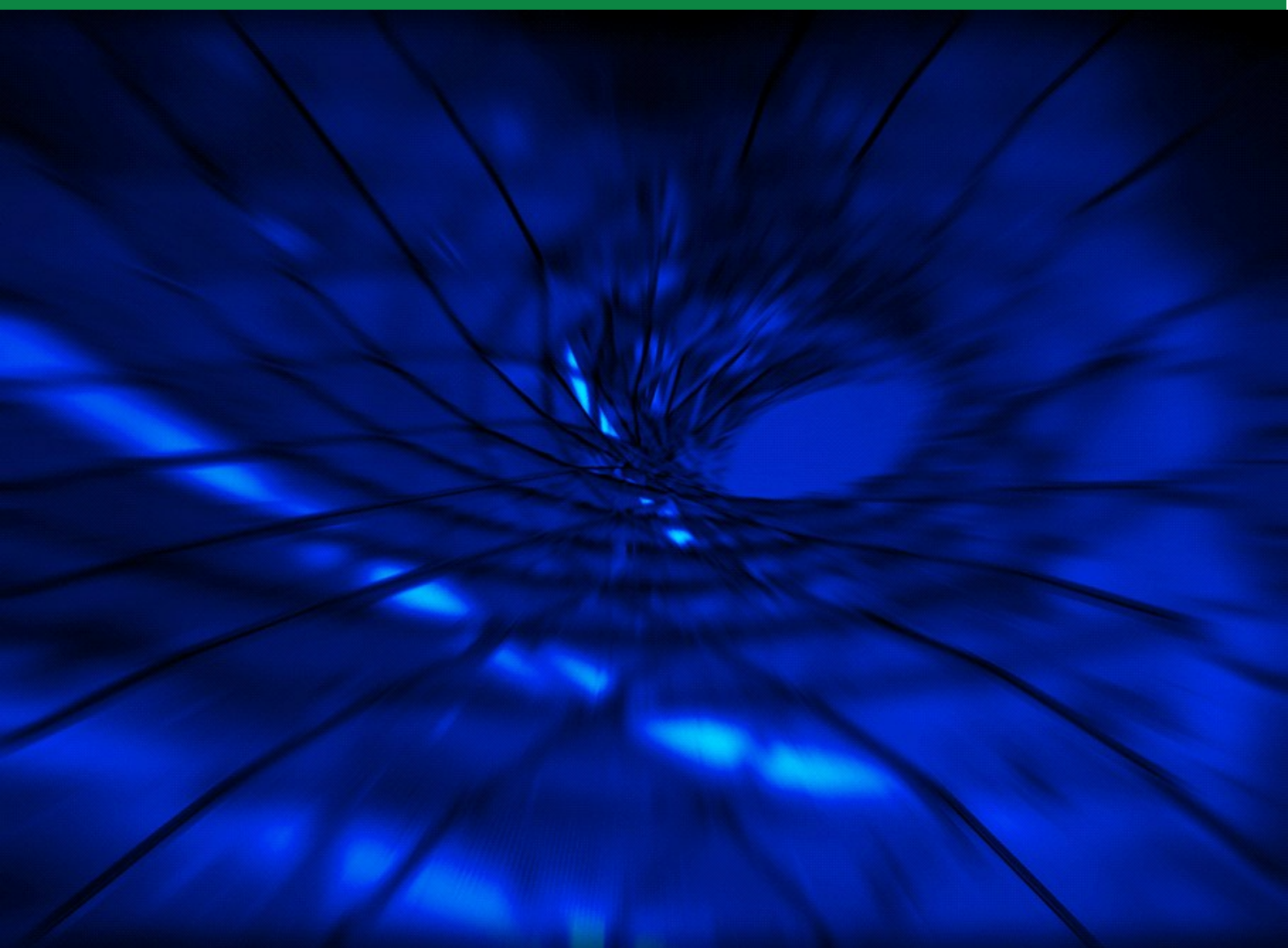


**LEIF MEJLBRO**

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# Topological and Metric Spaces, Banach Spaces and Bounded Operators

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Topological and Metric Spaces, Banach Spaces and Bounded Operators

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## Introduction

This is the second volume containing examples from FUNCTIONAL ANALYSIS. The topics here are limited to *Topological and metric spaces*, *Banach spaces* and *Bounded operators*.

Unfortunately errors cannot be avoided in a first edition of a work of this type. However, the author has tried to put them on a minimum, hoping that the reader will meet with sympathy the errors which do occur in the text.

Leif Mejlbro  
24th November 2009

# 1 Topological and metric spaces

## 1.1 Weierstraß's approximation theorem

**Example 1.1** Let  $\varphi \in C^1([0, 1])$ . It follows from Weierstraß's approximation theorem that  $B_{n,\varphi}(\theta)$  converges uniformly towards  $\varphi(\theta)$  and that  $B_{n,\varphi'}(\theta)$  converges uniformly towards  $\varphi'(\theta)$  on  $[0, 1]$ .

Prove that  $B'_{n,\varphi}(\theta) \rightarrow \varphi'(\theta)$  uniformly on  $[0, 1]$ .

HINT: First prove that  $B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta)$  converges uniformly towards 0 on  $[0, 1]$ .

Next prove that if  $\varphi \in C^\infty([0, 1])$ , then we have for every  $k \in \mathbb{N}$  that  $B_{n,\varphi}^{(n)}(\theta) \rightarrow \varphi^{(k)}(\theta)$  uniformly on  $[0, 1]$ .

NOTATION. We use here the notation

$$B_{n,\varphi}(\theta) = \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \theta^k (1-\theta)^{n-k}$$

for the so-called *Bernstein polynomials*.  $\diamond$

First write

$$\begin{aligned} B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta) &= \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \frac{d}{d\theta} \{ \theta^k (1-\theta)^{n-k} \} \\ &\quad - \sum_{k=0}^{n-1} \varphi'\left(\frac{k}{n-1}\right) \cdot \binom{n-1}{k} \cdot \theta^k (1-\theta)^{n-1-k}. \end{aligned}$$

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Here

$$\frac{d}{d\theta}\{\theta^k(1-\theta)^{n-k}\} = \begin{cases} n\theta^{n-1}, & \text{for } k = n, \\ k \cdot \theta^{k-1}(1-\theta)^{n-k} - (n-k)\theta^k(1-\theta)^{n-1-k}, & \text{for } 0 < k < n, \\ -n(1-\theta)^{n-1}, & \text{for } k = 0. \end{cases}$$

For  $0 < k < n$  we perform the calculation

$$\begin{aligned} \binom{n}{k} \frac{d}{d\theta} \{\theta^k(1-\theta)^{n-k}\} &= \frac{n!}{k!(n-k)!} \{k\theta^{k-1}(1-\theta)^{n-k} - (n-k)\theta^k(1-\theta)^{n-1-k}\} \\ &= \frac{n!}{(k-1)!(n-k)!} \theta^{k-1}(1-\theta)^{n-k} - \frac{n!}{k!(n-k-1)!} \theta^k(1-\theta)^{n-1-k} \\ &= n \binom{n-1}{k-1} \theta^{k-1}(1-\theta)^{n-k} - n \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Hence

$$\begin{aligned} B'_{n,\varphi}(\theta) &= \sum_{k=0}^n \varphi\left(\frac{k}{n}\right) \cdot \binom{n}{k} \cdot \frac{d}{d\theta} \{\theta^k(1-\theta)^{n-k}\} \\ &= \varphi(0) \cdot \{-n(1-\theta)^{n-1}\} + \varphi(1) \cdot n\theta^{n-1} + n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k-1} \theta^{k-1}(1-\theta)^{n-k} \\ &\quad - n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= n \{\varphi(1) \cdot \theta^{n-1} - \varphi(0) \cdot (1-\theta)^{n-1}\} + n \sum_{k=0}^{n-2} \varphi\left(\frac{k+1}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &\quad - n \sum_{k=1}^{n-1} \varphi\left(\frac{k}{n}\right) \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= n \sum_{k=0}^{n-1} \left\{ \varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right) \right\} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k} \\ &= \sum_{k=0}^{n-1} \frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}. \end{aligned}$$

Whence by insertion,

$$B'_{n,\varphi}(\theta) - B_{n-1,\varphi'}(\theta) = \sum_{k=0}^{n-1} \left\{ \frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) \right\} \cdot \binom{n-1}{k} \theta^k(1-\theta)^{n-1-k}.$$

We have assumed from the beginning that  $\varphi \in C^1([0,1])$ , thus

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) = \frac{1}{n} \varepsilon\left(\frac{1}{n}\right)$$

uniformly, so the remainder term is estimated uniformly independently of  $k$ . In fact, it follows from the Mean Value Theorem that

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} = \varphi'(\xi), \quad \text{for et passende } \xi \in \left] \frac{k}{n}, \frac{k+1}{n} \right[ ,$$

and as  $\frac{k}{n} - \frac{k}{n-1} = -\frac{k}{n(n-1)}$ , we get

$$\left| \frac{k}{n} - \frac{k}{n-1} \right| \leq \frac{1}{n-1},$$

and since  $\varphi'$  is continuous,

$$\varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) \rightarrow 0 \quad \text{ligeligt.}$$

From this follows precisely that

$$\frac{\varphi\left(\frac{k+1}{n}\right) - \varphi\left(\frac{k}{n}\right)}{\frac{1}{n}} - \varphi'\left(\frac{k}{n-1}\right) = \varphi'\left(\frac{k}{n}\right) - \varphi'\left(\frac{k}{n-1}\right) 0 \frac{1}{n} \varepsilon \left(\frac{1}{n}\right)$$

uniformly, and the claim is proved.

Finally, we get by induction that if  $\varphi \in C^k([0, 1])$ , then  $B_{n,\varphi}^{(k)}(\theta) \rightarrow \varphi^{(k)}(\theta)$  uniformly on  $[0, 1]$ .

**Example 1.2** Let  $\varphi$  be a real continuous function defined for  $x \geq 0$ , and assume that  $\lim_{x \rightarrow \infty} \varphi(x)$  exists (and is finite). Show that for  $\varepsilon > 0$  there are  $n \in \mathbb{N}$  and constants  $a_k$ ,  $k = 0, 1, \dots, n$ , such that

$$\left| \varphi(x) - \sum_{k=0}^n a_k e^{-kx} \right| \leq \varepsilon$$

for all  $x \geq 0$ .

First note that the range of  $e^{-x}$ ,  $x \in [0, \infty[$ , is  $]0, 1]$ , so we have  $t = e^{-x} \in ]0, 1]$ , thus  $x = \ln \frac{1}{t}$ . The function  $\psi(t)$ , given by

$$\psi(t) = \begin{cases} \varphi\left(\ln \frac{1}{t}\right) & \text{for } t \in ]0, 1], \\ \lim_{x \rightarrow \infty} \varphi(x) & \text{for } t = 0, \end{cases}$$

is continuous for  $t \in [0, 1]$ . It follows from Weierstraß's approximation theorem that there exists a polynomial  $\sum_{k=0}^n a_k t^k$ , such that

$$\left| \psi(t) - \sum_{k=0}^n a_k t^k \right| \leq \varepsilon \quad \text{for alle } t \in [0, 1].$$

Since  $\varphi(x) = \psi(e^{-x})$  for  $x \in [0, +\infty[$ , we conclude that

$$\left| \varphi(x) - \sum_{k=0}^n a_k e^{-kx} \right| \leq \varepsilon \quad \text{for every } x \in [0, +\infty[.$$



## 1.2 Topological and metric spaces

**Example 1.3** Let  $(M, d)$  be a metric space.

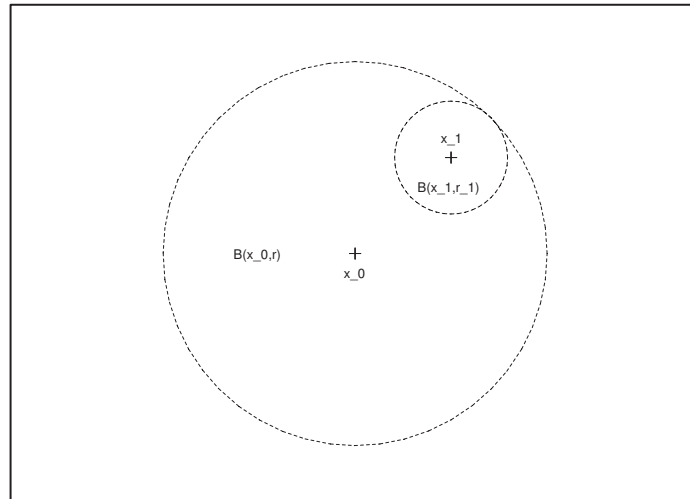
We define the open ball with centre  $x_0$  and radius  $r > 0$  by

$$B(x_0, r) = \{x \in M \mid d(x, x_0) < r\}.$$

We denote a subset  $A \subset M$  open, if there for any  $x_0 \in A$  is an open ball with centre  $x_0$  contained in  $A$ .

Show that an open ball is an open set.

Show that the open sets defined in this way is a topology on  $M$ .



Let  $x_1 \in B(x_0, r)$ , i.e.  $d(x_0, x_1) < r$ . Choose

$$r_1 = r - d(x_0, x_1) > 0.$$

We claim that

$$B(x_1, r_1) \subseteq B(x_0, r).$$

If  $x \in B(x_1, r_1)$ , then

$$d(x_1, x) < r_1 = r - d(x_0, x_1),$$

and it follows by the triangle inequality that

$$d(x_0, x) \leq d(x_0, x_1) + d(x_1, x) < d(x_0, x_1) + r - d(x_0, x_1) = r,$$

proving that  $x \in B(x_0, r)$ . This holds for every  $x \in B(x_1, r_1)$ , so we have proved with the chosen radius  $r_1$  that

$$B(x_1, r_1) \subseteq B(x_0, r),$$

hence every open ball is in fact an open set.

Then we shall prove that the system  $\mathcal{T}$  generated by all open balls is a topology. Thus a set  $T \in \mathcal{T}$  is characterized by the property that for every  $x \in T$  there exists an  $r > 0$ , such that  $B(x, r) \subseteq T$ .

- 1) It is trivial that  $M$  itself is an open set.  
That  $\emptyset$  is open follows from the formal definition:

$$\forall x_0 \in \emptyset \exists r \in \mathbb{R}_+ : B(x_0, r) \subseteq \emptyset.$$

Since there is no point in  $\emptyset$ , the condition is trivially fulfilled.

- 2) Let  $T = \bigcup_{j \in J} T_j$ , where all  $T_j \in \mathcal{T}$ . If  $x_0 \in T$ , then there exists a  $j \in J$ , such that  $x_0 \in T_j$ . Since  $T \in \mathcal{T}$ , there exists an  $r \in \mathbb{R}_+$ , such that

$$B(x_0, r) \subseteq T_j \subseteq T,$$

thus  $T \in \mathcal{T}$ .

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- 3) Let  $T = \bigcap_{j=1}^n T_j$ , where all  $T_j \in \mathcal{T}$ . If  $T = \emptyset$ , there is nothing to prove. Therefore, let  $x_0 \in T$ . Then  $x_0$  must lie in every  $T_j \in \mathcal{T}$ ,  $j = 1, \dots, n$ , so there are constants  $r_j \in \mathbb{R}_+$ ,  $j = 1, \dots, n$ , such that  $B(x_0, r_j) \subseteq T_j$ . Now put  $t = \min r_j \in \mathbb{R}_+$  (notice that there is only a finite number of  $r_j > 0$ ). Then

$$B(x_0, r) \subseteq B(x_0, r_j) \subseteq T_j \quad \text{for every } j = 1, \dots, n,$$

and hence also in the intersection,

$$B(x_0, r) \subseteq \bigcap_{j=1}^n T_j = T.$$

Using the definition of  $\mathcal{T}$  this means that  $T \in \mathcal{T}$ .

We have proved that  $\mathcal{T}$  is a topology.

**Example 1.4** Let  $(M, d)$  be a metric space. We say that a mapping  $T : M \rightarrow M$  is continuous in  $x_0 \in M$  if, for any  $\varepsilon > 0$  there is a  $\delta > 0$  such that for all  $x \in M$  we have

$$d(x_0, x) < \delta \implies d(Tx_0, Tx) < \varepsilon.$$

Show that  $T$  is continuous in  $x_0$  if and only if

$$x_n \rightarrow x_0 \implies Tx_n \rightarrow Tx_0.$$

Show that  $T$  is continuous if the open sets are defined as in EXAMPLE 1.3.

Recall that  $x_n \rightarrow x_0$  means that

$$(1) \quad \forall \delta \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(x_n, x_0) < \delta.$$

Assume that  $T$  is continuous in  $x_0 \in M$  and that  $x_n \rightarrow x_0$ . We shall prove that  $Tx_n \rightarrow Tx_0$ , i.e.

$$\forall \varepsilon \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon.$$

Let  $\varepsilon \in \mathbb{R}_+$  be arbitrary. Since  $T$  is continuous, we can find to this  $\varepsilon > 0$  a constant  $\delta = \delta(\varepsilon) \in \mathbb{R}_+$ , such that

$$(2) \quad \forall x \in M : d(x_0, x) < \delta \implies d(Tx_0, Tx) < \varepsilon.$$

Using that  $x_n \rightarrow x_0$ , we get by (1) an  $n_0 \in \mathbb{N}$  corresponding to  $\delta = \delta(\varepsilon)$  [in fact an  $n_0 \in \mathbb{N}$  corresponding to  $\varepsilon \in \mathbb{R}_+$ ], such that

$$\forall n \geq n_0 : d(x_n, x_0) < \delta = \delta(\varepsilon).$$

It follows from the continuity condition (2) that  $d(Tx_0, Tx_n) < \varepsilon$  for  $n \geq n_0$ , hence

$$\forall \varepsilon \in \mathbb{R}_+ \exists n_0 \in \mathbb{N} \forall n \geq n_0 : d(Tx_n, Tx_0) < \varepsilon,$$

and we have proved that if  $T$  is continuous in  $x_0 \in M$ , then

$$x_n \rightarrow x_0 \implies Tx_n \rightarrow Tx_0.$$

Then assume that  $T$  is *not* continuous at  $x_0 \in M$ , thus

$$(3) \exists \varepsilon \in \mathbb{R}_+ \forall \delta \in \mathbb{R}_+ \exists x \in M : d(x_0, x) < \delta \wedge d(Tx_0, Tx) \geq \varepsilon.$$

We shall prove that there exists a sequence  $(x_n)$ , such that  $x_n \rightarrow x_0$ , while  $Tx_n$  does not converge towards  $Tx_0$ .

Choose  $\varepsilon > 0$  as in (3). Putting  $\delta = \frac{1}{n}$  we get

$$\forall n \in \mathbb{N} \exists x_n \in M : d(x_0, x_n) < \frac{1}{n} \wedge d(Tx_0, Tx_n) \geq \varepsilon.$$

Then it follows that  $x_n \rightarrow x_0$  and  $Tx_n$  cannot be arbitrarily close to  $Tx_0$ , thus  $(Tx_n)$  does not converge towards  $Tx_0$ .

Assume that  $T^{\circ-1}(A)$  is open for every open set  $A$ . Choose  $x_0 \in M$  and  $A = B(Tx_0, \varepsilon)$ . Then  $A$  is open, so  $T^{\circ-1}(A)$  is open according to the assumption. It follows from  $x_0 \in T^{\circ-1}(A)$  that there is a  $\delta \in \mathbb{R}_+$ , such that

$$B(x_0, \delta) \subseteq T^{\circ-1}(A).$$

For every  $x_0 \in B(x_0, \delta)$ , thus  $d(x, x_0) < \delta$ , we get  $Tx \in B(Tx_0, \varepsilon)$ , hence  $d(Tx, Tx_0) < \varepsilon$ , and we have proved that  $T$  is continuous.

Conversely, assume that  $T$  is continuous, and let  $A$  be an open set, thus

$$\forall x_0 \in A \exists r \in \mathbb{R}_+ : d(x_0, x) < r \implies x \in A.$$

We shall prove that  $T^{\circ-1}(A)$  is open, i.e.

$$\forall y_0 \in T^{\circ-1}(A) \exists R \in \mathbb{R}_+ : B(y_0, R) \subseteq T^{\circ-1}(A).$$

This is done INDIRECTLY. *Assumem* that

$$\exists y_0 \in T^{\circ-1}(A) \forall R \in \mathbb{R}_+ : B(y_0, R) \setminus T^{\circ-1}(A) \neq \emptyset,$$

thus

$$\exists y_0 \in T^{\circ-1}(A) \forall R \in \mathbb{R}_+ \exists y \notin T^{\circ-1}(A) : d(y_0, y) < R.$$

Since  $T$  is continuous at  $y_0$ , it follows that

$$\forall r \in \mathbb{R}_+ \exists R \in \mathbb{R}_+ \forall y \in M : d(y_0, y) < R \implies d(Ty_0, Ty) = d(x_0, Ty) < r.$$

We conclude that  $Ty \in A$  contradicting that  $y \notin T^{\circ-1}(A)$ , and the claim is proved.

**Example 1.5** In a set  $M$  is given a function  $d'$  from  $M \times M$  to  $\mathbb{R}$  that satisfies

$$d'(x, y) = 0 \quad \text{if and only if} \quad x = y,$$

$$d'(x, y) \leq d'(z, x) + d'(z, y) \quad \text{for all } x, y, z \in M.$$

Show that  $(M, d')$  is a metric space.

If we choose  $z = y$  in the latter assumption and then use the former one, we get

$$d'(x, y) \leq d'(y, x) + d'(y, y) = d'(y, x) + 0 = d'(y, x),$$

proving that

$$d'(x, y) \leq d'(y, x) \quad \text{for all } x, y \in M.$$

By interchanging  $x$  and  $y$  we obtain the opposite inequality,  $d'(y, x) \leq d'(x, y)$ , hence

$$d'(x, y) = d'(y, x) \quad \text{for all } x, y \in M,$$

and  $d'$  is symmetric.

Using this result on the latter assumption we get the triangle inequality

$$d'(x, y) \leq d'(x, z) + d'(z, y).$$

It only remains to prove that  $d'(x, y) \geq 0$  for all  $x, y \in M$  in order to conclude that  $d'$  is a metric. This follows from

$$0 = d'(x, x) \leq d'(x, y) + d'(y, x) = 2d'(x, y),$$

so the two conditions of the example suffice for  $d'$  being a metric.

**Example 1.6** Let  $(M, d)$  be a metric space.

The diameter of a non-empty subset  $A$  of  $M$  is defined as

$$\delta(A) = \sup_{x, y \in A} d(x, y) \quad (\leq \infty).$$

Show that  $\delta(A) = 0$  if and only if  $A$  contains only one point.

If  $A = \{x\}$  only contains one point, then

$$\delta(A) = \sup_{x, y \in A} d(x, y) = d(x, x) = 0.$$

If  $A$  contains at least two points, choose  $x, y \in A$ , where  $x \neq y$ , from which we conclude that

$$\delta(A) = \sup_{t, z \in A} d(t, z) \geq d(x, y) > 0,$$

and the claim is proved.

**Example 1.7** Let  $(M, d)$  be a metric space. Show that  $d_1$  given by

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)} \quad \text{for } x, y \in M$$

is a metric on  $M$ .

Show that

$$\delta_1(A) = \sup_{x, y \in A} d_1(x, y) \leq 1$$

for all  $A \subset M$ .

Is it possible to find a subset  $A$  with  $\delta_1(A) = 1$ ?

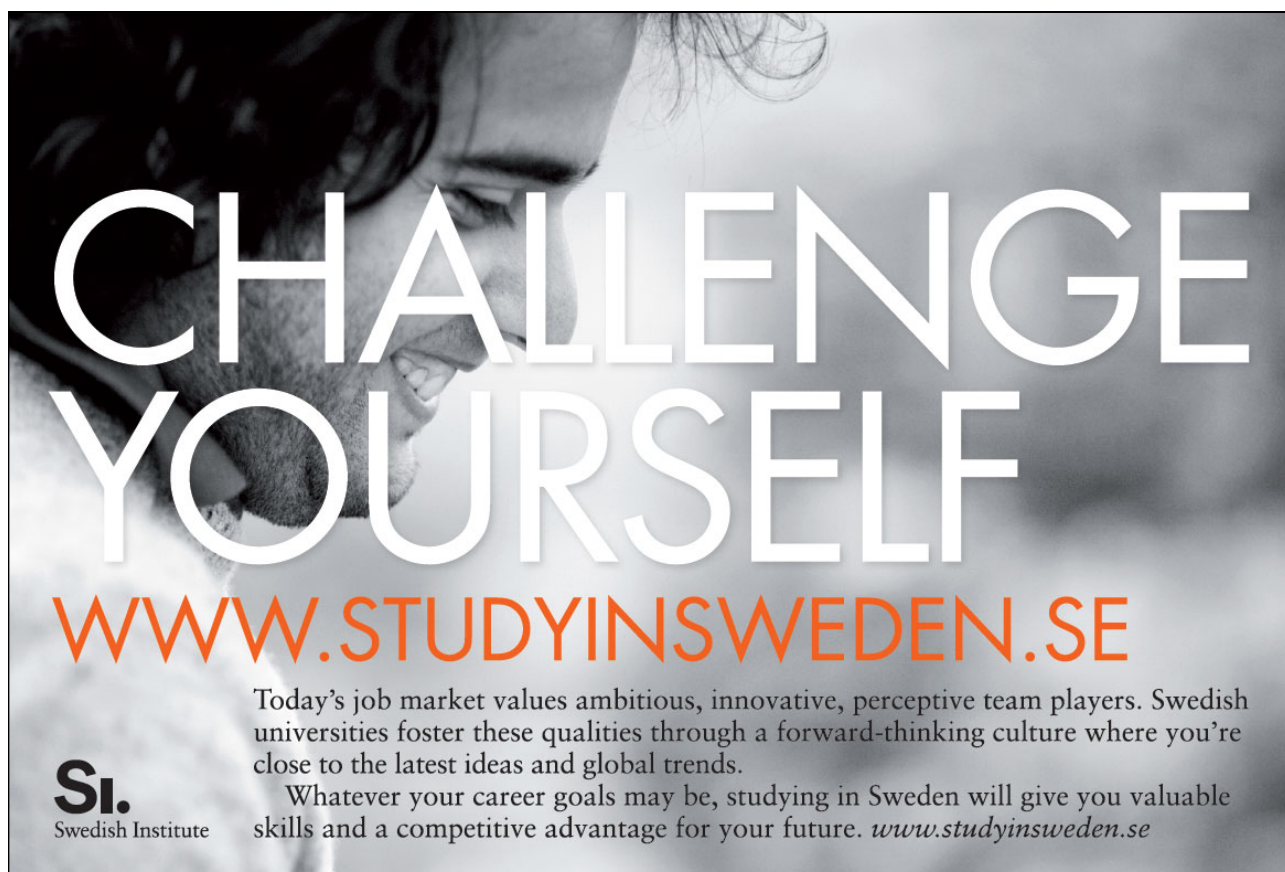
Show that  $d_1(x_n, x) \rightarrow 0$  if and only if  $d(x_n, x) \rightarrow 0$ .

1) We shall first prove that

$$d_1(x, y) = \frac{d(x, y)}{1 + d(x, y)}, \quad x, y \in M,$$

is a metric.

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- a) It is trivial that  $d_1(x, y) \geq 0$ , because  $d(x, y) \geq 0$ .  
 b) Then we see that  $d_1(x, y) = 0$ , if and only if the numerator  $d(x, y) = 0$ , i.e. if and only if  $x = y$ .  
 c) The condition  $d_1(x, y) = d_1(y, x)$  follows immediately from  $d(x, y) = d(y, x)$ .  
 d) It remains only to prove the triangle inequality

$$d_1(x, y) \leq d_1(x, z) + d_1(z, y).$$

Now  $d(x, y) \leq d(x, z) + d(z, y)$ , and the function

$$f(t) = \frac{t}{1+t} = 1 - \frac{1}{1+t}, \quad t \geq 0,$$

is increasing. Hence

$$\begin{aligned} d_1(x, y) &= \frac{d(x, y)}{1+d(x, y)} = f(d(x, y)) \\ &\leq f(d(x, z) + d(z, y)) = \frac{d(x, z) + d(z, y)}{1+d(x, z) + d(z, y)} \\ &= \frac{d(x, z)}{1+d(x, z) + d(z, y)} + \frac{d(z, y)}{1+d(x, z) + d(z, y)} \\ &\leq \frac{d(x, z)}{1+d(x, z)} + \frac{d(z, y)}{1+d(z, y)} \\ &= d_1(x, z) + d_1(z, y). \end{aligned}$$

Summing up, we have proved that  $d_1(x, y)$  is a metric on  $M$ .

- 2) It follows from

$$d_1(x, y) = \frac{d(x, y)}{1+d(x, y)} = 1 - \frac{1}{1+d(x, y)} \leq 1,$$

that

$$\delta_1(A) = \sup_{x, y \in A} d_1(x, y) \leq 1$$

for every subset  $A$ .

- 3) a) If the metric  $d$  is not bounded on  $M$ , then there are subsets  $A$ , such that  $\delta_1(A) = 1$ .  
 In fact, we choose to every  $n \in \mathbb{N}$  points  $x_n, y_n \in M$ , such that

$$d(x_n, y_n) \geq n - 1 \quad \text{for } n \in \mathbb{N}.$$

As mentioned previously,  $f(t) = \frac{t}{1+t}$  is increasing, so

$$d_1(x_n, y_n) = f(d(x, y)) \geq f(n - 1) = \frac{n - 1}{n} = 1 - \frac{1}{n}.$$

Putting

$$A = \{x_n \mid n \in \mathbb{N}\} \cup \{y_n \mid n \in \mathbb{N}\},$$

it follows that  $\delta_1(A) \geq 1 - \frac{1}{n}$  for every  $n \in \mathbb{N}$ , thus  $\delta_1(A) \geq 1$ . On the other hand, we have already proved that  $\delta_1(A) \leq 1$ , so we conclude that  $\delta_1(A) = 1$ .



b) If instead  $d$  is bounded on  $M$ , then  $M$  has itself a finite  $d$ -diameter,  $\delta(M) = c < \infty$ , and

$$\delta_1(M) = \frac{c}{1+c} = 1 - \frac{1}{1+c} < 1.$$

There are many examples of such metrics. The most obvious one is the well-known

$$d_0(x, y) = \begin{cases} 0 & \text{for } x = y, \\ 1 & \text{for } x \neq y, \end{cases}$$

where

$$\tilde{d}_0(x, y) = \begin{cases} 0 & \text{for } x = y, \\ \frac{1}{2} & \text{for } x \neq y. \end{cases}$$

We get another example by starting with the bounded  $d_1$  above. Then

$$d_2(x, y) = \frac{d_1(x, y)}{1 + d_1(x, y)} = \frac{d(x, y)}{1 + 2d(x, y)},$$

with  $\delta_2(A) \leq \frac{1}{2}$  for every subset  $A \subseteq M$ .

4) It follows from

$$d_1(x_n, x) = 1 - \frac{1}{1 + d(x_n, x)},$$

that the condition  $d_1(x_n, x) \rightarrow 0$  is equivalent with  $1 + d(x_n, x) \rightarrow 1$ , thus with  $d(x_n, x) \rightarrow 0$ , and the claim is proved.

**Example 1.8** Let  $(M_1, d_1)$  and  $(M_2, d_2)$  be metric spaces.

Show that  $M_1 \times M_2$  can be made into a metric space by the following definition of a metric  $d$ :

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2).$$

Show that also  $d^*$  given by

$$d^*((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$$

defines a metric on  $M_1 \times M_2$ .

1) Clearly,

$$d((x_1, x_2), (y_1, y_2)) \geq 0 \quad \text{and} \quad d^*((x_1, x_2), (y_1, y_2)) \geq 0.$$

2) If  $(x_1, x_2) = (y_1, y_2)$ , i.e.  $x_1 = y_1$  and  $x_2 = y_2$ , then

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{and} \quad d^*((x_1, x_2), (y_1, y_2)) = 0.$$

Conversely, if

$$d((x_1, x_2), (y_1, y_2)) = 0 \quad \text{or} \quad d^*((x_1, x_2), (y_1, y_2)) = 0,$$

then both

$$d_1(x_1, y_1) = 0 \quad \text{and} \quad d_2(x_2, y_2) = 0,$$

and it follows that  $x_1 = y_1$  and  $x_2 = y_2$ , and hence  $(x_1, x_2) = (y_1, y_2)$ .

3) The symmetry is obvious.

4) The triangle inequality holds for both  $d_1$  and  $d_2$ . Hence, it also holds for  $d$  and  $d^*$ . In fact,

$$\begin{aligned} d((x_1, x_2), (y_1, y_2)) &= d_1(x_1, y_1) + d_2(x_2, y_2) \\ &\leq \{d_1(x_1, z_1) + d_1(z_1, y_1)\} + \{d_2(x_2, z_2) + d_2(z_2, y_2)\} \\ &= \{d_1(x_1, z_1) + d_2(x_2, z_2)\} + \{d_1(z_1, y_1) + d_2(z_2, y_2)\} \\ &= d((x_1, x_2), (z_1, z_2)) + d((z_1, z_2), (y_1, y_2)), \end{aligned}$$

and

$$\begin{aligned} d^*((x_1, x_2), (y_1, y_2)) &= \max \{d_1(x_1, y_1), d_2(x_2, y_2)\} \\ &\leq \max \{d_1(x_1, z_1) + d_1(z_1, y_1), d_2(x_2, z_2) + d_2(z_2, y_2)\} \\ &\leq \max \{d_1(x_1, z_1), d_2(x_2, z_2)\} + \max \{d_1(z_1, y_1), d_2(z_2, y_2)\} \\ &= d^*((x_1, x_2), (z_1, z_2)) + d^*((z_1, z_2), (y_1, y_2)). \end{aligned}$$

**Example 1.9** Show that in any set  $M$  we can define a metric by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Then we call  $(M, d)$  for a discrete metric space.

Characterize the sequences in  $M$  where  $d(x_n, x) \rightarrow 0$ .

1) Clearly,  $d(x, y) \geq 0$ .

2) Clearly,  $d(x, y) = 0$ , if and only if  $x = y$ .

3) Clearly,  $d(x, y) = d(y, x)$ .

4) Finally, it is almost trivial that

$$d(x, y) \leq d(x, z) + d(z, y),$$

because the left hand side is always  $\leq 1$ . If the right hand side is  $< 1$ , then both  $d(x, z) = 0$  and  $d(z, y) = 0$ , and we infer that  $x = z$  and  $z = y$ , hence also  $x = y$ . This implies that the left hand side  $d(x, y) = 0$ , and the triangle inequality is fulfilled.

Summing up we have proved that  $(M, d)$  is a metric space.

If  $d(x_n, x) \rightarrow 0$ , then choose  $\varepsilon = \frac{1}{2}$ . There exists an  $n_0 \in \mathbb{N}$ , such that

$$d(x_n, x) < \varepsilon = \frac{1}{2} \quad \text{for } n \geq n_0.$$

This is only possible, if  $d(x_n, x) = 0$ , i.e. if

$$x_n = x \quad \text{for all } n \geq n_0.$$

We conclude that all the convergent sequences are constant eventually.

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**Example 1.10** Let  $(M, d)$  be a metric space and consider  $M$  as a topological space with the topology stemming from the open balls (the ball topology).

Recall that a set  $A$  is closed if  $M \setminus A$  is open.

Show that  $A \subset M$  is closed if and only if

$$x_n \in A, \quad x_n \rightarrow x \implies x \in A.$$

Show that if  $(M, d)$  is a complete metric space and  $A$  is a closed subset of  $M$ , then  $(A, d)$  is a complete metric space.

Assume that  $A$  is closed and let  $x_n \in A$  be a convergent sequence in  $M$ , i.e.  $x_n \rightarrow x \in M$ . We shall prove that  $x \in A$ .

INDIRECT PROOF. Assume that  $x \notin A$ , i.e.  $x \in M \setminus A$ , which is open.

There exists an  $r > 0$ , such that

$$B(x, r) \subseteq M \setminus A, \quad \text{i.e.} \quad B(x, r) \cap A = \emptyset.$$

Now,  $x_n \rightarrow x$ , so there exists an  $n_r \in \mathbb{N}$ , such that

$$d(x_n, x) < r \quad \text{for } n \geq n_r,$$

and we see that  $x_n \in B(x, r) \cap A = \emptyset$ , which is not possible. Hence our assumption is wrong, so we conclude that  $x \in A$ .

Conversely, assume for every convergent sequence  $(x_n) \subseteq A$  the limit point lies in  $A$ . We shall prove that  $A$  is closed, or equivalently that  $M \setminus A$  is open.

INDIRECT PROOF. Assume that  $M \setminus A$  is *not* open. There exists an  $x \in M \setminus A$ , such that

$$\forall r \in \mathbb{R}_+ \exists y \in A : d(x, y) < r.$$

If we put  $r = \frac{1}{n}$ ,  $n \in \mathbb{N}$ , with corresponding  $y = x_n$ , we define a sequence in  $A$ , which converges towards  $x$ , thus  $x \in A$  according to the assumption. This is contradicting the assumption that  $x \in M \setminus A$ . Hence this assumption must be wrong, and  $x \in A$  as requested.

Finally, assume that  $(M, d)$  is a *complete* metric space and that  $A$  is a *closed* subset of  $M$ . We shall prove that  $(A, d)$  is complete.

Let  $(x_n)$  be a Cauchy sequence on  $A$ . Then  $(x_n)$  is also a Cauchy sequence on the complete metric space  $M$ , thus  $(x_n)$  converges in  $M$  towards the limit  $x \in M$ . However,  $A$  is a closed subset, so it follows from the previous result that  $x \in A$ . We have proved that every Cauchy sequence  $(x_n)$  on  $A$  has a limit  $x \in A$ , which means that  $(A, d)$  is complete.

**Example 1.11** *Show that*

$$d(x, y) = |\arctan x - \arctan y|$$

*defines a metric on  $\mathbb{R}$ .*

The definition includes an absolute value, hence  $d(x, y) \geq 0$  for all  $x, y \in \mathbb{R}$ .

The function  $\arctan t$  is strictly increasing on  $\mathbb{R}$ , hence  $d(x, y) = 0$ , if and only if  $x = y$ .

Clearly,  $d(x, y) = d(y, x)$ .

The triangle inequality follows from

$$d(x, y) = |\arctan x - \arctan y| \leq |\arctan x - \arctan z| + |\arctan z - \arctan y| = d(x, z) + d(z, y).$$

**Example 1.12** *In  $\mathbb{R}^k$  we define*

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|,$$

$$d_2(x, y) = \left( \sum_{i=1}^k |x_i - y_i|^2 \right)^{\frac{1}{2}},$$

$$d_\infty(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|.$$

*Show that  $d_1$ ,  $d_2$  and  $d_\infty$  are metrics.*

*Show that*

$$d_\infty(x, y) \leq d_1(x, y) \leq k d_\infty(x, y),$$

*and find a similar inequality when  $d_1$  is replaced by  $d_2$ .*

*Show that if a sequence  $(x_n)$  converges to  $x$  in one of these metrics, then we have coordinate wise convergence:*

$$x_{ni} \rightarrow x_i \quad \text{for all } i = 1, 2, \dots, k.$$

We first prove that

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

is a metric:

- 1) Clearly,  $d_1(x, y) \geq 0$ .
- 2) Clearly,  $d_1(x, y) = 0$ , if and only if  $x = y$ .
- 3) Clearly,  $d_1(x, y) = d_1(y, x)$ .

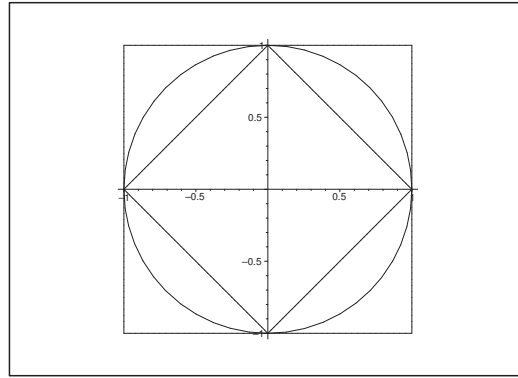


Figure 1: The three unit balls for  $d_1$  (innermost),  $d_2$  (the disc) and  $d_\infty$  (largest) in the case  $\mathbb{R}^2$ .

4) The triangle inequality follows by a small computation

$$\begin{aligned} d_1(x, y) &= \sum_{i=1}^k |x_i - y_i| \leq \sum_{i=1}^k \{|x_i - z_i| + |z_i - y_i|\} \\ &= \sum_{i=1}^k |x_i - z_i| + \sum_{i=1}^k |z_i - y_i| = d_1(x, z) + d_1(z, y). \end{aligned}$$

We have proved that  $d_1$  is a metric.

Then we prove that

$$d_2(x, y) = \left( \sum_{i=1}^k |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

is a metric. Again, the first three conditions are trivial. The triangle inequality,

$$\sqrt{\sum_{i=1}^k |x_i - y_i|^2} \leq \sqrt{\sum_{i=1}^k |x_i - z_i|^2} + \sqrt{\sum_{i=1}^k |z_i - y_i|^2}$$

is, however, more difficult to prove. There are several proofs of the triangle inequality of  $d_2$ . Here we shall not choose the most elegant one, but instead the intuitively most obvious one.

Put  $a_i = x_i - z_i$  and  $b_i = z_i - y_i$ ,  $i = 1, \dots, k$ . We shall prove that

$$\sqrt{\sum_{i=1}^k (a_i + b_i)^2} \leq \sqrt{\sum_{i=1}^k a_i^2} + \sqrt{\sum_{i=1}^k b_i^2}.$$

All terms are  $\geq 0$ , thus it is seen by squaring that we shall prove that

$$\sum_{i=1}^k a_i^2 + \sum_{i=1}^k b_i^2 + 2 \sum_{i=1}^k a_i b_i \leq \sum_{i=1}^k a_i^2 + \sum_{i=1}^k b_i^2 + 2 \sqrt{\sum_{i=1}^k \sum_{j=1}^k a_i^2 b_j^2},$$

which is reduced to the equivalent condition

$$\sum_{i=1}^k a_i b_i \leq \sqrt{\sum_{i=1}^k a_i^2} \cdot \sqrt{\sum_{j=1}^k b_j^2}.$$

The claim follows if we can prove the CAUCHY-SCHWARZ INEQUALITY

$$\left| \sum_{i=1}^k a_i b_i \right| \leq \sqrt{\sum_{i=1}^k a_i^2} \cdot \sqrt{\sum_{j=1}^k b_j^2}.$$

Another squaring shows that it suffices to prove that

$$\left( \sum_{i=1}^k a_i b_i \right)^2 \leq \sum_{i=1}^k a_i^2 \sum_{j=1}^k b_j^2,$$

i.e.

$$\sum_{i=1}^k a_i^2 b_i^2 + 2 \sum_{i=1}^{k-1} \sum_{j=i+1}^k a_i a_j b_i b_j \leq \sum_{i=1}^k a_i^2 b_i^2 + \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i^2 b_j^2 + a_j^2 b_i^2),$$

which again is equivalent with

$$0 \leq \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i^2 b_j^2 + a_j^2 b_i^2 - 2a_i a_j b_i b_j) = \sum_{i=1}^{k-1} \sum_{j=i+1}^k (a_i b_j - a_j b_i)^2.$$

The latter is clearly satisfied. Since we everywhere have computed “ $\Leftarrow$ ”, the claim is proved.

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Finally,

$$d_{\infty}(x, y) = \max_{1 \leq i \leq k} |x_i - y_i|$$

is a metric, because the first three conditions again are trivial, and the triangle inequality follows from

$$|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i| \quad \text{for every } i = 1, \dots, k,$$

thus

$$|x_i - y_i| \leq d_{\infty}(x, z) + d_{\infty}(z, y) \quad \text{for every } i = 1, \dots, k,$$

and by taking the maximum once more,

$$d_{\infty}(x, y) \leq d_{\infty}(x, z) + d_{\infty}(z, y).$$

We have now proved that  $d_1$ ,  $d_2$  and  $d_{\infty}$  are all metrics.

We can find  $j \in \{1, \dots, k\}$ , such that

$$\begin{aligned} d_{\infty}(x, y) &= \max_{1 \leq i \leq k} |x_i - y_i| = |x_j - y_j| \leq \sum_{i=1}^k |x_i - y_i| = d_1(x, y) \\ &\leq \sum_{i=1}^k \max |x_i - y_i| = k \cdot d_{\infty}(x, y). \end{aligned}$$

Analogously (with the same “maximal”  $j$ ),

$$\begin{aligned} d_{\infty}(x, y) &= \max_{1 \leq i \leq k} |x_i - y_i| = |x_j - y_j| = \sqrt{|x_j - y_j|^2} \\ &\leq \sqrt{\sum_{i=1}^k |x_i - y_i|^2} = d_2(x, y) \leq \sqrt{\sum_{i=1}^k \left\{ \max_{1 \leq i \leq k} |x_i - y_i| \right\}^2} \\ &= \sqrt{\sum_{i=1}^k \{d_{\infty}(x, y)\}^2} = \sqrt{k} \cdot d_{\infty}(x, y), \end{aligned}$$

and the wanted inequality becomes

$$d_{\infty}(x, y) \leq d_2(x, y) \leq \sqrt{k} \cdot d_{\infty}(x, y).$$

**Remark 1.1** A simple squaring shows that  $d_2(x, y) \leq d_1(x, y)$ , which can also be seen on the figure (the simple proof is left to the reader). This means that

$$d_{\infty}(x, y) \leq d_2(x, y) \leq d_1(x, y) \leq k \cdot d_{\infty}(x, y). \quad \diamond$$

Using that  $x_{ni} \rightarrow x_i$  for every  $i = 1, 2, \dots, k$ , if and only if  $d_{\infty}(x_n, x) \rightarrow 0$ , we conclude from the inequalities

$$d_{\infty}(x, y) \leq d_1(x, y) \leq k \cdot d_{\infty}(x, y),$$

$$d_{\infty}(x, y) \leq d_2(x, y) \leq \sqrt{k} \cdot d_{\infty}(x, y),$$

that this is fulfilled if and only if  $d_1(x_n, 0) \rightarrow 0$ , and if and only if  $d_2(x, y) \rightarrow 0$ .

**Example 1.13** Let  $c$  denote the set of convergent complex sequences  $x = (x_1, x_2, \dots)$ . Show that  $c$  is a complete metric space when equipped with the metric

$$d_\infty(x, y) = \sup_i |x_i - y_i|.$$

HINT: Show that the space of bounded complex sequences  $\ell^\infty$  is a complete space and show then that  $c$  is a closed subset, then apply Example 1.10.

Let  $x^n = (x_1^n, x_2^n, \dots)$ , where  $\lim_{i \rightarrow \infty} x_i^n$  exists, be a Cauchy sequence from  $c$ , thus

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : d(x^m, x^n) < \varepsilon.$$

This means that

$$\sup_i |x_i^m - x_i^n| < \varepsilon.$$

In particular,  $(x_i^n)_n$  is a Cauchy sequence on  $\mathbb{R}$  for every  $i$ , hence convergent,

$$\lim_{n \rightarrow \infty} x_i^n = x_i.$$

The *Hint* is *not* used, because it is not hard to prove directly that  $(x_i) \in c$ . It suffices to prove that  $(x_i)$  is a Cauchy sequence, i.e.

$$(4) \quad \forall \varepsilon > 0 \exists I \in \mathbb{N} \forall i, j \geq I : |x_i - x_j| < \varepsilon.$$

It follows from

$$|x_i - x_j| \leq |x_i - x_i^n| + |x_i^n - x_j^n| + |x_j^n - x_j|,$$

and  $(x_i^n)_n \rightarrow x_i$ , and even

$$\sup_i |x_i - x_i^n| \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

that

$$a) \quad \forall \varepsilon > 0 \exists N \forall n \geq N \forall i : |x_i - x_i^n| < \frac{\varepsilon}{3},$$

$$b) \quad \forall \varepsilon > 0 \forall n \exists I(n) \forall i, j \geq I(n) : |x_i^n - x_j^n| < \frac{\varepsilon}{3}.$$

First choose  $N$ , such that a) is fulfilled.

Then choose  $I = I(N)$ , such that b) is fulfilled for  $n = N$ .

If  $i, j \geq I = I(N)$ , then

$$\begin{aligned} |x_i - x_j| &\leq |x_i - x_i^N| + |x_i^N - x_j^N| + |x_j^N - x_j| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \end{aligned}$$

which is (4), and we have proved that  $(x_i)$  is a Cauchy sequence on  $\mathbb{R}$ , hence convergent. In particular,  $(x_i)$  is bounded, so  $(x_i) \in c$ , and  $c$  is complete.

**Example 1.14** In the set of bounded complex sequences  $\ell^\infty$  equipped with the metric from EXERCISE 12 we consider the sets  $c_0$  consisting of the sequences converging to 0 and  $c_{00}$  consisting of the sequences with only a finite number of elements different from 0. Investigate if  $c_0$  and/or  $c_{00}$  are closed subsets of  $\ell^\infty$ .

The sequence  $\left(\frac{1}{n}\right)$  belongs to  $\ell^\infty$ , though it does not belong to  $c_{00}$ . Choose

$$x^n = \left(1, \frac{1}{2}, \dots, \frac{1}{n}, 0, 0, \dots\right).$$

Then  $x^n \in c_{00}$  and  $x^n \rightarrow x = \left(\frac{1}{n}\right) \notin c_{00}$ , hence  $c_{00}$  is not closed.

Let  $x^n = (x_1^n, x_2^n, \dots) \in c_0$  be convergent in  $\ell^\infty$ , i.e.  $\lim_{i \rightarrow \infty} x_i^n = 0$  for every  $n$ . There exists an  $x \in \ell^\infty$ , such that

$$\forall \varepsilon > 0 \exists n_0 \forall n \geq n_0 : \|x - x^n\|_\infty = \sup_i |x_i - x_i^n| < \varepsilon.$$

We shall prove that  $\lim_{i \rightarrow \infty} x_i = 0$ . Now,

$$|x_i| \leq |x_i - x_i^n| + |x_i^n| \leq \|x - x^n\|_\infty + |x_i^n|.$$

First choose  $n$ , such that  $\|x - x^n\|_\infty < \frac{\varepsilon}{2}$ .

Then choose  $I$ , such that  $|x_i^n| < \frac{\varepsilon}{2}$  for every  $i \geq I$ . Summing up we get for all  $i \geq I$  that

$$|x_i| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We infer that  $\lim_{i \rightarrow \infty} x_i = 0$ , thus  $(x_i) \in c_0$ , and we have proved that  $c_0$  is a closed subset of  $\ell^\infty$ .

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### 1.3 Contractions

**Example 1.15** Consider the metric space  $(M, d)$ , where  $M = [1, \infty[$ , and  $d$  the usual distance. Let the mapping  $T : M \rightarrow M$  be given by

$$Tx = \frac{x}{2} + \frac{1}{x}.$$

Show that  $T$  is a contraction and find the minimal contraction constant  $\alpha$ . Find also the fixed point.

First compute

$$|Tx - Ty| = \left| \frac{x}{2} + \frac{1}{x} - \frac{y}{2} - \frac{1}{y} \right| = \left| \frac{x-y}{2} + \frac{1}{x} - \frac{1}{y} \right| = \left| \frac{x-y}{2} + \frac{y-x}{xy} \right| = |x-y| \cdot \left| \frac{1}{2} - \frac{1}{xy} \right|.$$

Now,  $x, y \geq 1$ , so  $0 < \frac{1}{xy} \leq 1$ , and the function

$$(x, y) \mapsto \frac{1}{2} - \frac{1}{xy}$$

has the range  $\left[-\frac{1}{2}, \frac{1}{2}\right]$ . We conclude that  $\alpha = \frac{1}{2}$ , so  $\frac{1}{2}$  is the smallest  $\alpha$ , for which

$$\left| \frac{1}{2} - \frac{1}{xy} \right| \leq \alpha.$$

The fixpoint satisfies the equation  $Tx = x$ , thus

$$x = \frac{x}{2} + \frac{1}{x}, \quad \text{hence } \frac{x}{2} = \frac{1}{x}, \quad \text{i.e. } x^2 = 2.$$

Since  $x \geq 1$ , the fixpoint must be  $x = \sqrt{2}$ , which also is easily seen by insertion.

Since  $\alpha = \frac{1}{2} < 1$ , it follows from the above that it is the only fixpoint.

**Example 1.16** A mapping  $T$  from a metric space  $(M, d)$  into itself is called a weak contraction if

$$d(Tx, Ty) < d(x, y),$$

for all  $x, y \in M$ ,  $x \neq y$ .

Show that  $T$  has at most one fixed point.

Show that  $T$  does not necessarily have a fixed point.

HINT: One should take  $Tx = x + \frac{1}{x}$  for  $x \geq 1$ .

Let  $T$  be a weak contraction, and assume that both  $x$  and  $y$  are fixpoints, i.e.  $Tx = x$  and  $Ty = y$ . If  $x \neq y$ , then

$$d(x, y) = d(Tx, Ty) < d(x, y),$$

which is not possible. Hence  $y = x$ , and there is at most one fixpoint.

Define  $Tx = x + \frac{1}{x}$  on  $[1, +\infty[$ . If  $x, y \in [1, +\infty[$ , then

$$|Tx - Ty| = \left| x + \frac{1}{x} - y - \frac{1}{y} \right| = \left| x - y + \frac{y - x}{xy} \right| = |x - y| \cdot \left| 1 - \frac{1}{xy} \right|.$$

It follows from  $0 < \frac{1}{xy} \leq 1$  for  $x, y \geq 1$ , that

$$|Tx - Ty| < |x - y| \quad \text{for } x \neq y,$$

and  $T$  is a weak contraction on  $[1, +\infty[$ .

The weak contraction  $Tx = x + \frac{1}{x}$  does not have a fixpoint, because  $Tx = x$  would imply that  $\frac{1}{x} = 0$ , which is not possible.

**Example 1.17** *It is very common in mathematical analysis to consider iterations of the form*

$$x_n = g(x_{n-1}),$$

where  $g$  is a  $C^1$ -function. Show that the sequence  $(x_n)$  is convergent for any choice of  $x_0$  if there is an  $\alpha$ ,  $0 < \alpha < 1$ , such that

$$|g'(x)| \leq \alpha$$

for all  $x \in \mathbb{R}$ .

It follows from the Mean Value Theorem that one to any  $x$  and  $y$  can find  $t = t(x, y)$  between  $x$  and  $y$ , such that

$$g(x) - g(y) = g'(t) \cdot (x - y),$$

thus

$$|g(x) - g(y)| = |g'(t)| \cdot |x - y| \leq \alpha |x - y|.$$

This proves that  $g$  is a contraction, and the claim follows from Banach's Fixpoint Theorem.

**Example 1.18** *To approximate the solution to an equation  $f(x) = 0$ , we bring the equation on the form  $x = g(x)$  and choose an  $x_0$  and use the iteration  $x_n = g(x_{n-1})$ . Assume that  $g$  is a  $C^1$ -function on the interval  $[x_0 - \delta; x_0 + \delta]$ , and that  $|g'(x)| \leq \alpha < 1$  for  $x \in [x_0 - \delta; x_0 + \delta]$ , and moreover*

$$|g(x_0) - x_0| \leq (1 - \alpha)\delta.$$

*Show that there is one and only one solution  $x \in [x_0 - \delta; x_0 + \delta]$  to the equation, and that  $x_n \rightarrow x$ .*

Noticing that  $|g'(x)| \leq \alpha < 1$  on the interval  $[x_0 - \delta; x_0 + \delta]$ , the claim follows from Banach's Fixpoint Theorem, if we only can prove that the iterative sequence  $(x_n)$  lies entirely in the interval  $[x_0 - \delta, x_0 + \delta]$ . We prove this by induction.

It is obvious that  $x_0 \in [x_0 - \delta, x_0, \delta]$ .

Assume that  $x_n \in [x_0 - \delta, x_0 + \delta]$ . Then we get for the following element  $x_{n+1} = g(x_n)$ ,

$$\begin{aligned} |x_{n+1} - x_0| &= |g(x_n) - x_0| \\ &\leq |g(x_n) - g(x_0)| + |g(x_0) - x_0| \\ &\leq \alpha |x_n - x_0| + (1 - \alpha)\delta \\ &\leq \alpha \delta + (1 - \alpha)\delta = \delta, \end{aligned}$$

proving that  $x_{n+1} \in [x_0 - \delta, x_0 + \delta]$ , and the claim follows.

**Example 1.19** Solve by iteration the equation  $f(x) = 0$  for  $f \in C^1([a, b])$ ,  $f(x) < 0 < f(b)$  and  $f'$  bounded and strictly positive in  $[a, b]$ .

HINT: Take  $g(x) = x - \lambda f(x)$  for a smart choice of  $\lambda$ .

Putting

$$g(x) = x - \lambda f(x), \quad \lambda \neq 0,$$

it follows that  $f(x) = 0$ , if and only if  $g(x) = x$ . Now,

$$g'(x) = 1 - \lambda f'(x) \quad \text{and} \quad 0 < k_1 \leq f'(x) \leq k_2,$$

so

$$1 - \lambda k_2 \leq g'(x) \leq 1 - \lambda k_1.$$

If we choose  $\lambda = \frac{1}{k_2}$ , then

$$0 \leq g'(x) \leq 1 - \frac{k_1}{k_2} = \alpha < 1,$$

and the mapping  $g : [a, b] \rightarrow [a, b]$  is increasing and a contraction, so it has by Banach's Fixpoint Theorem precisely one fixpoint in  $[a, b]$ .

**Example 1.20** Show that it is possible to solve the equation  $f(x)x^3 + x - 1 = 0$  by the iteration

$$x_n = g(x_{n-1}) = (1 + x_{n-1}^2)^{-1}.$$

Find  $x_1, x_2, x_3$  for  $x_0 = 1$ , and find an estimate for  $d(x, x_n)$ .

Let  $g(x) = \frac{1}{1 + x^2}$ . Then  $g(x) = x$  is equivalent with  $x = \frac{1}{1 + x^2}$ , thus  $x(1 + x^2) = 1$ , which we write as

$$f(x) = x^3 + x - 1 = 0,$$

i.e. exactly the equation we want to solve.

It follows from

$$g'(x) = -\frac{2x}{(1 + x^2)^2},$$

and

$$g''(x) = -\frac{2}{(1+x^2)^2} - 2x \cdot \frac{(-2) \cdot 2x}{(1+x^2)^3} = \frac{2}{(1+x^2)^3} \{-1 - x^2 + 4x^2\} = \frac{6\left(x^2 - \frac{1}{3}\right)}{(1+x^2)^3},$$

that  $g''(x) = 0$  for  $x = \pm \frac{1}{\sqrt{3}}$ . Since  $g'(x) \rightarrow 0$  for  $x \rightarrow \pm\infty$ , these points correspond to maximum and minimum for  $g'(x)$ , thus

$$|g'(x)| \leq \frac{2 \cdot \frac{1}{\sqrt{3}}}{\left(1 + \frac{1}{3}\right)^2} = \frac{\frac{2}{\sqrt{3}}}{\frac{16}{9}} = \frac{3\sqrt{3}}{8} = \alpha \leq 0.65,$$

and we have proved that  $g$  is a contraction, so the equation

$$f(x) = x^3 + x - 1 = 0$$

can be solved by the given iteration.

Let  $x_0 = 1$ . Then

$$x_1 = g(x_0) = \frac{1}{1+1} = \frac{1}{2},$$

$$x_2 = g\left(\frac{1}{2}\right) = \frac{1}{1 + \frac{1}{4}} = \frac{4}{5},$$

$$x_3 = g\left(\frac{4}{5}\right) = \frac{1}{1 + \frac{16}{25}} = \frac{25}{41}.$$



Finally,

$$|x - x_n| \leq \frac{\alpha^n}{1 - \alpha} \cdot |x_1 - x_0|,$$

so

$$|x - x_n| \leq \frac{\left(\frac{3\sqrt{3}}{8}\right)^n}{1 - \frac{3\sqrt{3}}{8}} \cdot \left(1 - \frac{1}{2}\right) = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{3\sqrt{3}}{8}\right)^n = \frac{4}{8 - 3\sqrt{3}} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}} < \frac{3}{2} \cdot \left(\frac{27}{64}\right)^{\frac{n}{2}}.$$

When we apply the iteration above on a pocket calculator, we get

$$x = 0.682\,327\,804.$$

**Remark 1.2** The iteration above can therefore be applied, though it is far from the fastest one. If the preset case we get by *Newton's iteration formula*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = \frac{2}{3}x_n + \frac{1}{3} \cdot \frac{3 - 2x_n}{3x_n^2 + 1},$$

from which already

$$x_4 = 0.682\,327\,804. \quad \diamond$$

**Example 1.21** A mapping  $T : \mathbb{R} \rightarrow \mathbb{R}$  satisfies a Lipschitz condition with constant  $k$ , if

$$|Tx - Ty| \leq k|x - y|, \quad \text{for all } x, y \in \mathbb{R}.$$

- 1) Is  $T$  a contraction?
- 2) If  $T$  is a  $C^1$ -function with bounded derivative, show that  $T$  satisfies a Lipschitz condition.
- 3) If  $T$  satisfies a Lipschitz condition, is  $T$  then a  $C^1$ -function with bounded derivative?
- 4) Assume that  $|Tx - Ty| \leq k|x - y|^\alpha$  for some  $\alpha > 1$ . Show that  $T$  is a constant.

- 1) If  $k \geq 1$ , then  $T$  is not necessarily a contraction.  
If instead  $0 \leq k < 1$ , then  $T$  is always a contraction.

- 2) It follows from the Mean Value Theorem that

$$|T(x) - T(y)| = |T'(t)| \cdot |x - y|,$$

where  $t = t(x, y)$  lies somewhere between  $x$  and  $y$ .

Since  $|T'(t)| \leq k$ , it is obvious that  $T$  fulfils a Lipschitz condition.

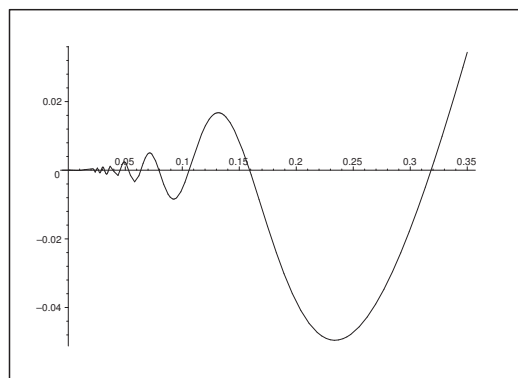


Figure 2: The graph of  $f(x) = x^2 \cdot \sin \frac{1}{x}$  for  $0 < x < 0.35$ .

3) The answer is “no”. Choose the function

$$f(x) = \begin{cases} x^2 \cdot \sin \frac{1}{x} & \text{for } x > 0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then  $f$  is differentiable with the derivative

$$f'(x) = \begin{cases} 2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x} & \text{for } x > 0, \\ \lim_{x \rightarrow 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0+} x \cdot \sin \frac{1}{x} = 0 & \text{for } x = 0, \\ 0 & \text{for } x < 0. \end{cases}$$

Choose  $x_0 > 0$ , such that  $f'(x_0) = 0$ , and put

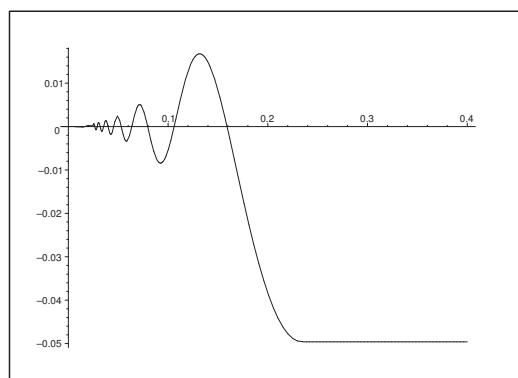


Figure 3: An example of a function  $T(x)$ .

$$T(x) = \begin{cases} f(x_0) & \text{for } x \geq x_0, \\ x^2 \cdot \sin \frac{1}{x} & \text{for } 0 < x < x_0, \\ 0 & \text{for } x \leq 0. \end{cases}$$

Then  $|T'(x)| \leq 2x_0 + 1$ , and  $T'(x)$  is defined everywhere, though not continuous for  $x = 0$ , where  $T'(x) = f'(x) = 2x \cdot \sin \frac{1}{x} - \cos \frac{1}{x}$  or  $0 < x < x_0$  does not have a limit value for  $x \rightarrow 0+$ . Thus we have constructed a mapping  $T \notin C^1$ , which satisfies a Lipschitz condition. (It is of course possible to construct far more complicated examples).

4) Assume that there exists an  $\alpha > 1$ , such that

$$|Tx - Ty| \leq k|x - y|^\alpha.$$

Then

$$0 \leq \left| \lim_{y \rightarrow x} \frac{Tx - Ty}{x - y} \right| \leq \lim_{y \rightarrow x} k \cdot \frac{|x - y|^\alpha}{|x - y|} = k \cdot \lim_{y \rightarrow x} |y - x|^{\alpha-1} = 0.$$

This proves that  $T$  is differentiable everywhere of the derivative 0. Then  $T$  is a constant.

**Example 1.22** Let  $T$  be a mapping from a complete metric space  $(M, d)$  into itself, and assume that there is a natural number  $m$  such that  $T^m$  is a contraction. Show that  $T$  has one and only one fixed point.

If  $T^m$  is a contraction, then  $T^m$  has a fixpoint  $x$ , thus  $T^m x = x$ . When we apply  $T$  on this equation, we get

$$T^{m+1}x = T^m(Tx) = Tx,$$

hence  $Tx$  is also a fixpoint of  $T^m$ .

Since  $T^m$  is a contraction, the fixpoint is unique, so  $Tx = x$ , and we have proved that  $x$  is a fixpoint for  $T$ .

Conversely, if  $x$  is a fixpoint for  $T$ , then  $x$  is also a fixpoint for  $T^m$ , because  $Tx = x$  implies that

$$T^m x = T^{m-1}(Tx) = T^{m-1}x = \dots = Tx = x.$$

We have assumed that  $T^m$  is a contraction, hence the fixpoint for  $T^m$  is unique. This is true for every fixpoint  $x$  for  $T$ , hence it must be unique.

**Example 1.23** We consider the metric space  $\mathbb{R}^k$  with the metric

$$d_1(x, y) = \sum_{i=1}^k |x_i - y_i|$$

and a mapping  $T : \mathbb{R}^k \rightarrow \mathbb{R}^k$  given by  $Tx = Cx + b$ , where  $C = (c_{ij})$  is a  $k \times k$  matrix and  $b \in \mathbb{R}^k$ .

Show that  $T$  is a contraction, if

$$\sum_{i=1}^k |c_{ij}| < 1 \quad \text{for all } j = 1, 2, \dots, k.$$

If we instead use the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^k |x_i - y_i|^2},$$

show that  $T$  is a contraction if

$$\sum_{i=1}^k \sum_{j=1}^k |c_{ij}|^2 < 1.$$

First note that the  $i$ -th coordinate of  $Tx$  is

$$(Tx)_i = \sum_{j=1}^k c_{ij}x_j + b_i, \quad i = 1, \dots, k.$$

Put  $y = Tx$  and  $w = Tz$  and

$$\alpha = \max_{1 \leq j \leq k} \sum_{i=1}^k |c_{ij}| < 1.$$

Then we get the estimates

$$\begin{aligned} d_1(Tx, Tz) &= \sum_{i=1}^k |y_i - w_i| = \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \right| \\ &\leq \sum_{i=1}^k \sum_{j=1}^k |c_{ij}| \cdot |x_j - z_j| \leq \alpha \sum_{j=1}^k |x_j - z_j| = \alpha \cdot d_1(x, z), \end{aligned}$$

and the condition  $\alpha = \max_{1 \leq j \leq k} \sum_{i=1}^k |c_{ij}| < 1$  assures that  $T$  is a contraction in  $(\mathbb{R}^k, d_1)$ .

If instead we consider the metric

$$d_2(x, y) = \sqrt{\sum_{i=1}^k |x_i - y_i|^2},$$

and assume that

$$\alpha^2 = \sum_{i=1}^k \sum_{j=1}^k |c_{ij}|^2 < 1,$$

then we get the following estimate

$$\begin{aligned} \{d_2(x, y)\}^2 &= \sum_{i=1}^k |y_i - w_i|^2 = \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \right|^2 \\ &= \sum_{i=1}^k \left| \sum_{j=1}^k c_{ij}(x_j - z_j) \cdot \sum_{\ell=1}^k c_{i\ell}(x_\ell - z_\ell) \right| \\ &\leq \sum_{i=1}^k \sum_{j=1}^k \sum_{\ell=1}^k |c_{ij}| \cdot |x_j - z_j| \cdot |c_{i\ell}| \cdot |x_\ell - z_\ell|. \end{aligned}$$

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Then apply

$$|ab| \leq \frac{1}{2} (a^2 + b^2),$$

which follows from the inequality  $(|a| - |b|)^2 = a^2 + b^2 - 2|ab| \geq 0$ .

If we put

$$a = |c_{i\ell}| \cdot |x_j - z_j| \quad \text{and} \quad b = |c_{ij}| \cdot |x_\ell - z_\ell|,$$

we get

$$\begin{aligned} \{d_2(y, w)\}^2 &= \sum_{i=1}^k \sum_{j=1}^k \sum_{\ell=1}^k \frac{1}{2} \{ |c_{i\ell}|^2 |x_j - z_j|^2 + |c_{ij}|^2 |x_\ell - z_\ell|^2 \} \\ &= \frac{1}{2} \sum_{i=1}^k \sum_{\ell=1}^k |c_{i\ell}|^2 \cdot \sum_{j=1}^k |x_j - z_j|^2 + \frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k |c_{ij}|^2 \cdot \sum_{\ell=1}^k |x_\ell - z_\ell|^2 \\ &\leq \frac{1}{2} \alpha^2 \{d_2(x, z)\}^2 + \frac{1}{2} \alpha^2 \{d_2(x, z)\}^2 = \alpha^2 \{d_2(x, z)\}^2. \end{aligned}$$

Since  $\alpha^2 < 1$ , and hence also  $0 \leq \alpha < 1$ , and

$$d_2(y, w) = d_2(Tx, Tz) \leq \alpha \cdot d_2(x, z),$$

we conclude that  $T$  is a contraction in  $(\mathbb{R}^k, d_2)$ .

**Example 1.24** In connection with Banach's Fixpoint Theorem, the inequality

$$d(x, x_n) \leq \frac{\alpha}{1 - \alpha} d(x_{n-1}, x_n)$$

is often mentioned. Prove this inequality.

Given that  $\alpha \in ]0, 1[$ , at  $Tx_n = x_{n+1}$ , and  $x_n \rightarrow x$ .

Choose to any  $\varepsilon \in \mathbb{R}_+$  an  $N$ , such that we for all  $p \geq N$  have  $d(x, x_p) < \varepsilon$ . If  $p \geq N$  and  $p \geq n + 1$ , then

$$\begin{aligned} d(x, x_n) &\leq d(x, x_p) + d(x_p, x_n) < \varepsilon + d(x_p, x_n) \\ &\leq \varepsilon + d(x_p, x_{p-1}) + d(x_{p-1}, x_{p-2}) + \cdots + d(x_{n+1}, x_n) \\ &= \varepsilon + d(Tx_{p-1}, Tx_{p-2}) + d(Tx_{p-2}, Tx_{p-3}) + \cdots + d(Tx_n, Tx_{n-1}) \\ &\leq \varepsilon + \alpha \cdot \frac{1 - \alpha^{p-n}}{1 - \alpha} \cdot d(x_{n-1}, x_n) \\ &\leq \varepsilon + \frac{\alpha}{1 - \alpha} \cdot d(x_{n-1}, x_n). \end{aligned}$$

This is true for every  $\varepsilon > 0$ , thus

$$d(x, x_n) \leq \frac{\alpha}{1 - \alpha} \cdot d(x_{n-1}, x_n).$$

**Example 1.25** Consider the matrix equation  $Ax + b = 0$ , where  $A = (a_{ij})_{i,j=1}^k$  (and the  $a_{ij}$  real). Put  $A = C - I$  and rewrite the equation as  $x = Cx + b$ .

If

$$(5) \sum_{j=1}^k |c_{ij}| < 1 \quad \text{for } i = 1, 2, \dots, k,$$

then there is a unique solution  $x$ , which can be found by iteration.

Prove that the condition (5) can be formulated as the following condition of the  $a_{ij}$ ,

$$a_{ii} < 0, \quad |a_{ii}| > \sum_{j=1, j \neq i}^k |a_{ij}|, \quad |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|,$$

for  $i = 1, 2, \dots, k$ .

We have  $a_{ij} = c_{ij} - \delta_{ij}$ , thus  $c_{ij} = \delta_{ij} + a_{ij}$ . In particular,  $c_{ii} = 1 + a_{ii}$ . Since

$$\sum_{j=1}^k |c_{ij}| < 1,$$

we have  $|c_{ii}| < 1$ , thus  $a_{ii} \in ]-2, 0[$ .

Furthermore,  $|c_{ij}| = |a_{ij}|$  for  $i \neq j$ , so

$$\sum_{j=1}^k |c_{ij}| = \sum_{j=1, j \neq i}^k |a_{ij}| + |1 + a_{ii}| < 1.$$

It follows that

$$\sum_{j=1}^k |a_{ij}| < 1 - |1 + a_{ii}| = 1 - |1 - |a_{ii}|| \leq 1.$$

If

$$|a_{ii}| \leq 1 \quad \left( < 2 - \sum_{j=1, j \neq i}^k |a_{ij}| \right),$$

then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1 - 1 + |a_{ii}| = |a_{ii}|.$$

If

$$|a_{ii}| > 1 \quad \left( > \sum_{j=1, j \neq i}^k |a_{ij}| \right),$$



then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1 - |a_{ii}| + 1 = 2 - |a_{ii}|,$$

hence by a rearrangement,

$$|a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|,$$

and we derive in both cases that

$$\sum_{j=1, j \neq i}^k |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|.$$

Conversely, assume that  $a_{ii} < 0$  and that

$$\sum_{j=1, j \neq i}^k |a_{ij}| < |a_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ij}|.$$

Then

$$\sum_{j=1, j \neq i}^k |a_{ij}| < 1.$$

If  $|a_{ii}| \leq 1$ , then

$$|a_{ii}| = 1 - 1 + |a_{ii}| = 1 - |1 - |a_{ii}|| = 1 - |1 + a_{ii}| = 1 - |c_{ii}|,$$

thus

$$\sum_{j=1, j \neq i}^k |a_{ij}| = \sum_{j=1, j \neq i}^k |c_{ij}| < 1 - |c_{ii}|,$$

and hence

$$\sum_{j=1}^k |c_{ij}| < 1.$$

If  $|a_{ii}| > 1$ , then

$$|a_{ii}| = 1 - 1 + |a_{ii}| = 1 + ||a_{ii}| - 1| = 1 + |a_{ii} + 1| = 1 + |c_{ii}|,$$

hence by insertion

$$1 + |c_{ii}| < 2 - \sum_{j=1, j \neq i}^k |a_{ijn}| = 2 - \sum_{j=1, j \neq i}^k |c_{ij}|,$$

follows by a rearrangement

$$\sum_{j=1}^k |c_{ij}| < 1.$$

## 1.4 Simple integral equations

**Example 1.26** Consider the Volterra integral equation:

$$x(t) - \mu \int_a^t k(t, s)x(s) ds = v(t), \quad t \in [a, b],$$

where  $v \in C([a, b])$ ,  $k \in C([a, b]^2)$  and  $\mu \in \mathbb{C}$ .

Show that the equation has a unique solution  $x \in C([a, b])$  for any  $\mu \in \mathbb{C}$ .

HINT: Write the equation  $x = Tx$  where

$$Tx = v(t) + \mu \int_a^t k(t, s)x(s) ds.$$

Take  $x_0 \in C([a, b])$  and define the iteration by  $x_{n+1} = Tx_n$ , then show by induction that

$$|T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \frac{(t-a)^m}{m!} d_\infty(x, y),$$

where  $c = \max |k|$ . Then show (by looking at  $d_\infty(T^m x, T^m y)$ ) that  $T^m$  is a contraction for some  $m$  and argue that  $T$  then must have a unique fixed point in the metric space  $(C([a, b]), d_\infty)$ .

Using the given definition of  $T$  we see that the equation is equivalent with  $Tx = x$ . Then

$$\begin{aligned} |Tx(t) - Ty(t)| &= |\mu| \cdot \left| \int_a^t k(t, s)x(s) ds - \int_a^t k(t, s)y(s) ds \right| = |\mu| \cdot \left| \int_a^t k(t, s) \cdot \{x(s) - y(s)\} ds \right| \\ &\leq |\mu| \cdot c \cdot d_\infty(x - y) \cdot \left| \int_a^t 1 ds \right| = |\mu|^1 \cdot c^1 \cdot \frac{(t-a)^1}{1!} d_\infty(x, y), \end{aligned}$$

which shows that the inequality above holds for  $m = 1$ .

Assume that for some  $m \in \mathbb{N}$ ,

$$(6) \quad |T^m x(t) - T^m y(t)| \leq |\mu|^m c^m \cdot \frac{(t-a)^m}{m!} d_\infty(x, y).$$

Then

$$\begin{aligned} |T^{m+1} x(t) - T^{m+1} y(t)| &= |\mu| \cdot \left| \int_a^t k(t, s)\{T^m x(s) - T^m y(s)\} ds \right| \\ &\leq |\mu| \cdot c \int_a^t |T^m x(s) - T^m y(s)| ds \\ &\leq |\mu| \cdot c \cdot |\mu|^m \cdot c^m \cdot d_\infty(x, y) \cdot \int_a^t \frac{(s-a)^m}{m!} ds \\ &= |\mu|^{m+1} c^{m+1} \cdot d_\infty(x, y) \cdot \left[ \frac{(s-a)^{m+1}}{(m+1)!} \right]_a^t \\ &= |\mu|^{m+1} c^{m+1} \cdot \frac{(t-a)^{m+1}}{(m+1)!} \cdot d_\infty(x, y), \end{aligned}$$

and (6) follows by induction for all  $m \in \mathbb{N}$ .

We infer from (6) that

$$d_{\infty}(T^m x, T^m y) \leq |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \cdot d_{\infty}(x, y).$$

Now

$$\sum_{m=0}^{\infty} |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} = \exp(|\mu| \cdot c \cdot (b-a))$$

is convergent, thus

$$|\mu|^m c^m \cdot \frac{(b-a)^m}{m!} \rightarrow 0 \quad \text{for } m \rightarrow \infty.$$

There exists in particular an  $M \in \mathbb{N}$ , such that

$$\alpha = |\mu|^m c^m \cdot \frac{(b-a)^m}{m!} < 1 \quad \text{for all } m \geq M.$$

Thus, if  $m \geq M$ , then  $T^m$  is a contraction, and  $T^m$  has a fixpoint  $x$ . An application of EXAMPLE 1.22 shows that  $x$  is also a fixpoint for  $T$ , and  $x$  is the unique fixpoint of  $T$ .

Let  $x_0 \in C^0([a, b])$ . Define by iteration  $x_{m+1} = Tx_m$ . Then  $x_m = T^m x_0$ . The sequence  $(x_{m \cdot n})$  converges towards  $x$ . The same does the sequence  $(x_{mn+j})$ , where  $j = 0, 1, \dots, m-1$ , because

$$x_{mn+j} = T^{mn} (T^j x_0) = T^{mn+j} x_0.$$

Summing up we conclude that  $(x_n)$  itself converges towards  $x$ , and the claim is proved.

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**Example 1.27** Solve by iteration the equation

$$f(t) = u(t) = \frac{1}{2} \int_0^1 e^{t-s} f(s) ds, \quad t \in [0, 1],$$

(where  $u$  is a given continuous function), by choosing  $f_0$  as  $u$ .  
Find in particular the solutions in the cases

$$u(t) = 1, \quad u(t) = t.$$

Then solve the equation directly (without using iteration), assuming that  $u \in C^1([0, 1])$ .

If we put  $f_0(t) = u(t)$ , then

$$f_1(t) = u(t) + \frac{1}{2} \int_0^1 e^{t-s} u(s) ds = u(t) + \frac{1}{2} \left\{ \int_0^1 e^{-s} u(s) ds \right\} \cdot e^t.$$

Putting  $a = \int_0^1 e^{-s} u(s) ds$ , we get

$$f(t) = u(t) + \frac{a}{2} e^t.$$

It follows that

$$\begin{aligned} f_2(t) &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} f_1(s) ds = u(t) + \frac{1}{2} e^t \left\{ \int_0^1 e^{-s} u(s) ds + \frac{a}{2} \int_0^1 e^{-s} e^s ds \right\} \\ &= u(t) + e^t \left\{ \frac{a}{2} + \frac{a}{4} \right\} = u(t) + \frac{3}{4} a \cdot e^t. \end{aligned}$$

We conclude from the structure

$$f(t) = u(t) + e^t \left\{ \frac{1}{2} \int_0^1 e^{-s} f(s) ds \right\},$$

that a solution must have the form  $f(t) = u(t) + c \cdot e^t$ . We therefore guess that the  $n$ -th iteration may be written

$$f_n(t) = u(t) + a \cdot k_n e^t.$$

We get by insertion

$$\begin{aligned} f_{n+1}(t) &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} f_n(s) ds \\ &= u(t) + \frac{1}{2} e^t \left\{ \int_0^1 e^{-s} u(s) ds + a \cdot k_n \int_0^1 e^{-s} e^s ds \right\} \\ &= u(t) + \frac{1}{2} a e^t \left\{ \frac{1}{2} + k_n \right\} = u(t) + a \left\{ \frac{1 + k_n}{2} \right\} e^t, \end{aligned}$$

and conclude that

$$k_{n+1} = \frac{1}{2} (1 + k_n).$$

If  $k_n \in [0, 1[$ , then it follows that  $k_n < k_{n+1} < 1$ , thus  $(k_n)$  is increasing and bounded. (Notice that  $k_1 = \frac{1}{2}$ ), thus it is convergent of the limit value  $k$ . We conclude from the equation of recursions that  $k = \frac{1}{2}(1 + k)$ , thus  $k = 1$ . Hence the solution is given by

$$f(t) = u(t) + e^t \int_0^1 e^{-s} u(s) ds.$$

CHECK. We get by insertion,

$$u(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds = u(t) + \frac{1}{2} e^t \int_0^1 e^{-s} u(s) ds + \frac{1}{2} e^t \int_0^1 e^{-s} u(s) ds = f(t),$$

proving that we have found a solution.  $\diamond$

If  $u(t) = 1$ , then

$$f(t) = 1 + e^t \int_0^1 e^{-s} ds = 1 + e^t [-e^{-s}]_0^1 = 1 + \left(1 - \frac{1}{e}\right) e^t.$$

If  $u(t) = t$ , then

$$f(t) = t + e^t \int_0^1 s e^{-s} ds = t + e^t [-s e^{-s} - e^{-s}]_0^1 = t + \left(1 - \frac{2}{e}\right) e^t.$$

As mentioned above the solution must have the form  $u(t) + c \cdot e^t$ . Then by insertion,

$$\begin{aligned} u(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds &= u(t) + \frac{1}{2} \int_0^1 e^{t-s} \{u(s) + c \cdot e^s\} ds \\ &= u(t) + \frac{1}{2} \left\{ \int_0^1 e^{-s} u(s) ds + c \right\} e^t = u(t) + c \cdot e^t = f(t), \end{aligned}$$

and we conclude that  $c = \int_0^1 e^{-s} u(s) ds$ .

If  $u \in C^1([0, 1])$ , then

$$f(t) = u(t) + \left\{ \frac{1}{2} \int_0^1 e^{-s} f(s) ds \right\} \cdot e^t \in C^1,$$

so we can ALTERNATIVELY solve the equation by differentiation with respect to  $t$ . It follows from

$$\frac{1}{2} \int_0^1 e^{t-s} f(s) ds = f(t) - u(t),$$

that

$$f'(t) = u'(t) + \frac{1}{2} \int_0^1 e^{t-s} f(s) ds = f(t) + u'(t) - u(t),$$

hence by a multiplication by  $e^{-t}$  follows by a rearrangement,

$$f'(t) e^{-t} - f(t) e^{-t} = \frac{d}{dt} \{e^{-t} f(t)\} = u'(t) e^{-t} - u(t) e^{-t} = \frac{d}{dt} \{e^{-t} u(t)\},$$

and we get by an integration

$$e^{-t}f(t) = e^{-t}u(t) + c,$$

hence

$$f(t) = u(t) + c \cdot e^t.$$

The constant  $c$  is determined as above. The latter variant is of course not the shortest one.

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**Example 1.28** Let  $C^0([a, b])$  be equipped with the metric

$$d(x, y) = \max_{t \in [a, b]} |x(t) - y(t)|.$$

We define an operator (a mapping)  $S$  by

$$Sx(t) = \int_a^b k(t, s)x(s) ds,$$

where  $k$  is a continuous function on  $[a, b] \times [a, b]$ . Let  $(x_n)$  be inductively given by

$$(7) \quad x_{n+1} = u + \mu Sx_n,$$

and put  $z_n = x_n - x_{n-1}$ . Prove that (7) equivalently can be written

$$(8) \quad z_{n+1} = \mu Sz_n.$$

Put  $x_0 = u$ , and prove that (7) implies the Neumann series

$$x = \lim_{n \rightarrow \infty} x_n = u + \mu Su + \mu^2 S^2 u + \cdots.$$

We note that

$$x_{n+1}(t) = u(t) + \mu \int_a^b k(t, s)x_n(s) ds = u(t) + \mu Sx_n(t).$$

Putting  $z_n = x_n - x_{n-1}$ , we get

$$\begin{aligned} z_{n+1} &= x_{n+1} - x_n = u + \mu Sx_n - u - \mu Sx_{n-1} \\ &= \mu S(x_n - x_{n-1}) = \mu Sz_n. \end{aligned}$$

If  $|\mu| < \frac{1}{(b-a)c}$ , then  $x_n \rightarrow x$ . It follows from

$$x_n = x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} + \cdots + x_1 - x_0 + x_0 = x_0 + z_1 + \cdots + z_n,$$

and

$$z_n = \mu Sz_{n-1} = \cdots = \mu^n S^n x_0,$$

that  $\sum_n z_n$  is convergent, and we have

$$x = \lim_{n \rightarrow \infty} x_n = u + \mu Su + \mu^2 S^2 u + \cdots.$$

**Example 1.29** *Solve*

$$x(t) - \mu \int_0^1 x(s) ds = 1$$

*by means of the Neumann series, where we assume that  $|\mu| < 1$ . Try also to solve the equation directly.*

In this case,  $u(t) = 1$  and  $k(t, s) = 1$ ,  $a = 0$  and  $b = 1$ , thus  $|\mu| < 1$  is a reasonable requirement (cf. EXAMPLE 1.28). It follows from EXAMPLE 1.28 that

$$x = 1 + \mu S + \mu^2 S^2 1 + \cdots.$$

We get from  $S1 = \int_0^1 1 ds = 1$ , that  $S^2 1 = 1$ . Then by induction,  $S^n 1 = 1$ , hence

$$x = 1 + \mu + \mu^2 + \cdots = \frac{1}{1 - \mu}.$$

We now solve the equation directly. It follows from the rearrangement

$$x(t) = 1 + \mu \int_0^1 x(s) dx$$

that  $x(t) = a$  must be a constant. Then by insertion,

$$a = 1 + \mu \cdot a,$$

hence

$$x(t) = a = \frac{1}{1 - \mu},$$

which apparently holds for every  $\mu \neq 1$ , and not just for  $|\mu| < 1$ .



## 2 Banach spaces

### 2.1 Simple vector spaces

**Example 2.1** In the vector space  $C([a, b])$  we consider the functions

$$e_0(t), e_1(t), \dots, e_n(t),$$

where  $e_j(t)$  is a polynomial of degree  $j$ , where  $j = 0, 1, \dots, n$ ,  
Show that  $e_0, e_1, \dots, e_n$  are linearly independent.

Since  $e_0(t) = e_0 \neq 0$ , we infer from  $a_0 e_0 = 0$  that  $a_0 = 0$ , and the claim is true for  $k = 0$ .

First let  $e_k(t) = t^k$ , and assume that the claim is true for  $k = 0, 1, \dots, n$ . Now let

$$a_0 + a_1 t + \dots + a_n t^n + a_{n+1} t^{n+1} \equiv 0 \quad \text{for } t \in [a, b].$$

We get by a differentiation,

$$a_1 + 2a_2 t + \dots + na_n t^{n-1} + (n+1)a_{n+1} t^n \equiv 0 \quad \text{for } t \in [a, b],$$

thus  $ka_k = 0$ ,  $k = 1, 2, \dots, n+1$ , according to the assumption of induction. We conclude that  $a_k = 0$  for  $k = 1, 2, \dots, n+1$ , which by insertion gives the condition  $a_0 = 0$ . Then it follows by induction that  $\{t^n \mid n \in \mathbb{N}_0\}$  are linearly independent.

Then let

$$e_k(t) = \sum_{j=0}^k e_{kj} t^j, \quad e_{kk} \neq 0,$$

and assume that

$$0 \equiv \sum_{k=0}^n a_k e_k(t) = \sum_{k=0}^n \sum_{j=0}^k a_k e_{kj} t^j = \sum_{j=0}^n \left\{ \sum_{k=j}^n a_k e_{kj} \right\} t^j.$$

It follows from the result above that

$$\sum_{k=j}^n a_k e_{kj} = 0 \quad \text{for } j = 0, 1, \dots, n.$$

We get for  $j = n$  that  $a_n e_{nn} = 0$ , and since  $e_{nn} \neq 0$ , we must have  $a_n = 0$ . Since  $e_{k,k+j} = 0$  for  $j \geq 1$ , the equation is reduced to

$$0 \equiv \sum_{j=0}^n \left\{ \sum_{k=j}^n a_k e_{kj} \right\} t^j = \sum_{j=0}^n \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j = \sum_{j=0}^{n-1} \left\{ \sum_{k=j}^{n-1} a_k e_{kj} \right\} t^j,$$

where we as before conclude that  $a_{n-1} = 0$ . Then by recursion,

$$a_{n-2} = \dots = a_1 = a_0 = 0.$$

**Example 2.2** Let  $U_1$  and  $U_2$  be subspaces of the vector space  $V$ . Show that  $U_1 \cap U_2$  is a subspace. Is  $U_1 \cup U_2$  always a subspace? If no, state conditions such that  $U_1 \cup U_2$  is a subspace.

If  $U_1$  and  $U_2$  are subspaces, then

$$\forall \lambda \forall u, v \in U_i : u + \lambda v \in U_i, \quad i = 1, 2.$$

If  $u, v \in U_1 \cap U_2$ , then in particular,  $u, v \in U_i, i = 1, 2$ , thus  $u + \lambda v \in U_i, i = 1, 2$ , according to the above. It follows that  $u + \lambda v \in U_1 \cap U_2$ , hence  $U_1 \cap U_2$  is also a subspace.

On the other hand,  $U_1 \cup U_2$  is rarely a subspace. E.g. the  $X$ -axis and the  $Y$ -axis are two subspaces in  $\mathbb{R}^2$ , and it is obvious that the union of the two axes is not a subspace.

The condition is that  $U_1 \subseteq U_2$ , or  $U_1 \supseteq U_2$ . In fact, if one of these conditions is satisfied, then it is obvious that  $U_1 \cup U_2 = U_i$ , where  $i$  is one of the numbers 1, 2.

If this condition is not fulfilled, then there exist

$$u_1 \in U_1 \setminus U_2 \quad \text{and} \quad u_2 \in U_2 \setminus U_1.$$

Assume that  $u_1 + u_2 \in U_1 \cup U_2$ , e.g.  $u_1 + u_2 \in U_1$ . Then  $u_2 = (u_1 + u_2) - u_1 \in U_1$  contradicting the assumption. Hence we conclude that  $u_1 + u_2 \notin U_1 \cup U_2$ , and  $U_1 \cup U_2$  is not a subspace.

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**Example 2.3** Let  $V$  denote the set of all real  $n \times n$  matrices.  
 Show that  $V$  with the usual scalar multiplication and addition is a vector space.  
 Is the set of all regular  $n \times n$ -matrices a subspace of  $V$ ?  
 Is the set of all symmetric  $n \times n$  matrices a subspace of  $V$ ?

The first question is trivial: Since  $0$  is the zero element, and since  $0$  is not regular, the set of all regular matrices is not a subspace.

The set of all symmetric matrices is of course a subspace. In fact, if  $(a_{ij})$  and  $(b_{ij})$  are symmetric, thus  $a_{ij} = a_{ji}$  and  $b_{ij} = b_{ji}$ , then

$$\lambda(a_{ij}) + (b_{ij}) = (\lambda a_{ij} + b_{ij}),$$

where

$$\lambda a_{ij} + b_{ij} = \lambda a_{ji} + b_{ji},$$

hence  $(\lambda a_{ij} + b_{ij})$  is again symmetric.

**Example 2.4** In the space  $C([a, b])$  we consider the sets

$U_1 =$  the set of polynomials defined on  $[a, b]$ .

$U_2 =$  the set of polynomials defined on  $[a, b]$  of degree  $\leq n$ .

$U_3 =$  the set of polynomials defined on  $[a, b]$  of degree  $= n$ .

$U_4 =$  the set of all  $f \in C([a, b])$  with  $f(a) = f(b) = 0$ .

$U_5 = C^1([a, b])$ .

Which of the  $U_i$ ,  $i = 1, 2, \dots, 5$ , are subspaces of  $C([a, b])$ ?

$U_1 =$  the set of all polynomials is a subspace.

$U_2 =$  the set of all polynomials of degree  $\leq n$  is a subspace.

$U_3 =$  the set of all polynomials of degree  $= n$  is not a subspace. E.g.  $0$  does not belong to  $U_3$ .

$U_4 =$  the set of all  $f \in C^0([a, b])$  with  $f(a) = f(b) = 0$  is a subspace.

$U_5 = C^1([a, b])$  is a subspace.

**Example 2.5** In  $C([-1, 1])$  we consider the sets  $U_1$  and  $U_2$  consisting of the odd and even functions in  $C([-1, 1])$ , respectively.

Show that  $U_1$  and  $U_2$  are subspaces and that  $U_1 \cap U_2 = \{0\}$ .

Show that every  $f \in C([-1, 1])$  can be written in the form  $f = f_1 + f_2$ , where  $f_1 \in U_1$  and  $f_2 \in U_2$ , and that this decomposition is unique.

If  $f, g$  are odd (even) functions, then  $f + \lambda g$  is again an odd (even) function. Hence  $U_1$  and  $U_2$  are subspaces.

If  $f \in U_1 \cap U_2$ , then both

$$f(-t) = f(t) \quad \text{and} \quad f(-t) = -f(t),$$

thus  $f(t) = -f(t)$  for all  $t$ , and we conclude that  $2f(t) \equiv 0$ . We conclude that  $f \equiv 0$ .

We see from

$$f(t) = \frac{f(t) + f(-t)}{2} + \frac{f(t) - f(-t)}{2},$$

where

$$\frac{f(t) + f(-t)}{2} \text{ is even, and } \frac{f(t) - f(-t)}{2} \text{ is odd,}$$

that such a splitting exists.

Assume that

$$f(t) = f_1(t) + f_2(t) = g_1(t) + g_2(t),$$

where  $f_1$  and  $g_1$  are odd, while  $f_2$  and  $g_2$  are even. Then

$$f_1(t) - g_1(t) = g_2(t) - f_2(t) \in U_1 \cap U_2 = \{0\},$$

hence  $f_1 - g_1 = 0$  and  $g_2 - f_2 = 0$ . We conclude that  $f_1 = g_1$  and  $f_2 = g_2$ , and the splitting is unique.

## 2.2 Normed spaces

**Example 2.6** In the space  $C^1([a, b])$  we have the norm

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|.$$

Show that we could take  $\sup_{t \in (a, b)} |f(t)|$  instead.

Show that  $C^1([a, b])$  with the sup-norm is not a Banach space.

Show that

$$\|f\|_\infty^* = \sup_{t \in [a, b]} |f(t)| + \sup_{t \in [a, b]} |f'(t)|$$

is also a norm on  $C^1([a, b])$  and that it is a Banach space with this norm.

Every  $f \in C^1([a, b])$  is continuous, so

$$\sup_{t \in [a, b]} |f(t)| = \sup_{t \in (a, b)} |f(t)|,$$

and we can use any of the two sup-norms.

It follows from Weierstraß's Approximation Theorem that the set  $\mathcal{P}$  of polynomials on  $[a, b]$  is dense in  $C^0([a, b])$  in the uniform norm. Since

$$\mathcal{P} \subset C^1([a, b]) \subset C^0([a, b])$$

and  $C^1([a, b]) \neq C^0([a, b])$ , we infer that  $C^1([a, b])$  cannot be complete, thus  $(C^1([a, b]), \|\cdot\|)$  is not a Banach space.

Then we shall prove that  $\|\cdot\|_\infty^*$  is a norm.

1) Clearly,  $\|f\|_\infty^* \geq 0$ .

2) If

$$\|f\|_\infty^* = \sup_{t \in [a, b]} |f(t)| + \sup_{t \in [a, b]} |f'(t)| = \|f\|_\infty + \|f'\|_\infty = 0,$$

then in particular  $\|f\| = 0$ , so  $f = 0$ , because  $f$  is continuous.

3)

$$\|\lambda f\|_\infty^* = \|\lambda f\|_\infty + \|\lambda f'\|_\infty = |\lambda|(\|f\|_\infty + \|f'\|_\infty) = |\lambda| \cdot \|f\|_\infty^*.$$

4)

$$\begin{aligned} \|f + g\|_\infty^* &= \|f + g\|_\infty + \|f' + g'\|_\infty \leq \|f\|_\infty + \|g\|_\infty + \|f'\|_\infty + \|g'\|_\infty \\ &= (\|f\|_\infty + \|f'\|_\infty) + (\|g\|_\infty + \|g'\|_\infty) = \|f\|_\infty^* + \|g\|_\infty^*. \end{aligned}$$

We have proved that  $\|\cdot\|_\infty^*$  is a norm on  $C^1([a, b])$ .

It “only” remains to prove that  $(C^1([a, b]), \|\cdot\|_\infty^*)$  is a Banach space.  
Let  $(f_n)$  be a Cauchy sequence, i.d.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \in \mathbb{N} : m, n \geq N \implies \|f_m - f_n\|_\infty^* < \varepsilon.$$

It follows from  $\|f\|_\infty^* = \|f\|_\infty + \|f'\|_\infty$ , that  $\|f\|_\infty \leq \|f\|_\infty^*$  and  $\|f'\|_\infty \leq \|f\|_\infty^*$ , thus  $(f_n)$  and  $(f'_n)$  are Cauchy sequences in the Banach space  $(C^0([a, b]), \|\cdot\|_\infty)$ . Hence there are continuous functions  $f, g \in C^0([a, b])$ , such that

$$f_n \rightarrow f \quad \text{and} \quad f'_n \rightarrow g.$$

Notice that it is not possible from this directly to conclude that

$$a) \ f \in C^1([a, b]), \quad b) \ f' = g.$$

A proof is required:

Define a function  $h \in C^1([a, b])$  by

$$h(x) = \int_a^x g(t) dt + f(a), \quad x \in [a, b].$$

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We shall prove that  $h(x) = f(x)$ . It suffices to prove that  $f_n \rightarrow h$  uniformly, because the limit function  $f \in C^0([a, b])$  is unique. From  $f_n \in C^1([a, b])$  follows that

$$f_n(x) = \int_a^x f'_n(t) dt + f_n(a), \quad x \in [a, b],$$

hence for every  $x \in [a, b]$ ,

$$\begin{aligned} |f_n(x) - h(x)| &= \left| \int_a^x f'_n(t) dt + f_n(a) - \int_a^x g(t) dt - f(a) \right| \\ &\leq \left| \int_a^x \{f'_n(t) - g(t)\} dt \right| + |f_n(a) - f(a)|. \end{aligned}$$

Let  $\varepsilon > 0$  be given. Since  $f_n(a) \rightarrow f(a)$ , and  $f'_n \rightarrow g$  uniformly for  $n \rightarrow +\infty$ , there exists an  $n_0 \in \mathbb{N}$ , such that for every  $n \geq n_0$ ,

$$|f_n(a) - f(a)| < \frac{\varepsilon}{2} \quad \text{and} \quad \sup_{t \in [a, b]} |f'_n(t) - g(t)| < \frac{\varepsilon}{2(b-a)}.$$

Therefore, if  $n \geq n_0$ , then for every  $x \in [a, b]$ ,

$$|f_n(x) - h(x)| < \left| \int_a^x \frac{\varepsilon}{2(b-a)} dt \right| + \frac{\varepsilon}{2} \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus

$$\|f_n - h\|_\infty < \varepsilon \quad \text{for all } n \geq n_0,$$

and we have proved that  $f_n \rightarrow h$  uniformly, hence  $f = h$ . Finally, since  $h' = g$ , the claim is proved.

**Example 2.7** Let  $f \in C([a, b])$  and consider the  $p$ -norms

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}, \quad p \geq 1,$$

and

$$\|f\|_\infty = \sup_{t \in [a, b]} |f(t)|.$$

Show that  $\|f\|_p \rightarrow \|f\|_\infty$  for  $p \rightarrow \infty$ .

The interval  $[a, b]$  is bounded, so

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}} \leq \left\{ \int_a^b \|f\|_\infty^p dt \right\}^{\frac{1}{p}} = \|f\|_\infty (b-a)^{\frac{1}{p}}.$$

The function  $f$  is continuous and  $[a, b]$  is compact, hence there exists a  $t_0 \in [a, b]$ , such that

$$|f(t_0)| = \|f\|_\infty.$$

To every  $\varepsilon > 0$  we can find an interval  $[c_\varepsilon, d_\varepsilon] \subseteq [a, b]$ ,  $c_\varepsilon < d_\varepsilon$  (independently of  $p$ ), such that

$$|f(t)| \geq (1 - \varepsilon)\|f\|_\infty \quad \text{for all } t \in [c_\varepsilon, d_\varepsilon].$$

Then we get the estimate

$$\begin{aligned} \|f\|_p &= \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}} \geq \left\{ \int_{c_\varepsilon}^{d_\varepsilon} |f(t)|^p dt \right\}^{\frac{1}{p}} \geq \left\{ (1 - \varepsilon)^p \|f\|_\infty^p \int_{c_\varepsilon}^{d_\varepsilon} dt \right\}^{\frac{1}{p}} \\ &= (1 - \varepsilon)\|f\|_\infty \cdot (d_\varepsilon - c_\varepsilon)^{\frac{1}{p}}. \end{aligned}$$

Summing up we get for every  $\varepsilon > 0$  that

$$(1 - \varepsilon)\|f\|_\infty \cdot (d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \leq \|f\|_p \leq \|f\|_\infty \cdot (b - a)^{\frac{1}{p}}.$$

If  $k > 0$  is kept fixed, we have  $k^{\frac{1}{p}} \rightarrow 1$  for  $p \rightarrow \infty$ . To every  $\varepsilon > 0$  there exists a  $P_\varepsilon > 0$ , such that for every  $p \geq P_\varepsilon$ ,

$$(d_\varepsilon - c_\varepsilon)^{\frac{1}{p}} \geq 1 - \varepsilon \quad \text{and} \quad (b - a)^{\frac{1}{p}} \leq 1 + \varepsilon,$$

hence

$$(1 - \varepsilon)^2 \|f\|_\infty \leq \|f\|_p \leq (1 + \varepsilon)\|f\|_\infty \quad \text{for every } p \geq P_\varepsilon.$$

This proves that  $\lim_{p \rightarrow +\infty} \|f\|_p$  exists and that

$$\lim_{p \rightarrow +\infty} \|f\|_p = \|f\|_\infty.$$

**Example 2.8** Let  $V$  be a normed vector space and let  $x_1, \dots, x_k$  be  $k$  linearly independent vectors from  $V$ . Show that there exists a positive constant  $m$ , such that for all scalars  $\alpha_i \in \mathbb{C}$ ,  $i = 1, \dots, k$ , we have

$$\|\alpha_1 x_1 + \dots + \alpha_k x_k\| \geq m(|\alpha_1| + \dots + |\alpha_k|).$$

*Indirect proof.* We assume that there exists a sequence  $(y_m)$ , where

$$y_m = \sum_{i=1}^k \beta_i^{(m)} x_i, \quad \text{where } \sum_{i=1}^k |\beta_i^{(m)}| = 1 \text{ for all } m \in \mathbb{N},$$

and where  $\|y_m\| \rightarrow 0$  for  $m \rightarrow +\infty$ . Under these assumptions we first notice that  $|\beta_i^{(m)}| \leq 1$ , such that  $\left(\beta_i^{(m)}\right)_{m=1}^{+\infty}$  is a bounded sequence of complex numbers. The complex numbers  $\mathbb{C}$  being complete in the absolute value, there exists a convergent subsequence

$$\left(\beta_1^{(m_j^1)}\right)_{j=1}^{+\infty} \quad \text{af} \quad \left(\beta_1^{(m)}\right).$$

The trick is first to thin out  $(\beta_2^{(m)})$  to the subsequence  $\left(\beta_1^{(m_j^1)}\right)$ , where  $(m_j^1)$  is given above.



Then thin it out once more to get a convergent subsequence

$$\left(\beta_2^{(m_j^2)}\right) \quad \text{of} \quad \left(\beta_2^{(m_j^1)}\right).$$

Because  $(m_j^2)$  is a subsequence of  $(m_j^1)$ , the subsequence  $\left(\beta_1^{(m_j^2)}\right)$  is also convergent.

Continue in this way. After  $k$  steps we have obtained a subsequence  $(m_j)$  from  $\mathbb{N}$ , such that

$$\left(\beta_i^{(m_j)}\right)_{j=1}^{+\infty} \quad \text{is convergent for all } i = 1, 2, \dots, k.$$

This means that  $(y_{m_j})$  is a convergent subsequence of  $(y_m)$ , hence

$$y_{m_j} \rightarrow y \quad \text{for } j \rightarrow +\infty,$$

and

$$y = \sum_{i=1}^k \beta_i x_i.$$

We conclude from

$$\sum_{i=1}^k |\beta_i| \geq \sum_{i=1}^k \left| \beta_i^{(m_j)} \right| - \sum_{i=1}^k \left| \beta_i^{(m_j)} - \beta_i \right| = 1 - \sum_{i=1}^k \left| \beta_i^{(m_j)} - \beta_i \right| \rightarrow 1, \quad \text{for } j \rightarrow +\infty,$$

and from the assumption that  $x_1, \dots, x_k$  are linearly independent that  $y \neq 0$ . This is contradicting the assumption that  $\|y_m\| \rightarrow 0$  for  $m \rightarrow +\infty$ .

We infer that if  $\sum_{i=1}^k |\beta_i| = 1$ , then there is a constant  $c > 0$ , such that

$$\left\| \sum_{i=1}^k \beta_i x_i \right\| \geq c.$$

We put for  $(\alpha_1, \dots, \alpha_k) \neq (0, \dots, 0)$ ,

$$\beta_i = \frac{\alpha_i}{|\alpha_1| + \dots + |\alpha_k|}.$$

Then the claim follows when we multiply by  $|\alpha_1| + \dots + |\alpha_k| \neq 0$ .

Finally, we notice that the case  $\alpha_1 = \dots = \alpha_k = 0$  follows trivially for quite other reasons.

**Example 2.9** Let  $V$  be a vector space and let  $\|\cdot\|$  and  $|||\cdot|||$  be two norms on  $V$ . The norms are said to be equivalent if there are positive constants  $m$  and  $M$  such that

$$m\|x\| \leq |||x||| \leq M\|x\|$$

for all  $x \in V$ .

Show that all norms on a finite dimensional vector space are equivalent.

Show that all equivalent norms define the same closed sets.

Let  $e_1, \dots, e_k$  be a basis for  $V$ . It follows from EXAMPLE 2.8 that there are constants  $c_1 > 0$  and  $c_2 > 0$ , such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c_1 \sum_{i=1}^k |\alpha_i| \quad \text{and} \quad \left| \left| \sum_{i=1}^k \alpha_i e_i \right| \right| \geq c_2 \sum_{i=1}^k |\alpha_i|.$$

Writing  $x = \sum_{i=1}^k \alpha_i e_i$ , we get

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^k \alpha_i e_i \right\| \leq \sum_{i=1}^k |\alpha_i| \cdot \|e_i\| \leq \max_{1 \leq i \leq k} \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \leq \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \left| \left| \sum_{j=1}^k \alpha_j e_j \right| \right| \\ &= \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot |||x||| \leq \frac{1}{c_2} \max_{1 \leq i \leq k} 1 \leq k \|e_i\| \cdot \sum_{j=1}^k |\alpha_j| \cdot |||e_j||| \\ &\leq \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \max_{1 \leq j \leq k} |||e_j||| \cdot \sum_{\ell=1}^k |\alpha_\ell| \leq \frac{1}{c_1} \cdot \frac{1}{c_2} \max_{1 \leq i \leq k} \|e_i\| \cdot \max_{1 \leq j \leq k} |||e_j||| \cdot |||x|||. \end{aligned}$$



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Thus we have proved that

$$\|x\| \leq a \cdot \|x\| \leq b \cdot \|x\|,$$

where

$$a = \frac{1}{c_2} \max_{i \leq i \leq k} \|e_i\| > 0 \quad \text{and} \quad b = a \cdot \frac{1}{c_1} \max_{1 \leq j \leq k} \|e_j\| > 0.$$

When we divide by  $a > 0$ , we get

$$m\|x\| = \frac{1}{a} \|x\| \leq \|x\| \leq \frac{b}{a} \|x\| = M\|x\|,$$

and we have proved that any two norms on a finite dimensional subspace are equivalent.

Since

$$m\|x\| \leq \|x\| \leq M\|x\|, \quad 0 < m \leq M,$$

and

$$\frac{1}{M} \|x\| \leq \|x\| \leq \frac{1}{m} \|x\|,$$

are equivalent, it suffices to prove that if  $U$  is closed with respect to  $\|\cdot\|$ , then  $U$  is also closed with respect to  $\|\cdot\|$ .

It is well-known (cf. EXAMPLE 1.10) that  $U$  is closed, if and only if

$$x_n \in U \text{ and } x_n \rightarrow x \implies x \in U.$$

Assume that  $U$  is closed with respect to  $\|\cdot\|$ , and let  $(x_n) \subseteq U$  be a sequence for which

$$\|x_n\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

thus  $(x_n)$  is convergent with respect to the norm  $\|\cdot\|$ . We shall prove that  $x \in U$ . However,

$$\|x_n - x\| \leq \frac{1}{m} \|x_n - x\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

so also  $x_n \rightarrow x$  with respect to the norm  $\|\cdot\|$ . It follows from the condition of EXAMPLE 1.10 (applied with respect to  $\|\cdot\|$ ) that  $x \in U$ , and the claim is proved.

**Example 2.10** *Show that a compact set in a normed vector space  $V$  is closed and bounded. If  $V$  is finite dimensional, show that a closed and bounded set is compact.*

Assume that  $U$  is compact in  $V$ , i.e. every sequence  $(x_n) \subseteq U$  has a subsequence  $(y_n)$ , which converges towards an element  $y$  in  $U$ . We shall prove that  $U$  is closed and bounded.

Assume that  $(x_n) \subseteq U$  is convergent in  $V$ , thus  $x_n \rightarrow x \in V$ . It follows from EXAMPLE 1.10 that  $U$  is closed, if we can prove that also  $x \in U$ .

According to the assumption there is a subsequence  $(y_n)$  of  $(x_n)$ , such that  $y_n \rightarrow y \in U$ . However, since  $x_n \rightarrow x$ , also  $y_n \rightarrow x$ , and since the limit value is unique in normed spaces, we conclude that  $x = y \in U$ , and it follows that  $U$  is closed.

Then we shall prove that if  $U$  is compact, then  $U$  is bounded. *Indirect proof.* Assume that  $U$  is unbounded. Let  $x_1 \in U$  be arbitrarily chosen. There exists an  $x_2 \in U$ , such that

$$\|x_2\| \geq 1 + \|x_1\|.$$

Choose inductively a sequence  $(x_n) \subseteq U$ , such that

$$\|x_{n+1}\| \geq 1 + \|x_n\|.$$

Then note that if  $x_n$  and  $x_{n+p}$ ,  $p \in \mathbb{N}$  are any two elements, then

$$\|x_{n+p}\| \geq 1 + \|x_{n+p-1}\| \geq 2 + \|x_{n+p-2}\| \geq \cdots \geq p + \|x_n\|,$$

hence

$$\|x_{n+p} - x_n\| \geq \|x_{n+p}\| - \|x_n\| \geq 0 \geq 1 \quad \text{for alle } p \in \mathbb{N},$$

proving that no subsequence of  $(x_n)$  is convergent, and  $U$  is not compact.

We get by contraposition that if  $U$  is compact, then  $U$  is bounded.

Assume now that  $V$  is finite dimensional and that  $U$  is bounded and closed. Let  $e_1, \dots, e_k$  denote a basis for  $V$ , and let the constant  $c > 0$  be chosen as in EXAMPLE 2.8, such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c(|\alpha_1| + \cdots + |\alpha_k|) = c \sum_{i=1}^k |\alpha_i|.$$

Let  $x_n \in U$ ,  $x_n = \sum_{i=1}^k \alpha_i^n e_i$ , be any sequence. It follows from  $U$  being bounded that  $\|x\| \leq B$  for every  $x \in U$ , i.e.

$$|\alpha_i| \leq \sum_{i=1}^k |\alpha_i| \leq \frac{1}{c} \left\| \sum_{i=1}^k \alpha_i e_i \right\| \leq \frac{B}{c}$$

for all  $i = 1, \dots, k$ . Hence the sequence  $(\alpha_1^n)_n$  is bounded, and it has therefore a convergent subsequence  $\left( \alpha_1^{n_j^1} \right)$ .

Since  $\left( \alpha_2^{n_j^1} \right)$  is a bounded sequence, it has a convergent subsequence  $\left( \alpha_2^{n_j^2} \right)$ , etc..

After  $k$  steps we have found a sequence  $(n_j)$ , for which  $(\alpha_i^{n_j})_j$  is convergent for  $j \rightarrow +\infty$  for every  $i = 1, \dots, k$ , of limit value  $\alpha_i$ .

Putting

$$y_j = \sum_{i=1}^k \alpha_i^{n_j} e_i,$$

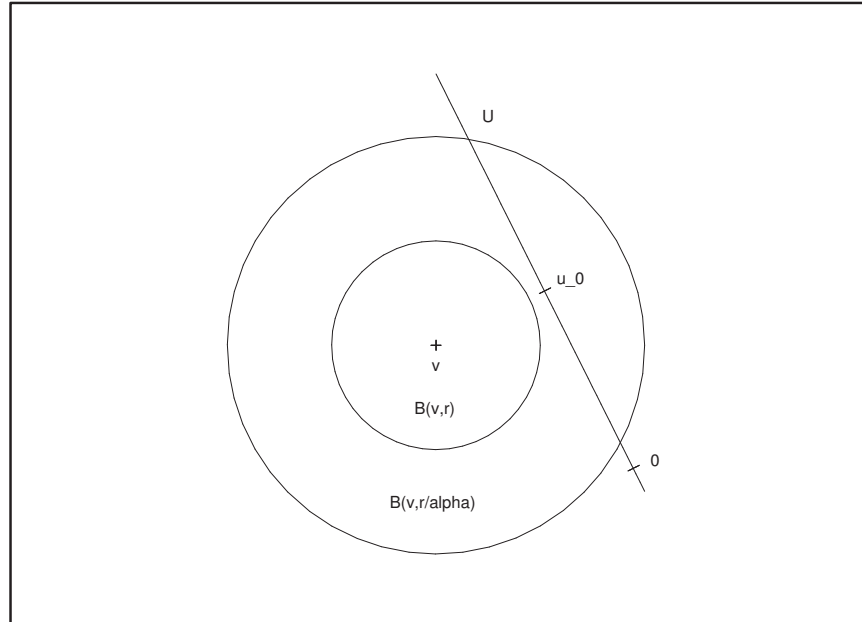
we get that  $(y_j)$  is convergent of limit

$$y_j \rightarrow y = \sum_{i=1}^k \alpha_i e_i.$$

Since  $y_j \in U$ , and  $U$  is closed, we get  $y \in U$  according to EXAMPLE 1.10, and the claim is proved.

**Example 2.11** *Riesz's lemma. Let  $V$  be a normed vector space and let  $U$  be a closed subspace of  $V$ ,  $U \neq V$ . Let  $\alpha$ ,  $0 < \alpha < 1$ , be given. Show that there is a  $v \in V$ , such that*

$$\|v\| = 1 \quad \text{and} \quad \|v - u\| \geq \alpha \quad \text{for all } u \in U.$$



It follows from  $U \neq V$ , that there exists a  $v \in V \setminus U$ .

The set  $U$  is closed, so  $V \setminus U$  is open. Hence there exists an  $r > 0$ , such that  $B(v, r) \cap U = \emptyset$ , where  $B(v, r)$  denotes the open ball of centre  $v$  and radius  $r$ . This means that

$$(9) \quad \|v - u\| \geq r \quad \text{for all } u \in U.$$

Choose  $r$  sufficiently large such that (cf. the figure)

$$B(v, r) \cap U = \emptyset \quad \text{and} \quad B\left(v, \frac{1}{\alpha} r\right) \cap U \neq \emptyset.$$

Then for every  $u_0 \in B\left(v, \frac{1}{\alpha} r\right) \cap U$ ,

$$(10) \quad r \leq \|v - u_0\| \leq \frac{1}{\alpha} r.$$

If we put

$$w = \frac{v - u_0}{\|v - u_0\|},$$

then  $\|w\| = 1$ .

We have for any  $u \in U$  that

$$\|w - u\| = \left\| \frac{v - u_0}{\|v - u_0\|} - u \right\| = \frac{1}{\|v - u_0\|} \|v - u_0 - \|v - u_0\| u\|.$$

Now  $u, u_0 \in U$ , and  $U$  is a subspace, hence  $u_0 + \|v - u_0\| u \in U$ . By applying (9) with  $u_0 + \|v - u_0\| u$  instead of  $u$ , it follows from (10) that

$$\|w - u\| = \frac{1}{\|v - u_0\|} \|v - (u_0 + \|v - u_0\| u)\| \geq \frac{r}{\|v - u_0\|} \geq \frac{r}{\frac{1}{\alpha} r} = \alpha.$$

We have proved that  $w \in V$  satisfies

$$\|w\| = 1 \quad \text{and} \quad \|w - u\| \geq \alpha \quad \text{for every } u \in U.$$

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**Example 2.12** In  $\ell^\infty$ , the vector space of bounded sequences, we consider the sets  $U_1$  and  $U_2$ , where  $U_1$  denotes the set of sequences with only finitely many elements different from 0 and  $U_2$  the set of sequences with all but the  $N$  first elements equal to 0.

Are  $U_1$  and/or  $U_2$  closed subspaces in  $\ell^\infty$ ?

Are  $U_1$  and/or  $U_2$  finite dimensional?

It follows from

$$x_n = \left(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}, 0, 0, \dots\right) \in U_1,$$

and

$$x_n \rightarrow \left(\frac{1}{k}\right)_{k \in \mathbb{N}} \notin U_1,$$

that  $U_1$  is not closed.

Of course  $U_1$  is a subspace, and since every *finite dimensional* subspace is closed (which  $U_1$  is not), we conclude that  $U_1$  is not finite dimensional.

On the other hand,  $U_2$  and  $\mathbb{R}^N$  are isomorphic, so  $U_2$  is a closed and finite dimensional vector space,  $\dim U_2 = N$ .

**Example 2.13** Let  $(V, \|\cdot\|)$  be a normed vector space, and let  $U$  be the unit ball,

$$U = \{x \in V \mid \|x\| \leq 1\}.$$

Prove that  $U$  is compact, if and only if  $V$  is finite dimensional.

Obviously,  $U$  is closed and bounded. If  $V$  is finite dimensional, then it follows from EXAMPLE 2.10 that  $U$  is compact. It remains to be proved that if  $U$  is compact, then  $V$  is finite dimensional.

INDIRECT PROOF. Assume that  $V$  is not finite dimensional. Choose any  $x_1 \in U$ , such that  $\|x_1\| = 1$ . Then  $x_1$  generates a subspace  $V_1$ . Then by Riesz's lemma (EXAMPLE 2.11) there exists an  $x_2 \in U$ , such that

$$\|x_2\| = 1 \quad \text{and} \quad \|x_2 - \lambda x_1\| \geq \frac{1}{2} \quad \text{for all } \lambda.$$

By induction, using Riesz's lemma in each step, we obtain a sequence  $x_n \in U$  of unit vectors,  $\|x_n\| = 1$ , such that

$$\left\|x_n - \sum_{j=1}^{n-1} \lambda_j x_j\right\| \geq \frac{1}{2} \quad \text{for any } \lambda_j.$$

We have in particular,

$$\|x_n - x_m\| \geq \frac{1}{2} \quad \text{for } n \neq m,$$

proving that  $(x_n)$  does not contain any convergent subsequence. Hence  $U$  is not compact.

We get by contraposition that if the unit ball  $U$  is compact, then the vector space  $V$  is finite dimensional.

**Example 2.14** Consider in  $\ell^p$  (where  $1 \leq p \leq +\infty$ ) the subspace  $U$  consisting of all sequences which are 0 eventually.

- 1) If  $1 \leq p < +\infty$ , is the subspace  $U$  then dense in  $\ell^p$ ?
- 2) If  $p = +\infty$ , is the subspace  $U$  then dense in  $\ell^\infty$ ?

1) The answer is 'yes'. In fact, if  $(x_j)_{j \in \mathbb{N}} \in \ell^p$ , then

$$\sum_{j=1}^{+\infty} |x_j|^p < +\infty.$$

To every  $\varepsilon > 0$  there is an  $N$ , such that

$$\sum_{j=N+1}^{+\infty} |x_j|^p < \varepsilon^p.$$

Putting  $x^N = (x_1, \dots, x_N, 0, 0, \dots) \in U$ , we get

$$\|x - x^N\|_p = \left\{ \sum_{j=N+1}^{+\infty} |x_j|^p \right\}^{\frac{1}{p}} < \{\varepsilon^p\}^{\frac{1}{p}} = \varepsilon.$$

2) In this case the answer is 'no'. In fact, if  $x = (1, 1, 1, \dots) \in \ell^\infty$ , then

$$\|x - y\|_\infty \geq 1 \quad \text{for every } y \in U.$$

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**Example 2.15** On  $C([a, b])$  we introduce the norm

$$\|f\|_p = \left\{ \int_a^b |f(t)|^p dt \right\}^{\frac{1}{p}}, \quad p \in ]1, +\infty[.$$

Let  $g \in C([a, b])$ , and let  $q$  be given by  $\frac{1}{p} + \frac{1}{q} = 1$ . Prove that we by

$$T_g f = \int_a^b f(t) \overline{g(t)} dt$$

define a linear functional on  $C([a, b])$ , and that

$$\|T_g\| = \|g\|_q \quad \left( = \left\{ \int_a^b |g(t)|^q dt \right\}^{\frac{1}{q}} \right).$$

Most of the claims have already been proved, included the estimate  $\|T_g\| \leq \|g\|_1$ . We shall only proof that we even get equality. The trick is to choose a suitable  $f \in C([a, b])$ . We have

$$T_g f = \int_a^b f(t) \overline{g(t)} dt.$$

Since  $g(t)$  is continuous, we get

$$g(t) = e^{i\varphi(t)} |g(t)|,$$

where  $\varphi(t)$  can be chosen continuous in every interval, in which  $g(t) \neq 0$ .

Choosing

$$f(t) = e^{i\varphi(t)} |g(t)|^{\frac{q}{p}},$$

$f$  is again continuous and

$$\|f\|_p^p = \int_a^b |g(t)|^q dt = \|g\|_q^q, \quad \text{thus} \quad \|f\|_p = \|g\|_q^{\frac{q}{p}} = \|g\|_q^{q-1},$$

and

$$\begin{aligned} T_g f &= \int_a^b f(t) \overline{g(t)} dt = \int_a^b e^{i\varphi(t)} |g(t)|^{\frac{q}{p}} e^{-i\varphi(t)} |g(t)| dt \\ &= \int_a^b |g(t)|^{\frac{q}{p}+1} dt = \int_a^b |g(t)|^{q(\frac{1}{p}+\frac{1}{q})} dt = \int_a^b |g(t)|^q dt \\ &= \|g\|_q^q = \|g\|_q \cdot \|g\|_q^{q-1} = \|g\|_q \cdot \|f\|_p. \end{aligned}$$

It follows from

$$|T_g f| = T_g f = \|g\|_q \|f\|_p \leq \|T_g\| \cdot \|f\|_p,$$

that  $\|g\|_q \leq \|T_g\|$ . Since already  $\|T_g\| \leq \|g\|_q$ , we must have  $\|T_g\| = \|g\|_q$ .

### 2.3 Banach spaces

**Example 2.16** Show that a closed subspace of a Banach space is itself a Banach space.

Let  $U$  be a closed subspace of a Banach space  $V$ . Since  $V$  is complete, it follows from EXAMPLE 1.10 that  $U$  is also complete, hence  $U$  is a Banach space.

**Example 2.17** Let  $V_i, i = 1, 2, \dots, n$ , be normed vector spaces, with norms  $\|\cdot\|_i, i = 1, 2, \dots, n$ . The product space  $V_1 \times V_2 \times \dots \times V_n = \bigotimes_{i=1}^n V_i$  is defined by

$$\bigotimes_{i=1}^n V_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in V_i, i = 1, 2, \dots, n\}.$$

In  $\bigotimes_{i=1}^n V_i$  we use coordinate wise addition:

$$(x_1, x_2, \dots, x_n) + (y_1, y_2, \dots, y_n) = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n),$$

and scalar multiplication:

$$\lambda(x_1, x_2, \dots, x_n) = (\lambda x_1, \lambda x_2, \dots, \lambda x_n),$$

and we define the norm by

$$\|(x_1, x_2, \dots, x_n)\| = \sum_{i=1}^n \|x_i\|_i.$$

Show that  $\bigotimes_{i=1}^n V_i$  with this norm is a normed vector space, and show that if all the spaces  $V_i$  with their respective norms are Banach spaces, then  $\bigotimes_{i=1}^n V_i$  is a Banach space.

We shall prove the claim by induction over  $n$ . For  $n = 1$  there is nothing to prove.

If  $n = 2$ , then clearly  $V_1 \times V_2$  is a vector space with the operations addition and scalar multiplication defined above. Then we shall prove that

$$\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2$$

is a norm.

Clearly,  $\|(x_1, x_2)\| \geq 0$ , and if  $\|(x_1, x_2)\| = \|x_1\|_1 + \|x_2\|_2 = 0$ , then both  $\|x_1\|_1 = 0$  and  $\|x_2\|_2 = 0$ , thus  $x_1 = 0$  og  $x_2 = 0$ .

Furthermore,

$$\|\lambda(x_1, x_2)\| = \|(\lambda x_1, \lambda x_2)\| = \|\lambda x_1\|_1 + \|\lambda x_2\|_2 = |\lambda|(\|x_1\|_1 + \|x_2\|_2) = |\lambda| \cdot \|(x_1, x_2)\|.$$

Finally,

$$\begin{aligned} \|(x_1, x_2) + (y_1, y_2)\| &= \|(x_1 + y_1, x_2 + y_2)\| = \|x_1 + y_1\|_1 + \|x_2 + y_2\|_2 \\ &\leq \|x_1\|_1 + \|y_1\|_1 + \|x_2\|_2 + \|y_2\|_2 \\ &= (\|x_1\|_1 + \|x_2\|_2) + (\|y_1\|_1 + \|y_2\|_2) \\ &= \|(x_1, x_2)\| + \|(y_1, y_2)\|, \end{aligned}$$

and we have proved that  $\|\cdot\|$  is a norm on  $V_1 \times V_2$ .

Then assume that both  $V_1$  and  $V_2$  are complete, and let  $((x_1^n, x_2^n))_n$  be a Cauchy sequence on  $V_1 \times V_2$ . It follows from

$$\|x_i^n - x_i^m\|_i \leq \|(x_1^n - x_1^m, x_2^n - x_2^m)\| = \|(x_1^n, x_2^n) - (x_1^m, x_2^m)\|, \quad t = 1, 2,$$

that  $(x_i^n)_n$  are Cauchy sequences on  $V_i$ ,  $i = 1, 2$ , hence convergent with limit values  $x_i$ ,  $i = 1, 2$ . By this construction we then get

$$\|(x_1, x_2) - (x_1^n, x_2^n)\| = \|x_1 - x_1^n\|_1 + \|x_2 - x_2^n\|_2 \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

proving that  $(x_1^n, x_2^n) \rightarrow (x_1, x_2) \in V_1 \times V_2$ . We have proved that  $V_1 \times V_2$  is complete, thus  $(V_1 \times V_2, \|\cdot\|)$  is a Banach space.

Assume that the claims are true for some  $n \in \mathbb{N}$  (this is true by the above for  $n = 1$  and for  $n = 2$ ), and consider  $\bigotimes_{i=1}^{n+1} U_i$ , where each  $U_i$  is a normed vector space (a Banach space). We define

$$V_1 = \bigotimes_{i=1}^n U_i \quad \text{and} \quad V_2 = U_{n+1}.$$

It follows from the assumption of the induction that  $(V_1, \|\cdot\|_n^*)$  is a normed vector space (or a Banach space) under the given assumptions, and the same is true for the space  $(V_2, \|\cdot\|_{n+1})$ . It only remains to notice that

$$\|(x_1, x_2, \dots, x_n)_n\|_n^* = \|x_1\|_1 + \|x_2\|_2 + \dots + \|x_n\|_n,$$

hence

$$\|(x_1, \dots, x_n, x_{n+1})\| = \|(x_1, \dots, x_n)_n\|_n^* + \|x_{n+1}\|_{n+1}.$$

It follows that  $\bigoplus_{i=1}^{n+1} U_i$  is a normed vector space (or a Banach space) under the given assumptions.

**Example 2.18** Assume that  $V$  and  $U$  are normed spaces and  $f : V \rightarrow U$  is a continuous mapping, and assume that  $X \subset V$  is a compact subset. Show that the image  $f(X) \subset U$  is compact. Show that a real function attains both maximum and minimum on a compact set.

There are several definitions of compactness. We shall here use *sequential compactness*, which is defined by  $X$  being sequential compact, if every sequence on  $X$  has a convergent subsequence.

We shall prove that if  $f : V \rightarrow U$  is continuous, and  $X \subset V$  is compact, then the image  $f(X) \subset U$  is also compact.

Let  $(y_n) \subset f(X)$  be any sequence on the image  $f(X)$ . There exists a sequence  $(x_n) \subset X$ , such that  $y_n = f(x_n)$  for every  $n \in \mathbb{N}$ . Since  $X$  is compact,  $(x_n)$  has a convergent subsequence  $(x'_n) \subseteq (x_n)$ , where  $x'_n \rightarrow x_0 \in X$  for  $n \rightarrow +\infty$ .

Now,  $f$  is continuous at  $x_0 \in X$ , so to every  $\varepsilon > 0$  there exists a  $\delta > 0$ , such that

$$\|f(x'_n) - f(x_0)\|_U < \varepsilon \quad \text{for } \|x'_n - x_0\|_V < \delta.$$

Then  $(x'_n) \rightarrow x_0$  implies that there exists an  $n_0 \in \mathbb{N}$ , such that

$$\|x'_n - x_0\|_V < \delta \quad \text{for all } n \geq n_0.$$

We have for the same  $n_0$  that

$$\|f(x'_n) - f(x_0)\|_U < \varepsilon \quad \text{for all } n \geq n_0,$$

which means that  $(f(x'_n))$  converges towards  $f(x_0)$ , thus every sequence  $(y_n) = (f(x_n)) \subseteq f(X)$  has a convergent subsequence  $(y'_n) = (f(x'_n))$ . Note for the limit point that  $f(x_0) \in f(X)$ .

Assume that  $f : X \rightarrow \mathbb{R}$  is continuous, where  $X$  is a compact subset of a normed space. It follows from the above that  $f(X) \subseteq \mathbb{R}$  is compact, thus closed and bounded in  $\mathbb{R}$ . In particular,  $f$  has both a maximum value and a minimum value.

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**Example 2.19** *Show that any finite dimensional subspace of a normed vector space is a Banach space.*

Let  $(V, \|\cdot\|)$  be the normed space, and let  $U$  be a finite dimensional subspace of  $V$ . Let  $e_1, \dots, e_k$ , denote a basis for  $U$ . It follows from EXAMPLE 2.8 that there exists a constant  $c > 0$  (corresponding to the basis  $e_1, \dots, e_k$ ), such that

$$\left\| \sum_{i=1}^k \alpha_i e_i \right\| \geq c(|\alpha_1| + \dots + |\alpha_k|).$$

Let  $x^n = \sum_{i=1}^k \alpha_i^n e_i$  denote a Cauchy sequence on  $U$ , thus

$$\forall \varepsilon > 0 \exists N \forall m, n \geq N : \|x^m - x^n\| = \left\| \sum_{i=1}^k (\alpha_i^m - \alpha_i^n) e_i \right\| < \varepsilon.$$

Then in particular,

$$|\alpha_i^m - \alpha_i^n| \leq \sum_{i=1}^k |\alpha_i^m - \alpha_i^n| \leq \frac{1}{c} \left\| \sum_{i=1}^k (\alpha_i^m - \alpha_i^n) e_i \right\| < \frac{\varepsilon}{c} \quad \text{for } m, n \geq N.$$

It follows that  $(\alpha_i^n)_n$  is a Cauchy sequence on  $\mathbb{C}$  for every  $i = 1, \dots, k$ , hence convergent,  $\alpha_i^n \rightarrow \alpha_i$  for  $n \rightarrow +\infty$ .

In this way we construct an element

$$x = \sum_{i=1}^k \alpha_i e_i \in U.$$

It remains to be proved that  $x^n \rightarrow x$  for  $n \rightarrow +\infty$ . However,

$$\|x - x^n\| = \left\| \sum_{i=1}^k (\alpha_i - \alpha_i^n) e_i \right\| \leq \sum_{i=1}^k |\alpha_i - \alpha_i^n| \cdot \|e_i\| \rightarrow 0 \quad \text{for } n \rightarrow +\infty,$$

because every term in the finite sum tends towards 0 for  $n \rightarrow +\infty$ . This proves that every finite dimensional subspace of a normed vector space is a Banach space.

**Example 2.20** Let  $V$  be a Banach space. A series  $\sum_{k=0}^{\infty} x_k$ ,  $x_k \in V$ , is convergent if the sequence  $(s_n)$ , where

$$s_n = \sum_{k=0}^n x_k,$$

is convergent in  $V$ .

Show that  $\sum_{k=0}^{\infty} \|x_k\| < \infty$  implies that  $\sum_{k=0}^{\infty} x_k$  is convergent.

Does the convergence of  $\sum_{k=0}^{\infty} x_k$  imply that  $\sum_{k=0}^{\infty} \|x_k\| < \infty$ ?

What if the space  $V$  is only assumed to be a normed space?

- 1) Given a Banach space  $V$ . It suffices to prove that  $(s_n)$  is a Cauchy sequence.

Let  $\varepsilon > 0$  be given. Since

$$\sum_{k=0}^{\infty} \|x_k\| < +\infty,$$

is finite, there exists an  $N$ , such that

$$\sum_{k=N}^{\infty} \|x_k\| < \varepsilon.$$

It holds for  $n > m \geq N$  that

$$\|s_n - s_m\| = \left\| \sum_{k=0}^n x_k - \sum_{k=0}^m x_k \right\| = \left\| \sum_{k=m+1}^n x_k \right\| \leq \sum_{k=m+1}^n \|x_k\| \leq \sum_{k=N}^{\infty} \|x_k\| < \varepsilon,$$

thus  $(s_n)$  is a Cauchy sequence in a Banach space, hence also convergent.

- 2) It is well-known that the claim does not hold in the simplest possible Banach space  $(\mathbb{R}, |\cdot|)$ , because there exist conditional convergent series like e.g.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2,$$

which are not absolutely convergent,

$$\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$$

- 3) This is not true, either. Denote by  $c$  the vector space consisting of real sequences  $(x_n)$ , where  $x_n = 0$  eventually, e.g. for  $n \geq N(x)$ . Choose as norm,

$$\|x\| = \sqrt{\sum_{n=1}^{\infty} x_n^2}.$$

Then  $c$  is dense in  $\ell^2$ , and  $c \neq \ell^2$ .

Choose  $x_n = \frac{1}{n} e_n$ . Then

$$\left\| \sum_{n=1}^{\infty} x_n \right\|^2 = \sum_{n=1}^{\infty} \|x_n\|^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6},$$

so  $\sum_{n=1}^{\infty} x_n \in \ell^2$ .

Clearly,  $\sum_{n=1}^{\infty} x_n$  is not zero, eventually, while all  $s_n = \sum_{k=1}^n x_k$  have this property. Hence

$$c \ni s_n \rightarrow \sum_{n=1}^{\infty} x_n \in \ell^2 \setminus c.$$

**Example 2.21** Let  $(V, \|\cdot\|)$  denote a normed space. Let  $V'$  denote the set of all bounded linear functionals on  $(V, \|\cdot\|)$ . The set  $V'$  is organized as a vector space by the operations

$$(f + g)(x) = f(x) + g(x), \quad \text{for all } x \in V,$$

$$(\alpha f)(x) = \alpha f(x), \quad \text{for all } x \in V,$$

and we introduce a norm on  $V'$  by

$$\|f\|' = \sum_{\|x\| \leq 1} |f(x)|.$$

Prove that  $(V', \|\cdot\|')$  is a Banach space. It is called the dual space  $V$ .

We shall first show that  $\|\cdot\|'$  is a norm on  $V'$ . It is obvious that  $\|f\|' \geq 0$ . If  $\|f\|' = 0$ , then

$$\sup_{\|x\| \leq 1} |f(x)| = 0.$$

Then we have  $\left\| \frac{x}{\|x\|} \right\| = 1$  for arbitrary  $x \neq 0$ , hence

$$|f(x)| = \left| f \left( \|x\| \cdot \frac{x}{\|x\|} \right) \right| = \|x\| \cdot \left| f \left( \frac{x}{\|x\|} \right) \right| = 0.$$

It follows from  $f(0) = 0$  that  $f(x) = 0$  for every  $x \in V$ , thus  $f \equiv 0$ . Furthermore,

$$\|\alpha f\|' = \sup_{\|x\| \leq 1} |\alpha f(x)| = |\alpha| \cdot \sup_{\|x\| \leq 1} |f(x)| = |\alpha| \cdot \|f\|',$$

and finally,

$$\begin{aligned} \|f + g\|' &= \sup_{\|x\| \leq 1} |f(x) + g(x)| \leq \sup_{\|x\| \leq 1} (|f(x)| + |g(x)|) \\ &\leq \sup_{\|x\| \leq 1} |f(x)| + \sup_{\|x\| \leq 1} |g(x)| = \|f\|' + \|g\|', \end{aligned}$$

and we have proved that  $\|\cdot\|'$  is a norm.

Assume that  $(f_n)$  is a Cauchy sequence on  $V'$ , i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|f_n - f_m\| < \varepsilon.$$

This means that

$$\|f_n - f_m\|' = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)| < \varepsilon \quad \text{for all } m, n \geq N,$$

i.e. we have for every  $x$ , for which  $\|x\| \leq 1$  that  $(f_n(x))$  is a Cauchy sequence in  $\mathbb{C}$ , hence convergent.

For any  $x \neq 0$  it follows that  $\frac{x}{\|x\|}$  is a unit vector, thus

$$\forall \varepsilon > 0 \exists N_x \in \mathbb{N} \forall m, n \geq N_x : \|f_n - f_m\|' < \frac{\varepsilon}{\|x\|},$$

which only means that

$$|f_n(x) - f_m(x)| = \|x\| \cdot \left| f_n\left(\frac{x}{\|x\|}\right) - f_m\left(\frac{x}{\|x\|}\right) \right| < \|x\| \cdot \frac{\varepsilon}{\|x\|} = \varepsilon,$$

so  $(f_n(x))$  is convergent for every  $x \in V \setminus \{0\}$ . If  $x = 0$ , we just get  $f_n(0) = 0 \rightarrow 0$  for  $n \rightarrow +\infty$ . If we put

$$f(x) = \lim_{n \rightarrow +\infty} f_n(x),$$

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then we have defined a functional on  $V$  for which in particular  $f(0) = 0$ . It remains only to prove that 1)  $f$  is linear, and at 2)  $f$  is bounded. However,

$$f(x + \lambda y) = \lim_{n \rightarrow +\infty} f_n(x + \lambda y) = \lim_{n \rightarrow +\infty} \{f_n(x) + \lambda f_n(y)\} = f(x) + \lambda f(y),$$

proving the linearity. Then

$$\begin{aligned} (11) \|f\|' &= \sup_{\|x\| \leq 1} |f(x)| = \sup_{\|x\| \leq 1} |f(x) - f_n(x) + f_n(x)| \\ &\leq \sup_{\|x\| \leq 1} |f(x) - f_n(x)| + \sup_{\|x\| \leq 1} |f_n(x)| \\ &= \sup_{\|x\| \leq 1} |f(x) - f_n(x)| + \|f_n\|'. \end{aligned}$$

Choose  $n$ , such that for all  $m \geq n$ ,

$$\|f_n - f_m\|' = \sup_{\|x\| \leq 1} |f_n(x) - f_m(x)| < 1.$$

Then  $f_m(x) \in B(f_n(x), 1)$  for every  $x$ , for which  $\|x\| \leq 1$ . Since  $f_m(x) \rightarrow f(x)$  for  $m \rightarrow +\infty$ , we have  $f(x) \in \overline{B(f_n(x), 1)}$ , so  $|f_n(x) - f(x)| \leq 1$  for all  $x$ , for which  $\|x\| \leq 1$ . From this we infer that

$$\sup_{\|x\| \leq 1} |f(x) - f_n(x)| \leq 1.$$

Therefore, if  $n$  is chosen as above, then it follows from (11) that  $\|f\|' \leq 1 + \|f_n\|'$ , hence  $f$  is bounded, and we have proved that every Cauchy sequence on  $V'$  is convergent, i.e.  $V'$  is a Banach space.

## 2.4 The Lebesgue integral

‘n

**Example 2.22** Let  $f \in L^1(\mathbb{R})$ .

- 1) Can we conclude that  $f(x) \rightarrow 0$  for  $|x| \rightarrow \infty$ ?
- 2) Can we find  $a, b \in \mathbb{R}$  such that  $|f(x)| \leq b$  for  $|x| \geq a$ ?

In both cases the answer is ‘no’. For example,  $g(x) = x \cdot 1_{\mathbb{Z}}(x)$  fulfils none of the conditions, and

$$\int |g(x)| dx = 0.$$

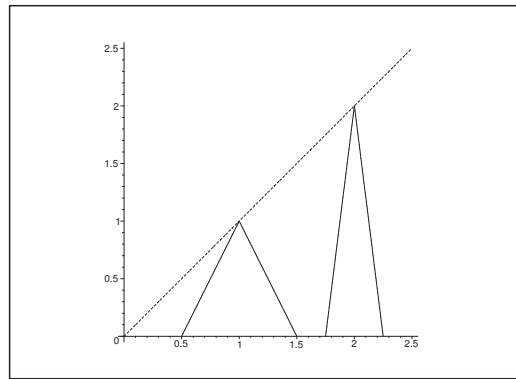


Figure 4: The graph of a *continuous* function  $f(x)$ , which does not fulfil the two requirements.

We shall now construct a function  $f$ , which is *continuous* and Lebesgue integrable, and which does not fulfil any of the two requirements above. Let

$$f(x) = \begin{cases} n & \text{for } x = n, & n \in \mathbb{N}, \\ 0 & \text{for } x = n \pm 2^{-n}, & n \in \mathbb{N}, \\ \text{piecewise linear,} & \text{otherwise.} \end{cases}$$

Clearly,  $f$  is continuous and satisfies neither (1) nor (2). We shall only prove that  $f$  is integrable. Now,  $f \geq 0$ , so

$$\int_{-\infty}^{+\infty} f(x) dx = \sum_{n=1}^{+\infty} \frac{1}{2} n \cdot 2 \cdot 2^{-n} = \sum_{n=1}^{+\infty} n 2^{-n} < +\infty,$$

and the claim is proved.

**Remark 2.1** For completeness we here add the full proof. We have

$$\begin{aligned} \sum_{n=1}^{+\infty} n \cdot 2^{-n} &= 2 \sum_{n=1}^{+\infty} n \cdot 2^{-(n+1)} = 2 \sum_{n=2}^{+\infty} (n-1) 2^{-n} = 2 \sum_{n=2}^{+\infty} n \cdot 2^{-n} - 2 \sum_{n=2}^{+\infty} 2^{-n} \\ &= 2 \sum_{n=1}^{+\infty} n \cdot 2^{-n} - 2 \cdot 1 \cdot 2^{-1} - \sum_{n=1}^{+\infty} 2^{-n} = 2 \sum_{n=1}^{+\infty} n \cdot 2^{-n} - 1 - 1, \end{aligned}$$

hence by a rearrangement,

$$\sum_{n=1}^{+\infty} n \cdot 2^{-n} = 2.$$

ALTERNATIVELY one may exploit that

$$\frac{d}{dz} \left( \frac{1}{1-z} \right) = \frac{1}{(1-z)^2} = \frac{d}{dz} \left( \sum_{n=0}^{+\infty} z^n \right) = \sum_{n=1}^{+\infty} n z^{n-1},$$

for  $|z| < 1$ . When we insert  $z = \frac{1}{2}$ , we easily get the result.  $\diamond$

**Example 2.23** *Prove that if  $f : \mathbb{R} \rightarrow \mathbb{R}$  is monotonous, then  $f$  has at most countably many points of discontinuity.*

We may assume that  $f$  is increasing, thus  $f(x) \geq f(y)$  for  $x > y$ . We may even restrict ourselves to the interval  $[0, 1]$ , because the number of intervals of the form  $[n, n+1]$ ,  $n \in \mathbb{Z}$ , is countable. This means that we may assume that  $f(x) = 0$  for  $x \leq 0$ , and  $f(x) = 1$  for  $x \geq 1$ .

Let  $\{x_j \mid j \in J\}$  be the set of all points of discontinuity in  $[0, 1]$ . Then to any  $x_j$  we can find an interval  $I_j$  with interior points on the  $Y$ -axis, such that  $f(x) \notin I_j$  for all  $x \in [0, 1]$ , i.e. one jumps over the values in  $I_j$  over.

Every  $I_j$  can be “numbered” by a rational number  $q_j \in I_j$ , because  $\mathbb{Q}$  is dense in  $\mathbb{R}$ . This means that  $\{x_j \mid j \in J\}$  contains just as many elements, as there are different elements in

$$\{q_j \mid j \in J\} \subseteq \mathbb{Q}.$$

Now,  $\mathbb{Q}$  is countable, so  $\{q_j \mid j \in J\}$  is countable, and thus  $\{x_j \mid j \in J\}$  is at most countable.

Define

$$f(x) = 2^{-n+1} \quad \text{for } x \in \left[ \frac{1}{n+1}, \frac{1}{n} \right], \quad n \in \mathbb{N}.$$

Then  $f$  is monotonous of the countably many points of discontinuity  $\left\{ \frac{1}{n} \mid n \in \mathbb{N} \setminus \{1\} \right\}$ , showing that there exist monotonous functions with a countable number of points of discontinuity.

An ALTERNATIVE proof is the following: We may as before assume that  $f$  is increasing on the interval  $[0, 1]$  with  $f(x) = 0$  for  $x \leq 0$  and  $f(x) = 1$  for  $x \geq 1$ .

If  $x_0$  is a point of discontinuity, then  $f(x) \leq f(x_0)$  for every  $x \leq x_0$ . Hence, if  $x_n \nearrow x_0$ , then  $(f(x_n))$  is an increasing bounded sequence of numbers, so  $(f(x_n))$  is convergent with the limit value  $c$ .

Let  $y_n \nearrow x_0$  be another such sequence of numbers. Then  $(f(y_n)) \rightarrow c'$ . We shall prove that  $c = c'$ . This is done INDIRECTLY.

Assume (e.g.) that  $c < c'$ , and let  $0 < \varepsilon < c' - c$ . Corresponding to this  $\varepsilon$  there exists an  $N$ , such that

$$|c' - f(y_n)| = c' - f(y_n) < \varepsilon \quad \text{for all } n \geq N.$$

To any  $y_n$  we can find an  $x_m$ , such that  $y_n < x_m < x_0$ , hence

$$f(y_n) \leq f(x_m) \quad [\leq c].$$

Then it follows that

$$\varepsilon < |c' - c| = c' - c = c' - f(y_n) + f(y_n) - c < \varepsilon + f(y_n) - c,$$

so  $f(y_n) - c > 0$ , and we have come to the contradiction

$$c < f(y_n) \leq f(x_m) \leq c \quad \text{for } n \geq N.$$

We therefore conclude that  $c' = c$ .

Since the limit value is the same, no matter how  $x_n \nearrow x_0$  is chosen, we conclude that

$$c = \lim_{x \rightarrow x_0^-} f(x).$$

We prove in a similar way that  $\lim_{x \rightarrow x_0^+} f(x)$  exists, and that these two values are different at any point of discontinuity.

Define the jump at a point of discontinuity  $x_0$  as

$$\sigma_0 = \lim_{x \rightarrow x_0^+} f(x) - \lim_{x \rightarrow x_0^-} f(x) > 0.$$

If  $x_0 < x_1$  are both points of discontinuity, then it follows from that the function is monotonous that

$$\lim_{x \rightarrow x_0^+} f(x) \leq \lim_{x \rightarrow x_1^-} f(x).$$

Let  $\{x_j \mid j \in J\}$  denote the set of point of discontinuity in  $[0, 1]$ . The image is contained in  $[0, 1]$ , hence

$$\sum_{x_j} \sigma_j \leq 1,$$

and the sum is finite. Every  $\sigma_j > 0$ , so the sum is at most countable, thus  $J \subseteq \mathbb{N}$ , and the claim is proved.

**Example 2.24** Prove that  $f(x) = \frac{|\sin x|}{x}$  is not Lebesgue integrable on  $[\pi, +\infty[$ , thus  $f \notin L^1([\pi, +\infty[)$ .

HINT: Consider

$$f_n(x) = \begin{cases} \frac{|\sin x|}{x}, & \pi \leq x \leq n\pi, \\ 0, & \text{otherwise,} \end{cases}$$

and exploit that  $f_n(x) \nearrow f(x)$  and  $\int_{\pi}^{\infty} f_n(x) dx \geq \frac{1}{3} \sum_{k=2}^n \frac{1}{k}$ .

Let  $f_n$  be given as above. Then clearly,

$$0 \leq f_n(x) \nearrow f(x).$$

Furthermore,

$$\begin{aligned}
 \int_{\pi}^{\infty} f_n(x) dx &= \int_{\pi}^{n\pi} \frac{|\sin x|}{x} dx = \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{|\sin x|}{x} dx \\
 &\geq \sum_{k=2}^n \int_{(k-1)\pi}^{k\pi} \frac{1}{k\pi} \cdot |\sin x| dx = \sum_{k=2}^n \frac{1}{k\pi} \left| \int_{(k-1)\pi}^{k\pi} \sin x dx \right| \\
 &= \sum_{k=2}^n \frac{1}{k\pi} \left| \int_0^{\pi} \sin x dx \right| = \sum_{k=2}^n \frac{2}{k\pi} = \frac{2}{\pi} \sum_{k=2}^n \frac{1}{k} \rightarrow +\infty \quad \text{for } n \rightarrow +\infty,
 \end{aligned}$$

and we infer that  $f$  is not Lebesgue integrable, i.e.  $f$  does not belong to  $L^1([\pi, \infty[)$ .

**Example 2.25** Give a simple proof of Hölder's inequality in the case of  $p = q = 2$  for the spaces of sequences.

We shall more precisely prove (Bohnenblust-Bunjakovski)-Cauchy-Schwarz-(Sobčyk)'s inequality

$$\sum_{i=1}^{+\infty} |x_i \bar{y}_i| = \sum_{i=1}^{\infty} |x_i| \cdot |y_i| \leq \|x\|_2 \cdot \|y\|_2,$$

if  $x, y \in \ell^2$ .

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Using that  $x + \lambda y \in \ell^2$  for every  $\lambda \in \mathbb{R}$ , we get

$$\begin{aligned} 0 &\leq \|x + \lambda y\|_2^2 = \sum_{i=1}^{+\infty} (x_i + \lambda y_i) \cdot (\bar{x}_i + \lambda \bar{y}_i) \\ &= \sum_{i=1}^{+\infty} \{ |x_i|^2 + \lambda^2 |y_i|^2 + \lambda \bar{x}_i y_i + \lambda x_i \bar{y}_i \} \\ &= \lambda^2 \sum_{i=1}^{+\infty} |y_i|^2 + \lambda \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\} + \sum_{i=1}^{+\infty} |x_i|^2, \end{aligned}$$

which we write in the form

$$\lambda^2 \cdot \|y\|_2^2 + \lambda \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\} + \|x\|_2^2 \geq 0.$$

This must hold for every real  $\lambda \in \mathbb{R}$ , so we must have

$$\begin{aligned} 0 &\geq B^2 - 4AC = \left\{ \sum_{i=1}^{\infty} \bar{x}_i y_i + \sum_{i=1}^{+\infty} x_i \bar{y}_i \right\}^2 - 4 \|x\|_2^2 \|y\|_2^2 \\ &= 4 \left( \operatorname{Re} \left\{ \sum_{i=1}^{+\infty} \bar{x}_i y_i \right\} - \{ \|x\|_2 \|y\|_2 \}^2 \right), \end{aligned}$$

hence

$$\left| \operatorname{Re} \left\{ \sum_{i=1}^{\infty} \bar{x}_i y_i \right\} \right| \leq \|x\|_2 \|y\|_2.$$

When  $x_i$  and  $y_i$  are all real, the inequality follows immediately.

In general,

$$\begin{aligned} \sum_{i=1}^{+\infty} |x_i \bar{y}_i| &= \sum_{i=1}^{+\infty} |x_i| \cdot |\bar{y}_i| \leq \|x\|_2 \cdot \|y\|_2 \\ &= \left\{ \sum_{i=1}^{+\infty} |x_i|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{i=1}^{\infty} |y_i|^2 \right\}^{\frac{1}{2}} = \|x\|_2 \|y\|_2, \end{aligned}$$

and the claim is proved.

**Example 2.26** Let  $w(t) \geq 0$  be a non-negative function on  $\mathbb{R}$ . We define a linear functional  $I_w$  by

$$I_w(f) = \int_{\mathbb{R}} f(t) w(t) dt,$$

for  $f w \in L^1(\mathbb{R})$ .

Assume that  $|f|^p w$  and  $|g|^q w$  are in  $L^1(\mathbb{R})$ , where  $f$  and  $g$  are (measurable) functions and  $1 < p, q < \infty$  with  $\frac{1}{p} + \frac{1}{q} = 1$ .

1. Show the generalized Hölder's inequality

$$|I_w(fg)| \leq \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}},$$

where the inequality for  $w = 1$  can be taken to be valid.

Now recall the Gamma function,

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \quad x > 0,$$

with the property  $\Gamma(x+1) = x\Gamma(x)$  for  $x > 0$ .

2. Use the generalized Hölder's inequality with

$$w(t) = t^{n-1} e^{-t}, \quad 0 < t < \infty, \quad \text{and} \quad p = q = 1,$$

to show that

$$\Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

3. Give a similar estimation of  $\Gamma(n+1)$  by taking

$$w(t) = t^{n-\frac{1}{2}} e^{-t}, \quad 0 < t < \infty, \quad \text{and} \quad p = q = 2,$$

and deduce that

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}, \quad n \in \mathbb{N}.$$

1) We get from  $w(t) \geq 0$  that both  $w^{1/p}$  and  $w^{1/q}$  are defined and that  $w^{1/p} \cdot w^{1/q} = w$ , and  $f \cdot w^{1/p} \in L^p(\mathbb{R})$  and  $g \cdot w^{1/q} \in L^q(\mathbb{R})$ . Applying the usual Hölder's inequality we get

$$\begin{aligned} |I_w(f \cdot g)| &= \left| \int_{-\infty}^{+\infty} f(t) g(t) w(t) dt \right| \leq \int_{-\infty}^{+\infty} |f(t) w^{\frac{1}{p}}(t)| \cdot |g(t) w^{\frac{1}{q}}(t)| dt \\ &\leq \left\{ \int_{-\infty}^{+\infty} |f(t)|^p w(t) dt \right\}^{\frac{1}{p}} \left\{ \int_{-\infty}^{+\infty} |g(t)|^q w(t) dt \right\}^{\frac{1}{q}} = \{I_w(|f|^p)\}^{\frac{1}{p}} \{I_w(|g|^q)\}^{\frac{1}{q}}, \end{aligned}$$

and we have proved the generalized Hölder's inequality.

- 2) Then apply this generalized inequality on  $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$  and  $g(t) = 1$ , and  $w(t) = t^{n-1} e^{-t} \cdot 1_{\mathbb{R}_+}(t)$ , we get

$$\begin{aligned}
 \Gamma\left(n + \frac{1}{2}\right) &= \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-1} e^{-t} dt \leq \{I_w(t)\}^{\frac{1}{2}} \{I_w(1)\}^{\frac{1}{2}} \\
 &= \left\{ \int_0^{+\infty} t \cdot t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} 1 \cdot t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \\
 &= \left\{ \int_0^{+\infty} t^n e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} t^{n-1} e^{-t} dt \right\}^{\frac{1}{2}} \\
 &= \{\Gamma(n+1)\}^{\frac{1}{2}} \{\Gamma(n)\}^{\frac{1}{2}} = \{n!(n-1)!\}^{\frac{1}{2}} = \left\{ \frac{(n!)^2}{n} \right\}^{\frac{1}{2}} = \frac{n!}{\sqrt{n}}.
 \end{aligned}$$

- 3) Finitely, let  $f(t) = \sqrt{t} \cdot 1_{\mathbb{R}_+}(t)$  and  $g(t) = 1$ , and  $w(t) = t^{n-\frac{1}{2}} e^{-t} \cdot 1_{\mathbb{R}_+}(t)$ . Then we get with  $p = q = 2$ ,

$$\begin{aligned}
 n! &= \Gamma(n+1) = \int_0^{+\infty} t^n e^{-t} dt = \int_0^{+\infty} \sqrt{t} \cdot 1 \cdot t^{n-\frac{1}{2}} e^{-t} dt \\
 &\leq \left\{ \int_0^{+\infty} t^{n+\frac{1}{2}} e^{-t} dt \right\}^{\frac{1}{2}} \left\{ \int_0^{+\infty} t^{n-\frac{1}{2}} e^{-t} dt \right\}^{\frac{1}{2}} \\
 &= \left\{ \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(n + \frac{1}{2}\right) \right\}^{\frac{1}{2}} = \left\{ \left(n + \frac{1}{2}\right) \left[ \Gamma\left(n + \frac{1}{2}\right) \right]^2 \right\}^{\frac{1}{2}} = \sqrt{n + \frac{1}{2}} \cdot \Gamma\left(n + \frac{1}{2}\right),
 \end{aligned}$$

and we have

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \Gamma\left(n + \frac{1}{2}\right) \leq \frac{n!}{\sqrt{n}}.$$

**Remark 2.2** Furthermore, if we use that  $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ , it follows from the functional equation that

$$\begin{aligned}
 \Gamma\left(n + \frac{1}{2}\right) &= \left(n - \frac{1}{2}\right) \Gamma\left(n - \frac{1}{2}\right) = \cdots = \left(n - \frac{1}{2}\right) \left(n - \frac{3}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\
 &= \frac{(2n-1)(2n-3) \cdots 3 \cdot 1}{2 \cdot 2 \cdots 2 \cdot 2} \sqrt{\pi} \\
 &= \frac{\sqrt{\pi}}{2^n} \cdot \frac{2n}{2n} \cdot \frac{2n-1}{1} \cdot \frac{2n-2}{2(n-1)} \cdots \frac{4}{2 \cdot 2} \cdot \frac{3}{1} \cdot \frac{2}{2 \cdot 1} \cdot \frac{1}{1} \\
 &= \frac{\sqrt{\pi}}{2^n} \cdot \frac{(2n)!}{2^n \cdot n!} = \frac{\sqrt{\pi}}{4^n} \binom{2n}{n} n!,
 \end{aligned}$$

hence by insertion

$$\frac{n!}{\sqrt{n + \frac{1}{2}}} \leq \frac{\sqrt{\pi}}{4^n} \binom{2n}{n} n! \leq \frac{n!}{\sqrt{n}},$$



thus

$$\frac{4^n}{\sqrt{\pi \left(n + \frac{1}{2}\right)}} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{\pi n}},$$

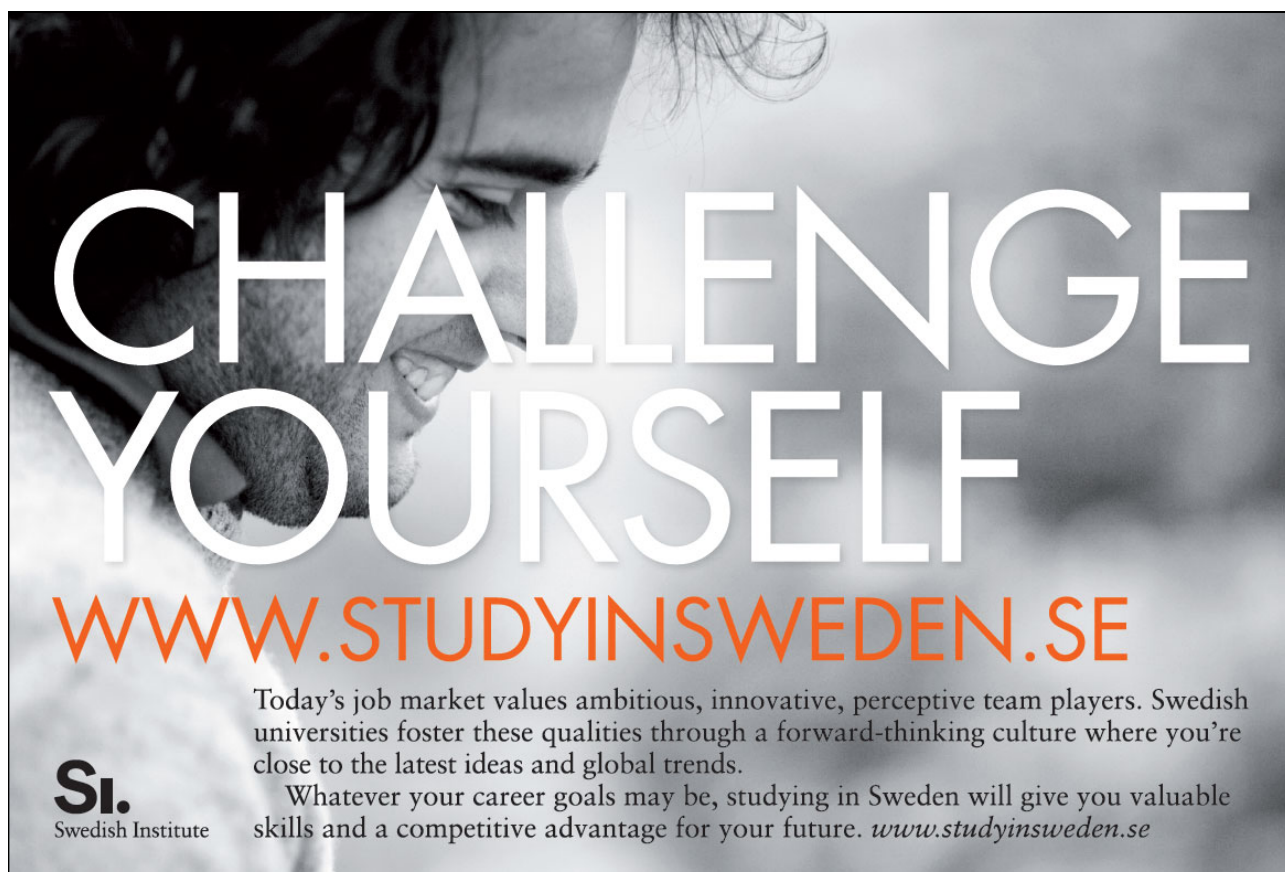
which is in agreement with Stirling's formula

$$n! \sim \sqrt{2\pi} \cdot n^{n+\frac{1}{2}} e^{-n},$$

because

$$\binom{2n}{n} = \frac{(2n)!}{(n!)^2} \sim \frac{\sqrt{2\pi} \cdot (2n)^{2n+\frac{1}{2}} e^{-2n}}{\left\{\sqrt{2\pi} \cdot n^{n+\frac{1}{2}} \cdot e^{-n}\right\}^2} = \frac{1}{\sqrt{2\pi}} \cdot \frac{(2n)^{2n+\frac{1}{2}}}{n^{2n+1}} = \frac{2^{2n}\sqrt{2}}{\sqrt{2\pi n}} = \frac{4^n}{\sqrt{\pi n}}. \quad \diamond$$

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**Example 2.27** *Let*

$$F = \{f \in C^2([0, 1]) \mid f(0) = f(1) = 0\} \subseteq L^2([0, 1]).$$

1) *Show that  $\|f'\|_2^2 \leq \|f\|_2 \cdot \|f''\|_2$  for  $f \in F$ .*

2) *Let  $f \in F$ . Show that  $|f(x)| \leq \|f'\|_2 \sqrt{x}$  for  $0 \leq x \leq 1$ , and deduce that*

$$\|f\|_2 \leq \frac{1}{\sqrt{2}} \|f'\|_2.$$

3) *Show that for  $f \in C^2([0, 1])$  with  $f(0) = f(1) = 0$  we have*

$$\|f'\|_2 \leq \frac{1}{\sqrt{2}} \|f''\|_2.$$

4) *Show by a counterexample that the result from question (3) is not valid for general  $f \in C^2([0, 1])$ .*

1) We deduce from  $f \in C^2([0, 1])$  and  $f(0) = f(1) = 0$  and a partial integration, followed by an application of the Cauchy-Schwarz inequality that

$$\begin{aligned} \|f'\|_2^2 &= \int_0^1 f'(t) \overline{f'(t)} dt = \left[ f(t) \overline{f'(t)} \right]_0^1 - \int_0^1 f(t) \overline{f''(t)} dt \\ &\leq 0 + \int_0^1 |f(t)| \cdot |f''(t)| dt \leq \|f\|_2 \cdot \|f''\|_2. \end{aligned}$$

2) From

$$f(x) = f(0) + \int_0^x f'(t) dt = \int_0^x 1_{[0, x]}(t) f'(t) dt$$

follows by Cauchy-Schwarz's inequality that

$$|f(x)| = \left| \int_0^1 1_{[0, x]}(t) f'(t) dt \right| \leq \|1_{[0, x]}\|_2 \cdot \|f'\|_2 = \sqrt{x} \cdot \|f'\|_2,$$

where we have used that

$$\|1_{[0, x]}\|_2 = \sqrt{\int_0^1 1_{[0, x]}(t) dt} = \sqrt{\int_0^x 1 dt} = \sqrt{x}.$$

3) Let  $f \in F$ . It follows from (1) and (2) that

$$\begin{aligned} \|f'\|_2^2 &\leq \|f\|_2 \cdot \|f''\|_2 = \left\{ \int_0^1 |f(x)|^2 dx \right\}^{\frac{1}{2}} \cdot \|f''\|_2 \leq \left\{ \int_0^1 x \|f'\|_2^2 dx \right\}^{\frac{1}{2}} \|f''\|_2 \\ &= \left\{ \int_0^1 x dx \right\}^{\frac{1}{2}} \|f'\|_2 \cdot \|f''\|_2 = \frac{1}{\sqrt{2}} \|f'\|_2 \|f''\|_2. \end{aligned}$$

If  $\|f'\|_2 = 0$ , the inequality is obvious.

If  $\|f'\|_2 > 0$ , we obtain the inequality when we divide by  $\|f'\|_2$ .

We derived the above by assuming that  $f \in F$ , thus  $f(0) = f(1) = 1$ .

Now, let  $f(0) = f(1) = c$ . Then  $f(x) - c \in F$ , hence

$$\|f'\|_2 = \|(f - c)'\|_2 \leq \frac{1}{\sqrt{2}} \|(f - c)''\|_2 = \frac{1}{\sqrt{2}} \|f''\|_2.$$

4) Finally, let  $f(x) = ax$ . Then  $f'(x) = a$  and  $f''(x) = 0$ , hence

$$\|f'\|_2 = |a| \quad \text{og} \quad \|f''\|_2 = 0,$$

and the inequality is not fulfilled for any  $a \neq 0$ .

**Example 2.28** 1) Let  $1 \leq p \leq q \leq \infty$ . Show that  $\ell^p \subset \ell^q$ .

2) Let  $1 \leq r < p < 2r$  and assume that the sequence  $(x_n)$  satisfies

$$\sum_{n=1}^{\infty} n |x_n|^p < \infty.$$

Show that  $(x_n) \in \ell^r$ .

1) If  $(x_n) \in \ell^p$ , then  $\sum_{n=1}^{+\infty} |x_n|^p < +\infty$ . In particular,  $x_n \rightarrow 0$  for  $n \rightarrow +\infty$ , hence there exists an  $N \in \mathbb{N}$ , such that  $|x_n| < 1$  for all  $n \geq N + 1$ .

For  $p = q$  there is nothing to prove. If  $1 \leq p < q < +\infty$ , then

$$\sum_{n=1}^{+\infty} |x_n|^q = \sum_{n=1}^N |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p \cdot |x_n|^{q-p} < \sum_{n=1}^N |x_n|^q + \sum_{n=N+1}^{+\infty} |x_n|^p < +\infty,$$

showing that  $(x_n) \in \ell^q$ .

If  $1 \leq p < q = +\infty$ , then clearly

$$\sup_{n \in \mathbb{N}} |x_n| \leq \max \{1, \sup \{|x_n| \mid n = 1, \dots, N\}\} < +\infty,$$

and we conclude that  $(x_n) \in \ell^\infty$ .

2) Then let  $1 \leq r < p < 2r$  and assume that

$$\sum_{n=1}^{+\infty} n |x_n|^p < +\infty.$$

Let  $0 < s < 1$ . We shall somehow way apply Hölder's inequality with  $\tilde{p} = \frac{1}{s} > 1$  and  $\tilde{q} = \frac{1}{1-s} > 1$ . The assumption shall also be applied later on, so we get by a reasonable rewriting and an application of Hölder's inequality,

$$\sum_{n=1}^{+\infty} |x_n|^r = \sum_{n=1}^{+\infty} \{n |x_n|^p\}^s \left\{ \frac{1}{n^s} |x_n|^{r-sp} \right\} \leq \left\{ \sum_{n=1}^{+\infty} n |x_n|^p \right\}^s \cdot \left\{ \sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}} |x_n|^{\frac{r-sp}{1-s}} \right\}^{1-s}.$$

By the assumption, the former factor is finite for every  $s \in ]0, 1[$ . The task is to choose  $s$  in this interval, such that the latter factor also becomes finite.

Using that  $2r > p$ , we get  $\frac{r-sp}{1-s} = 0$  for  $s = \frac{r}{p} > \frac{1}{2}$ . We get with this  $s$  that  $\alpha = \frac{s}{1-s} > 1$  and

$$\sum_{n=1}^{+\infty} n^{-\frac{s}{1-s}} |x_n|^{\frac{r-sp}{1-s}} = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} \cdot |x_n|^0 = \sum_{n=1}^{+\infty} \frac{1}{n^\alpha} < +\infty,$$

and the latter factor in the estimate above is finite for this particular  $s = \frac{r}{p} \in \left] \frac{1}{2}, 1 \right[$ . Now,  $s$  does not occur in the sum, we are estimating, so we conclude that

$$\sum_{n=1}^{+\infty} |x_n|^r < +\infty,$$

and we have proved that  $(x_n) \in \ell^r$ .

**Example 2.29** Define in  $\mathbb{R}^2$  the function

$$\|x\| = \|(x_1, x_2)\| = \left( \sqrt{|x_1|} + \sqrt{|x_2|} \right)^2.$$

Is it a norm?

Sketch the set  $\{(x_1, x_2) \mid \|(x_1, x_2)\| \leq 1\}$ .

First note that  $\|x\| = \|x\|_p$ , where  $p = \frac{1}{2} < 1$ .

The first two conditions of a norm are trivially fulfilled, so we shall only consider the triangle inequality. We shall prove that it is *not* satisfied. It suffices to find two vectors  $x$  and  $y$ , for which the triangle inequality does not hold.

Choose  $x = (1, 0)$  and  $y = (0, 1)$ . Then  $\|x\| = \|y\| = 1$ , and

$$\|x + y\| = \|(1, 1)\| = (\sqrt{1} + \sqrt{1})^2 = 4,$$

hence

$$\|x + y\| = 4 > 2 = \|x\| + \|y\|,$$

and the triangle inequality is not fulfilled, and  $\|\cdot\|$  is not a norm.

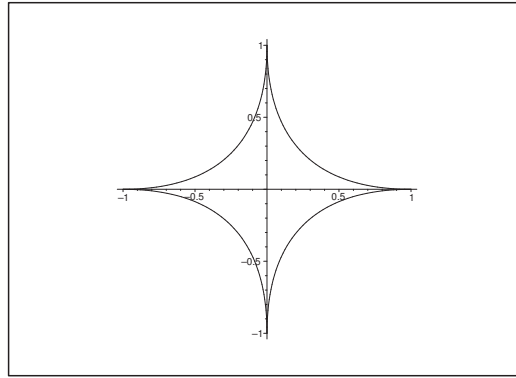


Figure 5: The unit “ball” corresponding to  $\|\cdot\|$ .

**Remark 2.3** It is not hard to prove that if  $\|\cdot\|$  is a norm, then the corresponding unit ball is convex. (However, not every convex set will induce a norm).

Since the set, which should be the unit ball clearly is *not* convex (cf. the figure),  $\|\cdot\|$  is *not* a norm.  $\diamond$

**Remark 2.4** Even if  $\|\cdot\|_{\frac{1}{2}}$  is not a norm in the usual sense, there exist some applications of it, e.g. in the theory of  $H^p$  spaces in Complex Function Theory, and the “norm” of such functions can nevertheless be given a reasonable interpretation.  $\diamond$

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### 3 Bounded operators

**Example 3.1** Let  $T$  be a linear operator from a normed space  $V$  into a normed space  $W$ .

Show that the image  $T(V)$  is a subspace of  $W$ .

Show that the kernel (or null-space)  $\ker(T)$  is a subspace of  $V$ .

If  $T$  is bounded, is it true that  $T(V)$  and/or  $\ker(T)$  are closed?

1) Let  $w_1, w_2 \in T(V) \subseteq W$ , and let  $\lambda$  be a scalar. We shall prove that

$$w_1 + \lambda w_2 \in T(V).$$

**Remark 3.1** It is here of paramount importance that the field of the scalars is the same both places. If e.g.  $T : V \rightarrow W$  is given by

$$Tx = x + i \cdot 0,$$

where  $V = (\mathbb{R}, +, \cdot, \|\cdot\|, \mathbb{R})$  and  $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$ , then  $T$  is linear, and  $T(V)$  is a subspace of the 2-dimensional space  $(\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{R})$  over  $\mathbb{R}$ . It is, however, not a subspace of the 1-dimensional space  $W = (\mathbb{C}, +, \cdot, \|\cdot\|, \mathbb{C})$  over  $\mathbb{C}$ , so the claim is not true in this case.  $\diamond$

It follows from the assumption  $w_1, w_2 \in T(V)$  that there exist  $v_1$  and  $v_2 \in V$ , such that  $w_1 = Tv_1$  and  $w_2 = Tv_2$ .

If we put  $v = v_1 + \lambda v_2 \in V$ , then

$$T(V) \ni Tv = T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = w_1 + \lambda w_2.$$

2) Now  $\ker(T) = \{v \in V \mid Tv = 0\}$ , and  $T$  is linear. Hence, if  $v_1, v_2 \in \ker(T)$ , and  $\lambda$  is a scalar, then

$$T(v_1 + \lambda v_2) = Tv_1 + \lambda Tv_2 = 0 + \lambda \cdot 0 = 0,$$

thus  $v_1 + \lambda v_2 \in \ker(T)$ , and  $\ker(T)$  is a subspace.

3) If  $T$  is bounded, then  $T$  is continuous. Now  $\{0\} \subset W$  is closed, so  $\ker(T) = T^{-1}(\{0\})$  is closed.

On the other hand,  $T(V)$  need not be closed, which is demonstrated by the example below.

Choose  $V = W = C^0([0, 1])$  with the norm  $\|\cdot\|_\infty$ , and let  $T : V \rightarrow W$  be given by

$$Tf(t) = \int_0^t f(s) ds, \quad t \in [0, 1].$$

Then  $T$  is bounded,

$$|Tf(t)| = \left| \int_0^t f(s) ds \right| \leq \int_0^t |f(s)| ds \leq \int_0^1 |f(s)| ds \leq 1 \cdot \|f\|_{\infty}, \quad t \in [0, 1],$$

hence

$$\|Tf\|_\infty \leq 1 \cdot \|f\|_\infty, \quad \|T\| \leq 1.$$

Furthermore,

$$T(V) = \{w \in C^1([0, 1]) \mid w(0) = 0\}$$

is dense in

$$\{w \in C^0([0, 1]) \mid w(0) = 0\} \subset W,$$

without being equal to it.

That  $T(V)$  is dense, is seen in the following way: Every polynomial of constant term 0 lies in  $T(V)$ . The claim then follows by a suitable variant of Weierstraß's Approximation Theorem.

There exist of course  $C^0$ -functions which are not of class  $C^1$ , hence  $T(V)$  is not equal to the smallest closed subspace

$$\{w \in C^0([0, 1]) \mid w(0) = 0\}$$

which contains  $T(V)$  (because  $T(V)$  is dense in this space).

**Example 3.2** In the Banach space  $\ell^p$ ,  $1 \leq p \leq \infty$ , we have a sequence  $(x_n)$  converging to an element  $x$ , where

$$x_n = (x_{n1}, x_{n2}, \dots) \quad \text{and} \quad x = (x_1, x_2, \dots).$$

Show that if  $x_n \rightarrow x$  in  $\ell^p$ , then  $x_{nk} \rightarrow x_k$  for all  $k \in \mathbb{N}$ .

If  $x_{nk} \rightarrow x_k$  for all  $k \in \mathbb{N}$ , is it true that  $x_n \rightarrow x$  in  $\ell^p$ ?

Let  $x_n \rightarrow x$  in  $\ell^p$ ,  $1 \leq p < \infty$ , thus  $\|x - x_n\|_p \rightarrow 0$  for  $n \rightarrow \infty$ , i.e.

$$\sum_{k=1}^{\infty} |x_k - x_{nk}|^p = \|x - x_n\|_p^p \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

If  $p = \infty$ , then  $x_n \rightarrow x$  in  $\ell^\infty$  means that

$$\|x - x_n\|_\infty = \sup_k |x_k - x_{nk}| \rightarrow 0 \quad \text{for } n \rightarrow \infty.$$

In both cases we get for every fixed  $k$  that

$$|x_k - x_{nk}| \leq \|x - x_n\|_p \rightarrow 0 \quad \text{for } n \rightarrow \infty,$$

thus  $x_{nk} \rightarrow x_k$  for  $n \rightarrow \infty$ , and the first claim is proved.

On the other hand, if  $x_{nk} \rightarrow x_k$  for every fixed  $k$ , then we cannot conclude that  $x_n \rightarrow x$  in  $\ell^p$ . Just choose

$$x_n = (\delta_{nk}) = (0, \dots, 0, 1, 0, \dots)$$

with 1 on place number  $n$ , and 0 otherwise.

We have for this sequence that  $x_{nk} \rightarrow 0$  for every fixed  $k$ , thus  $x = 0$ .

On the other hand,

$$\|x_n\|_p = \|x_n - 0\|_p = \left\{ \sum_{k=1}^{\infty} |\delta_{nk}|^p \right\}^{\frac{1}{p}} = 1 \quad \text{for } 1 \leq p < +\infty,$$

and

$$\|x_n\|_{\infty} = \|x_n - 0\|_{\infty} = 1,$$

so none of these sequences converges towards, i.e. the sequence does not converge in any  $\ell^p$ ,  $1 \leq p \leq +\infty$ .

**Example 3.3** Let  $T$  be a linear mapping from  $\mathbb{R}^m$  to  $\mathbb{R}^n$ , both equipped with the 2-norm. Let  $(a_{ij})$  denote a real  $n \times m$  matrix corresponding to  $T$ . Show that  $T$  is a bounded linear operator with  $\|T\|^2 \leq \sum_i \sum_j a_{ij}^2$ .

We get (cf. EXAMPLE 1.23)

$$\begin{aligned} \|Tx\|_2^2 &= \left\| \left( \sum_{j=1}^m a_{ij}x_j \right)_{i \in \mathbb{N}} \right\|_2^2 = \sum_{i=1}^n \left\{ \sum_{j=1}^m a_{ij}x_j \right\}^2 = \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ij}x_j a_{ik}x_k \\ &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m (a_{ij}x_k) \cdot (a_{ik}x_j). \end{aligned}$$

Then note that

$$|a_{ij}x_k| \cdot |a_{ik}x_j| \leq \frac{1}{2} a_{ij}^2 x_k^2 + \frac{1}{2} a_{ik}^2 x_j^2.$$

By insertion of this inequality,

$$\begin{aligned} \|Tx\|_2^2 &= \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m (a_{ij}x_k) \cdot (a_{ik}x_j) \leq \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ij}^2 x_k^2 + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m \sum_{k=1}^m a_{ik}^2 x_j^2 \\ &= 2 \cdot \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \cdot \sum_{k=1}^m x_k^2 = \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2 \cdot \|x\|_2^2. \end{aligned}$$

Since  $\|T\|^2$  is the smallest constant, for which we have such an estimate, we have

$$\|T\|^2 \leq \sum_{i=1}^n \sum_{j=1}^m a_{ij}^2.$$



**Example 3.4** Let  $T$  be a linear operator from a normed space  $V$  into a normed space  $W$ , and assume that  $V$  is finite dimensional. Show that  $T$  must be bounded.

The space  $V$  is finite dimensional, thus we can choose a basis  $e_1, \dots, e_n$  for  $V$ , where  $\|e_k\|_V = 1$ . Then for every  $v \in V$ ,

$$\begin{aligned}\|Tv\|_W &= \left\| T \left( \sum_{j=1}^n \lambda_j e_j \right) \right\|_W = \left\| \sum_{j=1}^n \lambda_j T e_j \right\|_W \leq \sum_{j=1}^n |\lambda_j| \cdot \|T e_j\|_W \\ &\leq \max \{ \|T e_j\|_W \mid j = 1, \dots, n \} \cdot \sum_{j=1}^n |\lambda_j|.\end{aligned}$$

If we can prove that there exists a constant  $c > 0$ , such that

$$(12) \quad \sum_{j=1}^n |\lambda_j| \leq c \left\| \sum_{j=1}^n \lambda_j e_j \right\|_V \quad \text{for every } \lambda_1, \dots, \lambda_n,$$

then

$$\|Tv\|_W \leq c \cdot \max_j \|T e_j\|_W \cdot \|v\|_V,$$

which shows that  $T$  is bounded

$$\|T\| \leq c \cdot \max_{j=1, \dots, n} \|T e_j\|_W.$$

We shall therefore only prove (12).

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INDIRECT PROOF. Assume that (12) does *not* hold, i.e. assume that

$$(13) \quad \forall N \in \mathbb{N} \exists \lambda_{N,1}, \dots, \lambda_{N,n} : \sum_{j=1}^n |\lambda_{N,j}| > N \left\| \sum_{j=1}^n \lambda_{N,j} e_j \right\|_V.$$

Due to the homogeneity we may assume that  $\lambda_{N,j}$  is chosen, such that

$$\sum_{j=1}^n |\lambda_{N,j}| = 1.$$

Then it follows from (13) that  $\|v_N\|_V \leq \frac{1}{N}$ , hence

$$v_N = \sum_{j=1}^n \lambda_{N,j} e_j \rightarrow 0 \quad \text{for } N \rightarrow \infty.$$

Now,  $e_1, \dots, e_n$  is a basis for  $V$ , hence  $\lambda_{N,j} \rightarrow 0$  for  $N \rightarrow \infty$  for every  $j = 1, \dots, n$ . In particular, there is an  $N_0 \in \mathbb{N}$ , such that for every  $N \geq N_0$  we have  $|\lambda_{N,j}| < \frac{1}{2n}$ . This gives us the following *contradiction*

$$1 = \sum_{j=1}^n |\lambda_{N,j}| < \sum_{j=1}^n \frac{1}{2n} = \frac{1}{2}.$$

We have now proved that (13) does *not* hold, hence (12) holds instead, and as proved previously (12) implies that  $T$  is bounded, and the claim is proved.

**Example 3.5** Let  $T$  be a linear operator from a finite dimensional vector space into itself. Show that  $T$  is injective if and only if  $T$  is surjective.

Let  $T : V \rightarrow V$  be linear, where  $\dim V = n$ . Let  $e_1, \dots, e_n$  form a basis. Now,  $T$  is linear, so  $T$  is injective, if and only if  $Tu = Tv$ , i.e.  $T(u-v) = 0$  implies that  $u = v$ , or put in another way,  $u-v = 0$ . Thus  $T$  is injective, if and only if

$$(14) \quad Tv = 0 \implies v = 0.$$

Now assume that  $T$  is injective. We shall prove that  $Te_1, \dots, Te_n \in V$  are linearly independent.

Assume that  $\lambda_1 Te_1 + \dots + \lambda_n Te_n = 0$ . Then by the linearity,

$$0 = \lambda_1 Te_1 + \dots + \lambda_n Te_n = T(\lambda_1 e_1 + \dots + \lambda_n e_n),$$

and we conclude using (14) that

$$\lambda_1 e_1 + \dots + \lambda_n e_n = 0.$$

Since  $e_1, \dots, e_n$  is a basis for  $V$ , we must have  $\lambda_1 = \dots = \lambda_n = 0$ , and it follows that  $Te_1, \dots, Te_n$  are  $n$  linearly independent vectors in the image  $T(V)$ . Then

$$n \geq \dim T(V) \geq n, \quad \text{thus} \quad \dim T(V) = n,$$

hence  $T(V) = V$ , and we have proved that  $T$  is surjective.

Assume conversely that  $T$  is surjective. To the basis formed by  $e_1, \dots, e_n \in V$  corresponds the vectors  $f_1, \dots, f_n \in V$ , where

$$Tf_1 = e_1, \quad \dots, \quad Tf_n = e_n.$$

If  $\lambda_1 f_1 + \dots + \lambda_n f_n = 0$ , then we conclude that

$$0 = T(\lambda_1 f_1 + \dots + \lambda_n f_n) = \lambda_1 Tf_1 + \dots + \lambda_n Tf_n = \lambda_1 e_1 + \dots + \lambda_n e_n.$$

Using again that  $e_1, \dots, e_n$  form a basis for  $V$ , we infer that  $\lambda_1 = \dots = \lambda_n = 0$ , which again implies that  $f_1, \dots, f_n$  form a basis for  $V$ .

If  $v = \lambda_1 f_1 + \dots + \lambda_n f_n$  satisfies  $Tv = 0$ , then

$$0 = Tv = T(\lambda_1 f_1 + \dots + \lambda_n f_n) = \lambda_1 Tf_1 + \dots + \lambda_n Tf_n = \lambda_1 e_1 + \dots + \lambda_n e_n,$$

and we infer again that  $\lambda_1 = \dots = \lambda_n = 0$ , hence  $v = 0$ , and (14) is fulfilled, so  $T$  is injective.

**Example 3.6** Let  $T$  be the linear mapping from  $C^\infty(\mathbb{R})$  into itself given by  $Tf = f'$ .

Show that  $T$  is surjective?

Is  $T$  injective?

Let  $f \in C^\infty(\mathbb{R})$ . Define  $g \in C^\infty(\mathbb{R})$  by

$$g(t) = \int_0^t f(s) ds, \quad t \in \mathbb{R}.$$

Clearly,  $Tg = f$ , so  $T(V) = C^\infty(\mathbb{R})$ , and  $T$  is surjective.

Define instead

$$g_1(t) = 1 + \int_0^t f(s) ds = 1 + g(t) \in C^\infty(\mathbb{R}).$$

Then

$$Tg_1 = f = Tg,$$

and since  $g_1 \neq g$ , it follows that  $T$  is not injective.

**Example 3.7** Let  $I = [a, b]$  be a bounded interval and consider the linear mapping  $T$  from  $C([a, b])$  into itself, given by

$$Tf(t) = \int_a^t f(s) ds.$$

We assume that  $C([a, b])$  is equipped with the sup-norm.

Show that  $T$  is bounded and find  $\|T\|$ .

Show that  $T$  is injective and find  $T^{-1} : T(C([a, b])) \rightarrow C([a, b])$ .

Is  $T^{-1}$  bounded?

When

$$Tf(t) = \int_a^t f(s) ds \quad \text{for } t \in [a, b],$$

then

$$|Tf(t)| = \left| \int_a^t f(s) ds \right| \leq \int_a^t |f(s)| ds \leq \|f\|_\infty \int_a^t ds = (t - a)\|f\|_\infty \leq (b - a) \cdot \|f\|_\infty,$$

thus

$$\|Tf\|_\infty \leq (b - a) \cdot \|f\|_\infty,$$

proving that  $T$  is bounded and  $\|T\| \leq b - a$ .

Choose  $f(t) = 1$  for every  $t \in [a, b]$ . Then  $\|f\|_\infty = 1$ , and

$$Tf(t) = \int_a^t ds = t - a \quad \text{for } t \in [a, b],$$

hence

$$\|Tf\|_\infty = \sup_{t \in [a, b]} (t - a) = b - a,$$

and we conclude that  $\|T\| \geq b - a$ , whence by the previously proved result,  $\|T\| = b - a$ .

Assume that

$$Tf(t) = \int_a^t f(s) ds \equiv 0.$$

Since  $f \in C([a, b])$ , we have  $Tf \in C^1([a, b])$  with

$$\frac{d}{dt} Tf(t) = f(t) \equiv 0,$$

which shows that  $f \equiv 0$ , so  $T$  is injective.

It follows from the above that  $T(C([a, b])) \subseteq C^1([a, b])$ . We get from  $Tf(a) = 0$  that even

$$T(C([a, b])) \subseteq \{g \in C^1([a, b]) \mid g(a) = 0\}.$$

Conversely, if  $g \in C^1([a, b])$  and  $g(a) = 0$ , then  $f = g' \in C([a, b])$ , and  $Tf = g$ , and the image becomes

$$T(C([a, b])) = \{g \in C^1([a, b]) \mid g(a) = 0\}.$$

Finally, it is immediately seen that

$$T^{-1} : T(C([a, b])) \rightarrow C([a, b])$$

is given by  $T^{-1}g = g'$ .

The operator  $T^{-1}$  is not bounded. We have e.g. that  $(t - a)^n \in T(C([a, b]))$ , and

$$\|(t - a)^n\|_{\infty} = \sup_{t \in [a, b]} |(t - a)^n| = (b - a)^n.$$

It follows from  $T^{-1}(t - a)^n = n(t - a)^{n-1}$  that

$$\|T^{-1}(t - a)^n\|_{\infty} = n(b - a)^{n-1} = \frac{n}{b - a} \|(t - a)^n\|_{\infty},$$

proving that there is no constant  $c > 0$ , such that

$$\|T^{-1}f\|_{\infty} \leq c \|f\|_{\infty}, \quad \text{for all } f \in T(C([a, b])),$$

and  $T$  is not bounded.

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**Example 3.8** Let  $T$  be a bounded linear operator from a normed vector space  $V$  into a normed vector space  $W$ , and assume that  $T$  is surjective. Assume that there is a  $c > 0$ , such that

$$\|Tx\| \geq c\|x\| \quad \text{for all } x \in V.$$

show that  $T^{-1}$  exists and that  $T^{-1} \in B(W, V)$ .

We require that  $T^{-1}$  exists, so we shall first prove that  $T$  is injective, i.e. if  $Tx = Ty$ , then  $x = y$ .

The mapping  $T$  is linear, so this is equivalent with that  $T(x - y) = 0$  implies that  $x - y = 0$ , or by a slight change of notation:

Assume that  $Tx = 0$ . Prove that  $x = 0$ .

When  $Tx = 0$ , then it follows from the assumption that

$$0 \leq \|x\| \leq \frac{1}{c} \|Tx\| = 0, \quad \text{thus } \|x\| = 0, \text{ hence } x = 0,$$

and the claim is proved.

We have proved that  $T$  is injective, thus  $T^{-1}$  exists. Now  $T(V) = W$ , so  $T^{-1} : W \rightarrow V$ , and  $T^{-1}$  is defined on all of  $W$ . It remains only to be proved that  $T^{-1}$  is bounded.

Let  $y \in W$ . Then  $x = T^{-1}y$  is defined. It follows from the assumption that

$$\|T^{-1}y\| = \|x\| \leq \frac{1}{c} \|Tx\| = \frac{1}{c} \|T(T^{-1}y)\| = \frac{1}{c} \|y\|,$$

which shows that  $T^{-1}$  is bounded,  $\|T^{-1}\| \leq \frac{1}{c}$ , and it follows that  $T^{-1} \in B(W, V)$ .

**Example 3.9** Let  $V$  and  $W$  be two normed spaces. Prove that  $B(V, W)$  is a normed vector space and that  $B(V, W)$  is a Banach space, if  $W$  is a Banach space.

It is well-known that  $B(V, W)$  is a vector space.

Define  $\|T\|$  by

$$\|T\| = \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\}.$$

Then clearly,  $\|T\| \geq 0$ . If  $T \neq 0$ , then there exists an  $x \in V$ , such that  $Tx \neq 0$ , and we conclude that  $\|T\| = 0$ , if and only if  $T = 0$ .

Furthermore,

$$\|\alpha T\| = \sup\{\|\alpha Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \sup\{\|Tx\|_W \mid \|x\|_V \leq 1\} = |\alpha| \cdot \|T\|.$$

Finally,

$$\begin{aligned} \|T_1 + T_2\| &= \sup\{\|(T_1 + T_2)x\|_W \mid \|x\|_V \leq 1\} \leq \sup\{\|T_1x\|_W + \|T_2x\|_W \mid \|x\|_V \leq 1\} \\ &\leq \sup\{\|T_1x\|_W \mid \|x\|_V \leq 1\} + \sup\{\|T_2x\|_W \mid \|x\|_V \leq 1\} = \|T_1\| + \|T_2\|, \end{aligned}$$

and we have proved that  $\|\cdot\|$  is a norm on  $B(V, W)$ , and  $B(V, W)$  is a normed vector space.

We now assume that  $W$  is a Banach space, thus every Cauchy sequence on  $W$  is convergent. We shall prove that  $B(V, W)$  becomes a Banach space with the norm introduced above. Let  $(T_n)$  be a Cauchy sequence on  $B(V, W)$ , i.e.

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|T_m - T_n\| < \varepsilon.$$

Then it follows from the definition that

$$\|T_m - T_n\| = \sup\{\|(T_m - T_n)x\|_W \mid \|x\|_V \leq 1\} = \sup\{\|T_m - T_n\|_W \mid \|x\|_V \leq 1\} < \varepsilon.$$

In particular, we have for every  $x \in V$ , for which  $\|x\|_V \leq 1$  that

$$\forall \varepsilon > 0 \exists N \in \mathbb{N} \forall m, n \geq N : \|T_m x - T_n x\|_W < \varepsilon,$$

which is the condition for  $(T_n x)$  being a Cauchy sequence on  $W$ . We assumed that  $W$  was a Banach space, so it is complete. This implies that  $(T_n x)$  is convergent, and it follows that  $(T_n(\lambda x)) = (\lambda T_n x)$  is also convergent in  $W$  for every  $\lambda$ , and the condition  $\|x\|_V \leq 1$  has become superfluous.

Define an operator  $T : V \rightarrow W$  by

$$Tx = \lim_{n \rightarrow +\infty} T_n x, \quad x \in V.$$

Then

$$T(x + \lambda y) = \lim_{n \rightarrow +\infty} T_n(x + \lambda y) = \lim_{n \rightarrow +\infty} \{T_n x + \lambda T_n y\} = \lim_{n \rightarrow +\infty} T_n x + \lambda \lim_{n \rightarrow +\infty} T_n y = Tx + \lambda Ty,$$

which shows that  $T$  is linear.

It remains to be proved that  $T \in B(V, W)$ , i.e. that  $T$  is bounded. If  $x \in V$  with  $\|x\|_V \leq 1$ , then

$$\|Tx\| = \left\| \lim_{n \rightarrow +\infty} T_n x \right\| \leq \sup_{n \in \mathbb{N}} \|T_n x\| \leq \sup_{n \in \mathbb{N}} \|T_n\|.$$

Since  $(T_n)$  is a Cauchy sequence, we have  $\sup_{n \in \mathbb{N}} \|T_n\| < +\infty$ , and we conclude that  $T \in B(V, W)$ . Thus we have proved that the Cauchy sequence  $(T_n) \subseteq B(V, W)$  converges towards  $T \in B(V, W)$ , and we have proved that  $B(V, W)$  is a Banach space.

**Example 3.10** Let  $S, T \in B(V, V)$ . Prove that the composite mapping  $ST$  (defined by  $(ST)x = S(Tx)$  for  $x \in V$ ) belongs to  $B(V, V)$ , and that

$$\|ST\| \leq \|S\| \cdot \|T\|.$$

When  $S, T \in B(V, V)$ , the composition  $ST$  is defined (and linear) on all of  $V$ . We shall only prove that  $ST$  is bounded. Now, for every  $x \in V$ ,

$$\|(ST)x\|_V = \|S(Tx)\|_V \leq \|S\| \cdot \|Tx\|_V \leq \|S\| \cdot \|T\| \cdot \|x\|_V,$$

so

$$\|ST\| = \sup\{\|(ST)x\|_V \mid \|x\|_V \leq 1\} \leq \sup\{\|S\| \cdot \|T\| \cdot \|x\|_V \mid \|x\|_V \leq 1\} = \|S\| \cdot \|T\|.$$

**Example 3.11** Let  $V$  be a Banach space and let  $T \in B(V)$  be such that  $T^{-1}$  exists and belongs to  $B(V)$ .

Show that if  $\|T\|$  and  $\|T^{-1}\| \leq 1$ , then

$$\|T\| = \|T^{-1}\| = 1,$$

and  $\|Tf\| = \|f\|$  for all  $f \in V$ .

It follows from the assumptions that  $T$  is bijective,

$$(15) \quad Tf = g, \quad T^{-1}g = f.$$

We first prove that

$$\|Tf\| = \|f\| \quad \text{for every } f \in V.$$

This follows from

$$\|Tf\| \leq \|T\| \cdot \|f\| = \|f\| = \|T^{-1}f\| \leq \|T^{-1}\| \cdot \|g\| = \|g\| = \|Tf\|.$$

Hence we must have equality everywhere, and in particular,

$$\|Tf\| = \|f\| \quad \text{for all } f \in V,$$

and

$$\|T^{-1}g\| = \|g\| \quad \text{for all } g \in V.$$

Finally, we get

$$\|T\| = \sum \{\|Tf\| \mid \|f\| = 1\} = \sup\{\|f\| \mid \|f\| = 1\} = 1,$$

and

$$\|T^{-1}\| = \sup\{\|T^{-1}g\| \mid \|g\| = 1\} = \sup\{\|g\| \mid \|g\| = 1\} = 1.$$

**Example 3.12** Let  $H$  denote a Hilbert space and let  $T \in B(H)$  and assume that there is a positive  $c$  such that

$$|(Tx, x)| \geq c \|x\|^2 \quad \text{for all } x \in H.$$

Show that  $T^{-1}$  exists and belongs to  $B(H)$ .

Assume that  $Tx = 0$ . Then

$$0 = |(Tx, x)| \geq c \|x\|^2 \geq 0,$$

from which we conclude that  $x = 0$ , and we have proved that  $T$  is injective, so  $T^{-1}$  exists.



If  $x = T^{-1}y$  for some  $y \in H$ , then it follows from the estimate

$$c \|x\|^2 = c \|T^{-1}y\|^2 \leq |(y, T^{-1}y)| \leq \|y\| \cdot \|T^{-1}y\|,$$

that  $\|T^{-1}\| \leq \frac{1}{c}$ , so  $T^{-1}$  is bounded on the image  $T(H)$ .

It remains to prove that the image  $T(H)$  is all of  $H$ . Let  $x \perp T(H)$ . Then we get again that

$$0 = |(Tx, x)| \geq c \|x\|^2,$$

which proves that  $x = 0$  is the only vector, which is perpendicular to the image, so  $\overline{T(H)} = H$ . Since  $T^{-1}$  is bounded, it has a continuous extension to  $\overline{T(H)} = H$ , and it follows that  $T^{-1} \in B(H)$ .

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**Example 3.13** Let  $p > 1$  and let  $f(x, t) \geq 0$  be a (measurable) function on  $\mathbb{R}^2$  such that

$$g(t) = \left\{ \int_{\mathbb{R}} f(x, t) dx \right\}^{p-1}$$

exists.

1) Put  $q = \frac{p}{p-1}$  and show that

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p \leq \|g\|_q \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

2) Let  $f(x, t)$  be a (measurable) function on  $\mathbb{R}^2$  such that the function

$$x \mapsto \|f(x, \cdot)\|_p$$

belongs to  $L^1(\mathbb{R})$ . Use question 1 to show the inequality

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx,$$

first for  $p > 1$ , and then for  $p = 1$ .

3) Let  $g \in L^p(\mathbb{R})$  and  $h \in L^1(\mathbb{R})$ . We define the convolution  $g \star h$  by

$$g \star h(t) = \int_{\mathbb{R}} g(t-x) h(x) dx.$$

Show that convolution with an  $L^1(\mathbb{R})$ -function is a linear and bounded mapping from  $L^p(\mathbb{R})$  into  $L^p(\mathbb{R})$  for any  $p > 1$ .

1) We get

$$\begin{aligned} \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dx \right\}^p dt = \int_{\mathbb{R}} g(t) \left\{ \int_{\mathbb{R}} f(x, t) dx \right\} dt \\ &= \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} g(t) \cdot f(x, t) dt \right\} dx \leq \int_{\mathbb{R}} \|g\|_q \|f(x, \cdot)\|_p dx \\ &= \|g\|_q \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx. \end{aligned}$$

2) We may of course assume that  $f(x, t) \geq 0$ , because we can in general replace  $f$  by  $|f|$ , which gives a more “narrow” estimate. Then we can use the result from 1.

Let  $p > 1$ . Then

$$\begin{aligned} \|g\|_q &= \left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, t) dx \right)^{(p-1) \cdot \frac{p}{p-1}} dt \right\}^{\frac{p-1}{p}} = \left( \left\{ \int_{\mathbb{R}} \left( \int_{\mathbb{R}} f(x, t) dx \right)^p dt \right\}^{\frac{1}{p}} \right)^{p-1} \\ &= \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^{p-1}, \end{aligned}$$

which inserted into the result of 1) gives

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p^p \leq \left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_o^{p-1} \cdot \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

Since  $p > 1$ , this is reduced to

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx.$$

When  $p = 1$ , then we get instead by interchanging the order of integration

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_1 = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dx \right\} dt = \int_{\mathbb{R}} \left\{ \int_{\mathbb{R}} f(x, t) dt \right\} dx = \int_{\mathbb{R}} \|f(x, \cdot)\|_1 dt.$$

For a general  $f$  we get

$$\left\| \int_{\mathbb{R}} f(x, \cdot) dx \right\|_1 \leq \left\| \int_{\mathbb{R}} |f(x, \cdot)| dx \right\|_p \leq \int_{\mathbb{R}} \|f(x, \cdot)\|_p dx,$$

because  $\| |f(x, \cdot)| \|_p = \|f(x, \cdot)\|_p$ .

3) Given  $h \in L^1(\mathbb{R})$ - Define an operator  $T$  by

$$Tg(x) = g \star h(x),$$

for the  $g \in L^p(\mathbb{R})$ ,  $p > 1$ , for which this expression makes sense. Then clearly,  $T$  is linear.

Let  $g \in L^p(\mathbb{R})$ . Using 2) above we get the following estimate, where we allow ourselves to write  $\|g \star h\|$  before we have proved that it makes sense,

$$\begin{aligned} \|Tg\|_p &= \|g \star h\|_p = \left\| \int_{\mathbb{R}} g(\star - x) h(x) dx \right\|_p \\ &\leq \int_{\mathbb{R}} \|g(\star - x)\|_p \cdot h(x) dx = \|g\|_p \cdot \|h\|_1 < \infty. \end{aligned}$$

This estimate shows that  $g \star h \in L^p(\mathbb{R})$  is defined and that the mapping  $T$  is bounded of norm  $\|T\| \leq \|h\|_1$ .

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