Questions

Any questions to exercises and homeworks from last time?

Table of Content

The Fast Fourier Transform

- The FFT is an algorithm to compute the discrete time Fourier transform (DTFT) efficiently.
- The DTFT is the Fourier transform for equally spaced sampled complex data (in time and frequency domain)
- The DTFT is often used to find an approximation to the power spectral density

$$P(\psi) = \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp(-\mathrm{j}2\pi\psi n) \right|^2$$

at discrete sampling points k in the frequecy domain

$$P[k] = \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(-\mathrm{j}2\pi \frac{k\, n}{N}\right) \right|^2 \quad \text{for} \quad k = 0, \ldots, N-1$$

The Fast Fourier Transform in Python

- Implementation of the FFT in libraries may differ in pre-factor (1, $\frac{1}{N}$, $\frac{1}{2\pi}$, or square-roots of them), exponent ($\pm 2\pi$ or ω), and start index (0 or $-\frac{N}{2}$), and the lacking of the absolute value.
- In Python, the FFT is usually calculated as

$$X[k] = \sum_{n=0}^{N-1} x[n] \exp\left(-2\pi \mathrm{j} \frac{n\,k}{N}\right) \qquad k = 0, \ldots, N-1$$

  which is just the sum-term of the power spectrum $P[k]$.
- Symmetric Fourier transform (start index at $-N/2$):

$$X_{\text{sym}} = X[k] \exp\left(-2\pi \mathrm{j} \frac{k}{N} \frac{N-1}{2}\right)$$

- The FFT is not actually implemented in Python due to speed.
- FFT modules in Python just link to a compiled library.

FFT Libraries in Python

- Most common wrapper modules for the FFT are:
  - `numpy.fft` https://docs.scipy.org/doc/numpy-1.15.0/reference/routines.fft.html
  - `scipy.fftpack` https://docs.scipy.org/doc/scipy/reference/fftpack.html
  - `pyfftw` https://github.com/pyFFTW/pyFFTW
- Functions provided by those modules have nearly the same arguments, thus modules are interchangeable.
- Speed of the FFT can vary heavily with
  - utilized library (optimizations and multithreading),
  - data type (single, or double precision), and
  - number of samples (different algorithms perform differently)
- `scipy` and `numpy` work out of the box and try to use Intel's Math Kernel Library (MKL).
- FFTW is usually the fastest but needs extra steps on Windows, as Windows does not ship with a C compiler.

Partial Spectrum DFT Algorithms in Python

Goertzel-Algorithm

- To calculate the DFT for an arbitrary list of frequencies.
- Faster than FFT for low number of desired frequencies.
- No module in python but implementation is trivial.

Chirp-z Transform

- Also called chirp-transform algorithm (CTA), or zoom-FFT
- Based on Bluestein FFT (for arbitrary data length)
- CZT didn't made it into scipy/numpy due to numerical problems with corner cases.

  `https://github.com/scipy/scipy/issues/4288`
- Available at `https://gist.github.com/ewmoore/6d321c1dbb5cea9bfaeb`

Windowing

- There is no unwindowed FFT!

$$P_{\mathrm{rect}}[k] = \left| \frac{1}{N} \sum_{n=0}^{N-1} x[n] \exp\left(-\mathrm{j}2\pi \frac{n\,k}{N} \right) \right|^2$$

$$P_{\mathrm{wind}}[k] = \left| \frac{1}{\sum_{n=0}^{N-1} a[n]} \sum_{n=0}^{N-1} a[n] x[n] \exp\left(-\mathrm{j}2\pi \frac{n\,k}{N} \right) \right|^2$$

- Different windows produce different the PSF.
- Allows to trade beam-width for side-lobe level, e.g., increases dynamic range but reduces resolution.
- Note that the sum can still be evaluated using an FFT library!

Windowing using Python

- Three modules provide window functions:
  - `numpy`
  - `scipy.signal.windows`
  - `spectrum.window`

- `numpy` provides some basic window functions with

  `np.bartlett`, `np.blackman`, `np.hamming`, `np.hanning`, `np.kaiser`

```
import numpy as np
wind=np.hanning(128)
```

- `scipy.signal.windows` provide most commonly used window functions with the option for symmetric windows.

  https://docs.scipy.org/doc/scipy/reference/signal.windows.html

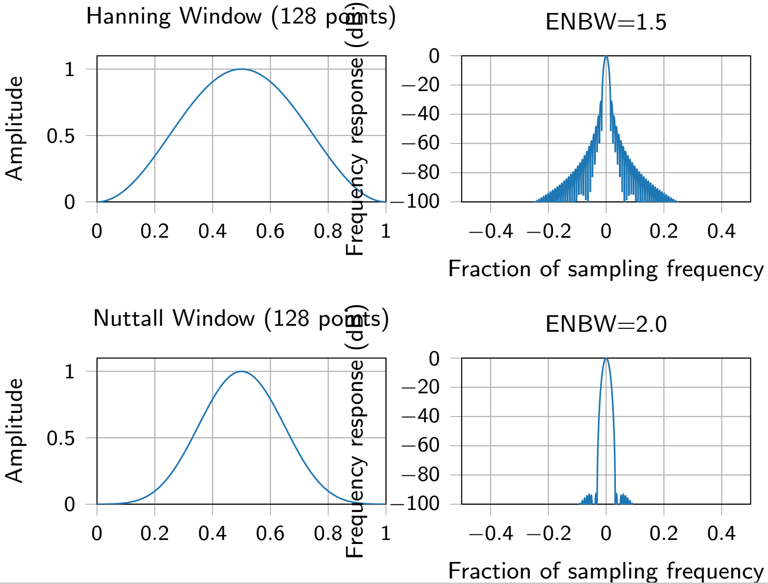
```
import scipy.signal.windows as windows
wind=windows.nuttall(N, sym=True)
```

Windowing using Python

- `spectrum.window` provides many tools for spectrum estimation, like functions for generating and analysis of a wider range of window functions. https://github.com/cokelaer/spectrum

```python
import spectrum
import matplotlib.pyplot as plt
from matplotlib2tikz import save as tikz_save

# generate the samples
wind_samples=spectrum.window.window_nuttall(128)

# generate window object
for name in ('hanning', 'nuttall'):
    wind=spectrum.Window(128,name)      # generate a window object
    wind_samples=wind.data              # get the data

    plt.figure()
    wind.plot_time_freq()               # plot the window function

    tikz_save('Window_'+name+'.tikz');
    plt.savefig('Window_'+name+'.png', dpi=150)
```

Windowing using Python



Sums of Sinusoidal Signals With Normalized Frequency

- Sum of M sinusoidal signal at sampling indicess n:

$$s_{\text{real}}[n] = \sum_{m=0}^{M} A_m \sin\left(2\pi\psi_m n + \phi_m\right)$$

$$s_{\text{analytic}}[n] = \sum_{m=0}^{M} A_m \exp(\mathrm{j}2\pi\psi_m n + \mathrm{j}\phi_m)$$

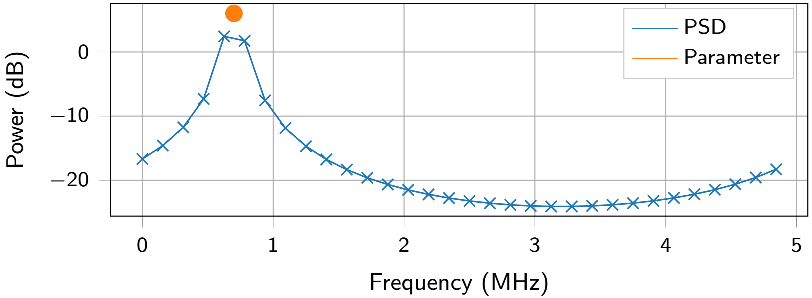
- Normalized notations to become independent of sample rate.
- Here: $\psi_m = f_m / T_\mathrm{s}$ with

ψ_m	normalized frequency
f_m	frequency in Hz
T_s	sampling frequency in Hz

- $\psi_m = 0$ is DC, $\psi_m = 0.5$ is half the sampling frequency (cf. Nyquist–Shannon sampling theorem)
- Note: $\sin x = \frac{1}{2\mathrm{j}}\left(\exp(\mathrm{j}x) - \exp(-\mathrm{j}x)\right)$

Exercise: Visualization of the FFT Spectrum

- Consider 32 samples of a complex signal taken at a sampling rate of 5 MHz with $A = 2$, $\psi = 0.14$, and $\phi = 0$.
- Estimate and plot the power spectrum using the FFT.
  - Plot in dB over MHz
  - What happens if ψ is a multiple of $1/N$, e.g, $\psi = 0.125$?
  - Hint: take a look at the keyword parameter `endpoint` if you use linspace to construct the frequency axis.



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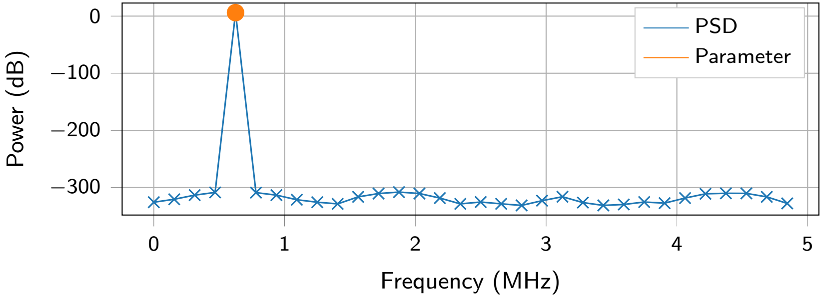


Homework: Visualization of the FFT Spectrum

- Consider 128 samples of a real signal taken at a sampling rate of 2 MHz, containing three sinusoidals with

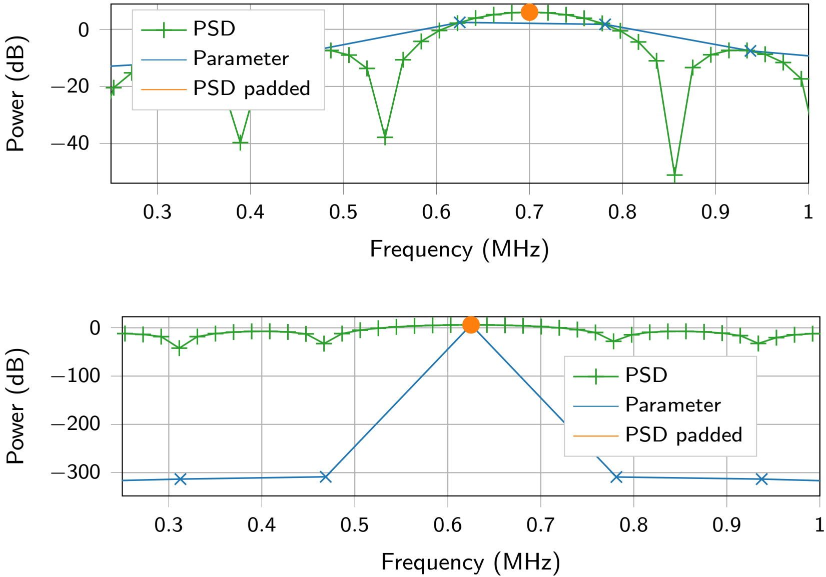
```
Am   = [1.0, 0.5, 1e-4, 0.001];
psim = [0.1, 0.1+2.5/N, 0.17, 0.21];
phim = [0.0, 0.0, 2.0, 0.0];
```

- Estimate the power spectrum by using the FFT.
  - Plot it in dB (normalized to `Am[0]`) over frequency in MHz.
  - Use `xlim` and `ylim` to zoom the plot to the interesting region 0.1 to 0.5 MHz and -100 to 10 dB.
  - Mark the magnitudes and frequencies in the spectrum plot.
- Plot the power spectrum for three different window functions: rect, hanning and nuttall.
- The ideal Fourier transform would produce 4 peaks. Explain:
  - Why (in terms of shape of the main- and sidelobes) peaks are not visible/distinct for certain window functions?
  - Why do the tips of the DFT/FFT peaks not meet the sinusoidal parameters?

Sinusoidal Parameter Estimation Using the FFT

- Idea: Location of the maximum in the power spectrum corresponds to the parameters of the strongest sinusoid.
- Zero padding
  - Padding the signal with zeros prior the FFT produces a finer spectral grid.
  - The function call to `FFT` has a parameter for zero padding. NO manual concatenate needed.
  - The pulse-spread-function does not change!
  - Thus, discretization errors are reduced, resolution is not increased, however memory consumption is increased.
- AWGN introduces additional errors. Similar to the AWGN peak detection from the correlation exercise.

Zero-Padding Example



Parabola Fit

- The frequency discretization of the DTFT is—one of many—sources for an inaccurate frequency estimation using the argmax method.
- The main lobe of the PSD (squared magnitude spectrum) is well approximated by a 2nd-order polynomial for most window functions.

$$P(\psi) = a_0 + a_1\psi + a_2\psi^2$$

- The location of the maximum of the 2nd polynomial can be found analytical without discretization.

$$0 = \left.\frac{\mathrm{d}P(\psi)}{\mathrm{d}\psi}\right|_{\psi=\psi_{\mathrm{max}}} \quad \Rightarrow \quad \psi_{\mathrm{max}} = -\frac{a_1}{2a_2} \quad F_{\mathrm{max}} = a_0 - \frac{a_1^2}{4a_2}$$

- No parabola/linear fit for phase due to wrapping on 2π. Use DFT/goertzel ones ψ_{max} found.

Calculate Parameters of the Polynomial

- Tricks for derivation:
  - center the system of coordinates around 0 frequency
  - FFT bins are all equally spaced by d_f, i.e $\psi \in \{0, -d_\mathrm{f}, d_\mathrm{f}\}$

$$\begin{pmatrix} P_\mathrm{L} \\ P_\mathrm{C} \\ P_\mathrm{R} \end{pmatrix} = \begin{pmatrix} 1 & -d_\mathrm{f} & d_\mathrm{f}^2 \\ 1 & 0 & 0 \\ 1 & d_\mathrm{f} & d_\mathrm{f}^2 \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

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- Polynomial Parameters:

$$a_0 = P_\mathrm{C} \qquad a_1 = \frac{P_\mathrm{R} - P_\mathrm{L}}{2d_\mathrm{f}} \qquad a_2 = \frac{P_\mathrm{L} - 2P_\mathrm{C} + P_\mathrm{R}}{2d_\mathrm{f}^2}$$

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- Result:

$$\psi_\text{offset} = -\frac{a_1}{2a_2} = \frac{d_\mathrm{f}(P_\mathrm{R} - P_\mathrm{L})}{2\,(P_\mathrm{L} - 2P_\mathrm{C} + P_\mathrm{L})}$$

$$F_\text{max} = \frac{-2P_\mathrm{R}(4P_\mathrm{C} + P_\mathrm{L}) + (P_\mathrm{L} - 4P_\mathrm{C})^2 + P_\mathrm{R}^2}{16P_\mathrm{C} - 8(P_\mathrm{L} + P_\mathrm{R})}$$

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- Zero-padding is still required but to a less extent.

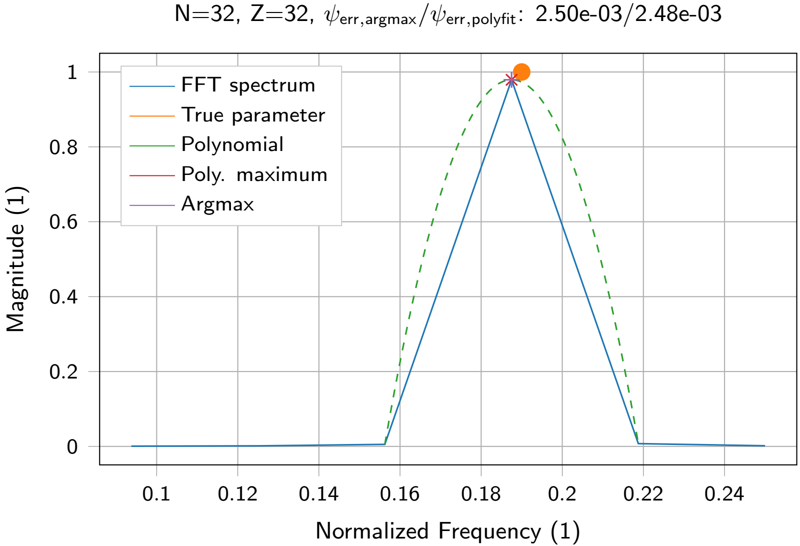
Exercise: Calculate Polynomial Fit

```python
def calc_polyfit_correction(spectrum_excert, df):
    """
    Parameters
    ----------
    spectrum_excert: list of three floats
        left, center and right magnitude value

    df: float
        frequency difference of two consecutive FFT-bins

    Returns
    -------
    offset_frequency: float
        Offset towards the location of Fc

    Fmax: magnitude at the maximum

    a: list of three floats
        Parameters of the polynomial.

    """
```

Exercise: Test Polynomial Fit

- Generate and plot the FFT based spectrum estimate for a sinusoid `A=[1]; psi=[0.19]; phi=[0.85]; N=32; Z=4*N`.
- Apply the method `calc_polynomial_fit` from the exercise before.
- Annotate the plot with
  - a marker for the sinusoidal parameters,
  - the maximum using argmax,
  - the maximum obtained by polynomial fit, and
  - the polynom in a region of ± 1 FFT bin around the maximum.
- Hints:
  - `numpy.polyval` can help drawing the polynomial curve
  - The FFT spectrum is periodic, thus getting the left and right value might need index wrapping.
    - If argmax produced index 0, negative indexing allows to get the left value with `index-1`.
    - If argmax produced index $Z-1$, obtaining the right value might produce `index is out of bounds` error. Check for this case or use the modulo operation % to avoid it.

Solution to the Polyfit Excersice



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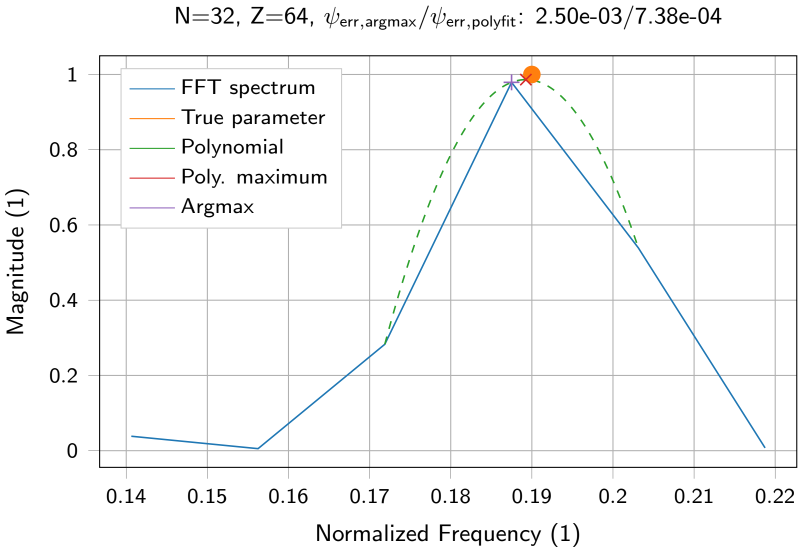

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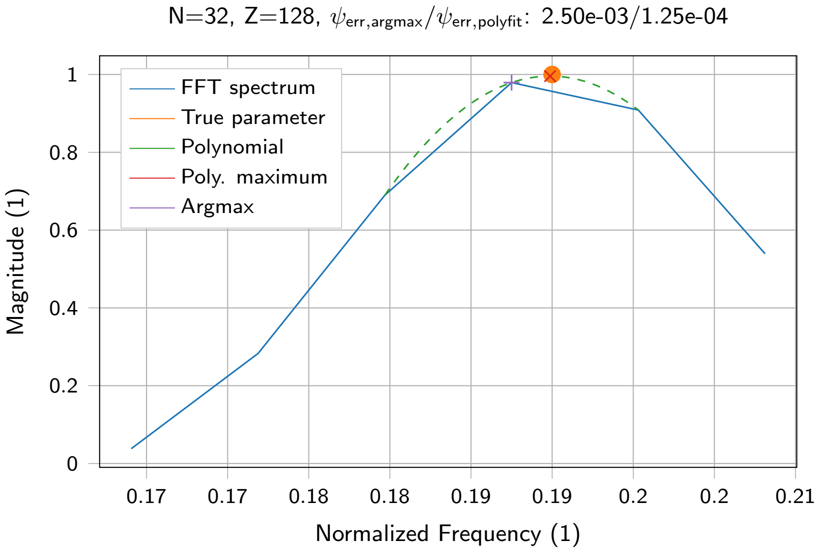

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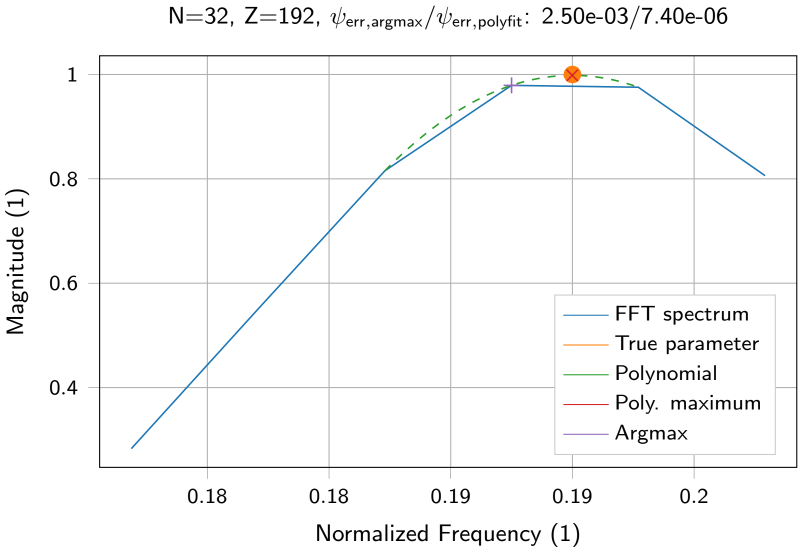

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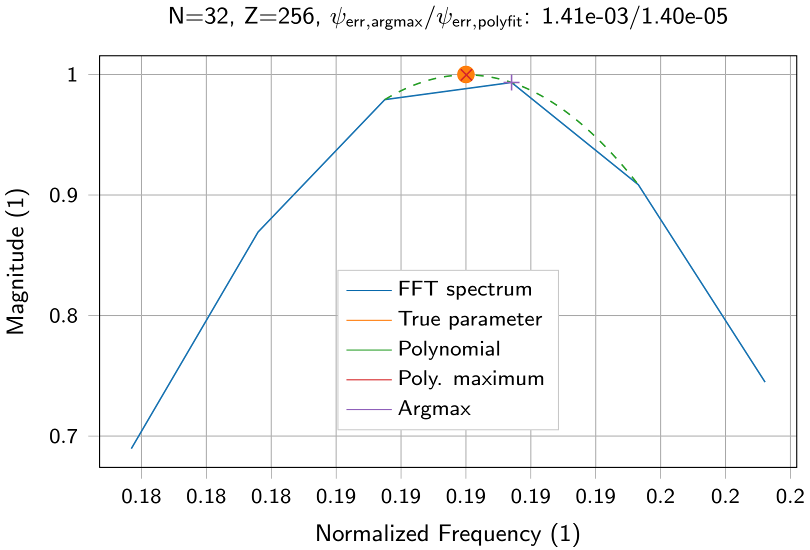

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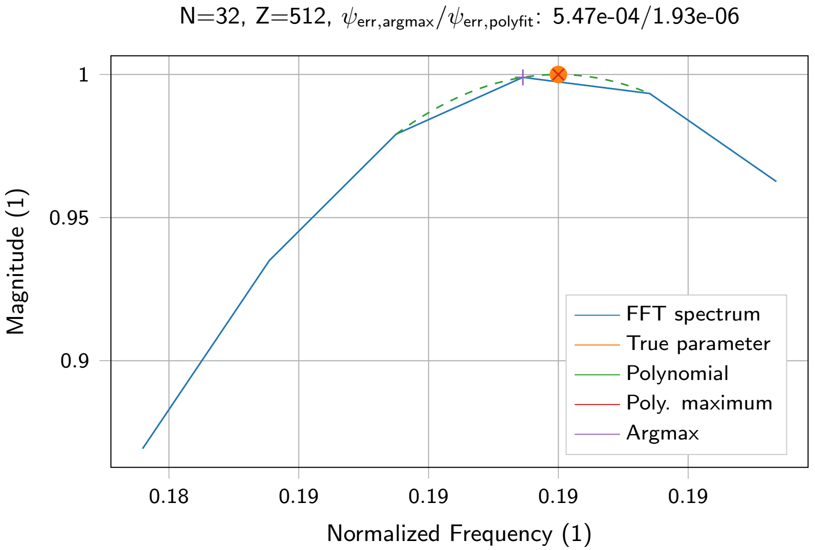

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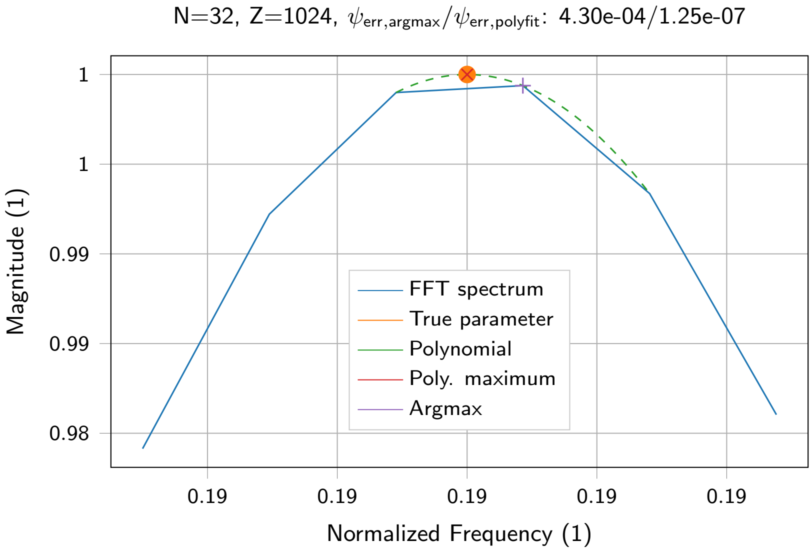

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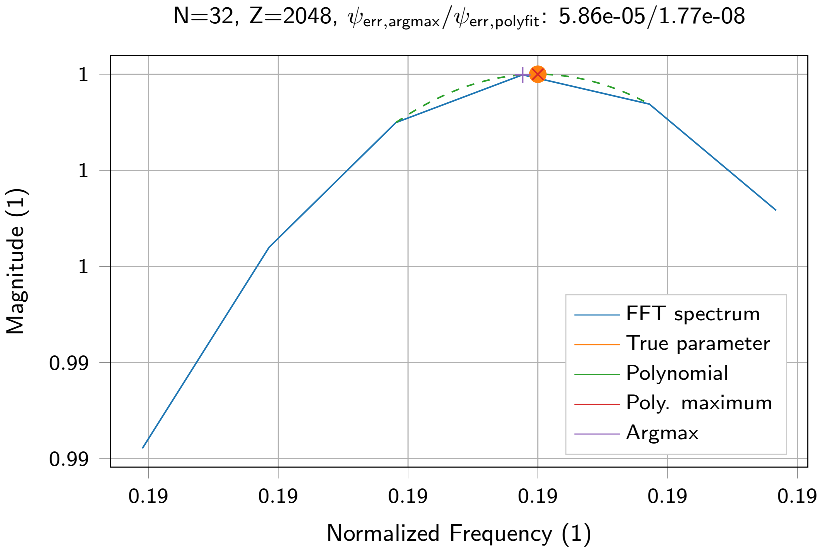
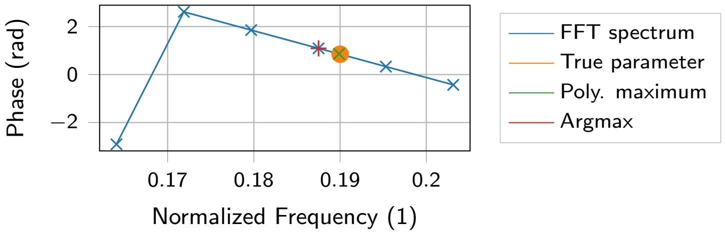


Homework: Phase estimate

- Implement either the DFT sum or the Goertzel algorithm to be able to estimate a certain spectral bin.
- Use the DFT sum or the Goertzel algorithm to estimate the phase at the frequency obtained by the parabola fit.
- Plot the phase spectrum and mark the true and the estimated phase.

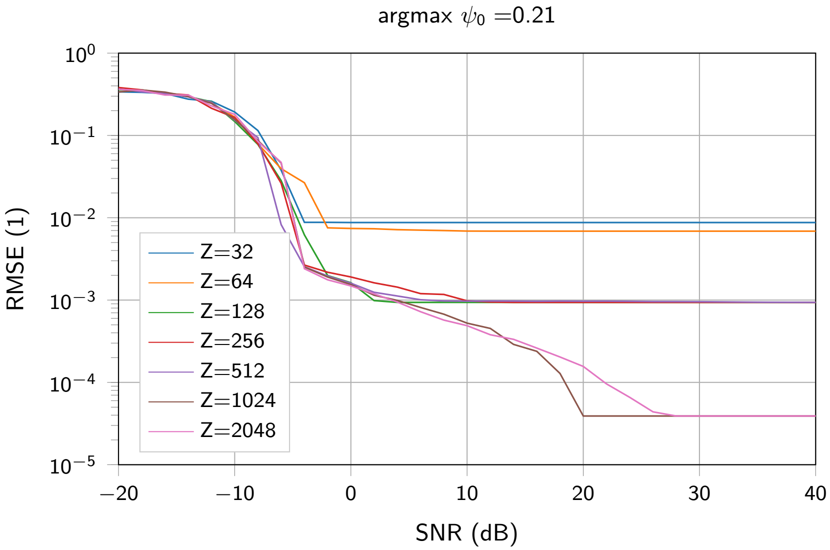
N=32, Z=128, $\phi_{\mathrm{err,argmax}}/\phi_{\mathrm{err,polyfit}}$: 2.43e-01/1.22e-02



Homework: RMSE of Argmax, and Polyfit

- Consider a signal $s_{\mathrm{RX}} = s_{\mathrm{IF}} + n$ of length $N = 32$, where s_{IF} is a complex cisoid with $A = 1$, $\psi = 0.21$, $\phi = 0$ and n is zero-mean complex white Gaussian noise.
- Plot the RMSE (in logarithmic scale) of frequency estimation over SNR (-20 to 40 dB in steps of 2 dB) using the argmax method and the polyfit method for $Z_{\mathrm{f}} \in \{1, 2, 4, 8, 16, 32, 64\}$.
- Hints:
  - $\mathrm{SNR} = \frac{A^2}{2\sigma^2}$
  - Z is the zero-padding factor, i.e. a signal with N should produce $Z = Z_{\mathrm{f}} N$ FFT-bins.
  - Use at least 1000 different noise realizations for each SNR to get smooth curves for the RMSE.

Expected Plots for Homework



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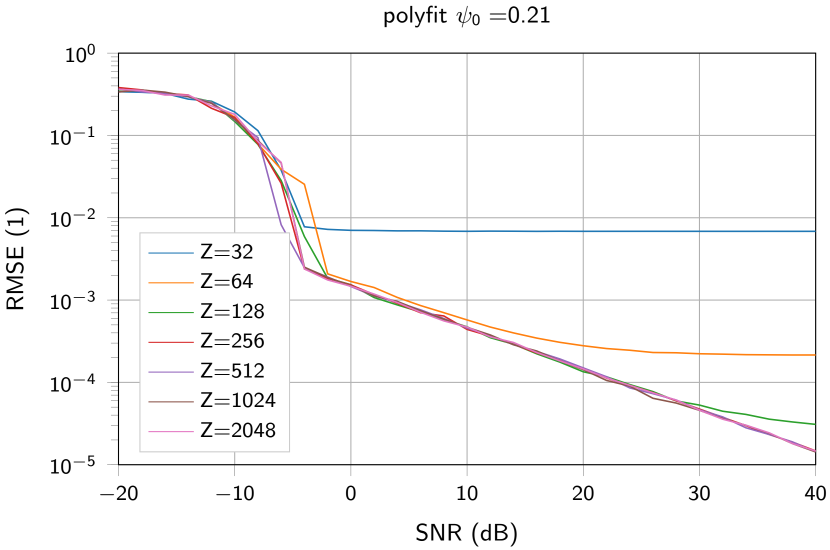

Homework: RMSE of Argmax, and Polyfit, cntd.

- Why is a higher zero-padding factor more beneficial for the argmax method than for polyfit?
- Why does the polyfit method not decrease the RMSE (compared to argmax) for $Z < 4$?
- Why can the argmax method show the same RMSE (at high SNR) for different values of Z?
- Why does the achievable RMSE change when using the argmax method (consider $\psi \in \{0.21, 0.22, 0.23\}$)?