} + r \frac{\partial z}{\partial r} \frac{\partial P_n}{\partial P_n} \partial z = x \frac{\partial P_n}{\partial x} + y \frac{\partial P_n}{\partial y} + z \frac{\partial P_n}{\partial z} = n P_n.$$

Example 3.6 *Given the functions*

$$X(u) = \ln(2 + u), \quad u > -2, \quad and \quad f(x, y) = y^3 e^x, \quad (x, u) \in \mathbb{R}^2,$$

and a C^1-function $Y(x)$, $x \in A$, of which we only know that

$$0 \in A \qquad Y(0) = \pi, \qquad Y'(0) = 2.$$

Considering the composite function $F(u) = f(X(u), Y(u))$ we shall find the derivative $F'(0)$.

A Determination of a derivative, where we apparently are missing some information.

D Analyze the chain rule.

I We get by the chain rule

$$\begin{aligned} F'(u) &= \frac{\partial f}{\partial x} \frac{dX}{du} + \frac{\partial f}{\partial y} \frac{dY}{du} = y^3 e^x \cdot \frac{1}{2 + u} + 3y^2 e^x \cdot Y'(u) \\ &= Y(u)^3 \cdot (2 + u) \cdot \frac{1}{2 + u} + 3Y(u)^2 \cdot (2 + u) \cdot Y'(u) \\ &= Y(u)^3 + 3Y(u)^2 \cdot Y(u) \cdot (2 + u). \end{aligned}$$

Putting $u = 0$ we get

$$F'(0) = Y(0)^3 + 3Y(0)^2 \cdot Y'(0) \cdot 2 = \pi^3 + 6\pi^2 \cdot 2 = \pi^3 + 12\pi^2 = \pi^2(12 + \pi).$$

Example 3.7 . *Find the derivative of the function*

$$F(u) = \text{Arcsin}\left(\frac{\sin u \cos u}{\sqrt{2 + \cos^2 u}}\right), \qquad u \in \mathbb{R}$$

by putting $F(u) = f(X(u), Y(u))$, where

$$f(x, y) = \text{Arcsin}\left(\frac{x}{\sqrt{y}}\right).$$

A The chain rule.

D Identify $X(u)$ and $Y(u)$, and use the chain rule.

I We shall clearly choose

$$x = X(u) = \sin u \cdot \cos u, \quad \text{and} \quad y = Y(u) = 2 + \cos^2 u.$$

First calculate

$$\frac{dx}{du} = \cos^2 u - \sin^2 u = \cos 2y,$$

$$\frac{dy}{du} = -2\sin u \cdot \cos u = -\sin 2u,$$

together with

$$\frac{\partial f}{\partial x} = \frac{1}{\sqrt{1 - \dfrac{x^2}{y}}} \cdot \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{y - x^2}} = \frac{1}{\sqrt{2 + \cos^2 u - \sin^2 u \cdot \cos^2 u}} = \frac{1}{\sqrt{2 + \cos^4 u}},$$

and

$$\frac{\partial f}{\partial y} = \frac{1}{\sqrt{1 - \dfrac{x^2}{y}}} \cdot \left(-\frac{1}{2}\frac{x}{y\sqrt{y}}\right) = -\frac{x}{2y} \cdot \frac{1}{\sqrt{y - x^2}} = -\frac{\sin u \cdot \cos u}{4 + 2\cos^2 u} \cdot \frac{1}{\sqrt{2 + \cos^4 u}}.$$

Then by the chain rule

$$
\begin{aligned}
F'(u) &= \frac{\partial f}{\partial x} \cdot \frac{dx}{du} + \frac{\partial f}{\partial y} \cdot \frac{dy}{du} \\
&= \frac{\cos^2 u - \sin^2 u}{\sqrt{2 + \cos^4 u}} - \frac{\sin u \cos u}{4 + 2\cos^2 u} \cdot \frac{1}{\sqrt{2 + \cos^4 u}}\,(-2\sin u\cos u) \\
&= \frac{1}{\sqrt{2 + \cos^4 u}}\left\{\cos^2 u - \sin^2 u + \frac{2\sin^2 u\cos^2 u}{4 + 2\cos^2 u}\right\} \\
&= \frac{1}{\sqrt{2 + \cos^4 u}}\left\{\cos^2 u - \frac{2\sin^2 u}{2 + \cos^2 u}\right\}.
\end{aligned}
$$

C By the traditional calculation we get

$$
\begin{aligned}
F'(u) &= \frac{1}{\sqrt{1 - \dfrac{\sin^2 u\cos^2 u}{2 + \cos^2 u}}}\left\{\frac{\cos^2 u - \sin^2 u}{\sqrt{2 + \cos^2 u}} - \frac{1}{2}\cdot\frac{\sin u\cos u\cdot(-2\sin u\cos^2 u)}{(2 + \cos^2 u)\sqrt{2 + \cos^2 u}}\right\} \\
&= \frac{1}{\sqrt{2 + \cos^2 u - \sin^2 u\cos^2 u}}\left\{\cos^2 u - \sin^2 u + \frac{\sin^2 u\cos^2 u}{2 + \cos^2 u}\right\} \\
&= \frac{1}{\sqrt{2 + \cos^4}}\left\{\cos^2 u + \sin^2 u\cdot\frac{-2 - \cos^2 u + \cos^2 u}{2 + \cos^2 u}\right\} \\
&= \frac{1}{\sqrt{2 + \cos^4 u}}\left\{\cos^2 u - \frac{2\sin^2 u}{2 + \cos^2 u}\right\}.
\end{aligned}
$$

The two results agree.

4 Directional derivative

Example 4.1 *Find in each of the following cases the directional derivative of the given function* $f : \mathbb{R}^3 \to \mathbb{R}$ *in the point given by the index 0 in the direction of the vector* **v**.

1) $f(x, y, z) = x + 2xy - 3y^2$, $(x_0, y_0, z_0) = (1, 2, 1)$, $\mathbf{v} = (3, 4, 0)$.

2) $f(x, y, z) = ze^x \cos(\pi y)$, $(x_0, y_0, z_0) = (0, -1, 1)$, $\mathbf{v} = (-1, 2, 1)$.

3) $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $(x_0, y_0, z_0) = (1, 1, 0)$, $\mathbf{v} = (1, -1, 2)$.

4) $f(x, y, z) = xy + yz + xz$, $(x_0, y_0, z_0) = (1, 2, 3)$, $\mathbf{v} = (1, 1, 1)$.

A Directional derivative.

D Insert into the formula

$$f'\left(\mathbf{x}; \frac{\mathbf{v}}{|\mathbf{v}|}\right) = \frac{\mathbf{v}}{|\mathbf{v}|} \cdot \bigtriangledown f(\mathbf{x}),$$

where we *must* remember to norm **v**.

I 1) Here,

$$\bigtriangledown f(x, y) = (1 + 2y, 2x - 6y) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{3^2 + 4^2} = 5,$$

so

$$\begin{aligned}
f'\left((1, 2); \left(\frac{3}{5}, \frac{4}{5}\right)\right) &= \frac{1}{5}(3, 4) \cdot (1 + 4, 2 - 12) = \frac{1}{5}(3, 4) \cdot (5, -10) \\
&= (3, 4) \cdot (1, -2) = -5.
\end{aligned}$$

2) Here,

$$\bigtriangledown f(x, y, z) = \left(ze^x \cos(\pi y), -\pi z e^x \sin(\pi y), e^x \cos(\pi z)\right),$$

and $|\mathbf{v}| = \sqrt{1 + 4 + 1} = \sqrt{6}$, so

$$\begin{aligned}
f'\left((0, -1, 1); \frac{1}{\sqrt{6}}(-1, 2, 1)\right) &= \frac{1}{\sqrt{6}}(-1, 2, 1) \cdot (1 \cdot e^0 \cdot (-1), 0, -1) \\
&= \frac{1}{\sqrt{6}}\{(-1)^2 + 0 - 1\} = 0.
\end{aligned}$$

3) Here

$$\bigtriangledown f(x, y, z) = (2x, 4y, 6z) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{6},$$

so

$$f'\left((1, 1, 0); \frac{1}{\sqrt{6}}(1, -1, 2)\right) = \frac{1}{\sqrt{6}}(1, -1, 2) \cdot (2, 4, 0) = \frac{1}{\sqrt{6}}(2 - 4) = -\frac{2}{\sqrt{6}} = -\sqrt{\frac{2}{3}}.$$

4) Here

$$\bigtriangledown f(x, y, z) = (y + z, x + z, x + y) \qquad \text{and} \qquad |\mathbf{v}| = \sqrt{3},$$

so

$$f'\left((1, 2, 3); \frac{1}{\sqrt{3}}(1, 1, 1)\right) = \frac{1}{\sqrt{3}} \cdot 2[x + y + z]_{(x,y,z)=(1,2,3)} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

Example 4.2 *Find in each of the following cases the directional derivative of the function f at the point given by the index 0 in the direction of the point given by the index 1.*

1) $f(x, y, z) = xyz + \dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x}$ *defined for* $xyz \neq 0$ *from* $(x_0, y_0, z_0) = (1, -1, 1)$ *to* $(x_1, y_1, z_1) = (3, 1, 2)$.

2) $f(x, y, z) = 2x^3y - 3y^2z$ *defined in* $\mathbb{R}^3$ *from* $(x_0, y_0, z_0) = (1, 2, -1)$ *to* $(x_1, y_1, z_1) = (3, -1, 5)$.

3) $f(x, y, z) = x \ln(1 + e^{yz})$ *defined in* $\mathbb{R}^3$ *from* $(x_0, y_0, z_0) = (1, 1, 0)$ *to* $(x_1, y_1, z_1) = (0, 0, -1)$.

A Directional derivative.

D Calculate $\bigtriangledown f(x_0, y_0, z_0)$ and find the unit vector $\mathbf{e}$.

I 1) Here

$$\bigtriangledown f = \left(yz + \frac{1}{y} - \frac{z}{x^2}, xz - \frac{x}{y^2} + \frac{1}{z}, xy - \frac{y}{z^2} + \frac{1}{x} \right),$$

so

$$\bigtriangledown f(1, -1, 1) = (-1 - 1 - 1, 1 - 1 + 1, -1 + 1 + 1) = (-3, 1, 1).$$

Furthermore,

$$\mathbf{v} = (3, 1, 2)_{(}1, -1, 1) = (2, 2, 1), \qquad \text{where } |\mathbf{v}| = \sqrt{2^2 + 2^2 + 1^2} = 3,$$

so

$$f' \left((1, -1, 1); \frac{1}{3}(2, 2, 1) \right) = \frac{1}{3}(2, 2, 1) \cdot (-3, 1, 1) = \frac{1}{3}\{-6 + 2 + 1\} = -1.$$

2) Here

$$\bigtriangledown f = \left(6x^2y, 2x^3 - 6yz, -3y^2 \right),$$

so

$$\bigtriangledown f(1, 2, -1) = (6 \cdot 1^2 \cdot 2, 2 - 6 \cdot 2 \cdot (-1), -3 \cdot 2^2) = (12, 14, -12).$$

Furthermore,

$$\mathbf{v} = (3, -1, 5) - (1, 2, -1) = (2, -3, 6), \qquad \text{where } |\mathbf{v}| = \sqrt{2^2 + 3^2 + 6^2} = 7,$$

so

$$f' \left((1, 2, -1); \frac{1}{7}(2, -3, 6) \right) = \frac{1}{7}(2, -3, 6) \cdot (12, 14, -12) = \frac{1}{7}\{-48 - 42\} = -\frac{90}{7}.$$

3) Here

$$\bigtriangledown f = \left(\ln(1 + e^{yz}), \frac{xze^{yz}}{1 + e^{yz}}, \frac{xye^{yz}}{1 + e^{yz}} \right),$$

so

$$\nabla f(1,1,0) = \left(\ln 2, 0, \frac{1}{2} \right).$$

Furthermore,

$$\mathbf{v} = (0,0,-1) - (1,1,0) = (-1,-1,-1), \qquad \text{where } |\mathbf{v}| = \sqrt{3},$$

so

$$f'\left((1,1,0); \frac{1}{\sqrt{3}}(-1,-1,-1) \right) = -\frac{1}{\sqrt{3}} \left(\frac{1}{2} + \ln 2 \right).$$

Example 4.3 *Given the function*

$$f(x, y, z) = \text{Arctan}\left(x + \frac{1}{y}\right) + \sinh\left(z^2 - 1\right), \qquad y < 0.$$

Find the direction in which the directional derivation of f at the point $(1, -1, 1)$ is smallest, and indicate this minimum.

A Directional derivative.

D First calculate $\nabla f(1, -1, 1)$. Then conclude that the direction must be

$$\mathbf{e} = -\frac{\nabla f}{\| \nabla f \|}.$$

I We get by differentiation

$$\nabla f = \left(\frac{1}{1 + \left(x + \dfrac{1}{y}\right)^2}, \frac{-\dfrac{1}{y^2}}{1 + \left(x + \dfrac{1}{y}\right)^2}, 2z\cosh(z^2 - 1) \right),$$

hence

$$\nabla f(1, -1, 1) = (1, -1, 2) \qquad \text{where} \qquad \| \nabla f(1, -1, 1) \| = \sqrt{6}.$$

Using the direction

$$\mathbf{e} = -\frac{1}{\sqrt{6}}(1, -1, 2) = -\frac{\nabla f(1, -1, 1)}{\| \nabla f(1, -1, 1) \|}$$

we get the directional derivative

$$f'((1, -1, 1); \mathbf{e}) = \mathbf{e} \cdot \nabla f(1, -1, 1) = -\frac{\| \nabla f(1, -1, 1) \|^2}{\| \nabla f(1, -1, 1) \|} = -\| \nabla f(1, -1, 1) \| = -\sqrt{6}.$$

Example 4.4 *Let f be a C^1-function of two variables. We sketch from a fixed point (x_0, y_0) in any direction the corresponding directional derivative of f at the point (x_0, y_0). Prove that we by this procedure obtain two circles which are tangent to each other at the point (x_0, y_0), and find the centres of these circles.*

A "Theoretical" example concerning the directional derivative.

D Without loss of generality we may assume that $(x_0, y_0) = (0, 0)$. Calculate

$$f'(\mathbf{0}; \mathbf{e})\mathbf{e} \qquad \text{or} \qquad |f'(\mathbf{0}; \mathbf{e}|\mathbf{e}$$

for every unit vector $\mathbf{e}$.
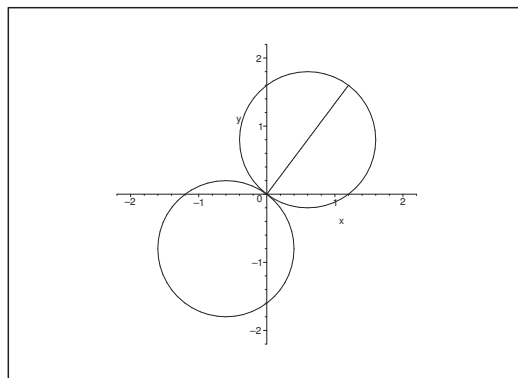
Figure 22: The sketched diameter is $\bigtriangledown f(0,0)$.

I We can obviously assume that $(x_0, y_0) = (0,0)$.

Then let

$$\bigtriangledown f(\mathbf{0}) = \left(\frac{\partial f}{\partial x}(\mathbf{0}), \frac{\partial f}{\partial y}(\mathbf{0}) \right) := (a,b).$$

Any unit vector can be written in the form

$$\mathbf{e}(\varphi) = (\cos \varphi, \sin \varphi), \qquad \varphi \in [0, 2\pi[,$$

so

(2) $f'(\mathbf{0}; \mathbf{e}(\varphi))\mathbf{e}(\varphi) = (a \cos \varphi + b \sin \varphi)(\cos \varphi, \sin \varphi) = (x(\varphi), y(\varphi)),$

where

$$x(\varphi) = a \cos^2 \varphi + b \sin \varphi \cos \varphi = \frac{1}{2}\{a \cos 2\varphi + b \sin 2\varphi\} + \frac{a}{2}$$

and

$$y(\varphi) = a \sin \varphi \cos \varphi + b \sin^2 \varphi = \frac{1}{2}\{a \sin 2\varphi - b \cos 2\varphi\} + \frac{b}{2}.$$

Hence

$$\left\{x(\varphi) - \frac{a}{2}\right\}^2 + \left\{y(\varphi) - \frac{b}{2}\right\}^2 = \frac{1}{4}\{a^2 + b^2\},$$

and the centre lies at the point

$$\left(\frac{a}{2}, \frac{b}{2} \right) = \frac{1}{2} \bigtriangledown f(\mathbf{0}),$$

and the radius is $\frac{1}{2}| \bigtriangledown f(\mathbf{0})|$.

We have above calculated the signed directional derivative. If we instead interpret (2) as

$$|f'(\mathbf{0}; \mathbf{e}(\varphi))|\mathbf{e}(\varphi),$$

then we obtain the cirle which is the mirror image in $\mathbf{0}$.

Example 4.5 *Find the directional derivative of the funkction*

$$f(x, y, z) = y\sqrt{1 + x^2 z^2}, \qquad (x, y, z) \in \mathbb{R}^3,$$

at the point $(\sqrt{2}, 1, 2)$ *in the direction towards the point* $(\sqrt{2}, 2, 2 + \sqrt{3})$.

A Directional derivative.

D Find the unit vector in the direction and apply the formula of the directional derivative.

I The direction is

$$\mathbf{v} = (\sqrt{2}, 2, 2 + \sqrt{3}) - (\sqrt{2}, 1, 2) = (0, 1, \sqrt{3}) = 2\left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right),$$

hence $\|\mathbf{v}\| = 2$, and $\mathbf{e} = \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Then the directional derivative is

$$f'\left((\sqrt{2}, 1, 2)\,;\, \left(0, \frac{1}{2}, \frac{\sqrt{3}}{2}\right)\right)$$

$$= 0 \cdot \frac{\partial f}{\partial x}(\sqrt{2}, 1, 2) + \frac{1}{2} \frac{\partial f}{\partial y}(\sqrt{2}, 1, 2) + \frac{\sqrt{3}}{2} \frac{\partial f}{\partial z}(\sqrt{2}, 1, 2)$$

$$= \frac{1}{2}\left[\sqrt{1 + x^2 z^2}\right]_{(\sqrt{2}, 1, 2)} + \frac{\sqrt{3}}{2}\left[\frac{x^2 yz}{\sqrt{1 + x^2 z^2}}\right]_{(\sqrt{2}, 1, 2)}$$

$$= \frac{1}{2}\sqrt{1 + 2 \cdot 4} + \frac{\sqrt{3}}{2} \cdot \frac{2 \cdot 1 \cdot 2}{\sqrt{1 + 2 \cdot 4}} = \frac{3}{2} + \frac{2\sqrt{3}}{3} = \frac{9 + 4\sqrt{3}}{6}.$$

Example 4.6 *Given the function*

$$f(x, y, z) = 2x + 2y^2 z + xy^2 z, \qquad (x, y, z) \in \mathbb{R}^3.$$

Find $(\bigtriangledown f)(1, -1, 2)$, *and then the unit vector* $\mathbf{e}$, *for which*

$$f'((1, -1, 2); \mathbf{e})$$

is as large as possible.

A Gradient and directional derivative.

D Just calculate.

I The gradient is

$$\bigtriangledown f = (2 + y^2 z, 4yz + 2xyz, 2y^2 + xy^2),$$

hence

$$(\bigtriangledown f)(1, -1, 2) = (2 + (-1)^2 \cdot 2, -4 \cdot 2 + 2 \cdot 1(-1) \cdot 2, 2 + 1) = (4, -12, 3)$$

where the maximum is

$$\| \bigtriangledown f(1,-1,2)\| = \sqrt{16+144+9} = \sqrt{169} = 13 = f'((1,-1,2);\mathbf{e})$$

attained for

$$\mathbf{e} = \left(\frac{4}{13}, -\frac{12}{13}, \frac{3}{13} \right).$$

5   Partial derivatives of higher order

Example 5.1 *Find in each of the following cases the first and the second differential for the function f at the point which is indicated with the index 0.*

1) $f(x, y) = x \exp(y^2 - 1)$ *in* $\mathbb{R}^2$ *from* $(x_0, y_0) = (1, 1)$.

2) $f(x, y) = \text{Arctan}(x + y) + \ln(1 + x)$ *for* $x > -1$ *from* $(x_0, y_0) = (0, 1)$.

3) $f(x, y) = (x^2 + y^2) \ln(x^2 + y^2)$ *in* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *from* $(x_0, y_0) = (0, 1)$.

4) $f(x, y) = \sqrt{x^2 + y^2}$ *i* $\mathbb{R}^2 \setminus \{\mathbf{0}\}$ *from* $(x_0, y_0) = (3, 4)$.

A First and second differential.

D First calculate the partial derivatives.

I It is obvious that $f(x, y)$ is of class C^∞ in the domain in all four cases.

1) The partial derivatives are here

$$\frac{\partial f}{\partial x} = \exp(y^2 - 1), \qquad \frac{\partial f}{\partial y} = 2xy \exp(y^2 - 1),$$

$$\frac{\partial^2 f}{\partial x^2} = 0, \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 2x \exp(y^2 - 1),$$

$$\frac{\partial^2 f}{\partial y^2} = 2x \exp(y^2 - 1) + 4xy^2 \exp(y^2 - 1) = 2x(1 + 2y^2) \exp(y^2 - 1),$$

so

$$df((1,1), \mathbf{h}) = \nabla f(1, 1) \cdot \mathbf{h} = (1, 2) \cdot (h_x, h_y) = h_x + 2h_y = \text{``} dx + 2dy \text{''}$$

and

$$\begin{aligned}
d^2 f((1,1); \mathbf{h}) &= \frac{\partial^2 f}{\partial x^2}(1, 1) h_x^2 + 2 \frac{\partial^2 f}{\partial x \partial y}(1, 1) h_x h_y + \frac{\partial^2 f}{\partial y^2}(1, 1) h_y^2 \\
&= 0 \cdot h_x^2 + 2 \cdot 2 h_x h_y + 2(1 + 2)h_y^2 \\
&= 4h_x h_y + 6h_y^2 = \text{``} 4dxdy + 6(dy)^2 \text{''}.
\end{aligned}$$

2) Here

$$\frac{\partial f}{\partial x} = \frac{1}{1 + (x + y)^2} + \frac{1}{1 + x}, \qquad \frac{\partial f}{\partial x}(0, 1) = \frac{1}{2} + 1 = \frac{3}{2},$$

$$\frac{\partial f}{\partial y} = \frac{1}{1 + (x + y)^2}, \qquad \frac{\partial f}{\partial y}(0, 1) = \frac{1}{2},$$

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2(x + y)}{\{1 + (x + y)^2\}^2} - \frac{1}{(1 + x)^2}, \quad \frac{\partial^2 f}{\partial x^2}(0, 1) = -\frac{2}{2^2} - 1 = -\frac{3}{2},$$

$$\frac{\partial^2 f}{\partial x}\partial y = -\frac{2(x + y)}{\{1 + (x + y)^2\}^2}, \qquad \frac{\partial^2 f}{\partial x \partial y}(0, 1) = -\frac{1}{2},$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{2(x + y)}{\{1 + (x + y)^2\}^2}, \qquad \frac{\partial^2 f}{\partial y^2}(0, 1) = -\frac{1}{2}.$$

Then by insertion,

$$df((0,1); \mathbf{h}) = \frac{3}{2} h_x + \frac{1}{2} h_y = \text{``} \frac{3}{2} dy + \frac{1}{2} dy \text{''},$$

and

$$d^2 f((0,1); \mathbf{h}) = -\frac{3}{2} h_x^2 - h_x h_y - \frac{1}{2} h_y^2 = \text{``} -\frac{3}{2}(dx)^2 - dxdy - \frac{1}{2}(dy)^2 \text{''}.$$

3) Here

$$\frac{\partial f}{\partial x} = 2x \ln(x^2 + y^2) + 2x, \qquad\qquad \frac{\partial f}{\partial x}(0,1) = 0,$$

$$\frac{\partial f}{\partial y} = 2y \ln(x^2 + y^2) + 2y, \qquad\qquad \frac{\partial f}{\partial y}(0,1) = 2,$$

$$\frac{\partial^2 f}{\partial x^2} = 2\ln(x^2 + y^2) + \frac{4x^2}{x^2 + y^2} + 2; \quad \frac{\partial^2 f}{\partial x^2}(0,1) = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{4xy}{x^2 + y^2}, \qquad\qquad \frac{\partial^2 f}{\partial x \partial y}(0,1) = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = 2\ln(x^2 + y^2) + \frac{4y^2}{x^2 + y^2} + 2; \quad \frac{\partial^2 f}{\partial y^2}(0,1) = 6.$$

We see that

$$df((0,1); \mathbf{h}) = 2h_y = \text{``} 2dy \text{''}$$

and

$$d^2 f((0,1); \mathbf{h}) = 2h_x^2 + 6h_y^2 = \text{``} 2(dx)^2 + 6(dy) \text{''}.$$

4) Here

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad\qquad\qquad \frac{\partial f}{\partial x}(3,4) = \frac{3}{5},$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \qquad\qquad\qquad \frac{\partial f}{\partial y}(3,4) = \frac{4}{5},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\sqrt{x^2 + y^2}} - \frac{x^2}{(\sqrt{x^2 + y^2})^3} = \frac{y^2}{(\sqrt{x^2 + y^2})^3}, \quad \frac{\partial^2 f}{\partial x^2}(3,4) = \frac{16}{125},$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\frac{xy}{(\sqrt{x^2 + y^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial x \partial y}(3,4) = -\frac{12}{125},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{(\sqrt{x^2 + y^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial y^2}(3,4) = \frac{9}{125},$$

hence

$$df((3,3); \mathbf{h}) = \frac{3}{5} h_x + \frac{4}{5} h_y = \text{``} \frac{3}{5} dx + \frac{4}{5} dy \text{''},$$

and

$$d^2 f((3,3); \mathbf{h}) = \frac{16}{125} h_x^2 - \frac{24}{125} h_x h_y + \frac{9}{125} h_y^2.$$

Example 5.2 *Let the function* $f : \mathbb{R}^2 \to \mathbb{R}$ *be given by*

$$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2 + y^2}, & (x,y) \neq (0,0), \\ 0, & (x,y) = (0,0). \end{cases}$$

1) Prove that f has partial derivatives of first order at every point of the plane.

2) Prove that the mixed derivatives f''_{xy} and f''_{yx} both exist at the point $(0,0)$, though

$$f''_{xy}(0,0) \neq f''_{yx}(0,0).$$

3) Find $f''_{xy}(x,y)$ for $(x,y) \neq (0,0)$, and prove that this function does not have any limit for $(x,y) \to (0,0)$.

A Partial derivatives of first and second order.

D Discuss the existence of f'_x and f'_y; then calculate $f''_{xy}(0,0)$ and $f_{yx}(0,0)$ *at the point* $(0,0)$. Finally, calculate $f''_{x,y}(x,y)$ in general and switch to polar coordinates.

I 1) When $(x,y) \neq (0,0)$, we see that $f(x,y)$ is a quotient of two polynomials where the denominator is > 0. Accordingly the partial derivatives of $f(x,y)$ exist of any order when $(x,y) \neq (0,0)$. We get for $(x,y) \neq (0,0)$ that

$$\frac{\partial f}{\partial x} = \frac{y^3}{x^2 + y^2} - \frac{2x^2 y^3}{(x^2 + y^2)^2} = \frac{y^3(y^2 - x^2)}{(x^2 + y^2)^2}$$

and

$$\frac{\partial f}{\partial y} = \frac{3xy^2}{x^2 + y^2} - \frac{2xy^4}{(x^2 + y^2)^2} = \frac{xy^2(3x^2 + y^2)}{(x^2 + y^2)^2}.$$

We find at $(0,0)$

$$f(x,0) - f(0,0) = 0 = f(0,y) - f(0,0),$$

so we conclude that

$$\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0.$$

Summarizing we see that the partial derivatives of first order exist everywhere in $\mathbb{R}^2$.

2) Then it follows from the expressions of $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ that

$$\frac{\partial f}{\partial x}(0, y) - \frac{\partial f}{\partial x}(0,0) = \frac{y^5}{y^4} - 0 = y,$$

and

$$\frac{\partial f}{\partial y}(x, 0) - \frac{\partial f}{\partial y}(0,0) = 0.$$

We conclude that

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = \lim_{y \to 0} \frac{1}{y} \left\{ \frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0) \right\} = \lim_{y \to 0} \frac{y}{y} = 1$$

and

$$\frac{\partial^2 f}{\partial y} \partial x (0,0) = \lim_{x \to 0} \frac{1}{x} \left\{ \frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0) \right\} = 0,$$

so both

$$\frac{\partial^2 f}{\partial x \partial y}(0,0) = 1 \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x}(0,0) = 0$$

exist and yet they are different.

3) It follows from 1) that

$$\frac{\partial f}{\partial x} = \frac{y^3(y^2 - x^2)}{(x^2 + y^2)^2} = \frac{y^5 - y^3 x^2}{(x^2 + y^2)^2},$$

so

$$\begin{aligned}
\frac{\partial^2 f}{\partial x \partial y} &= \frac{5y^4 - 3y^2 x^2}{(x^2 + y^2)^2} - 2 \cdot \frac{2y}{(x^2 + y^2)^3} \cdot (y^5 - y^3 x^2) \\
&= \frac{y^2}{(x^2 + y^2)^2}(5y^2 - 3x^2) - \frac{4y^4}{(x^2 + y^2)^3} \cdot (y^2 - x^2).
\end{aligned}$$

When we switch to polar coordinates $x = \varrho \cos \varphi$, $y = \varrho \sin \varphi$, we get

$$\begin{aligned}
f''_{xy}(x,y) &= \frac{\varrho^2 \sin^2 \varphi}{\varrho^4}(5\varrho^2 \sin^2 \varphi - 3\varrho^2 \cos^2 \varphi) - \frac{\varrho^4 \sin^4 \varphi}{\varrho^6}(\varrho^2 \sin^2 \varphi - \varrho^2 \cos^2 \varphi) \\
&= \sin^2 \varphi(5 \sin^2 \varphi - 3 \cos^2 \varphi) - \sin^4 \varphi(\sin^2 \varphi - \cos^2 \varphi) \\
&= \sin^2 \varphi \left(4\sin^2 \varphi - 4\cos^2 \varphi + \sin^2 \varphi + \cos^2 \varphi + \left(\frac{1 - \cos 2\varphi}{2} \right)^2 \cos 2\varphi \right) \\
&= \sin^2 \varphi \left(-4 \cos 2\varphi + 1 + \frac{1}{4} \left\{ 1 - 2 \cos 2\varphi + \cos^2 2\varphi \right\} \cos 2\varphi \right).
\end{aligned}$$

This expression is not constant in φ (the latter factor is a polynomial of third degree in $\cos 2\varphi$), hence the limit does not exist when $\varrho \to 0$, and there are no further conditions on φ.

Example 5.3 *Find in each of the following cases the partial derivatives of first and second order of the given function $f : \mathbb{R}^2 \to \mathbb{R}$.*

1) $f(x,y) = \sin(x^2 y^3)$.

2) $f(x,y) = \sin(\cos(2x - 3y))$.

3) $f(x,y) = \sqrt{1 + x^2 + y^2}$.

4) $f(x,y) = \ln(1 + \cos^2(xy))$.

5) $f(x,y) = \exp(x + xy - 2y)$.

6) $f(x,y) = \text{Arctan}(x - y)$.

A Partial derivatives of first and second order of C^∞-functions.

D Differentiate.

I 1) When $f(x, y) = \sin(x^2 y^3)$, then

$$\frac{\partial f}{\partial x} = 2xy^3 \cos(x^2 y^3) \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2 y^2 \cos(x^2 y^3),$$

whence

$$\frac{\partial^2 f}{\partial x^2} = 2y^3 \cos(x^2 y^3) - 4x^2 y^6 \sin(x^2 y^3),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6xy^2 \cos(x^2 y^3) - 6x^3 y^5 \sin(x^2 y^3)$$

$$\frac{\partial^2 f}{\partial y^2} = 6x^2 y \cos(x^2 y^3) - 9x^4 y^4 \sin(x^2 y^3).$$

2) When $f(x, y) = \sin(\cos(2x - 3y))$, then

$$\frac{\partial f}{\partial x} = -2\sin(2x - 3y) \cdot \cos(\cos(2x - 3y)),$$

$$\frac{\partial f}{\partial y} = 3\sin(2x - 3y) \cdot \cos(\cos(2x - 3y)),$$

whence

$$\frac{\partial^2 f}{\partial x^2} = -4\cos(2x - 3y)\cos(\cos(2x - 3y)) - 4\sin^2(2x - 3y)\sin(\cos(2x - 3y)),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 6\cos(2x - 3y)\cos(\cos(2x - 3y)) + 6\sin^2(2x - 3y)\sin(\cos(2x - 3y)),$$

$$\frac{\partial^2 f}{\partial y^2} = -9\cos(2x - 3y)\cos(\cos(2x - 3y)) - 9\sin^2(2x - 3y)\sin(\cos(2x - 3y)).$$

3) When $f(x, y) = \sqrt{1 + x^2 + y^2}$, then

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{1 + x^2 + y^2}}, \qquad \frac{\partial f}{\partial y} = \frac{y}{\sqrt{1 + x^2 + y^2}},$$

whence

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{\sqrt{1 + x^2 + y^2}} - \frac{x^2}{(\sqrt{1 + x^2 + y^2})^3} = \frac{1 + y^2}{(\sqrt{1 + x^2 + y^2})^3},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{xy}{(\sqrt{1 + x^2 + y^2})^3},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{1 + x^2}{(\sqrt{1 + x^2 + y^2})^3}.$$

4) When $f(x,y) = \ln(1 + \cos^2(xy))$, then

$$
\frac{\partial f}{\partial x} = -\frac{2y\sin(xy)\cos(xy)}{1+\cos^2(xy)} = -\frac{y\sin(2xy)}{1+\cos^2(xy)},
$$

$$
\frac{\partial f}{\partial y} = -\frac{2x\sin(xy)\cos(xy)}{1+\cos^2(xy)} = -\frac{x\sin(2xy)}{1+\cos^2(xy)},
$$

and accordingly

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= -\frac{2y^2\cos(2xy)}{1+\cos^2(xy)} + \frac{y\sin(2xy)}{\{1+\cos^2(xy)\}^2}\{-2\cos(xy)\sin(xy)\}y \\[2mm]
&= -2y^2\frac{(\cos^2(xy)-\sin^2(xy))(1+\cos^2(xy)) - 2\sin^2(xy)\cos^2(xy)}{\{1+\cos^2(xy)\}^2} \\[2mm]
&= -2y^2\frac{\cos^2(xy)+\cos^4(xy)-\sin^2(xy)-2\sin^2(xy)\cos^2(xy)}{\{1+\cos^2(xy)\}^2} \\[2mm]
&= -2y^2\frac{-2\cos^2(xy)+4\cos^4(xy)-\sin^2(xy)}{\{1+\cos^2(xy)\}^2} \\[2mm]
&= 2y^2\frac{1+\cos^2(xy)-4\cos^4(xy)}{\{1+\cos^2(xy)\}^2}.
\end{aligned}
$$

Due to the symmetry in x and y we by interchanging the letters

$$
\frac{\partial^2 f}{\partial y^2} = 2x^2\frac{1+\cos^2(xy)-4\cos^4(xy)}{\{1+\cos^2(xy)\}^2}.
$$

Finally,

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial^2 f}{\partial y\partial x} &= -\frac{\sin(2xy)}{1+\cos^2(xy)} - \frac{2xy\cos(2xy)}{1+\cos^2(xy)} \\[2mm]
&\qquad\qquad + \frac{x\sin(2xy)}{\{1+\cos^2(xy)\}^2}\{-2\cos(xy)\sin(xy)\cdot y\} \\[2mm]
&= -\frac{\sin(2xy)+2xy\cos(2xy)}{1+\cos^2(xy)} - \frac{xy\sin^2(2xy)}{\{1+\cos^2(xy)\}^2}.
\end{aligned}
$$

5) When $f(x,y) = \exp(x+xy-2y)$, then

$$
\frac{\partial f}{\partial x} = (1+y)\exp(x+xy-2y) = (1+y)f(x,y),
$$

$$
\frac{\partial f}{\partial y} = (x-2)\exp(x+xy-2y) = (x-2)f(x,y),
$$

whence

$$\frac{\partial^2 f}{\partial x^2} \;=\; (1+y)^2 f(x,y) = (1+y)^2 \exp(x+xy-2y),$$

$$\frac{\partial^2 f}{\partial x \partial y} \;=\; \frac{\partial^2 f}{\partial y \partial x} = 1 \cdot f(x,y) + (1+y)(x-2)f(x,y)$$

$$\;=\; (x+xy-2y-1)\exp(x+xy-2y),$$

$$\frac{\partial^2 f}{\partial y^2} \;=\; (x-2)^2 f(x,y) = (x-2)^2 \exp(x+xy-2y).$$

6) When $f(x,y) = \text{Arctan}(x-y)$, we get

$$\frac{\partial f}{\partial x} = \frac{1}{1+(x-y)^2}, \qquad \frac{\partial f}{\partial y} = -\frac{1}{1+(x-y)^2},$$

hence

$$\frac{\partial^2 f}{\partial x^2} = -\frac{2(x-y)}{\{1+(x-y)^2\}^2},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{2(x-y)}{1+(x-y)^2\}^2},$$

$$\frac{\partial^2 f}{\partial y^2} = -\frac{2(x-y)}{\{1+(x-y)^2\}^2}.$$

Example 5.4 *Prove in each of the following cases that the given function f satisfies the given differential equation everywhere in its domain. In some of the cases there occur some constants α, β, γ; check if these can be chosen freely. Notice that the variables are not x, y or z in all cases.*

1) *Prove that the function $\ln \sqrt{x^2 + y^2}$, defined in $\mathbb{R}^2 \setminus \{\mathbf{0}\}$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

2) *Prove that the function $e^{\alpha x} \cos(\alpha y)$, defined in $\mathbb{R}^2$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

3) *Prove that the function $e^{-t}(\cos x + \sin y)$, defined in $\mathbb{R}^3$, fulfils the differential equation*

$$\frac{\partial f}{\partial t} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

4) *Prove that the function $\sin(\alpha x) \sin(\beta y) \sin(\gamma \sqrt{\alpha^2 + \beta^2}\, t)$, defined in $\mathbb{R}^3$, fulfils the differential equation*

$$\frac{1}{\gamma^2} \frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 t}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

5) *Prove that the function $\dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, defined in $\mathbb{R}^3 \setminus \{\mathbf{0}\}$, fulfils the differential equation*

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0.$$

6) *Prove that the function $t^\alpha \exp\left(-\dfrac{r^2}{4t}\right)$, defined for $t > 0$, fulfils the differential equation*

$$r^2 \frac{\partial f}{\partial t} = \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right).$$

A Partial differential equations.

D Differentiate the given function and put it into the differential equation.

I 1) When $f(x, y) = \ln \sqrt{x^2 + y^2} = \dfrac{1}{2} \ln(x^2 + y^2)$, we get

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2}, \qquad \frac{\partial f}{\partial y} = \frac{y}{x^2 + y^2},$$

hence

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{x^2 + y^2} - \frac{2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, \qquad \frac{\partial^2 f}{\partial y^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

Then by insertion

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2} + \frac{x^2 - y^2}{(x^2 + y^2)^2} = 0,$$

and the equation is fulfilled.

2) Here

$$\frac{\partial f}{\partial x} = \alpha e^{\alpha x} \cos(\alpha y), \qquad \frac{\partial f}{\partial y} = -\alpha e^{\alpha x} \sin(\alpha y),$$

and

$$\frac{\partial^2 f}{\partial x^2} = \alpha^2 e^{\alpha x} \cos(\alpha y), \qquad \frac{\partial^2 f}{\partial y^2} = -\alpha^2 e^{\alpha x} \cos(\alpha y).$$

Then by insertion into the differential equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = \alpha^2 \left\{ e^{\alpha x} \cos(\alpha y) - e^{\alpha x} \cos(\alpha y) \right\} = 0.$$

The equation is satisfied, and we can choose any α.

3) Here

$$\frac{\partial f}{\partial t} = -e^{-t}(\cos x + \sin y).$$

Then

$$\frac{\partial^2 f}{\partial x^2} = -e^{-t} \cos x \quad \text{og} \quad \frac{\partial^2 f}{\partial y^2} = -e^{-t} \sin y,$$

and the differential equation is fulfilled

4) Here

$$\frac{1}{\gamma^2} \frac{\partial^2 f}{\partial t^2} = -(\alpha^2 + \beta^2) f(x, y, t),$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\alpha^2 f(x, y, t), \qquad \frac{\partial^2 f}{\partial y^2} = -\beta^2 f(x, y, t),$$

and the differential equation is fulfilled.

We must require that $\gamma \neq 0$. Notice that when $\gamma = 0$, then $f(x, y, t) \equiv 0$, while $\dfrac{1}{\gamma^2} \dfrac{\partial^2 f}{\partial t^2}$ is not defined.

5) When $f(x, y, z) = \dfrac{1}{\sqrt{x^2 + y^2 + z^2}}$, we have

$$\frac{\partial f}{\partial x} = -\frac{x}{(x^2 + y^2 + x^2)^{3/2}},$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\frac{1}{(x^2 + y^2 + z^2)^{3/2}} + \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}.$$

Due to the symmetry we get by interchanging the letters

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x^2 + 2y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}, \qquad \frac{\partial^2 f}{\partial z^2} = \frac{-x^2 - y^2 + 2z^2}{(x^2 + y^2 + z^2)^{5/2}},$$

thus

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}} + \frac{-x^2 + 2y^2 - z^2}{(x^2 + x^2 + z^2)^{5/2}} + \frac{-x^2 - y^2 + 2z^2}{(x^2 + +y^2 + z^2)^{5/2}} = 0.$$

The equation is satisfied.

6) When $f(r, t) = t^\alpha \exp\left(-\dfrac{r^2}{4t}\right)$, we get

$$\frac{\partial f}{\partial t} = \alpha t^{\alpha-1} \exp\left(-\frac{r^2}{4t}\right) + t^\alpha \exp\left(-\frac{r^2}{4t}\right) \cdot \frac{r^2}{4t^2} = \left\{\alpha t^{\alpha-1} + \frac{1}{4} r^2 t^{\alpha-2}\right\} \exp\left(-\frac{r^2}{4t}\right),$$

and accordingly

$$(3) \quad r^2 \frac{\partial f}{\partial t} = \left\{\alpha r^2 t^{\alpha-1} + \frac{1}{4} r^4 t^{\alpha-2}\right\} \exp\left(-\frac{r^2}{4t}\right).$$

Furthermore,

$$\frac{\partial f}{\partial r} = t^\alpha \exp\left(-\frac{r^2}{4t}\right) \cdot \left(-\frac{r}{2t}\right) = -\frac{1}{2} t^{\alpha-1} r \exp\left(-\frac{r^2}{4t}\right),$$

so

$$r^2 \frac{\partial f}{\partial r} = -\frac{1}{2} t^{\alpha-1} r^3 \exp\left(-\frac{r^2}{4t}\right)$$

and

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = -\frac{3}{2} t^{\alpha-1} r^2 \exp\left(-\frac{r^2}{4t}\right) + \frac{1}{4} r^2 t^{\alpha-2} \exp\left(-\frac{r^2}{4t}\right)$$

$$(4) \qquad\qquad = \left\{-\frac{3}{2} t^{\alpha-1} r^2 + \frac{1}{4} r^4 t^{\alpha-2}\right\} \exp\left(-\frac{r^2}{4t}\right).$$

By comparison we see that (3) and (4) only equals each other when $\alpha = -\dfrac{3}{2}$, corresponding to the fact that only

$$f(r,t) = t^{-3/2} \exp\left(-\frac{r^2}{4t}\right), \qquad t > 0,$$

of the given set of functions are solutions of

$$r^2 \frac{\partial f}{\partial t} = \frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right).$$

Example 5.5 *A C^2-function f in two variables satisfies the partial differential equation*

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial y^2} = 0.$$

Introduce the new variables $u = x + y$ and $v = x - y$, and prove that the function

$$f(u, v) = f\left(\frac{u + v}{2}, \frac{u - v}{2}\right)$$

fulfils the equation

$$\frac{\partial^2 g}{\partial u \partial v} = 0.$$

Furthermore, prove that it follows from

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

that

$$\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.$$

A Transform of the variables in partial differential equations.

D Follow the given guidelines.

I When

$$f(u, v) = f\left(\frac{u + v}{2}, \frac{u - v}{2}\right), \qquad x = \frac{u + v}{2}, \quad y = \frac{u - v}{2},$$

then

$$(5) \quad \frac{\partial g}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v} = \frac{1}{2}\frac{\partial f}{\partial x} - \frac{1}{2}\frac{\partial f}{\partial y} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y}\right\},$$

hence

$$\frac{\partial^2 g}{\partial u \partial v} = \frac{1}{2}\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial x}{\partial u} - \frac{1}{2}\frac{\partial^2 f}{\partial y^2} \cdot \frac{\partial y}{\partial u} = \frac{1}{4}\left\{\frac{\partial^2}{\partial x^2} - \frac{\partial^2 f}{\partial y^2}\right\} = 0$$

using the assumption.

Assume that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

We perform the following calculation

$$\frac{\partial g}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u} = \frac{1}{2}\left\{\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\right\},$$

thus

$$
\begin{aligned}
\frac{\partial^2 g}{\partial u^2} &= \frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial y}{\partial u} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial u} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial u}\right\} \\
&= \frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}\right\} = \frac{1}{2}\frac{\partial^2 f}{\partial x\partial y}.
\end{aligned}
$$

Finally, we get from (5.5)

$$
\begin{aligned}
\frac{\partial^2 g}{\partial v^2} &= \frac{1}{2}\left\{\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y\partial x}\cdot\frac{\partial y}{\partial v} + \frac{\partial^2 f}{\partial x\partial y}\cdot\frac{\partial x}{\partial v} + \frac{\partial^2 f}{\partial y^2}\cdot\frac{\partial y}{\partial v}\right\} \\
&= \frac{1}{4}\left\{\frac{\partial^2 f}{\partial x^2} - 2\frac{\partial^2 f}{\partial x\partial y} + \frac{\partial^2 f}{\partial y^2}\right\} = -\frac{1}{2}\frac{\partial^2 f}{\partial x\partial y},
\end{aligned}
$$

so by adding,

$$
\frac{\partial^2 g}{\partial u^2} + \frac{\partial^2 g}{\partial v^2} = 0.
$$

This is a well-known trick in the theory of partial differential equations.

Download free books at BookBooN.com

6 Taylor's formula for functions of several variables

Example 6.1 *Given the function*

$$f(x,y) = \exp(x + xy - 2u), \qquad (x,y) \in \mathbb{R}^2.$$

Find the approximating polynomial of at most second degree $P(x,y)$ and $Q(x,y)$ from the points of expansion $(0,0)$ and $(1,1)$ respectively. Calculate the values $P\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ and $Q\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$; compare these with the value $f\left(\dfrac{1}{2}, \dfrac{1}{2}\right)$ found on a pocket calculator.

A Approximating polynomials.

D Differentiate and apply a formula.

I For $f(x,y) = \exp(x + xy - 2y)$ we get

$$\frac{\partial f}{\partial x} = (1 + y)\exp(x + xu - 2y), \qquad \frac{\partial f}{\partial y} = (x - 2)\exp(x + xy - 2y),$$

and

$$\frac{\partial^2 f}{\partial x^2} = (1 + y)^2 \exp(x + xy - 2y),$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = (x + xy - 2y - 1)\exp(x + xy - 2y),$$

$$\frac{\partial^2 f}{\partial y^2} = (x - 2)^2 \exp(x + xy - 2y).$$

1) When the point of expansion is $(0,0)$ we get the coefficients

$$f(0,0) = 1, \quad f'_x(0,0) = 1, \quad f'_y(0,0) = -2,$$

$$f''_{xx}(0,0) = 1, \quad f''_{xy}(0,0) = f''_{yx}(0,0) = -1, \quad f''_{yy}(0,0) = 4,$$

and accordingly,

$$\begin{aligned}
P(x,y) &= f(0,0) + f'_x(0,0) \cdot x + f'_y(0,0) \cdot y \\
&\quad + \frac{1}{2}\left\{ f''_{xx}(0,0) \cdot x^2 + 2f''_{xy}(0,0) \cdot xy + f''_{yy}(0,0) \cdot y^2 \right\} \\
&= 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2.
\end{aligned}$$

Alternatively,

$$\exp(t) = 1 + t + \frac{1}{2}t^2 + \cdots,$$

so if we write $t = x - 2y + xy$ and include every term of higher degree than 2 in the dots, we get

$$\exp(x + xy - 2y) = 1 + \{x - 2y + xy\} + \frac{1}{2}\{x - 2y + xy\}^2 + \cdots$$

$$= 1 + x - 2y + xy + \frac{1}{2}(2 - 2y)^2 + \cdots$$

$$= 1 + x - 2y + xy + \frac{1}{2}x^2 - 2xy + 2y^2 + \cdots$$

$$= 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2 + \cdots.$$

As mentioned above the dots indicate the terms of higher degree than 2. We get the wanted approximating polynomial by deleting the dots, i.e.

$$P(x, y) = 1 + x - 2y + \frac{1}{2}x^2 - xy + 2y^2.$$

2) When the point of expansion is $(1, 1)$ we get the coefficients

$$f(1,1) = 1, \quad f'_x(1,1) = 2, \quad f'_y(1,1) = -1,$$

$$f''_{xx}(1,1) = 4, \quad f''_{xy}(1,1) = f''_{yx}(1,1) = -1, \quad f''_{yy}(1,1) = 1,$$

so

$$
\begin{aligned}
Q(x, y) &= f(1,1) + f'_x(1,1)(x-1) + f'_y(1,1)(y-1) \\
&\quad + \frac{1}{2}f''_{xx}(1,1)(x-1)^2 + f''_{xy}(1,1)(x-1)(y-1) + \frac{1}{2}f''_{yy}(1,1)(y-1)^2 \\
&= 1 + 2(x-1) - (y-1) + 2(x-1)^2 - (x-1)(y-1) + \frac{1}{2}(y-1)^2.
\end{aligned}
$$

REMARK. The variables in $Q(x, y)$ *ought to be* $(x - 1, y - 1)$ and *not* (x, y). The reason is that the approximating polynomial $Q(x, y)$ supplies us with the best approximation in the *neightbourhood* of the point $(1, 1)$, which means that for numerical reasons should *not* expand from the fairly distant point $(0, 0)$. ◊

The polynomial can also in this case be found *alternatively*. Since the point of expansion is $(1, 1)$, we introduce the new variables $(h, k) = (x - 1, y - 1)$, which are small in the neighbourhood of $(1, 1)$. Hence, $(x, y) = (h + 1, k + 1)$. Then

$$\exp(x + xy - 2y) = \exp(h + 1 + (h+1)(k+1) - 2(k+1))$$

$$= \exp(1 + h + 1 + h + k + hk - 2 - 2k) = \exp(2h - k + hk)$$

$$= 1 + \{2h - k + hk\} + \frac{1}{2!}\{2h - k + hk\}^2 + \cdots$$

$$= 1 + 2h - k + hk + \frac{1}{2}(2h - k)^2 + \cdots$$

$$= 1 + 2h - k + hk + 2h^2 - 2hk + \frac{1}{2}k^2 + \cdots,$$

where the dots as usual indicate terms of higher degree. Thus

$$
\begin{aligned}
Q(x, y) &= 1 + 2h - k + 2h^2 - hk + \frac{1}{2}k^2 \\
&= 1 + 2(x-1) - (y-1) + 2(x-1)^2 - (x-1)(y-1) + \frac{1}{2}(y-1)^2.
\end{aligned}
$$

3) We evaluate

$$P\left(\frac{1}{2},\frac{1}{2}\right) = 1 + \frac{1}{2} - 2 \cdot \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2}\right)^2 - \frac{1}{2} \cdot \frac{1}{2} + 2 \cdot \left(\frac{1}{2}\right)^2$$

$$= 1 + \frac{1}{2} - 1 + \frac{1}{8} - \frac{1}{4} + \frac{1}{2} = \frac{7}{8} = 0,875,$$

and

$$Q\left(\frac{1}{2},\frac{1}{2}\right) = 1 + 2 \cdot \left(-\frac{1}{2}\right) - \left(-\frac{1}{2}\right) + 2\left(-\frac{1}{2}\right)^2 - \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{2}\right)^2$$

$$= 1 - 1 + \frac{1}{2} + \frac{1}{2} - \frac{1}{4} + \frac{1}{8} = \frac{7}{8} = 0.875.$$

Finally, we get by using a pocket calculator

$$f\left(\frac{1}{2},\frac{1}{2}\right) = \exp\left(\frac{1}{2} + \frac{1}{4} - 1\right) = \exp\left(-\frac{1}{4}\right) \approx 0,779.$$

The approximations have a relatively large error (approx. 12 %). This is caused by the fact that the point $\left(\dfrac{1}{2},\dfrac{1}{2}\right)$ is fairly distant from both points of expansions.

Example 6.2 *Let $f \in C^2(A)$, where A is an open subset of $\mathbb{R}^2$. Prove that for $(x,y) \in A$ and $|h|$ sufficiently small,*

$$4h^2 f''_{xy}(x,y) = \{f(x+h,y+h)+f(x-h,y-h)-f(x+h,y-h)-f(x-h,y+h)\}+\varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2} \to 0$ for $h \to 0$. When we neglect $\varepsilon(h)$ we get an approximative expression of $f''_{xy}(x,y)$, which can be applied in numerical calculations.
Set up analogous formulæ for $f''_{xx}(x,y)$ and $f''_{yy}(x,y)$.

A Approximating polynomials.

D Calculate the approximating polynomial for $f(x+h,y+k)$. Replace (h,k) by $(\pm h, \pm h)$ (all four combinations) and compare.

I We know already that

$$f(x+h,y+k) = f + f'_x \cdot h + f'_y \cdot k + \frac{1}{2}\left\{f''_{xx}h^2 + 2f''_{xy}hk + f''_{yy}k^2\right\} + \varepsilon(h,k),$$

where $\varepsilon(h,k)/(h^2+k^2) \to 0$ for $(h,k) \to 0$, and where we have used the shorthand f, f'_x, etc. instead of the total expression $f(x,y)$, $f'_x(x,y)$, etc. in all details.

By successively replacing (h,k) by (h,h), $(-h,-h)$, $(h,-h)$ and $(-h,h)$ we get

$$f(x+h,y+h) = f + f'_x \cdot h + f'_y \cdot h + \frac{1}{2}f''_{xx} \cdot h^2 + f''_{xy} \cdot h^2 + \frac{1}{2}f''_{yy} \cdot h^2 + \varepsilon_1(h),$$

$$f(x-h,y-h) = f - f'_x \cdot h - f'_y \cdot h + \frac{1}{2}f''_{xx} \cdot h^2 + f''_{xy} \cdot h^2 + \frac{1}{2}f''_{yy} \cdot h^2 + \varepsilon_2(h),$$

$$f(x+h, y-h) = f + f'_x \cdot h - f'_y \cdot h + \frac{1}{2} f''_{xx} \cdot h^2 - f''_{xy} \cdot h^2 + \frac{1}{2} f''_{yy} \cdot h^2 + \varepsilon_3(h),$$

$$f(x-h, y+h) = f - f'_x \cdot h + f'_y \cdot h + \frac{1}{2} f''_{xx} \cdot h^2 - f''_{xy} \cdot h^2 + \frac{1}{2} f''_{yy} \cdot h^2 + \varepsilon_4(h),$$

where $\dfrac{\varepsilon_i(h)}{h^2} \to 0$ for $h \to 0$.

It follows that

$$f(x+h, y+h) + f(x-h, y-h) - f(x+h, y-h) - f(x-h, y+h)$$
$$= 0 \cdot f + 0 \cdot f'_x \cdot h + 0 \cdot f'_y \cdot h + 0 \cdot f''_{xx} \cdot h^2 + 4 f''_{xy} \cdot h^2 + 0 \cdot f''_{yy} \cdot h^2 + \varepsilon(h),$$

hence by a rearrangement

$$4 f''_{xy}(x, y) h^2 = \{ f(x+h, y+h) + f(x-h, y-y) - f(x+h, y-h) - f(x-h, y+h) \} + \varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2} \to 0$ for $h \to 0$, and the claim is proved.

REMARK. This formula is useful in numerical calculations of $f''_{xy}(x,y)$, when we know the values of $f(x+mh, y+nh)$, $m, n \in \mathbb{Z}$. $\Diamond$

If we instead put $k = 0$, we get

$$f(x+h, y) = f(x,y) + f'_x(x,y)h + \frac{1}{2}f''_{xx}(x,y)h^2 + \varepsilon_1(h),$$

$$f(x-h, y) = f(x,y) - f'_x(x,y)h + \frac{1}{2}f''_{xx}(x,y)h^2 + \varepsilon_2(h),$$

hence by adding,

$$f(x+h, y) + f(x-h, y) = 2f(x,y) + f''_{xx}(x,y) \cdot h^2 + \varepsilon(h),$$

and by a rearrangement,

$$h^2 f''_{xx}(x,y) = \{f(x+h, y) - 2f(x,y) + f(x-h, y)\} + \varepsilon(h).$$

Analogously,

$$h^2 f''_{yy}(x,y) = \{f(x, y+h) - 2f(x,y) + f(x, y-h)\} + \varepsilon(h),$$

where $\dfrac{\varepsilon(h)}{h^2} \to 0$ for $h \to 0$.

Example 6.3 *Find the approximating polynomial of at most second degree of the given functions in the given points of expansion:*

1) *The function* $\ln\{(x+1)^2 + (y-1)^2\}$, *defined in* $\mathbb{R}^2 \setminus \{(-1, 1)\}$, *from the point* $(0, 0)$.

2) *The function* $\sqrt{x^2 + y^2}$, *defined in* $\mathbb{R}^2 \setminus \{(0,0)\}$ *from the point* $(3, 4)$.

3) *The function* $\text{Arctan}\ \dfrac{y}{x}$, *defined for* $x > 0$ *from the point* $(1, \sqrt{3})$.

4) *The function* $\sqrt[5]{x^2 + 2y^3}$, *defined for* $x^2 + 2y^3 > 0$ *from* $(4, 2)$.

5) *The function* $x^3 + xy - 12x - 6y$, *defined in* $\mathbb{R}^2$ *from* $(1, 3)$.

6) *The function* $\sqrt{x^2 + y^2 + z^2}$, *defined in* $\mathbb{R}^3 \setminus \{(0, 0, 0)\}$ *from the point* $(3, 6, 6)$.

7) *The function* $\sin(x - y) + z(x + y) - 2x + 1$, *defined in* $\mathbb{R}^3$ *from* $(0, 0, 1)$.

8) *The function* $(\cosh x) \cdot \sin(x - y - 2z)$, *defined in* $\mathbb{R}^3$ *from* $\left(0, \dfrac{\pi}{2}, 0\right)$.

A Approximating polynomials of at most second degree.

D Use preferably the standard method, i.e. differentiate and apply a formula. Note the standard scheme in each case.

In some cases it is possible instead to use standard Taylor series.

I 1) The function $f(x, y) = \ln\{(x+1)^2 + (y-1)^2\}$ is of class C^∞ in the given domain, and

$$f(x, y) = \ln\{(x+1)^2 + (y-1)^2\}, \qquad\qquad f(\mathbf{0}) = \ln 2,$$

$$\frac{\partial f}{\partial x} = \frac{2(x+1)}{(x+1)^2 + (y-1)^2}, \qquad\qquad \frac{\partial f}{\partial x}(\mathbf{0}) = 1,$$

$$\frac{\partial f}{\partial y} = \frac{2(y-1)}{(x+1)^2 + (y-1)^2}, \qquad\qquad \frac{\partial f}{\partial y}(\mathbf{0}) = -1,$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{1}{(x+1)^2 + (y-1)^2} - \frac{4(x+1)^2}{\{(x+1)^2 + (y-1)^2\}^2}, \quad \frac{\partial^2 f}{\partial x^2}(\mathbf{0}) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{4(x+1)(y-1)}{\{(x+1)^2 + (y-1)^2\}^2}, \qquad \frac{\partial^2 f}{\partial x \partial y}(\mathbf{0}) = 1,$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{2}{(x+1)^2 + (y-1)^2} - \frac{4(y-1)^2}{\{(x+1)^2 + (y-1)^2\}^2}, \quad \frac{\partial^2 f}{\partial y^2}(\mathbf{0}) = 0.$$

The coefficients of the approximating polynomial are the numbers in the right hand column. We get by insertion,

$$\begin{aligned}
P_2(x, y) &= f(\mathbf{0}) + \left\{ \frac{\partial f}{\partial x}(\mathbf{0}) \cdot (x - 0) + \frac{\partial f}{\partial y}(\mathbf{0}) \cdot (y - 0) \right\} \\
&\quad + \frac{1}{2!} \left\{ \frac{\partial^2 f}{\partial x^2}(\mathbf{0})(x-0)^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\mathbf{0})(x-0)(y-0) + \frac{\partial^2 f}{\partial y^2}(\mathbf{0})(y-0)^2 \right\} \\
&= \ln 2 + x - y + \frac{1}{2} \cdot 2xy = \ln 2 + x - y + xy.
\end{aligned}$$

2) The function is of course also defined at $(0, 0)$, but it is only of class C^∞ in $\mathbb{R} \setminus \{(0, 0)\}$. Using the same procedure as before we get for the point $(3, 4)$,

$$f(x, y) = \sqrt{x^2 + y^2}, \qquad\qquad f(3, 4) = 5,$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2}}, \qquad\qquad \frac{\partial f}{\partial x}(3, 4) = \frac{3}{5},$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2}}, \qquad\qquad \frac{\partial f}{\partial y}(3, 4) = \frac{4}{5},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2}{(\sqrt{x^2 + y^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial x^2}(3, 4) = \frac{16}{125},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{xy}{(\sqrt{x^2 + y^2})^3}, \qquad \frac{\partial^2 f}{\partial x \partial y}(3, 4) = -\frac{12}{125},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2}{(\sqrt{x^2 + y^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial y^2}(3, 4) = \frac{9}{125}.$$

By choosing $(x_1, y_1) = (x - x_0, y - y_0) = (x - 3, y - 4)$ as our new variables we get

$$
\begin{aligned}
P_2(x, y) &= 5 + \frac{3}{5}(x - 3) + \frac{4}{5}(y - 4) \\
&\quad + \left\{ \frac{16}{125}(x-3)^2 - \frac{12}{125} \cdot 2(x-3)(y-4) + \frac{9}{125}(y-4)^2 \right\} \\
&= 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) + \frac{8}{125}(x-3)^2 - \frac{12}{125}(x-3)(y-4) + \frac{9}{250}(y-4)^2,
\end{aligned}
$$

which can be reduced to

$$
P_2(x, y) = 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) + \frac{1}{250}\left\{4(x-3) - 3(y-4)\right\}^2.
$$

3) The function is of class C^∞ in the given domain (and of course also defined for $x < 0$; but this case is not at all relevant here). We have as before

$$
\begin{aligned}
f(x, y) &= \operatorname{Arctan} \frac{y}{x}, & f(1, \sqrt{3}) &= \frac{\pi}{3}, \\[2mm]
\frac{\partial f}{\partial x} &= \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}, & \frac{\partial f}{\partial x}(1, \sqrt{3}) &= -\frac{\sqrt{3}}{4}, \\[2mm]
\frac{\partial f}{\partial y} &= \frac{x}{x^2 + y^2}, & \frac{\partial f}{\partial y}(1, \sqrt{3}) &= \frac{1}{4}, \\[2mm]
\frac{\partial^2 f}{\partial x^2} &= \frac{2xy}{(x^2 + y^2)^2}, & \frac{\partial^2 f}{\partial x^2}(1, \sqrt{3}) &= \frac{\sqrt{3}}{8}, \\[2mm]
\frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial^2 f}{\partial y \partial x} = \frac{y^2 - x^2}{(x^2 + y^2)^2}, & \frac{\partial^2 f}{\partial x \partial y}(1, \sqrt{3}) &= \frac{1}{8}, \\[2mm]
\frac{\partial^2 f}{\partial y^2} &= -\frac{2xy}{(x^2 + y^2)^2}, & \frac{\partial^2 f}{\partial y^2}(1, \sqrt{3}) &= -\frac{\sqrt{3}}{8}.
\end{aligned}
$$

The approximating polynomial from $(1, \sqrt{3})$ is

$$
\begin{aligned}
P_2(x, y) &= \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x - 1) + \frac{1}{4}(y - \sqrt{3}) \\
&\quad + \frac{\sqrt{3}}{16}(x - 1)^2 + \frac{1}{8}(x - 1)(y - \sqrt{3}) - \frac{\sqrt{3}}{16}(y - 1)^2,
\end{aligned}
$$

which can be reduced to

$$
\begin{aligned}
P_2(x, y) &= \frac{\pi}{3} - \frac{\sqrt{3}}{4}(x - 1) + \frac{1}{4}(y - \sqrt{3}) \\
&\quad + \frac{\sqrt{3}}{16}\left\{(x-1) + \sqrt{3}(y - \sqrt{3})\right\}\left\{(x-1) - \frac{1}{\sqrt{3}}(y - \sqrt{3})\right\}.
\end{aligned}
$$

4) We see that when $x^2 + 2y^3 > 0$, then the function is of class C^∞. We calculate as before,

$$f(x, y) = (x^2 + 2y^3)^{1/5}, \qquad\qquad f(4, 2) = 2,$$

$$\frac{\partial f}{\partial x} = \frac{2}{5} x(x^2 + 2y^3)^{-4/5}, \qquad\qquad \frac{\partial f}{\partial x}(4, 2) = \frac{1}{10},$$

$$\frac{\partial f}{\partial y} = \frac{6}{5} y^2 (x^2 + 2y^3)^{-4/5}, \qquad\qquad \frac{\partial f}{\partial y}(4, 2) = \frac{3}{10},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2}{5}(x^2 + 2y^3)^{-4/5} - \frac{16}{25} x^2 (x^2 + 2y^3)^{-9/5}, \qquad\qquad \frac{\partial^2 f}{\partial x^2}(4, 2) = \frac{1}{200},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{48}{25} xy^2 (x^2 + 2y^3)^{-9/5}, \qquad\qquad \frac{\partial^2 f}{\partial x \partial y}(4, 2) = -\frac{3}{50},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{12}{5} y(x^2 + 2y^3)^{-4/5} - \frac{144}{25} y^4 (x^2 + 2y^3)^{-9/5}, \qquad\qquad \frac{\partial^2 f}{\partial y^2}(4, 2) = \frac{3}{25}.$$

The approximating polynomial from $(4, 2)$ is

$$P_2(x, y) = 2 + \frac{1}{10}(x - 4) + \frac{3}{10}(y - 2) + \frac{1}{400}(x - 4)^2 - \frac{3}{50}(x - 4)(y - 2) + \frac{3}{50}(y - 2)^2.$$

5) When one is asked to find the approximating polynomial for

$$f(x, y) = x^3 + xy^2 - 12x - 6y$$

of at most second degree from $(1, 3)$, it is tempting just to remove the terms $x^3 + xy^2$, which are of third degree. This is, however, not the right procedure, because the point of expansion is *not* $(0, 0)$, but translated to $(1, 3)$.

In order to explain what is going on we shall first apply the rather elaborate standard procedure, for later to give an alternative. We get by the standard procedure,

$$f(x, y) = x^3 + xy^2 - 12x - 6y, \quad f(1, 3) = -20,$$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 - 12, \qquad \frac{\partial f}{\partial x}(1, 3) = 0,$$

$$\frac{\partial f}{\partial y} = 2xy - 6, \qquad \frac{\partial f}{\partial y}(1, 3) = 0,$$

$$\frac{\partial^2 f}{\partial x^2} = 6x, \qquad \frac{\partial^2 f}{\partial x^2}(1, 3) = 6,$$

$$\frac{\partial^2 f}{\partial x \partial y} = 2y, \qquad \frac{\partial^2 f}{\partial x \partial y}(1, 3) = 6,$$

$$\frac{\partial^2 f}{\partial y^2} = 2x, \qquad \frac{\partial^2 f}{\partial y^2}(1, 3) = 2.$$

Then the approximating polynomial is

$$P_2(x, y) = -20 + 3(x - 1)^2 + 6(x - 1)(y - 3) + (y - 3)^2,$$

where we of course use $(x - 1, y - 3)$ as the new (and more correct) variables.

ALTERNATIVELY we start by introducing $(x_1, y_1) = (x - 1, y - 3)$ as our new variables, i.e. $(x, y) = (x_1 + 1, y_1 + 3)$. Then by insertion,

$$
\begin{aligned}
f(x, y) &= x^3 + xy^2 - 12x - 6y \\[1mm]
&= (x_1 + 1)^3 + (x_1 + 1)(y_1 + 3)^2 - 12(x_1 + 1) - 6(y_1 + 3) \\[1mm]
&= x_1^3 + 3x_1^2 + 3x_1 + 1 + (x_1 + 1)(y_1^2 + 6y_1 + 9) - 12x_1 - 12 - 6y_1 - 18 \\[1mm]
&= x_1^3 + 3x_1^2 - 9x_1 - 6y_1 - 29 + x_1 y_1^2 + 6x_1 y_1 + 9x_1 + y_1^2 + 6y_1 + 9 \\[1mm]
&= -20 + 3x_1^2 + 6x_1 y_1 + y_1^2 + \left\{ x_1^3 + x_1 y_1^2 \right\}.
\end{aligned}
$$

The approximative polynomial from $(1, 3)$ is then obtained by deleting all terms of degree > 2 in (x_1, y_1), thus

$$\begin{aligned} P_2(x, y) &= -20 + 3x_1^2 + 6x_1y_1 + y_1^2 \\ &= -20 + 3(x-1)^2 + 6(x-1)(y-3) + (y-3)^2. \end{aligned}$$

6) The function is of class C^∞ for $(x, y, z) \neq (0, 0, 0)$. We use the same method as before, only supplied with an extra variable. (Notice the systematics).

$$f(x, y, z) = \sqrt{x^2 + y^2 + z^2}, \qquad\qquad f(3, 6, 6) = 9,$$

$$\frac{\partial f}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \qquad\qquad \frac{\partial f}{\partial x}(3, 6, 6) = \frac{1}{3},$$

$$\frac{\partial f}{\partial y} = \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \qquad\qquad \frac{\partial f}{\partial y}(3, 6, 6) = \frac{2}{3},$$

$$\frac{\partial f}{\partial z} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \qquad\qquad \frac{\partial f}{\partial z}(3, 6, 6) = \frac{2}{3},$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial x^2}(3, 6, 6) = \frac{8}{81},$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{x^2 + z^2}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial y^2}(3, 6, 6) = \frac{5}{81},$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad\qquad \frac{\partial^2 f}{\partial z^2}(3, 6, 6) = \frac{5}{81},$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = -\frac{xy}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad \frac{\partial^2 f}{\partial x \partial y}(3, 6, 6) = -\frac{2}{81},$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = -\frac{xz}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad \frac{\partial^2 f}{\partial x \partial z}(3, 6, 6) = -\frac{2}{81},$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = -\frac{yz}{(\sqrt{x^2 + y^2 + z^2})^3}, \qquad \frac{\partial^2 f}{\partial y \partial z}(3, 6, 6) = -\frac{4}{81}.$$

From this we get the approximating polynomial from $(3, 6, 6)$,

$$\begin{aligned} P_2(x, y, z) &= 9 + \frac{1}{1!}\left\{ \frac{1}{3}(x-6) + \frac{2}{3}(y-6) + \frac{2}{3}(z-6) \right\} \\ &\quad + \frac{1}{2!}\left\{ \frac{8}{81}(x-3)^2 + \frac{5}{81}(y-6)^2 + \frac{5}{81}(z-6)^2 \right\} \\ &\quad - \frac{2}{2!}\left\{ \frac{2}{81}(x-3)(y-6) + \frac{4}{81}(y-6)(z-6) + \frac{2}{81}(z-6)(x-3) \right\} \\ &= 9 + \frac{1}{3}(x-3) + \frac{2}{3}(y-6) + \frac{2}{3}(z-6) + \frac{4}{81}(x-3)^2 + \frac{5}{162}(y-6)^2 \\ &\quad + \frac{5}{162}(z-6)^2 - \frac{2}{81}(x-3)(y-6) - \frac{4}{81}(y-6)(z-z) - \frac{2}{81}(z-6)(x-3). \end{aligned}$$

7) Using the same method as above we get

$$f(x, y, z) = \sin(x - y) + z(x + y) - 2x + 1, \qquad f(0, 0, 1) = 1,$$

$$\frac{\partial f}{\partial x} = \cos(x - y) + z - 2, \qquad\qquad \frac{\partial f}{\partial x} = 0,$$

$$\frac{\partial f}{\partial y} = -\cos(x - y) + z, \qquad\qquad \frac{\partial f}{\partial y}(0, 0, 1) = 0,$$

$$\frac{\partial f}{\partial z} = x + y, \qquad\qquad \frac{\partial f}{\partial z}(0, 0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial x^2} = -\sin(x - y), \qquad\qquad \frac{\partial^2 f}{\partial x^2}(0, 0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = -\sin(x - y), \qquad\qquad \frac{\partial^2 f}{\partial y^2}(0, 0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial z^2} = 0, \qquad\qquad \frac{\partial^2 f}{\partial z^2}(0, 0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \sin(x - y), \qquad\qquad \frac{\partial^2 f}{\partial x \partial y}(0, 0, 1) = 0,$$

$$\frac{\partial^2 f}{\partial x \partial z} = \frac{\partial^2 f}{\partial z \partial x} = 1, \qquad\qquad \frac{\partial^2 f}{\partial x \partial z}(0, 0, 1) = 1,$$

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = 1, \qquad\qquad \frac{\partial^2 f}{\partial y \partial z}(0, 0, 1) = 1.$$

Accordingly, the approximating polynomial from $(0, 0, 1)$ is

$$P_2(x, y, z) = 1 + \frac{1}{1!} \cdot 0 + \frac{2}{2!}\{(x-0)(z-1) + (y-0)(z-1)\} = 10(x + y)(z - 1).$$

We note that thee natural parameters are here $(x, y, z - 1)$.

ALTERNATIVELY we exploit that $(x - 0) - (y - 0) = x - y$ is the approximating polynomial for $\sin(x - y)$ of at most second degree, and since the rest is a polynomial of second degree in (x, y, z), we get

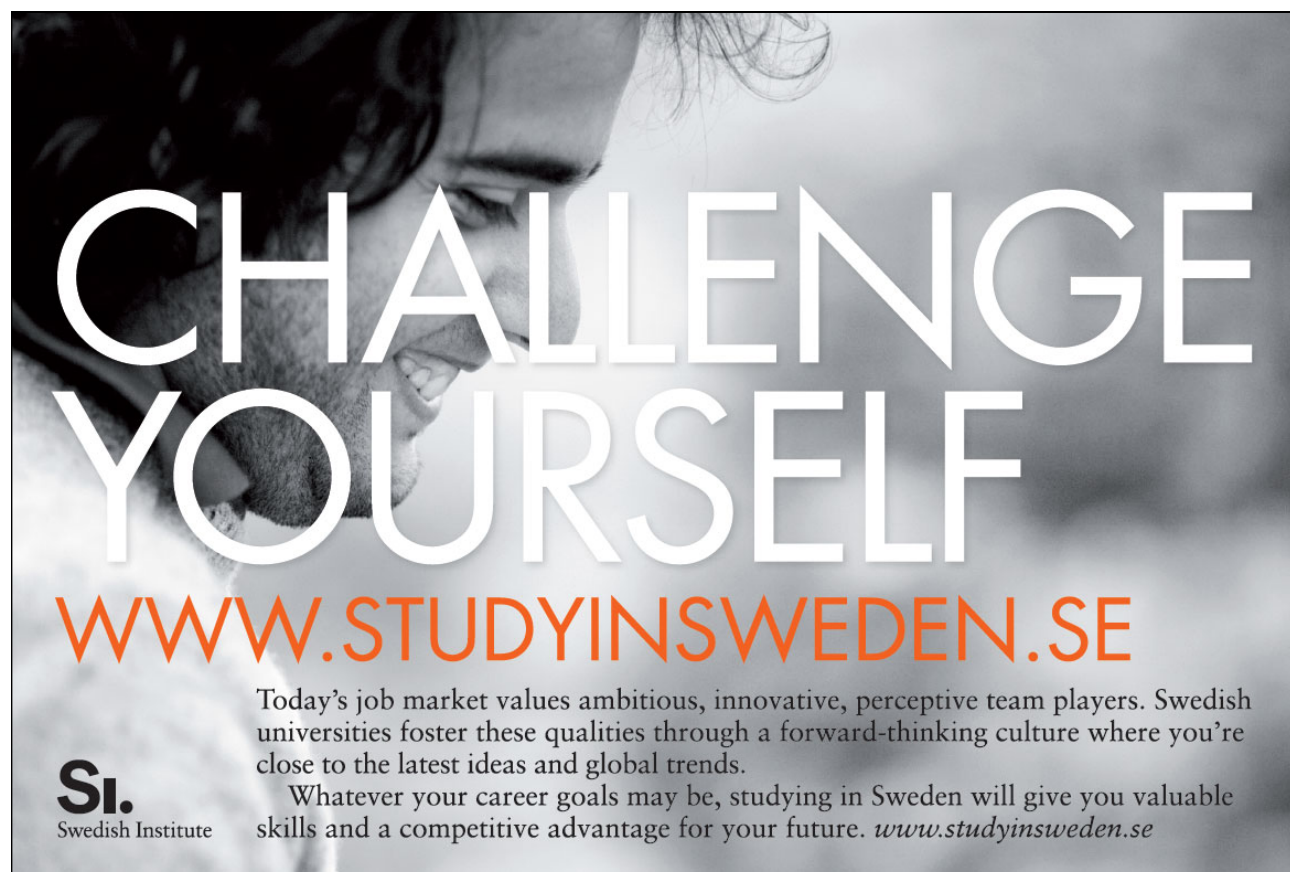$$P_2(x, y, z) = x - y + z(x + y) - 2x + 1 = z(x + y) - (x + y) + 1 = 1 + (x + y)(z - 1).$$

8) We first rewrite the expression and use series expansions,

$$
\begin{aligned}
f(x,y,z) &= \cosh x \cdot \sin(x-y-2z) = \cosh x \cdot \sin\left(x - \left(y - \frac{\pi}{2}\right) - 2z - \frac{\pi}{2}\right) \\
&= -\cosh x \cdot \cos\left(x - \left(y - \frac{\pi}{2}\right) - 2z\right) \\
&= -\left\{1 + \frac{1}{2}x^2 + \cdots\right\}\left\{1 - \frac{1}{2}\left[x - \left(y - \frac{\pi}{2}\right) - 2z\right]^2 + \cdots\right\} \\
&= -1 - \frac{1}{2}x^2 + \frac{1}{2}\left\{x - \left(y - \frac{\pi}{2}\right) - 2z\right\}^2 + \cdots \\
&= -1 + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + 4z^2 - x\left(y - \frac{\pi}{2}\right) + 2\left(y - \frac{\pi}{2}\right)z - 2xz + \cdots,
\end{aligned}
$$

where the dots denote terms of higher degree. The approximating polynomial is obtained by removing these dots:

$$
\begin{aligned}
P_2(x,y,z) &= -1 - \frac{1}{2}x^2 + \frac{1}{2}\left\{x - \left(y - \frac{\pi}{2}\right) - 2z\right\}^2 \\
&= -1 - \frac{1}{2}\left\{2x - \left(y - \frac{\pi}{2}\right) - 2z\right\}\left\{\left(y - \frac{\pi}{2}\right) + 2z\right\} \\
&= -1 + \frac{1}{2}\left(y - \frac{\pi}{2}\right)^2 + 4z^2 - x\left(y - \frac{\pi}{2}\right) + 2\left(y - \frac{\pi}{2}\right)z - 2xz.
\end{aligned}
$$

ALTERNATIVELY we get by the standard method,

$$f(x,y,z) = \cosh x \cdot \sin(x-y-2z), \qquad\qquad f\left(0,\frac{\pi}{2},0\right) = -1,$$

$$\frac{\partial f}{\partial x} = \sinh x \sin(x-y-2z) + \cosh x \cos(x-y-2z), \qquad \frac{\partial f}{\partial x}\left(0,\frac{\pi}{2},0\right) = 0,$$

$$\frac{\partial f}{\partial y} = -\cosh x \cos(x-y-2z), \qquad\qquad \frac{\partial f}{\partial y}\left(0,\frac{\pi}{2},0\right) = 0,$$

$$\frac{\partial f}{\partial z} = -2\cosh x \cos(x-y-2z), \qquad\qquad \frac{\partial f}{\partial z}\left(0,\frac{\pi}{2},0,\right) = 0,$$

$$\frac{\partial^2 f}{\partial x^2} = 2\sinh x \cos(x-y-2z), \qquad\qquad \frac{\partial^2 f}{\partial x^2}\left(0,\frac{\pi}{2},0\right) = 0,$$

$$\frac{\partial^2 f}{\partial y^2} = \cosh x \sin(x-y-2z), \qquad\qquad \frac{\partial^2 f}{\partial y^2}\left(0,\frac{\pi}{2},0\right) = 1,$$

$$\frac{\partial^2 f}{\partial z^2} = -4\cosh x \sin(x-y-2z), \qquad\qquad \frac{\partial^2 f}{\partial z^2}\left(0,\frac{\pi}{2},0\right) = 4,$$

$$\frac{\partial^2 f}{\partial x \partial y} = -\sinh x \cos(x-y-2z) + \cosh x \sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial x \partial y}\left(0,\frac{\pi}{2},0\right) = -1,$$

$$\frac{\partial^2 f}{\partial y \partial z} = -2\cosh x \sin(x-y-2z), \qquad\qquad \frac{\partial^2 f}{\partial y \partial z}\left(0,\frac{\pi}{2},0\right) = 2,$$

$$\frac{\partial^2 f}{\partial z \partial x} = -2\sinh x \cos(x-y-2z) + 2\cosh x \sin(x-y-2z), \qquad \frac{\partial^2 f}{\partial z \partial x}\left(0,\frac{\pi}{2},0\right) = -2.$$

Hence, the approximating polynomial from $\left(0,\dfrac{\pi}{2},0\right)$ is

$$
\begin{aligned}
P_2(x,y,z) &= -1 + \frac{1}{1!}\{0\} + \frac{1}{2!}\left\{\left(y-\frac{\pi}{2}\right)^2 + 4z^2\right\} \\
&\quad + \frac{2}{2!}\left\{-x\left(y-\frac{\pi}{2}\right) + 2\left(y-\frac{\pi}{2}\right)z - 2xz\right\} \\
&= -1 + \frac{1}{2}\left(y-\frac{\pi}{2}\right)^2 + 4z^2 - x\left(y-\frac{\pi}{2}\right) + 2z\left(y-\frac{\pi}{2}\right) - 2xz.
\end{aligned}
$$

Example 6.4 *Find approximating values of the following expressions by using the approximating polynomials of at most second degree from* **Example 6.3**. *Compare with the values which we get by using a pocket calculator instead.*

1) *The length L of the diagonal in a rectangle of edge lengths 2.9 and 4.2.*

2) *The length L of the space diagonal in a rectangular box of edge lengths 3.03 and 5.98 and 6.01.*

3) $\sqrt[5]{3.8^2 + 2 \cdot 2.1^3}$.

A Approximating values.

D Identify the corresponding function f. Apply the approximations found in **Example 6.3**. Compare the results with a calculation of a pocket calculator.

I 1) By using a pocket calculator we find that the length is

$$L = \sqrt{2.9^2 + 4.2^2} \approx 5.103\,920.$$

The corresponding function is $f(x, y) = \sqrt{x^2 + y^2}$, expanded from $(3, 4)$.

According to **Example 6.3.2** the approximation is given by

$$(6) \qquad P_2(z, y) \;=\; 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) + \frac{8}{125}(x-3)^2 - \frac{12}{125}(x-3)(y-4) + \frac{9}{250}(y-4)^2$$

$$(7) \qquad \qquad \;=\; 5 + \frac{3}{5}(x-3) + \frac{4}{5}(y-4) + \frac{1}{250}\{4(x-3) - 3(y-4)\}^2.$$

Since $x - 3 = -\dfrac{1}{10}$ and $y - 4 = \dfrac{1}{5} = \dfrac{2}{10}$, it follows from (6) that

$$\begin{aligned}
P_2(2, 9; 4, 2) &= 5 + \frac{6}{10}\left(-\frac{1}{10}\right) + \frac{8}{10} \cdot \frac{2}{10} + \frac{64}{1000}\left(-\frac{1}{10}\right)^2 - \frac{96}{1000}\left(-\frac{1}{10}\right)\frac{2}{10} + \frac{36}{1000}\left(\frac{2}{10}\right)^2 \\
&= 5 - \frac{6}{100} + \frac{16}{100} + \frac{1}{100\,000}(64 + 192 + 144) \\
&= 5 + \frac{1}{10} + \frac{400}{100\,000} = 5.104.
\end{aligned}$$

If we instead use (7), we get by somewhat simpler calculations,

$$\begin{aligned}
P_2(2, 9; 4, 2) &= 5 + \frac{6}{10}\left(-\frac{1}{10}\right) + \frac{8}{10} \cdot \frac{2}{10} + \frac{4}{1000}\left\{-\frac{4}{10} - 3 \cdot \frac{2}{10}\right\}^2 \\
&= 5 + \frac{1}{10} + \frac{4}{1000} = 5.104.
\end{aligned}$$

By comparison we see that the relative error is $< 1.6 \cdot 10^{-3}\%$.

2) A calculation on a pocket calculator shows that the length is

$$L = \sqrt{3.03^2 + 5.98^2 + 6.01^2} \approx 9.003\,410.$$

The corresponding function is $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, expanded from the point $(3, 6, 6)$.
According to **Example 6.3.6** the corresponding approximation is given by

$$\begin{aligned}
P_2(x, y, z) &= 9 + \frac{1}{3}(x-3) + \frac{2}{3}(y-6) + \frac{2}{3}(x-6) + \frac{4}{81}(x-3)^2 + \frac{5}{162}(y-6)^2 + \frac{5}{162}(z-6)^2 \\
&\quad - \frac{2}{81}(x-3)(y-6) - \frac{4}{81}(y-6)(z-6) - \frac{2}{81}(z-6)(x-3).
\end{aligned}$$

When r $(x, y, z) = (3.03; 5.98; 6.01)$, we have $x - 3 = \frac{3}{100}$ and $y - 6 = -\frac{2}{100}$ and $z - 6 = \frac{1}{100}$. Then we get the approximate value by insertion,

$$P_2(3.03; 5.98; 6.01) = 9 + \frac{1}{3} \cdot \frac{3}{100} + \frac{2}{3}\left(-\frac{2}{100}\right) + \frac{2}{3} \cdot \frac{1}{100}$$

$$+ \frac{4}{81}\left(\frac{3}{100}\right)^2 + \frac{5}{162}\left(-\frac{2}{100}\right)^2 + \frac{5}{162}\left(\frac{1}{100}\right)^2$$

$$- \frac{2}{81} \cdot \frac{3}{100}\left(-\frac{2}{100}\right) - \frac{4}{81}\left(-\frac{2}{100}\right) \cdot \frac{1}{100} - \frac{2}{81}\left(\frac{1}{100}\right)\left(\frac{3}{100}\right)$$

$$= 9 + \frac{1}{300}(3 - 4 + 2) + \frac{1}{162 \cdot 10\,000}(2 \cdot 4 \cdot 9 + 5 \cdot 405 + 24 + 16 - 12)$$

$$= 9 + \frac{1}{300} + \frac{1}{1\,620\,000}(72025 + 28)$$

$$= 9 + \frac{1}{300} + \frac{125}{162 \cdot 10\,000}$$

$$= 9 + \frac{1}{900}\left(1 + \frac{5}{216}\right) = 9 + \frac{221}{64800}$$

$$\approx 9.003\,410 \quad (!).$$

The error is invisible here, in particular because the value found on a pocket calculator is also an approximate value.

3) We get by means of a pocket calculator

$$\sqrt[5]{3.8^2 + 2 \cdot 2.21^3} \approx 2., 011\,883.$$

The corresponding function is $f(x, y) = \sqrt[5]{x^2 + 2y^3}$, expanded from the point $(4, 2)$.

We get from **Example 6.3.4** the approximation

$$P_2(x, y) = 2 + \frac{1}{10}(x - 4) + \frac{3}{10}(y - 2) + \frac{1}{400}(x - 4)^2 - \frac{3}{50}(x - 4)(y - 2) + \frac{3}{50}(y - 2)^2.$$

Since $x - 4 = -\frac{2}{10}$ and $y - 2 = \frac{1}{10}$, it follows by insertion that

$$P_2(3.8; 2.1) = 2 - \frac{2}{100} + \frac{3}{100} + \frac{1}{400} \cdot \frac{4}{100} - \frac{3}{50}\left(-\frac{2}{100}\right) + \frac{3}{50} \cdot \frac{1}{100}$$

$$= 2 + \frac{1}{100} + \frac{1}{10\,000}(1 + 12 + 6) = 2 + \frac{1}{100} + \frac{19}{10\,000} = 2.0119.$$

A comparison shows that this is a very accurate approximation.

Example 6.5 *A function $f \in C^\infty(\mathbb{R}^2)$ satisfies the equations*

$$f(x,0) = e^x, \qquad f_y'(x,y) = 2y\, f(x,y).$$

Find the approximating polynomial of at most second degree for the function f with $(0,0)$ as the point of expansion.

A Approximating polynomial from apparently very vague assumptions.

D Find the constants by using the definition of partial differentiability.

I Since $f \in C^\infty$, we are allowed to interchange the order of the differentiations, whenever it is necessary. By using the standard method we get

$$f(x,0) = e^x, \qquad\qquad\qquad f(0,0) = 1$$

$$f_x'(x,0) = e^x, \qquad\qquad\qquad f_x'(0,0) = 1,$$

$$f_y'(x,y) = 2y\, f(x,y), \qquad\qquad f_y'(0,0) = 0,$$

$$f_{xx}''(x,0) = e^x, \qquad\qquad\qquad f_{xx}''(0,0) = 1,$$

$$f_{xy}''(x,y) = 2y\, f_x'(x,y), \qquad\qquad f_{xy}''(0,0) = 0,$$

$$f_{yy}''(x,y) = 2f(x,y) + 4y^2 f(x,y), \qquad f_{yy}''(0,0) = 2.$$

We get the approximating polynomial,

$$P_2(x,y) = 1 + 1 \cdot x + 0 \cdot y + \frac{1}{2} \cdot 1 \cdot x^2 + 0 \cdot xy + \frac{1}{2} \cdot 2y^2 = 1 + x + \frac{1}{2}x^2 + y^2.$$

C It is actually possible to determine $f(x,y)$ uniquely from the given information. In fact, if we divide the latter equation by $f(x,y) \neq 0$, then

$$\frac{f_y'(x,y)}{f(x,y)} = \frac{\partial}{\partial y} \ln|f(x,y)| = 2y.$$

When we integrate with respect to y we get with some arbitrary function $\varphi(x)$ in x that $\ln|f(x,y)| = y^2 + \varphi(x)$. Hence there exists a function $\Phi(x)$, such that

$$f(x,y) = \Phi(x) \cdot \exp(y^2).$$

We put $y = 0$. Then it follows from the former of the given equations that

$$f(x,0) = e^x = \Phi(x).$$

Hence

$$\begin{aligned}
f(x,y) &= \exp(x + y^2) = 1 + \{x + y^2\} + \frac{1}{2}\{x + y^2\}^2 + \cdots \\
&= 1 + x + y^2 + \frac{1}{2}x^2 + \cdots .
\end{aligned}$$

It follows immediately that the approximating polynomial is

$$P_2(x,y) = 1 + x + \frac{1}{2}x^2 + y^2,$$

and we have tested our result. $\Diamond$

Example 6.6 *Indicate on a figure the domain of the function*

$$f(x, y) = \ln \left\{ \left(4y - y^2 - x \right) \sqrt{x} \right\}.$$

Then find the approximating polynomial of at most first degree for f at the point of expansion $(2, 1)$.

A Domain and approximating polynomial.
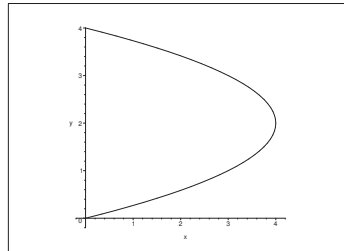
D Check where $f(x, y)$ is defined.

Figure 23: The domain of $f(x, y)$.

I The logarithm is only defined on the set of positive numbers, so $(4y - y^2 - x)\sqrt{x}$ must be defined and positive. In particular, $x > 0$ and $4y - y^2 - x > 0$, so

$$0 < 4y - y^2 - x = 4 - (4 - 4y + y^2) - x = 4 - (y - 2)^2 - x,$$

and thus

$$0 < x < 4 - (y - 2)^2.$$

The domain is bounded of the Y axis and the parabola of the equation $x = 4 - (y - 2)^2$.

By the rearrangement

$$f(x, y) = \ln\left(4y - y^2 - x\right) + \frac{1}{2}\ln x \qquad \text{for } (x, y) \in D,$$

we get

$$f(2, 1) = \ln(4 - 1 - 2) + \frac{1}{2}\ln 2$$

and

$$\frac{\partial f}{\partial x} = -\frac{1}{4y - y^2 - x} + \frac{1}{2}\cdot\frac{1}{x}, \qquad \frac{\partial f}{\partial x}(2, 1) = -1 + \frac{1}{4} = -\frac{3}{4},$$

and

$$\frac{\partial f}{\partial y} = \frac{4 - 2y}{4y - y^2 - x}, \qquad \frac{\partial f}{\partial y}(2, 1) = 2,$$

hence

$$P_1(x, y) = \frac{1}{2}\ln 2 - \frac{3}{4}(x - 2) + 2(y - 1).$$

Example 6.7 *It is well-known that an equation like*

$$f(x, y) = 0$$

under suitable circumstances can be solved with respect to one of its variables, and one has e.g. $y = Y(x)$, and then a differentiation of $f(x, y) = 0$ with respect to x gives a formula of the derivative:

$$Y'(x) = -\frac{f'_x(x, Y(x))}{f'_y(x, Y(x))}.$$

Prove by a similar procedure the formula

$$Y''(x) = -\frac{f''_{yy}(x, Y(x))\{Y'(x)\}^2 + 2f''_{xy}(x, Y(x))Y'(x) + f''_{xx}(x, Y(x))}{f'_y(x, Y(x))}.$$

This formula holds under the assumptions that the denominator is different from zero, and that both f and Y are C^2-functions.

A Implicit given function.

D Differentiate $f(x, Y(x)) = 0$ twice with respect to x.

I Under the given assumptions we get by an implicit differentiation (i.e. in fact the chain rule) that

$$
\begin{aligned}
0 &= \frac{d}{dx} f(x, Y(x)) \\
&= f'_x(x, Y(x)) \frac{dx}{dx} + f'_y(x, Y(x)) \frac{dY}{dx} \\
&= f'_y(x, Y(x)) \cdot Y'(x) + f'_x(x, Y(x)),
\end{aligned}
$$

hence by another differentiation

$$
\begin{aligned}
0 &= f'_y(x, Y(x)) Y''(x) + f''_{xy}(x, Y(x)) Y'(x) + f''_{yy}(x, Y(x)) \{Y'(x)\}^2 \\
&\quad + f''_{xx}(x, Y(x)) + f''_{xy}(x, Y(x)) Y'(x) \\
&= f'_y(x, Y(x)) Y''(x) + f''_{xx}(x, Y(x)) + 2f''_{xy}(x, Y(x)) Y'(x) + f''_{yy}(x, Y(x)) \{Y'(x)\}^2.
\end{aligned}
$$

When we divide by $f'_y(x, Y(x)) \neq 0$ and rearrange we obtain the searched formula.

Example 6.8 *Given the function*

$$f(x, y) = y^3 \cos x + y + x - 2, \qquad (x, y) \in \mathbb{R}^2.$$

1. *Solve the equation $f(0, y) = 0$.*

Then we get the information that the equation $f(x, y) = 0$ in a neighbourhood of the point $(0, 1)$ defines y uniquely as a function of x, i.e. $y = Y(x)$.

2. *Find $Y(0)$, and then find $Y'(0)$ and $Y''(0)$ by using the formulæ from **Example 6.7**. Find the approximating polynomial of at most second degree for Y with the point of expansion $x_0 = 0$.*

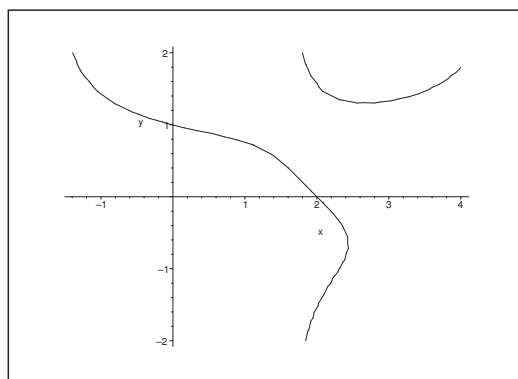**A** Implicit given function.

D Use the guidelines.

Figure 24: The graph of the equation $y^3 \cos x + y + x - 2 = 0$.

I 1) First solve the equation

$$0 = f(0, y) = y^3 + y - 2.$$

It is obvious that $y = 1$ is a solution. Since

$$f(0, y) = y^3 + y - 2 = y^3 - y + 2(y - 1) = (y - 1)(y^2 + 2),$$

it follows that $y = 1$ is the only real solution.

2) Then clearly $Y(0) = 1$. Furthermore,

$$f'_x(x, y) = -y^3 \sin x + 1, \qquad f_x(0, 1) = 1,$$

$$f'_y(x, y) = 3y^2 \cos x + 1, \qquad f'_y(0, 1) = 4,$$

$$f''_{xx}(x, y) = -y^3 \cos x, \qquad f''_{xx}(0, 1) = -1,$$

$$f''_{xy}(x, y) = -3y^2 \sin x, \qquad f''_{xy}(0, 1) = 0,$$

$$f''_{yy}(x, y) = 6y \cos x, \qquad f''_{yy}(0, 1) = 6.$$

Using the formulæ of **Example 6.7** we get

$$Y'(0) = -\frac{f_x'(0,1)}{f_y'(0,1)} = -\frac{1}{4}$$

and

$$
\begin{aligned}
Y''(0) &= -\frac{f_{yy}''(0,1)\{Y'(0)\}^2 + 2f_{xy}''(0,1)\cdot Y'(0) + f_{xx}''(0,1)}{f_y'(0,1)} \\
&= -\frac{6\cdot(-\frac{1}{4})^2 + 2\cdot 0\cdot(-\frac{1}{4}) - 1}{4} = -\frac{\frac{6}{16}-1}{4} = -\frac{1}{4}\left(\frac{3}{8}-1\right) = \frac{5}{32}.
\end{aligned}
$$

We get in particular the approximating polynomial of at most second degree,

$$P_2(x) = Y(0) + \frac{1}{1!}\,Y'(0)\cdot(x-x_0) + \frac{1}{2!}\,Y''(0)\cdot(x-x_0)^2 = 1 - \frac{1}{4}\,x + \frac{5}{64}\,x^2.$$

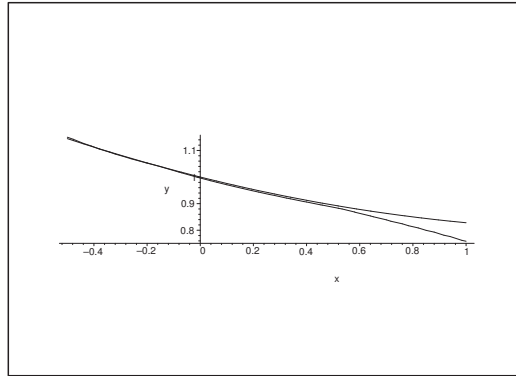It is seen on the figure that the approximation is very accurate in the neighbourhood of $(0,1)$.

Figure 25: The graphs of $f(x, y) = 0$ and the approximating polynomial from $(0, 1)$.

Example 6.9 *Write Taylor's formula for a C^2-function f, where we choose successively the vector of increase (h_x, h_y) as*

$$(h, 0), \qquad (0, h), \qquad (-h, 0) \quad or \quad (0, -h).$$

Explain why $\bigtriangledown^2 f(x, y)$ is a measure of how much $f(x, y)$ deviates from the average of the values of the function in the four neighbouring points. Prove in particular that an harmonic *function f approximately fulfils*

$$f(x, y) = \frac{1}{4}\{f(x + h, y) + f(x, y + h) + f(x - h, y) + f(x, y - h)\}.$$

Derive an analogous result in the case where one consider the four neighbouring points for which (h_x, h_y) is equal to

$$(h, h), \qquad (h, -h), \qquad (-h, h) \quad or \quad (-h, -h).$$

A Taylor's formula; approxiamtion of the average.

D Start by writing down Taylor's formula, and then make the analysis from this.

I Taylor's formula is

$$
\begin{aligned}
f\left(x + h_x, y + h_y\right) \;=\;\; & f(x, y) + h_x f'_x(x, y) + h_y f'_y(x, y) \\
& + \frac{1}{2}\left\{h_x^2 f''_{xx}(x, y) + 2h_x h_y f''_{xy}(x, y) + h_y^2 f''_{yy}(x, y)\right\} \\
& + \varepsilon\left(h_x, h_y\right) \cdot \left(h_x^2 + h_y^2\right),
\end{aligned}
$$

where $\varepsilon\left(h_x, h_y\right) \to 0$ for $\left(h_x, h_y\right) \to (0, 0)$.

We get in particular,

$$f(x + h, y) = f(x, y) + h\, f'_x(x, y) + \frac{1}{2}\, h^2 f''_{xx}(x, y) + \varepsilon(h)\, h^2,$$

$$f(x - h, y) = f(x, y) - h\, f'_x(x, y) + \frac{1}{2}\, h^2 f''_{xx}(x, y) + \varepsilon(h)\, h^2,$$

$$f(x, y+h) = f(x,y) + h\, f_y'(x,y) + \frac{1}{2}\, h^2 f_{yy}''(x,y) + \varepsilon(h)\, h^2,$$

$$f(x, y-h) = f(x,y) - h\, f_y'(x,y) + \frac{1}{2}\, h^2 f_{yy}''(x,y) + \varepsilon(h)\, h^2.$$

The average is

$$
\begin{aligned}
\mathcal{M}f((x,y); h) &= \frac{1}{4}\{f(x+h,y) + f(x-h,y) + f(x,y+h) + f(x,y-h)\} \\
&= f(x,y) + \frac{1}{4}\, h^2 \left\{ f_{xx}''(x,y) + f_{yy}''(x,y) \right\} + \varepsilon(h)\, h^2 \\
&= f(x,y) + \frac{h^2}{4}\, \nabla^2 f(x,y) + \varepsilon(h)\, h^2.
\end{aligned}
$$

Then by a rearrangement

$$f(x,y) = \mathcal{M}f((x,y); h) - \frac{1}{4}\, h^2\, \nabla^2 f(x,y) + \varepsilon(h)\, h^2,$$

so in this sense $\nabla^2 f(x,y)$ is a measure of the deviation of the average from the value of the function.

If f is harmonic then $\nabla^2 f(x,y) = 0$, so

$$f(x,y) = \mathcal{M}f((x,y); h) + \varepsilon(h)\, h^2,$$

and we see that the average is a good approximation.

If we instead choose $(h_x, h_y) = (\pm h, \pm h)$ with all four possible combinations of the sign, then by letting f_x' etc. be a shorthand of $f_x'(x,y)$, etc.,

$$f(x+h,y+h) = f(x,y) + h\{f_x' + f_y'\} + \frac{1}{2}\, h^2\{f_{xx}'' + 2f_{xy}'' + f_{yy}''\} + \varepsilon(h)\, h^2,$$

$$f(x-h,y-h) = f(x,y) - h\{f_x' + f_y'\} + \frac{1}{2}\, h^2\{f_{xx}'' + 2f_{xy}'' + f_{yy}''\} + \varepsilon(h)\, h^2,$$

$$f(x+h,y-h) = f(x,y) + h\{f_x' - f_y'\} + \frac{1}{2}\, h^2\{f_{xx}'' - 2f_{xy}'' + f_{yy}''\} + \varepsilon(h)\, h^2,$$

$$f(x-h,y+h) = f(x,y) - h\{f_x' - f_y'\} + \frac{1}{2}\, h^2\{f_{xx}'' - 2f_{xy}'' + f_{yy}''\} + \varepsilon(h)\, h^2.$$

Here the average is

$$
\begin{aligned}
\tilde{\mathcal{M}}f((x,y); h) &= \frac{1}{4}\{f(x+h,y+h) + f(x-h,y-h) + f(x+h,y-h) + f(x-,y+h)\} \\
&= f(x,y) + \frac{1}{2}\, h^2\, \nabla^2 f(x,y) + \varepsilon(h) \cdot h^2,
\end{aligned}
$$

hence by a rearrangement

$$f(x,y) = \tilde{\mathcal{M}}f((x,y); h) - \frac{1}{2}\, h^2\, \nabla^2 f(x,y) + \varepsilon(h)\, h^2.$$

We get the same conclusion as above, since the only difference is the factor $\frac{1}{2}$ instead of $\frac{1}{4}$.

If f is harmonic, we also get in this case that

$$f(x,y) = \tilde{\mathcal{M}}f((x,y);h) + \varepsilon(h)\,h^2,$$

and we see again that the average is a very good approximation.

Example 6.10 *Find the approximating polynomial of at most second degree of the function*

$$f(x,y) = x\,\sinh(x+2y), \qquad (x,y) \in \mathbb{R}^2$$

expanded from the point $(x,y) = (2,-1)$.

A Approximating polynomial.

D Either use Taylor's formula or known series expansions.

I First method. First calculate

$$f(x,y) = x\,\sinh(x+2y), \qquad\qquad f(2,-1) = 0,$$

$$f_x'(x,y) = \sinh(x+2y) + x\,\cosh(x+2y), \qquad f_x'(2,-1) = 2,$$

$$f_y'(x,y) = 2x\,\cosh(x+2y), \qquad\qquad f_y'(2,-1) = 4,$$

$$f_{xx}''(x,y) = 2\cosh(x+2y) + x\,\sinh(x+2y), \qquad f_{xx}''(2,-1) = 2,$$

$$f_{xy}''(x,y) = 2\cosh(x+2y) + 2x\,\sinh(x+2y), \qquad f_{xy}''(2,-1) = 2,$$

$$f_{yy}''(x,y) = 4x\,\sinh(x+2y), \qquad\qquad f_{yy}''(2,-1) = 0.$$

By means of the second column we get the coefficients of the Taylor expansion, hence

$$
\begin{aligned}
P_2(x,y) &= 0 + \frac{1}{1!}\{2(x-2)+4(y+1)\} + \frac{1}{2!}\{2(x-2)^2 + 2\cdot 2(x-2)(y+1) + 0\} \\[2mm]
&= 2(x-2) + 4(y+1) + (x-2)^2 + 2(x-2)(y+1).
\end{aligned}
$$

Second method. First change variables by putting $x = 2 + \xi$ and $y = -1 + \eta$. Then by insertion followed by known series expansions, in which terms of higher order are written as dots,

$$
\begin{aligned}
f(x,y) &= x\,\sinh(x+2y) = (2+\xi)\sinh(\xi+2\eta) \\[2mm]
&= (2+\xi)\{(\xi+2\eta) + \cdots\} \\[2mm]
&= 2\xi + 4\eta + \xi^2 + 2\xi\eta + \cdots \\[2mm]
&= 2(x-2) + 4(y+1) + (x-2)^2 + 2(x-2)(y+1) + \cdots,
\end{aligned}
$$

hence

$$P_2(x, y) = 2(x - 2) + 4(y + 1) + (x - 2)^2 + 2(x - 2)(y + 1).$$

REMARK. Of numerical reasons one shall *always* in examples of approximating polynomials use the variables $\mathbf{x} - \mathbf{x}_0$, here $(x - 2, y + 1)$, because the expansion is bound to the point $\mathbf{x}_0$, here $(2, -1)$. Many textbooks erroneously "reduce" further to the variables (x, y). ◊

Example 6.11 *Find the approximating polynomial of at most second degree for the function*

$$g(x, y) = \sqrt{4 - 2x^2 - y^2}, \qquad 2x^2 + y^2 < 4,$$

with the point of expansion $(1, 1)$.

A Approximating polynomial.

D Either use Taylor's formula, or rewrite $g(x, y)$ as some known function for which we know the Taylor series.

Figure 26: Part of the graph of $g(x, y)$.

I If we put $z = g(x, y) = \sqrt{4 - 2x^2 - y^2} \geq 0$, it follows by a squaring and a rearrangement that the equation of the surface can also be written

$$\left(\frac{x}{\sqrt{2}}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{2}\right)^2 = 1, \qquad z \geq 0,$$

i.e. the graph is the upper half of an ellipsoidal surface of centrum $(0, 0, 0)$ and half axes $\sqrt{2}$, 2 and 2.

First method. Clearly, the function $g(x, y)$ is of class C^∞ in the domain, where the point $(1, 1)$ lies. Then calculate

$$g(x, y) = \sqrt{4 - 2x^2 - y^2}, \qquad\qquad g(1, 1) = 1,$$

$$g'_x(x, y) = -\frac{2x}{\sqrt{4 - 2x^2 - y^2}}, \qquad\qquad g'_x(1, 1) = -2,$$

$$g'_y(x, y) = -\frac{y}{\sqrt{4 - 2x^2 - y^2}}, \qquad\qquad g'_y(1, 1) = -1,$$

$$g''_{xx}(x, y) = -\frac{2}{\sqrt{4 - 2x^2 - y^2}} - \frac{4x^2}{(\sqrt{4 - 2x^2 - y^2})^3}, \quad g''_{xx}(1, 1) = -6,$$

$$g''_{xy}(x, y) = -\frac{2xy}{(\sqrt{4 - 2x^2 - y^2})^3}, \qquad\qquad g''_{xy}(1, 1) = -2,$$

$$g''_{yy}(x, y) = -\frac{1}{\sqrt{4 - 2x^2 - y^2}} - \frac{y^2}{(\sqrt{4 - 2x^2 - y^2})^3}, \quad g''_{yy}(1, 1) = -2.$$

Then the approximating polynomial is according to Taylor's formula and the right hand column

$$
\begin{aligned}
P_2(x, y) &= 1 - 2(x-1) - (y-1) + \frac{1}{2}\left\{-6(x-1)^2 - 2 \cdot 2(x-1)(y-1) - 2(y-1)^2\right\} \\
&= 1 - 2(x-1) - (y-1) - 3(x-1)^2 - 2(x-1)(y-1) - (y-1)^2.
\end{aligned}
$$

Second method. First introduce some new variables by $x = 1 + \xi$ and $y = 1 + \eta$. Then by insertion and introduction of a known series expansion for $\sqrt{1+t}$, where the dots as usual indicate terms of higher order,

$$
\begin{aligned}
g(x,y) &= \sqrt{4 - 2(1+\xi)^2 - (1+\eta)^2} = \sqrt{1 - 4\xi + 2\xi^2 - 2\eta - \eta^2} \\[2mm]
&= 1 - \frac{1}{2}\left(4\xi + 2\eta + 2\xi^2 + \eta^2\right) - \frac{1}{8}\left(4\xi + 2\eta + \cdots\right)^2 + \cdots \\[2mm]
&= 1 - 2\xi - \eta - \xi^2 - \frac{1}{2}\eta^2 - \frac{1}{8}\left(16\xi^2 + 16\xi\eta + 4\eta^2\right) + \cdots \\[2mm]
&= 1 - 2\xi - \eta - 3\xi^2 - 2\xi\eta - \eta^2 + \cdots,
\end{aligned}
$$

and we conclude that the approximating polynomial is

$$
\begin{aligned}
P_2(x,y) &= 1 - 2\xi - \eta - 3\xi^2 - 2\xi\eta - \eta^2 \\
&= 1 - 2(x-1) - (y-1) - 3(x-1)^2 - 2(x-1)(y-1) - (y-1)^2.
\end{aligned}
$$

Example 6.12 *Find the approximating polynomial of at most second degree of the function*

$$
f(x,y) = \ln x + \exp(xy - 2), \qquad (x,y) \in \mathbb{R}_+ \times \mathbb{R},
$$

expanded from the point $(x,y) = (1,2)$.

A Approximating polynomial.

D Either calculate the Taylor coefficients, or use some known series expansions.

I First method. *The standard method.* Clearly, $f \in C^\infty(\mathbb{R}_+ \times \mathbb{R})$ and $(1,2) \in \mathbb{R}_+ \times \mathbb{R}$. Then by differentiation,

$$
\begin{aligned}
&f(x,y) = \ln x + \exp(xy - 2), && f(1,2) = 1, \\[2mm]
&f'_x(x,y) = \frac{1}{x} + y\,\exp(xy - 2), && f'_x(1,2) = 3, \\[2mm]
&f'_y(x,y) = x\,\exp(xy - 2), && f'_y(1,2) = 1, \\[2mm]
&f''_{xx}(x,y) = -\frac{1}{x^2} + y^2\,\exp(xy - 2), && f''_{xx}(1,2) = 3, \\[2mm]
&f''_{xy}(x,y) = f''_{yx}(x,y) = (1 + xy)\exp(xy - 2), && f''_{xy}(1,2) = 3, \\[2mm]
&f''_{yy}(x,y) = x^2\,\exp(xy - 2), && f''_{yy}(1,2) = 1.
\end{aligned}
$$

The approximating polynomial of at most second degree is

$$
\begin{aligned}
P_2(x,y) &= f(1,2) + f'_x(1,2) \cdot (x-1) + f'_y(1,2) \cdot (y-2) \\
&\quad + \frac{1}{2}\left\{ f''_{xx}(1,2)\,(x-1)^2 + 2f''_{xy}(1,2)\,(x-1)(y-2) + f''_{yy}(1,2)\,(y-2)^2 \right\} \\
&= 1 + 3(x-1) + (y-2) + \frac{3}{2}(x-1)^2 + 3(x-1)(y-2) + \frac{1}{2}(y-2)^2.
\end{aligned}
$$

Second method. *Suitable series expansions of known standard functions.* First rewrite $f(x, y)$ as a function of the translated variables $(x-1, y-2)$, which are zero at the point of expansion $(1, 2)$. Then

$$
\begin{aligned}
f(x, y) &= \ln x + \exp(xy - 2) \\
&= \ln(1 + (x-1)) + \exp\{(x-1)(y-2) + 2x + y - 4\} \\
&= \ln\{1 + (x-1)\} + \exp\{(x-1)(y-2) + 2(x-1) + (y-2)\} \\
&= \ln\{1 + (x-1)\} + \exp\{(x-1)(y-2)\} \cdot \exp\{2(x-1)\} \cdot \exp(y-2).
\end{aligned}
$$

By means of known series expansions for $\ln(1 + t)$ and $\exp(t)$, where we remove all terms of degree higher than 2 in $x - 1$ and $y - 2$, we get

$$
\begin{aligned}
f(x,y) \;=\;& \ln\{1+(x-1)\}+\exp\{(x-1)(y-2)\}\cdot\exp\{2(x-1)\}\cdot\exp(y-2) \\[2mm]
\;=\;& (x-1)-\frac{1}{2}\,(x-1)^2+\cdots \\[2mm]
& +\{1+(x-1)(y-2)+\cdots\}\{1+2(x-1)+2(x-1)^2+\cdots\}\cdot\Big\{1+(y-2)+\frac{1}{2}(y-2)^2+\cdots\Big\} \\[2mm]
\;=\;& (x-1)-\frac{1}{2}(x-1)^2+(x-1)(y-2)+1+(y-2)+\frac{1}{2}(y-2)^2 \\[2mm]
& +2(x-1)+2(x-1)(y-2)+2(x-1)^2+\cdots \\[2mm]
\;=\;& \frac{3}{2}(x-1)^2+3(x-1)(y-2)+\frac{1}{2}(y-2)^2+3(x-1)+(y-2)+1+\cdots,
\end{aligned}
$$

and we conclude that

$$
P_2(x,y)=1+3(x-1)+(y-2)+\frac{3}{2}(x-1)^2+3(x-1)(y-2)+\frac{1}{2}(y-2)^2.
$$

Example 6.13 *1) Sketch the domain D of*

$$f(x, y) = \sqrt{2 + x - y} + \ln\left(4 - x^2 - y^2\right).$$

2) Check whether D is open or closed or none of the kind.

3) Find the approximating polynomial of at most first degree for f with $(1, -1)$ as point of expansion.

4) Find the domain E of the vector field

$$\mathbf{V}(x, y) = \left(\sqrt{2 + x - y}, \sqrt{y} + \ln\left(4 - x^2 - y^2\right)\right).$$

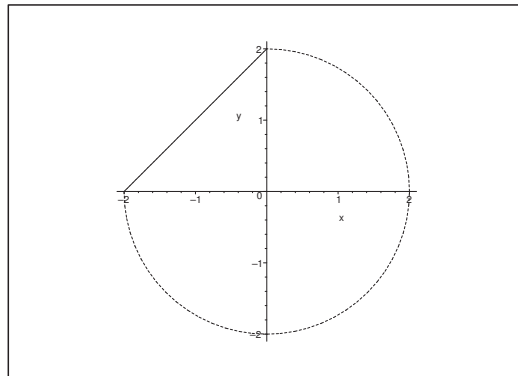**A** Domains, open and closed sets, approximating polynomial.

D Standard task.

Figure 27: The domain D.

I 1) The function $f(x, y) = \sqrt{2 + x - y} + \ln\left(4 - x^2 - y^2\right)$ is defined for

$$2 + x - y \geq 0 \qquad \text{and} \qquad 4 - x^2 - y^2 > 0,$$

hence for

$$y \leq x + 2 \qquad \text{and} \qquad x^2 + y^2 < 4 = 2^2.$$

2) The set D is neither open nor closed.

3) The approximating polynomial from $(1, -1)$.

First variant. It follows from

$$f(x, y) = \sqrt{2 + x - y} + \ln\left(4 - x^2 - y^2\right), \qquad f(1, -1) = 2 + \ln 2,$$

$$f_x'(x, y) = \frac{1}{2}\frac{1}{\sqrt{2 + x - y}} - \frac{2x}{4 - x^2 - y^2}, \qquad f_x'(1, -1) = -\frac{3}{4},$$

$$f_y'(x, y) = -\frac{1}{2}\frac{1}{\sqrt{2 + x - y}} - \frac{2y}{4 - x^2 - y^2}, \qquad f_y'(1, -1) = \frac{3}{4},$$

that

$$P_1(x, y) = 2 + \ln 2 - \frac{3}{4}(x - 1) + \frac{3}{4}(y + 1).$$

Second variant. If we put $x = x_1 + 1$ and $y = y_1 - 1$, then we get by series expansions,

$$
\begin{aligned}
f(x,y) &= \sqrt{2+x-y} + \ln(4 - x^2 - y^2) \\[2mm]
&= \sqrt{2+x_1+1-y_1+1} + \ln(4 - \{x_1+1\}^2 - \{y_1-1\}^2) \\[2mm]
&= \sqrt{4+x_1-y_1} + \ln(2 - 2x_1 + 2y_1 - x_1^2 - y_1^2) \\[2mm]
&= 2\sqrt{1 + \frac{x_1}{4} - \frac{y_1}{4}} + \ln 2 + \ln\left(1 - x_1 + y_1 - \frac{1}{2}\, x_1^2 - \frac{1}{2}\, y_1^2\right) \\[2mm]
&= 2\left\{1 + \frac{1}{2}\left(\frac{x_1}{4} - \frac{y_1}{4}\right) + \cdots\right\} + \ln 2 + \left\{-x_1 + y_1 - \frac{1}{2}\, x_1^2 - \frac{1}{2}\, y_1^2\right\} + \cdots \\[2mm]
&= 2 + \ln 2 + \frac{1}{4}\, x_1 - \frac{1}{4}\, y_1 - x_1 + y_1 + \cdots,
\end{aligned}
$$

where the dots as usual indicate terms of higher order. We conclude that

$$
P_1(x,y) = 2 + \ln 2 - \frac{3}{4}\, x_1 + \frac{3}{4}\, y_1 = 2 + \ln 2 - \frac{3}{4}\,(x-1) + \frac{3}{4}\,(y+1).
$$
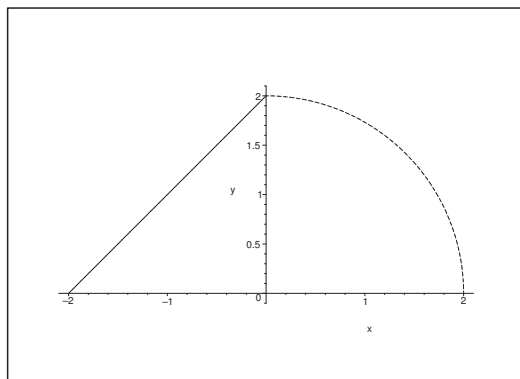

Figure 28: The domain E of the vector field $\mathbf{V}$.

4) The domain E of the vector field consists of the points in D, for which $\sqrt{y}$ is also defined, so we must also require that $y \geq 0$.

Example 6.14 *1) Sketch the domain D of*

$$f(x,y) = e^{x+y} + \ln\left(4 - x^2 - 4y^2\right).$$

2) Check if D is open or closed of none of the kind.

3) Find the approximating polynomial of at most second degree for f with $(0,0)$ as point of expansion.

A Domain and approximating polynomial for a function.

D Analyze where each subfunction is defined. Then the approximating polynomial is either found by means of known series expansions or by calculating the Taylor coefficients.
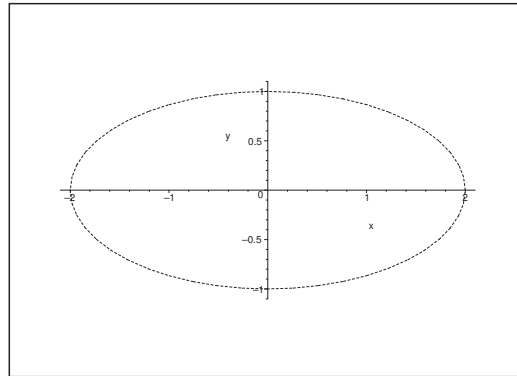

Figure 29: The domain D is the open ellipsoidal disc.

I 1) The function e^{x+y} is defined for every $(x,y) \in \mathbb{R}^2$.
   The function $\ln\left(4 - x^2 - 4y^2\right)$ is defined, if and only if $4 - x^2 - 4y^2 > 0$, i.e. if and only if

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{1}\right)^2 < 1.$$

   The domain is the open ellipsoidal disc of centrum $(0,0)$ and half axes 2 and 1, cf. the figure.
   2) As mentioned above in 1), the set D is open.
   3) **First variant.** *Known series expansions.*
      Let $(x,y) \in K(\mathbf{0};1) \subset D$, and let dots denote terms of higher degree than 2. Then

$$\begin{aligned}
f(x,y) &= e^{x+y} + 2\ln 2 + \ln\left(1 - \frac{1}{4}\left(x^2 + 4y^2\right)\right) \\
&= 1 + \frac{1}{1!}(x+y) + \frac{1}{2!}(x+y)^2 + \cdots + 2\ln 2 - \frac{1}{4}\left(x^2 + 4y^2\right) + \cdots \\
&= 1 + 2\ln 2 + x + y + \frac{1}{2}x^2 + xy + \frac{1}{2}y^2 - \frac{1}{4}x^2 - y^2 + \cdots \\
&= 1 + 2\ln 2 + x + y + \frac{1}{4}x^2 + xy - \frac{1}{2}y^2 + \cdots.
\end{aligned}$$

   The approximating polynomial of at most second degree from $(0,0)$ is

$$P_2(x,y) = 1 + 2\ln 2 + x + y + \frac{1}{4}x^2 + xy - \frac{1}{2}y^2.$$

Second variant. *Taylor expansion.*

We get by successive differentiation

$$f(x,y) = e^{x+y} + \ln(4 - x^2 - 4y^2), \qquad\qquad f(0,0) = 1 + \ln 4 = 1 + 2\ln 2,$$

$$f_x'(x,y) = e^{x+y} - \frac{2x}{4 - x^2 - yy^2}, \qquad\qquad f_x'(0,0) = 1,$$

$$f_y'(x,y) = e^{x+y} - \frac{8y}{4 - x^2 - 4y^2}, \qquad\qquad f_y'(0,0) = 1,$$

$$f_{xx}''(x,y) = e^{x+y} - \frac{2}{4 - x^2 4y^2} - \frac{4x^2}{(4 - x^2 - 4y^2)^2}, \qquad f_{xx}''(0,0) = 1 - \frac{2}{4} = \frac{1}{2},$$

$$f_{yy}''(x,y) = e^{x+y} - \frac{8}{4 - x^2 - 4y^2} - \frac{64y^2}{(4 - x^2 - 4y^2)^2}, \qquad f_{yy}''(0,0) = 1 - \frac{8}{4} = -1,$$

$$f_{xy}''(x,y) = e^{x+y} - \frac{16xy}{(4 - x^2 - 4y^2)^2}, \qquad\qquad f_{xy}''(0,0) = 1.$$

Hence

$$P_2(x, y) = 1 + 2 \ln 2 + x + y + \frac{1}{4} x^2 - \frac{1}{2} y^2 + xy.$$

REMARK. The expressions of the second derivative may occur in several variants:

a)

$$f''_{xx}(x, y) = e^{x+y} - \frac{8 + 2x^2 - 8y^2}{(4 - x^2 - 4y^2)^2}, \quad f''_{xx}(0, 0) = 1 - \frac{8}{16} = \frac{1}{2},$$

$$f''_{yy}(x, y) = e^{x+y} - \frac{32 - 8x^2 + 32y^2}{(4 - x^2 - 4y^2)^2}, \quad f''_{yy}(0, 0) = 1 - \frac{32}{16} = -1,$$

together with the more elegant version, where dots denote terms which will become zero by the insertion of $(x, y) = (0, 0)$:

b)

$$f''_{xx}(x, y) = e^{x+y} - \frac{2}{4 - x^2 - 4y^2} + \cdots, \quad f''_{xx}(0, 0) = 1 - \frac{2}{4} = \frac{1}{2},$$

$$f''_{yy}(x, y) = e^{x+y} - \frac{8}{4 - x^2 - 4y^2} + \cdots, \quad f''_{yy}(0, 0) = 1 - \frac{8}{4} = -1,$$

$$f''_{xy}(x, y) = e^{x+y} + \cdots, \quad\quad\quad\quad\quad f''_{xy}(0, 0) = 1.$$

Example 6.15 *Given the function*

$$f(x, y) = e^{xy} + (2 - x)e^y - 2ey, \quad (x, y) \in \mathbb{R}^2.$$

Find the approximating polynomial of at most second degree for f with $(1, 1)$ as point of expansion.

A Approximating polynomial.

D The function is clearly of class C^∞. Either calculate the Taylor coefficients, or use known series expansions.

I First method. *Calculation of the Taylor coefficients.*

We get by mechanical computations,

$$f(x, y) = e^{xy} + (2 - x)e^y - 2ey, \quad f(1, 1) = 0,$$

$$f'_x(x, y) = y\, e^{xy} - e^y, \quad\quad\quad\quad\quad f'_x(1, 1) = 0,$$

$$f'_y(x, y) = x\, e^{xy} + (2 - x)e^y - 2e, \quad f'_y(1, 1) = 0,$$

$$f''_{xx}(x, y) = y^2 e^{xy}, \quad\quad\quad\quad\quad\quad f''_{xx}(1, 1) = e,$$

$$f''_{xy}(x, y) = e^{xy} + xy\, e^{xy} - e^y, \quad\quad f''_{xy}(1, 1) = e,$$

$$f''_{yy}(x, y) = x^2 e^{xy} + (2 - x)e^y, \quad\quad f''_{yy}(1, 1) = 2e.$$

Then the approximating polynomial of at most second degree for f from $(1,1)$ is

$$
\begin{aligned}
P_2(x,y) &= f(1,1) + \frac{1}{1!}\left\{ f'_x(1,1)\cdot(x-1) + f'_y(1,1)\cdot(y-1) \right\} \\[2mm]
&\quad + \frac{1}{2!}\left\{ f''_{xx}(1,1)\cdot(x-1)^2 + f''_{xy}(1,1)\cdot(x-1)(y-1) + f''_{yy}(1,1)\cdot(y-1)^2 \right\} \\[2mm]
&= \frac{e}{2}(x-1)^2 + e(x-1)(y-1) + e(y-1)^2.
\end{aligned}
$$

Second method. *Application of known series expansions.*

When we translate

$$
x_1 = x-1, \quad y_1 = y-1, \quad \text{i.e.} \quad x = x_1+1, \quad y = y_1+1,
$$

to the point of expansion and use known series expansions up to the second degree (and where terms of higher degrees are indicated by dots) we get

$$
\begin{aligned}
f(x,y) &= e^{xy} + (2-x)e^y - 2ey \\
&= \exp((x_1+1)(y_1+1)) + (1-x_1)\exp(y_1+1) - 2e(y_1+1) \\
&= \exp(1+x_1+y_1+x_1y_1) + e(1-x_1)\exp(y_1) - 2e - 2ey_1 \\
&= e\left\{ \exp(x_1+y_1)\cdot\exp(x_1y_1) + (1-x_1)\exp(y_1) - 2 - 2y_1 \right\},
\end{aligned}
$$

and hence

$$
\begin{aligned}
f(x,y) &= e\left(\left\{ 1 + x_1 + y_1 + \frac{1}{2}(x_1+y_1)^2 + \cdots \right\}\{1 + x_1y_1 + \cdots\} \right. \\
&\qquad \left. + (1-x_1)\left\{ 1 + y_1 + \frac{1}{2}y_1^2 + \cdots \right\} - 2 - 2y_1 \right) \\[2mm]
&= e\left\{ 1 + x_1 + y_1 + \frac{1}{2}x_1^2 + x_1y_1 + \frac{1}{2}y_1^2 + x_1y_1 + \cdots \right. \\
&\qquad \left. + 1 + y_1 + \frac{1}{2}y_1^2 - x_1 - x_1y_1 + \cdots - 2 - 2y_1 \right\} \\[2mm]
&= e\left\{ \frac{1}{2}x_1^2 + x_1y_1 + y_1^2 + \cdots \right\}.
\end{aligned}
$$

The dots indicate terms of higher degree, so we conclude that the approximating polynomial of at most second degree with $(1,1)$ as point of expansion is

$$
P_2(x,y) = \frac{e}{2}x_1^2 + ex_1y_1 + ey_1^2 = \frac{e}{2}(x-1)^2 + e(x-1)(y-1) + e(y-1)^2.
$$

Example 6.16 *Given the function*

$$f(x, y) = \sqrt{1 - 2x - y} + \ln(1 - 2y + x), \qquad (x, y) \in D.$$

1) Find the domain D.

2) Sketch D.

3) Check if D is

   a) open,

   b) closed,

   c) bounded,

   d) star shaped.

4) Find the approximating polynomial ofat most second degree for f with the point of expansion $(0, 0)$.

A Domain; approximating polynomial.

D Analyze each part of the function separately and take the intersections of all these domains. Then use either known series expansions, of calculate the Taylor coefficients.
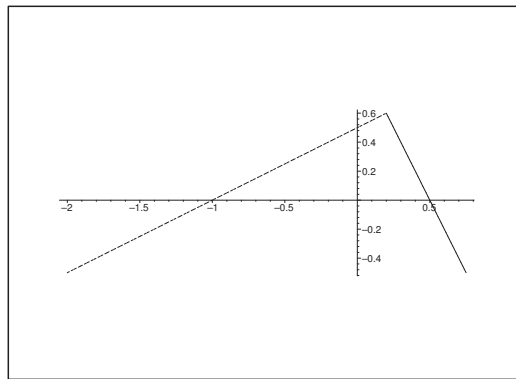

Figure 30: The domain D is the angular space inclusive the fully drawn boundary curve and exclusive the dotted boundary curve. The domain in unbounded downwards.

I 1.–3. The function is defined when

$$1 - 2x - y \geq 0 \qquad \text{and} \qquad 1 - 2y + x > 0,$$

i.e. when

$$y \leq 1 - 2x \qquad \text{og} \qquad y < \frac{1}{2}(x + 1),$$

or written in another way,

$$x \leq \frac{1}{2}(1 - y) \qquad \text{and} \qquad x > 2y - 1.$$

Since the lines intersect at $\left(\dfrac{1}{5}, \dfrac{3}{5}\right)$, the domain can be written

$$D = \left\{ (x, y) \ \middle| \ y < \frac{3}{5},\ 2y - 1 < x \le \frac{1}{2}\left(1 - y\right) \right\}.$$

Note that D is the intersection of an open and a closed half plane.

We see immediately that

1) D is not open, because a part of the boundary, though not the total boundary, lies in D,
2) D is not closed, because a part of the boundary, though not the total boundary, lies outside D,
3) D is not bounded. The whole of the negative Y axis lies in D.
4) Since D is the intersection of two convex sets, it is itself convex and therefore also starshaped with respect to any point in D.

4. We have here two variants.

First variant. *The standard method.* It follows from the computations

$$f(x,y) = \sqrt{1-2x-y} + \ln(1-2y+x), \qquad\qquad f(0,0) = 1,$$

$$f_x'(x,y) = -\frac{1}{\sqrt{1-2x-y}} + \frac{1}{1-2y+x}, \qquad\qquad f_x'(0,0) = 0,$$

$$f_y'(x,y) = -\frac{1}{2}\,\frac{1}{\sqrt{1-2x-y}} - \frac{2}{1-2y+x}, \qquad\qquad f_y'(0,0) = -\frac{5}{2},$$

$$f_{xx}''(x,y) = \frac{1}{(1-2x-y)^{3/2}} - \frac{1}{(1-2y+x)^2}, \qquad f_{xx}''(0,0) = -2,$$

$$f_{xy}''(x,y) = -\frac{1}{2}\,\frac{1}{(1-2x-y)^{3/2}} - \frac{4}{(1-2y+x)^2}, \quad f_{yy}''(0,0) = -\frac{17}{4},$$

that the approximating polynomial of at most second degree from $(0,0)$ is

$$
\begin{aligned}
P_2(x,y) \;=\;& f(0,0) + f_x'(0,0)\cdot x + f_y'(0,0)\cdot y \\
& + \frac{1}{2}\left\{ f_{xx}''(0,0)\cdot x^2 + 2f_{xy}''(0,0)\cdot xy + f_{yy}''(0,0)\cdot y^2 \right\} \\
=\;& 1 - \frac{5}{2}\,y - x^2 + \frac{3}{2}\,xy - \frac{17}{8}\,y^2.
\end{aligned}
$$

Second variant. *Known series expansions.* It is well-known that

$$\sqrt{1+t} = 1 + \left(\begin{array}{c}\frac{1}{2}\\1\end{array}\right)t + \left(\begin{array}{c}\frac{1}{2}\\2\end{array}\right)t^2 + \cdots = 1 + \frac{1}{2}\,t - \frac{1}{8}\,t^2 + \cdots,$$

and

$$\ln(1+u) = u - \frac{1}{2}\,u^2 + \cdots.$$

If we put

$$t = -(2x+y) = -2x-y \qquad \text{and} \qquad u = x - 2y,$$

then both t and u are for the first degree in (x,y), and the approximating polynomial of at most second degree is

$$
\begin{aligned}
P_2(x,y) \;=\;& 1 + \frac{1}{2}\,t - \frac{1}{8}\,t^2 + u - \frac{1}{2}u^2 \\
=\;& 1 - \frac{1}{2}\,(2x+y) - \frac{1}{8}\,(2x+y)^2 + x - 2y - \frac{1}{2}\,(x-2y)^2 \\
=\;& 1 - x - \frac{1}{2}\,y + x - 2y - \frac{1}{8}\,(4x^2+4xy+y^2) - \frac{1}{2}\,(x^2-4xy+4y^2) \\
=\;& 1 - \frac{5}{2}\,y - x^2 + \frac{3}{2}\,xy - \frac{17}{8}\,y^2.
\end{aligned}
$$

Example 6.17 *1) Sketch the domain D of the function*

$$f(x, y) = \ln\left(4 - x^2 - y^2\right) - \sqrt{5 - 4x} + y^2.$$

2) Check if D is open or closed or none of the kind.

3) Compute the gradient $\bigtriangledown f$.

4) Find the approximating polynomial of at most first degree for the function f, when the point $(1, \sqrt{2})$ is used as point of expansion.

A Domain, gradient, approximating polynomial.

D Treat every subfunction separately. The approximating polynomial can then be found in several ways.
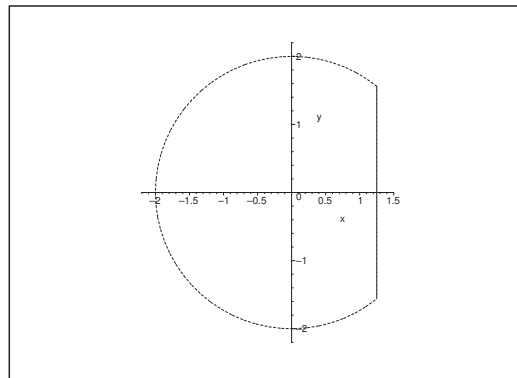

Figure 31: The domain D.

I 1) The function $\ln\left(4 - x^2 - y^2\right)$ is defined for $4 - x^2 - y^2 > 0$, i.e. for $x^2 + y^2 < 4 = 2^2$, which describes the open disc of centrum $(0, 0)$ and radius 2.

The function $\sqrt{5 - 4x}$ is defined for $5 - 4x \geq 0$, i.e. in the closed half space $x \leq \dfrac{5}{4}$.

Now y^2 is defined for every $(x, y) \in \mathbb{R}^2$, so the domain D is the intersection of the two sets mentioned above,

$$D = \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; x^2 + y^2 < 2^2, \; x \leq \frac{5}{4} \right\}.$$

2) The set D is neither open (a part of the boundary, $x = \dfrac{5}{4}$, lies in D) nor closed (another part, the circular ar, does not lie in D).

3) The gradient is calculated straight away,

$$\bigtriangledown f(x, y) = \left(-\frac{2x}{4 - x^2 - y^2} + \frac{2x}{\sqrt{5 - 4x}}, \; -\frac{2y}{4 - x^2 - y^2} + 2y \right).$$

Notice that

$$\nabla f(1, \sqrt{2}) = \left(-\frac{2}{1} + \frac{2}{1}, -\frac{2\sqrt{2}}{1} + 2\sqrt{2} \right) = \mathbf{0},$$

so $(1, \sqrt{2})$ is a stationary point of f.

4) **First variant.** The approximating polynomial of at most first degree with $(1, \sqrt{2})$ as point of expansion is according to 3) given by,

$$
\begin{aligned}
P_1(x, y) &= f(1, \sqrt{2}) + \nabla f(1, \sqrt{2}) \cdot (x - 1, y - \sqrt{2}) = f(1, \sqrt{2}) + 0 \\[2mm]
&= \ln(4 - 1 - 2) - \sqrt{5 - 4} + 2 = 1.
\end{aligned}
$$

Second variant. If we put $x = s + 1$ and $y = t + \sqrt{2}$, it follows by insertion and by using known series expansions that

$$
\begin{aligned}
f(x, y) &= \ln\left(4 - x^2 - y^2\right) - \sqrt{5 - 4x} + y^2 \\[1mm]
&= \ln\left(4 - (s + 1)^2 - (t + \sqrt{2})^2\right) - \sqrt{5 - 4(s + 1)} + (t + \sqrt{2})^2 \\[1mm]
&= \ln\left(1 - 2s - 2\sqrt{2}\,t - s^2 - t^2\right) - \sqrt{1 - 4s} + 2 + 2\sqrt{2}\,t + t^2 \\[1mm]
&= -2s - 2\sqrt{2}\,t + \cdots - \left\{ 1 - \frac{1}{2} \cdot 4s + \cdots \right\} + 2 + 2\sqrt{2}\,t + \cdots \\[1mm]
&= 1 + \cdots,
\end{aligned}
$$

where the dots as usual denote terms of degree ≥ 2.

The approximating polynomial of at most first degree from $(1, \sqrt{2})$ is therefore the constant

$$P_1(x, y) = 1.$$

REMARK. There is nothing unusual in the fact that the approximating polynomial of at most first degree is a constant, i.e. a degerenated polynomial of degree zero. $\Diamond$