Answer Key for Major Task Part (2)

Part (A): Laplace Transform

I) Find Laplace transform of:
1) $e^{-5t}(\cosh 2t + \sinh 6t)$

This problem requires the application of the First Shifting Theorem, which states that $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$, where $F(s) = \mathcal{L}\{f(t)\}$.

Step 1: Define $f(t) = \cosh 2t + \sinh 6t$ and find its Laplace transform, $F(s)$.

Using the linearity of the Laplace transform:

$$\mathcal{L}\{\cosh 2t + \sinh 6t\} = \mathcal{L}\{\cosh 2t\} + \mathcal{L}\{\sinh 6t\}$$

From the table of standard Laplace transforms, we know:

$$\mathcal{L}\{\cosh at\} = \frac{s}{s^2-a^2} \quad \text{and} \quad \mathcal{L}\{\sinh at\} = \frac{a}{s^2-a^2}$$

Applying these formulas for $a = 2$ and $a = 6$:

$$\mathcal{L}\{\cosh 2t\} = \frac{s}{s^2-4}$$

$$\mathcal{L}\{\sinh 6t\} = \frac{6}{s^2-36}$$

Therefore, the transform $F(s)$ is:

$$F(s) = \frac{s}{s^2-4} + \frac{6}{s^2-36}$$

Step 2: Apply the First Shifting Theorem for the term e^{-5t}, which corresponds to $a = -5$.

The required transform is $F(s-(-5)) = F(s+5)$.

Substitute s with $s+5$ in the expression for $F(s)$:

$$\mathcal{L}\{e^{-5t}(\cosh 2t + \sinh 6t)\} = \frac{s+5}{(s+5)^2-4} + \frac{6}{(s+5)^2-36}$$

$\boxed{\mathcal{L}\{e^{-5t}(\cosh 2t + \sinh 6t)\} = \frac{s+5}{(s+5)^2-4} + \frac{6}{(s+5)^2-36}}$

2) $e^{-3t} \cos 4t \sin 2t$

This problem is solved by first using a trigonometric product-to-sum identity and then applying the First Shifting Theorem.

Step 1: Simplify the trigonometric product $\cos 4t \sin 2t$. We use the identity:

$$\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$$

Letting $A = 4t$ and $B = 2t$:

$$\cos 4t \sin 2t = \frac{1}{2}[\sin(4t+2t) - \sin(4t-2t)] = \frac{1}{2}[\sin 6t - \sin 2t]$$

The function to transform is now $e^{-3t}\left(\frac{1}{2}[\sin 6t - \sin 2t]\right)$.

Step 2: Find the Laplace transform of $f(t) = \frac{1}{2}[\sin 6t - \sin 2t]$, which we will call $F(s)$.

Using the linearity of the Laplace transform and the standard transform $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$:

$$\mathcal{L}\{f(t)\} = \frac{1}{2}(\mathcal{L}\{\sin 6t\} - \mathcal{L}\{\sin 2t\})$$

$$F(s) = \frac{1}{2}\left(\frac{6}{s^2+6^2} - \frac{2}{s^2+2^2}\right) = \frac{1}{2}\left(\frac{6}{s^2+36} - \frac{2}{s^2+4}\right)$$

$$F(s) = \frac{3}{s^2+36} - \frac{1}{s^2+4}$$

Step 3: Apply the First Shifting Theorem for the term e^{-3t}, which corresponds to $a = -3$.

The required transform is $F(s-(-3)) = F(s+3)$.

Substitute s with $s+3$ in the expression for $F(s)$:

$$\mathcal{L}\{e^{-3t} \cos 4t \sin 2t\} = \frac{3}{(s+3)^2+36} - \frac{1}{(s+3)^2+4}$$

$\boxed{\mathcal{L}\{e^{-3t} \cos 4t \sin 2t\} = \frac{3}{(s+3)^2+36} - \frac{1}{(s+3)^2+4}}$

3) $\frac{\cos t - \cos 3t}{t}$

To find the Laplace transform of a function divided by t, we use the "Integration of Transform" theorem. This theorem states that if $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(u)\,du$$

This is valid if the limit $\lim_{t\to 0^+} \frac{f(t)}{t}$ exists.

Step 1: Define $f(t) = \cos t - \cos 3t$ and find its Laplace transform $F(s)$.

Using the linearity of the Laplace transform and the standard result $\mathcal{L}\{\cos at\} = \frac{s}{s^2+a^2}$:

$$F(s) = \mathcal{L}\{\cos t\} - \mathcal{L}\{\cos 3t\} = \frac{s}{s^2+1} - \frac{s}{s^2+9}$$

Step 2: Apply the Integration of Transform theorem by integrating $F(u)$ from s to ∞.

$$\mathcal{L}\left\{\frac{\cos t - \cos 3t}{t}\right\} = \int_s^\infty \left(\frac{u}{u^2+1} - \frac{u}{u^2+9}\right)du$$

Step 3: Evaluate the integral.

The integral can be computed as follows:

$$\begin{aligned}\int_s^\infty \left(\frac{u}{u^2+1} - \frac{u}{u^2+9}\right)du &= \left[\frac{1}{2}\ln(u^2+1) - \frac{1}{2}\ln(u^2+9)\right]_s^\infty \\ &= \frac{1}{2}\left[\ln\left(\frac{u^2+1}{u^2+9}\right)\right]_s^\infty \\ &= \lim_{b\to\infty}\left[\frac{1}{2}\ln\left(\frac{b^2+1}{b^2+9}\right)\right] - \frac{1}{2}\ln\left(\frac{s^2+1}{s^2+9}\right)\end{aligned}$$

The limit term evaluates to zero:

$$\lim_{b\to\infty}\frac{1}{2}\ln\left(\frac{1+1/b^2}{1+9/b^2}\right) = \frac{1}{2}\ln(1) = 0$$

The final result is:

$$0 - \frac{1}{2}\ln\left(\frac{s^2+1}{s^2+9}\right) = -\frac{1}{2}\ln\left(\frac{s^2+1}{s^2+9}\right) = \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+1}\right)$$

$$\boxed{\mathcal{L}\left\{\frac{\cos t - \cos 3t}{t}\right\} = \frac{1}{2}\ln\left(\frac{s^2+9}{s^2+1}\right)}$$

4) $t \sin 2t \sinh 2t$

This problem is solved using the "Differentiation of Transform" theorem, which states that if $\mathcal{L}\{f(t)\} = F(s)$, then:

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

For $n = 1$, this simplifies to $\mathcal{L}\{tf(t)\} = -\frac{d}{ds} F(s)$.

Step 1: Let $f(t) = \sin 2t \sinh 2t$ and find its Laplace transform $F(s)$.

First, express $\sinh 2t$ in its exponential form: $\sinh 2t = \frac{e^{2t} - e^{-2t}}{2}$.

$$f(t) = \sin 2t \left(\frac{e^{2t} - e^{-2t}}{2} \right) = \frac{1}{2} e^{2t} \sin 2t - \frac{1}{2} e^{-2t} \sin 2t$$

We find the Laplace transform of $f(t)$ using linearity and the First Shifting Theorem on the base transform $\mathcal{L}\{\sin 2t\} = \frac{2}{s^2+4}$.

$$\begin{aligned} F(s) &= \frac{1}{2}\mathcal{L}\{e^{2t} \sin 2t\} - \frac{1}{2}\mathcal{L}\{e^{-2t} \sin 2t\} \\ &= \frac{1}{2}\left(\frac{2}{(s-2)^2+4} - \frac{2}{(s+2)^2+4} \right) \\ &= \frac{1}{s^2-4s+8} - \frac{1}{s^2+4s+8} \end{aligned}$$

Step 2: Apply the differentiation theorem $\mathcal{L}\{tf(t)\} = -\frac{dF}{ds}$.

$$\begin{aligned} \mathcal{L}\{tf(t)\} &= -\frac{d}{ds}\left(\frac{1}{s^2-4s+8} - \frac{1}{s^2+4s+8} \right) \\ &= -\left[-\frac{2s-4}{(s^2-4s+8)^2} - \left(-\frac{2s+4}{(s^2+4s+8)^2} \right) \right] \\ &= \frac{2s-4}{(s^2-4s+8)^2} - \frac{2s+4}{(s^2+4s+8)^2} \end{aligned}$$

$\mathcal{L}\{t \sin 2t \sinh 2t\} = \frac{2s-4}{(s^2-4s+8)^2} - \frac{2s+4}{(s^2+4s+8)^2}$

5) $te^{2t}(\cos 3t)^2$

This problem combines the First Shifting Theorem and the Differentiation of Transform theorem. A direct approach is to handle the $t(\cos 3t)^2$ part first, then apply the shift.

Step 1: Let $h(t) = t(\cos 3t)^2$. We find its Laplace transform, $H(s)$.

First, simplify $(\cos 3t)^2$ with the power-reduction identity $\cos^2 A = \frac{1}{2}(1 + \cos 2A)$.

$$(\cos 3t)^2 = \frac{1}{2}(1 + \cos 6t)$$

Thus, $h(t) = t \cdot \frac{1}{2}(1 + \cos 6t) = \frac{1}{2}t + \frac{1}{2}t\cos 6t$.

Step 2: Find the Laplace transform $H(s) = \mathcal{L}\{h(t)\}$.

$$H(s) = \frac{1}{2}\mathcal{L}\{t\} + \frac{1}{2}\mathcal{L}\{t\cos 6t\}$$

The transform of the first term is $\mathcal{L}\{t\} = \frac{1}{s^2}$.

For the second term, we apply the differentiation theorem $\mathcal{L}\{tf(t)\} = -F'(s)$ with $f(t) = \cos 6t$:

$$\mathcal{L}\{t\cos 6t\} = -\frac{d}{ds}\mathcal{L}\{\cos 6t\} = -\frac{d}{ds}\left(\frac{s}{s^2+36}\right)$$

$$-\frac{d}{ds}\left(\frac{s}{s^2+36}\right) = -\frac{(s^2+36)(1) - s(2s)}{(s^2+36)^2} = -\frac{s^2+36-2s^2}{(s^2+36)^2} = \frac{s^2-36}{(s^2+36)^2}$$

Combining these results gives $H(s)$:

$$H(s) = \frac{1}{2}\left[\frac{1}{s^2} + \frac{s^2-36}{(s^2+36)^2}\right]$$

Step 3: Apply the First Shifting Theorem to find $\mathcal{L}\{e^{2t}h(t)\}$, which is $H(s-2)$.

We replace every s in $H(s)$ with $s-2$:

$$\mathcal{L}\{te^{2t}(\cos 3t)^2\} = \frac{1}{2}\left[\frac{1}{(s-2)^2} + \frac{(s-2)^2-36}{((s-2)^2+36)^2}\right]$$

$\mathcal{L}\{te^{2t}(\cos 3t)^2\} = \frac{1}{2}\left[\frac{1}{(s-2)^2} + \frac{(s-2)^2-36}{((s-2)^2+36)^2}\right]$

6) $\sin^2 4t \sinh^2 2t$

To find the Laplace transform, we first use power-reduction identities to simplify the function into a sum of terms whose transforms are known or can be easily derived.

Step 1: Apply the power-reduction identities for sine and hyperbolic sine.

$$\sin^2 A = \frac{1}{2}(1 - \cos 2A) \quad \text{and} \quad \sinh^2 B = \frac{1}{2}(\cosh 2B - 1)$$

For our function, with $A = 4t$ and $B = 2t$:

$$\sin^2 4t = \frac{1}{2}(1 - \cos 8t)$$

$$\sinh^2 2t = \frac{1}{2}(\cosh 4t - 1)$$

Step 2: Expand the product of these two expressions.

$$\begin{aligned} f(t) = \sin^2 4t \sinh^2 2t &= \left[\frac{1}{2}(1 - \cos 8t)\right]\left[\frac{1}{2}(\cosh 4t - 1)\right] \\ &= \frac{1}{4}(1 - \cos 8t)(\cosh 4t - 1) \\ &= \frac{1}{4}(\cosh 4t - 1 - \cos 8t \cosh 4t + \cos 8t) \end{aligned}$$

Step 3: Find the Laplace transform of each term using linearity.

$$\mathcal{L}\{f(t)\} = \frac{1}{4}[\mathcal{L}\{\cosh 4t\} - \mathcal{L}\{1\} + \mathcal{L}\{\cos 8t\} - \mathcal{L}\{\cos 8t \cosh 4t\}]$$

The standard transforms are:

$$\mathcal{L}\{\cosh 4t\} = \frac{s}{s^2 - 16}, \quad \mathcal{L}\{1\} = \frac{1}{s}, \quad \mathcal{L}\{\cos 8t\} = \frac{s}{s^2 + 64}$$

For the product term $\mathcal{L}\{\cos 8t \cosh 4t\}$, we express $\cosh 4t$ in exponential form $\cosh 4t = \frac{e^{4t} + e^{-4t}}{2}$ and apply the First Shifting Theorem:

$$\begin{aligned} \mathcal{L}\{\cos 8t \cosh 4t\} &= \mathcal{L}\left\{\cos 8t\left(\frac{e^{4t} + e^{-4t}}{2}\right)\right\} \\ &= \frac{1}{2}[\mathcal{L}\{e^{4t}\cos 8t\} + \mathcal{L}\{e^{-4t}\cos 8t\}] \\ &= \frac{1}{2}\left[\frac{s-4}{(s-4)^2 + 64} + \frac{s+4}{(s+4)^2 + 64}\right] \end{aligned}$$

Step 4: Combine all the transformed terms to get the final answer.

$$\mathcal{L}\{f(t)\} = \frac{1}{4}\left[\frac{s}{s^2 - 16} - \frac{1}{s} + \frac{s}{s^2 + 64} - \frac{1}{2}\left(\frac{s-4}{(s-4)^2 + 64} + \frac{s+4}{(s+4)^2 + 64}\right)\right]$$

$$\boxed{\mathcal{L}\{\sin^2 4t \sinh^2 2t\} = \frac{1}{4}\left[\frac{s}{s^2-16} - \frac{1}{s} + \frac{s}{s^2+64} - \frac{s-4}{2((s-4)^2+64)} - \frac{s+4}{2((s+4)^2+64)}\right]}$$

7) $e^{2t}(t-1)^2$

This problem is solved efficiently by first finding the Laplace transform of $(t-1)^2$ and then applying the First Shifting Theorem.

Step 1: Define $f(t) = (t-1)^2$. Expand the polynomial.

$$f(t) = t^2 - 2t + 1$$

Step 2: Find the Laplace transform of $f(t)$, denoted as $F(s)$.

Using the linearity of the Laplace transform and the standard formula $\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$:

$$\begin{aligned} F(s) = \mathcal{L}\{t^2 - 2t + 1\} &= \mathcal{L}\{t^2\} - 2\mathcal{L}\{t\} + \mathcal{L}\{1\} \\ &= \frac{2!}{s^3} - 2 \cdot \frac{1!}{s^2} + \frac{0!}{s^1} \\ &= \frac{2}{s^3} - \frac{2}{s^2} + \frac{1}{s} \end{aligned}$$

Step 3: Apply the First Shifting Theorem, $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$, with $a = 2$.

We replace every s in $F(s)$ with $s - 2$.

$$\mathcal{L}\{e^{2t}(t-1)^2\} = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}$$

$\boxed{\mathcal{L}\{e^{2t}(t-1)^2\} = \frac{2}{(s-2)^3} - \frac{2}{(s-2)^2} + \frac{1}{s-2}}$

II) Find the inverse Laplace transform of:

1) $\frac{s^2}{(s^2+9)^2}$

This problem is efficiently solved using the Convolution Theorem, which states that $\mathcal{L}^{-1}\{F(s)G(s)\} = (f * g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$.

Step 1: Decompose the function into a product of two simpler transforms, $F(s)$ and $G(s)$.

$$\frac{s^2}{(s^2+9)^2} = \left(\frac{s}{s^2+9}\right) \cdot \left(\frac{s}{s^2+9}\right)$$

Let $F(s) = G(s) = \frac{s}{s^2+9}$.

Step 2: Find the corresponding time-domain functions, $f(t)$ and $g(t)$.

From the standard transform pair $\mathcal{L}\{\cos(at)\} = \frac{s}{s^2+a^2}$, with $a = 3$, we get:

$$f(t) = g(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+9}\right\} = \cos(3t)$$

Step 3: Apply the Convolution Theorem by evaluating the convolution integral.

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+9)^2}\right\} = \int_0^t \cos(3\tau)\cos(3(t-\tau))d\tau$$

Step 4: Simplify the integrand using the product-to-sum identity $\cos A \cos B = \frac{1}{2}[\cos(A-B) + \cos(A+B)]$.

Let $A = 3\tau$ and $B = 3t - 3\tau$. Then $A - B = 6\tau - 3t$ and $A + B = 3t$.

$$\cos(3\tau)\cos(3t-3\tau) = \frac{1}{2}[\cos(6\tau - 3t) + \cos(3t)]$$

Step 5: Integrate the simplified expression from 0 to t.

$$\begin{aligned}
\int_0^t \frac{1}{2}[\cos(6\tau-3t)+\cos(3t)]d\tau &= \frac{1}{2}\left[\frac{\sin(6\tau-3t)}{6} + \tau\cos(3t)\right]_0^t \\
&= \frac{1}{2}\left[\left(\frac{\sin(3t)}{6} + t\cos(3t)\right) - \left(\frac{\sin(-3t)}{6} + 0\right)\right] \\
&= \frac{1}{2}\left[\frac{\sin(3t)}{6} + t\cos(3t) + \frac{\sin(3t)}{6}\right] \quad (\text{since } \sin(-x) = -\sin(x)) \\
&= \frac{1}{2}\left[\frac{2\sin(3t)}{6} + t\cos(3t)\right] \\
&= \frac{1}{6}\sin(3t) + \frac{1}{2}t\cos(3t)
\end{aligned}$$

$\mathcal{L}^{-1}\left\{\frac{s^2}{(s^2+9)^2}\right\} = \frac{1}{2}t\cos(3t) + \frac{1}{6}\sin(3t)$

2) $\frac{s+5}{(s+4)(s^2+4s+13)}$

To find the inverse Laplace transform, we use partial fraction decomposition.

Step 1: Decompose the expression. The quadratic factor $s^2+4s+13$ is irreducible as its discriminant $4^2-4(13)<0$.

$$\frac{s+5}{(s+4)(s^2+4s+13)} = \frac{A}{s+4} + \frac{Bs+C}{s^2+4s+13}$$

Step 2: Solve for the coefficients A, B, and C by clearing the denominator.

$$s+5 = A(s^2+4s+13) + (Bs+C)(s+4)$$

Set $s=-4$ to find A:

$$-4+5 = A(16-16+13) \implies 1 = 13A \implies A = \frac{1}{13}$$

Expand and equate coefficients for B and C:

$$s+5 = (A+B)s^2 + (4A+4B+C)s + (13A+4C)$$

From the s^2 term: $0 = A+B \implies B = -A = -\frac{1}{13}$.

From the constant term: $5 = 13A+4C \implies 5 = 13(\frac{1}{13})+4C \implies 5 = 1+4C \implies C=1$.

The decomposition is:

$$\frac{1}{13}\cdot\frac{1}{s+4} + \frac{-\frac{1}{13}s+1}{s^2+4s+13} = \frac{1}{13}\left(\frac{1}{s+4} + \frac{-s+13}{s^2+4s+13}\right)$$

Step 3: Rewrite the quadratic term by completing the square and adjusting the numerator.

$$s^2+4s+13 = (s+2)^2+9 = (s+2)^2+3^2$$

$$\frac{-s+13}{(s+2)^2+3^2} = \frac{-(s+2)+2+13}{(s+2)^2+3^2} = -\frac{s+2}{(s+2)^2+3^2} + \frac{15}{(s+2)^2+3^2} = -\frac{s+2}{(s+2)^2+3^2} + 5\cdot\frac{3}{(s+2)^2+3^2}$$

Step 4: Find the inverse Laplace transform of the complete expression.

$$\begin{aligned} f(t) &= \mathcal{L}^{-1}\left\{\frac{1}{13}\left(\frac{1}{s+4}\right) + \frac{1}{13}\left(-\frac{s+2}{(s+2)^2+3^2} + 5\cdot\frac{3}{(s+2)^2+3^2}\right)\right\} \\ &= \frac{1}{13}e^{-4t} - \frac{1}{13}\mathcal{L}^{-1}\left\{\frac{s+2}{(s+2)^2+3^2}\right\} + \frac{5}{13}\mathcal{L}^{-1}\left\{\frac{3}{(s+2)^2+3^2}\right\} \\ &= \frac{1}{13}e^{-4t} - \frac{1}{13}e^{-2t}\cos(3t) + \frac{5}{13}e^{-2t}\sin(3t) \end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{s+5}{(s+4)(s^2+4s+13)}\right\} = \frac{1}{13}e^{-4t} + \frac{e^{-2t}}{13}(5\sin(3t)-\cos(3t))$$

> 3) $\frac{2}{s^3(s^2+1)}$

The inverse Laplace transform can be found using the Convolution Theorem.

Step 1: Decompose the function into a product $F(s)G(s)$.

$$\frac{2}{s^3(s^2+1)} = \left(\frac{2}{s^3}\right) \cdot \left(\frac{1}{s^2+1}\right)$$

Let $F(s) = \frac{2}{s^3}$ and $G(s) = \frac{1}{s^2+1}$.

Step 2: Find the inverse transforms $f(t)$ and $g(t)$.

$$f(t) = \mathcal{L}^{-1}\left\{\frac{2!}{s^{2+1}}\right\} = t^2$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1^2}\right\} = \sin(t)$$

Step 3: Apply the Convolution Theorem: $\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)g(t-\tau)d\tau$.

$$\mathcal{L}^{-1}\left\{\frac{2}{s^3(s^2+1)}\right\} = \int_0^t \tau^2 \sin(t-\tau)d\tau$$

Step 4: Evaluate the integral using integration by parts twice.

First pass: Let $u = \tau^2$ and $dv = \sin(t-\tau)d\tau$. Then $du = 2\tau d\tau$ and $v = \cos(t-\tau)$.

$$\int \tau^2 \sin(t-\tau)d\tau = \tau^2 \cos(t-\tau) - \int 2\tau \cos(t-\tau)d\tau$$

Second pass on the remaining integral: Let $u = 2\tau$ and $dv = \cos(t-\tau)d\tau$. Then $du = 2d\tau$ and $v = -\sin(t-\tau)$.

$$\int 2\tau \cos(t-\tau)d\tau = -2\tau \sin(t-\tau) - \int -2\sin(t-\tau)d\tau = -2\tau \sin(t-\tau) + 2\cos(t-\tau)$$

Combining these, the antiderivative is:

$$\tau^2 \cos(t-\tau) - [-2\tau \sin(t-\tau) + 2\cos(t-\tau)] = (\tau^2 - 2)\cos(t-\tau) + 2\tau \sin(t-\tau)$$

Step 5: Evaluate the definite integral from 0 to t:

$$\begin{aligned}
&[(\tau^2-2)\cos(t-\tau) + 2\tau\sin(t-\tau)]_0^t \\
&= [(t^2-2)\cos(0) + 2t\sin(0)] - [(0-2)\cos(t) + 0 \cdot \sin(t)] \\
&= [t^2-2] - [-2\cos(t)] \\
&= t^2 - 2 + 2\cos(t)
\end{aligned}$$

$\boxed{\mathcal{L}^{-1}\left\{\frac{2}{s^3(s^2+1)}\right\} = t^2 - 2 + 2\cos(t)}$

III) Use Laplace transform to solve the following equations:

1) $3y' - 4y = \sin(2t)$, $y(0) = \frac{1}{3}$

We solve the initial value problem by applying the Laplace transform to the differential equation.

Step 1: Take the Laplace transform of both sides.

$$\mathcal{L}\{3y' - 4y\} = \mathcal{L}\{\sin(2t)\}$$

$$3\mathcal{L}\{y'\} - 4\mathcal{L}\{y\} = \mathcal{L}\{\sin(2t)\}$$

Step 2: Substitute the Laplace transform rules for derivatives and functions. Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$3(sY(s) - y(0)) - 4Y(s) = \frac{2}{s^2+4}$$

Step 3: Substitute the initial condition $y(0) = \frac{1}{3}$ and algebraically solve for $Y(s)$.

$$3\left(sY(s) - \frac{1}{3}\right) - 4Y(s) = \frac{2}{s^2+4}$$

$$3sY(s) - 1 - 4Y(s) = \frac{2}{s^2+4}$$

$$Y(s)(3s-4) = 1 + \frac{2}{s^2+4} = \frac{s^2+4+2}{s^2+4} = \frac{s^2+6}{s^2+4}$$

$$Y(s) = \frac{s^2+6}{(3s-4)(s^2+4)}$$

Step 4: Decompose $Y(s)$ using partial fractions.

$$\frac{s^2+6}{(3s-4)(s^2+4)} = \frac{A}{3s-4} + \frac{Bs+C}{s^2+4}$$

Multiplying by the denominator gives:

$$s^2 + 6 = A(s^2+4) + (Bs+C)(3s-4)$$

To find A, set $s = \frac{4}{3}$:

$$\left(\frac{4}{3}\right)^2 + 6 = A\left(\left(\frac{4}{3}\right)^2 + 4\right) \implies \frac{16}{9} + 6 = A\left(\frac{16}{9} + 4\right) \implies \frac{70}{9} = A\left(\frac{52}{9}\right) \implies A = \frac{35}{26}$$

To find B and C, expand and equate the coefficients of the powers of s:

$$s^2 + 6 = (A+3B)s^2 + (-4B+3C)s + (4A-4C)$$

From s^2: $1 = A + 3B \implies 1 = \frac{35}{26} + 3B \implies B = -\frac{3}{26}$.

From the constant term: $6 = 4A - 4C \implies 6 = 4(\frac{35}{26}) - 4C \implies C = -\frac{2}{13}$.

Thus, $Y(s)$ becomes:

$$Y(s) = \frac{35/26}{3s-4} + \frac{-3/26s - 2/13}{s^2+4} = \frac{35}{78}\frac{1}{s-4/3} - \frac{3}{26}\frac{s}{s^2+4} - \frac{2}{13}\frac{1}{s^2+4}$$

Step 5: Find the inverse Laplace transform of each term to obtain $y(t)$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{35}{78}\frac{1}{s-4/3} - \frac{3}{26}\frac{s}{s^2+4} - \frac{1}{13}\frac{2}{s^2+4}\right\}$$

$$y(t) = \frac{35}{78}e^{4t/3} - \frac{3}{26}\cos(2t) - \frac{1}{13}\sin(2t)$$

$y(t) = \frac{35}{78}e^{4t/3} - \frac{3}{26}\cos(2t) - \frac{1}{13}\sin(2t)$

2) $y'' - 3y' + 2y = 0$, $y(0) = 0$, $y'(0) = 3$

We solve this initial value problem by applying the Laplace transform.

Step 1: Take the Laplace transform of the equation.

$$\mathcal{L}\{y''\} - 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} = \mathcal{L}\{0\}$$

Step 2: Use the properties of the Laplace transform for derivatives. Let $Y(s) = \mathcal{L}\{y(t)\}$.

$$[s^2Y(s) - sy(0) - y'(0)] - 3[sY(s) - y(0)] + 2Y(s) = 0$$

Step 3: Substitute the initial conditions $y(0) = 0$ and $y'(0) = 3$.

$$[s^2Y(s) - s(0) - 3] - 3[sY(s) - 0] + 2Y(s) = 0$$

$$s^2Y(s) - 3 - 3sY(s) + 2Y(s) = 0$$

Step 4: Solve for $Y(s)$.

$$Y(s)(s^2 - 3s + 2) = 3$$

$$Y(s) = \frac{3}{s^2 - 3s + 2} = \frac{3}{(s-1)(s-2)}$$

Step 5: Decompose $Y(s)$ using partial fractions.

$$\frac{3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

Multiplying by the denominator gives $3 = A(s-2) + B(s-1)$.

Setting $s = 1$ yields $3 = A(-1) \implies A = -3$.

Setting $s = 2$ yields $3 = B(1) \implies B = 3$.

So, $Y(s) = \frac{-3}{s-1} + \frac{3}{s-2}$.

Step 6: Compute the inverse Laplace transform to find $y(t)$.

$$y(t) = \mathcal{L}^{-1}\left\{\frac{-3}{s-1}\right\} + \mathcal{L}^{-1}\left\{\frac{3}{s-2}\right\}$$

$$y(t) = -3e^t + 3e^{2t}$$

$\boxed{y(t) = 3e^{2t} - 3e^t}$

3) $\frac{dx}{dt} - 3y = 6e^t$, $\frac{dy}{dt} + 3x = 0$, $x(0) = y(0) = 0$

We solve this system of linear differential equations by applying the Laplace transform to each equation.

Step 1: Transform the system into the s-domain. Let $X(s) = \mathcal{L}\{x(t)\}$ and $Y(s) = \mathcal{L}\{y(t)\}$.

The transform of the first equation is:

$$\mathcal{L}\{\frac{dx}{dt}\} - 3\mathcal{L}\{y\} = \mathcal{L}\{6e^t\} \implies sX(s) - x(0) - 3Y(s) = \frac{6}{s-1}$$

The transform of the second equation is:

$$\mathcal{L}\{\frac{dy}{dt}\} + 3\mathcal{L}\{x\} = \mathcal{L}\{0\} \implies sY(s) - y(0) + 3X(s) = 0$$

Step 2: Substitute the initial conditions $x(0) = 0$ and $y(0) = 0$.

$$(1) \quad sX(s) - 3Y(s) = \frac{6}{s-1}$$
$$(2) \quad 3X(s) + sY(s) = 0$$

Step 3: Solve the resulting system of algebraic equations for $X(s)$ and $Y(s)$.

From equation (2), we express $Y(s)$ in terms of $X(s)$: $Y(s) = -\frac{3}{s}X(s)$.

Substitute this into equation (1):

$$sX(s) - 3\left(-\frac{3}{s}X(s)\right) = \frac{6}{s-1} \implies X(s)\left(s + \frac{9}{s}\right) = \frac{6}{s-1}$$
$$X(s)\left(\frac{s^2+9}{s}\right) = \frac{6}{s-1} \implies X(s) = \frac{6s}{(s-1)(s^2+9)}$$

Now, find $Y(s)$:

$$Y(s) = -\frac{3}{s}\left(\frac{6s}{(s-1)(s^2+9)}\right) = \frac{-18}{(s-1)(s^2+9)}$$

Step 4: Use partial fraction decomposition to find the inverse transforms.

For $X(s)$: $\frac{6s}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$. Solving gives $A = \frac{3}{5}, B = -\frac{3}{5}, C = \frac{27}{5}$.

$$X(s) = \frac{3}{5}\frac{1}{s-1} - \frac{3}{5}\frac{s}{s^2+9} + \frac{27}{5}\frac{1}{s^2+9}$$

For $Y(s)$: $\frac{-18}{(s-1)(s^2+9)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+9}$. Solving gives $A = -\frac{9}{5}, B = \frac{9}{5}, C = -\frac{9}{5}$.

$$Y(s) = -\frac{9}{5}\frac{1}{s-1} + \frac{9}{5}\frac{s}{s^2+9} - \frac{9}{5}\frac{1}{s^2+9}$$

Step 5: Find the inverse Laplace transforms of $X(s)$ and $Y(s)$.

$$x(t) = \frac{3}{5}e^t - \frac{3}{5}\cos(3t) + \frac{27}{5\cdot 3}\sin(3t) = \frac{3}{5}(e^t - \cos(3t) + 3\sin(3t))$$

$$y(t) = -\frac{9}{5}e^t + \frac{9}{5}\cos(3t) - \frac{9}{5\cdot 3}\sin(3t) = -\frac{3}{5}(3e^t - 3\cos(3t) + \sin(3t))$$

$x(t) = \frac{3}{5}(e^t - \cos(3t) + 3\sin(3t))$
$y(t) = -\frac{3}{5}(3e^t - 3\cos(3t) + \sin(3t))$

4) $x(t) + 2\int_0^t x(u)\cos(t-u)du = 9e^{2t}$

This is an integral equation of the convolution type.

Step 1: Identify the convolution and take the Laplace transform of the entire equation.

The integral term is the convolution of $x(t)$ and $\cos(t)$, i.e., $x(t) * \cos(t)$. Applying the Laplace transform to the equation:

$$\mathcal{L}\{x(t)\} + 2\mathcal{L}\{x(t) * \cos(t)\} = \mathcal{L}\{9e^{2t}\}$$

Using the Convolution Theorem, $\mathcal{L}\{f * g\} = F(s)G(s)$, we get:

$$X(s) + 2X(s)\mathcal{L}\{\cos(t)\} = \frac{9}{s-2}$$

Step 2: Substitute the known transform for $\cos(t)$ and solve for $X(s)$.

$$X(s) + 2X(s)\left(\frac{s}{s^2+1}\right) = \frac{9}{s-2}$$

$$X(s)\left(1 + \frac{2s}{s^2+1}\right) = X(s)\left(\frac{s^2+2s+1}{s^2+1}\right) = X(s)\frac{(s+1)^2}{s^2+1} = \frac{9}{s-2}$$

$$X(s) = \frac{9(s^2+1)}{(s-2)(s+1)^2}$$

Step 3: Use partial fraction decomposition for $X(s)$.

$$\frac{9(s^2+1)}{(s-2)(s+1)^2} = \frac{A}{s-2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$$

Multiplying by the denominator gives $9(s^2+1) = A(s+1)^2 + B(s-2)(s+1) + C(s-2)$.

Setting $s = 2$ yields $9(5) = A(3)^2 \implies 45 = 9A \implies A = 5$.

Setting $s = -1$ yields $9(2) = C(-3) \implies 18 = -3C \implies C = -6$.

Equating the coefficients of s^2 gives $9 = A + B \implies 9 = 5 + B \implies B = 4$.

$$X(s) = \frac{5}{s-2} + \frac{4}{s+1} - \frac{6}{(s+1)^2}$$

Step 4: Find the inverse Laplace transform of $X(s)$ to get $x(t)$.

$$x(t) = \mathcal{L}^{-1}\left\{\frac{5}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{4}{s+1}\right\} - 6\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}$$

The first two terms are standard exponentials. The third term uses the first shifting theorem on $\mathcal{L}^{-1}\{1/s^2\} = t$.

$$x(t) = 5e^{2t} + 4e^{-t} - 6te^{-t}$$

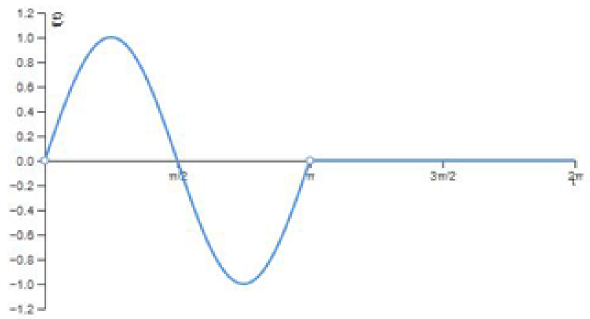
$\boxed{x(t) = 5e^{2t} + e^{-t}(4-6t)}$

IV) Sketch the graph for the following functions and find its transformation:

> 1) $f(t) = \begin{cases} \sin 2t & \text{when } 0 < t < \pi \\ 0 & \text{when } t > \pi \end{cases}$

Step 1: Sketch the graph of the function.

The function $f(t)$ consists of one full cycle of $\sin(2t)$ over the interval $[0, \pi]$ and is zero thereafter.



Step 2: Express the function using the Heaviside step function.

The function is active from $t = 0$ until $t = \pi$. This can be written as the product of $\sin(2t)$ and a "window" function created by Heaviside functions.

$$f(t) = \sin(2t)[H(t) - H(t - \pi)] = \sin(2t) - \sin(2t)H(t - \pi)$$

Step 3: Find the Laplace transform.

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{\sin(2t)\} - \mathcal{L}\{\sin(2t)H(t - \pi)\}$$

The first term is a standard transform: $\mathcal{L}\{\sin(2t)\} = \frac{2}{s^2+4}$.

For the second term, we use the Second Shifting Theorem: $\mathcal{L}\{g(t)H(t - a)\} = e^{-as}\mathcal{L}\{g(t + a)\}$. Here, $g(t) = \sin(2t)$ and $a = \pi$.

We find $g(t + a)$:

$$g(t + \pi) = \sin(2(t + \pi)) = \sin(2t + 2\pi) = \sin(2t)$$

Therefore, the transform of the second part is:

$$\mathcal{L}\{\sin(2t)H(t - \pi)\} = e^{-\pi s}\mathcal{L}\{\sin(2t)\} = e^{-\pi s}\frac{2}{s^2 + 4}$$

Step 4: Combine the results.

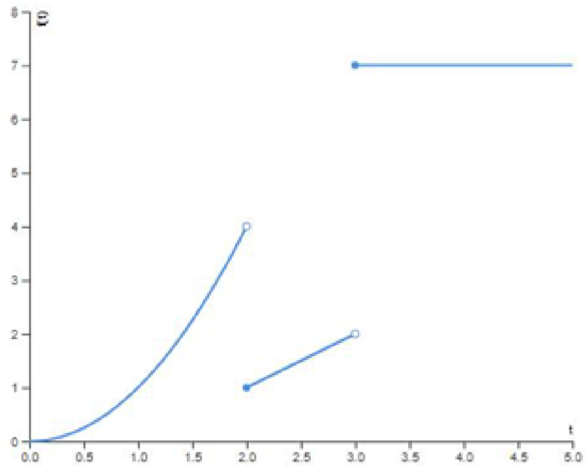
$$\mathcal{L}\{f(t)\} = \frac{2}{s^2 + 4} - e^{-\pi s}\frac{2}{s^2 + 4} = \frac{2(1 - e^{-\pi s})}{s^2 + 4}$$

$\boxed{\mathcal{L}\{f(t)\} = \frac{2(1-e^{-\pi s})}{s^2+4}}$

2) $f(t) = \begin{cases} t^2 & \text{when } 0 < t < 2 \\ t-1 & \text{when } 2 < t < 3 \\ 7 & \text{when } t > 3 \end{cases}$

Step 1: Sketch the graph of the function.

The function is defined piecewise, consisting of a parabolic section, a linear section, and a constant section.



Step 2: Express $f(t)$ using Heaviside step functions.

We represent the function by turning on and off each piece at the appropriate time:

$$f(t) = t^2[H(t) - H(t-2)] + (t-1)[H(t-2) - H(t-3)] + 7H(t-3)$$

Distribute and collect terms with common Heaviside functions:

$$f(t) = t^2 + (t - 1 - t^2)H(t-2) + (-(t-1) + 7)H(t-3)$$

$$f(t) = t^2 + (-t^2 + t - 1)H(t-2) + (8-t)H(t-3)$$

Step 3: Prepare for the Second Shifting Theorem by expressing the coefficients in terms of $t - a$.

For $H(t-2)$: let $t = u + 2$. The coefficient is
$-(u+2)^2 + (u+2) - 1 = -(u^2 + 4u + 4) + u + 1 = -u^2 - 3u - 3$. Let this be $g_1(t-2)$.

For $H(t-3)$: let $t = u + 3$. The coefficient is $8 - (u+3) = 5 - u$. Let this be $g_2(t-3)$.

The function becomes $f(t) = t^2 + g_1(t-2)H(t-2) + g_2(t-3)H(t-3)$ where $g_1(t) = -t^2 - 3t - 3$ and $g_2(t) = 5 - t$.

Step 4: Apply the Laplace transform using linearity and the Second Shifting Theorem.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{t^2\} + \mathcal{L}\{g_1(t-2)H(t-2)\} + \mathcal{L}\{g_2(t-3)H(t-3)\} \\ &= \mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{g_1(t)\} + e^{-3s}\mathcal{L}\{g_2(t)\} \\ &= \mathcal{L}\{t^2\} + e^{-2s}\mathcal{L}\{-t^2 - 3t - 3\} + e^{-3s}\mathcal{L}\{5 - t\} \\ &= \frac{2}{s^3} + e^{-2s}\left(-\frac{2}{s^3} - \frac{3}{s^2} - \frac{3}{s}\right) + e^{-3s}\left(\frac{5}{s} - \frac{1}{s^2}\right) \end{aligned}$$

$F(s) = \frac{2}{s^3} - e^{-2s}\left(\frac{2}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right) + e^{-3s}\left(\frac{5}{s} - \frac{1}{s^2}\right)$

(i) Find F(s).

The function $f(t)$ is periodic with period $P = 2T$. The Laplace transform of a periodic function is given by the formula:

$$F(s) = \frac{1}{1 - e^{-Ps}} \int_0^P e^{-st} f(t) dt$$

Step 1: Apply the formula with $P = 2T$ and the given function definition.

$$F(s) = \frac{1}{1 - e^{-2Ts}} \left(\int_0^T e^{-st}(a) dt + \int_T^{2T} e^{-st}(0) dt \right)$$

Step 2: Evaluate the integral.

$$\int_0^T a e^{-st} dt = a \left[-\frac{1}{s} e^{-st} \right]_0^T = -\frac{a}{s}(e^{-sT} - e^0) = \frac{a}{s}(1 - e^{-sT})$$

Step 3: Substitute the integral result into the formula for $F(s)$ and simplify.

$$F(s) = \frac{1}{1 - e^{-2Ts}} \cdot \frac{a}{s}(1 - e^{-sT})$$

Using the difference of squares factorization $1 - e^{-2Ts} = (1 - e^{-sT})(1 + e^{-sT})$, we simplify the expression:

$$F(s) = \frac{a(1 - e^{-sT})}{s(1 - e^{-sT})(1 + e^{-sT})} = \frac{a}{s(1 + e^{-sT})}$$

$\boxed{F(s) = \frac{a}{s(1+e^{-sT})}}$

(ii) Solve: $y'' + k^2 y = f(t)$; $y(0) = y'(0) = 0$

We solve this initial value problem using the Laplace transform.

Step 1: Take the Laplace transform of the differential equation.

Let $Y(s) = \mathcal{L}\{y(t)\}$. Applying the transform and substituting the initial conditions gives:

$$\mathcal{L}\{y''\} + k^2\mathcal{L}\{y\} = \mathcal{L}\{f(t)\}$$
$$(s^2Y(s) - sy(0) - y'(0)) + k^2Y(s) = F(s)$$
$$(s^2 + k^2)Y(s) = F(s)$$

Step 2: Substitute the expression for $F(s)$ from part (i) and solve for $Y(s)$.

$$Y(s) = \frac{1}{s^2+k^2}F(s) = \frac{1}{s^2+k^2} \cdot \frac{a}{s(1+e^{-sT})} = \frac{a}{s(s^2+k^2)(1+e^{-sT})}$$

Step 3: Express $Y(s)$ as a series using the geometric series expansion $\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n$.

$$Y(s) = \frac{a}{s(s^2+k^2)}\sum_{n=0}^{\infty}(-1)^n e^{-nsT}$$

Step 4: Find the inverse transform of the base function $G(s) = \frac{a}{s(s^2+k^2)}$.

Using partial fraction decomposition: $G(s) = \frac{a}{k^2}\left(\frac{1}{s} - \frac{s}{s^2+k^2}\right)$.

The inverse transform $g(t) = \mathcal{L}^{-1}\{G(s)\}$ is:

$$g(t) = \mathcal{L}^{-1}\left\{\frac{a}{k^2}\frac{1}{s} - \frac{a}{k^2}\frac{s}{s^2+k^2}\right\} = \frac{a}{k^2}(1-\cos(kt))$$

Step 5: Find the inverse transform of the full expression for $Y(s)$ using the Second Shifting Theorem.

The theorem states $\mathcal{L}^{-1}\{e^{-cs}G(s)\} = g(t-c)H(t-c)$. Applying this to the series:

$$\begin{aligned} y(t) &= \mathcal{L}^{-1}\left\{\sum_{n=0}^{\infty}(-1)^n G(s)e^{-nsT}\right\} \\ &= \sum_{n=0}^{\infty}(-1)^n \mathcal{L}^{-1}\{G(s)e^{-nsT}\} \\ &= \sum_{n=0}^{\infty}(-1)^n g(t-nT)H(t-nT) \end{aligned}$$

Substituting the expression for $g(t)$ yields the final solution.

$y(t) = \frac{a}{k^2}\sum_{n=0}^{\infty}(-1)^n[1-\cos(k(t-nT))]H(t-nT)$

Part (B): Fourier Series

> **Find Fourier series of f(x) and graph the periodic continuation of f(x):**
> 1) $f(x) = x \quad -\pi < x < \pi$

Step 1: Analyze the function and determine its properties.

The function $f(x) = x$ is defined on the interval $(-\pi, \pi)$. The period of the periodic continuation is $P = \pi - (-\pi) = 2\pi$, so the half-period is $T = \pi$.

We test for symmetry by evaluating $f(-x)$:

$$f(-x) = (-x) = -x = -f(x)$$

Since $f(-x) = -f(x)$, the function is an **odd function**. Consequently, its Fourier series will only contain sine terms, and the coefficients a_0 and a_n are zero.

$$a_0 = 0, \quad a_n = 0 \quad \text{for } n \geq 1$$

The Fourier series has the form:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right) = \sum_{n=1}^{\infty} b_n \sin(nx)$$

Step 2: Calculate the b_n coefficients.

For an odd function, the formula for b_n is:

$$b_n = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{n\pi x}{T}\right)dx$$

Substituting $f(x) = x$ and $T = \pi$:

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)dx$$

We evaluate this integral using integration by parts, $\int u\,dv = uv - \int v\,du$, with $u = x$ and $dv = \sin(nx)dx$. This gives $du = dx$ and $v = -\frac{\cos(nx)}{n}$.

$$\begin{aligned} b_n &= \frac{2}{\pi}\left[-\frac{x}{n}\cos(nx)\Big|_0^{\pi} - \int_0^{\pi}\left(-\frac{\cos(nx)}{n}\right)dx\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n}\int_0^{\pi}\cos(nx)dx\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n + \frac{1}{n^2}[\sin(nx)]_0^{\pi}\right] \\ &= \frac{2}{\pi}\left[-\frac{\pi}{n}(-1)^n + 0\right] \\ &= -\frac{2}{n}(-1)^n = \frac{2}{n}(-1)^{n+1} \end{aligned}$$

Step 3: Write the Fourier series.

Substituting the expression for b_n into the series form:

$$f(x) = \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin(nx)$$

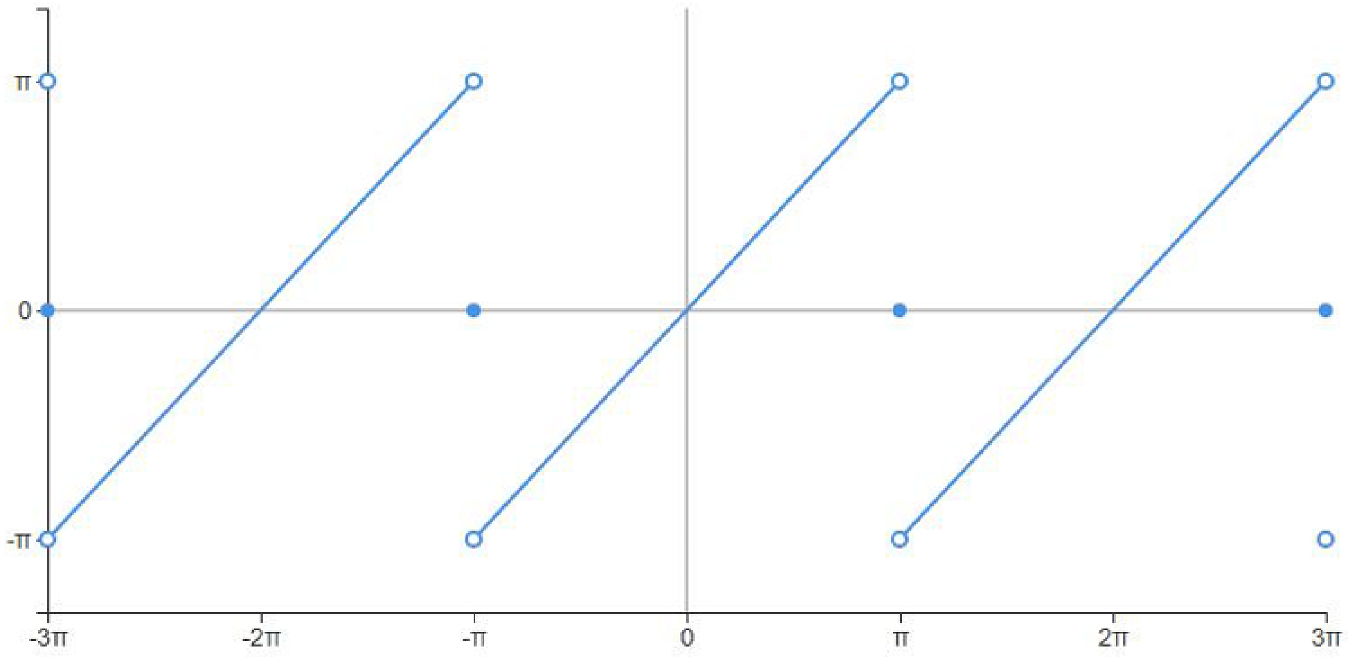
The first few terms are:

$$f(x) = 2\sin(x) - \sin(2x) + \frac{2}{3}\sin(3x) - \frac{1}{2}\sin(4x) + \ldots$$

The Fourier series for $f(x) = x$ on $(-\pi, \pi)$ is $f(x) = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(nx)$.

Step 4: Graph the periodic continuation.

The graph is a sawtooth wave, repeating every 2π. At the points of discontinuity, $x = (2k+1)\pi$, the series converges to the midpoint of the jump, which is $\frac{\pi+(-\pi)}{2} = 0$.



> 2) $f(x) = x^2 \quad -\pi < x < \pi$ and then find $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

This problem requires us to first find the Fourier series for $f(x) = x^2$ on the interval $(-\pi, \pi)$ and then use this series to determine the value of the given infinite sum.

Part (i): Find the Fourier series of $f(x)$

Step 1: Analyze the function's symmetry.

The function $f(x) = x^2$ is defined on $(-\pi, \pi)$. The period is $P = 2\pi$, so the half-period is $T = \pi$.

Since $f(-x) = (-x)^2 = x^2 = f(x)$, the function is **even**. This implies that its Fourier series will be a pure cosine series, meaning all b_n coefficients are zero.

$$b_n = 0 \quad \text{for } n \geq 1$$

The series will have the form: $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$.

Step 2: Calculate the coefficient a_0.

$$a_0 = \frac{2}{T} \int_0^T f(x)dx = \frac{2}{\pi} \int_0^{\pi} x^2 dx = \frac{2}{\pi} \left[\frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left(\frac{\pi^3}{3} \right) = \frac{2\pi^2}{3}$$

Step 3: Calculate the coefficients a_n.

$$a_n = \frac{2}{T} \int_0^T f(x) \cos(nx) dx = \frac{2}{\pi} \int_0^{\pi} x^2 \cos(nx) dx$$

We evaluate this integral using integration by parts twice.

$$\begin{aligned} a_n &= \frac{2}{\pi} \left[\left(\frac{x^2}{n} \sin(nx) \right)_0^{\pi} - \int_0^{\pi} \frac{2x}{n} \sin(nx) dx \right] \\ &= \frac{2}{\pi} \left[0 - \frac{2}{n} \int_0^{\pi} x \sin(nx) dx \right] \\ &= -\frac{4}{n\pi} \left[\left(-\frac{x}{n} \cos(nx) \right)_0^{\pi} - \int_0^{\pi} \left(-\frac{\cos(nx)}{n} \right) dx \right] \\ &= -\frac{4}{n\pi} \left[-\frac{\pi}{n} \cos(n\pi) + \frac{1}{n^2} [\sin(nx)]_0^{\pi} \right] \\ &= -\frac{4}{n\pi} \left[-\frac{\pi}{n} (-1)^n + 0 \right] = \frac{4(-1)^n}{n^2} \end{aligned}$$

Step 4: Construct the Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos(nx)$$

Part (ii): Find the value of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$

Step 5: Evaluate the series at a specific point.

To find the sum, we can evaluate the Fourier series at a point where $f(x)$ is continuous. Let's choose $x = 0$, where $f(0) = 0$.

Substituting $x = 0$ into the series:

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(0)$$

$$-\frac{\pi^2}{3} = 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\frac{\pi^2}{12}$$

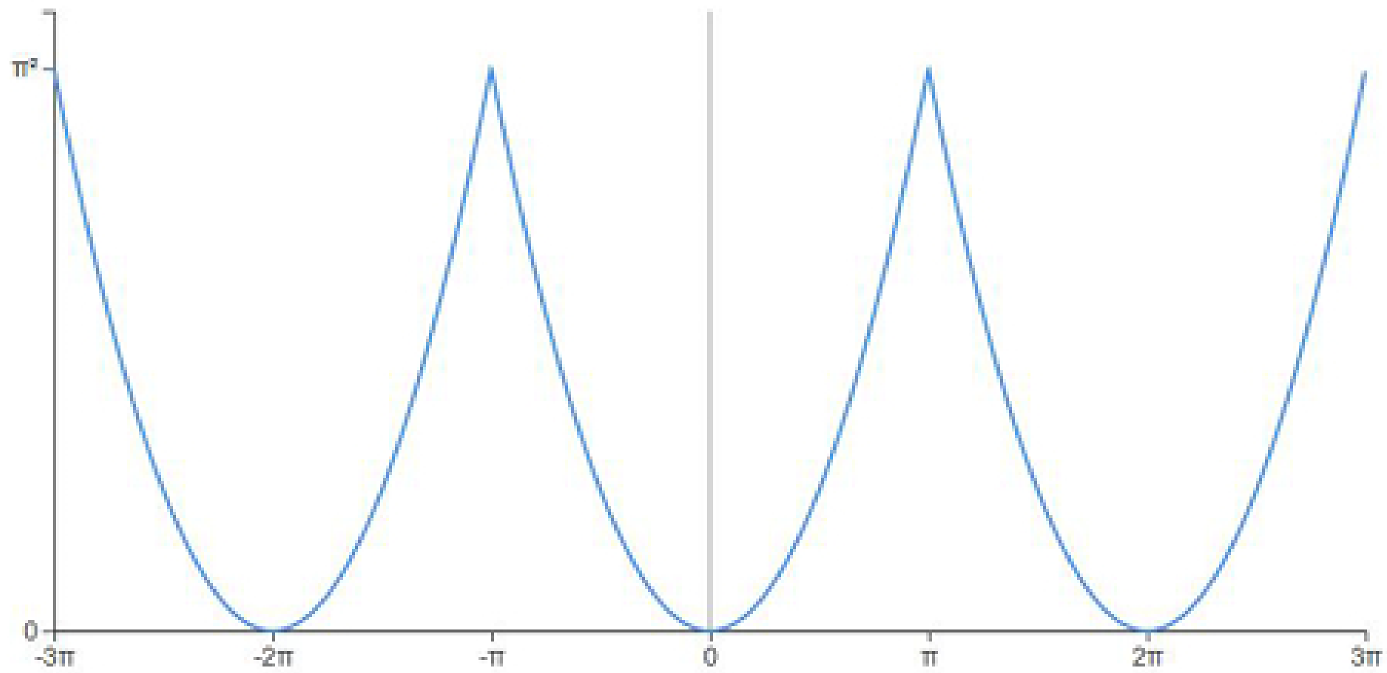
The sum we need to find is $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$. We can relate this to our result:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} = -\left(-\frac{\pi^2}{12} \right) = \frac{\pi^2}{12}$$

The value of the sum is $\frac{\pi^2}{12}$.

Graph of the Periodic Continuation

The periodic extension of $f(x) = x^2$ on $(-\pi, \pi)$ results in a continuous, repeating parabolic wave.



> 3) $f(x) = \begin{cases} 1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$

Step 1: Analyze the function and its properties.

The function is defined on the interval $(-\pi, \pi)$, giving a period of 2π and a half-period $T = \pi$. To check for symmetry, we evaluate $f(-x)$ for $x > 0$:

If $x \in (0, \pi)$, then $-x \in (-\pi, 0)$, so $f(-x) = -1$. Since $f(x) = 1$ for $x \in (0, \pi)$, we have $f(-x) = -f(x)$.

The function is **odd**. Consequently, its Fourier series will be a sine series, with $a_0 = 0$ and $a_n = 0$.

The series has the form $f(x) = \sum_{n=1}^{\infty} b_n \sin(nx)$.

Step 2: Calculate the b_n coefficients.

For an odd function, the formula is:

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(nx)dx = \frac{2}{\pi}\int_0^{\pi}(1)\sin(nx)dx$$

$$b_n = \frac{2}{\pi}\left[-\frac{\cos(nx)}{n}\right]_0^{\pi} = -\frac{2}{n\pi}[\cos(n\pi) - \cos(0)] = -\frac{2}{n\pi}[(-1)^n - 1]$$

This simplifies based on whether n is even or odd:

- If n is even, $(-1)^n = 1$, so $b_n = 0$.
- If n is odd, $(-1)^n = -1$, so $b_n = -\frac{2}{n\pi}[-1-1] = \frac{4}{n\pi}$.

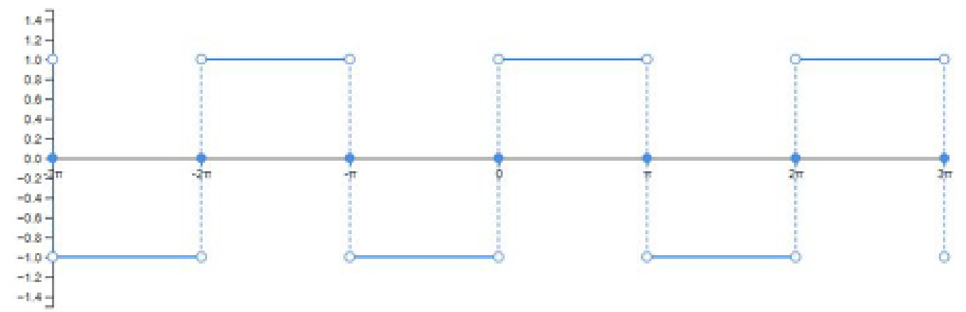
Step 3: Write the Fourier series.

The series includes only terms for odd n, which can be represented by letting $n = 2k - 1$.

$$f(x) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi}\sin((2k-1)x)$$

$f(x) = \frac{4}{\pi}\left(\sin(x) + \frac{1}{3}\sin(3x) + \frac{1}{5}\sin(5x) + \dots\right)$

Graph of the Periodic Continuation: The periodic continuation of this function is a square wave.



4) $f(x) = x^4 - 2\pi^2 x^2 \quad -\pi < x < \pi$. Then, find $\sum_{n=1}^{\infty} \frac{1}{n^4}$

We first find the Fourier series for $f(x)$ and then use it to evaluate the sum.

Step 1: Analyze function properties.

The function is defined on $(-\pi, \pi)$, so $T = \pi$. Since $f(-x) = (-x)^4 - 2\pi^2(-x)^2 = x^4 - 2\pi^2 x^2 = f(x)$, the function is **even**, which means $b_n = 0$ for all n, and we find a cosine series.

Step 2: Calculate the Fourier coefficients.

First, a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(x^4 - 2\pi^2 x^2)dx = \frac{2}{\pi}\left[\frac{x^5}{5} - \frac{2\pi^2 x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left(\frac{\pi^5}{5} - \frac{2\pi^5}{3}\right) = -\frac{14\pi^4}{15}$$

Next, a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi}(x^4 - 2\pi^2 x^2)\cos(nx)dx = \frac{2}{\pi}\left(\int_0^{\pi} x^4\cos(nx)dx - 2\pi^2\int_0^{\pi} x^2\cos(nx)dx\right)$$

Using integration by parts multiple times, we find the values of these definite integrals:

$$\int_0^{\pi} x^2\cos(nx)dx = \frac{2\pi(-1)^n}{n^2}$$

$$\int_0^{\pi} x^4\cos(nx)dx = \frac{4\pi^3(-1)^n}{n^2} - \frac{24\pi(-1)^n}{n^4}$$

Substituting these into the expression for a_n:

$$a_n = \frac{2}{\pi}\left[\left(\frac{4\pi^3(-1)^n}{n^2} - \frac{24\pi(-1)^n}{n^4}\right) - 2\pi^2\left(\frac{2\pi(-1)^n}{n^2}\right)\right] = \frac{2}{\pi}\left[-\frac{24\pi(-1)^n}{n^4}\right] = \frac{-48(-1)^n}{n^4} = \frac{48(-1)^{n+1}}{n^4}$$

Step 3: Construct the Fourier series.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) = -\frac{7\pi^4}{15} + 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(nx)$$

Step 4: Use the series to find $\sum_{n=1}^{\infty}\frac{1}{n^4}$.

To isolate the desired sum, we can evaluate the series at a point. Let's use $x = 0$, where $f(0) = 0$ and the function is continuous.

$$0 = -\frac{7\pi^4}{15} + 48\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4}\cos(0) \implies \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \frac{7\pi^4}{15\cdot 48} = \frac{7\pi^4}{720}$$

This gives the value of the alternating series. To find the sum of $1/n^4$, we relate it to the alternating series sum. Let $S = \sum_{n=1}^{\infty}\frac{1}{n^4}$.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^4} = \sum_{n=1}^{\infty}\frac{1}{n^4} - 2\sum_{k=1}^{\infty}\frac{1}{(2k)^4} = S - 2\left(\frac{1}{16}\right)\sum_{k=1}^{\infty}\frac{1}{k^4} = S - \frac{1}{8}S = \frac{7}{8}S$$

Now we solve for S:

$$\frac{7}{8}S = \frac{7\pi^4}{720} \implies S = \frac{8}{7}\cdot\frac{7\pi^4}{720} = \frac{8\pi^4}{720} = \frac{\pi^4}{90}$$

$\boxed{\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}}$

5) $f(x) = x|x| \quad -1 < x < 1$

Step 1: Analyze the function and its properties.

The function is defined on the interval $(-1, 1)$, so the period of its periodic continuation is $P = 2$, and the half-period is $T = 1$.

We can express the function piecewise:

$$f(x) = \begin{cases} x^2 & \text{if } 0 \le x < 1 \\ -x^2 & \text{if } -1 < x < 0 \end{cases}$$

To determine symmetry, we evaluate $f(-x)$:

$$f(-x) = (-x)|-x| = -x|x| = -f(x)$$

The function is **odd**. Therefore, its Fourier series will be a pure sine series, with coefficients $a_0 = 0$ and $a_n = 0$.

The series will have the form: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{T}\right) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$.

Step 2: Calculate the b_n coefficients.

For an odd function, the formula for b_n is:

$$b_n = \frac{2}{T}\int_0^T f(x)\sin(n\pi x)dx = 2\int_0^1 x^2\sin(n\pi x)dx$$

We evaluate this integral using integration by parts. The antiderivative of $x^2\sin(n\pi x)$ is:

$$\int x^2\sin(n\pi x)dx = -x^2\frac{\cos(n\pi x)}{n\pi} + 2x\frac{\sin(n\pi x)}{(n\pi)^2} + 2\frac{\cos(n\pi x)}{(n\pi)^3}$$

Evaluating this from 0 to 1:

$$\begin{aligned}\int_0^1 x^2\sin(n\pi x)dx &= \left[-x^2\frac{\cos(n\pi x)}{n\pi} + 2x\frac{\sin(n\pi x)}{(n\pi)^2} + 2\frac{\cos(n\pi x)}{(n\pi)^3}\right]_0^1 \\ &= \left(-\frac{\cos(n\pi)}{n\pi} + \frac{2\sin(n\pi)}{(n\pi)^2} + \frac{2\cos(n\pi)}{(n\pi)^3}\right) - \left(0 + 0 + \frac{2\cos(0)}{(n\pi)^3}\right) \\ &= -\frac{(-1)^n}{n\pi} + 0 + \frac{2(-1)^n}{(n\pi)^3} - \frac{2}{(n\pi)^3} \\ &= \frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{(n\pi)^3}\end{aligned}$$

Therefore, the coefficient b_n is:

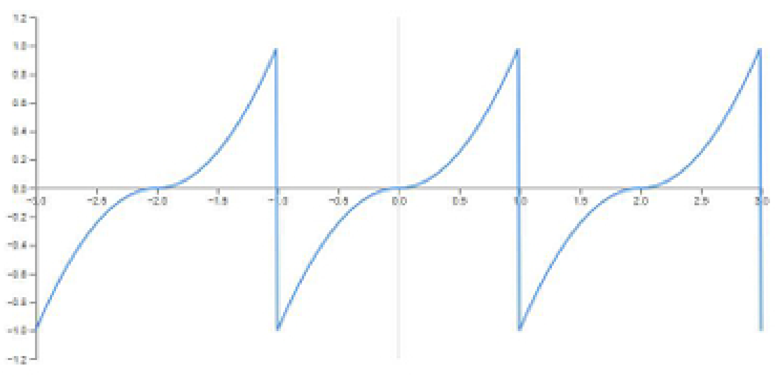
$$b_n = 2\left(\frac{(-1)^{n+1}}{n\pi} + \frac{2((-1)^n - 1)}{(n\pi)^3}\right)$$

Step 3: Write the Fourier series.

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n - 1)}{(n\pi)^3}\right)\sin(n\pi x)$$

$\boxed{f(x) = \sum_{n=1}^{\infty}\left[\frac{2(-1)^{n+1}}{n\pi} + \frac{4((-1)^n-1)}{(n\pi)^3}\right]\sin(n\pi x)}$

Graph of the Periodic Continuation:



> 6) $f(x) = \begin{cases} e^{-x} & 0 < x < 2 \\ 1 & -2 < x < 0 \end{cases}$

Step 1: Analyze function properties and set up the Fourier series.

The function is defined on $(-2, 2)$. The period is $P = 4$, so the half-period is $T = 2$. The function is neither even nor odd, so we must compute all Fourier coefficients a_0, a_n, and b_n.

The Fourier series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{2}\right) + b_n \sin\left(\frac{n\pi x}{2}\right) \right]$$

Step 2: Calculate the coefficient a_0.

$$\begin{aligned} a_0 &= \frac{1}{T} \int_{-T}^{T} f(x) dx = \frac{1}{2} \int_{-2}^{2} f(x) dx \\ &= \frac{1}{2} \left[\int_{-2}^{0} (1) dx + \int_{0}^{2} e^{-x} dx \right] \\ &= \frac{1}{2} \left([x]_{-2}^{0} + [-e^{-x}]_0^2 \right) \\ &= \frac{1}{2} \left((0 - (-2)) + (-e^{-2} - (-1)) \right) = \frac{3 - e^{-2}}{2} \end{aligned}$$

Step 3: Calculate the coefficients a_n.

$$a_n = \frac{1}{T} \int_{-T}^{T} f(x) \cos\left(\frac{n\pi x}{T}\right) dx = \frac{1}{2} \left[\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} e^{-x} \cos\left(\frac{n\pi x}{2}\right) dx \right]$$

The first integral is zero: $\int_{-2}^{0} \cos\left(\frac{n\pi x}{2}\right) dx = \left[\frac{2}{n\pi} \sin\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} = 0$.

For the second integral, we use the standard formula for $\int e^{ax} \cos(bx) dx$:

$$\int_0^2 e^{-x} \cos\left(\frac{n\pi x}{2}\right) dx = \left[\frac{e^{-x}}{(-1)^2 + (\frac{n\pi}{2})^2} \left(-\cos\left(\frac{n\pi x}{2}\right) + \frac{n\pi}{2} \sin\left(\frac{n\pi x}{2}\right) \right) \right]_0^2$$

$$= \frac{e^{-2}}{1 + (\frac{n\pi}{2})^2} (-\cos(n\pi)) - \frac{e^0}{1 + (\frac{n\pi}{2})^2} (-\cos(0)) = \frac{-e^{-2}(-1)^n + 1}{1 + (n\pi/2)^2} = \frac{4(1 - e^{-2}(-1)^n)}{4 + n^2\pi^2}$$

Therefore, $a_n = \frac{1}{2} \cdot \frac{4(1-e^{-2}(-1)^n)}{4+n^2\pi^2} = \frac{2(1-e^{-2}(-1)^n)}{4+n^2\pi^2}$.

Step 4: Calculate the coefficients b_n.

$$b_n = \frac{1}{2} \left[\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right) dx + \int_{0}^{2} e^{-x} \sin\left(\frac{n\pi x}{2}\right) dx \right]$$

First integral: $\int_{-2}^{0} \sin\left(\frac{n\pi x}{2}\right) dx = \left[-\frac{2}{n\pi} \cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{0} = -\frac{2}{n\pi}(1 - (-1)^n)$.

Second integral, using $\int e^{ax} \sin(bx) dx$:

$$\int_0^2 e^{-x} \sin\left(\frac{n\pi x}{2}\right) dx = \left[\frac{e^{-x}}{1 + (\frac{n\pi}{2})^2} \left(-\sin\left(\frac{n\pi x}{2}\right) - \frac{n\pi}{2} \cos\left(\frac{n\pi x}{2}\right) \right) \right]_0^2$$

$$= \frac{e^{-2}}{1 + (\frac{n\pi}{2})^2} \left(0 - \frac{n\pi}{2} \cos(n\pi)\right) - \frac{1}{1 + (\frac{n\pi}{2})^2} \left(0 - \frac{n\pi}{2}\right) = \frac{\frac{n\pi}{2}(1 - e^{-2}(-1)^n)}{1 + (n\pi/2)^2} = \frac{2n\pi(1 - e^{-2}(-1)^n)}{4 + n^2\pi^2}$$

Combining the parts for b_n:

$$b_n = \frac{1}{2} \left[-\frac{2(1 - (-1)^n)}{n\pi} + \frac{2n\pi(1 - e^{-2}(-1)^n)}{4 + n^2\pi^2} \right] = \frac{(-1)^n - 1}{n\pi} + \frac{n\pi(1 - e^{-2}(-1)^n)}{4 + n^2\pi^2}$$

Step 5: Write the complete Fourier series.

$$f(x) = \frac{3-e^{-2}}{4} + \sum_{n=1}^{\infty} \left(\frac{2(1-e^{-2}(-1)^n)}{4+n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) + \left[\frac{(-1)^n - 1}{n\pi} + \frac{n\pi(1-e^{-2}(-1)^n)}{4+n^2\pi^2} \right] \sin\left(\frac{n\pi x}{2}\right) \right)$$

Problem 7

> 7) Expand $f(x) = 1 - x$ for $0 < x < 1$ as: (i) Fourier sine series (ii) Fourier cosine series. Complete the function definition. Graph the periodic continuation of $f(x)$.

(i) Fourier Sine Series

To obtain the Fourier sine series, we construct the odd periodic extension of $f(x)$. The fundamental interval is chosen as $[-1, 1]$, which gives a half-period of $L = 1$.

The odd extension, $f_{\text{odd}}(x)$, is defined by:

$$f_{\text{odd}}(x) = \begin{cases} f(x) & 0 < x < 1 \\ -f(-x) & -1 < x < 0 \end{cases} = \begin{cases} 1 - x & 0 < x < 1 \\ -(1 - (-x)) & -1 < x < 0 \end{cases} = \begin{cases} 1 - x & 0 < x < 1 \\ -1 - x & -1 < x < 0 \end{cases}$$

For an odd function, the Fourier coefficients a_n (including a_0) are zero. The series is thus a pure sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

The coefficients b_n are calculated using the formula:

$$b_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx = 2 \int_0^1 (1 - x) \sin(n\pi x) dx$$

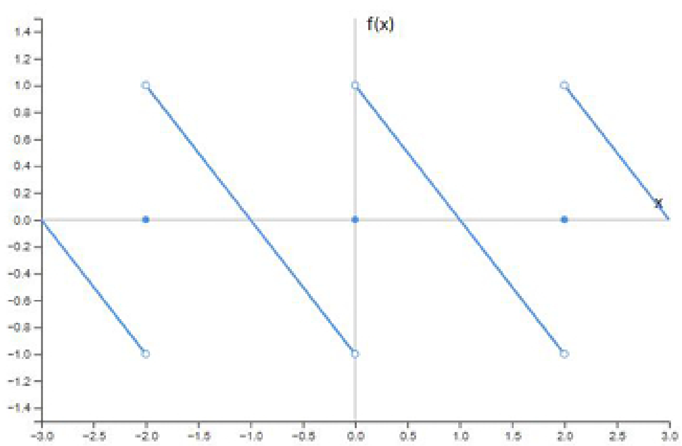
We evaluate the integral using integration by parts, $\int u\, dv = uv - \int v\, du$. Let $u = 1 - x$ and $dv = \sin(n\pi x)dx$. Then $du = -dx$ and $v = -\frac{1}{n\pi}\cos(n\pi x)$.

$$\begin{aligned} b_n &= 2\left(\left[(1-x)\left(-\frac{\cos(n\pi x)}{n\pi}\right)\right]_0^1 - \int_0^1 \left(-\frac{\cos(n\pi x)}{n\pi}\right)(-dx)\right) \\ &= 2\left[-\frac{(1-x)\cos(n\pi x)}{n\pi}\right]_0^1 - \frac{2}{n\pi}\int_0^1 \cos(n\pi x)dx \\ &= 2\left((0) - \left(-\frac{(1)\cos(0)}{n\pi}\right)\right) - \frac{2}{n\pi}\left[\frac{\sin(n\pi x)}{n\pi}\right]_0^1 \\ &= 2\left(\frac{1}{n\pi}\right) - \frac{2}{(n\pi)^2}(\sin(n\pi) - \sin(0)) \\ &= \frac{2}{n\pi} \end{aligned}$$

Therefore, the Fourier sine series for $f(x) = 1 - x$ on the interval $(0, 1)$ is:

$$\boxed{f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\sin(n\pi x) = \frac{2}{\pi}\left(\sin(\pi x) + \frac{1}{2}\sin(2\pi x) + \frac{1}{3}\sin(3\pi x) + \cdots\right)}$$

Graph of the Odd Periodic Continuation of f(x)



(ii) Fourier Cosine Series

(ii) Fourier Cosine Series

To obtain the Fourier cosine series, we construct the even periodic extension of $f(x)$. The half-period is again $L = 1$
.

The even extension, $f_{\text{even}}(x)$, is defined by:

$$f_{\text{even}}(x) = \begin{cases} f(x) & 0 < x < 1 \\ f(-x) & -1 < x < 0 \end{cases} = \begin{cases} 1 - x & 0 < x < 1 \\ 1 - (-x) & -1 < x < 0 \end{cases} = \begin{cases} 1 - x & 0 < x < 1 \\ 1 + x & -1 < x < 0 \end{cases}$$

This function is equivalent to $f_{\text{even}}(x) = 1 - |x|$ on $[-1, 1]$.

For an even function, the Fourier coefficients b_n are zero. The series takes the form:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x)$$

First, we calculate the average value coefficient, a_0:

$$a_0 = \frac{2}{L}\int_0^L f(x)dx = 2\int_0^1 (1-x)dx = 2\left[x - \frac{x^2}{2}\right]_0^1 = 2\left(1 - \frac{1}{2}\right) = 1$$

The constant term in the series is $\frac{a_0}{2} = \frac{1}{2}$.

Next, we calculate a_n for $n \geq 1$:

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 2\int_0^1 (1-x)\cos(n\pi x)dx$$

Using integration by parts with $u = 1 - x$ and $dv = \cos(n\pi x)dx$, we have $du = -dx$ and $v = \frac{1}{n\pi}\sin(n\pi x)$.

$$\begin{aligned} a_n &= 2\left(\left[(1-x)\frac{\sin(n\pi x)}{n\pi}\right]_0^1 - \int_0^1 \frac{\sin(n\pi x)}{n\pi}(-dx)\right) \\ &= 2[0 - 0] + \frac{2}{n\pi}\int_0^1 \sin(n\pi x)dx \\ &= \frac{2}{n\pi}\left[-\frac{\cos(n\pi x)}{n\pi}\right]_0^1 \\ &= -\frac{2}{(n\pi)^2}[\cos(n\pi) - \cos(0)] \\ &= -\frac{2}{n^2\pi^2}((-1)^n - 1) \end{aligned}$$

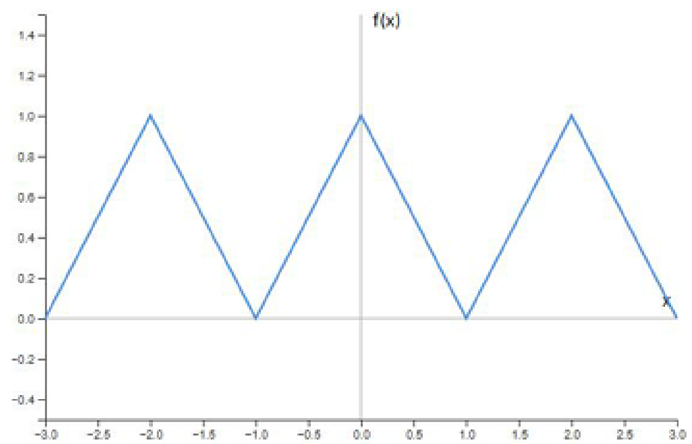
The value of a_n depends on the parity of n:

$$a_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n^2\pi^2} & \text{if } n \text{ is odd} \end{cases}$$

We can write the series using only odd integers by letting $n = 2k - 1$. The Fourier cosine series is:

$$\boxed{f(x) = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{4}{(2k-1)^2\pi^2}\cos((2k-1)\pi x) = \frac{1}{2} + \frac{4}{\pi^2}\left(\cos(\pi x) + \frac{1}{9}\cos(3\pi x) + \frac{1}{25}\cos(5\pi x) + \cdots\right)}$$

Graph of the Even Periodic Continuation of f(x)



Part (C): Partial Differential Equations

Problem 1

> 1 - Solve the following initial and boundary value problem $u_t = 4u_{xx}$,
>
> Where $u(0,t) = u(4,t) = 0$, $u(x,0) = 5\sin(\pi x)$.

This problem describes heat conduction in a one-dimensional rod of length $L = 4$. The governing equation is the heat equation $u_t = c^2 u_{xx}$, with a thermal diffusivity of $c^2 = 4$, so $c = 2$.

The boundary conditions are $u(0,t) = u(4,t) = 0$, indicating that the ends of the rod are kept at a constant temperature of 0.

The general solution for the heat equation with these homogeneous Dirichlet boundary conditions is given by the series:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

Substituting $L = 4$ and $c = 2$, the solution becomes:

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right) e^{-\left(\frac{2n\pi}{4}\right)^2 t} = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right) e^{-\frac{n^2\pi^2}{4} t}$$

To find the coefficients A_n, we apply the initial condition $u(x,0) = 5\sin(\pi x)$. At $t = 0$:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right)$$

We equate this to the given initial condition:

$$5\sin(\pi x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{4}\right) = A_1 \sin\left(\frac{\pi x}{4}\right) + A_2 \sin\left(\frac{2\pi x}{4}\right) + A_3 \sin\left(\frac{3\pi x}{4}\right) + A_4 \sin\left(\frac{4\pi x}{4}\right) + \cdots$$

By comparing the terms, we see that the initial condition is already in the form of a Fourier sine series. We match the argument of the sine function: $\pi x = \frac{n\pi x}{4}$. This implies $n = 4$.

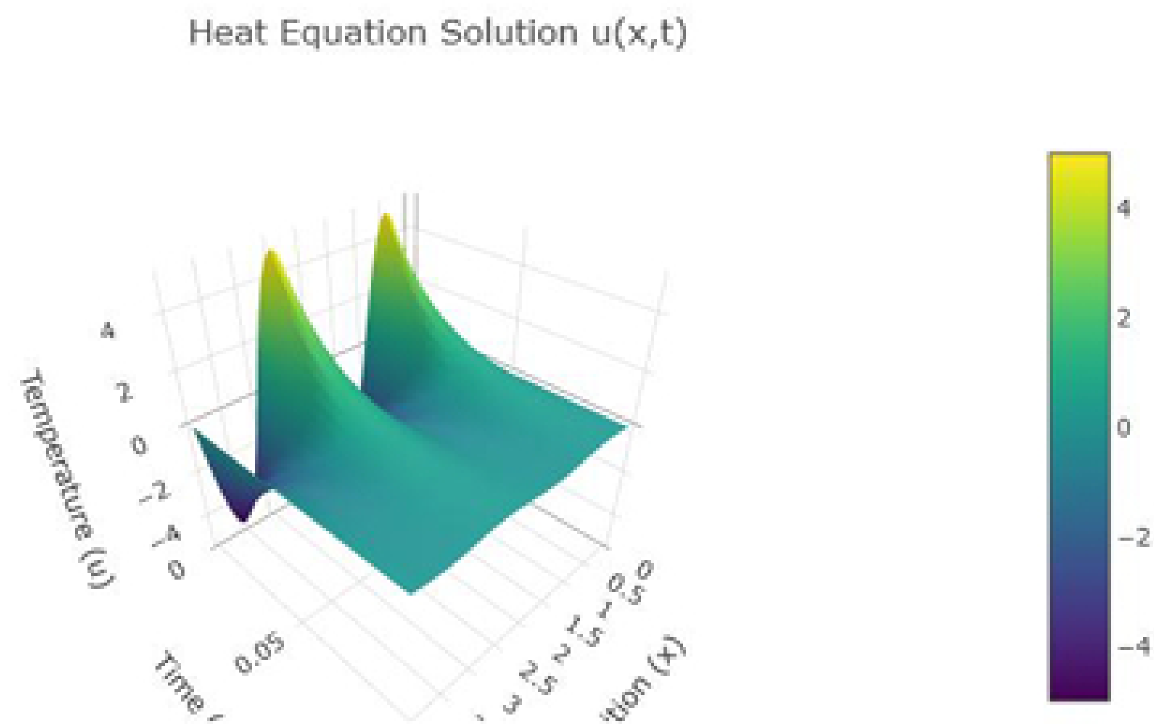
Therefore, only the coefficient for $n = 4$ is non-zero, with a value of $A_4 = 5$. All other coefficients $A_n = 0$ for $n \neq 4$.

Substituting this back into the general solution gives the specific solution:

$$u(x,t) = A_4 \sin\left(\frac{4\pi x}{4}\right) e^{-\frac{4^2\pi^2}{4} t}$$

$\boxed{u(x,t) = 5\sin(\pi x) e^{-4\pi^2 t}}$

3D Visualization of the Heat Distribution u(x,t)



Problem 2

> 2 - Use the method of separation to solve the one dimensional equation $u_{tt} = u_{xx}$, for $0 < x < \frac{\pi}{2}, t > 0$
>
> with the following boundary conditions: $u(0,t) = u(\frac{\pi}{2},t) = 0$;
>
> and the initial conditions: $u(x,0) = 2x$ and $u_t(x,0) = 0$.

This is a wave equation problem on a string of length $L = \pi/2$. The wave speed is $c^2 = 1$, so $c = 1$. The ends of the string are fixed.

The general solution for the wave equation with these homogeneous Dirichlet boundary conditions is:

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi x}{L}\right)\left[A_n \cos\left(\frac{cn\pi t}{L}\right) + B_n \sin\left(\frac{cn\pi t}{L}\right)\right]$$

Substituting $L = \pi/2$ and $c = 1$, we get $\frac{n\pi}{L} = \frac{n\pi}{\pi/2} = 2n$. The solution form becomes:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(2nx)\left[A_n \cos(2nt) + B_n \sin(2nt)\right]$$

We now apply the initial conditions.

Initial Position: $u(x,0) = 2x$

At $t = 0$, the general solution simplifies to:

$$u(x,0) = \sum_{n=1}^{\infty} A_n \sin(2nx)$$

We must find the coefficients A_n for the Fourier sine series of $f(x) = 2x$ on $[0, \pi/2]$.

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{2}{\pi/2}\int_0^{\pi/2}(2x)\sin(2nx)dx = \frac{8}{\pi}\int_0^{\pi/2} x\sin(2nx)dx$$

Using integration by parts, $\int u\,dv = uv - \int v\,du$, with $u = x$ and $dv = \sin(2nx)dx$.

$$\begin{aligned} A_n &= \frac{8}{\pi}\left(\left[-\frac{x}{2n}\cos(2nx)\right]_0^{\pi/2} - \int_0^{\pi/2} -\frac{1}{2n}\cos(2nx)dx\right) \\ &= \frac{8}{\pi}\left(-\frac{\pi/2}{2n}\cos(n\pi) + \frac{1}{2n}\left[\frac{1}{2n}\sin(2nx)\right]_0^{\pi/2}\right) \\ &= \frac{8}{\pi}\left(-\frac{\pi}{4n}(-1)^n + \frac{1}{4n^2}(\sin(n\pi) - 0)\right) \\ &= \frac{8}{\pi}\left(-\frac{\pi}{4n}(-1)^n\right) = -\frac{2}{n}(-1)^n = \frac{2(-1)^{n+1}}{n} \end{aligned}$$

Initial Velocity: $u_t(x,0) = 0$

First, differentiate the general solution with respect to t:

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(2nx)\left[-2nA_n \sin(2nt) + 2nB_n \cos(2nt)\right]$$

At $t = 0$:

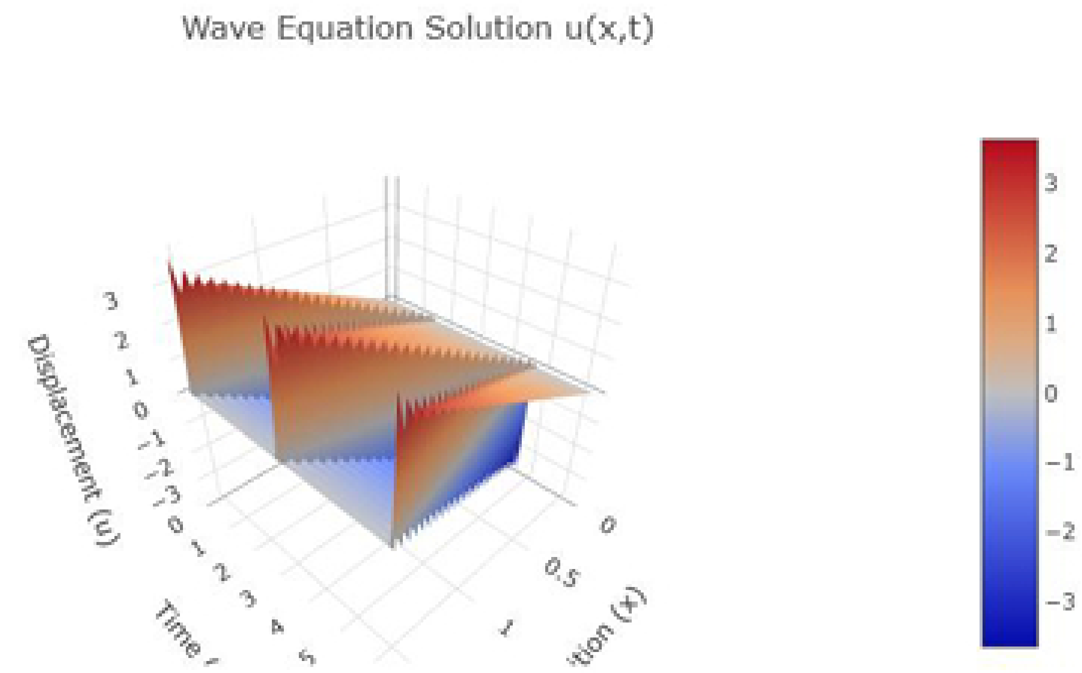
$$u_t(x,0) = \sum_{n=1}^{\infty} \sin(2nx)[2nB_n] = 0$$

For this series to be zero for all x, each coefficient must be zero. Thus, $2nB_n = 0$, which implies $B_n = 0$ for all n.

Substituting the coefficients A_n and B_n into the general solution yields the final answer:

$\boxed{u(x,t) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(2nx)\cos(2nt)}$

3D Visualization of the Wave Propagation u(x,t)



Problem 3

3 - For the initial value problem (IVP): $y' = 1 + \frac{y}{x}$, $1 \le x \le 2$, $y(1) = 2$, obtain an approximation of $y(2)$ using:

i. The analytical solution of the IVP,
ii. Euler's method, $h = 0.25$
iii. The Fourth-Order-Runge-Kutta method, $h = 0.5$

For each of the numerical estimates (ii) and (iii), determine the relative error based on (i).

i) Analytical Solution

The differential equation $y' = 1 + \frac{y}{x}$ can be rewritten as a first-order linear equation: $y' - \frac{1}{x}y = 1$. Here, $P(x) = -1/x$ and $Q(x) = 1$.

The integrating factor is $\mu(x) = e^{\int P(x)dx} = e^{\int -1/x dx} = e^{-\ln|x|} = e^{\ln(x^{-1})} = \frac{1}{x}$ for $x > 0$.

Multiplying the standard form by $\mu(x)$ gives $\frac{1}{x}y' - \frac{1}{x^2}y = \frac{1}{x}$, which is $\frac{d}{dx}\left(\frac{1}{x}y\right) = \frac{1}{x}$.

Integrating both sides with respect to x:

$$\frac{1}{x}y = \int \frac{1}{x}dx = \ln|x| + C$$
$$y(x) = x(\ln|x| + C)$$

Using the initial condition $y(1) = 2$:

$$2 = 1(\ln(1) + C) \implies 2 = 0 + C \implies C = 2$$

The analytical solution is $y(x) = x(\ln x + 2)$.

The exact value at $x = 2$ is:

$$y(2) = 2(\ln 2 + 2) \approx 2(0.693147 + 2) = 5.386294$$

ii) Euler's Method, $h = 0.25$

The iterative formula for Euler's method is $y_{i+1} = y_i + hf(x_i, y_i)$, where $f(x, y) = 1 + y/x$.

With $x_0 = 1, y_0 = 2, h = 0.25$, we perform 4 steps to reach $x = 2$.

1. **Step 1 ($i = 0$):** $x_0 = 1, y_0 = 2$
   $y_1 = 2 + 0.25(1 + 2/1) = 2 + 0.75 = 2.75$
2. **Step 2 ($i = 1$):** $x_1 = 1.25, y_1 = 2.75$
   $y_2 = 2.75 + 0.25(1 + 2.75/1.25) = 2.75 + 0.25(3.2) = 3.55$
3. **Step 3 ($i = 2$):** $x_2 = 1.5, y_2 = 3.55$
   $y_3 = 3.55 + 0.25(1 + 3.55/1.5) \approx 3.55 + 0.841667 = 4.391667$
4. **Step 4 ($i = 3$):** $x_3 = 1.75, y_3 \approx 4.391667$
   $y_4 = 4.391667 + 0.25(1 + 4.391667/1.75) \approx 4.391667 + 0.877381 = 5.269048$

The approximation of $y(2)$ using Euler's method is $\boxed{5.269048}$.

iii) Fourth-Order-Runge-Kutta (RK4) Method, $h = 0.5$

The RK4 formulas are: $y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$. Note the formula provided in the training data is $y_{i+1} = y_i + (\omega_1 + 2\omega_2 + 2\omega_3 + \omega_4)/6$ where $\omega_i = h \cdot k_i$. For consistency I will use k_i to represent the slopes as standard. The formula becomes $y_{i+1} = y_i + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$. Let's use the worksheet's notation ω_i: $y_{i+1} = y_i + \frac{1}{6}(\omega_1 + 2\omega_2 + 2\omega_3 + \omega_4)$ where $\omega_i = hf(\ldots)$.

We perform 2 steps to reach $x = 2$.

Step 1: From $x = 1$ to $x = 1.5$ ($x_0 = 1, y_0 = 2, h = 0.5$)

$$\omega_1 = 0.5 \cdot f(1, 2) = 0.5(1 + 2/1) = 1.5$$
$$\omega_2 = 0.5 \cdot f(1 + 0.25, 2 + \omega_1/2) = 0.5 \cdot f(1.25, 2.75) = 0.5(1 + 2.75/1.25) = 1.6$$
$$\omega_3 = 0.5 \cdot f(1 + 0.25, 2 + \omega_2/2) = 0.5 \cdot f(1.25, 2.8) = 0.5(1 + 2.8/1.25) = 1.62$$
$$\omega_4 = 0.5 \cdot f(1 + 0.5, 2 + \omega_3) = 0.5 \cdot f(1.5, 3.62) = 0.5(1 + 3.62/1.5) \approx 1.706667$$
$$y_1 = 2 + \frac{1}{6}(1.5 + 2(1.6) + 2(1.62) + 1.706667) \approx 3.607778$$

Step 2: From $x = 1.5$ to $x = 2.0$ ($x_1 = 1.5, y_1 \approx 3.607778, h = 0.5$)

$$\omega_1 = 0.5 \cdot f(1.5, 3.607778) \approx 1.702593$$
$$\omega_2 = 0.5 \cdot f(1.75, 3.607778 + \omega_1/2) \approx 1.77402$$
$$\omega_3 = 0.5 \cdot f(1.75, 3.607778 + \omega_2/2) \approx 1.78423$$
$$\omega_4 = 0.5 \cdot f(2.0, 3.607778 + \omega_3) \approx 1.84800$$
$$y_2 = 3.607778 + \frac{1}{6}(1.702593 + 2(1.77402) + 2(1.78423) + 1.84800) \approx 5.385652$$

The approximation of $y(2)$ using the RK4 method is $\boxed{5.385652}$.

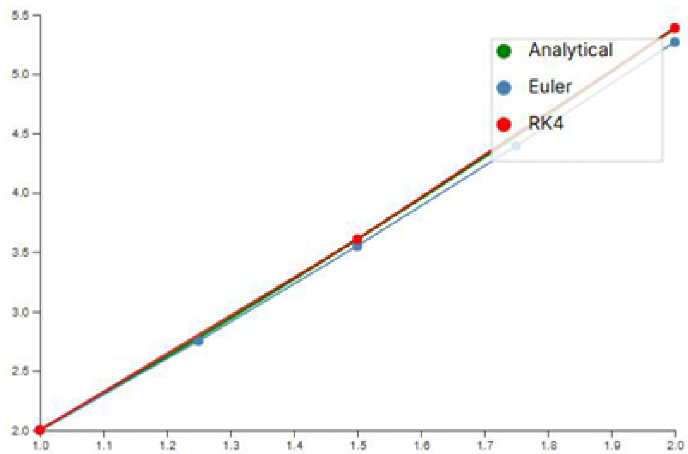
Relative Error Analysis

The exact value is $y_{\text{exact}} \approx 5.386294$.

- **Euler's Method:**
  Relative Error = $\left|\frac{y_{\text{exact}} - y_{\text{Euler}}}{y_{\text{exact}}}\right| = \left|\frac{5.386294 - 5.269048}{5.386294}\right| \approx 0.021767$, or $\boxed{2.18\%}$.
- **RK4 Method:**
  Relative Error = $\left|\frac{y_{\text{exact}} - y_{\text{RK4}}}{y_{\text{exact}}}\right| = \left|\frac{5.386294 - 5.385652}{5.386294}\right| \approx 0.000119$, or $\boxed{0.012\%}$.

The RK4 method is significantly more accurate than Euler's method, even with a larger step size.

Comparison of Analytical and Numerical Solutions

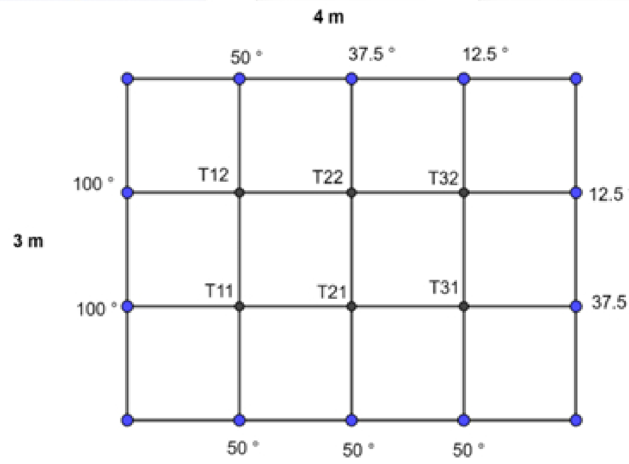


Problem 4: A plate $4.0m \times 3.0m$ is subjected to temperatures as shown in the Figure. Use the finite difference method to solve the PDE:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

with the grid and the boundary conditions shown in the figure.

Constraints: Take $T_{21} = 0.72T_{11}$ and $T_{31} = 0.47T_{11}$.



1. Finite Difference Formulation

The problem asks us to solve the steady-state heat equation (Laplace's equation) on a 2D grid. The finite difference approximation for the second derivative is used to discretize the domain. For a uniform grid with spacing $\Delta x = \Delta y$, the temperature at any interior node (i, j) is the arithmetic mean of its four neighbors:

$$T_{i,j} = \frac{T_{i+1,j} + T_{i-1,j} + T_{i,j+1} + T_{i,j-1}}{4}$$

$$4T_{i,j} - T_{i+1,j} - T_{i-1,j} - T_{i,j+1} - T_{i,j-1} = 0$$

We define the grid indices such that i represents the column (x-direction) and j represents the row (y-direction). Based on the figure, the internal unknown nodes are:

- Row 2 ($j = 2$): T_{12}, T_{22}, T_{32}
- Row 1 ($j = 1$): T_{11}, T_{21}, T_{31}

2. Establishing Equations for Each Node

We write the nodal equations for the four primary unknowns ($T_{11}, T_{12}, T_{22}, T_{32}$), substituting the known boundary conditions and the given constraints ($T_{21} = 0.72T_{11}$ and $T_{31} = 0.47T_{11}$).

Node (1,1) [Bottom-Left Internal]:
Neighbors: Left (100°), Right (T_{21}), Top (T_{12}), Bottom (50°).

$$4T_{11} = 100 + T_{21} + T_{12} + 50$$

Substitute $T_{21} = 0.72T_{11}$:

$$4T_{11} = 150 + 0.72T_{11} + T_{12}$$

$$(4 - 0.72)T_{11} - T_{12} = 150$$

$$3.28T_{11} - T_{12} = 150 \quad \ldots \text{(Eq. I)}$$

Node (1,2) [Top-Left Internal]:
Neighbors: Left (100°), Right (T_{22}), Top (50°), Bottom (T_{11}).

$$4T_{12} = 100 + T_{22} + 50 + T_{11}$$

$$4T_{12} - T_{22} - T_{11} = 150 \quad \ldots \text{(Eq. II)}$$

Node (1,2) [Top-Left Internal]:
Neighbors: Left (100°), Right (T_{22}), Top (50°), Bottom (T_{11}).

$$4T_{12} = 100 + T_{22} + 50 + T_{11}$$
$$4T_{12} - T_{22} - T_{11} = 150 \quad \ldots(\text{Eq. II})$$

Node (3,2) [Top-Right Internal]:
Neighbors: Left (T_{22}), Right (12.5°), Top (12.5°), Bottom (T_{31}).

$$4T_{32} = T_{22} + 12.5 + 12.5 + T_{31}$$

Substitute $T_{31} = 0.47T_{11}$:

$$4T_{32} - T_{22} - 0.47T_{11} = 25 \quad \ldots(\text{Eq. III})$$

Node (2,2) [Top-Center Internal]:
Neighbors: Left (T_{12}), Right (T_{32}), Top (37.5°), Bottom (T_{21}).

$$4T_{22} = T_{12} + T_{32} + 37.5 + T_{21}$$

Substitute $T_{21} = 0.72T_{11}$:

$$4T_{22} - T_{12} - T_{32} - 0.72T_{11} = 37.5 \quad \ldots(\text{Eq. IV})$$

3. Solving the System

We will express T_{12}, T_{22}, T_{32} in terms of T_{11} and solve Eq. IV.

Step A: From Eq. I

$$T_{12} = 3.28T_{11} - 150$$

Step B: Substitute T_{12} into Eq. II to find T_{22}

$$4(3.28T_{11} - 150) - T_{22} - T_{11} = 150$$
$$13.12T_{11} - 600 - T_{11} - T_{22} = 150$$
$$12.12T_{11} - 750 = T_{22}$$

Step C: Substitute T_{22} into Eq. III to find T_{32}

$$4T_{32} - (12.12T_{11} - 750) - 0.47T_{11} = 25$$
$$4T_{32} - 12.59T_{11} + 750 = 25$$
$$4T_{32} = 12.59T_{11} - 725$$
$$T_{32} = 3.1475T_{11} - 181.25$$

Step D: Substitute all expressions into Eq. IV to solve for T_{11}

$$4(12.12T_{11} - 750) - (3.28T_{11} - 150) - (3.1475T_{11} - 181.25) - 0.72T_{11} = 37.5$$

Grouping the terms by T_{11} and constants:

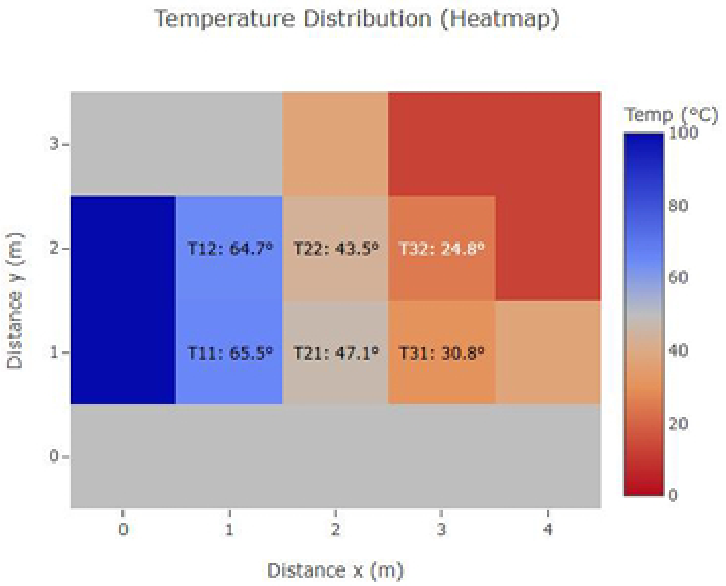
$$(48.48 - 3.28 - 3.1475 - 0.72)T_{11} + (-3000 + 150 + 181.25) = 37.5$$
$$41.3325T_{11} - 2668.75 = 37.5$$
$$41.3325T_{11} = 2706.25$$
$$T_{11} \approx 65.47^\circ$$

$$T_{11} \approx 65.47^\circ$$

4. Calculating Remaining Temperatures

Using $T_{11} = 65.47$:

- $T_{12} = 3.28(65.47) - 150 = \mathbf{64.74}^\circ$
- $T_{22} = 12.12(65.47) - 750 = \mathbf{43.50}^\circ$
- $T_{32} = 3.1475(65.47) - 181.25 = \mathbf{24.82}^\circ$
- $T_{21} = 0.72(65.47) = \mathbf{47.14}^\circ$
- $T_{31} = 0.47(65.47) = \mathbf{30.77}^\circ$



Final Answer:

The steady-state temperatures at the internal grid points are approximately:

Node	Temperature (°C)
T_{11}	65.47
T_{21}	47.14
T_{31}	30.77
T_{12}	64.74
T_{22}	43.50
T_{32}	24.82