MISSPECIFICATION AVERSE PREFERENCES*

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Job Market Paper

November 11, 2025
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Abstract. We study a decision maker who approaches an uncertain decision problem by formulating a set of plausible probabilistic models of the environment but is aware that these models are only stylized and incomplete approximations. The agent is effectively facing two layers of uncertainty. Not only is the decision maker uncertain regarding what model in this set has the best fit (ambiguity), but she is also concerned that the best-fit model itself might be a poor description of the environment (model misspecification). We develop an axiomatic foundation for a general class of preferences that capture concern toward these two layers of uncertainty and allow us to compare individuals' degrees of aversion to model misspecification and ambiguity independently of each other.

*I am particularly indebted to David Dillenberger, Kevin He, and George Mailath for their valuable guidance. I am also grateful to Aislinn Bohren, Simone Cerreia-Vioglio, Roberto Corrao, Tommaso Denti, Luigi Falasconi, Jesus Fernandez-Villaverde, Peter Klibanoff, Giacomo Lanzani, Fabio Maccheroni, Julien Manili, Massimo Marinacci, Sujoy Mukerji, Guillermo Ordonez, Andrew Postlewaite, Alex Sawyer, Ana Sofia Teles, Juuso Toikka, Gerelt Tserenjigmid, Mu Zhang, and seminar participants at UPenn and RUD 2025 for helpful comments and suggestions.
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1. Introduction

Economic agents often employ simplified and stylized descriptions of the complex environment they face to help guide their decisions. This implies that model misspecification is a pervasive phenomenon affecting many decision problems. For example, a policymaker might have an incorrect description of how the economy would respond to a fiscal or monetary stimulus, or a company's marketing department might have a wrong assessment of how demand would react to changes in the price of a product. As a result, a growing literature studies the implications of using misspecified models in the context of decision making and strategic interaction (see Section 1.1 for a comprehensive literature review). A common assumption in this literature is that once agents have settled on using a specific statistical model of the environment, they disregard the possibility of it being misspecified and act in a fully Bayesian fashion, evaluating alternatives by their expected utility with respect to that model. However, sophisticated enough agents should realize that their model is only a simplified approximation of reality. As suggested by Hansen and Sargent (2001), an economic agent who is concerned with acting on the basis of an incorrectly formulated model should make decisions that are *robust*; that is, policies that work reasonably well across all models that are close enough to the reference model. Following this idea, the first axiomatic treatment of decision criteria featuring misspecification aversion has been proposed by Cerreia-Vioglio, Hansen, Maccheroni, and Marinacci (2025). Moreover, Lanzani (2025) axiomatizes a special case of preferences that are robust to misspecification and uses it to study learning under endogenous misspecification concerns.

In this paper, we provide an axiomatic foundation of a general class of preferences that are averse to the possibility of misspecification. The main contribution of our model is a way of meaningfully disentangling misspecification aversion from the more commonly studied aversion to ambiguity. We adopt a version of the Anscombe-Aumann framework where uncertainty is captured by a set of states of the world Ω, and the decision maker (henceforth, DM) needs to choose an act f that maps states of the world to outcomes. The DM does not know the true *data-generating process (DGP)* governing the environment, but she has statistical information in her possession. This is given by a set $\mathcal{M}$ of distributions over states of the world. Each *model* $m \in \mathcal{M}$ can be interpreted as an alternative plausible hypothesis regarding the DGP. Being aware that models are only imperfect and stylized descriptions of the real environment, the DM might become concerned that, in fact, no hypothesized model in $\mathcal{M}$ is an accurate approximation of the DGP; or, in other words, that the true DGP is not contained in

$\mathcal{M}$. Moreover, the DM also has at her disposal a best-fit map, identifying the model that is the best approximation of the true DGP based on different state realizations. This perspective immediately reveals two distinct ways uncertainty enters the decision problem:

- *ambiguity:* the agent lacks information to formulate and commit to a single prior over the set of probabilistic models $\mathcal{M}$,
- *model misspecification:* the agent worries that the entire modeling frame is too narrow and that none of the candidate probabilistic models in $\mathcal{M}$ is a good description of the environment.

As an illustration, consider an investor who needs to decide how much of her wealth to allocate to a stock with an uncertain return versus a safe, government bond. To face this portfolio investment problem, the agent postulates that the stock return is normally distributed with unknown mean and variance, not because she actually believes this assumption to be correct, but because it makes the problem tractable and easier to analyze. In this scenario, the investor faces ambiguity if she is unable to pin down a unique prior over the mean and variance parameters of the hypothesized normal stock returns. On the other hand, the agent also faces model misspecification concerns if she worries that the distribution of stock returns might not be normal after all, but display thicker tails or some degree of skewness.

As highlighted by the illustration, ambiguity and model misspecification are conceptually distinct phenomena; the first one is generated by the uncertainty among the different probabilistic models in the hypothesized set $\mathcal{M}$, while the second pertains to the concern that the correct description of the environment faced by the agent might lie well outside of the scope of the set of hypothesized probabilistic models. The approach developed in this paper enables a meaningful disentangling of model misspecification concerns from attitudes toward ambiguity, without requiring stark assumptions about the latter. This differs from the previous decision-theoretic literature that, in order to study misspecification, shut down interesting ambiguity attitudes by assuming the DM to be either infinitely ambiguity averse or ambiguity neutral. We are, instead, able to accommodate misspecification aversion while allowing for complete flexibility in the DM's ambiguity attitudes (aversion, neutrality, or indifference). This clarifies that misspecification aversion and ambiguity attitudes are two distinct behavioral phenomena that can, in principle, lead to different choice behavior.

Specifically, we show that, in this framework, the DM evaluates each uncertain alternative according to the following two-step procedure. First, if the DM were told sufficient information to determine that a distribution $m \in \mathcal{M}$ is the best-fit model, she would evaluate an act f according to a quasiconcave misspecification certainty equivalent $V^m(f) = I(u(f), m)$ of the utility act induced by f, given the utility function over outcomes u. We interpret this misspecification certainty equivalent as the certain utility level that the DM would be willing to accept in order to eliminate the uncertainty stemming from the possibility of model misspecification. Specifically, the quasiconcavity property of this certainty equivalent captures the DM's preference for hedging against this uncertainty and, therefore, her concerns for model misspecification. In the main specification of our preferences, this misspecification certainty equivalent takes the form of a misspecification-robust criterion:

$$(1) \qquad V^m(f) = \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p, m)\}$$

where $c(\cdot, m)$ is an index of misspecification aversion. That is, even conditional on observing sufficient information to determine that m is the best model, the DM would not completely trust it out of misspecification concerns. Therefore, in evaluating an act f, she would also take into account other distributions p outside of $\mathcal{M}$ that are not too far apart in a probabilistic sense from m. The index $c(\cdot, m)$ captures the DM's confidence in the model m. An important special case is given by $c(\cdot, m) = \lambda R(\cdot||m)$, where R is the relative entropy and $\lambda > 0$ is a parameter of misspecification aversion. When the DM's concern for misspecification is high, λ is low, and therefore, she would give preference to acts that perform robustly well across a larger set of models around m. In the extreme case of λ approaching infinity, the DM becomes misspecification neutral and evaluates acts according to their expected utility given m.

Second, being also uncertain about the identity of the best-fit model, the DM aggregates together the misspecification-robust evaluations:

$$(2) \qquad V(f) = \hat{I}\left(\left(V^m(f)\right)_{m \in \mathcal{M}}\right) = \hat{I}\left(\min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p, \cdot)\}\right)$$

where $\hat{I} : \mathbb{R}^{\mathcal{M}} \to \mathbb{R}$ is an aggregator capturing the DM's attitudes toward the ambiguity regarding what model is the best-fit one. We show that we can interpret $\hat{I}$ as a utility certainty equivalent of the uncertain (because of ambiguity) profile of misspecification-robust evaluations; that is, the certain utility level that the DM would be willing to accept in order to eliminate ambiguity about which of the hypothesized probabilistic models has the best fit.

We illustrate how our framework distinguishes concern toward misspecification from attitudes toward ambiguity. We show that we can rank two agents in terms of their degree of misspecification aversion by only comparing their misspecification index c

(without imposing any mutual restrictions on their aggregators $\hat{I}$) and, similarly, we can rank agents in terms of their attitudes toward ambiguity by only comparing their aggregator $\hat{I}$ (without imposing any mutual restrictions on their misspecification aversion indexes). Specifically, DM1 is more misspecification averse than DM2 if and only if $c_1(\cdot, m) \leq c_2(\cdot, m)$ for all hypothesized models $m \in \mathcal{M}$. On the other hand, DM1 is more averse to ambiguity than DM2 if and only if $\hat{I}_1 \leq \hat{I}_2$. That is, the first individual is more ambiguity averse if she is willing to accept lower certainty equivalents than the second to eliminate the ambiguity regarding the identity of the best-fit model.

We provide explicit axiomatizations of tractable functional forms of the aggregator $\hat{I}$. In particular, we discuss the following two special cases. First, we show that if the DM confronts the uncertainty regarding the identity of the best-fit model according to the expected utility tenets, she aggregates the misspecification-robust evaluations in a Bayesian fashion:

$$V_{\phi,\mu}(f) = \int_{\mathcal{M}} \phi\left(\min_{p\in\Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p,m)\}\right) d\mu(m) \tag{3}$$

where μ is the DM's subjective prior over the set of models $\mathcal{M}$ and ϕ captures the DM's attitudes toward ambiguity. This criterion was first proposed by Cerreia-Vioglio et al. (2025) without providing an explicit axiomatization. Moreover, when the DM is neutral toward ambiguity and shows a uniform concern for misspecification, this criterion becomes the average robust control representation axiomatized by Lanzani (2025).[1] If, instead, the DM is misspecification neutral, that is, when $c(\cdot, m)$ assigns an infinite penalization to any probability model different from m itself, this criterion becomes the well-known smooth ambiguity model of Klibanoff, Marinacci, and Mukerji (2005). Second, we show that if the DM is cautious and evaluates the uncertainty about the best-fit model according to a worst-case scenario approach, then the aggregator takes on a maxmin form and we obtain the cautious criterion introduced by Cerreia-Vioglio et al. (2025):

$$V_{min}(f) = \min_{p\in\Delta(\Omega)} \left\{\mathbb{E}_p[u(f)] + \min_{m\in\mathcal{M}} c(p,m)\right\}. \tag{4}$$

Before moving on to discuss the related literature, we provide a brief overview of the two central axioms that underpin our decision criterion. The first, which we call *misspecification aversion*, requires that even after conditioning on the event that a given $m \in \mathcal{M}$ is the best-fit model, the DM's preferences still do not necessarily satisfy

[1] To be precise, we would also need to impose that the conditional misspecification-robust evaluations are the multiplier preferences proposed by Hansen and Sargent (2001) and axiomatized by Strzalecki (2011).

full-fledged independence, but display instances of uncertainty aversion. Intuitively, suppose the DM had sufficient information to determine that m is the best approximation in $\mathcal{M}$. If she were completely certain that the true DGP is included in $\mathcal{M}$, she would conclude as a matter of fact that m *is* the correct description of the environment and evaluate uncertain alternatives according to their expected utility given m. The fact that even after m is revealed to be the best-fit model, the DM's preferences might still exhibit violations of independence and a preference for hedging against the residual uncertainty reflects her mistrust of the best-fit model m and, thereby, a concern that the set of hypothesized models is misspecified. The second axiom is *consistency*. It requires that the DM prefers an act f to g whenever f is preferred to g conditional on each $m \in \mathcal{M}$. This axiom connects the subjective preferences of the DM to the statistical information encoded in the set of models $\mathcal{M}$. It captures the idea that even if aware of the possibility of misspecification, the DM still puts substantive trust in the set of models. If they provide a unanimous ranking of two alternatives, after taking into account misspecification concerns, then the DM's preferences comply with that ranking.

In the last section, I illustrate that misspecification aversion can have qualitatively different implications compared to ambiguity aversion only by revisiting the monopoly pricing example discussed in Ball and Kattwinkel (2024). Assuming that the monopolist's preferences are represented by the criterion (4), we show that the payoff guarantee of the optimal posted price is robust to perturbations of the monopolist's posited set of distributions over buyers' valuations if the monopolist exhibits some degree of concern for misspecification.

The rest of the paper is structured as follows. Section 1.1 discusses the relevant literature. Section 2 lays out the decision framework and the notions of the hypothesized set of models and best-fit map. Section 3 introduces and discusses the axioms characterizing the misspecification averse preferences. Section 4 states and discusses the representation results. Section 5 illustrates the implications of misspecification aversion in the context of a monopoly pricing example. Section 6 concludes. All proofs can be found in the Appendix.

1.1. **Related Literature.**

Preferences and Statistical Information.

This paper is related to the literature connecting statistical information to choice behavior (see, for example, Amarante (2009), Al-Najjar and De Castro (2014), Epstein and Seo (2010), and Klibanoff, Mukerji, and Seo (2014)). Within this class, the closest paper to ours is Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2013). Building on their setup, we incorporate misspecification aversion in the preferences of a DM using exogenous, statistical information to inform her choices. In their case, since the DM does not care about model misspecification, preferences conditional on a model m are expected utility. Therefore, their consistency axiom requires the DM's preferences to comply whenever two acts are unanimously ranked according to their expected utility:

$$\forall m \in \mathcal{M},\ \mathbb{E}_m[u(f)] \geq \mathbb{E}_m[u(g)] \implies f \succsim g.$$

They show that this implies their representation only depends on the profile of expected utility evaluations $(\mathbb{E}_m[u(f)])_{m\in\mathcal{M}}$, so that preferences are represented via an aggregator of the map $m \mapsto \mathbb{E}_m[u(f)]$. In our case, however, even after observing the missing information sufficient to pin down a unique best-fit model $m \in \mathcal{M}$, the DM, out of misspecification concerns, would only trust m to be the best approximation to the DGP, but not necessarily the correct one. Therefore, her preferences conditional on m are not necessarily expected utility, but can still display a preference for robustness across models that are in a vicinity of m. In Theorem 1, we show that the representation of our class of misspecification averse preferences only depends on the profile of misspecification certainty equivalents $(I(u(f), m))_{m\in\mathcal{M}}$, so that the representation can be expressed as an ambiguity certainty equivalent aggregator $\hat{I}$ of the map $m \mapsto I(u(f), m)$. The fact that this map is no longer linear in the models $m \in \mathcal{M}$ is one of the main technical difficulty that we deal with in this paper.[2] Moreover, we show that also in our case, axioms on preferences over acts can be translated into properties of the ambiguity certainty equivalent $\hat{I}$ without having to resort to second-order acts. This allows us to readily axiomatize tractable and more easily interpretable functional forms of the aggregator $\hat{I}$.

This paper is also related to the recent axiomatization by Denti and Pomatto (2022) of identifiable smooth ambiguity preferences. Without positing an exogenous set of probabilistic models, and abstracting from misspecification concerns, they find conditions under which preferences are represented by the smooth ambiguity criterion, where the beliefs involved in the representation are identifiable; that is, they are completely orthogonal for some kernel κ. In this paper, we start with the DM having an exogenously given set of models and a best-fit map, connect the DM's preferences to

[2]In this respect, this paper is also related to Mu, Pomatto, Strack, and Tamuz (2021). In a different context, they show that monotone additive statistics can be represented as averages of CARA certainty equivalents.

this statistical models by consistency, but allow the DM to display aversion toward misspecification.

Decision Criteria incorporating Misspecification Concerns.

Following the spirit of the robust control model pioneered by Hansen and Sargent (2001), the first paper to propose and axiomatize preferences that display aversion to model misspecification is Cerreia-Vioglio et al. (2025). They first axiomatize the criterion (4) in a two-preference setup à la Gilboa, Maccheroni, Marinacci, and Schmeidler (2010). The DM has an objectively rational preference that satisfies weak certainty independence but is incomplete, and a subjectively rational preference that is complete, but satisfies independence only on constant acts. These two preferences are linked via two axioms originated in Gilboa et al. (2010). The first is consistency. It requires that the subjectively rational preference is a completion of the objectively rational. The second is that the DM exercises caution; that is, if the objectively rational preference is not confident enough to rank an uncertain act over a deterministic one, then the deterministic act is chosen by the subjectively rational preference (when in doubt, go with the certain alternative). Moreover, the two preferences are informed by the set of probabilistic models via coherence requirements analogous to those given in this paper. They also propose a foundation for a more general aggregator of the misspecification-robust evaluations in a setup involving a two-preference family indexed by varying sets of posited models. In this context, they also discuss the Bayesian version of the aggregator (3). Lanzani (2025) also adopts the view of Cerreia-Vioglio et al. (2013) by considering states of the world that describe both the realization of the payoff-relevant state and the distribution over such payoff states. They assume that the DM has variational preferences and obtain a special case of (3), which they call the average robust control criterion, by imposing that preferences on bets over models satisfy the sure-thing principle and uncertainty neutrality (thus obtaining an affine ϕ). Moreover, they propose axioms that characterize the asymptotic behavior of the index of misspecification concern when the DM's preferences evolve in reaction to the arrival of new information. We show (Theorems 4 and 5) that the criteria introduced by Cerreia-Vioglio et al. (2025) as well as the average robust control criterion of Lanzani (2025) fall within the general class of misspecification averse preferences studied in this paper. In particular, Cerreia-Vioglio et al. (2025)'s main criterion and Lanzani (2025)'s criterion represent two opposite ends of the spectrum; the cautious robust criterion (4) displays an extreme form of ambiguity aversion, while the average robust control criterion is neutral toward ambiguity. One contribution of our paper is to allow more flexible attitudes toward ambiguity while proposing a way to disentangle those from the degree of misspecification aversion. This is reflected in the fact that the representation parameters capturing model misspecification aversion (the index c) and ambiguity aversion (the aggregator $\hat{I}$) are independent of each other. Finally, a recent paper by Bonaglia and Dedola (2025) proposes a methodology to identify endogenously from the preferences of a misspecification concerned DM the set of probabilistic models $\mathcal{M}$.

Learning with Misspecified Models.

Starting with Esponda and Pouzo (2016), many papers have examined the asymptotic behavior of actions and beliefs when agents take repeated decisions in a stochastic environment of which they have a possibly incorrect or only partial understanding (see, for instance, Frick, Iijima, and Ishii, 2022; Fudenberg, Lanzani, and Strack, 2021). In all these models, agents are not concerned about misspecification and are expected utility maximizers. A key result is that misspecification is asymptotically persistent and thus matters in shaping agents' behavior and beliefs, even when agents collect many observations generated by the true data-generating process. A different strand of the literature allows agents to realize that their model is misspecified and switch to a competing alternative (see, for example, Ba (2021), Fudenberg and Lanzani (2023), and He and Libgober (2021)). The main difference from the misspecification averse preferences axiomatized in this paper is that agents act in a fully Bayesian fashion once they have selected one of the competing models on the basis of a statistical fitness test.

2. Decision Framework

We begin by describing the decision environment faced by the DM. Uncertainty is described by a state space Ω endowed with a countably generated sigma-algebra $\mathcal{G}$. Let X be the space of *consequences*, a convex subset of a topological vector space. The DM needs to choose *acts*, that is, functions $f : \Omega \to X$ mapping states to consequences that are measurable with respect to $\mathcal{G}$. We say that an act f is bounded if there exists a finite subset K of X such that $f(\omega)$ is in the convex hull of K for all $\omega \in \Omega$. We denote by $\mathcal{F}$ the set of all bounded acts and by $\mathcal{F}_0$ the subset of acts that are simple. Abusing notation, we denote by $x \in X$ also the constant act yielding consequence x in each state $\omega \in \Omega$. For each $f, f' \in \mathcal{F}$ and $\alpha \in [0,1]$, the convex combination $\alpha f + (1-\alpha) f'$ is the act given by:

$$(\alpha f + (1-\alpha) f')(\omega) \coloneqq \alpha f(\omega) + (1-\alpha) f'(\omega)$$

for all $\omega \in \Omega$. Given any $E \in \mathcal{G}$ and simple acts $f, g \in \mathcal{F}$, let fEg be the act taking value $f(\omega)$ if $\omega \in E$ and value $g(\omega)$ if $\omega \in \Omega \setminus E$. If $\mathcal{E}$ is a sub-sigma-algebra of $\mathcal{G}$, denote by $\mathcal{F}(\mathcal{E})$ the subset of acts in $\mathcal{F}$ that are measurable with respect to $\mathcal{E}$.

Let $\succsim$ be a preference relation over $\mathcal{F}$. Denote by $\succ$ and $\sim$ respectively the asymmetric and symmetric part of $\succsim$. An event $E \in \mathcal{G}$ is *null* if for all acts $f, f' \in \mathcal{F}$, $f|_{\Omega \setminus E} = \tilde{f}|_{\Omega \setminus E}$ implies that $f \sim f'$.

2.1. **Probabilistic Models and Best-Fit Map.** Let $\Delta \coloneqq \Delta(\Omega, \mathcal{G})$ denote the space of countably additive probability measures on $(\Omega, \mathcal{G})$. Endow Δ with the natural sigma-algebra $\mathcal{D}$ generated by the family of evaluation maps and any subset of Δ, with its relative sigma-algebra.[3]

We assume that the DM is equipped with a set $\mathcal{M} \subseteq \Delta$ of probability distributions over states of the world that, given some external information, she believes are plausible descriptions of the uncertain environment she is facing. In keeping with the classical setup of Wald (1950), each model $m \in \mathcal{M}$ can be interpreted as an alternative hypothesis regarding the DGP, based on substantive motivations, like scientific theories and empirical evidence. The models in $\mathcal{M}$ are sometimes referred to in the literature (see, for example, Hansen and Sargent (2022) and Cerreia-Vioglio et al. (2025)) as *structured*, to remark their special status in the eye of the DM as opposed to other distributions outside $\mathcal{M}$. Following Box (1976, 1979) and Cox (1995)'s idea that models are only approximations, we do not assume that the set of models includes the DGP, that is, the true probability law governing state uncertainty. Moreover, we allow for the possibility that the DM is aware of this fact, and perceives the possibility that her set of models might be *misspecified*.

Our aim in this paper is to discern between ambiguity about which probabilistic model is the best approximation to the DGP and concern about misspecification, that is, the fact that no hypothesized model is an accurate approximation of the DGP. Uncertainty about models is usually motivated in terms of "lack of information" preventing the DM from selecting the best one. Following Cerreia-Vioglio et al. (2013),[4] we formalize this missing information via the idea of sufficient statistics and information (Dynkin, 1978). Specifically, we assume that the measurable space of states of the world $(\Omega, \mathcal{G})$ and the posited set of probabilistic models $\mathcal{M}$ admit a *best-fit map* $\mathfrak{q} : \Omega \to \mathcal{M}$ such that

$$m(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}) = 1 \text{ for all } m \in \mathcal{M}. \tag{5}$$ [5]

[3]Appendix A provides rigorous definitions of the mathematical concepts and details regarding the notation.

[4]See also Amarante (2009), Al-Najjar and De Castro (2014), Epstein and Seo (2010), and Klibanoff et al. (2014) for related approaches and Denti and Pomatto (2022) for a discussion of this condition.

[5]The requirement that each model $m \in \mathcal{M}$ is selected by the best-fit map with probability one according to m is equivalent to the notion of sufficient statistics introduced by Dynkin (1978) and is related to the strong law of large numbers. In mathematics and probability, this property is what

Specifically, we interpret the event $E^m = \{\omega : \mathfrak{q}(\omega) = m\}$ that model m is selected by the map $\mathfrak{q}$ as the event that m is the probabilistic model within $\mathcal{M}$ that closest resembles the true closes resembles the correct distribution over the states of the world. Then, the sigma-algebra $\mathcal{A}$ generated by the best-fit map $\mathfrak{q}$ can be interpreted as representing the *sufficient* information to determine which model in the family $\mathcal{M}$ is the closest approximation of the true DGP, i.e., has the best fit. Condition (5) is the well-known assumption that the DM's family of probabilistic models is *point-identified* whenever it is not misspecified. Indeed, this condition only requires that if a model $m \in \mathcal{M}$ turned out to be, indeed, the correct DGP, then the best-fit map would select model m with probability one. We interpret this framework as follows. Suppose that the well-specified description of the environment is given by the set of models $\mathcal{P}$ and a map $p : \omega \mapsto p^\omega \in \Delta$ such that $P(\{\omega \in \Omega : p^\omega = P\}) = 1$ for all $P \in \mathcal{P}$. The interpretation is that the realization of ω also pins down what is the true DGP p^ω, so that the sigma-algebra of events that makes p^ω measurable captures the sufficient information to determine what is the true probability law over states. Moreover, the statement that $p^\omega = P$ with probability one according to P is a requirement that the correct description of the environment is not contradictory; that is, whenever P is the true DGP, then it is selected with probability one by the map p^ω.[6] However, the DM posits a misspecified set of models $\mathcal{M} \subseteq \mathcal{P}$ that does not necessarily include all models in $\mathcal{P}$. Now suppose that the DM tried to estimate the best-fit model within the hypothesized set $\mathcal{M}$. Since the DM has posited a misspecified set of models, the insight from Berk (1966) suggests that she would asymptotically select from $\mathcal{M}$ the closest model to P; that is, the minimizer (assumed unique) $q^*(P) \in \mathcal{M}$ solving $\min_{m\in\mathcal{M}} D(P||m)$, where $D(\cdot||\cdot)$ is an appropriate measure of fit. Define the function $\mathfrak{q}(\omega) := q^*(p^\omega)$. Then, the set E^m would, indeed, represent the event that m has the best fit and the sigma-algebra $\mathcal{A}$ generated by $\mathfrak{q}$ would encode the information needed to determine which of the hypothesized models has the best fit. Moreover, for all $m \in \mathcal{M}$ we would, indeed, have that $m\left(\{\omega : \mathfrak{q}(\omega) = m\}\right) = 1$. It is in this sense that we interpret $\mathfrak{q}$ as a *best-fit map* and the information in $\mathcal{A}$ as the sufficient information to determine the best approximation of the DGP among those in $\mathcal{M}$. That is, if the DM were able to observe ω, she would infer that the model $m_\omega \equiv \mathfrak{q}(\omega)$ is the model that closest resembles the true DGP.

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is known as complete orthogonality of the set of probability models $\mathcal{M}$. See, for example, Mauldin, Preiss, and Weizsacker (1983) and Weis (1984).

[6] We can see the analogy to the strong law of large numbers if we interpret each ω as the realization of an infinite sequence of random variables and p^ω as the limit of a consistent estimator.

Example 1: Consider the portfolio decision example mentioned in the introduction, and consider an investor who needs to decide at an initial period how much to invest in a stock versus a safe perpetuity paying a fixed amount at every future period. At each period t, a stock return $s_t \in \mathbb{R}$ is realized. Then, from the investor's perspective at the initial period, a state of the world is an infinite sequence of stock return realizations, and the state space is given by the sequence space $\Omega = \mathbb{R}^{\mathbb{N}}$. In order to solve the portfolio problem, the investor postulates that the stock returns are iid normally distributed over time, with unknown mean $\mu \in \mathbb{R}$ and variance $\sigma^2 \in \mathbb{R}_{++}$. The investor makes this assumption not because she believes it to be inherently correct, but because it provides an easy-to-use and tractable model to solve the investment problem. In this scenario, the set of probabilistic models posited by the investor is

$$\mathcal{M} = \{m_{\mu,\sigma^2} = \times_{n=1}^{\infty} \Phi_{\mu,\sigma^2} : \Phi_{\mu,\sigma^2} \sim N(\mu, \sigma^2),\ \mu \in \mathbb{R},\ \sigma^2 > 0\}.$$

Suppose, however, that the correct description of the environment is that the stock return is iid distributed according to a generalized normal distribution p_θ, parametrized by three parameters $\theta = (\theta_1, \theta_2, \theta_3) \in \mathbb{R} \times \mathbb{R}^2_{++}$ respectively capturing the mean, the variance, and the tail thickness of the distribution. The agent attempts to parse the uncertainty regarding the mean and variance of the (misspecified) normal family of models he posited by considering the asymptotic behavior of the MLE estimates from the observations of stock returns accumulating over time. She, then, constructs her best-fit map $\mathfrak{q} : \Omega \to \mathbb{R} \times \mathbb{R}_{++}$ as the limit of these estimates. Given the insights from White (1982), we know that with probability one according to the correct iid generalized normal DGP with parameter θ, this limit would be equal to the unique minimizer $(\hat{\mu}, \hat{\sigma}^2)$ solving $\min_{\mu,\sigma^2} R(p_\theta, \Phi_{\mu,\sigma^2})$, which, in fact, would exactly match the mean and variance of the true DGP: $(\hat{\mu}, \hat{\sigma}^2) = (\theta_1, \theta_2)$.[7] We can thus interpret the iid normal model with mean θ_1 and variance θ_2 as the best approximation of the true DGP. This would be true whenever the generalized normal DGP has the first two parameter components equal to (θ_1, θ_2), no matter the value of the third, tail thickness parameter θ_3 (including the case where the DPG is actually iid normal, corresponding to the case $\theta_3 = 2$). Thus, we can interpret the event $\{\omega : \mathfrak{q}(\omega) = (\theta_1, \theta_2)\}$ as the event that the iid normal model with mean θ_1 and variance θ_2 is the model within the hypothesized set that has the best-fit and $\mathfrak{q}$ as a best-fit map.

[7] Here R is the relative entropy; that is, $R(q||p) = \int_\Omega \ln \frac{dq}{dp} dq$ if $q \ll p$ and equal to ∞ otherwise.

3. Basic Conditions, Coherence, and Consistency

In the sequel, we fix a measurable state space $(\Omega, \mathcal{G})$ and a set of probabilistic models $\mathcal{M}$, admitting a best-fit map $\mathfrak{q}$ satisfying the properties outlined in the previous section.

3.1. **Basic Conditions.**

We assume that the DM's preferences satisfy the axioms of Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio (2011).

Axiom 1 (Basic Conditions):

(i) Weak Order. *$\succsim$ is complete and transitive.*

(ii) Monotonicity. *For all $f, f' \in \mathcal{F}$, if $f(\omega) \succsim f'(\omega)$ for all $\omega \in \Omega$, then $f \succsim f'$.*

(iii) Mixture Continuity. *If $f, f', f'' \in \mathcal{F}$, the sets $\{\alpha \in [0,1] : \alpha f' + (1-\alpha) f'' \succsim f\}$ and $\{\alpha \in [0,1] : f \succsim \alpha f' + (1-\alpha) f''\}$ are both closed.*

(iv) Risk Independence. *For all $x, y, z \in X$ and $\alpha \in [0,1]$,*

$$x \succsim y \iff \alpha x + (1-\alpha) z \succsim \alpha y + (1-\alpha) z \ .$$

(v) Unboundedness. *There exist $x, y \in X$ such that $x \succ y$ and for all $\alpha \in (0,1)$, there are $z, z' \in X$ such that*

$$\alpha z + (1-\alpha) y \succ x \succ y \succ \alpha x + (1-\alpha) z' \ .$$

The first four requirements guarantee that the preferences are a continuous and monotone weak order satisfying independence when restricted to constant acts. Then, the theorem of Herstein and Milnor (1953) implies that the preferences are represented on X by an affine utility u. If we interpret the mixture space X as the set of simple lotteries over outcomes, these axioms imply that the DM evaluates lotteries - i.e., constant acts not involving uncertainty about the state of the world - according to their objective expected utility. The last requirement is mostly for technical convenience, and it guarantees that the utility over consequences u will be unbounded above and below.

Finally, the next axiom guarantees that preferences are robust to small perturbations and guarantees the countable additivity of the subjective probabilities.

Axiom 2 (Monotone Continuity): *For all $f, f' \in \mathcal{F}$ and $x \in X$, for all $(E_n)_{n \in \mathbb{N}} \subseteq \mathcal{G}$ such that $E_1 \supseteq E_2 \supseteq \cdots$ and $\bigcap_{n \in \mathbb{N}} E_n = \emptyset$, if $f \succ f'$, then, there exists $n_0 \in \mathbb{N}$ such that $x E_{n_0} f \succ f'$.*

3.2. **Coherence and Consistency.**

For each model $m \in \mathcal{M}$, recall that we defined $E^m := \mathfrak{q}^{-1}(m) \in \mathcal{A}$ as the set of states of the world for which the best-fit map would imply that m is the best approximation of the DGP.

The following axiom captures the idea that the preferences of the DM are coherent with the statistical framework embodied by the set of models and the best-fit map.

Axiom 3 (Coherence):

(i) For all models $m \in \mathcal{M}$, E^m is nonnull and for all $f, g, h, h' \in \mathcal{F}$,

$$fE^m h \succsim gE^m h \iff fE^m h' \succsim gE^m h'.$$

(ii) For all $m \in \mathcal{M}$ and $f, g, h \in \mathcal{F}$,

$$f = g \quad a.e.\ [m] \implies fE^m h \sim gE^m h\ .$$

(iii) For all $x \in X$ and $f \in \mathcal{F}$, the set $\{m \in \mathcal{M} : fE^m x \succsim x\}$ is measurable.

Coherence requires that the DM's preferences are adapted to the statistical information implied by the hypothesized models and the best-fit map. Point *(i)* requires that for each model m, the preferences of the DM deem possible the event that m is indeed the best approximation of the true DGP.[8] Moreover, the second part of the first point requires that each event E^m satisfies Savage's *sure-thing principle*. This implies that the DM is able to identify the event that each model is the best approximation of the DGP and make conditional assessments of the acts based on this event. In particular, this guarantees that we can define nontrivial preferences $\succsim^m$ conditional on a model $m \in \mathcal{M}$ being the best approximation to the true DGP in an unambiguous way by requiring that for all $f, g \in \mathcal{F}$,

$$f \succsim^m g \iff \exists h \in \mathcal{F},\ fE^m h \succsim gE^m h\ .$$

The second requirement ensures that the preferences of the DM incorporate the information provided by the best-fit map and are coherent with the selected best-fit model. If two acts are equal with probability one according to a model $m \in \mathcal{M}$, the fact that the DM pays special attention to the model when it is the best-fit one suggests that she will be indifferent between them conditional on the event that m is, indeed, the best approximation.

Finally, the last point is a measurability requirement of preferences with respect to the sufficient sigma-algebra. To summarize, coherence implies that each model $m \in \mathcal{M}$ induces a well-defined and nontrivial conditional preference $\succsim^m$ that ranks as indifferent acts that are equal with probability one according to m and changes in a measurable fashion with respect to the models.

[8] We can think of this as a parsimony requirement: if the DM thought that a model could never be the best-fit one, then she might just as well drop it altogether.

The next axiom is key in tying together the DM's subjective preferences with the set of models and the conditional preferences they induce.

Axiom 4 (Consistency): *For all $f, f', g \in \mathcal{F}$,*

$$(\forall m \in \mathcal{M}, \quad fE^m g \succsim f'E^m g) \implies f \succsim f' \ .$$

This assumption is analogous to the consistency axiom introduced in Gilboa et al. (2010) and Cerreia-Vioglio et al. (2013). We can think of the set of models $\mathcal{M}$ as identifying an objective preference over acts. If an act f dominates act f' conditional on each model $m \in \mathcal{M}$, then f is objectively preferred by the DM to f'. Consistency requires that the subjective preferences of the DM are informed by the objective preferences.

3.3. **Misspecification Aversion.** We next state the axiom characterizing the DM's aversion to misspecification. It is a model-conditional version of the uncertainty aversion axiom introduced by Schmeidler (1989). That is, the preferences after conditioning on the event that model $m \in \mathcal{M}$ is the best-fit one do not need to necessarily satisfy full-fledged independence, but still display a preference for hedging against the uncertainty due to model misspecification.

Axiom 5 (Misspecification Aversion): *For all models $m \in \mathcal{M}$, $f, g \in \mathcal{F}$, and $\alpha \in (0,1)$,*

$$f \sim gE^m f \implies \alpha f + (1-\alpha) gE^m f \succsim f \ .$$

We interpret Axiom 5 as capturing the idea that the DM is aware that the set of models is possibly misspecified and is concerned about it. Recall that we interpret ambiguity as the lack of information needed to pin down a unique probability distribution over states of the world. Now, suppose the DM was able to observe sufficient information to determine that a model m is the best-fit among all those in $\mathcal{M}$. If she were entirely sure that the true DGP is included in $\mathcal{M}$, she should conclude as a matter of fact that m *is* the correct description of the uncertainty about the states. In this case, there is no reason why the DM's preferences should exhibit any uncertainty aversion; instead, they should behave according to the subjective expected utility tenets. It follows that any residual uncertainty aversion can only be ascribed to the DM's concern that her set of probabilistic models is misspecified. This is precisely the content of Axiom 5. The fact that the DM, even after being told that m is the best-fit model, still exhibits violations of independence and a preference for hedging against the residual uncertainty reveals the DM's concern for model misspecification.

Axiom 4 and 5 epitomize the perspective that we have taken in identifying the DM's misspecification aversion. Specifically, in our approach, the DM perceives her concerns for misspecification model m by model m. This is precisely the content of the misspecification aversion axiom, which requires the preferences of the DM, conditional on each given model m having the best-fit, to be averse to the uncertainty experienced by the DM because of the possibility that the given model m is not an accurate description of the environment. Having incorporated misspecification concerns about each hypothesized model, the DM lets the misspecification averse model-conditional rankings $\succsim^m$ inform her subjective preferences $\succsim$ via the axiom of consistency. It is only at this point, having already taken care of misspecification concerns "pointwise", that the DM incorporates attitudes toward the ambiguity about the identity of the best-fit model.

To summarize, we define the preferences under analysis as a binary relation satisfying all the axioms discussed so far.

Definition 1 (Misspecification Averse Preferences): A preference relation $\succsim$ on $\mathcal{F}$ is said to be *Misspecification Averse* if it satisfies Axioms 1, 2, 3, 4, and 5.

Before proceeding to the representation results in the next section, it is worth noting that at this point, no stance has been taken on the attitudes of the DM toward ambiguity. Indeed, our approach enables us to incorporate the DM's concerns about misspecification, while still accommodating fully flexible attitudes toward ambiguity. The following definition categorizes the DM's attitudes toward ambiguity in terms of her hedging behavior regarding the uncertainty about which model within $\mathcal{M}$ has the best fit.

Definition 2 (Ambiguity Attitudes): We say that the DM's preference $\succsim$ is:

- *ambiguity averse* if $f \sim g$ implies that $\alpha f + (1-\alpha)g \succsim f$ for all $f, g \in \mathcal{F}(\mathcal{A})$ and $\alpha \in (0,1)$;
- *ambiguity neutral* if $f \sim g$ implies that $\alpha f + (1-\alpha)g \sim f$ for all $f, g \in \mathcal{F}(\mathcal{A})$ and $\alpha \in (0,1)$;
- *ambiguity loving* if $f \sim g$ implies that $f \succsim \alpha f + (1-\alpha)g$ for all $f, g \in \mathcal{F}(\mathcal{A})$ and $\alpha \in (0,1)$.

Since this uncertainty about the identity of the best-fit model is captured by the events in the sigma-algebra $\mathcal{A}$, the ambiguity attitudes of the DM are completely encoded in her preferences over acts that are measurable with respect to $\mathcal{A}$. Thus, the DM exhibits ambiguity aversion if her preferences are uncertainty averse over $\mathcal{F}(\mathcal{A})$; that is, if she displays a preference for hedging against the uncertainty regarding the identity of the best-fit model. Conversely, the DM's preferences are ambiguity loving if the opposite is true; that is, she dislikes hedging against this uncertainty.

4. Representation of Misspecification Averse Preferences

Before stating the main representation result regarding the representation of misspecification averse preferences, we introduce a few definitions. We say that $I : B(\mathcal{G})\times\mathcal{M} \to \mathbb{R}$ is a *family of misspecification averse certainty equivalents* if it is measurable in the second argument and for all $m \in \mathcal{M}$, $I(\cdot, m)$ is monotone, normalized, continuous, quasiconcave, and satisfies monotone convergence. On the other hand, we say that $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ is an *ambiguity certainty equivalent aggregator* if it is monotone, normalized, continuous, and satisfies monotone convergence.[9]

Theorem 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$. The following are equivalent:*

(i) $\succsim$ is a misspecification averse preference relation,

(ii) there exist an affine and surjective utility function $u : X \to \mathbb{R}$, a family of misspecification averse certainty equivalents $I : B(\mathcal{G}) \times \mathcal{M} \to \mathbb{R}$ and an ambiguity certainty equivalent aggregator $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$, such that for all $m \in \mathcal{M}$,

$$f \succsim^m g \iff I(u(f), m) \geq I(u(g), m)$$

and

$$f \succsim g \iff \hat{I}\left(I(u(f), \cdot)\right) \geq \hat{I}\left(I(u(g), \cdot)\right) \tag{6}$$

for all $f, g \in \mathcal{F}$. Moreover, for all $m \in \mathcal{M}$, $\varphi = \varphi'$ a.e. $[m] \implies I(\varphi, m) = I(\varphi', m)$ for all $\varphi, \varphi' \in B(\mathcal{G})$,

Moreover, u is unique up to positive affine transformations, and I and $\hat{I}$ are unique given u.

We can interpret the representation of the misspecification averse preferences characterized in Theorem 1 as a two-step procedure to evaluate an act. First, if the DM were told that m is the best-fit model within $\mathcal{M}$; that is, conditional on the event E^m, she would evaluate an act f according to the misspecification certainty equivalent $I(u(f), m)$, where u is a utility function over consequences. In other words, the quantity $I(u(f), m)$ represents the certain utility level that the DM would be willing to accept to eliminate uncertainty, when she is sure that m is the best-fit model. Since conditional on knowing that model m has the best-fit, the residual uncertainty can only

[9] We denote by $B(\mathcal{G})$ the set of measurable and bounded functions $\varphi : \Omega \to \mathbb{R}$. Similarly, we denote by $B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ the set of measurable and bounded functions $\varphi : \mathcal{M} \to \mathbb{R}$. In the remainder of the paper, continuity is always understood with respect to convergence in the sup norm. Moreover, a certainty equivalent satisfies monotone convergence if it is also continuous with respect to monotone sequences of functions that converge only pointwise.

be ascribed to the possibility of misspecification, the fact that the certainty equivalent $I(\cdot, m)$ is quasiconcave, and not just the expected value of the utility act $u(f)$ according to model m, reveals that the DM is concerned about misspecification. Indeed, each function $I(\cdot, m)$ can be seen as a non-linear expectation with respect to the best-fit model $m \in \mathcal{M}$. While it fails to be linear, it satisfies many other characteristic properties of expectations, like monotonicity, normalization, and the same evaluation of functions that are almost surely equal according to m. In particular, for each function φ, the random variable $\omega \mapsto I(\varphi, \mathfrak{q}(\omega))$ can be understood as a non-linear common conditional expectation of φ given the sigma-algebra $\mathcal{A}$ for the family $I(\cdot, m)$.[10]

Given the representation of the model-conditional preferences, we are able to associate with each act $f \in \mathcal{F}$ a function $m \mapsto I(f, m)$ that maps each hypothesized model m to the misspecification certainty equivalent of act f conditional on m being the best-fit model. The axiom of consistency then implies that if $I(f, m) \geq I(g, m)$ for all $m \in \mathcal{M}$, the DM is confident that f is better than g and thus $f \succsim g$. As remarked in Cerreia-Vioglio et al. (2025), this exemplifies the special status of the hypothesized models over distributions that are not in $\mathcal{M}$. If the misspecification certainty equivalents according to each model m rank unanimously an act over another, this is sufficient for the DM to decide to pick the first one. However, in general, the set of models will not provide a unanimous ranking of every pair of acts and, therefore, in the second step, the DM aggregates the misspecification certainty equivalents of an act f according to the aggregator $\hat{I}$, which is an ambiguity certainty equivalent capturing attitudes toward the ambiguity about the identity of the best-fit model. The fact that no stance has been taken on the ambiguity attitudes of the DM is reflected by the fact that the representation imposes no restriction on the curvature of the aggregator $\hat{I}$. The following result, then, readily clarifies how the aggregator $\hat{I}$ encodes the ambiguity attitudes of the DM.

Corollary 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$ and $\succsim$ is a misspecification averse preference. Then:*

- *$\succsim$ is ambiguity averse if and only if $\hat{I}$ is quasiconcave;*
- *$\succsim$ is ambiguity neutral if and only if $\hat{I}$ is linear;*
- *$\succsim$ is ambiguity loving if and only if $\hat{I}$ is quasiconvex.*

As we now make precise via a comparative statics exercise, the representation of the misspecification averse preferences obtained in Theorem 1 completely disentangles misspecification concerns from attitudes toward ambiguity. Misspecification aversion is

[10] That is, the analogue of the law of iterated expectations holds: $I\left(I(\varphi, \mathfrak{q}(\cdot)), m\right) = I(\varphi, m)$ for all $m \in \mathcal{M}$.

entirely captured by the family of misspecification averse certainty equivalents $I(\cdot, m)$, independently of the shape of the ambiguity aggregator. Attitudes toward ambiguity are, instead, entirely captured by the aggregator $\hat{I}$, independently of the family of misspecification averse certainty equivalents I. To this end, we adapt the approach to defining comparative uncertainty aversion introduced by Ghirardato and Marinacci (2002) to the present context. Given two preferences $\succsim_1$ and $\succsim_2$, we say that $\succsim_1$ is *more misspecification averse* than $\succsim_2$ if for all $m \in \mathcal{M}$, $f \in \mathcal{F}$ and $x \in X$,

$$fE^m x \succsim_1 x \implies fE^m x \succsim_2 x \; . \tag{7}$$

The idea behind this notion is that, also in this case, constant acts are unaffected by the possibility that the set of hypothesized models is misspecified, since they are non-stochastic and, therefore, their evaluation does not depend on the probabilistic assessment of state uncertainty. Therefore, if it is true that after conditioning on any given model $m \in \mathcal{M}$, a DM is not concerned enough about misspecification to choose a constant act over an uncertain one, a fortiori, that should also be true for a less misspecification averse DM.

On the other hand, we say that $\succsim_1$ is *more ambiguity averse* than $\succsim_2$ if for all $f \in \mathcal{F}(\mathcal{A})$ and $x \in X$,

$$f \succsim_1 x \implies f \succsim_2 x. \tag{8}$$

The intuition for this definition is that acts that are measurable with respect to $\mathcal{A}$ are exactly those acts that are only affected by the uncertainty regarding what is the best approximation among the set of hypothesized models, but not by misspecification concerns regarding the accuracy of the models in approximating the true DGP (notice that they need to be constant on each event E^m). Therefore, the definition above states that if $\succsim_1$ is more averse to ambiguity than $\succsim_2$ then, whenever ambiguity considerations are not enough for DM1 to prefer the certain outcome x to the act f that is affected by ambiguity about the best-fit model, then definitely they should not be enough for the less averse DM2. We, then, have the following result characterizing the comparative statics of misspecification aversion and ambiguity attitudes in terms of the elements of the misspecification averse representation.

PROPOSITION 1: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$ and $\succsim_1$ and $\succsim_2$ are two misspecification averse preference relations. Then:*

(i) $\succsim_1$ is more misspecification averse than $\succsim_2$ if and only if u_2 is a positive affine transformation of u_1 and, after normalizing $u_1 = u_2$, $I_1(\cdot, m) \leq I_2(\cdot, m)$ for all models $m \in \mathcal{M}$.

(ii) $\succsim_1$ *is more ambiguity averse than* $\succsim_2$ *if and only if* u_2 *is a positive affine transformation of* u_1 *and, after normalizing* $u_1 = u_2$, $\hat{I}_1 \leq \hat{I}_2$.

The first part of the result states that $\succsim_1$ is more averse to model misspecification than $\succsim_2$ if, conditional on each given model m having the best fit, DM1 is willing to accept lower certainty equivalents than DM2 as compensation for acts that are affected by the possibility of misspecification. Similarly, $\hat{I}_1$ and $\hat{I}_2$ can be interpreted as certainty equivalents of uncertain bets on the likelihood of which model is the best-fit one. The result can then be taken as stating that $\succsim_1$ is more averse to ambiguity than $\succsim_2$ if DM1 is willing to accept lower certainty equivalents than DM2 as compensation for uncertain bets over the likelihood of the best approximation in $\mathcal{M}$. In this sense, Proposition 1 allows us to interpret the aggregator $\hat{I}$ as incorporating the DM's attitudes toward uncertainty about the identity of the best-fit model. Proposition 1, thus, clarifies how representation (6) achieves a separation of attitudes regarding the ambiguity about the identity of the best-fit model and misspecification concerns. Attitudes toward ambiguity are captured by the aggregator $\hat{I}$, while the misspecification certainty equivalent $I(\cdot, m)$ is an index of the degree of aversion to the possibility that the set of hypothesized models is misspecified.

In the remainder of the paper, we will focus on a special class of misspecification averse certainty equivalents that allows us to interpret the DM's concerns about misspecification in terms of a robust approach to model misspecification. This special case is obtained by adding more structure to the preferences conditional on each model m having the best fit. Before introducing the additional axiom, define for each given probability distribution $p \in \Delta$ and for each simple act f the "average" of f according to the probability model p:

$$\mathbb{E}_p[f] \coloneqq \sum_{x \in X} xp\left(f^{-1}(x)\right).$$

Notice that since f has a finite image and X is convex, $\mathbb{E}_p[f] \in X$ and is the constant act that would be indifferent to f in the eyes of an Anscombe-Aumann EU maximizer who holds belief p over the state space Ω.

Axiom 6 (Variational Misspecification):

(i) $\mathcal{M}$-Weak Certainty Independence. *For all* $m \in \mathcal{M}$, $f, f' \in \mathcal{F}$, $x, y \in X$, *and* $\alpha \in (0,1)$,

$$\alpha f + (1-\alpha)x \succsim \alpha f' E^m f + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succsim \alpha f' E^m f + (1-\alpha)y\ .$$

(ii) *For all $m \in \mathcal{M}$, if $p(E^m) = 1$ but $p \neq m$, then there exist $f \in \mathcal{F}_0$ and $x \in X$ such that $fE^m x \succsim x$ but $x \succ \mathbb{E}_p[fE^m x]$.*

The first part of the axiom is a model-conditional version of the axiom characterizing the variational preferences of Maccheroni, Marinacci, and Rustichini (2006). That is, the preferences after conditioning on the event that the model $m \in \mathcal{M}$ is the best-fit one satisfy a weaker form of independence, weak certainty independence, but they still do not need to satisfy full-fledged independence because of misspecification concerns.

The second part of the axiom clarifies the interpretation of m as having a special status in the eyes of the DM compared to other models not in $\mathcal{M}$. In principle, any p assigning probability one to E^m is not contradicted by observing evidence E^m. However, since p is not selected by $\mathfrak{q}$, the DM's preferences display instances of "incoherence" with it. That is, the DM prefers a (possibly) uncertain alternative f to the constant act x conditional on E^m even if x is strictly preferred to the p-average of f.[11]

Armed with this additional axiom, we have the following result.

Theorem 2: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$. The following are equivalent:*

(i) *$\succsim$ is a misspecification averse preference relation satisfying Axiom 6,*

(ii) *there exist an affine and surjective utility function $u : X \to \mathbb{R}$, a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$,[12], and an ambiguity certainty equivalent aggregator $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$, such that for all $f, g \in \mathcal{F}$,*

$$f \succsim g \iff \hat{I}\left(I(u(f), \cdot)\right) \geq \hat{I}\left(I(u(g), \cdot)\right)$$

and for all $m \in \mathcal{M}$, $\succsim^m$ is represented by

$$I(u(f), m) = \min_{p \ll m} \left\{ \int_{\Omega} u(f) dp + c(p, m) \right\} \quad \text{for all } f \in \mathcal{F}. \tag{9}$$

Moreover, u is unique up to positive affine transformations, and c and $\hat{I}$ are unique given u.

We can interpret the functional form (9) in terms of a robust approach to the possibility of misspecification. Suppose that the DM has observed sufficient information to determine that m is the best-fit model she has available. Because of the possibility of misspecification, in evaluating an act f conditional on this information, the DM forms a robust evaluation of the act f by taking into consideration also probability

[11] This last requirement is not strictly needed to obtain the representation and only clarifies the interpretation of each m being the unique reference model for the DM after observing the event E^m. In particular, this implies that the misspecification index $c(\cdot, m)$ is uniquely minimized at m.

[12] See Appendix A for a rigorous definition of the notion of statistical distance.

distributions outisde of $\mathcal{M}$. The statistical distance $c(\cdot, m)$ captures how distant in a statistical sense a distribution p is from the best-fit model m. Specifically, since the DM is concerned that m might not be an accurate approximation of the true DGP, she also takes into account other models p that are not too far apart from m. The index $c(\cdot, m)$ captures exactly the DM's confidence in the best-fit model m. When $c(\cdot, m)$ is (uniformly) lower, the DM potentially takes into account a larger set of models around m in evaluating an act; this reflects a lower trust in m or, conversely, a higher aversion to misspecification. An important and tractable case is when the misspecification index takes the form $c(\cdot, m) = \lambda R(\cdot||m)$ for all hypothesized models $m \in \mathcal{M}$, where $\lambda > 0$ is a parameter of misspecification aversion and R is the relative entropy. In this case, the misspecification concern is proportional to the relative entropy with respect to the best-fit model, and it is uniform across models in $\mathcal{M}$ (see Lanzani (2025)), where a higher aversion toward misspecification is captured by a lower parameter λ. We now make this intuition about the statistical distance $c(\cdot, m)$ being an index of the degree of misspecification aversion precise via a comparative statics exercise.

Corollary 2: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$ and $\succsim_1$ and $\succsim_2$ are two misspecification averse preference relations satisfying Axiom 6. Then, $\succsim_1$ is more misspecification averse than $\succsim_2$ if and only if u_2 is a positive affine transformation of u_1 and, after normalizing $u_1 = u_2$, $c_1(\cdot, m) \leq c_2(\cdot, m)$ for all models $m \in \mathcal{M}$.*

Before proceeding to study special cases of the aggregator $\hat{I}$, we provide a partial converse showing that if preferences are represented by the criterion given in Theorem 2, then there exists a best-fit map $\hat{\mathfrak{q}}$ with respect to which the preferences satisfy the coherence, consistency, and misspecification aversion axioms. To state the result, we introduce the following definition. The aggregator $\hat{I}$ is *strongly monotone* if for all $\xi_1, \xi_2 \in B(\mathcal{M})$ such that $\xi_1 > \xi_2$[13], then $\hat{I}(\xi_1) > \hat{I}(\xi_2)$.

Proposition 2: *Assume that $(\Omega, \mathcal{G})$ is a standard Borel space and that $\mathcal{M}$ is a measurable subset of Δ (not assumed to admit a best-fit map). Suppose that a preference relation $\hat{\succsim}$ is represented by the criterion characterized in Theorem 2 and that for all $\xi \in B_0(\mathcal{M})$ such that $0 \leq \xi \leq 1$ there exists $\varphi \in B_0(\mathcal{G})$ such that $0 \leq \varphi \leq 1$ and $\xi(m) = I(\varphi, m)$ for all $m \in \mathcal{M}$. Then, there exists a best-fit map $\hat{\mathfrak{q}} : \Omega \to \mathcal{M}$ such that $m\left(\hat{\mathfrak{q}}^{-1}(m)\right) = 1$ for all $m \in \mathcal{M}$ and if we define $\hat{E}^m = \hat{\mathfrak{q}}^{-1}(m)$, then*

$$f = g \quad a.e.\ [m] \implies f\hat{E}^m h \hat{\sim} g\hat{E}^m h$$

[13] Recall that $\xi_1 > \xi_2$ if $\xi_1 \geq \xi_2$ and $\xi_1(m) > \xi_2(m)$ for some $m \in \mathcal{M}$.

for all $f, g, h \in \mathcal{F}$. If, furthermore, $\hat{I}$ is strongly monotone, then $\hat{\succsim}$ satisfies Axioms 3, 4, and 5 given the best-fit map $\hat{\mathfrak{q}}$.

The abstract form of $\hat{I}$ in the general representation of Theorem 1 is due to the fact that no behavioral assumptions regarding the independence properties of the preference relation $\succsim$ have been made other than independence on constant acts. The following results show how we can explicitly obtain specific shapes of the aggregator $\hat{I}$ by imposing additional behavioral axioms on a suitable subset of acts. Indeed, we have argued how the sigma-algebra $\mathcal{A}$ generated by the best-fit map $\mathfrak{q}$ can be viewed as capturing the sufficient information needed to determine which model in $\mathcal{M}$ is the best approximation available to the DM. Therefore, we can interpret the set $\mathcal{F}(\mathcal{A})$ of acts that are measurable with respect to $\mathcal{A}$ as bets over the identity of the best-fit model. Since the aggregator $\hat{I}$ only captures the DM's aversion toward ambiguity, it is reasonable to expect that its form will only depend on the DM's preferences over this subset of acts. The following results confirm this intuition, as we are, indeed, able to characterize special cases of the aggregator by imposing suitable axioms on $\mathcal{F}(\mathcal{A})$.

First of all, we study the case where the aggregator also takes a variational form. To that end, we introduce the following axiom, which requires the preferences over $\mathcal{F}(\mathcal{A})$ to satisfy Maccheroni et al. (2006)'s weak certainty independence axiom as well as Schmeidler (1989)'s uncertainty aversion.

Axiom 7 ($\mathcal{A}$-Variational):

- $\mathcal{A}$-Weak Certainty Independence. *For all $f, f' \in \mathcal{F}(\mathcal{A})$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha f' + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succsim \alpha f' + (1-\alpha)y \ .$$

- $\mathcal{A}$-Uncertainty Aversion. *For all $f, f' \in \mathcal{F}(\mathcal{A})$ and $\alpha \in (0,1)$,*

$$f \sim f' \implies \alpha f' + (1-\alpha)f \succsim f \ .$$

We have the following result.

Theorem 3: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$. The following are equivalent:*

(i) $\succsim$ is a misspecification averse preference relation satisfying Axioms 6 and 7,

(ii) there exist an affine and surjective utility function $u : X \to \mathbb{R}$, a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$, a grounded, lower semicontinuous and

convex function $\kappa : \Delta(\mathcal{M}) \to [0, \infty]$ such that $\succsim$ is represented by:

$$\min_{\nu \in \Delta(\mathcal{M})} \left\{ \int_{\mathcal{M}} \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p,m)\} \, d\nu(m) + \kappa(\nu) \right\} \tag{10}$$

and $\min_p \{\mathbb{E}_p[u(f)] + c(p,m)\}$ *represents $\succsim_m$ for all $m \in \mathcal{M}$.*
Moreover, u is unique up to positive affine transformations, and c and κ are unique given u.

In representation (10), the DM's aversion toward ambiguity manifests itself in the impossibility for the DM of formulating a unique prior ν over models in $\mathcal{M}$. The convex cost $\kappa : \Delta(\mathcal{M}) \to [0, +\infty]$ is an index of aversion toward ambiguity. Specifically, a lower κ corresponds to a higher degree of aversion toward ambiguity. Note that the functional form of the aggregator characterized in Theorem 3 also covers the seminal maxmin expected utility model of Gilboa and Schmeidler (1989). Indeed, if we let $k(\nu) = 0$ for all ν in a compact and convex set of priors $C \subseteq \Delta(\mathcal{M})$ over the set of hypothesized models and equal to ∞ otherwise,[14] criterion (10) would take the shape:

$$\min_{\nu \in C} \int_{\mathcal{M}} \min_{p \in \Delta(\Omega)} \{\mathbb{E}_p[u(f)] + c(p,m)\} \, d\nu(m)$$

This criterion extends the celebrated multiple priors model of Gilboa and Schmeidler (1989) to incorporate robustness to model misspecification. The DM's ambiguity aversion manifests itself in the DM's inability to pin down a unique prior over the set of models $\mathcal{M}$. Instead, the DM is only able to form a set of possible priors C and then takes the worst-case expected value of the misspecification-robust certainty equivalents out of ambiguity aversion. Another particularly tractable case of criterion (10) is when both c and κ are proportional to the relative entropy. As shown by Strzalecki (2011)'s axiomatization of the multiplier preferences introduced by Hansen and Sargent (2001), this turns out to be the case when the preferences satisfy Savage's sure-thing principle

Axiom 8 (Savage's P2. Sure-Thing Principle): *For all $E \in \mathcal{G}$ and $f, g, h, h' \in \mathcal{F}$, $fEh \succsim gEh$ if and only if $fEh' \succsim gEh'$.*

We, then, have the following corollary.

Corollary 3: *Suppose that $|\mathcal{M}| > 3$. Then, $\succsim$ is a misspecification averse preference satisfying Axioms 6, 7, and 8 if and only if there exist an affine and surjective utility function $u : X \to \mathbb{R}$, misspecification parameters $\lambda_m \in [0, \infty]$ for all $m \in \mathcal{M}$,*

[14]This shake pf the cost κ can be obtained by imposing that the preferencessatisfy a stronger form of indipendence, namely the certainty independence axiom introduced by Gilboa and Schmeidler (1989). We do not state the formal result in the interest of brevity.

an ambiguity parameter $\zeta \in [0,\infty]$ *, and a reference prior* $\mu \in \Delta(\mathcal{M})$ *such that* $\succsim$ *is represented by:*

$$\min_{\nu \in \Delta(\mathcal{M})} \left\{ \int_{\mathcal{M}} \min_{p \in \Delta(\Omega)} \left\{ \mathbb{E}_p[u(f)] + \lambda_m R(p||m) \right\} d\nu(m) + \zeta R(\nu||\mu) \right\} \tag{11}$$

and $\min_p \{\mathbb{E}_p[u(f)] + \lambda_m R(p||m)\}$ *represents* $\succsim_m$ *for all* $m \in \mathcal{M}$.

Moreover, u *is unique up to positive affine transformations,* μ *is unique, and* $(\lambda_m)_{m\in\mathcal{M}}$ *and* ζ *are unique given* u.

In the representation (11), the relative entropy $R(p||m)$ measures the distance of p from model m and λ_m captures the degree of concern that model m is not a good approximation. When λ_m is lower, the DM reveals a higher concern for misspecification. Similarly, the relative entropy $R(\nu||\mu)$ measures the distance of ν from the reference prior μ, and ζ captures the DM's aversion to ambiguity.

The next result provides a foundation for a Bayesian version of the misspecification averse preferences also discussed in Cerreia-Vioglio et al. (2025), where the DM forms a subjective belief capturing her uncertainty regarding the identity of the best-fit model in $\mathcal{M}$. We introduce the following axiom.

Axiom 9 ($\mathcal{A}$-SEU):

(i) $\mathcal{A}$-Tradeoff Consistency. *For all nonnull events* $E, A \in \mathcal{A}$*, for all* $x, y, z, w \in X$*, and* $f, g \in \mathcal{F}(\mathcal{A})$*, if* $xEf \succsim yEg$*,* $zEf \succsim wEg$*, and* $xAf \succsim yAg$*, then* $zAf \succsim wAg$.

(ii) $\mathcal{A}$-S-Continuity. *For all finite partitions* $\{E_1,\dots,E_n\} \subseteq \mathcal{A}$*, for all* $x, y \in X$*, and* $h \in \mathcal{F}(\mathcal{A})$*, the sets* $\{(\alpha_1,\dots,\alpha_n) \in [0,1]^n : \sum_{i=1}^n \chi_{E_i}[\alpha_i x + (1-\alpha_i)y] \succsim f\}$ *and* $\{(\alpha_1,\dots,\alpha_n) \in [0,1]^n : f \succsim \sum_{i=1}^n \chi_{E_i}[\alpha_i x + (1-\alpha_i)y]\}$ *are closed.*

This axiom adapts to our context the Tradeoff Consistency and S-Continuity axioms introduced by Wakker (2013) to axiomatize subjective expected utility with arbitrary state spaces. We then have the following characterization of the Bayesian aggregator.

Theorem 4: *Suppose* $(\Omega, \mathcal{G}, \mathcal{M})$ *admits a best-fit map* $\mathfrak{q}$ *and* $|\mathcal{M}| > 2$*. The following are equivalent:*

(i) $\succsim$ *is a misspecification averse preference relation satisfying Axioms 6 and 9,*

(ii) there exist an affine and surjective utility function $u : X \to \mathbb{R}$*, a convex statistical distance* $c : \Delta \times \mathcal{M} \to [0,\infty]$*, a strictly increasing and continuous function* $\phi : \mathbb{R} \to \mathbb{R}$ *and a prior* $\mu \in \Delta(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ *such that* $\succsim$ *is represented by:*

$$\int_{\mathcal{M}} \phi \left(\min_{p \in \Delta} \int_{\Omega} u(f)dp + c(p,m) \right) d\mu(m) \tag{12}$$

and $\min_p \{\mathbb{E}_p[u(f)] + c(p,m)\}$ *represents* $\succsim_m$ *for all* $m \in \mathcal{M}$.

Moreover, u is unique up to positive affine transformations, c is unique given u, ϕ is unique up to positive affine transformations given u, and μ is unique.

As before, the DM's concern for misspecification is captured by the fact that even conditioning on the information revealing that m is the best-fit model, she still takes into account other distributions that are close enough to m. In this case, the perception of uncertainty regarding the identity of the best-fit model in the absence of the information in the sufficient sigma-algebra $\mathcal{A}$ and the attitudes toward this uncertainty are captured, respectively, by the Bayesian prior μ over the set of hypothesized models and the index of ambiguity attitudes ϕ. The subjective belief μ quantifies what models the DM considers more likely to be good approximations of the true DGP. The curvature of ϕ captures the attitudes exhibited by the DM toward the ambiguity about the best-fit model. Specifically, ϕ is concave (convex) if and only if the DM is ambiguity averse (loving). The Bayesian criterion (12) can be seen as an extension of the smooth ambiguity model of Klibanoff et al. (2005) to incorporate misspecification concerns. We can recover the smooth ambiguity model by letting the misspecification aversion index c go to infinity (except on the diagonal, where it is always 0). This is equivalent to taking a limit case where the DM is neutral to misspecification. As already remarked in the introduction, this criterion becomes the average robust control criterion axiomatized by Lanzani (2025) when the DM is neutral toward the ambiguity regarding the identity of the best-fit model. This would, indeed, imply that the index ϕ is affine. The relative entropy formulation of the misspecification aversion index $c(\cdot, m) = \lambda R(\cdot||m)$ could be obtained by imposing Savage's sure-thing principle as discussed above.

Finally, the following theorem shows that the main criterion axiomatized in Cerreia-Vioglio et al. (2025) can arise as a special case of the representation in Theorem 2, when we assume that preferences exhibit a cautious attitude with respect to the uncertainty about the best-fit model.

Axiom 10 ($\mathcal{M}$-Caution): *For all $f \in \mathcal{F}$ and $x \in X$,*

$$\exists m \in \mathcal{M},\ x \succ f E^m x \implies x \succsim f.$$

This axiom is the conceptual analogue in our framework to the caution axiom in Gilboa et al. (2010). Indeed, the set of hypothesized models induces a (typically incomplete) dominance relation $\succsim_{\mathcal{M}}$, where for all $f, g \in \mathcal{F}$,

$$f \succsim_{\mathcal{M}} g \iff \forall m \in \mathcal{M},\ f \succsim^m g.$$

If $f \succsim_{\mathcal{M}} g$, this means that f is better than g according to each model $m \in \mathcal{M}$ after taking into account misspecification concerns. Because the DM trusts the set of models, when $f \succsim_{\mathcal{M}} g$, the DM is confident that f is better than g. Then, Axiom 10 can be rewritten as the requirement that if $f \not\succsim_{\mathcal{M}} x$, then $x \succsim f$. The interpretation is that if the DM is not sure that the uncertain act f is better than the constant (and therefore unaffected by uncertainty considerations) act x, then she should behave cautiously and prefer the certain act over the uncertain one. We also impose the following technical axiom.

Axiom 11 ($\mathcal{M}$-Lower Semicontinuity): *For all $x \in X$ and $f \in \mathcal{F}$, the set $\{m \in \mathcal{M} : x \succsim fE^m x\}$ is closed.*

This axiom is a strengthening of requirement (*iii*) in the axiom of Coherence (it requires closedness and not measurability only), and it is only needed to ensure that minima are achieved in the criterion. The following result shows that $\mathcal{M}$-Caution delivers the criterion of Cerreia-Vioglio et al. (2025).

Theorem 5: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ admits a best-fit map $\mathfrak{q}$ and $\mathcal{M}$ is compact.*[15] *The following are equivalent:*

(i) $\succsim$ is a misspecification averse preference relation satisfying Axioms 6, 10, and 11,

(ii) there exist an affine and surjective utility function $u : X \to \mathbb{R}$ and a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$ such that $\succsim$ is represented by:

$$V(f) = \min_{p \in \Delta} \int_{\Omega} u(f)dp + \min_{m \in \mathcal{M}} c(p, m) \tag{13}$$

and $\min_p \{\mathbb{E}_p[u(f)] + c(p, m)\}$ represents $\succsim_m$ for all $m \in \mathcal{M}$.

Moreover, u is unique up to positive affine transformations, and c is unique given u.

Notice that $\min_{m \in \mathcal{M}} c(m', m) = 0$ for all models $m' \in \mathcal{M}$. Therefore, $C_{\mathcal{M}}(\cdot) := \min_{m \in \mathcal{M}} c(\cdot, m)$ can be seen as a statistical distance between probability distributions and the set of hypothesized models $\mathcal{M}$ capturing the degree of misspecification concern of the DM, when she takes a worst-case scenario approach to the ambiguity regarding what is the best-fit model.

5. Misspecification Aversion and Robust Monopoly Pricing

In order to illustrate the different implications that misspecification aversion can have compared to ambiguity aversion, we revisit the monopoly pricing example from Ball and Kattwinkel (2024). There is a seller (the DM) who wants to sell a single good to a buyer. The seller does not know the buyer's valuation $\omega \in \Omega = [0, \bar{\theta}]$ for the good. The seller is only informed that the buyer's median valuation is $\gamma \in (0, \bar{\theta})$, so

[15] As for Axiom 11, closedness of $\mathcal{M}$ is only needed to ensure that minima are achieved.

that the set of plausible probability distributions $\mathcal{M}$ entertained by the DM contains all probability measures having median equal to γ. If the seller picks a price $P \in \mathbb{R}_+$, only buyers having a valuation above the price buy the good. Thus, each price induces an act $f_P(\omega) = P\mathbf{1}\{\omega \geq P\}$. The payoff guarantee of a price $P \geq 0$ is given by the worst-case expected profits:

$$\inf_{m \in \mathcal{M}} \mathbb{E}_m[f_P] = \inf_{m \in \mathcal{M}} P\ m(\{\omega \geq P\})\ .$$

The payoff guarantee of a price P is said to be robust if

$$\liminf_n \mathbb{E}_{q_n}[f_P] \geq \inf_{m \in \mathcal{M}} \mathbb{E}_m[f_P]$$

for all $m \in \mathcal{M}$ and sequences $(q_n)_n$ in $\Delta(\Omega)$ converging to m. Ball and Kattwinkel (2024) show that the maxmin optimal price maximizing the guarantee is given by $P^* = \gamma$ but that, however, the payoff guarantee of P^* is not robust. We argue that a seller who is concerned about the possibility that the set $\mathcal{M}$ is misspecified would instead choose a price inducing a robust payoff guarantee. Indeed, suppose that the seller's preferences are given by the cautious version of the misspecification averse criterion:

$$V^\lambda(f_P) = \inf_{m \in \mathcal{M}} \inf_{q \in \Delta(\Omega)} \mathbb{E}_q[f_P] + \lambda c_\eta(q, m)$$

with

$$c_\eta(q, m) = W_1(q, m) + \eta R(q||m)$$

where W_1 is the Wasserstein distance and $\eta > 0$.[16] As discussed at length in the previous section, we interpret λ as the index of misspecification concern. In particular, the case $\lambda = \infty$ corresponds to the absence of misspecification concern and, therefore, to the maximization of the payoff guarantee. We have the following result.

Proposition 3: *Fix $\gamma \in (0, \bar{\theta})$ and $\eta < \gamma/2$. For all $\lambda \in (0, \infty)$, we have that the solution to $\max_{P \geq 0} V^\lambda(f_P)$ is given by*

$$\hat{P}(\lambda) = \frac{\lambda\gamma + \lambda\eta \log(1 + 2/\lambda)}{1 + \lambda} \in (0, \gamma)$$

Moreover, $\partial \hat{P}(\lambda)/\partial\lambda > 0$ and

$$\lim_{\lambda \to 0} \hat{P}(\lambda) = 0 \quad \textit{and} \quad \lim_{\lambda \to \infty} \hat{P}(\lambda) = \gamma.$$

[16]To obtain the result, we could have chosen $c(q, m) = W_1(q, m)$. The perturbation by the relative entropy is only added to ensure that c_η satisfies all the properties outlined in the representation theorems.

In this case, in choosing the optimal price, the misspecification concerned seller anticipates that the set of plausible models $\mathcal{M}$ might not contain the correct distribution of the buyer's valuations and, therefore, chooses a price lower than the maxmin optimal one. It is easy to see that this fact readily implies that the payoff guarantee of the price $\hat{P}(\lambda)$ is robust for all $\lambda < \infty$. Thus, even a minimal degree of misspecification concern on the part of the seller solves the issue of non-robustness of the payoff guarantee, while still allowing the DM to achieve a guarantee arbitrarily close to the one implied by the maxmin optimal price.

6. Conclusion

This paper provides an axiomatic foundation of general preferences that are misspecification averse. We study a framework where the DM formulates a possibly misspecified set of models that she considers plausible descriptions of the environment. We introduce the notion of a best-fit map that identifies the most suitable approximation of the true DGP based on (in principle) observable states. This allows us to discern between the DM's concern about the set of models being misspecified and negative attitudes toward the uncertainty about what hypothesized models are more likely to be the best description of the environment. The main result is that the DM's preferences are a monotone aggregation of misspecification-robust evaluations based on each model. As we saw in the paper, this representation achieves a separation of attitudes toward ambiguity, captured by the aggregator, and misspecification concerns, captured by the misspecification-robust conditional evaluations. Specific shapes of the aggregator can be obtained by imposing additional suitable behavioral axioms on the DM's preferences. We show that the important decision criteria recently introduced in the literature by Cerreia-Vioglio et al. (2025) and Lanzani (2025) fall within the general class of misspecification averse preferences we studied. In particular, we provide specific axioms to obtain the Bayesian aggregator and the cautious criterion from the general case.

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Appendix

Appendix A. Mathematical Preliminaries

A.1. **Basic Notions.** Given an arbitrary measurable space $(Y, \mathcal{Y})$, we denote by $\Delta(Y, \mathcal{Y})$ the space of countably additive probability measures on $(Y, \mathcal{Y})$. Sometimes, we will omit making explicit reference to the sigma-algebra whenever no ambiguities can arise. Since both these spaces can be identified with subsets of the dual space of $B_0(Y, \mathcal{Y})$, the space of $\mathcal{Y}$-measurable simple functionals mapping Y to the real line, endowed with the supnorm $||\cdot||_\infty$, we endow them with the weak* topology. We endow $\Delta(Y, \mathcal{Y})$ with the Borel sigma-algebra generated by this topology; which is the same as the natural sigma-algebra $\mathcal{D}^{Y,\mathcal{Y}}$ generated by the family of evaluation maps:

$$\forall E \in \mathcal{Y}, \quad E^* : \Delta(Y, \mathcal{Y}) \to \mathbb{R},\ p \mapsto p(E)\ .$$

and any subset $\mathcal{Q}$ of Δ, with the relative sigma-algebra $\mathcal{D}^{Y,\mathcal{Y}}_{\mathcal{M}} := \mathcal{D}^{Y,\mathcal{Y}} \cap \mathcal{M}$. Moreover, denote by $B(Y, \mathcal{Y})$ the set of bounded $\mathcal{Y}$-measurable functionals from Y to $\mathbb{R}$. We know that $B(Y, \mathcal{Y})$ is the supnorm closure of $B_0(Y, \mathcal{Y})$.

Given a nonempty subset $\tilde{B}$ of $B(Y, \mathcal{Y})$, a functional $\Psi : \tilde{B} \to \mathbb{R}$ is said to be a *niveloid* if for all $\varphi, \varphi', \in \tilde{B}$,

$$\Psi(\varphi) - \Psi(\varphi') \leq \sup(\varphi - \varphi')$$

A niveloid is Lipschitz continuous with respect to the supnorm. Indeed:

$$\Psi(\varphi) - \Psi(\varphi') \leq \sup(\varphi - \varphi') \leq |\sup(\varphi - \varphi')| \leq \sup|\varphi - \varphi'| = ||\varphi - \varphi'||_\infty$$
$$\Psi(\varphi') - \Psi(\varphi) \leq \sup(\varphi' - \varphi) \leq |\sup(\varphi' - \varphi)| \leq \sup|\varphi' - \varphi'| = ||\varphi - \varphi'||_\infty$$

so that $|\Psi(\varphi) - \Psi(\varphi')| \leq ||\varphi - \varphi'||_\infty$ for all $\varphi, \varphi' \in \tilde{B}$. Moreover, the functional Ψ is said to be *normalized* if $\Psi(k) = k$ for all $k \in \mathbb{R}$ such that $k \in \tilde{B}$, where we identify each real number with the constant function yielding it everywhere. Finally, the functional Ψ is said to be *monotone* if whenever $\varphi, \varphi' \in \tilde{B}$ and $\varphi \geq \varphi'$, then $\Psi(\varphi) \geq \Psi(\varphi')$.[17] We say that Ψ is *monotone continuous* if for all $\varphi, \varphi' \in \tilde{B}$ and $k \in \tilde{B}$, for all monotone sequences $(E_n)_n \in \mathcal{Y}$ such that $E_n \downarrow \emptyset$, if $\Psi(\varphi) > \Psi(\varphi')$, then there exists $n_0 \in \mathbb{N}$ such that $\Psi(k\chi_{E_{n_0}} + \varphi\chi_{E^c_{n_0}}) > \Psi(\varphi')$.

We define on $B(Y, \mathcal{Y})$ the *lattice operations* $\vee$ and $\wedge$ as follows: for all $\varphi, \varphi' \in B(Y, \mathcal{Y})$, $(\varphi \vee \varphi')(\omega) = \max\{\varphi(y), \varphi'(y)\}$ and $(\varphi \wedge \varphi')(\omega) = \min\{\varphi(y), \varphi'(y)\}$ for all $y \in Y$. We say that a nonempty subset L of $\subseteq B(Y, \mathcal{Y})$ is a *lattice* if for all $\varphi, \varphi' \in L$,

[17] See Maccheroni et al. (2006) and Cerreia-Vioglio, Maccheroni, Marinacci, and Rustichini (2014) for an in-depth discussion of niveloids and their properties.

$\varphi \vee \varphi', \varphi \wedge \varphi' \in L$. If $(\varphi_n)_N$ is a sequence of functions in $\subseteq B(Y, \mathcal{Y})$ and $\varphi \in B(Y, \mathcal{Y})$, we write $\varphi_n \to \varphi$ to mean that $(\varphi_n)_n$ converges uniformly to φ. If we want to stress that the uniformly convergent sequence is monotone, we write $\varphi_n \nearrow \varphi$ if $\varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $\varphi_n \searrow \varphi$ if $\varphi_n \geq \varphi_{n+1}$ for all $n \in \mathbb{N}$. Finally, we write $\varphi_n \uparrow \varphi$ if $\varphi_n \leq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $(\varphi_n)_n$ converges pointwise to φ and, similarly, $\varphi_n \downarrow \varphi$ if $\varphi_n \geq \varphi_{n+1}$ for all $n \in \mathbb{N}$ and $(\varphi_n)_n$ converges pointwise to φ.

A.2. **Probabilities and Statistical Distances.** We now discuss some basic mathematical notions about probabilities and statistical distances. Fix an arbitrary measurable space $(Y, \mathcal{Y})$. For any $p, q \in \Delta(Y, \mathcal{Y})$, we write $p \ll q$ to denote that p is *absolutely continuous* with respect to q. Moreover, if $q \in \Delta(Y, \mathcal{Y})$ and f and g are $\mathcal{Y}$-measurable functions mapping Y to some arbitrary set, we write $f = g$ *a.e.* $[q]$ whenever $q(\{y \in Y : f(y) = g(y)\}) = 1$. As it is standard in measure-theoretic contexts, we assume throughout the convention $0 \cdot \infty = 0$. If f is a function mapping Y to some measurable space, we denote by $\sigma(f)$ the sigma-algebra generated by f.

Given a convex subset C of $\Delta(Y, \mathcal{Y})$ and an extended real valued function $\varphi : C \to \bar{\mathbb{R}}$, we denote by $\operatorname{dom} \varphi$ the effective domain of φ, that is the subset of its domain on which φ takes on finite values; that is, $\operatorname{dom} \varphi \coloneqq \{p \in C : |\varphi(p)| < \infty\}$. Moreover, we say such function φ to be *grounded* if $\inf_{p \in C} \varphi(p) = 0$. Fix a subset $\mathcal{Q} \subseteq \Delta(Y, \mathcal{Y})$ of countably additive probability measures. A function $c : \Delta(Y, \mathcal{Y}) \times \mathcal{Q} \to [0, \infty]$ is said to be a *statistical distance* if it satisfies the following two properties:

(i) for each $q \in \mathcal{M}$, $p = q$ implies $c(p, q) = 0$,
(ii) $c(\cdot, q)$ is lower semicontinuous for all $q \in \mathcal{Q}$.

Furthermore, a statistical distance c is convex if the section $c(\cdot, q)$ is a convex function for each $q \in \mathcal{Q}$ and is said to be a *divergence* if for all $q \in \mathcal{Q}$, $p \in \operatorname{dom} c(\cdot, q)$ implies that $p \ll q$.

A.3. **Structured Spaces.** Say that the triple $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space if $(\Omega, \mathcal{G})$ is a measure space, where $\mathcal{G}$ is a countably generated sigma-algebra and $\mathcal{M} \subseteq \Delta(\mathcal{G}) \coloneqq \Delta(\Omega, \mathcal{G})$ is a set of models admitting a best-fit map $\mathfrak{q}$ with sufficient sigma-algebra $\mathcal{A}$. Given a structured space, let $\mathcal{D} \coloneqq \mathcal{D}^{\Omega, \mathcal{G}}$ and $\mathcal{D}_{\mathcal{M}} \coloneqq \mathcal{D}^{\Omega, \mathcal{G}}_{\mathcal{M}}$ respectively the natural sigma-algebra on $\Delta(\mathcal{G})$ and the relative sigma-algebra on $\mathcal{M}$. Throughout the section, fix a structured space. In particular, recall that $E^m \coloneqq \mathfrak{q}^{-1}(m)$ and $m(E^m) = 1$ for all $m \in \mathcal{M}$. Denote by Λ the set of all the events in $\mathcal{G}$ that have probability either 0 or 1 according to all models $m \in \mathcal{M}$:

$$\Lambda \coloneqq \{E \in \mathcal{G} : \forall m \in \mathcal{M}, m(E) = 1 \text{ or } m(E) = 0\}.$$

Lemma B.1: *The sigma-algebra generated by* $\mathfrak{q}$ *is in* Λ*:* $\mathcal{A} = \sigma(\mathfrak{q}) \subseteq \Lambda$*. In particular,* $m(E) \in \{0,1\}$ *for all* $E \in \mathcal{A}$ *and model* $m \in \mathcal{M}$*.*

Proof of Lemma B.1: By definition of the sigma-algebra $\mathcal{D}$, $\sigma(\mathfrak{q})$ is generated by the class:

$$\mathcal{C} := \left\{ \mathfrak{q}^{-1}\left(\{p \in \Delta(\mathcal{G}) : p(E) \le r\}\right) : r \in [0,1],\ E \in \mathcal{G} \right\} .$$

Then, take any $r \in [0,1]$ and $E \in \mathcal{G}$. We have that for any $m \in \mathcal{M}$,

$$\begin{aligned} m\left(\mathfrak{q}^{-1}\left(\{p \in \Delta(\mathcal{G}) : p(E) \le r\}\right)\right) &= m\left(\{\omega \in \Omega : \mathfrak{q}^{\omega}(E) \le r\}\right) \\ &= m\left(\{\omega \in \Omega : \mathfrak{q}^{\omega}(E) \le r\} \cap E^m\right) \\ &= \begin{cases} 1 & \text{if } m(E) \le r \\ 0 & \text{if } m(E) > r \end{cases}, \end{aligned}$$

and, therefore, $\mathfrak{q}^{-1}\left(\{p \in \Delta(\mathcal{G}) : p(E) \le r\}\right) \in \Lambda$, showing that $\mathcal{C} \subseteq \Lambda$.

It is clear that $\Omega, \emptyset \in \Lambda$ and that if $E \in \Lambda$, then $\Omega \setminus E \in \Lambda$. Moreover, if we take $(E)_{n\in\mathbb{N}} \subseteq \Lambda$, for each $m \in \mathcal{M}$, we have either of two cases. If $m(E_n) = 0$ for all $n \in \mathbb{N}$, then:

$$m(\cup_{n\in\mathbb{N}} E_n) \le \sum_{n\in\mathbb{N}} m(E_n) = 0 \implies m(\cup_{n\in\mathbb{N}} E_n) = 0.$$

If, instead, there exists $k \in \mathbb{N}$ such that $m(E_k) = 1$, then:

$$m(\cup_{n\in\mathbb{N}} E_n) \ge m(E_k) = 1 \implies m(\cup_{n\in\mathbb{N}} E_n) = 1.$$

It follows that $\cup_{n\in\mathbb{N}} E_n \in \Lambda$. We can, thus, conclude that Λ is a sigma-algebra containing $\mathcal{M}$ and, therefore, $\sigma(\mathfrak{q}) = \sigma(\mathcal{C}) \subseteq \Lambda$. ■

Suppose that $u : X \to \mathbb{R}$ is an affine and surjective function. If $\mathcal{E}$ is a sub-sigma-algebra of $\mathcal{G}$, we can define the operator $u : \mathcal{F}(\mathcal{E}) \to B(\mathcal{E})$ as follows: for each $f \in \mathcal{F}(\mathcal{E})$,

$$u(f)(\omega) = u(f(\omega))$$

for all $\omega \in \Omega$.

Lemma B.2: *Suppose* u *is affine and surjective. Then,* $u : \mathcal{F}(\mathcal{E}) \to B(\mathcal{E})$ *is an affine operator. Moreover,* $\{u(f) : f \in \mathcal{F}_0(\mathcal{E})\} = B_0(\mathcal{E})$ *and* $\{u(f) : f \in \mathcal{F}(\mathcal{E})\} = B(\mathcal{E})$*.*

Proof: Take any $f \in \mathcal{F}_0(\mathcal{E})$. Then, there exists a finite, measurable partition of Ω, $(E_i)_{i=1}^k \subseteq \mathcal{E}$, and consequences $(x_i)_{i=1}^k \subseteq X$ such that $f = \sum_{i=1}^k \chi_{E_i} x_i$. Then, for all E_i and for all $\omega \in E_i$,

$$u(f)(\omega) = u(f(\omega)) = u(x_i)$$

and therefore, $u(f) = \sum_{i=1}^{k} \chi_{E_i} u(x_i)$. Therefore, $u(f) \in B_0(\mathcal{E})$ for all $f \in \mathcal{F}(\mathcal{E})$ so that the operator is well-defined and $\{u(f) : f \in \mathcal{F}(\mathcal{E})\} \subseteq B_0(\mathcal{E})$. Moreover, take $\alpha \in (0,1)$ and $f, f' \in \mathcal{F}(\mathcal{E})$. We have that for all $\omega \in \Omega$,

$$\begin{aligned}
u(\alpha f + (1-\alpha)f')(\omega) &= u((\alpha f(\omega) + (1-\alpha)f'(\omega)) \\
&= \alpha u(f(\omega)) + (1-\alpha)u(f'(\omega)) \\
&= \alpha u(f)(\omega) + (1-\alpha)u(f')(\omega)
\end{aligned}$$

proving affinity. Finally, take any $\varphi \in B_0(\mathcal{E})$. Then, there exist a finite, measurable partition of Ω, $(E_i)_{i=1}^{k} \subseteq \mathcal{E}$, and reals $(r_i)_{i=1}^{k} \subseteq \mathbb{R}$ such that $\varphi = \sum_{i=1}^{k} \chi_{E_i} r_i$. Since $\operatorname{Im} u = \mathbb{R}$, for each r_i we can pick $x_i \in X$ such that $r_i = u(x_i)$. Setting $f = \sum_{i=1}^{k} \chi_{E_i} x_i$ we can see that $\varphi = u(f)$ and $\varphi \in \mathcal{F}_0(\mathcal{E})$. This shows that $B_0(\mathcal{E}) \subseteq \{u(f) : f \in \mathcal{F}_0(\mathcal{E})\}$.

Take now $f \in \mathcal{F}$. Then, we can find a finite set $X_0 \subseteq X$ such that $\operatorname{Im} f \subseteq \operatorname{co} X_0$. Moreover, the latter set is compact and u is continuous on co_X since it is affine. It follows that $u(f)$ is measurable and $\min \operatorname{co} X_0 \leq u(f(\omega)) \leq \max \operatorname{co} X_0$ for all $\omega \in \Omega$. We conclude that $u(f) \in B(\mathcal{E})$. As for the other direction, take $\varphi \in B(\mathcal{E})$. Then, we can find $k < K$ such that $k \leq \varphi \leq K$. Since u is surjective, we can pick $x_k, x_K \in X$ such that $u(x_k) = k$ and $u(x_K) = K$. Then, define the function $\alpha_\varphi : \Omega \to [0,1]$ as

$$\alpha_\varphi(\omega) = \frac{\varphi(\omega) - k}{K - k}$$

and notice that it is also measurable. Define $f_\varphi(\omega) = x_k + \alpha_\varphi(\omega)(x_K - x_k)$ and notice that it is also measurable. Finally, using affinity of u, we obtain that for all $\omega \in \Omega$,

$$u(f_\varphi) = u(x_k) + \alpha(\omega)_\varphi(u(x_K) - u(x_k)) = k + \frac{\varphi(\omega) - k}{K - k}(K - k) = \varphi(\omega).$$

We conclude that $B(\mathcal{E}) \subseteq \{u(f) : f \in \mathcal{F}(\mathcal{E})\}$. ∎

Appendix B. Auxiliary Results

We say that a binary relation $\succsim$ over $\mathcal{F}$ is *solvable* if, for each act $f \in \mathcal{F}$, there exists a constant act $x_f \in X$ such that $x_f \sim f$. We call such (possibly non-unique) act the *certainty equivalent* of f. Next, we show that a preference relation that satisfies Axiom 1 is solvable.

Lemma B.3: *Suppose that $\succsim$ is a preference relation on $\mathcal{F}$ satisfying Axiom 1. Then, $\succsim$ is solvable.*

Proof of Lemma B.3: Fix any $f \in \mathcal{F}$. By definition, we can find a finite set $X_0 \subseteq X$ such that $\operatorname{Im} f \subseteq \operatorname{co} X_0$. Then, we can pick x^* and x_* in X_0 such that $x^* \succsim x \succsim x_*$ for all $x \in X_0$. Axiom 1 then implies that for all $\omega \in \Omega$, $x^* \succsim f(\omega) \succsim x_*$.

By Axiom 1.ii, this implies that $x^* \succsim f \succsim x_*$. Now, $\{\alpha \in [0,1] : \alpha x^* + (1-\alpha)x_* \succsim f\}$ and $\{\alpha \in [0,1] : f \succsim \alpha x^* + (1-\alpha)x_*\}$ are closed by mixture continuity and are non-empty, since the first one contains 1 and the second one contains 0. Moreover, by completeness of $\succsim$, their union is the whole $[0,1]$. Since the closed, unit interval is connected, such sets must have a non-empty intersection. This shows the existence of $x_f \in X$ such that $x_f \sim f$. ■

We proceed by defining the preferences conditional on a given model $m \in \mathcal{M}$ being the best-fit model and show that they inherit some properties from the unconditional preferences. Let us first recall the following axioms characterizing the variational preferences axiomatized by Maccheroni et al. (2006).

Axiom B.1 (Variational):

- Weak Certainty Independence. *For all $f, f' \in \mathcal{F}$, $x, y \in X$, and $\alpha \in (0,1)$,*

$$\alpha f + (1-\alpha)x \succsim \alpha f' + (1-\alpha)x \implies \alpha f + (1-\alpha)y \succsim \alpha f' + (1-\alpha)y \ .$$

- Uncertainty Aversion. *For all $f, f' \in \mathcal{F}$ and $\alpha \in (0,1)$,*

$$f \sim f' \implies \alpha f' + (1-\alpha)f \succsim f \ .$$

Lemma B.4: *Suppose that $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and that the preference relation $\succsim$ satisfies Axioms 1, 2, 3, 4, and 5. For all $m \in \mathcal{M}$, define $\succsim^m$ as follows: for all $f, f' \in \mathcal{F}$,*

$$f \succsim^m f' \iff \exists g \in \mathcal{F},\ fE^m g \succsim f'E^m g.$$

Then, $\succsim^m$ is well-defined, satisfies Axiom 1, 2, and B.1 and coincides with $\succsim$ when restricted to constant acts in X.

Proof of Lemma B.4: Fix any $m \in \mathcal{M}$ and consider $\succsim^m$ as defined in Equation B.4. We show that this is a well-defined binary relation over $\mathcal{F}$. Indeed, suppose that for $f, f' \in \mathcal{F}$, there exists some $g \in \mathcal{F}$ such that $fE^m g \succsim f'E^m g$. Then, Axiom 3 implies that $fE^m h \succsim f'E^m h$ for all $h \in \mathcal{F}$. Therefore, in the following, we just fix a $g \in \mathcal{F}$ and notice that $f \succsim^m f' \iff fE^m g \succsim f'E^m g$. Moreover, note that for any $f, f', g \in \mathcal{F}$ and $\alpha \in [0,1]$, $(\alpha f + (1-\alpha)f')E^m g = \alpha(fE^m g) + (1-\alpha)(f'E^m g)$. Indeed, if $\omega \in E^m$:

$$\begin{aligned}
((\alpha f + (1-\alpha)f')E^m g)(\omega) &= (\alpha f + (1-\alpha)f')(\omega) \\
&= \alpha f(\omega) + (1-\alpha)f'(\omega) \\
&= \alpha(fE^m g)(\omega) + (1-\alpha)(f'E^m g)(\omega) \\
&= (\alpha(fE^m g) + (1-\alpha)(f'E^m g))(\omega)
\end{aligned}$$

and, if $\omega \in \Omega \setminus E^m$:

$$\begin{aligned}
((\alpha f + (1-\alpha) f')E^m g)(\omega) &= g(\omega) \\
&= \alpha g(\omega) + (1-\alpha) g(\omega) \\
&= \alpha (f E^m g)(\omega) + (1-\alpha)(f' E^m g)(\omega) \\
&= (\alpha (f E^m g) + (1-\alpha)(f' E^m g))(\omega) \ .
\end{aligned}$$

Step 1: Weak Order. Take any $f, f' \in \mathcal{F}$. Then, since $\succsim$ is complete, it follows that either $fE^m g \succsim f' E^m g$ or $f' E^m g \succsim f E^m g$. That is, either $f \succsim^m f'$ or $f' \succsim^m f$, showing that $\succsim^m$ is complete. Moreover, suppose that there are $f, f', f'' \in \mathcal{F}$ such that $f \succsim^m f'$ and $f' \succsim^m f''$. Then, $fE^m g \succsim f' E^m g$ and $f' E^m g \succsim f'' E^m g$. Since $\succsim$ is transitive, it follows that $fE^m g \succsim f'' E^m g$ and, therefore, that $f \succsim^m f''$. This shows that $\succsim^m$ is also transitive.

Step 2: Mixture Continuity. Take any $f, f', f'' \in \mathcal{F}$. We show that $\{\alpha \in [0,1] : \alpha f' + (1-\alpha) f'' \succsim^m f\}$ is closed. Indeed, take any $\alpha_0 \in [0,1]$ and let $g = f$:

$$\begin{aligned}
&\alpha_0 \in \{\alpha \in [0,1] : \alpha f' + (1-\alpha) f'' \succsim^m f\} \\
\iff &\alpha_0 f' + (1-\alpha_0) f'' \succsim^m f \\
\iff &(\alpha_0 f' + (1-\alpha_0) f'')E^m g \succsim f E^m g \\
\iff &\alpha_0 (f' E^m f) + (1-\alpha_0)(f'' E^m f) \succsim f \\
\iff &\alpha_0 \in \{\alpha \in [0,1] : \alpha (f' E^m f) + (1-\alpha)(f'' E^m f) \succsim f\}
\end{aligned}$$

so that $\{\alpha \in [0,1] : \alpha f' + (1-\alpha) f'' \succsim^m f\} = \{\alpha \in [0,1] : \alpha (f' E^m f) + (1-\alpha)(f'' E^m f) \succsim f\}$ and the latter is closed by Axiom 1. By an analogous argument, it follows that also $\{\alpha \in [0,1] : f \succsim^m \alpha f' + (1-\alpha) f''\}$ is closed. Hence, $\succsim^m$ satisfies mixture continuity.

Step 3: Weak Certainty Independence. Take any $f, f' \in \mathcal{F}$, $x, y \in X$ and $\alpha \in (0,1)$. Then,

$$\alpha f + (1-\alpha) x \succsim^m \alpha f' + (1-\alpha) x \implies [\alpha f + (1-\alpha) x] E^m g \succsim [\alpha f' + (1-\alpha) x] E^m g$$

and letting $g = \alpha f + (1-\alpha) x$ this implies that:

$$\begin{aligned}
\alpha f + (1-\alpha) x &\succsim [\alpha f' + (1-\alpha) x] E^m [\alpha f + (1-\alpha) x] \\
&= \alpha f' E^m f + (1-\alpha) x.
\end{aligned}$$

But, then, by Axiom 5,

$$\begin{aligned}\alpha f + (1-\alpha)y &\succsim \alpha f' E^m f + (1-\alpha)y \\ &= [\alpha f' + (1-\alpha)y]E^m[\alpha f + (1-\alpha)y],\end{aligned}$$

which, then, implies that $\alpha f + (1-\alpha)y \succsim^m \alpha f' + (1-\alpha)y$. If follows that $\succsim^m$ satisfies Weak Certainty Independence. A fortiori, it satisfies Risk Independence.

Step 4: Non-triviality. Since E^m is nonnull, there must exist $f, f', g \in \mathcal{F}$ such that $fE^m g \succ f'E^m g$. Since f and f' are finite-valued, we can pick $x, y \in X$ so that $x \succsim f(\omega)$ and $f'(\omega) \succsim y$ for all $\omega \in E^m$. But then, monotonicity implies that

$$xE^m g \succsim fE^m g \succ f'E^m g \succsim yE^m g$$

and, by transitivity, $xE^m g \succ yE^m g$ so that $x \succ^m y$. It follows that $\succsim^m$ is non-trivial.

Step 5. $\succsim^m|_X = \succsim_X$. By Axiom 1, $\succsim$ is a non-trivial weak order satisfying mixture continuity and independence when restricted to X. By Steps 1-4, the same is true for $\succsim^m$. Then, by Herstein and Milnor (1953), there exist affine functions $u, u_m : X \to \mathbb{R}$ such that u represents $\succsim|_X$ and u_m represents $\succsim^m|_X$. Moreover, since both $\succsim$ and $\succsim^m$ are non-trivial, u and u_m are non-constant. Now, take any $x, y \in X$ such that $x \succsim y$. Then, for all $\omega \in \Omega$,

$$\begin{aligned}\omega \in E^m &\implies (xE^m g)(\omega) = x \succsim y = (yE^m g)(\omega) \\ \omega \in \Omega \setminus E^m &\implies (xE^m g)(\omega) = g(\omega) = (yE^m g)(\omega)\end{aligned}$$

so that, since $\succsim$ satisfies reflexivity and monotonicity by Axiom 1, $xE^m g \succsim yE^m g$ and, therefore, $x \succsim^m y$. Thus, for all $x, y \in X$:

$$\begin{aligned}u(x) \geq u(y) &\implies x \succsim|_X y \\ &\implies x \succsim y \\ &\implies x \succsim^m y \\ &\implies x \succsim^m|_X y \\ &\implies u_m(x) \geq u_m(y) \ .\end{aligned}$$

By Corollary B.3 in Ghirardato, Maccheroni, and Marinacci (2004), there exists $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $u = au_m + b$. This implies the claim.

Step 6: Monotonicity. Take $f, f' \in \mathcal{F}$ and assume that $f(\omega) \succsim^m f'(\omega)$ for all $\omega \in \Omega$. Since by Step 4, $\succsim^m|_X = \succsim_X$, it is also the case that $f(\omega) \succsim f'(\omega)$ for all $\omega \in \Omega$. Then, since $\succsim$ satisfies Axiom 1, reflexivity and monotonicity imply that $fE^m g \succsim f'E^m g$ and, therefore, $f \succsim^m f'$, proving the statement.

Step 7: Unboundedness. This follows immediately by Step 5.

Step 8. Uncertainty Aversion

Take any $f, f' \in \mathcal{F}$ and $\alpha \in (0,1)$ and suppose that $f \sim^m f'$. Then, taking $g = f$ in the definition of $\succsim^m$ and since $\succsim$ satisfies Axiom 5, we have

$$\begin{aligned} f \sim^m f' &\implies f \sim f' E^m f \\ &\implies \alpha f + (1-\alpha) f' E^m f \succsim f \\ &\implies [\alpha f + (1-\alpha) f'] E^m f \succsim f E^m f \\ &\implies \alpha f + (1-\alpha) f' \succsim^m f \end{aligned}$$

showing that $\succsim^m$ satisfies Uncertainty Aversion.

Step 9: Monotone Continuity.

Take any $f, f' \in \mathcal{F}$ such that $f \succ^m f'$, $x \in X$, and $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{G}$ such that $A_1 \supseteq A_2 \supseteq \cdots$ and $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$. Taking $g = f$ in the definition of $\succsim^m$, we have that $f \succ f' E^m f$. Moreover, for each $n \in \mathbb{N}$, let $E_n \coloneqq A_n \cap E^m$ and observe that $E_n = A_n \cap E^m \supseteq A_{n+1} \cap E^m = E_{n+1}$ and

$$\bigcap_{n\in\mathbb{N}} E_n = \bigcap_{n\in\mathbb{N}} (A_n \cap E^m) = (\bigcap_{n\in\mathbb{N}} A_n) \cap E^m = \emptyset \cap E^m = \emptyset.$$

Since $\succsim$ satisfies Axiom 2, we can find $n_0 \in \mathbb{N}$ such that $x E_{n_0} f \succ f' E^m f$. Moreover,

$$\begin{aligned} \omega \in E_{n_0} = A_{n_0} \cap E^m &\implies (x E_{n_0} f)(\omega) = x = ((x A_{n_0} f) E^m f)(\omega), \\ \omega \in E^m \setminus A_{n_0} &\implies (x E_{n_0} f)(\omega) = f(\omega) = ((x A_{n_0} f) E^m f)(\omega), \\ \omega \notin E^m &\implies (x E_{n_0} f)(\omega) = f(\omega) = ((x A_{n_0} f) E^m f)(\omega). \end{aligned}$$

Therefore, $(x A_{n_0} f) E^m f = x E_{n_0} f \succ f' E^m f$ which implies that $x A_{n_0} f \succ^m f'$ as we wanted to show. ■

Appendix C. Structured Functionals

Throughout the section, assume that $(\Omega, \mathcal{G})$ is a measurable space, $\mathcal{M}$ is a set of models admitting a best-fit map $\mathfrak{q}$ generating the sigma-algebra $\mathcal{A}$, and that there exists a family of operators $I^m : B(\mathcal{G}) \to \mathbb{R}$ for each $m \in \mathcal{M}$, satisfying the following properties:

(i) I^m is normalized, monotone, and continuous for all $m \in \mathcal{M}$;
(ii) for all $m \in \mathcal{M}$, $\varphi = \psi$ a.e. $[m]$ implies $I^m(\varphi) = I^m(\psi)$ for all $\varphi, \psi \in B(\mathcal{G})$;
(iii) for all $\varphi \in B(\mathcal{G})$, $m \mapsto I^m(\varphi)$ is measurable.

We begin by showing that for all $m \in \mathcal{M}$, I^m is linear when restricted to $B(\mathcal{A})$.

Lemma B.5: *For all $m \in \mathcal{M}$, the restriction of I^m to $B(\mathcal{A})$ is linear. In particular, I^m is Lipschitz continuous of order 1 on $B(\mathcal{A})$.*

Proof of Lemma B.5: Fix $m \in \mathcal{M}$ arbitrarily. First of all, we show that I^m is additive on $B(\mathcal{A})$. Since $B_0(\mathcal{A})$ is dense in $B(\mathcal{A})$, we can find sequences $(\varphi_n)_{n\in\mathbb{N}}$, $(\psi_n)_{n\in\mathbb{N}} \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi$ and $\psi_n \to \psi$ in the supnorm. Fix any $n \in \mathbb{N}$. Then, there exists a partition $(E_i^n)_{i=1}^{k_n} \subseteq \mathcal{A}$ such that $\varphi_n = \sum_{i=1}^{k_n} r_i^n \chi_{E_i^n}$ and $\psi_n = \sum_{i=1}^{k_n} \tilde{r}_i^n \chi_{E_i^n}$ for reals $(r_i^n)_{i=1}^{k_n}, (\tilde{r}_i^n)_{i=1}^{k_n} \subseteq \mathbb{R}$. By Lemma B.1, for each $m \in \mathcal{M}$, there exists a unique $j_n(m) \in \{1, \ldots, k_n\}$ such that $m(E_{j_n(m)}^n) = 1$ and $m(E_i^n) = 0$ for all $i \neq j_n(m)$. Therefore, for all $m \in \mathcal{M}$, $\varphi_n = r_{j_n(m)}^n$ and $\psi_n = \tilde{r}_{j(m)}^n$ a.e.$[m]$ and, similarly $\varphi_n + \psi_n = r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n$, so that

$$\begin{aligned} I^m(\varphi_n + \psi_n) &= I^m(r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n) \\ &= r_{j_n(m)}^n + \tilde{r}_{j_n(m)}^n \\ &= I^m(r_{j_n(m)}^n) + I^m(\tilde{r}_{j_n(m)}^n) = I^m(\varphi_n) + I^m(\psi^n). \end{aligned}$$

We conclude that $I^m(\varphi_n + \psi_n) = I^m(\varphi_n) + I^m(\psi_n)$ for all $n \in \mathbb{N}$. Since I^m is continuous with respect to supnorm convergence, taking limits, we conclude that $I^m(\varphi + \psi) = I^m(\varphi) + I^m(\psi)$.

We now show that I^m is homogeneous. Take $\varphi \in B(\mathcal{A})$ and $\kappa \in \mathbb{R}$. As before, we can find a sequence $(\varphi_n)_{n\in\mathbb{N}} \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi$ in the supnorm. Notice also that $||\kappa\varphi_n - \kappa\varphi||_\infty = |\kappa|||\varphi_n - \varphi||_\infty \to 0$. Fix any n and pick a partition $(E_i^n)_{i=1}^{k_n} \subseteq A$ such that $\varphi_n = \sum_{i=1}^{k_n} r_i^n \chi_{E_i^n}$ for reals $(r_i^n)_{i=1}^{k_n} \subseteq \mathbb{R}$. For each $m \in \mathcal{M}$, Lemma B.1 implies that there exists a unique $j_n(m) \in \{1, \ldots, k_n\}$ such that $m(E_{j_n(m)}^n) = 1$ and $m(E_i^n) = 0$ for all $i \neq j_n(m)$. Therefore, for all $m \in \mathcal{M}$, $\varphi_n = r_{j_n(m)}^n$ a.e. $[m]$ and, similarly $\kappa\varphi_n = \kappa r_{j_n(m)}^n$ a.e. $[m]$, so that

$$I^m(\kappa\varphi_n) = I^m(\kappa r_{j_n(m)}^n) = \kappa r_{j_n(m)}^n = \kappa I^m(r_{j_n(m)}^n) = \kappa I^m(\varphi_n)$$

Therefore, $I^m(\kappa\varphi_n) = \kappa I^m(\varphi_n)$ for all $n \in \mathbb{N}$ and taking limits and by continuity of I^m we conclude that $I^m(\kappa\varphi) = \kappa I^m(\varphi)$.

■

The following lemma shows that we are able to find a non-linear conditional expectation given $\mathcal{A}$ that is common to all hypothesized models $m \in \mathcal{M}$.

Lemma B.6: *The map $m \mapsto I^m(\varphi)$ is bounded and there exists a non-linear common conditional expectation of $(I^m)_{m\in\mathcal{M}}$ given $\mathcal{A}$. This is a map $I_\mathcal{A} : B(\mathcal{G}) \to \mathbb{R}^\Omega$ such that for all $\varphi \in B(\mathcal{G})$, $I_\mathcal{A}(\varphi)$ is in $B(\mathcal{A})$, $I_\mathcal{A}(\varphi)(\omega) = I^{\mathfrak{q}(\omega)}(\varphi)$ for all $\omega \in \Omega$ and for all*

$A \in \mathcal{A}$ *and* $m \in \mathcal{M}$,

$$I^m\left(I_{\mathcal{A}}(\varphi)\chi_A\right) = I^m(\varphi\chi_A).$$

PROOF OF LEMMA B.6: First, we show that $m \mapsto I^m(\varphi)$ is bounded. Indeed, take $\varphi \in B(\Omega, \mathcal{G})$. Then, there exist $k, K \in \mathbb{R}$ such that $k \leq \varphi \leq K$. Since for each $m \in \mathcal{M}$, I^m is normalized and monotone, we have that

$$k = I^m(k) \leq I^m(\varphi) \leq I^m(K) = K$$

proving boundedness.

Fix any $\varphi \in B(\Omega, \mathcal{G})$. Since $m \mapsto I^m(\varphi)$ is bounded and $\mathcal{D}_{\mathcal{M}}$-measurable, it follows that the composition

$$\begin{array}{rccccc} I^{\mathfrak{q}(\cdot)}(\varphi): & (\Omega, \mathcal{A}) & \to & (\mathcal{M}, \mathcal{D}_{\mathcal{M}}) & \to & (\mathbb{R}, \mathcal{B}(\mathbb{R})) \\ & \omega & \mapsto & \mathfrak{q}(\omega) & \mapsto & I^{\mathfrak{q}(\omega)}(\varphi) \end{array}$$

is a $\mathcal{A}$-measurable and bounded functional. Obtain $I_{\mathcal{A}}(\varphi)$ by defining $I_{\mathcal{A}}(\varphi)(\omega) = I^{\mathfrak{q}(\omega)}\varphi$ for all $\omega \in \Omega$. It is easy to see that $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$. Moreover, take any $A \in \mathcal{A}$ and fix $m \in \mathcal{M}$ arbitrarily. We know that $m(E^m) = 1$, so that $m(\Omega \setminus E^m) = 0$, where we recall that $E^m = \{\omega \in \Omega : \mathfrak{q}(\omega) = m\}$. Moreover, since $A \in \Lambda$ by Lemma B.1, we have that either $m(A) = 1$ or $m(A) = 0$. In any case, this implies that

$$m(A \cap E^m) = m(A)m(E^m) = p(A) = m(A).$$

Now, notice that $I_{\mathcal{A}}(\varphi)(\omega) = I^m(\varphi)$ for all $\omega \in E^m$ and, therefore, $I_{\mathcal{A}}(\varphi)\chi_A = I^m(\varphi)$ a.e. $[m]$ if $m(A) = 1$ and $I_{\mathcal{A}}(\varphi)\chi_A = 0$ a.e. $[m]$ if $m(A) = 0$. Similarly, $\varphi\chi_A = \varphi$ a.e. $[m]$ if $m(A) = 1$ and $\varphi\chi_A = 0$ a.e. $[m]$ if $m(A) = 0$. Then, by property *(ii)* of I^m, it follows that

$$\begin{aligned} m(A) = 1 &\implies I^m(I_{\mathcal{A}}(\varphi)\chi_A) = I^m(I^m(\varphi)) = I^m(\varphi) = I^m(\varphi\chi_A) \\ m(A) = 0 &\implies I^m(I_{\mathcal{A}}(\varphi)\chi_A) = I^m(0) = 0 = I^m(\varphi\chi_A) \end{aligned}$$

.

■

Notice that for each $\varphi \in B(\mathcal{G})$, we can see $I^m(\varphi)$ as a function from models to $\mathbb{R}$:

$$I(\varphi, \cdot) : \mathcal{M} \to \mathbb{R}, \quad m \mapsto I(\varphi, m) \coloneqq I^m(\varphi).$$

Define the operator $T : B(\Omega, \mathcal{G}) \to \mathbb{R}^{\mathcal{M}}$ such that for all $\varphi \in B(\Omega, \mathcal{G})$,

$$T(\varphi)(m) = I(\varphi, m)$$

for all $m \in \mathcal{M}$. By Lemma B.6, we have that $\operatorname{Im} T \subseteq B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. Moreover, we have the following result.

Lemma B.7: *$T : B(\Omega, \mathcal{G}) \to B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is Lipschitz continuous of order 1 with respect to supnorm convergence and is additive and homogeneous on $B(\mathcal{A})$.*

Proof of Lemma B.7: Take $\varphi, \psi \in B(\mathcal{A})$ and $k \in \mathbb{R}$. By Lemma B.5, I^m is linear when restricted to $B(\mathcal{A})$ for all $m \in \mathcal{M}$. Therefore:

$$T(\varphi + \kappa\psi)(m) = I^m(\varphi + \kappa\psi) = I^m(\varphi) + \kappa I^m(\psi) = T(\varphi)(m) + \kappa T(\psi)(m)$$

for all $m \in \mathcal{M}$. Thus, $T(\varphi + \kappa\psi) = T(\varphi) + \kappa T(\psi)$, showing linearity. Moreover, since each I^m is Lipschitz continuous of order 1 on $B(\mathcal{A})$, we have that:

$$|T(\varphi)(m) - T(\psi)(m)| = |I^m(\varphi) - I^m(\psi)| \leq ||\varphi - \psi||_\infty$$

and, therefore,

$$||T(\varphi) - T(\psi)||_\infty = \sup_{m \in \mathcal{M}} |T(\varphi)(m) - T(\psi)(m)| \leq ||\varphi - \psi||_\infty$$

showing that T is also Lipschitz continuous of order 1 on $B(\mathcal{A})$.

■

Lemma B.8: *Let $T(B(\mathcal{A}))$ and $T(B_0(\mathcal{A}))$ be the images through T of $B(\mathcal{A})$ and $B_0(\mathcal{A})$ respectively. Then, $T(B(\mathcal{A})) = \operatorname{Im} T$ and $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$. Moreover, T preserves lattice operations when restricted to $B(\mathcal{A})$. In particular, $\operatorname{Im} T$ is a lattice.*

Proof of Lemma B.8: It is clear that

$$\begin{aligned} T(B(\mathcal{A})) &= \{I(\varphi, \cdot) : \varphi \in B(\Omega, \mathcal{A})\} \\ &\subseteq \{I(\varphi, \cdot) : \varphi \in B(\Omega, \mathcal{G})\} = \operatorname{Im} T \end{aligned}$$

since $\mathcal{A}$ is a a sub-sigma-algebra of $\mathcal{G}$. As for the reverse inclusions, take any $\xi \in \operatorname{Im} T$ and let $\varphi_\xi \in B(\mathcal{G})$ be such that $\xi = T(\varphi_\xi)$. Then, by Lemma B.6, $I_{\mathcal{A}}(\varphi_\xi) \in B(\mathcal{A})$ and for all $m \in \mathcal{M}$,

$$T(I_{\mathcal{A}}(\varphi_\xi))(m) = I^m\left(I_{\mathcal{A}}(\varphi_\xi)\right) = I^m(\varphi_\xi) = T(\varphi_\xi)(m) = \xi(m),$$

so that $\xi \in T(B(\mathcal{A}))$, showing that $\operatorname{Im} T \subseteq T(B(\mathcal{A}))$. Next, we show that $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$. Take $\xi \in \operatorname{Im} T$ and a corresponding $\varphi_\xi \in B(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ (which exists given what shown above). Since $B_0(\mathcal{A})$ is supnorm dense in $B(\mathcal{A})$, we can find a sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $||\varphi_n - \varphi_\xi||_\infty \to 0$. Define $\xi_n = T(\varphi_n)$ for each $n \in \mathbb{N}$ and note that $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$. We show that ξ_n converges to ξ in the supnorm. Indeed, by Lemma B.7, T is Lipschitz of order 1 and, therefore

$$||\xi - \xi_n||_\infty = ||T(\varphi) - T(\varphi_n)||_\infty \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Finally, we show that T preserves lattice operations on $B(\mathcal{A})$. Indeed, pick $\varphi, \tilde{\varphi} \in B(\mathcal{A})$ arbitrarily. Since $B_0(\mathcal{A})$ is supnorm dense in $B(\mathcal{A})$, we can take sequences $(\varphi)_n, (\tilde{\varphi}_n)_n \subseteq B_0(\mathcal{A})$ such that $||\varphi - \varphi_n||_\infty, ||\tilde{\varphi} - \tilde{\varphi}_n||_\infty \to 0$. For each $n \in \mathbb{N}$, we can find a finite partition $(E_n^i)_{i=1}^k$ and reals $(r_n^i)_{i=1}^k$, $(\tilde{r}_n^i)_{i=1}^k$ such that:

$$\varphi_n = \sum_{i=1}^k \chi_{E_n^i} r_n^i, \quad \tilde{\varphi}_n = \sum_{i=1}^k \chi_{E_n^i} \tilde{r}_n^i.$$

Fix any $m \in \mathcal{M}$. By Lemma B.1, for each $n \in \mathbb{N}$, there is a unique E_n^l in the partition such that $m(E_n^l) = 1$. Therefore, $\varphi_n = r_n^l$ and $\tilde{\varphi}_n = \tilde{r}_n^l$ a.e. $[m]$, so that $I^m(\varphi_n) = I^m(r_n^l) = r_n^l$ and $I^m(\tilde{\varphi}_n) = I^m(\tilde{r}_n^l) = \tilde{r}_n^l$ for all $n \in \mathbb{N}$. Clearly, it is also the case that $\varphi_n \vee \tilde{\varphi}_n = r_n^l \vee \tilde{r}_n^l$ a.e. $[m]$ so that $I^m(\varphi_n \vee \tilde{\varphi}_n) = I^m(r_n^l \vee \tilde{r}_n^l) = r_n^l \vee \tilde{r}_n^l$ for all $n \in \mathbb{N}$. Therefore:

$$I^m(\varphi_n \vee \tilde{\varphi}_n) = r_n^l \vee \tilde{r}_n^l = I^m(\varphi_n) \vee I^m(\tilde{\varphi}_n)$$

for all $n \in \mathbb{N}$. Since lattice operations are continuous and I^m is Lipschitz, taking limits, it follows that

$$T(\varphi \vee \tilde{\varphi})(m) = I^m(\varphi \vee \tilde{\varphi}) = I^m(\varphi) \vee I^m(\tilde{\varphi}) = T(\varphi)(m) \vee T(\tilde{\varphi})(m).$$

Since m was chosen arbitrarily, we can conclude that $T(\varphi \vee \tilde{\varphi}) = T(\varphi) \vee T(\tilde{\varphi})$. That $\operatorname{Im} T$ is a lattice follows from the fact that $\operatorname{Im} T = T(B(\mathcal{A}))$ and $T|_{B(\mathcal{A})}$ preserves lattice operations. ■

Recall that $B_0(\mathcal{D}_\mathcal{M}) \coloneqq B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ and $B(\mathcal{D}_\mathcal{M}) \coloneqq B(\mathcal{M}, \mathcal{D}_\mathcal{M})$ are, respectively, the spaces of simple and bounded functions on the set of models $\mathcal{M}$ measurable with respect to $\mathcal{D}_\mathcal{M}$. The following result shows that these spaces can be covered by applying the operator T respectively to $B_0(\mathcal{A})$ and $B(\mathcal{A})$. Further, characteristic functions of sets in $\mathcal{D}_\mathcal{M}$ can be recovered by applying the operator T to characteristic functions of sets in $\mathcal{A}$.

Lemma B.9: $\operatorname{Im} T = B(\mathcal{M}, \mathcal{D}_\mathcal{M})$. *Moreover,* $T(B_0(\mathcal{A})) = B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ *and* $T(\{\chi_E : E \in \mathcal{A}\}) = \{\chi_D : D \in \mathcal{D}_\mathcal{M}\}$. *Moreover, for each* $\xi \in B_0(\mathcal{M}, \mathcal{D}_\mathcal{M})$ *such that* $0 \leq \xi \leq 1$, *there is* $\varphi \in B(\mathcal{A})$ *with* $0 \leq \varphi \leq 1$ *such that* $\xi = T(\varphi)$.

Proof of Lemma B.9: We prove the results via a series of steps.

Step (i). For all $E \in \mathcal{A}$, *there exists* $D_E \in \mathcal{D}_\mathcal{M}$ *such that* $T(\chi_E) = \chi_{D_E}$.

Proof: Take any $E \in \mathcal{A}$. By Lemma B.1, $E \in \Lambda$ and, therefore, for all $m \in \mathcal{M}$, either $m(E) = 1$ or $m(E) = 0$. But then for all $m \in \mathcal{M}$:

$$m(E) = 1 \implies \chi_E = 1 \text{ a.e. } [m] \implies T(\chi_E)(m) = I^m(\chi_E) = I^m(1) = 1,$$
$$m(E) = 0 \implies \chi_E = 0 \text{ a.e. } [m] \implies T(\chi_E)(m) = I^m(\chi_E) = I^m(0) = 0.$$

Therefore, $\operatorname{Im} T(\chi_E) \in \{0, 1\}$. Moreover, by Lemma **??**, $D_E \coloneqq [T(\chi_E)]^{-1}(\{1\}) \in \mathcal{D}_{\mathcal{M}}$ and $T(\chi_E) = \chi_{D_E}$ as we wanted to show. $\square$

Step (ii). For all $D \in \mathcal{D}_{\mathcal{M}}$, there exists $E^D \in \mathcal{A}$ such that $T(\chi_{E^D}) = \chi_D$.

Proof: Take any $D \in \mathcal{D}_{\mathcal{M}}$ and let $E^D = \mathfrak{q}^{-1}(D)$. Since the space is structured, $E^D \in \mathcal{A}$ and $m(E^D) = 1$ if $m \in D$ and $m(E^D) = 0$ if $m \in \mathcal{M} \setminus D$. But then for all $m \in \mathcal{M}$:

$$m \in D \implies \chi_{E^D} = 1 \text{ a.e. } [m] \implies T(\chi_{E^D})(m) = I^m(\chi_{E^D}) = I^m(1) = 1,$$
$$m(E) \in \mathcal{M} \setminus D \implies \chi_{E^D} = 0 \text{ a.e. } [m] \implies T(\chi_{E^D})(m) = I^m(\chi_{E^D}) = I^m(0) = 0,$$

and we can, thus, conclude that $T(\chi_{E^D}) = \chi_D$. $\square$

Steps (i) and (ii) together imply that $T(\{\chi_E : E \in \mathcal{A}\}) = \{\chi_D : D \in \mathcal{D}_{\mathcal{M}}\}$.

Step (iii). $T(B_0(\mathcal{A})) \subseteq B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$.

Proof: Take $\xi \in T(B_0(\mathcal{A}))$. By definition, there exists $\varphi_\xi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$. Then, there exists a partition $(E_i)_{i=1}^k \subseteq \mathcal{A}$ and reals $(r_i)_{i=1}^k$ such that $\varphi_\xi = \sum_{i=1}^k \chi_{E_i} r_i$. By Step (i), we have that for each $i = 1, \ldots, k$, we can find $D_{E_i} \in \mathcal{D}_{\mathcal{M}}$ such that $T(\chi_{E_i}) = \chi_{D_{E_i}}$. Moreover, since for all $i = 1, \ldots, k$, $E_i \in \mathcal{A} \subseteq \Lambda$ by Lemma B.1, either $m(E_i) = 1$ or $m(E_i) = 0$ for each $m \in \mathcal{M}$. It follows that for each m, there is a unique element in the partition E_{j_m} such that $m(E_{j_m}) = 1$ and $m(E_i) = 0$ if $i \neq j_m$. Then, for each $m \in \mathcal{M}$,

$$\varphi_\xi = r_{j_m} \text{ a.e. } [m] \implies T(\varphi_\xi)(m) = I^m(\varphi_\xi) = I^m(r_{j_m}) = r_{j_m}$$

and, since $\chi_{E_{j_m}} = 1$ a.e. $[m]$ and $\chi_{E_i} = 0$ a.e. $[m]$ for $i \neq j_m$,

$$\chi_{D_{E_{j_m}}}(m) = T(\chi_{E_{j_m}})(m) = I^m(\chi_{E_{j_m}}) = I^m(1) = 1 \implies m \in D_{E_{j_m}}$$
$$\forall i \neq j_m, \quad \chi_{D_{E_i}}(m) = T(\chi_{E_i})(m) = I^m(\chi_{E_i}) = I^m(0) = 0 \implies m \notin D_{E_i}.$$

It follows that $\xi = T(\varphi_\xi) = \sum_{i=1}^k \chi_{D_{E_i}} r_i \in B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. $\square$

Step (iv). $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \subseteq T(B_0(\mathcal{A}))$, In particular, for all $D \in \mathcal{D}_{\mathcal{M}}$, there exists $E^D \in \mathcal{A}$ such that $\chi_D = T(\chi_{E^D})$.

Proof: Take any $\xi \in B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. By definition, there exists a partition $(D_i)_{i=1}^k \subseteq \mathcal{D}_{\mathcal{M}}$ of $\mathcal{M}$ and reals $(r_i)_{i=1}^k$ such that $\xi = \sum_{i=1}^k \chi_{D_i} r_i$. By Step (ii), for each $i = 1, \ldots, k$, we can find $E^{D_i} \in \mathcal{A}$ such that $\chi_{D_i} = T(\chi_{E^{D_i}})$. Define $\varphi_\xi \coloneqq \sum_{i=1}^k \chi_{E^{D_i}} r_i$. Clearly, $\varphi_\xi \in B_0(\mathcal{A})$. Moreover, for each $m \in \mathcal{M}$, let D_{j_m} be the unique element of the partition such that $m \in D_{j_m}$. We know by Lemma B.1 that since $E^{D_{j_m}} \in \mathcal{A}$, $m(E^{D_{j_m}}) \in \{0, 1\}$. If $m(E^{D_{j_m}}) = 0$, then $\chi_{E^{D_{j_m}}} = 0$ a.e. $[m]$ and, therefore, $T(\chi_{E^{D_{j_m}}})(m) = I^m(\chi_{E^{D_{j_m}}}) = I^m(0) = 0 \neq \chi_{D_{j_m}}(m) = 1$, a contradiction. We conclude that $m(E^{D_{j_m}}) = 1$ so that $\varphi_\xi = r_{j_m}$ a.e. $[m]$. Therefore,

$$T(\varphi_\xi)(m) = I^m(\varphi_\xi) = I^m(r_{j_m}) = r_{j_m} = r_{j_m}\chi_{D_{j_m}}(m) = \xi(m).$$

for all $m \in \mathcal{M}$. It follows that $T(\varphi_\xi) = \xi$, showing that $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \subseteq T(B_0(\mathcal{A}))$. $\square$

Step (iii) and (iv) imply that $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) = T(B_0(\mathcal{A}))$. Then, we have the following chain of inclusions:

$$B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \subseteq T(B_0(\mathcal{A})) \subseteq B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}).$$

Moreover, $B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is supnorm dense in $B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ and by Lemma B.8, $T(B_0(\mathcal{A}))$ is supnorm dense in $\operatorname{Im} T$. Taking the supnorm closure of the previous chain of inclusions, we obtain that:

$$B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) = \operatorname{cl} B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \subseteq \operatorname{cl} T(B_0(\mathcal{A})) = \operatorname{Im} T \subseteq \operatorname{cl} B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) = B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$$

and, therefore, we can conclude that $\operatorname{Im} T = B(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$.

The last part of the result follows by steps *iii* and *iv* and by Lemma B.7. ■

Lemma B.10:

(i) *If $\xi, \xi' \in B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ are such that $\xi \geq \xi'$, then there exist $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ such that $\varphi_\xi \geq \varphi_{\xi'}$ and $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$.*

(ii) *If $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ is an increasing (decreasing) sequence uniformly bounded above (below) by a constant K, there exists an increasing (decreasing) sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $\xi_n = T(\varphi_n)$ and $\varphi_n \leq K$ $(\varphi_n \geq K)$ for all $n \in \mathbb{N}$.*

(iii) *If $\xi \in \operatorname{Im} T$ and $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ such that $\xi_n \uparrow \xi$ $(\xi_n \downarrow \xi)$, then we can find an increasing (decreasing) sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ and $\varphi \in B(\mathcal{A})$ such that $\varphi_n \uparrow \varphi$ $(\varphi_n \downarrow \varphi)$, $\xi = T(\varphi)$, and $\xi_n = T(\varphi_n)$ for all $n \in \mathbb{N}$. Moreover, if $K \in \mathbb{R}$ and $\xi \leq K$ $(\xi \geq K)$, then $\varphi \leq K$ $(\varphi \geq K)$.*

(iv) *If $\xi, \xi' \in \operatorname{Im} T$ are such that $\xi \geq \xi'$, then there exist $\varphi_\xi, \varphi_{\xi'} \in B(\mathcal{A})$ such that $\varphi_\xi \geq \varphi_{\xi'}$ and $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$.*

Proof of Lemma B.10: We prove the lemma in a number of steps.

Proof of (i): Take $\xi, \xi' \in T(B_0(\mathcal{A}))$ such that $\xi \geq \xi'$. By definition, we can pick $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ and $\xi' = T(\varphi_{\xi'})$. Moreover, we can find a partition $(E_i)_{i=1}^n \subseteq \mathcal{A}$ of Ω and reals $(r_i)_{i=1}^n, (r_i')_{i=1}^n$ such that

$$\varphi_\xi = \sum_{i=1}^n \chi_{E_i} r_i, \quad \varphi_{\xi'} = \sum_{i=1}^n \chi_{E_i} r_i'.$$

Take an element E_k in the partition. If $m(E_k) = 0$ for all $m \in \mathcal{M}$, we can assume wlog that $r_k = r_k'$. Indeed, for all $m \in \mathcal{M}$, $\varphi_{\xi'} = \sum_{i \neq k} \chi_{E_i} r_i' + \chi_{E_k} r_k$ a.e. $[m]$, this implies

$$\xi' = T(\varphi_{\xi'})(m) = I^m(\varphi_{\xi'}) = I^m(\sum_{i \neq k} \chi_{E_i} r_i' + \chi_{E_k} r_k) = T(\sum_{i \neq k} \chi_{E_i} r_i' + \chi_{E_k} r_k)(m).$$

If there exists $m \in \mathcal{M}$ such that $m(E_k) \neq 0$, then $m(E_k) = 1$ since $E_k \in \mathcal{A} \subseteq \Lambda$ by Lemma B.1. Therefore, $\varphi_\xi = r_k$ and $\varphi_{\xi'} = r_k'$ a.e. $[m]$ and, therefore:

$$r_k = I^m(r_k) = I^m(\varphi_\xi) = T(\varphi_\xi)(m) = \xi(m),$$
$$r_k' = I^m(r_k') = I^m(\varphi_{\xi'}) = T(\varphi_{\xi'})(m) = \xi'(m),$$

and, we conclude that $r_k = \xi(m) \geq \xi'(m) = r_k'$. We have thus shown that $r_i \geq r_i'$ for all $i = 1, \ldots, n$. Hence, it follows that $\varphi_\xi \geq \varphi_{\xi'}$. It is then immediate to see that since each I^m is normalized, if $\xi \leq K$ for some K in $\mathbb{R}$, we can find $\varphi_\xi \in B_0(\mathcal{A})$ such that $\varphi_\xi \leq K$ and $\xi = T(\varphi_\xi)$. $\square$

Proof of (ii): Take a sequence $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ and $K \in \mathbb{R}$ such that $\xi_n \leq \xi_{n+1} \leq K$ for all $n \in \mathbb{N}$. By Step (i), we can find a sequence $\varphi_{\xi_n} \in B_0(\mathcal{A})$ such that $\xi_n = T(\varphi_{\xi_n})$ and $\varphi_{\xi_n} \leq K$ for all $n \in \mathbb{N}$. However, this sequence is not necessarily increasing. Then, define for each $n \in \mathbb{N}$, $\varphi_n(\omega) = \sup_{k \leq n} \varphi_{\xi_k}(\omega)$ for all ω. Notice that $\varphi_n : \Omega \to \mathbb{R}$ is well-defined and in $B_0(\mathcal{A})$. Moreover, the sequence $(\varphi_n)_n$ so constructed is increasing and uniformly bounded above by K. Moreover, since T preserves lattice operations by Lemma B.8, we have that for each $n \in \mathbb{N}$,

$$T(\varphi_n) = T\left(\sup_{k \leq n} \varphi_{\xi_k}\right) = \sup_{k \leq n} T\left(\varphi_{\xi_k}\right) = \sup_{k \leq n} \xi_k = \xi_n,$$

where the last equality follows from the fact that $(\xi_n)_n$ is a monotonically increasing sequence. $\square$

Proof of (iii): Take a sequence $(\xi_n)_n \subseteq T(B_0(\mathcal{A}))$ and $\xi \in \operatorname{Im} T$ such that $\xi_n \uparrow \xi$. Since ξ is bounded, $K_0 = \sup_{m \in \Omega} \xi$ is finite. Moreover, we have that $\xi_n \leq \xi \leq K_0$ for all $n \in \mathbb{N}$. By by point (ii), we can find an increasing sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $\xi_n = T(\varphi_n)$ and $\varphi_n \leq K_0$ for all $n \in \mathbb{N}$. Since for each $\omega \in \Omega$, $(\varphi_n(\omega))_n$ is a monotonically increasing sequence of numbers bounded above by K_0, it converges to some $\lim_n \varphi_n(\omega) \leq K_0$. Therefore, the pointwise limit $\varphi := \lim_n \varphi_n$ is well-defined, it is in $B(\mathcal{A})$, and it is uniformly bounded above by K_0. Moreover, we have that for all $n \in \mathbb{N}$,

$$k = \min_{\omega\in\Omega} \varphi_1(\omega) \leq \varphi_1 \leq \varphi_n \leq K_0 \implies ||\varphi_n||_\infty \leq \max\{|k|, |K_0|\}.$$

Therefore, $(\varphi_n)_n$ is uniformly bounded in the norm. Moreover, for each $m \in \mathcal{M}$, Thereom 13 in Maccheroni et al. (2006) and Proposition 5 in Cerreia-Vioglio et al. (2014), imply that I^m has the Lebesgue property. Therefore:

$$T(\varphi)(m) = I^m(\varphi) = I^m(\lim_n \varphi_n) = \lim_n I^m(\varphi_n) = \lim_n \xi_n(m) = \xi(m).$$

It is immediate to see that for all $k \in \mathbb{R}$ such that $\xi \leq K$, $K \geq K_0$ and, therefore, $\varphi \leq K$. $\square$

Proof of (iv): Take $\xi_1, \xi_2 \in B(\mathcal{M}, \mathcal{D}_\mathcal{M})$ such that $\xi_1 \geq \xi_2$. Define $\tilde{\xi} = \xi_1 - \xi_2$ and notice that $\tilde{\xi} \geq 0$. By point (iii), we can find $\tilde{\varphi} \in B(\mathcal{A})$ such that $\tilde{\varphi} \geq 0$ and $\tilde{\xi} = T(\tilde{\varphi})$. Moreover, we can take $\varphi_2 \in B(\mathcal{A})$ such that $\xi_2 = T(\varphi_2)$. Then, define $\varphi_1 := \varphi_2 + \tilde{\varphi} \in B(\mathcal{A})$. Clearly, $\varphi_1 \geq \varphi_2$ and since T is linear on $B(\mathcal{A})$, we have that

$$T(\varphi_1) = T(\varphi_2 + \tilde{\varphi}) = T(\varphi_2) + T(\tilde{\varphi}) = \xi_2 + \tilde{\xi} = \xi_1$$

as we wanted to show. $\square$

This concludes the proof of the lemma. $\blacksquare$

Proposition B.1: *The following are equivalent:*

(i) $I : B(\mathcal{A}) \to \mathbb{R}$ *is normalized, monotone, and such that for all* $\varphi, \varphi' \in B_0(\mathcal{A})$*,*

$$(\forall m \in \mathcal{M},\ I^m(\varphi) \geq I^m(\psi)) \implies I(\varphi) \geq I(\psi).$$

(ii) there exists a normalized and monotone functional $\hat{I} : B_0(\mathcal{M}, \mathcal{D}_\mathcal{M}) \to \mathbb{R}$ *such that for all* $\varphi \in B_0(\mathcal{A})$*,*

$$I(\varphi) = \hat{I}(T(\varphi)).$$

Moreover, $\hat{I}$ *is unique and*

- $\hat{I}$ *is continuous if and only if* I *is continuous.*
- $\hat{I}$ *is quasiconcave if and only if* I *is quasiconcave.*
- $\hat{I}$ *is monotone continuous if and only if* I *is monotone continuous.*
- $\hat{I}$ *is quasiconcave if and only if* I *is quasiconcave.*
- $\hat{I}$ *is translation invariant if and only if* I *is translation invariant.*

Proof of Proposition B.1:
(i) implies (ii). Define $\hat{I} : B(\mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ as follows: for all $\xi \in B(\mathcal{D}_{\mathcal{M}})$,

$$\hat{I}(\xi) = I(\varphi_\xi),$$

where $\varphi_\xi \in B(\mathcal{A})$ is chosen so that $\xi = T(\varphi_\xi)$.

Step 1: $\hat{I}$ is well-defined. Pick $\xi \in B(\mathcal{D}_{\mathcal{M}})$ arbitrarily. That a $\varphi_\xi \in B(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ exists follows from Lemma B.9. Moreover, suppose there are two $\varphi, \psi \in B(\mathcal{A})$ such that $T(\varphi)(m) = I^m(\varphi) = \xi(m) = I^m(\psi) = T(\psi)(m)$ for all $m \in \mathcal{M}$. Then, by assumption, it must be the case that $I(\varphi) = I(\psi)$, showing that $\hat{I}$ is well-defined.

Step 2: $\hat{I}$ is normalized. Take any $k \in \mathbb{R}$. Then, since each I^m is normalized, it follows that $k = I^m(k) = T(k)(m)$ for all $m \in \mathcal{M}$. By definition, it follows that $\hat{I}(k) = I(k) = k$, where the last equality follows from the assumption that I is normalized. This proves the step.

Step 3: $\hat{I}$ is monotone. Take $\xi, \xi' \in B(\mathcal{D}_{\mathcal{M}})$ such that $\xi \geq \xi'$. By Lemma B.9, we can find $\varphi_\xi, \varphi_{\xi'} \in B(\mathcal{A})$ such that $\varphi_\xi \geq \varphi_{\xi'}$ and $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$. Since I is monotone

$$\hat{I}(\xi) = \hat{I}(T(\varphi_\xi)) = I(\varphi_\xi) \geq I(\varphi_{\xi'}) = \hat{I}(T(\varphi_{\xi'})) = \hat{I}(\xi')$$

showing that also $\hat{I}$ is monotone.

Step 4: $\hat{I}$ is unique. Suppose there is another $\tilde{I} : B(\mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that $I(\varphi) = \tilde{I}(T(\varphi))$ for all $\varphi \in B(\mathcal{A})$. Then, take any $\xi \in B(\mathcal{D}_{\mathcal{M}})$. By Lemma B.9, there exists $\varphi_\xi \in B(\mathcal{A})$ and such that $\xi = T(\varphi_\xi)$. Then,

$$\tilde{I}(\xi) = \tilde{I}(T(\varphi_\xi)) = I(\varphi_\xi) = \hat{I}(T(\varphi_\xi)) = \hat{I}(\xi).$$

It follows that $\tilde{I} = \hat{I}$.

Step 5: $\hat{I}$ is continuous. Suppose that I is continuous. Fix any $\xi, \xi' \in B(\mathcal{D}_{\mathcal{M}})$ and $c \in \mathbb{R}$. First we show that the set $L = \{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$ is closed. If it is empty, it is closed. If it is nonempty, take any sequence $(\alpha_n)_n$ in L such that $\alpha_n \to \alpha_0$. By Lemma B.9, we can pick $\varphi, \varphi' \in B(\mathcal{A})$ such that $\xi = T(\varphi)$ and $\xi' = T(\varphi')$. Lemma B.5 implies that

$$\begin{aligned} \forall n \in \mathbb{N}, \quad & \alpha_n\xi + (1-\alpha_n)\xi' = \alpha_n T(\varphi) + (1-\alpha_n)T(\varphi') = T(\alpha_n\varphi + (1-\alpha_n)\varphi') \\ & \alpha_0\xi + (1-\alpha_0)\xi' = \alpha_0 T(\varphi) + (1-\alpha_0)T(\varphi') = T(\alpha_0\varphi + (1-\alpha_0)\varphi') \end{aligned}$$

Therefore, by definition of $\hat{I}$ and continuity of I:

$$\begin{aligned} c &\geq \liminf_n \hat{I}(\alpha_n \xi + (1-\alpha_n)\xi') \\ &= \liminf_n I(\alpha_n \varphi + (1-\alpha_n)\varphi') \\ &= I(\alpha_0 \varphi + (1-\alpha_0)\varphi') \\ &= \hat{I}(\alpha_0 \xi + (1-\alpha_0)\xi') \end{aligned}$$

and, therefore, $\alpha_0 \in \{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$, showing that this set is closed. By a symmetric argument, we can show that $\{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \geq c\}$ is also closed. Since this holds for all $\xi, \xi' \in B(\mathcal{D}_\mathcal{M})$ and $c \in \mathbb{R}$, and $\hat{I}$ is monotone by Step 3, Proposition 43 in Cerreia-Vioglio et al. (2011) implies that $\hat{I}$ is continuous.

Step 6: $\hat{I}$ is quasiconcave. Fix any $\alpha \in \mathbb{R}$. We show that the set $U_c = \{\xi \in B(\mathcal{D}_\mathcal{M}) : \hat{I}(\xi) \geq c\}$ is convex. If it is empty, this is vacuously true. Suppose it is nonempty. Take $\xi_1, \xi_2 \in U_c$ and $\alpha \in [0,1]$. By Lemma B.9, we can pick $\varphi_1, \varphi_2 \in B(\mathcal{A})$ such that $\xi_1 = T(\varphi_1)$ and $\xi_2 = T(\varphi_2)$. Notice that $I(\varphi_1) = \hat{I}(\xi_1) \geq c$ and $I(\varphi_2) = \hat{I}(\xi_2) \geq c$. Since I is quasiconcave, it follows that $I(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq c$. Moreover, Lemma B.5 implies that $T(\alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha\xi_1 + (1-\alpha)\xi_2$. Then:

$$\hat{I}(\alpha\xi_1 + (1-\alpha)\xi_2) = I(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq c$$

and, therefore, $\alpha\xi_1 + (1-\alpha)\xi_2 \in U_c$, showing convexity. Since c was arbitrarily chosen, we conclude that $\hat{I}$ is quasiconcave.

Step 7: $\hat{I}$ is monotone continuous Take $\xi, \xi' \in B(\mathcal{D}_\mathcal{M})$ and $k \in \mathbb{R}$, a monotone sequence $(D_n)_n$ in $D_\mathcal{M}$ such that $D_n \downarrow \emptyset$, and assume that $\hat{I}(\xi) > \hat{I}(\xi')$. Then, we can find $\varphi, \varphi' \in B(\mathcal{A})$ such that $\xi = T(\varphi)$ and $T(\varphi') = \xi'$. It follows that $I(\varphi) = \hat{I}(\xi) > \hat{I}(\xi') = I(\varphi')$. Let $E_n \coloneqq \mathfrak{q}^{-1}(D_n) \in \mathcal{A}$ and notice that $E_n \downarrow \emptyset$. Therefore, there exists n_0 such that $I(kE_{n_0}\varphi) > I(\varphi')$. Since $E_{n_0} \in \mathcal{A}$, for all $m \in \mathcal{M}$, $m(E_{n_0}) \in \{0,1\}$ and

$$\begin{aligned} m(E_{n_0}) = 1 &\implies kE_{n_0}\varphi = k \ m\text{-a.e.} \implies I^m(kE_{n_0}\varphi) = I^m(k) = k \\ m(E_{n_0}) = 0 &\implies kE_{n_0}\varphi = \varphi \ m\text{-a.e.} \implies I^m(kE_{n_0}\varphi) = I^m(\varphi) = \xi(m) \end{aligned}$$

Moreover, notice that $m(E_{n_0}) = 1$ if and only if $m \in D_{n_0}$ and $m(E_{n_0}) = 0$ if and only if $m \notin D_{n_0}$. Therefore, $kD_{n_0}\xi = T(kE_{n_0}\varphi)$ and we can conclude that $\hat{I}(kD_{n_0}\xi) = I(kE_{n_0}\varphi) > I(\varphi') = \hat{I}(\xi')$ as we wanted to show.

Step 8: $\hat{I}$ is translation invariant. Take $\xi \in B(\mathcal{D}_\mathcal{M})$ and $k \in \mathbb{R}$. Then, we can find $\varphi \in B(\mathcal{A})$ such that $\xi = T(\varphi)$. Since I is translation invariant and normalized, $I(\varphi + k) = I(\varphi) + k$. On the other hand, since each I^m is translation invariant,

$I^m(\varphi + k) = I^m(\varphi) + k$ for all $m \in \mathcal{M}$ and, thus, $T(\varphi + k) = T(\varphi) + k = \xi + k$. Then:

$$\hat{I}(\xi + k) = \hat{I}(T(\varphi + k)) = I(\varphi + k) = I(\varphi) + k = \hat{I}(\xi) + k$$

as we wanted to show.

(ii) implies (i).

Suppose there exists a normalized, monotone, and continuous functional $\hat{I} : B(\mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that for all $\varphi \in B(\mathcal{A})$, $I(\varphi) = \hat{I}(T(\varphi))$.

Step 1: I is normalized.

Take $k \in \mathbb{R}$. Since $\hat{I}$ is normalized, we have that $\hat{I}(k) = k$. Moreover, $T(k)(m) = I^m(k) = k$ for all $m \in \mathcal{M}$. Therefore, $I(k) = \hat{I}(T(k)) = \hat{I}(k) = k$, showing that I is normalized.

Step 2: I is monotone.

Take $\varphi, \varphi' \in B(\mathcal{A})$ such that $\varphi \geq \varphi'$. For all $m \in \mathcal{M}$, I^m is monotone and, therefore, $T(\varphi)(m) = I^m(\varphi) \geq I^m(\varphi') = T(\varphi')(m)$. But, then, since $\hat{I}$ is monotone

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi'),$$

showing that I is monotone.

Step 3: If $\varphi, \varphi' \in B(\mathcal{A})$ and $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$, then $I(\varphi) \geq I(\varphi')$.

Take any two $\varphi, \varphi' \in B(\mathcal{A})$ and assume that $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$. Then, $T(\varphi) \geq T(\varphi')$ and, therefore, since $\hat{I}$ is monotone:

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi').$$

Step 4: I is continuous. Take a sequence $(\varphi_n)_n$ in $B(\mathcal{A})$ such that $\varphi_n \to \varphi \in B(\mathcal{A})$ uniformly. Since for each $m \in \mathcal{M}$, I^m is Lipschitz continuous, it follows that for all m, $|I^m(\varphi_n) - I^m(\varphi)| \leq ||\varphi - \varphi_n||_\infty$ so that:

$$||T(\varphi_n) - T(\varphi)||_\infty \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Thus, $T(\varphi_n)$ converges uniformly to $T(\varphi)$ and by Lemma B.9, $T(\varphi_n), T(\varphi) \in B(\mathcal{D}_{\mathcal{M}})$. Therefore, by continuity of $\hat{I}$, we have that:

$$I(\varphi_n) = \hat{I}(T(\varphi_n)) \to \hat{I}(T(\varphi)) = I(\varphi)$$

showing that I is continuous.

Step 5: I is quasiconcave. Suppose $\hat{I}$ is quasiconcave. Take $\varphi_1, \varphi_2 \in B(\mathcal{A})$ and $\alpha \in [0, 1]$. Since I^m is concave, it follows that

$$I^m(\alpha\varphi_1 + (1 - \alpha)\varphi_2) \geq \alpha I^m(\varphi_1) + (1 - \alpha)I^m(\varphi_2)$$

for all $m \in \mathcal{M}$. Therefore, since $\hat{I}$ is monotone and quasiconcave,

$$\begin{aligned} I(\alpha\varphi_2 + (1-\alpha)\varphi_2) &= \hat{I}(T(\alpha\varphi_1 + (1-\alpha)\varphi_2)) \\ &\geq \hat{I}(\alpha T(\varphi_1) + (1-\alpha)T(\varphi_2)) \\ &\geq \min\{\hat{I}(T(\varphi_1)), \hat{I}(T(\varphi_2))\} = \min\{I(\varphi_1), I(\varphi_2)\} \end{aligned}$$

showing that I is quasiconcave.

Step 6: I is monotone continuous. Take $\varphi, \varphi' \in B(\mathcal{A})$ and $k \in \mathbb{R}$, a monotone sequence $(E_n)_n$ in $\mathcal{A}$ such that $E_n \downarrow \emptyset$, and assume that $I(\varphi) > I(\varphi')$. Then, $\hat{I}(T(\varphi)) = I(\varphi) > I(\varphi') = \hat{I}(T(\varphi'))$. Notice that for each $n \in \mathbb{N}$, $E_n \in \mathcal{A}$ and, therefore, $m(E_n) \in \{0,1\}$ for all $m \in \mathcal{M}$. Then, let $D_n = \{m \in \mathcal{M} : m(E_n) > \frac{1}{2}\}$ and notice that $m \in D_n$ if and only if $m(E_n) = 1$ and $m \notin D_n$ if and only if $m(E_n) = 0$. Clearly, D_n is a decreasing sequence of sets. We show that $\cap_n D_n = \emptyset$. Take any $m \in \mathcal{M}$. Since m is countably additive, by continuity of finite measures, it must be the case that $m(E_n) \to 0$. However, since $m(E_n) \in \{0,1\}$ for all $n \in \mathbb{N}$, this implies that there is a N such that $m(E_n) = 0$ for all $n > N$. This implies that $m \notin D_n$ for $n > N$ and, therefore, $m \notin \cap_n D_n$. It follows that $D_n \downarrow \emptyset$. Since $\hat{I}$ is monotone continuous, there exists a n_0 such that $\hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) > \hat{I}(T(\varphi'))$. Finally note that for all $m \in \mathcal{M}$,

$$\begin{aligned} m \in D_{n_0} &\implies m(E_{n_0}) = 1 \implies I^m(\chi_{E_{n_0}} k + \chi_{E^c_{n_0}} \varphi) = I^m(k) = k \\ m \in D^c_{n_0} &\implies m(E_{n_0}) = 0 \implies I^m(\chi_{E_{n_0}} k + \chi_{E^c_{n_0}} \varphi) = I^m(\varphi) = T(\varphi)(m). \end{aligned}$$

Hence, $T(\chi_{E_{n_0}} k + \chi_{E^c_{n_0}} \varphi) = \chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)$ and, therefore,

$$\begin{aligned} I(\chi_{E_{n_0}} k + \chi_{E^c_{n_0}} \varphi) &= \hat{I}(T((\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi)) \\ &= \hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) \\ &> \hat{I}(T(\varphi')) = I(\varphi') \end{aligned}$$

as we wanted to show.

Step 7: I is translation invariant. Take $\varphi \in B(\mathcal{A})$ and $k \in \mathbb{R}$. Since I^m is translation invariant, $I^m(\varphi + k) = I^m(\varphi) + k$ for all $m \in \mathcal{M}$. That is, $T(\varphi + k) = T(\varphi) + k$. Since $\hat{I}$ is translation invarian and normalized,

$$I(\varphi + k) = \hat{I}(T(\varphi + k)) = \hat{I}(T(\varphi) + k) = \hat{I}(T(\varphi)) + k = I(\varphi) + k$$

as we wanted to show. ■

Proof of Proposition B.1:
(i) implies (ii). Define $\hat{I} : B_0(\mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ as follows: for all $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$,

$$\hat{I}(\xi) = I(\varphi_\xi),$$

where $\varphi_\xi \in B_0(\mathcal{A})$ is chosen so that $\xi = T(\varphi_\xi)$.

Step 1: $\hat{I}$ is well-defined. Pick $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$ arbitrarily. That a $\varphi_\xi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi_\xi)$ exists follows from Lemma B.9. Moreover, suppose there are two $\varphi, \psi \in B_0(\mathcal{A})$ such that $T(\varphi)(m) = I^m(\varphi) = \xi(m) = I^m(\psi) = T(\psi)(m)$ for all $m \in \mathcal{M}$. Then, by assumption, it must be the case that $I(\varphi) = I(\psi)$, showing that $\hat{I}$ is well-defined.

Step 2: $\hat{I}$ is normalized. Take any $k \in \mathbb{R}$. Then, since each I^m is normalized, it follows that $k = I^m(k) = T(k)(m)$ for all $m \in \mathcal{M}$. By definition, it follows that $\hat{I}(k) = I(k) = k$, where the last equality follows from the assumption that I is normalized. This proves the step.

Step 3: $\hat{I}$ is monotone. Take $\xi, \xi' \in \operatorname{Im} T$ such that $\xi \geq \xi'$. By Lemma B.9, $\xi, \xi' \in T(B_0(\mathcal{A}))$ and, therefore, Lemma B.10 implies that we can find $\varphi_\xi, \varphi_{\xi'} \in B_0(\mathcal{A})$ such that $\varphi_\xi \geq \varphi_{\xi'}$ and $\xi = T(\varphi_\xi)$, $\xi' = T(\varphi_{\xi'})$. Since I is monotone

$$\hat{I}(\xi) = \hat{I}(T(\varphi_\xi)) = I(\varphi_\xi) \geq I(\varphi_{\xi'}) = \hat{I}(T(\varphi_{\xi'})) = \hat{I}(\xi')$$

showing that also $\hat{I}$ is monotone.

Step 4: $\hat{I}$ is unique. Suppose there is another $\tilde{I} : B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that $I(\varphi) = \tilde{I}(T(\varphi))$ for all $\varphi \in B_0(\mathcal{A})$. Then, take any $\xi \in B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. By Lemma B.9, there exists $\varphi_\xi \in B_0(\mathcal{A})$ and such that $\xi = T(\varphi_\xi)$. Then,

$$\tilde{I}(\xi) = \tilde{I}(T(\varphi_\xi)) = I(\varphi_\xi) = \hat{I}(T(\varphi_\xi)) = \hat{I}(\xi).$$

It follows that $\tilde{I} = \hat{I}$.

Step 5: $\hat{I}$ is continuous. Suppose that I is continuous. Fix any $\xi, \xi' \in B_0(\mathcal{D}_{\mathcal{M}})$ and $c \in \mathbb{R}$. First we show that the set $\{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$ is closed. If it is empty, it is closed. If it is nonempty, take any sequence $(\alpha_n)_n \subseteq L$ such that $\alpha_n \to \alpha_0$. By Lemma B.9, we can pick $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $\xi' = T(\varphi')$. Moreover, we can pick we a finite partition $(E_i)_{i=1}^k$ and reals $(r_i)_{i=1}^k$, $(r'_i)_{i=1}^k$ such that:

$$\varphi = \sum_{i=1}^k \chi_{E_i} r_i, \quad \varphi' = \sum_{i=1}^k \chi_{E_i} r'_i.$$

Fix any $m \in \mathcal{M}$. Then, there is a unique E_{j_m} such that $m(E_{j_m}) = 1$ and $m(E_i) = 0$ if $i \neq j_m$. Therefore, it follows that for all $n \in \mathbb{N}$,

$$I^m(\alpha_n\varphi + (1-\alpha_n)\varphi') = \alpha_n r_{j_m} + (1-\alpha_n)r'_{j_m} = \alpha_n I^m(\varphi) + (1-\alpha_n)I^m(\varphi'),$$
$$I^m(\alpha_0\varphi + (1-\alpha_0)\varphi') = \alpha_0 r_{j_m} + (1-\alpha_0)r'_{j_m} = \alpha_0 I^m(\varphi) + (1-\alpha_0)I^m(\varphi').$$

Since $m \in \mathcal{M}$ was arbitrarily chosen, it follows that:

$$\forall n \in \mathbb{N}, \quad \alpha_n\xi + (1-\alpha_n)\xi = \alpha_n T(\varphi) + (1-\alpha_n)T(\varphi') = T(\alpha_n\varphi + (1-\alpha_n)\varphi')$$
$$\alpha_0\xi + (1-\alpha_0)\xi = \alpha_0 T(\varphi) + (1-\alpha_0)T(\varphi') = T(\alpha_0\varphi + (1-\alpha_0)\varphi')$$

Therefore, by definition of $\hat{I}$ and continuity of I:

$$\begin{aligned} c &\geq \liminf_n \hat{I}(\alpha_n\xi + (1-\alpha_n)\xi') \\ &= \liminf_n I(\alpha_n\varphi + (1-\alpha_n)\varphi') \\ &= I(\alpha_0\varphi + (1-\alpha_0)\varphi') \\ &= \hat{I}(\alpha_0\xi + (1-\alpha_0)\xi') \end{aligned}$$

and, therefore, $\alpha_0 \in \{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \leq c\}$, showing that this set is closed. By a symmetric argument, we can show that $\{\alpha \in [0,1] : \hat{I}(\alpha\xi + (1-\alpha)\xi') \geq c\}$ is also closed. Siince this holds for all $\xi, \xi' \in B_0(\mathcal{D}_\mathcal{M})$ and $c \in \mathbb{R}$, and $\hat{I}$ is monotone by Step 3, Proposition 43 in Cerreia-Vioglio et al. (2011) implies that $\hat{I}$ is continuous.

Step 6: $\hat{I}$ is quasiconcave. Fix any $\alpha \in \mathbb{R}$. We show that the set $U_c = \{\xi \in B_0(\mathcal{D}_\mathcal{M}) : \xi \geq c\}$ is convex. If it is empty, this holds vacuously true. Suppose it is nonempty. Take $\xi_1, \xi_2 \in U_c$ and $\alpha \in [0,1]$. By Lemma B.9, we can pick $\varphi_1, \varphi_2 \in B_0(\mathcal{A})$ such that $\xi_1 = T(\varphi_1)$ and $\xi_2 = T(\varphi_2)$. Notice that $I(\varphi_1) = \hat{I}(\xi_1) \geq c$ and $I(\varphi_1) = \hat{I}(\xi_1) \geq c$. Since I is quasiconcave, it follows that $I(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq c$. Now, pick a partition $\{E_i\}_{i=1}^k \subseteq \mathcal{F}$ and profiles of scalars $(r_i^1)_{i=1}^k, (r_i^2)_{i=1}^k \subseteq \mathbb{R}$ such that $\varphi_1 = \sum_{i=1}^k \chi_{E_i} r_i^1$ and $\varphi_2 = \sum_{i=2}^k \chi_{E_i} r_i^2$. Fix $m \in \mathcal{M}$. Since the partition is in $\mathcal{A}$, there is a unique j_m such that $m(E_{j_m}) = 1$ and $m(E_i) = 0$ if $i \neq j_m$. Therefore,

$$I^m(\alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha r^1_{j_m} + (1-\alpha)r^2_{j_m} = \alpha I^m(\varphi_1) + (1-\alpha)I^m(\varphi_2) = \alpha\xi_1(m) + (1-\alpha)\xi_2(m)$$

Therefore, we can conclude that $T(\alpha\varphi_1 + (1-\alpha)\varphi_2) = \alpha\xi_1 + (1-\alpha)\xi_2$. Then:

$$\hat{I}(\alpha\xi_1 + (1-\alpha)\xi_2) = I(\alpha\xi_1 + (1-\alpha)\xi_2) \geq c$$

and, therefore, $\alpha\xi_1 + (1-\alpha)\xi_2 \in U_c$, showing convexity. Since c was arbitrarily chosen, we conclude that $\hat{I}$ is quasiconcave.

Step 7: $\hat{I}$ is monotone continuous Take $\xi, \xi' \in B_0(\mathcal{D}_{\mathcal{M}})$ and $k \in \mathbb{R}$, a monotone sequence $(D_n)_n \in D_{\mathcal{M}}$ such that $D_n \downarrow \emptyset$, and assume that $\hat{I}(\xi) > \hat{I}(\xi')$. Then, we can find $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $T(\varphi') = \xi'$. It follows that $I(\varphi) = \hat{I}(\xi) > \hat{I}(\xi') = I(\varphi')$. Let $E_n \coloneqq \mathfrak{q}^{-1}(D_n) \in \mathcal{A}$ and notice that $E_n \downarrow \emptyset$. Therefore, there exists n_0 such that $I(kE_{n_0}\varphi) > I(\varphi')$. Since $E_{n_0} \in \mathcal{A}$, for all $m \in \mathcal{M}$, $m(E_{n_0}) \in \{0, 1\}$ and

$$m(E_{n_0}) = 1 \implies kE_{n_0}\varphi = k \text{ a.e. } [m] \implies I^m(kE_{n_0}\varphi) = I^m(k) = k$$
$$m(E_{n_0}) = 0 \implies kE_{n_0}\varphi = \varphi \text{ a.e. } [m] \implies I^m(kE_{n_0}\varphi) = I^m(\varphi) = \xi(m)$$

Moreover, notice that $m(E_{n_0}) = 1$ if and only if $m \in D_{n_0}$ and $m(E_{n_0}) = 0$ if and only if $m \notin D_{n_0}$. Therefore, $kD_{n_0}\xi = T(kE_{n_0}\varphi)$ and we can conclude that $\hat{I}(kD_{n_0}\xi) = I(kE_{n_0}\varphi) > I(\varphi') = \hat{I}(\xi')$ as we wanted to show.

Step 8: $\hat{I}$ is translation invariant. Take $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$ and $k \in \mathbb{R}$. Then, we can find $\varphi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$. Since I is translation invariant and normalized, $I(\varphi + k) = I(\varphi) + k$. On the other hand, since each I^m is translation invariant, $I^m(\varphi + k) = I^m(\varphi) + k$ for all $m \in \mathcal{M}$ and, thus, $T(\varphi + k) = T(\varphi) + k = \xi + k$. Then:

$$\hat{I}(\xi + k) = \hat{I}(T(\varphi + k)) = I(\varphi + k) = I(\varphi) + k = \hat{I}(\xi) + k$$

as we wanted to show.

(ii) implies (i).

Suppose there exists a normalized, monotone, and continuous functional $\hat{I} : B_0(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that for all $\varphi \in B_0(\mathcal{A})$, $I(\varphi) = \hat{I}(T(\varphi))$.

Step 1: I is normalized.

Take $k \in \mathbb{R}$. Since $\hat{I}$ is normalized, we have that $\hat{I}(k) = k$. Moreover, $T(k)(m) = I^m(k) = k$ for all $m \in \mathcal{M}$. Therefore, $I(k) = \hat{I}(T(k)) = \hat{I}(k) = k$, showing that I is normalized.

Step 2: I is monotone.

Take $\varphi, \varphi' \in B_0(\mathcal{A})$ such that $\varphi \geq \varphi'$. For all $m \in \mathcal{M}$, I^m is monotone and, therefore, $T(\varphi)(m) = I^m(\varphi) \geq I^m(\varphi') = T(\varphi')(m)$. But, then, since $\hat{I}$ is monotone

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi'),$$

showing that I is monotone.

Step 3: If $\varphi, \varphi' \in B_0(\mathcal{A})$ and $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$, then $I(\varphi) \geq I(\varphi')$.

Take any two $\varphi, \varphi' \in B_0(\mathcal{A})$ and assume that $I^m(\varphi) \geq I^m(\varphi')$ for all $m \in \mathcal{M}$. Then, $T(\varphi) \geq T(\varphi')$ and, therefore, since $\hat{I}$ is monotone:

$$I(\varphi) = \hat{I}(T(\varphi)) \geq \hat{I}(T(\varphi')) = I(\varphi').$$

Step 4: I is continuous. Take a sequence $(\varphi_n)_n \subseteq B_0(\mathcal{A})$ such that $\varphi_n \to \varphi \in B_0(\mathcal{A})$ uniformly. Since for each $m \in \mathcal{M}$, I^m is Lipschitz continuous, it follows that for all m, $|I^m(\varphi_n) - I^m(\varphi)| \leq ||\varphi - \varphi_n||_\infty$ so that:

$$||T(\varphi_n) - T(\varphi)||_\infty \leq ||\varphi - \varphi_n||_\infty \to 0.$$

Thus, $T(\varphi_n)$ converges uniformly to $T(\varphi)$ and by Lemma B.9, $T(\varphi_n), T(\varphi) \in B_0(\mathcal{D}_\mathcal{M})$. Therefore, by continuity of $\hat{I}$, we have that:

$$I(\varphi_n) = \hat{I}(T(\varphi_n)) \to \hat{I}(T(\varphi)) = I(\varphi)$$

showing that I is continuous.

Step 5: I is quasiconcave. Suppose $\hat{I}$ is quasiconcave. Take $\varphi_1, \varphi_2 \in B_0(\mathcal{A})$ and $\alpha \in [0,1]$. Since I^m is concave, it follows that

$$I^m(\alpha\varphi_1 + (1-\alpha)\varphi_2) \geq \alpha I^m(\varphi_1) + (1-\alpha) I^m(\varphi_2)$$

for all $m \in \mathcal{M}$. Therefore, since $\hat{I}$ is monotone and quasiconcave,

$$\begin{aligned} I(\alpha\varphi_2 + (1-\alpha)\varphi_2) &= \hat{I}(T(\alpha\varphi_1 + (1-\alpha)\varphi_2)) \\ &\geq \hat{I}(\alpha T(\varphi_1) + (1-\alpha)T(\varphi_2)) \\ &\geq \min\{\hat{I}(T(\varphi_1)), \hat{I}(T(\varphi_2))\} = \min\{I(\varphi_1), I(\varphi_2)\} \end{aligned}$$

showing that I is quasiconcave.

Step 6: I is monotone continuous. Take $\varphi, \varphi' \in B_0(\mathcal{A})$ and $k \in \mathbb{R}$, a monotone sequence $(E_n)_n \in \mathcal{A}$ such that $E_n \downarrow \emptyset$, and assume that $I(\varphi) > I(\varphi')$. Then, $\hat{I}(T(\varphi)) = I(\varphi) > I(\varphi') = \hat{I}(T(\varphi'))$. Notice that for each $n \in \mathbb{N}$, $E_n \in \mathcal{A}$ and, therefore, $m(E_n) \in \{0,1\}$ for all $m \in \mathcal{M}$. Then, let $D_n = \{m \in \mathcal{M} : m(E_n) > \frac{1}{2}\}$ and notice that $m \in D_n$ if and only if $m(E_n) = 1$ and $m \notin D_n$ if and only if $m(E_n) = 0$. Clearly, D_n is a decreasing sequence of sets. We show that $\cap_n D_n = \emptyset$. Take any $m \in \mathcal{M}$. Since m is countably additive, by continuity of finite measures, it must be the case that $m(E_n) \to 0$. However, since $m(E_n) \in \{0,1\}$ for all $n \in \mathbb{N}$, this implies that there is a N such that $m(E_n) = 0$ for all $n > N$. This implies that $m \notin E_n$ for $n > N$ and, therefore, $m \notin \cap_n D_n$. It follows that $D_n \downarrow \emptyset$. Since $\hat{I}$ is monotone continuous, there exists a n_0 such that $\hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) > \hat{I}(T(\varphi'))$. Finally note that for all

$m \in \mathcal{M}$,

$$m \in D_{n_0} \implies m(D_{n_0}) = 1 \implies I^m(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = I^m(k) = k$$
$$m \in D^c_{n_0} \implies m(D_{n_0}) = 0 \implies I^m(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = I^m(\varphi) = T(\varphi)(m).$$

Hence, $T(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) = \chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)$ and, therefore,

$$\begin{aligned} I(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi) &= \hat{I}(T((\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} \varphi)) \\ &= \hat{I}(\chi_{D_{n_0}} k + \chi_{D^c_{n_0}} T(\varphi)) \\ &> \hat{I}(T(\varphi')) = I(\varphi') \end{aligned}$$

as we wanted to show.

Step 7: I is translation invariant. Take $\varphi \in B_0(\mathcal{A})$ and $k \in \mathbb{R}$. Since I^m is translation invariant, $I^m(\varphi + k) = I^m(\varphi) + k$ for all $m \in \mathcal{M}$. That is, $T(\varphi + k) = T(\varphi) + k$. Since $\hat{I}$ is translation invarian and normalized,

$$I(\varphi + k) = \hat{I}(T(\varphi + k)) = \hat{I}(T(\varphi) + k) = \hat{I}(T(\varphi)) + k = I(\varphi) + k$$

as we wanted to show. ■

Appendix D. Proof of the Main Results

Proof of Theorem 1: *(i) implies (ii).*

Suppose that $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and the preference relation $\succsim$ satisfies Axioms 1, 2, 3, 4, and 5. Since $\succsim$ is a non-trivial, algebraically continuous weak order satisfying independence when restricted to constant acts, we know by Herstein and Milnor (1953) that there exists an affine and non-constant function $u : X \to \mathbb{R}$ representing $\succsim$ over X. Moreover, such u is cardinally unique. Next, we show that $\operatorname{Im} u = \mathbb{R}$. Clearly, being u affine and X convex, $\operatorname{Im} u$ must be an interval. Pick $x, y \in X$ such that $x \succsim y$ and a monotonically decreasing sequence $(\alpha_n)_n \subseteq [0, 1]$ such that $\alpha_n \to 0$. Then, by unboundedness, for each $n \in \mathbb{N}$, there exists $z_n, z'_n \in X$ such that:

$$\alpha_n z_n + (1 - \alpha_n) y \succ x \succ y \succ \alpha_n z'_n + (1 - \alpha_n) x$$

Since u represents $\succsim$ on X and is affine, this implies:

$$\alpha_n u(z_n) + (1 - \alpha_n) u(y) > u(x) > u(y) > \alpha_n u(z'_n) + (1 - \alpha_n) u(x)$$

and, rearranging:

$$u(z_n) > \frac{u(x) - u(y)}{\alpha_n} + u(y) \quad \text{and} \quad u(z'_n) < -\frac{u(x) - u(y)}{\alpha_n} + u(x)$$

for all $n \in \mathbb{N}$. Therefore, $(u(z_n))_n$ and $(u(z'_n))_n$ are sequences in $\operatorname{Im} u$, the first monotonically increasing and diverging to $+\infty$, the second monotonically decreasing and diverging to $-\infty$. This implies that $\operatorname{Im} u = \mathbb{R}$.

Now, fix $m \in \mathcal{M}$ and define $\succsim^m$ as in Lemma B.4. The lemma implies that $\succsim^m |_X = \succsim|_X$. Therefore, $\succsim^m$ is represented by u when restricted to constant acts in X. Define the functional $I^m : B(\mathcal{G}) \to \mathbb{R}$ as follows: for each $\varphi \in B(\mathcal{G})$, $I^m(\varphi) = u(x_{f_\varphi})$ where $f_\varphi \in \mathcal{F}$ is chosen such that $\varphi = u(f_\varphi)$ and $x_{f_\varphi} \sim^m f_\varphi$. This functional is well-defined by Lemmas B.4 and B.2. Moreover, define $V^m(f) := I_0^m \circ u : \mathcal{F} \to \mathbb{R}$. Again, by Lemma B.2, V^m is a well-defined functional over $\mathcal{F}$. Moreover, it represents $\succsim^m$. Indeed, for any $f, f' \in \mathcal{F}$:

$$
\begin{aligned}
f \succsim^m f' &\iff x_f \succsim^m x_{f'} \\
&\iff u(x_f) \geq u(x_{f'}) \\
&\iff I^m(u(f)) \geq I^m(u(f')) \\
&\iff V^m(f) \geq V^m(f') \ .
\end{aligned}
$$

We first show that I^m is monotone, normalized, continuous, and quasiconcave.

Normalization. Take $k \in \mathbb{R}$. Since $\operatorname{Im} u = \mathbb{R}$, we can find $x^k \in X$ such that $u(x^k) = k$. Then:

$$I^m(k) = u(x^k) = k$$

showing that I^m is normalized.

Monotonicity. Take $\varphi, \psi \in B(\mathcal{G})$ and assume that $\varphi \geq \psi$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi$ and $u(f_\psi) = \psi$. Then, for all $\omega' \in \Omega$,

$$u(f_\varphi(\omega')) = u(f_\varphi)(\omega') = \varphi(\omega') \geq \psi(\omega') = u(f_\psi)(\omega') = u(f_\psi(\omega'))$$

and, therefore, $f_\varphi(\omega) \succsim^m f_\psi(\omega)$. Then, since by Lemma B.4, $\succsim^m$ satisfies monotonicity and transitivity, $f_\varphi \succsim^m f_\psi$ and, therefore, $x_{f_\varphi} \succsim^m x_{f_\psi}$. We can, thus, conclude that

$$I^m(\varphi) = u(x_{f_\varphi}) \geq u(x_{f_\psi}) = I^m(\psi)$$

which proves the claim.

Quasiconcavity. Take any $\varphi, \psi \in B(\mathcal{G})$ such that $I^m(\varphi) = I^m(\psi)$ and $\alpha \in (0,1)$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then:

$$V^m(f_\varphi) = I_0^m(u(f_\varphi)) = I_0^m(\varphi) = I_0^m(\psi) = I_0^m(u(f_\psi)) = V^m(f_\psi)$$

so that $f_\varphi \sim^m f_\psi$. Since $\succsim^m$ satisfies Axiom B.1, uncertainty aversion implies that

$$\alpha f_\varphi + (1-\alpha) f_\psi \succsim^m f_\psi$$

and, therefore:

$$\begin{aligned} I^m(\alpha\varphi + (1-\alpha)\psi) &= I^m(\alpha u(f_\varphi) + (1-\alpha)u(f_\psi)) \\ &= I^m(u(\alpha f_\varphi + (1-\alpha)f_\psi)) \\ &= V^m(\alpha f_\varphi + (1-\alpha)f_\psi) \\ &\geq V^m(f_\psi) \\ &= I^m(u(f_\psi)) = I^m(\psi) \end{aligned}$$

proving the claim.

Continuity. It follows by a routine argument.

$\varphi = \psi$ *a.e.* $[m]$ *implies* $I^m(\varphi) = I^m(\psi)$.

Take $\varphi, \psi \in B(\mathcal{G})$ and assume that $\varphi = \psi$ a.e. $[m]$. Then, we can find a set E with $m(E) = 1$ such that $\varphi(\omega) = \psi(\omega)$ for all $\omega \in E$. Take $f_\varphi, f_\psi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi$ and $u(f_\psi) = \psi$. Let $\tilde{f}_\psi = f_\varphi E f_\psi$. It is clear that $u(\tilde{f}_\psi) = \psi$ and $f_\varphi(\omega) = \tilde{f}_\psi(\omega)$ for all $\omega \in E$. Axiom 3, then, implies that $f_\varphi \sim^m \tilde{f}_\psi$ and we condlude that

$$I^m(\varphi) = I^m(u(f_\varphi)) = I^m(u(\tilde{f}_\psi)) = I(\psi)$$

as we wanted to show.

$m \mapsto I^m(\varphi)$ *is measurable for all* $\varphi \in B(\mathcal{G})$. Take any real number $r \in \mathbb{R}$. We want to show that $\{m \in \mathcal{M} : I^m(\varphi) > r\}$ is a measurable set in $\mathcal{D}_\mathcal{M}$. Since u is surjective, take x_r such that $u(x_r) = r$. Moreover, by Lemma B.2, we can pick f_φ such that $u(f_\varphi) = \varphi$. Then, we have:

$$\begin{aligned} \{m \in \mathcal{M} : I^m(\varphi) > r\} &= \{m \in \mathcal{M} : I^m(u(f_\varphi)) > u(x_r)\} \\ &= \{m \in \mathcal{M} : f_\varphi E^m x_r \succsim x_r\} \end{aligned}$$

and the latter is measurable since $\succsim$ satisfies Coherence.

We already know that $\succsim$ is represented by u when restricted to constant acts. Define the functional $I : B(\mathcal{G}) \to \mathbb{R}$ such that for each $\varphi \in B(\mathcal{G})$, $I(\varphi) \coloneqq u(x_{f_\varphi})$, where $f_\varphi \in \mathcal{F}$ is chosen so that $\varphi = u(f_\varphi)$. By Lemma B.2, such act f_φ exists for all $\varphi \in B(\mathcal{G})$, while the certainty equivalent $x_{f_\varphi} \sim f_\varphi$ exists by Lemma B.3. Moreover, for any $\varphi \in B(\mathcal{G})$, if there are two $f_\varphi, f'_\varphi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi = u(f'_\varphi)$, we then have that since u represents $\succsim$ over X,

$$\begin{aligned} u(f_\varphi)(\omega) = u(f'_\varphi)(\omega) &\implies u(f_\varphi(\omega)) = u(f'_\varphi(\omega)) \\ &\implies f_\varphi(\omega) \sim f'_\varphi(\omega) \end{aligned}$$

for all $\omega \in \Omega$. By Axiom 1.(ii) of monotonicity, it follows that $f_\varphi \sim f'_\varphi$ and, by transitivity, that $x_{f_\varphi} \sim x_{f'_\varphi}$. Therefore, we can conclude that $u(x_{f_\varphi}) = u(x_{f'_\varphi})$, showing that I is a well-defined functional on $B(\mathcal{G})$. It is easily seen that such functional is also normalized, monotone, and continuous.[18] Moreover, it is monotone continuous.

Define the function $V := I \circ u : \mathcal{F} \to \mathbb{R}$. For all $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim f' &\iff x_{f'} \succsim x_{f'} \\ &\iff V(f) = I(u(f)) = u(x_f) \geq u(x_{f'}) = I(u(f)) = V(f') \ . \end{aligned}$$

This shows that V represents $\succsim$ on $\mathcal{F}$.

Moreover, let $I_{\mathcal{A}}$ be the generalized conditional expectation as in Lemma B.6. Take now $\varphi, \psi \in B(\mathcal{G})$ such that $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then, $I^m(u(f_\varphi)) \geq I^m(u(f_\psi))$ for all $m \in \mathcal{M}$ so that $f_\varphi \succsim^m f_\psi$ for all $m \in \mathcal{M}$. Consistency implies that $f_\varphi \succsim f_\psi$. Therefore:

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) \geq V(f_\psi) = I(u(f_\psi)) \geq I(\psi) \ .$$

By this fact and since I is monotone, normalized, and continuous, by Lemma B.10, there exists a unique monotone, normalized, continuous functional $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ such that $I(\varphi) = \hat{I}(T(\varphi))$ for all $\varphi \in B(\mathcal{A})$. Moreover, since I is monotone continuous, so is $\hat{I}$. Take now any $\varphi \in B(\mathcal{G})$. Since $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$, we also know that $I^m(I_{\mathcal{A}}(\varphi)) = I^m(\varphi)$ for all $m \in \mathcal{M}$. By what shown above, $I(\varphi) = I(I_{\mathcal{A}}(\varphi))$ and, therefore,

$$I(\varphi) = I(I_{\mathcal{A}}(\varphi)) = \hat{I}(T(I_{\mathcal{A}}(\varphi))) = \hat{I}(T(\varphi)).$$

By letting $I(\cdot, m) = I^m$ for all $m \in \mathcal{M}$, we conclude that for all $f, g \in \mathcal{F}$,

$$f \succsim g \iff I(u(f)) \geq I(u(g)) \iff \hat{I}(I(u(f), \cdot)) \geq \hat{I}(I(u(g), m)).$$

Uniqueness follows by Lemma B.1 and routine arguments.

(ii) implies (i). It follows by routine arguments. ■

Proof of Proposition 1: Suppose that $\succsim_1$ and $\succsim_2$ are two misspecification averse preferences represented respectively by $(\hat{I}_1, u_1, I_1)$ and $(\hat{I}_2, u_2, I_2)$ as in Theorem 1.

Point (i). Suppose that u_1 is a positive affine transformation of u_2 and that $I_1(\cdot, m) \leq I_2(\cdot, m)$ for all $m \in \mathcal{M}$. Without loss of generality, assume that $u_1 = u_2 = u$. Fix any $m \in \mathcal{M}$ and take any $f \in \mathcal{F}$ and $x \in X$ such that $fE^m x \succsim_1^m x$. Then, $f \succsim_1^m x$ and,

[18]See for example the proof of Theorem 1 (Omnibus) in the working paper version of Cerreia-Vioglio, Maccheroni, and Marinacci (2022).

therefore, $I_1(u(f), m) \geq u(x)$. Then:

$$I_2(u(f), m) \geq I_1(u(f), m) \geq u(x)$$

so that $f \succsim_2^m x$, and, therefore, $fE^m x \succsim_2 x$.

As for the other direction, note that Equation 7 and nontriviality imply that u_2 is a positive affine transformation of u_1. Without loss of generality, set $u_1 = u_2 = u$. Fix any $m \in \mathcal{M}$ and take $\varphi \in B(\mathcal{G})$. Let $f \in \mathcal{F}$ be such that $u(f) = \varphi$ and $x \in X$ such that $f \sim_1^m x$. Then, condition 7 implies that $f \succsim_2^m x$, so that

$$I_1^m(\varphi) = I_1^m(u(f)) = u(x) \leq I_2^m(u(f)) = I_2^m(\varphi).$$

Therefore, $I_1^m(\varphi) \leq I_2^m(\varphi)$ for all $\varphi \in B(\mathcal{G})$.

Point (ii). Suppose that $u_1 = u_2 = u$ and that $\hat{I}_1 \leq \hat{I}_2$. Take any $f \in \mathcal{F}(\mathcal{A})$ and $x \in X$ and assume that $f \succsim_1 x$. Since f is measurable with respect to $\mathcal{A}$, for each $m \in \mathcal{M}$, f must be constant on E^m and, therefore coherence and normalization imply:

$$I_1(u(f), m) = I_1\left(u(f)\chi_{E^m}, m\right) = u(f|_{E^m}) = I_2\left(u(f)\chi_{E^m}, m\right) = I_2(u(f), m).$$

Then, we have that:

$$u(x) \leq \hat{I}_1\left(I_1(u(f), \cdot)\right) \leq \hat{I}_2\left(I_1(u(f), \cdot)\right) = \hat{I}_2\left(I_2(u(f), \cdot)\right)$$

so that $f \succsim_2 x$.

As for the other direction, equation (8) and nontriviality automatically imply that u_2 is a positive affine transformation of u_1. Assume that $u_1 = u_2 = u$ and take $\xi \in B(\mathcal{D}_{\mathcal{M}})$. Then, by Lemmas B.9 and B.2, there exists $f \in \mathcal{F}(\mathcal{A})$ such that $\xi = I_1(u(f), \cdot)$. By the same argument given above, it is also the case that $\xi = I_2(u(f), \cdot)$. Take $x \in X$ such that $f \sim_1 x$. Then, condition (8) implies that $f \succsim_2 x$. Therefore:

$$\hat{I}_1(\xi) = \hat{I}_1(I_1(u(f), \cdot)) = u(x) \leq \hat{I}_2(I_2(u(f), \cdot)) = \hat{I}_2(\xi).$$

Thus, $\hat{I}_1(\xi) \leq \hat{I}_2(\xi)$ for all $\xi \in B(\mathcal{D}_{\mathcal{M}})$.

■

Proof of Theorem 2: *(i) implies (ii).*

Suppose that $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and the preference relation $\succsim$ satisfies Axioms 1, 2, 3, 4, 5, and 6. By Theorem 1, we can find $\hat{I} : B(\mathcal{M}, \mathcal{D}_{\mathcal{M}}) \to \mathbb{R}$ and $I : B(\mathcal{G}) \times \mathcal{M} \to \mathbb{R}$ satisfying the properties stated in the theorem and such that $f \succsim g$ if and only if $\hat{I}(I(u(f), \cdot)) \geq \hat{I}(I(u(g), \cdot))$ and $f \succsim^m g$ if and all if $I(u(f), m) \geq I(u(g), m)$ for all $f, g \in \mathcal{F}$ and $m \in \mathcal{M}$.

Let $I^m = I(\cdot, m)$ for all $m \in \mathcal{M}$. We show that I^m is translation invariant. Take any $\varphi, \psi \in B(\mathcal{G})$ and $k, r \in \mathbb{R}$. By Lemma B.2 and surjectivity, we can find $f_\varphi, f_\psi \in \mathcal{F}$ and $x^k, x^r \in X$ such that $u(f_\varphi) = \varphi$, $u(f_\psi) = \psi$, $u(x^k) = k$, and $u(x^r) = r$. Now, for any $\alpha \in (0,1)$, since u is an affine operator, we have for each $\xi \in \{\varphi, \psi\}$, $l \in \{k, r\}$,

$$u(\alpha f_\xi + (1-\alpha)x^l) = \alpha u(f_\xi) + (1-\alpha)u(x^l) = \alpha\xi + (1-\alpha)l \ .$$

Then, applying Axiom 6,

$$\begin{aligned}
& I^m\left(\alpha\varphi + (1-\alpha)k\right) = I^m\left(\alpha\psi + (1-\alpha)k\right) \\
\implies & I^m\left(u(\alpha f_\varphi + (1-\alpha)x^k)\right) = I^m\left(u(\alpha f_\psi + (1-\alpha)x^k)\right) \\
\implies & \alpha f_\varphi + (1-\alpha)x^k \sim^m \alpha f_\psi + (1-\alpha)x^k \\
\implies & \alpha f_\varphi + (1-\alpha)x^r \sim^m \alpha f_\psi + (1-\alpha)x^r \\
\implies & I^m\left(u(\alpha f_\varphi + (1-\alpha)x^r)\right) = I^m\left(u(\alpha f_\psi + (1-\alpha)x^r)\right) \\
\implies & I^m\left(\alpha\varphi + (1-\alpha)r\right) = I^m\left(\alpha\psi + (1-\alpha)r\right) \ .
\end{aligned}$$

Then, for any $\varphi', \psi' \in B(\mathcal{G})$ and $k', r' \in \mathbb{R}$, by letting $\varphi = \varphi'/\alpha$, $\psi = \psi'/\alpha$, $k = k'/(1-\alpha)$, and $r = r'/(1-\alpha)$ in the previous implication:

$$I^m\left(\varphi' + k'\right) = I^m\left(\psi' + k'\right) \implies I^m\left(\varphi' + r'\right) = I^m\left(\psi' + r'\right) \ .$$

Then, take any $\bar{f} \in B(\mathcal{G})$ and $l \in \mathbb{R}$. Normalization implies that $I^m(\bar{f}) = I^m(I^m(\bar{f}))$. By what is shown above, this implies:

$$I^m(\bar{f} + l) = I^m(I^m(\bar{f}) + l) = I^m(\bar{f}) + l$$

proving the claim.

It follows that I^m is normalized, monotone, continuous, quasiconcave, and translation invariant. By Theorem 4 in Cerreia-Vioglio et al. (2014), it follows that I^m is a normalized and concave niveloid. Then, by Lemma 26 in Maccheroni et al. (2006), there exists a grounded, lower semicontinuous and convex function $c^m : \Delta \to [0,1]$ such that:

$$\begin{aligned}
I^m(\varphi) &= \min_{p' \in \Delta(\mathcal{G})} \left\{ \int_\Omega \varphi dp' + c^m(p') \right\} \\
c^m(p) &= \sup_{\varphi' \in B(\mathcal{G})} \left\{ I^m(\varphi') - \int_\Omega \varphi' dp \right\}
\end{aligned} \tag{14}$$

for all $\varphi \in B(\mathcal{G})$ and $p \in \Delta(\mathcal{G})$. Then, define $c(\cdot, m) \coloneqq c^m(\cdot)$ for all $m \in \mathcal{M}$. We have that for each $m \in \mathcal{M}$ and for each $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim^m f' &\iff V^m(f) \geq V^m(f') \\ &\iff I^m(u(f)) \geq I^m(u(f')) \\ &\iff \min_{p\in\Delta(\mathcal{G})} \left\{ \int_\Omega u(f)dp + c(p,m) \right\} \geq \min_{p\in\Delta(\mathcal{G})} \left\{ \int_\Omega u(f')dp + c(p,m) \right\}, \end{aligned}$$

proving the representation in (9). We only need to check that $c(\cdot, m)$ is finite only on probabilities that are absolutely continuous with respect to m. This is the content of the next lemma.

LEMMA B.11: *For all $m \in \mathcal{M}$, if $p \in \operatorname{dom} c(\cdot, m)$, then $p \ll m$ and $c(p, m) = 0$ if and only if $p = m$. In particular, c is a convex divergence.*

PROOF OF LEMMA B.11: Fix any $m \in \mathcal{M}$.

We first show that if $p \in \operatorname{dom} c(\cdot, m)$, then p is absolutely continuous with respect to m. Suppose there exists a model $m \in \mathcal{M}$ and a $\hat{p} \in \operatorname{dom} c(\cdot, m)$ that is not absolutely continuous with respect to m. We show that $\succsim$ would violate Coherence. Indeed, we can find a measurable set $E \in \mathcal{G}$ such that $m(E) = 0$ but $\hat{p}(E) > 0$. Consider the sequence of functions $(\varphi_n)_{n\in\mathbb{N}} \subseteq B(\mathcal{G})$ such that for each $n \in \mathbb{N}$, $\varphi_n = -n\chi_E$. Since $m(E) = 0$, $\varphi_n = 0$ a.e. $[m]$ and, therefore, $I^m(\varphi_n) = I^m(0) = 0$ for any $n \in \mathbb{N}$ Since $\hat{p} \in \operatorname{dom} c(\cdot, m)$, $c(\hat{p}, m) < \infty$, so that there exists $N \in \mathbb{N}$ large enough such that $c(\hat{p}, m) < N \cdot \hat{p}(E)$. Therefore,

$$\begin{aligned} I^m(\varphi_N) &= \min_{p\in\Delta} \left\{ \int_\Omega \varphi_N dp + c(p,m) \right\} \\ &= \min_{p\in\Delta} \left\{ \int_E -N \ dp + c(p,m) \right\} \\ &= \min_{p\in\Delta} \{ -N \ p(E) + c(p,m) \} \\ &\leq -N \ \hat{p}(E) + c(\hat{p},m) \\ &< 0 \end{aligned}$$

which is a contradiction. We now show that $c(p, m) = 0$ if and only if $p = m$. Let $P_0 \coloneqq \{p_0 \in \Delta(\Omega) : c(p_0, m) = 0\}$. First of all, P_0 is non-empty because $c(\cdot, m)$ is grounded. Moreover, $P_0 \subseteq \{p_0 \in \Delta(\Omega) : p_0 \ll m\}$ by what just shown above. Take $p_0 \ll m$ such that $p_0 \neq m$. Then, by Coherence there must exist $f \in \mathcal{F}$ such that $fE^m x \succsim x$, but $x \succ \int_\Omega f dp_0$. But, then,

$$\int_\Omega u(f)dp_0 + c(p_0,m) \geq \min_{p\in\Delta(\Omega)} \left\{ \int_\Omega u(f) + c(p,m) \right\} \geq u(x) > u\left(\int_\Omega u(f)dp_0 \right) = \int_\Omega u(f)dp_0$$

which implies that $c(p_0, m) > 0$. Since this holds for all $p_0 \ll m$ sich that $p_0 \neq m$, it must be the case that $\emptyset \neq P_0 \subseteq \{m\}$. That is, $c(p, m) = 0$ if and only if $p = m$.

□

Finally, as far as uniqueness, that u is cardinally unique follows from Herstein and Milnor (1953). Moreover, the uniqueness of c given u is guaranteed by the fact that $\succsim^m$ is an unbounded variational preference and Proposition 6 in Maccheroni et al. (2006). ■

Proof of Corollary 2: The result follows by Proposition 1 and noticing that for each $m \in \mathcal{M}$ and $\varphi \in B(\mathcal{G})$, equation (14) implies that $I_1(\varphi, m) \leq I_2(\varphi, m)$ if and only if $c_1(\cdot, m) \leq c_2(\cdot, m)$. ■

Proof of Proposition 2: Suppose the assumptions of the theorem are satisfied. Pick any $D \in \mathcal{D}_{\mathcal{M}}$. We want to show that there exists a $E^D \in \mathcal{G}$ such that $I^m(\chi_{E^D}) = \chi_D(m)$ for all $m \in \mathcal{M}$. By assumption, there exists $\varphi \in B_0(\mathcal{G})$ such that $0 \leq \varphi \leq 1$ and $I^m(\varphi) = \chi_D(m)$ for all $m \in \mathcal{M}$. Let $E^D = \{\omega \in \Omega : \varphi(\omega) > 0\}$ which clearly is in $\mathcal{G}$. Also notice that $\chi_D \geq \varphi$. Then,using monotonicity, if $m \in D$,

$$1 = \chi_D(m) = I^m(\varphi) \leq I^m(\chi_{E^D}) = \min_p p(E) + c(p, m) \leq m(E^D) \leq 1$$

showing that $\chi_D(m) = 1 = I^m(\chi_{E^D})$. On the other hand, suppose that $m \notin D$. Then

$$0 = \chi_D(m) = I^m(\varphi) = \min_{p \ll m} \int \varphi dp + c(p, m) = \int \varphi d\hat{p} + c(\hat{p}, m)$$

where $\hat{p}$ it the probability where the minimum in the above equation is attained. Then, since $\varphi \geq 0$ and $c(\hat{p}, m) \geq 0$, it must be the case that $\int \varphi d\hat{p} = 0$ which in turn implies that $\hat{p}(E^D) = 0$ and $c(\hat{p}, m) = 0$. Since the latter is uniquely minimized at m, it follows that $m = \hat{p}$ and, therefore, $m(E^D) = 0$. Thus,

$$I^m(\chi_{E^D}) = \min_{p \ll m} p(E^D) + c(p, m) = \min_{p \ll m} 0 + c(p, m) = 0 = \chi_D(m)$$

Since $(\Omega, \mathcal{G})$ is a standard Borel space and $\mathcal{M}$ is a measurable subset of $\mathcal{D}$, then also $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is a standard Borel space[19]. Thus, we can find a sequence $(D_n)_n \subseteq D_{\mathcal{M}}$ that separates points in $\mathcal{M}$. That is, if $m \neq m'$ for some $m, m' \in \mathcal{M}$, there exists D_n such that $m \in D_n$ and $m' \notin D_n$. By defining $\alpha_{\mathcal{M}} : \mathcal{M} \to \{0, 1\}^{\mathbb{N}}$, as $\alpha_{\mathcal{M}}(m) = (\chi_{D_n}(m))_n$, the fact that $(D_n)_n$ separates points in $\mathcal{M}$ implies that $\alpha_{\mathcal{M}}$ is injective. It is easy to see that it is also measurable. Since $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$ is a standard Borel space, the inverse $\alpha_{\mathcal{M}}^{-1} : \operatorname{Im} \alpha_{\mathcal{M}} \to \mathcal{M}$ exists and is also measurable. Furthermore, by what is shown above, for each $n \in \mathbb{N}$, we can find $E_n \in \mathcal{G}$ such that $I^m(E_n) = \chi_{D_n}(m)$ for all $m \in \mathcal{M}$.

[19] See Theorems 17.23-17.24 and Corollary 13.4 in Kechris (2012)

Define similarly $\alpha_\Omega : \Omega \to \{0,1\}^{\mathbb{N}}$ as $\alpha_\Omega(\omega) = (\chi_{E_n}(\omega))_n$, which is also a measurable function. Fix arbitrarily $m_0 \in \mathcal{M}$ and define $\mathfrak{q} : \Omega \to \mathcal{M}$ as

$$\mathfrak{q}(\omega) = \begin{cases} \alpha_{\mathcal{M}}^{-1} \circ \alpha_\Omega(\omega) & \text{if } \alpha_\Omega(\omega) \in \operatorname{Im} \alpha_{\mathcal{M}} \\ m_0 & \text{otherwise.} \end{cases}$$

Clearly, $\mathfrak{q}$ is measurable. We only need to show that $m(\mathfrak{q}^{-1}(m)) = 1$ for all $m \in \mathcal{M}$. Then, fix $m \in \mathcal{M}$. Take $n \in \mathbb{N}$ such that $m \in D_n$. Then, $\chi_{D_n}(m) = 1$ and, therefore,

$$1 \geq m(E_n) = m(E_n) + c(m,m) \geq \min_{p \ll m} p(E_n) + c(p,m) = I^m(\chi_{E_n}) = \chi_{D_n}(m) = 1$$

which implies that $m(E_n) = 1$. On the other hand, take $n \in \mathbb{N}$ such that $m \notin D_n$. Then $\chi_{D_n}(m) = 0$ and, therefore,

$$0 = \chi_{D_n}(m) = I^m(\chi_{E_n}) = \min_{p \ll m} p(E_n) + c(p,m) = \underbrace{\hat{p}(E_n)}_{\geq 0} + \underbrace{c(\hat{p},m)}_{\geq 0}$$

where $\hat{p}$ attains the minimum in the problem $\min_{p \ll m} p(E_n) + c(p,m)$. Then, $\hat{p}(E_n) = 0$ and $c(\hat{p},m) = 0$. But since $c(\cdot,m) \geq 0$ is uniquely minimized at m, it follows that $m = \hat{p}$ and, therefore, $m(E_n) = 0$ and, thereby, $m(\Omega \setminus E_n) = 1$. With this, notice that

$$\begin{aligned}
\mathfrak{q}^{-1}(m) &\supseteq \{\omega \in \Omega : \mathfrak{q}(\omega) = m\} \\
&= \{\omega \in \Omega : \alpha_\Omega(\omega) = \alpha_{\mathcal{M}}(m)\} \\
&= \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m) \text{ for all } n \in \mathbb{N}\} \\
&= \bigcap_{n \in \mathbb{N}} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \\
&= \left(\bigcap_{n : m \in D_n} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \right) \bigcap \left(\bigcap_{n : m \notin D_n} \{\omega \in \Omega : \chi_{E_n}(\omega) = \chi_{D_n}(m)\} \right) \\
&= (\cap_{n : m \in D_n} E_n) \bigcap (\cap_{n : m \notin D_n} \Omega \setminus E_n)
\end{aligned}$$

and, therefore,

$$1 \geq m\left(\mathfrak{q}^{-1}(m)\right) \geq m\left((\cap_{n : m \in D_n} E_n) \cap (\cap_{n : m \notin D_n} \Omega \setminus E_n)\right) = 1$$

implying the result.

Define $\hat{E}^m = \hat{\mathfrak{q}}^{-1}(m)$ for all m. Fix $m_0 \in \mathcal{M}$ and take $f_1, f_2 \in \mathcal{F}$ and fix any $g \in \mathcal{M}$. Note that since $m_0(\hat{E}^{m_0}) = 1$ and $m(\hat{E}^{m_0}) = 0$ whenever $m \neq m_0$, then $u(f_i \hat{E}^{m_0} g) = u(f_i)$ a.e.$[m_0]$ and $u(f_i \hat{E}^{m_0} g) = u(g)$ a.e. $[m]$ whenever $m \neq m_0$ for $i = 1,2$. This implies that $I^{m_0}(u(f_i \hat{E}^{m_0} g)) = I^{m_0}(u(f_i))$ for $i = 1,2$ and $I^m(u(f_1 \hat{E}^{m_0} g)) =$

$I^m(u(f_2\hat{E}^{m_0}g))$ for $m \neq m_0$. Then

$$\begin{aligned} I^{m_0}(u(f_1)) = I^{m_0}(u(f_2)) &\implies I(u(f_1\hat{E}^{m_0}g),\cdot) = I(u(f_2\hat{E}^{m_0}g),\cdot) \\ &\implies \hat{I}\left(I(u(f_1\hat{E}^{m_0}g)\right) = \hat{I}\left(I(u(f_2\hat{E}^{m_0}g)\right) \\ &\implies f_1\hat{E}^{m_0}g \sim f_2\hat{E}^{m_0}g \end{aligned}$$

and

$$\begin{aligned} I^{m_0}(u(f_1)) > I^{m_0}(u(f_2)) &\implies I(u(f_1\hat{E}^{m_0}g),\cdot) > I(u(f_2\hat{E}^{m_0}g),\cdot) \\ &\implies \hat{I}\left(I(u(f_1\hat{E}^{m_0}g)\right) > \hat{I}\left(I(u(f_2\hat{E}^{m_0}g)\right) \\ &\implies f_1\hat{E}^{m_0}g \succ f_2\hat{E}^{m_0}g \end{aligned}$$

This implies that $\forall f_1, f_2, g \in \mathcal{F}$, $f_1\hat{E}^{m_0}g \succsim f_1\hat{E}^{m}g$ if and only if $I^{m_0}(u(f_1)) \geq I^{m_0}(u(f_2))$ as we wanted to show. Checking the axioms is now routine. ■

Appendix E. Proof of the Special Cases

Proof of Theorem 3: *(i) implies (ii).* The result follows from Proposition B.1, Maccheroni et al. (2006), and an analogous argument to that of Theorem 1. ■

Proof of Corollary 3: The result follows from an easy application of Theorem 3, Strzalecki (2011), and standard arguments. ■

The next Lemma is key in proving Theorem 4.

Lemma B.12: *Suppose $\succsim$ is a misspecification averse preference that satisfies Axiom 9. There exist an affine and surjective $u : X \to \mathbb{R}$, a strictly increasing and concave $\phi : \mathbb{R} \to \mathbb{R}$, and $\nu \in \Delta(\Omega, \mathcal{A})$ such that for all $f, g \in \mathcal{F}_0(\mathcal{A})$,*

$$f \succsim g \iff \phi^{-1}\left(\int_\Omega \phi(u(f))d\nu\right) \geq \phi^{-1}\left(\int_\Omega \phi(u(g))d\nu\right) .$$

Moreover, ν is unique, u is unique up to positive affine transformations, and ϕ is unique up to positive affine transformations given u.

Proof of Lemma B.12: By Herstein and Milnor (1953), there exists an affine $u : X \to \mathbb{R}$ representing $\succsim$ on X. Since $\succsim$ is unbounded, the argument in the proof of Theorem 1 shows that u must be surjective. Define the binary relation $\succsim^\dagger$ on $B_0(\mathcal{A})$ as follows. For all $\varphi, \psi \in B_0(\mathcal{A})$, $\varphi \succsim^\dagger \psi$ if and only if $f_\varphi \succsim f_\psi$ for some $f_\varphi, f_\psi \in \mathcal{F}_0(\mathcal{A})$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$.

Step 0. $\succsim^\dagger$ is well-defined. That we can find $f_\varphi \in \mathcal{F}_0(\mathcal{A})$ such that $\varphi = u(f)$ for all $\varphi \in B_0(\mathcal{A})$ follows from Lemma B.2. Take $\varphi, \psi \in B_0(\mathcal{A})$ and suppose there are $f_\varphi, f'_\varphi, f_\psi, f'_\psi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi = u(f'_\varphi)$ and $u(f_\psi) = \psi = u(f'_\psi)$. Then, since u

represents $\succsim$ on X and by monotonicity,

$$[\forall \omega \in \Omega,\ u(f_\varphi(\omega)) = u(f'_\varphi(\omega))] \implies [\forall \omega \in \Omega,\ f_\varphi(\omega) \sim f'_\varphi(\omega)] \implies f_\varphi \sim f'_\varphi$$
$$[\forall \omega \in \Omega,\ u(f_\psi(\omega)) = u(f'_\psi(\omega))] \implies [\forall \omega \in \Omega,\ f_\psi(\omega) \sim f'_\psi(\omega)] \implies f_\psi \sim f'_\psi$$

We conclude that $f_\varphi \succsim f_\psi$ if and only if $f'_\varphi \succsim f'_\psi$, showing that $\succsim^\dagger$ is well-defined.

Step 1. $\succsim^\dagger$ is complete and transitive. Take $\varphi, \psi \in B_0(\mathcal{A})$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}_0(\mathcal{A})$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Since $\succsim$ is complete, either $f_\varphi \succsim f_\psi$, and then $\varphi \succsim^\dagger \psi$, or $f_\psi \succsim f_\varphi$, and then $\psi \succsim^\dagger \varphi$. This shows that $\succsim^\dagger$ is complete. As for transitivity, take $\varphi_1, \varphi_2, \varphi_3 \in B_0(\mathcal{A})$ such that $\varphi_1 \succsim^\dagger \varphi_2$ and $\varphi_2 \succsim^\dagger \varphi_3$. Then, there exist $f_i \in \mathcal{F}_0(\mathcal{A})$ such that $\varphi_i = u(f_i)$ for each $i \in \{1, 2, 3\}$ and $f_1 \succsim f_2$ and $f_2 \succsim f_3$. Since $\succsim$ is transitive, $f_1 \succsim f_3$ and, therefore, $\varphi_1 \succsim^\dagger \varphi_3$.

Step 2. For all $E \in \mathcal{A}$, E is $\succsim$-null if and only if E is $\succsim^\dagger$-null. Suppose that E is $\succsim$-null. Take any $\varphi, \psi \in B_0(\mathcal{A})$. Then, we can find f_φ, f_ψ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Since E is $\succsim$-null, $f_\varphi E f_\psi \sim f_\psi$. Finally, note that $u(f_\varphi E f_\psi) = \varphi E \psi$ and, therefore, $varphi E\psi \sim^\dagger \psi$. On the other hand, suppose that $E \in \mathcal{A}$ is $\succsim^\dagger$-null. Then, for any $fmg \in \mathcal{F}_0(\mathcal{A})$, we have that $u(fEg) = u(f)Eu(g) \sim^\dagger u(g)$. It follows that $fEg \sim g$.

Step 3. Tradeoff Consistency Take $A, E \in \mathcal{A}$ that are nonnull for $\succsim^\dagger$, $r_1, r_2, t_1, t_2 \in \mathbb{R}$, $\varphi, \psi \in B_0(\mathcal{A})$, and assume that $r_1 A\varphi \succsim^\dagger t_1 A\psi$, $r_2 Af\varphi \succsim^\dagger t_2 A\psi$, and $r_1 E\varphi \succsim^\dagger t_1 E\psi$. Take now $x_1, x_2, y_1, y_2 \in X$ such that $r_i = u(x_i)$ and $t_i = u(y_i)$ for $i \in \{1, 2\}$ and f_φ, f_ψ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. It follows that $x_1 A f_\varphi \succsim^\dagger y_1 A f_\psi$, $x_2 A f_\varphi \succsim^\dagger y_2 A f_\psi$, and $x_1 E f_\varphi \succsim^\dagger y_1 E f_\psi$. Moreover, by Step 2, we also know that E and A are nonnull for $\succsim$. By Axiom, we conclude that $x_2 E f_\varphi \succsim^\dagger y_2 E f_\psi$ and, therefore, $r_2 E\varphi \succsim^\dagger t_2 E\psi$. This proves the step.

Step 3. S-Continuity Fix a finite, measurable partition $\{E_1, \dots E_n\}$ and take $\varphi = \sum_{i=1}^n \chi_{E_i} r_i \in B_0(\mathcal{A})$, with $r_i \in \mathbb{R}$ for all $i \in \{1, \dots, n\}$. We want to show that

$$A \coloneqq \{(t_1, \dots t_n) \in \mathbb{R}^n : \sum_{i=1}^n \chi_{E_i} t_i \succsim^\dagger \varphi\} \quad \text{and} \quad B \coloneqq \{(t_1, \dots t_n) \in \mathbb{R}^n : \varphi \succsim^\dagger \sum_{i=1}^n \chi_{E_i} t_i\}$$

are closed in $\mathbb{R}^n$. First of all, since u is surjective, we can pick $z_1, \dots, z_n \in X$ such that $r_i = u(z_i)$ for all $i \in \{1, \dots, n\}$. Then, if we let $f = \sum_{i=1}^n \chi_i z_i$, we see that $f \in \mathcal{F}$ and $\varphi = u(f)$. By Lemma, we can find $z \in X$ such that $f \sim z$ and, therefore, $\varphi \sim^\dagger r_\varphi \coloneqq u(z)$. To show that A is closed, take a sequence $(t^k)_k$ in $\mathbb{R}^n$ such that $t^k \to t \in \mathbb{R}^n$ and $\sum_{i=1}^n \chi_{E_i} t_i^k \succsim^\dagger \varphi$ for all $k \in \mathbb{N}$. We want to show that $\sum_{i=1}^n \chi_{E_i} t_i \succsim^\dagger \varphi$. Since $t^k \to t$ in the product topology, the sequence $(t^k)_k$ is supnorm bounded and, therefore, we can find $l, L \in \mathbb{R}$ such that $l \leq t_i^k \leq L$ for all $i \in \{1, \dots n\}$ and $k \in \mathbb{N}$.

In particular, for each $i \in \{1, \dots n\}$ and $k \in \mathbb{N}$, we can find $\alpha_i^k \in [0,1]$ such that $t_i^k = \alpha_i^k L + (1-\alpha_i^k)l$. Similarly, we can find $\alpha \in [0,1]^n$ such that $t_i = \alpha_i L + (1-\alpha_i)l$ for all $i \in \{1, \dots, n\}$. It is easy to see that $\alpha^k \to \alpha$. Moreover, Since u is surjective, we can find $x, y \in X$ such that $u(x) = L$ and $u(y) = l$ and, since u is affine, for all $i \in \{1, \dots n\}$ it holds that

$$u(\alpha_i^k x + (1-\alpha_i^k)y) = \alpha_i^k u(x) + (1-\alpha_i^k)u(y) = \alpha_i^k L + (1-\alpha_i^k)l = t_i^k$$

for all $k \in \mathbb{N}$ and, similarly, that $t_i = \alpha_i u(x) + (1-\alpha_i)u(y)$. Then:

$$\begin{aligned} \sum_{i=1}^n \chi_{E_i} t_i^k \succsim^\dagger \varphi &\iff \sum_{i=1}^n \chi_{E_i} t_i^k \succsim^\dagger r_\varphi \\ &\iff \sum_{i=1}^n \chi_{E_i} u(\alpha_i^k x + (1-\alpha_i^k)y) \succsim^\dagger u(z) \\ &\iff \sum_{i=1}^n \chi_{E_i} [\alpha_i^k x + (1-\alpha_i^k)y] \succsim z \end{aligned}$$

for all $k \in \mathbb{N}$. By S-continuity, we conclude that $\sum_{i=1}^n \chi_{E_i}[\alpha_i x + (1-\alpha_i)y] \succsim z$ and, therefore, $\sum_{i=1}^n \chi_{E_i} t_i \succsim^\dagger r_\varphi \sim^\dagger \varphi$. It follows that A is closed. An analogous argument shows that also B is closed.

Clearly, $\mathbb{R}$ is connected and separable. Moreover, $\succsim^\dagger$ is a weak order on $B_0(\mathcal{A})$ that satisfies S-Continuity and Tradeoff Consistency. By Theorem V.3.4 in Wakker, there exists a finitely additive probability ν and a continuous function $\phi : \mathbb{R} \to \mathbb{R}$ such that $J(\varphi) = \int \phi(\varphi) d\nu$ represents $\succsim^\dagger$ on $B_0(\mathcal{A})$. Moreover, by assumption, there are two $\succsim$-nonnull, disjoint events in $\mathcal{A}$. By the previous step, these two events are also $\succsim^\dagger$-nonnull. Then, Observation V.3.4' in Wakker implies that ν is unique and ϕ is cardinally unique. Monotone continuity implies that ν is countably additive.

In particular, for all $f, g \in \mathcal{F}_0(\mathcal{A})$,

$$f \succsim g \iff u(f) \succsim^\dagger u(g) \iff \int \phi(u(f(\omega))) d\nu(\omega) \geq \int \phi(u(g(\omega))) d\nu(\omega)$$

It is evident that ϕ is strictly increasing, while concavity follows from uncertainty aversion. ■

Lemma B.13: *Suppose $(\Omega, \mathcal{G}, \mathcal{M})$ is a structured space and there exist a utility function $u : X \to \mathbb{R}$, a convex statistical distance $c : \Delta \times \mathcal{M} \to [0, \infty]$, a strictly increasing and continuous function $\phi : \operatorname{Im} u \to \mathbb{R}$ and a prior $\mu \in \Delta(\mathcal{M}, \mathcal{D}_\mathcal{M})$ such that $\succsim$ is represented on $\mathcal{F}$ by*

$$\int_\mathcal{M} \phi\left(I^m(u(f))\right) d\mu(m)$$

where I^m is defined as in (9). Then, there exists a probability measure $\nu \in \Delta(\Omega, \mathcal{A})$ such that the restriction of $\succsim$ to $\mathcal{F}(\mathcal{A})$ is represented by

$$\int_\Omega \phi(u(f))d\nu.$$

Moreover, ν is nonatomic if μ is nonatomic.

Proof of Lemma B.13: Suppose the premise holds and define the following measure: for all $A \in \mathcal{A}$,

$$\nu(A) = \int_{\mathcal{M}} m(A) d\mu(m)$$

and notice that $\nu \in \Delta(\Omega, \mathcal{A})$ and $\nu(\Omega_0) = 1$. Moreover, for all $D \in \mathcal{D}_{\mathcal{M}}$, since

$$m \in D \implies m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \geq m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}\right) = 1,$$
$$m \notin D \implies m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \leq 1 - m\left(\{\omega \in \Omega : \mathfrak{q}(\omega) = m\}\right) = 0$$

then,

$$\begin{aligned} \nu \circ \mathfrak{q}^{-1}(D) &= \nu\left(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\}\right) \\ &= \int_{\mathcal{M}} m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) \\ &= \int_D m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) + \int_{\mathcal{M}\setminus D} m(\{\omega \in \Omega : \mathfrak{q}(\omega) \in D\})\, d\mu(m) \\ &= \int_D 1 d\mu(m) = \mu(D). \end{aligned}$$

Therefore, for any $\psi \in B_0(\Omega, \mathcal{A})$, we have that

$$\begin{aligned} \int_{\mathcal{M}} \phi\left(I^m(\psi)\right) d\mu(m) &= \int_{\mathcal{M}} \phi\left(I^m(\psi)\right) d(\nu \circ \mathfrak{q}^{-1})(m) \\ &= \int_{\Omega_0} \phi\left(I^{\mathfrak{q}(\omega)}(\psi)\right) d\nu(\omega) \\ &= \int_{\Omega} \phi\left(I_{\mathcal{A}}(\psi)(\omega)\right) d\nu(\omega) \\ &= \int_{\Omega} \phi\left(\psi\right) d\nu. \end{aligned}$$

where we apply the change of variable formula and $I_{\mathcal{A}}$ is the generalized common conditional expectation of $(I^m)_{m\in\mathcal{M}}$ given $\mathcal{A}$ of Lemma B.6. It follows that for all $f, g \in \mathcal{A}$, $f \succsim f$ if and only if

$$\int_\Omega \phi(u(f))d\nu \geq \int_\Omega \phi(u(g))d\nu.$$

as we wanted to show.

Furthermore, assume that μ is notatomic. We show that also ν is non-atomic. To this end, take $E \in \mathcal{A}$ such that $\nu(A) > 0$. Then, there exists by Lemma B.9 a set

$D_E \in \mathcal{D}_\mathcal{M}$ such that $I^m(\chi_E) = \chi_{D_E}(m)$ for all $m \in \mathcal{M}$. Then,

$$\mu(D_E) = \int_\mathcal{M} \chi_{D_E}(m)d\mu(m) = \int_\mathcal{M} I^m(\chi_E)d\mu(m) = \int_\mathcal{M} m(E)d\mu(m) = \nu(E) > 0$$

where we use the fact that $m(E) \in \{0, 1\}$ for all $m \in \mathcal{M}$. Since μ is nonatomic, there exists a subset $D_0 \subseteq D_E$ in $\mathcal{D}_\mathcal{M}$ such that $0 < \mu(D_0) < \mu(D_E)$. Again by Lemma B.9, we can find $E^{D_0} \in \mathcal{A}$ such that $\chi_{D_0}(m) = I^m(\chi_{E^{D_0}})$ for all $m \in \mathcal{M}$. Then, let $E_0 \coloneqq E \cap E^{D_0} \subseteq E$. We have that for all $m \in \mathcal{M}$,

$$\chi_{D_0} = \chi_{D_0}\ \chi_{D_E} = I^m(\chi_{E^{D_0}})\ I^m(\chi_E) = I^m(\chi_{E_0})$$

where again we use Lemma B.1. Therefore,

$$\nu(E_0) = \int_\mathcal{M} m(E_0)d\mu(m) = \int_\mathcal{M} I^m(\chi_{E_0})d\mu(m) = \int_\mathcal{M} \chi_{D_0}d\mu(m) = \mu(D_0)$$

so that $0 < \nu(E_0) < \nu(E)$, proving that ν is nonatomic. ■

Proof of Theorem 4: *(i) implies (ii).*

We know that $\succsim$ is represented by u when restricted to constant acts. Define the functional $I : B(\mathcal{G}) \to \mathbb{R}$ such that for each $\varphi \in B(\mathcal{G})$, $I(\varphi) \coloneqq u(x_{f_\varphi})$, where $f_\varphi \in \mathcal{F}$ is chosen so that $\varphi = u(f_\varphi)$. By Lemma B.2, such act f_φ exists for all $\varphi \in B(\mathcal{G})$, while the certainty equivalent $x_{f_\varphi} \sim f_\varphi$ exists by Lemma B.3. Moreover, for any $\varphi \in B(\mathcal{G})$, if there are two $f_\varphi, f'_\varphi \in \mathcal{F}$ such that $u(f_\varphi) = \varphi = u(f'_\varphi)$, we then have that since u represents $\succsim$ over X,

$$\begin{aligned} u(f_\varphi)(\omega) = u(f'_\varphi)(\omega) &\implies u(f_\varphi(\omega)) = u(f'_\varphi(\omega)) \\ &\implies f_\varphi(\omega) \sim f'_\varphi(\omega) \end{aligned}$$

for all $\omega \in \Omega$. By Axiom 1.(ii) of monotonicity, it follows that $f_\varphi \sim f'_\varphi$ and, by transitivity, that $x_{f_\varphi} \sim x_{f'_\varphi}$. Therefore, we can conclude that

$$I(\varphi) = u(x_{f_\varphi}) = u(x_{f'_\varphi}) = I(\varphi)$$

showing that I is a well-defined functional on $B(\mathcal{G})$. It is easily seen that such functional is also normalized, monotone, and continuous.[20]

Define the function $V \coloneqq I \circ u : \mathcal{F} \to \mathbb{R}$. For all $f, f' \in \mathcal{F}$,

$$\begin{aligned} f \succsim f' &\iff x_f \succsim x_{f'} \\ &\iff V(f) = I(u(f)) = u(x_f) \geq u(x_{f'}) = I(u(f')) = V(f')\ . \end{aligned}$$

[20]See for example the proof of Theorem 1 (Omnibus) in the working paper version of Cerreia-Vioglio et al. (2022).

This shows that V represents $\succsim$ on $\mathcal{F}$. Moreover, by Theorem 2, for each $m \in \mathcal{M}$, $\succsim^m$ is represented by $I^m \circ u$, where $I^m : B(\mathcal{G}) \to \mathbb{R}$ is as defined in (14). Moreover, let $I_{\mathcal{A}}$ be the generalized conditional expectation from Lemma B.6. Take now $\varphi, \psi \in B_0(\mathcal{G})$ such that $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$. By Lemma B.2, we can find $f_\varphi, f_\psi \in \mathcal{F}$ such that $\varphi = u(f_\varphi)$ and $\psi = u(f_\psi)$. Then, $I^m(u(f_\varphi)) \geq I^m(u(f_\psi))$ for all $m \in \mathcal{M}$ so that $f_\varphi \succsim^m f_\psi$ for all $m \in \mathcal{M}$. Consistency implies that $f_\varphi \succsim f_\psi$. Therefore:

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) \geq V(f_\psi) = I(u(f_\psi)) \geq I(\psi) \ .$$

Moreover, by Lemma B.12, there exist an unbounded and affine $\tilde{u} : X \to \mathbb{R}$, a strictly increasing $\phi : \mathbb{R} \to \mathbb{R}$, and a probability $\nu \in \Delta(\Omega, \mathcal{A})$ such that the restriction of $\succsim$ to $\mathcal{F}_0(\mathcal{A})$ is represented by the functional:

$$f \mapsto \phi^{-1}\left(\int_\Omega \phi(\tilde{u}(f))d\nu\right) \ .$$

Moreover, since $\Omega \setminus \Omega_0$ is null, $\nu(\Omega \setminus \Omega_0) = 0$. Without loss of generality, we can assume that $\tilde{u} = u$ and normalize $\phi(0) = 0$ and $\phi(1) = 1$. Now, define the map $J : B(\mathcal{A}) \to \mathbb{R}$ such that

$$J(\varphi) = \phi^{-1}\left(\int_\Omega \phi(\varphi)d\nu\right)$$

for all $\varphi \in B(\mathcal{A})$. Since ϕ is continuous and strictly increasing, J is well-defined, normalized, and continuous. Moreover, for all $f, g \in \mathcal{F}(\mathcal{A})$,

$$f \succsim g \iff J(u(f)) \geq J(u(g)) \ .$$

Moreover, take any $\varphi \in B_0(\mathcal{A})$. By Lemma B.2, we can choose $f_\varphi \in \mathcal{F}(\mathcal{A})$ such that $\varphi = u(f_\varphi) = f_\varphi$. Then, since both V and $J \circ u$ represent $\succsim$ on $\mathcal{F}(\mathcal{A})$,

$$I(\varphi) = I(u(f_\varphi)) = V(f_\varphi) = u(x_{f_\varphi}) = J\left(u(f_\varphi)\right) = J(\varphi) \ .$$

We conclude that $I(\varphi) = J(\varphi)$ for all $\varphi \in B_0(\mathcal{A})$. Take now any $\varphi \in B_0(\mathcal{G})$. Since $I_{\mathcal{A}}(\varphi) \in B(\mathcal{A})$ and $B_0(\mathcal{A})$ is dense in $B(\mathcal{A})$, we can pick sequences $(\psi_n^l)_{n\in\mathbb{N}}, (\psi_n^u)_{n\in\mathbb{N}} \in B_0(\mathcal{A})$ such that $\psi_n^l \nearrow I_{\mathcal{A}}(\varphi)$ and $\psi_n^u \searrow I_{\mathcal{A}}(\varphi)$ uniformly. Fix any $m \in \mathcal{M}$. Since I^m is monotone, we have that for all $n \in \mathbb{N}$:

$$I^m(\psi_n^l) \leq I^m(I_{\mathcal{A}}(\varphi)) \leq I^m(\psi_n^u) \ .$$

By Proposition B.6, we also have that $I^m(I_{\mathcal{A}}(\varphi)) = I^m(\varphi)$ and, therefore, we have that for all $n \in \mathbb{N}$,

$$I^m(\psi_n^l) \leq I^m(\varphi) \leq I^m(\psi_n^u) \ .$$

Since m was chosen arbitrarily, this holds for all $m \in \mathcal{M}$. This and the fact that I and J coincide on $B_0(\mathcal{A})$ imply that for all $n \in \mathbb{N}$:

$$J(\psi_n^l) = I(\psi_n^l) \leq I(\varphi) \leq I(\psi_n^u) = J(\psi_n^u)$$

Passing to the limit and using the fact that J is continuous, we obtain that:

$$J(I_{\mathcal{A}}(\varphi)) \leq I(\varphi) \leq J(I_{\mathcal{A}}(\varphi)) \ .$$

That is:

$$\begin{aligned} I(\varphi) &= J(I_{\mathcal{A}}(\varphi)) \\ &= \phi^{-1}\left(\int_{\Omega} \phi\left(I_{\mathcal{A}}(\varphi)\right) \nu(d\tilde{\omega})\right) \\ &= \phi^{-1}\left(\int_{\Omega_0} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} \varphi \ dp + c(p, \mathfrak{q}(\tilde{\omega}))\right\}\right) \nu(d\tilde{\omega})\right) \ . \end{aligned}$$

Finally, since $\mathfrak{q}_0 = \mathfrak{q}|_{\Omega_0}$ is a measurable transformation from $(\Omega_0, \mathcal{A}_0)$ to $(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$, define the image measure $\mu \coloneqq \nu \circ \mathfrak{q}_0^{-1} \in \Delta^{\sigma}(\mathcal{M}, \mathcal{D}_{\mathcal{M}})$. Then, by Theorem 16.23 in Billingsley (1995):

$$\begin{aligned} I(\varphi) &= \phi^{-1}\left(\int_{\Omega_0} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} \varphi \ dp + c(p, \mathfrak{q}(\tilde{\omega}))\right\}\right) d\nu(\tilde{\omega})\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} \varphi \ dp + c(p, \mathfrak{q}(\tilde{\omega}))\right\}\right) d(\nu \circ \mathfrak{q}^{-1})(m)\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} \varphi \ dp + c(p, m)\right\}\right) d\mu(m)\right) \ . \end{aligned}$$

But, then, $\succsim$ is represented on $\mathcal{F}$ by

$$V(f) = I(u(f)) = \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\min_{p\in\Delta}\left\{\int_{\Omega} u(f) \ dp + c(p, m)\right\}\right) d\mu(m)\right)$$

as we wanted to show. Next, we show that if $\chi_D = T(\chi_E)$ for $E \in \mathcal{A}$ and $D \in \mathcal{D}_{\mathcal{M}}$, then $\nu(E) = \mu(D)$. Indeed,

$$\begin{aligned} \phi^{-1}(\nu(E)) &= \phi^{-1}\left(\int_{\Omega_0} \phi(\chi_E) d\nu\right) \\ &= J(\chi_{E^D}) = J(I_{\mathcal{A}}(\chi_E)) = I(\chi_E) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(I(\chi_E, m)\right) d\mu(m)\right) \\ &= \phi^{-1}\left(\int_{\mathcal{M}} \phi\left(\chi_D\right) d\mu(m)\right) \\ &= \phi^{-1}(\mu(D)). \end{aligned}$$

and since ϕ^{-1} is strictly increasing, this implies that $\nu(E) = \mu(D)$.

Uniqueness follows by standard arguments.

(ii) implies (i). It is clear that $\succsim$ satisfies Axioms 1-5 are satisfied. Moreover, there exists a probability measure $\nu \in \Delta(\Omega, \mathcal{A})$ such that the restriction of $\succsim$ to $\mathcal{F}(\mathcal{A})$ is represented by the functional

$$\int_{\Omega} \phi(u(f))d\nu.$$

This implies that $\succsim$ satisfies Wakker's axioms when restricted to $\mathcal{F}(\mathcal{A})$. ■

Proof of Theorem 5: *(i) implies (ii).* We know that there exists an affine $u : X \to \mathbb{R}$ and a normalized, monotone, continuous, and quasiconcave functional $I : B_0(\mathcal{G}) \to \mathbb{R}$ such that $\succsim$ is represented by $I \circ u$ on $\mathcal{F}$. By Theorem 2, we know that for each $m \in \mathcal{M}$, there exists I^m given as in (9) such that $I^m \circ u$ represents $\succsim^m$ on $\mathcal{F}$. By consistency, we also know that for all $\varphi, \psi \in B_0(\mathcal{G})$, $I^m(\varphi) \geq I^m(\psi)$ for all $m \in \mathcal{M}$ implies that $I(\varphi) \geq I(\psi)$. Therefore, by Proposition B.1, there exists a unique normalized, monotone, and continuous $\hat{I} : B_0(\mathcal{D}_{\mathcal{M}})$ such that $\hat{I}(I(\varphi, \cdot)) = I(\varphi)$ for all $\varphi \in B_0(\mathcal{G})$. Moreover, $\hat{I}$ is quasiconcave and monotone continuous. Take $\xi \in B_0(\mathcal{D}_{\mathcal{M}})$. By Lemma B.9, we can find a $\varphi \in B_0(\mathcal{A})$ such that $\xi = T(\varphi)$ and $f \in B_0(\mathcal{G})$ such that $\varphi = u(f)$. Notice that since there exists a K such that $\xi(m) \geq K$ for all $m \in \mathcal{M}$, $r_0 := \inf_{m\in\mathcal{M}} \xi(m) \geq K$ and, therefore, $r_0 \in \mathbb{R}$. Pick $r > r_0$. Then, we can find $x_0, x \in X$ such that $r_0 = u(x_0)$ and $r = u(x)$. Take a sequence $(\alpha_n) \in (0, 1)$ such that $\alpha_n \downarrow 0$ and let $x_n = \alpha_n x + (1 - \alpha_n)x_0$. Fix any $n \in \mathbb{N}$. By affinity of u,

$$u(x_n) = \alpha_n u(x)+(1-\alpha_n)u(x_0) = \alpha_n r+(1-\alpha_n)r_0 > r_0 = \inf_{m\in\mathcal{M}} \xi(m) = \inf_{m\in\mathcal{M}} I(u(f), m).$$

Therefore, there exists $m_n \in \mathcal{M}$ such that $u(x_n) > I(u(f), m_n)$. This implies that $x_n \succ_m f$ and, therefore, Caution implies that $x_n \succsim f$. That is,

$$\alpha_n r + (1 - \alpha_n)r_0 = u(x_n) \geq I(u(f)) = I(\varphi) = \hat{I}(\xi).$$

This holds for all $n \in \mathbb{N}$ and passing to the limit, we obtain $r_0 \geq \hat{I}(\xi)$. On the other hand, we have that for all $m \in \mathcal{M}$, $r_0 = \inf_{m'\in\mathcal{M}} \xi(m') \leq \xi(m)$ and, therefore, since $\hat{I}$ is normalized and monotone, $r_0 = \hat{I}(r_0) \leq \hat{I}(\xi)$. It follows that $\hat{I}(\xi) = r_0 = \inf_{m\in\mathcal{M}} \xi(m)$. Therefore, $\hat{I}$. Now, for all $\xi, \xi' \in B(\mathcal{D}_{\mathcal{M}})$,

$$\hat{I}(\xi) - \hat{I}(\xi') = \inf_{m\in\mathcal{M}} \xi(m) - \inf_{m\in\mathcal{M}} \xi'(m) \leq \inf_{m\in\mathcal{M}} (\xi(m) - \xi(m)).$$

Thus, $\hat{I}$ is a niveloid, and it is, therefore, Lipschitz continuous. It follows that it admits a unique, monotone, and continuous extension to $B(\mathcal{D}_{\mathcal{M}})$, wich, abusing notation, we also denote $\hat{I}$. Then, pick any $\xi \in B(\mathcal{D}_{\mathcal{M}})$. Since $B_0(\mathcal{D}_{\mathcal{M}})$ is dense in $B(\mathcal{D}_{\mathcal{M}})$, we can find two sequences $(\xi_n^u)_n, (\xi_n^l)_n$ such that $\xi_n^u \searrow \xi$ and $\xi_n^l \nearrow \xi$. Since $\hat{I}$ is monotone, we have that for all $n \in \mathbb{N}$, $\xi_n^l \le \xi \le \xi_n^u$ and, therefore,

$$\hat{I}(\xi_n^l) = \inf_{m\in\mathcal{M}} \xi_n^l(m) \le \inf_{m\in\mathcal{M}} \xi(m) \le \inf_{m\in\mathcal{M}} \xi_n^u(m) = \hat{I}(\xi_n^u).$$

Since $\hat{I}$ is continuous, passing to the limit, we obtain that $\hat{I}(\xi) = \inf_{m\in\mathcal{M}} \xi$. Therefore, we have that for all $\varphi \in B_0(\mathcal{G})$,

$$\begin{aligned}\hat{I}(I(\varphi,\cdot)) &= \inf_{m\in\mathcal{M}} \min_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + c(p,m)\\ &= \inf_{p\in\Delta(\mathcal{G})} \inf_{m\in\mathcal{M}} \int_\Omega \varphi dp + c(p,m)\\ &= \inf_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \inf_{m\in\mathcal{M}} c(p,m).\end{aligned}$$

Suppose that in addition $\mathcal{M}$ is closed and $\succsim$ satisfies. Fix any $\varphi \in B(\mathcal{G})$. Take any $r \in \mathbb{R}$ and pick $f \in \mathcal{F}$ and $x_r \in X$ such that $\varphi = u(f)$ and $r = u(x_r)$. Then:

$$\begin{aligned}\{m \in \mathcal{M} : I(\varphi, m) \le r\} &= \{m \in \mathcal{M} : I^m(u(f)) \le u(x_r)\}\\ &= \{m \in \mathcal{M} : x_r \succsim^m f\}\end{aligned}$$

and the latter is closed by axiom. Therefore, $m \mapsto I(\varphi, m)$ is lower semicontinuous. Therefore, the functional $\tilde{I}_\varphi : \Delta(\mathcal{G} \times \mathcal{M} \to \mathbb{R}$ defined as $\tilde{I}_\varphi(p,m) := I(\varphi,m) - \int_\Omega \varphi dp$ is lower semicontinuous in (p,m). Then, since

$$c(p,m) = \sup_{\varphi\in B_0(\mathcal{G})} \left\{ I(\varphi,m) - \int_\Omega \varphi dp \right\} = \sup_{\varphi\in B_0(\mathcal{G})} \tilde{I}_\varphi(p,m)$$

for all $(p,m) \in \Delta \times \mathcal{M}$ and by the theorem of the maximum (see Aliprantis and Border (2007), Lemma 17.29), we can conclude that c is lower semicontinuous in (p,m). Then applying Aliprantis and Border (2007), Lemma 17.30 twice, we obtain that $\inf_{m\in\mathcal{M}} c(\cdot,m) = \min_{m\in\mathcal{M}} c(\cdot,m)$ is lower semicontinuous and, therefore,

$$\begin{aligned}\hat{I}(I(\varphi,\cdot)) &= \inf_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \inf_{m\in\mathcal{M}} c(p,m)\\ &= \min_{p\in\Delta(\mathcal{G})} \int_\Omega \varphi dp + \min_{m\in\mathcal{M}} c(p,m).\end{aligned}$$

Since $\varphi \in B_0(\mathcal{G})$ was arbitrarily chosen, we conclude that this holds everywhere on $B_0(\mathcal{G})$. Therefore, for all $f, g \in \mathcal{F}$,

$$\begin{aligned}f \succsim g &\iff I(u(f)) \ge I(u(g))\\ &\iff \hat{I}(I(u(f),\cdot)) \ge \hat{I}(I(u(g),\cdot))\\ &\iff \min_{p\in\Delta(\mathcal{G})} \int_\Omega u(f)dp + \min_{m\in\mathcal{M}} c(p,m) \ge \min_{p\in\Delta(\mathcal{G})} \int_\Omega u(g)dp + \min_{m\in\mathcal{M}} c(p,m)\end{aligned}$$

as we wanted to show. ■

Appendix F. Proofs for the Application to Robust MD

Proof of Proposition 3: For all $p \geq 0$, define the function

$$\Phi_p(z, \alpha) = pz + \lambda\eta\left((1-z)\log\frac{(1-z)}{\alpha} + z\log\frac{z}{1-\alpha}\right) + \lambda(\gamma - p)\left[\frac{1}{2} - z\right]^+$$

for all $z \in [0, 1]$ and $\alpha \in [0, 1/2]$.

Pick any $0 < p \leq \gamma$ and consider $m \in \mathcal{M}$. Let $A = [0, p)$. Since γ is a median of m, we have that $m([\gamma, \infty)) \geq 1/2$ and, therefore,

$$m(A) = 1 - m([p, \infty)) \leq 1 - m([\gamma, \infty)) \leq \frac{1}{2}$$

Put $\alpha_m = m(A)$ and note that $\alpha_m \in [0, 1/2]$. Fix $q \ll m$ and put $z_q = q(A^c)$. By the data-processing inequality, we have

$$\begin{aligned} R(q||m) &\geq R(q|_{\sigma(\{A\})}||m_{\sigma(\{A\})}) \\ &= q(A)\log\frac{q(A)}{m(A)} + q(A^c)\log\frac{q(A^c)}{m(A^c)} \\ &= (1-z_q)\log\frac{1-z_q}{\alpha_m} + z_q\log\frac{z_q}{1-\alpha_m} \end{aligned}$$

On the other hand,

$$\begin{aligned} W_1(q, m) &= \inf_{\pi \in \Pi(q,m)} \int |\omega_1 - \omega_2| d\pi(\omega_1, \omega_2) \\ &\geq \inf_{\pi \in \Pi(q,m)} \int_{\omega_1 \leq p, \omega_2 \geq \gamma} |\omega_1 - \omega_2| d\pi(\omega_1, \omega_2) \\ &\geq \inf_{\pi \in \Pi(q,m)} \int_{\omega_1 \leq p, \omega_2 \geq \gamma} (\gamma - p) d\pi(\omega_1, \omega_2) \\ &= (\gamma - p) \inf_{\pi \in \Pi(q,m)} \pi(\{\omega_1 \leq p, \omega_2 \geq \gamma\}) \end{aligned}$$

where recall that $\gamma - p \geq 0$. Moreover, for any $\pi \in \Pi(q, m)$, we have

$$\begin{aligned} \pi(\{\omega_1 \leq p, \omega_2 \geq \gamma\}) &= \pi(\{\omega_2 \geq \gamma\}) - \pi(\{\omega_2 \geq \gamma\} \cap \{\omega_1 > p\}) \\ &\geq \pi(\{\omega_2 \geq \gamma\}) - \pi(\{\omega_1 \geq p\}) \\ &= m([\gamma, \infty)) - q([p, \infty)) \\ &\geq \frac{1}{2} - z_q \end{aligned}$$

and, therefore,

$$W_1(q,m) \geq (\gamma - p)\left[\frac{1}{2} - z_q\right]^+$$

Putting the pieces together, we have that:

$$\begin{aligned} pq(A_p^c) + \lambda c_\eta(q,m) &\geq pq(A_p^c) + \lambda\eta\left(q(A_p)\log\frac{q(A_p)}{m(A_p)} + q(A_p^c)\log\frac{q(A_p^c)}{m(A_p^c)}\right) + \lambda\left(\gamma - p\right)\left[\frac{1}{2} - q(A_p^c)\right]^+ \\ &\geq \inf_{\alpha\in[0,\frac{1}{2}]}\inf_{z\in[0,1]}\Phi(z,\alpha) \end{aligned}$$

Let $\hat{\Phi}_p(\alpha) = \min_{z\in[0,1]}\Phi_p(z,\alpha)$. For each given $\alpha \in (0,\frac{1}{2})$, $\Phi_p(z,\alpha)$ is convex in z and strictly convex on both $(0,\frac{1}{2})$ and on $(\frac{1}{2},1)$. It follows that the problem admits a unique minimizer $\hat{z}(\alpha)$ for all $\alpha \in (0,\frac{1}{2})$ which is either equal to $\frac{1}{2}$ or interior in either $(0,\frac{1}{2}$ or in $(\frac{1}{2},1)$. Now, consider $\alpha \in (0,\alpha)$. If $\hat{z}(\alpha) \in (0,\frac{1}{2}]$, then $\hat{z}(\alpha) \leq \frac{1}{2} < 1-\alpha$. If, instead, $\hat{z}(\alpha) \in (\frac{1}{2},1)$, it needs to satisfy the FOC:

$$\begin{aligned} \frac{\partial\Phi_p(\hat{z}(\alpha),\alpha)}{\partial z} &= p + \lambda\eta\left[\log\frac{\hat{z}(\alpha)}{1-\hat{z}(\alpha)} + \log\frac{\alpha}{1-\alpha}\right] = 0 \\ \implies \frac{\hat{z}(\alpha)}{1-\hat{z}(\alpha)} &= \exp\left\{-\frac{p}{\lambda\eta} - \log\frac{\alpha}{1-\alpha}\right\} = e^{-\frac{p}{\lambda\eta}}\,\frac{1-\alpha}{\alpha} < \frac{1-\alpha}{\alpha} \\ \implies \hat{z}(\alpha) &< 1-\alpha \end{aligned}$$

We conclude that $\hat{z}(\alpha) < 1-\alpha$ for all $\alpha \in (0,\frac{1}{2})$. Now, for $\alpha \in (0,\frac{1}{2})$, applying the envelope theorem we obtain that:

$$\frac{d\hat{\Phi}_p(\alpha)}{d\alpha} = \frac{\partial\Phi_p(\hat{z}(\alpha),\alpha)}{\partial\alpha} = \frac{\hat{z}(\alpha) - (1-\alpha)}{\alpha(1-\alpha)} < 0$$

Moreover, for $\alpha = 0$, we have that $\Phi_p(z,0) = +\infty$ if $z \neq 1$ and $\Phi_p(z,0) = p$ if $z = 1$. It follows that $\hat{z}(\alpha) = 1$. Moreover, for all $\alpha \in (0,\frac{1}{2})$,

$$0 \leq \hat{\Phi}_p(\alpha) \leq \Phi_p(1,\alpha) = p - \log(1-\alpha)$$

and, therefore, taking limits $\lim_{\alpha\downarrow 0}\hat{\Phi}_p(\alpha) \leq p$. We conclude that

$$\inf_{\alpha\in[0,\frac{1}{2}]}\inf_{z\in[0,1]}\Phi_p(z,\alpha) = \inf_{\alpha\in[0,\frac{1}{2}]}\hat{\Phi}(\alpha) = \hat{\Phi}\left(\frac{1}{2}\right) = \min_{z\in[0,1]}\Phi_p\left(z,\frac{1}{2}\right) = \min_{z\in[0,\frac{1}{2}]}\Phi_p\left(z,\frac{1}{2}\right)$$

where the last equality follows from the fact that we showed above that $\hat{z}(\frac{1}{2}) \leq 1-\frac{1}{2} = \frac{1}{2}$. We conclude that

$$pq(A_p^c) + \lambda c_\eta(q,m) \geq \min_{z\in[0,\frac{1}{2}]}\Phi_p\left(z,\frac{1}{2}\right)$$

Since m and $q \ll m$ were chosen arbitrarily, this holds for all $m \in \mathcal{M}$ and $q \ll m$ and, therefore, we conclude that

$$V(f_p) = \inf_{m \in \mathcal{M}} \inf_{q \ll m} pq(A_p^c) + \lambda c_\eta(q,m) \geq \min_{z \in [0,\frac{1}{2}]} \Phi_p\left(z, \frac{1}{2}\right)$$

Moreover, fix any $z \in [0, \frac{1}{2}]$ and take $m_\varepsilon = \frac{1}{2}\delta_{p-\varepsilon} + \frac{1}{2}\delta_\gamma$ and $q_\varepsilon = (1-z)\delta_{p-\varepsilon} + z\delta_\gamma$ for $0 < \varepsilon < p$. Then:

$$R(q_\varepsilon, m_\varepsilon) = (1-z)\log\frac{1-z}{1/2} + z\log\frac{z}{1/2}$$

and

$$W_1(q_\varepsilon, m_\varepsilon) = \int |F_{q_\varepsilon}(t) - F_{m_\varepsilon}(t)|dt = \left(\frac{1}{2} - z\right)(\gamma - p + \varepsilon)$$

so that

$$V(f_p) \leq pz + \lambda\left[\left(\frac{1}{2} - z\right)(\gamma - p + \varepsilon)\right] + \lambda\eta\left[(1-z)\log\frac{1-z}{1/2} + z\log\frac{z}{1/2}\right]$$

Taking limits as $\varepsilon \downarrow 0$, we ge that

$$V(f_p) \leq \Phi_p(z, \alpha) \leq \min_{z \in [0,\frac{1}{2}]} \Phi_p\left(z, \frac{1}{2}\right)$$

showing that, indeed,

$$V(f_p) = \min_{z \in [0,\frac{1}{2}]} \Phi_p\left(z, \frac{1}{2}\right)$$

where recall that

$$\begin{aligned}
\Phi_p\left(z, \frac{1}{2}\right) &= pz + \lambda\eta\left[(1-z)\log 2(1-z) + z\log 2z\right] + \lambda(\gamma - p)\left(\frac{1}{2} - z\right) \\
&= \left[(1+\lambda)p - \lambda\gamma\right]z + \lambda\eta\left[(1-z)\log(1-z) + z\log z\right] + \lambda\eta\log 2 + \lambda\frac{\gamma - p}{2}.
\end{aligned}$$

Now,

$$\frac{\partial\Phi_p(z, 1/2)}{\partial z} = (1+\lambda)p - \lambda\gamma + \lambda\eta\log\frac{z}{1-z}$$

and, therefore,

$$\frac{\partial\Phi_p(z, 1/2)}{\partial z} < 0 \iff z < \frac{1}{1 + \exp\{\frac{(1+\lambda)p - \lambda\gamma}{\lambda\eta}\}}$$

Notice that the RHS is above 1/2 if and only if

$$\frac{(1+\lambda)p - \lambda\gamma}{\lambda\eta} \leq 0 \iff p \leq \frac{\lambda}{1+\lambda}\gamma$$

Thus, we have two cases:

$$p \leq \frac{\lambda}{1+\lambda}\gamma \implies \hat{z} = \frac{1}{2}$$
$$p > \frac{\lambda}{1+\lambda}\gamma \implies \hat{z} = \frac{1}{1+\exp\{\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}\}}$$

For the second case, since FOC must hold, we have that $(1+\lambda)p - \lambda\gamma = \lambda\eta \log \frac{1-\hat{z}}{\hat{z}}$ and, therefore,

$$\begin{aligned}
[(1+\lambda)p-\lambda\gamma]\,\hat{z} + \lambda\eta\,[(1-\hat{z})\log(1-\hat{z}) + \hat{z}\log\hat{z}] &= \lambda\eta\left[\hat{z}\log\frac{1-\hat{z}}{\hat{z}} + (1-\hat{z})\log(1-\hat{z}) + \hat{z}\log\hat{z}\right] \\
&= \lambda\eta\log(1-\hat{z}) \\
&= \lambda\eta\log\frac{1}{1+\exp\{-\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}\}} \\
&= -\lambda\eta\log\left(1+e^{-\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}}\right)
\end{aligned}$$

and, therefore,

$$V(f_p) = \begin{cases} \frac{p}{2} & \text{if } 0 < p \leq \frac{\lambda}{1+\lambda}\gamma \\ \lambda\eta\left[\log 2 - \log\left(1+e^{-\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}}\right)\right] + \lambda\frac{\gamma-p}{2} & \text{if } \frac{\lambda}{1+\lambda}\gamma < p \leq \gamma \end{cases}$$

Note first that for any $p \geq 0$, we have that for any $m \in \Updownarrow$ and $q \ll m$,

$$pq(\omega \geq p) + c_\eta(q,m) \geq 0$$

so that $V(f_p) \geq 0$. If $p = 0$, choose for example $m_\gamma = q_\gamma = \frac{1}{2}\delta_0 + \frac{1}{2}\delta_\gamma$ and notice that

$$0q_\gamma(\omega \geq p) + c_\eta(q_\gamma, m_\gamma) = 0$$

so that we can conclude that $V(f_0) = 0$. Similarly, for $p > \gamma$,

$$pq(\omega \geq q_\gamma) + c_\eta(q_\gamma, m_\gamma) = 0$$

and, therefore, $V(f_p) = 0$ for all $p > \gamma$.

We conclude that

$$V(f_p) = \begin{cases} \frac{p}{2} & \text{if } 0 \leq p \leq \frac{\lambda}{1+\lambda}\gamma \\ \lambda\eta\left[\log 2 - \log\left(1+e^{-\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}}\right)\right] + \lambda\frac{\gamma-p}{2} & \text{if } \frac{\lambda}{1+\lambda}\gamma < p \leq \gamma \\ 0 & \text{if } p > \gamma \end{cases}$$

We can, therefore, restrict attention to $p \in [0, \gamma]$. First of all, take the derivative for $\frac{\lambda}{1+\lambda}\gamma < p < \gamma$:

$$\frac{\partial V(f_p)}{\partial p} = \frac{1+\lambda}{1+e^{\frac{(1+\lambda)p-\lambda\gamma}{\lambda\eta}}} - \frac{\lambda}{2}$$

which is positive if and only if

$$p \leq \frac{\lambda\gamma + \lambda\eta\log(1+2/\lambda)}{1+\lambda}$$

Notice that

$$\frac{\lambda\gamma + \lambda\eta\log(1+2/\lambda)}{1+\lambda} < \gamma \iff \lambda\eta\log(1+2/\lambda) < \gamma$$

Moreover, since $\lambda\log(1+2/\lambda) < 2$ for all $\lambda > 0$ and $\eta < \gamma/2$, it is, indeed, the case that $\lambda\eta\log(1+2/\lambda) < \gamma$ and, therefore, conclude that the solution is

$$\hat{p}(\lambda) = \frac{\lambda\gamma + \lambda\eta\log(1+2/\lambda)}{1+\lambda} \in (0, \gamma).$$

Moreover:

$$\begin{aligned} \frac{d\hat{p}(\lambda)}{d\lambda} &= \frac{\left(\gamma + \eta\log(1+2/\lambda) - \frac{2\eta}{2+\lambda}\right)(1+\lambda) - (\lambda\gamma + \lambda\eta\log(1+2/\lambda))}{(1+\lambda)^2} \\ &= \frac{\gamma + \eta\log(1+2/\lambda) - \eta\frac{2+2\lambda}{2+\lambda}}{(1+\lambda)^2} \end{aligned}$$

Let $\psi(\lambda) = \gamma + \eta\log(1+2/\lambda) - \eta\frac{2+2\lambda}{2+\lambda}$ and notice that

$$\frac{d\psi(\lambda)}{d\lambda} = -\frac{2\eta}{\lambda+\lambda^2} - \frac{2\eta}{(2+\lambda)^2} < 0$$

for all $\lambda > 0$ and that

$$\lim_{\lambda\to 0}\psi(\lambda) = \infty \quad \text{and} \quad \lim_{\lambda\to\infty}\psi(\lambda) = \gamma - 2\eta > 0$$

It follows that $\frac{d\hat{p}(\lambda)}{d\lambda} > 0$ for all $\lambda > 0$. Finally,

$$\lim_{\lambda\to 0}\hat{p}(\lambda) = 0 \quad \text{and} \quad \lim_{\lambda\to\infty}\hat{p}(\lambda) = \gamma.$$

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