Theory of Matrices

3.1 Introduction

The concept of matrix was first proposed by Arthur Cayley in 1860. Today, it is an essential tool for the development of modern science and technology. There are various applications of matrices in the field of arts (e.g., logic programming, precedence grammar of a language, etc.), sciences (e.g., quantum mechanics, chemical bonds, etc.), business (e.g., pay-off matrix, optimization, etc.), computer science (e.g., computer memory, digital image, pattern recognition, etc.), engineering and technology (e.g., production engineering, structural engineering, control engineering, etc.), and so on. Specifically, the entire computer software is developed based on the concept of matrices. So, matrix algebra is an important part in the curriculum of engineering students.

In this chapter, some basic and advanced concepts of matrices are discussed.

3.2 Definitions of Matrices and Determinants

Let us define the terms *matrix* and *determinants* with their basic components.

3.2.1 Matrix

Matrix A *matrix* is a rectangular array or arrangement of numbers or functions enclosed by a pair of round brackets ( ) or a pair of square brackets [ ] or || ||.

Notes

1. Normally, a matrix is denoted by (bold and italicized) capital letters such as $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, ...
2. Plural of matrix is matrices.

Illustration 3.1 An illustration of matrices with different notations is shown below:

$$\boldsymbol{A} = \begin{pmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{pmatrix} = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{bmatrix},$$

$$\boldsymbol{B} = \begin{Vmatrix} 2 & 0 \\ -1.1 & 5.2 \end{Vmatrix}, \ C = \begin{pmatrix} 2+i & 0+i\sqrt{3} \\ -1.1 & 5.2 \end{pmatrix}$$

Different components such as rows, columns, the (i, j)th element, diagonal elements, off-diagonal elements, leading/principal diagonal elements, etc. of a matrix $\boldsymbol{A}$ of size $m \times n$, i.e., $\boldsymbol{A}_{m\times n}$ are shown in Fig. 3.1.

3.2.2 Types of Matrices

Square Matrix A matrix with equal number of rows and columns is called a *square matrix.*

Illustration 3.2 Let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{bmatrix}.$$

Here, the number of rows and columns are both 3. So, it is a square matrix of size (or order) 3.

Row and Column Matrix A matrix with only one row is called a *row matrix.* Similarly, a matrix with only one column is called a *column matrix.*

Illustration 3.3 Let $\begin{bmatrix} 2 & 3.33 & 4 & 7.9 \end{bmatrix}$ be a row matrix of size 4. It is basically a matrix of size 1×4. Similarly,

$$\begin{bmatrix} 2 \\ 5 \\ 7 \\ 1 \end{bmatrix}$$

is a column matrix of size 4. So, it is a matrix of size 4×1.

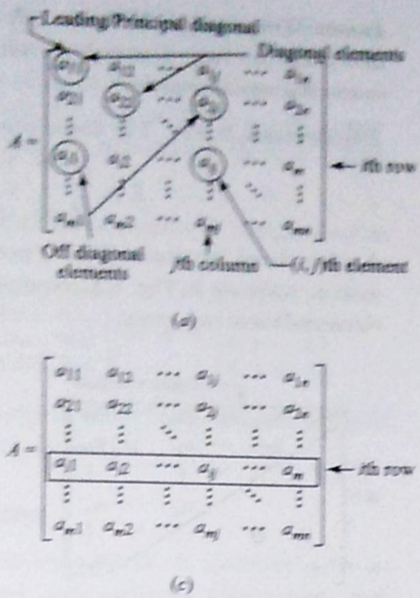

Fig. 3.1 Different components of a matrix.

Identity Matrix A square matrix is called an *identity matrix* or a *unit matrix* if its non-diagonal elements are all zeros and diagonal elements are all 1. An identity matrix is denoted by I.

Illustration 3.4 Let

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

be a matrix of size 3. Here I is a square matrix and all the off-diagonal elements are 0 but all the diagonal elements are 1. So, it is an identity matrix of size 3.

Unity Matrix A matrix $U = [a_{ij}]_{m \times n}$ is called a *unity matrix* if $a_{ij} = 1$ for $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

Illustration 3.5 Let

$$U = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Here, all the elements of U are 1. So, it is a unity matrix.

Diagonal Matrix A square matrix is called a *diagonal matrix* if its off-diagonal elements are all zero, i.e., $A = [a_{ij}]_{n \times n}$, where

$$a_{ij} = \begin{cases} a_{ii}, & i = j \\ 0, & i \neq j \end{cases}$$

A diagonal matrix is also represented by $A = \text{dig}[a_{11}, a_{22}, \ldots, a_{nn}]$ or $A = \text{dig}\,[a_{11}\ a_{22}\ \cdots\ a_{nn}]$.

Illustration 3.6 Let

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$$

be a square matrix of size 3. Here, all the off-diagonal elements are zero. So, it is a diagonal matrix of size 3.

Scalar Matrix A square matrix $A = [a_{ij}]_{n \times n}$ is called a *scalar matrix* if

$$a_{ij} = \begin{cases} c, & i = j \\ 0, & i \neq j \end{cases}$$

i.e., a diagonal matrix whose diagonal elements are all equal, is called a *scalar matrix*.

Illustration 3.7 Let

$$A = \begin{bmatrix} c & 0 & \cdots & 0 \\ 0 & c & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & c \end{bmatrix}$$

be a matrix. Here, all the off-diagonal elements are 0 and all the diagonal elements are equal to a constant c. So, it is a scalar matrix.

Note that $A = c\,I$.

Zero Matrix or *Null Matrix* A matrix is called a *zero* or a *null matrix* if all elements of this matrix are zeros. Normally it is denoted by O.

Illustration 3.8 Let

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

be a matrix. Here, all the elements are zeros. So, it is a zero or null matrix of size 3.

Tridiagonal Matrix A matrix is called a *tridiagonal matrix* if all elements of this matrix except the leading diagonal and its neighboring elements are zero.

Illustration 3.9 Let a matrix

$$T = \begin{bmatrix} a_1 & c_1 & 0 & 0 & 0 & 0 & 0 \\ b_1 & a_2 & c_2 & 0 & 0 & 0 & 0 \\ 0 & b_2 & a_3 & c_3 & 0 & 0 & 0 \\ 0 & 0 & b_3 & a_4 & c_4 & 0 & 0 \\ 0 & 0 & 0 & b_4 & a_5 & c_5 & 0 \\ 0 & 0 & 0 & 0 & b_5 & a_6 & c_6 \\ 0 & 0 & 0 & 0 & 0 & b_6 & a_7 \end{bmatrix}$$

Here, elements other than the leading diagonal, upper, and lower diagonal elements are all zero. So, this $\boldsymbol{T}$ is a tridiagonal matrix.

☞ ***Note*** Tridiagonal matrices are very useful in structural engineering and also in solving difference equations and recursive relations.

Band Matrix A matrix is called a *band matrix* if the significant elements are in a band and the other elements are zero.

Illustration 3.10 Let a matrix

$$B = \begin{bmatrix} a_1 & c_1 & 0 & 0 & d_1 & 0 & 0 \\ b_1 & a_2 & c_2 & 0 & 0 & d_2 & 0 \\ 0 & b_2 & a_3 & c_3 & 0 & 0 & d_3 \\ e_1 & 0 & b_3 & a_4 & c_4 & 0 & 0 \\ f_1 & e_2 & 0 & b_4 & a_5 & c_5 & 0 \\ 0 & f_2 & e_3 & 0 & b_5 & a_6 & c_6 \\ 0 & 0 & f_3 & e_4 & 0 & b_6 & a_7 \end{bmatrix}$$

Here, all significant elements (i.e., elements other than 0) are in the form of bands. So, $\boldsymbol{B}$ is a band matrix.

☞ ***Note*** Band matrices are very useful in structural engineering.

Triangular Matrix A matrix is called a *triangular matrix* if the elements either below or above the leading diagonal are all zeros.

Illustration 3.11 Two matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ described as

$$\boldsymbol{U} = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 5 & 12 \\ 0 & 0 & 10 \end{bmatrix} \text{ or } \boldsymbol{L} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 7 & 12 & 10 \end{bmatrix}$$

are called triangular matrices since the lower diagonal elements of $\boldsymbol{U}$ or upper diagonal elements of $\boldsymbol{L}$ are all zeros.

Upper Triangular Matrix A matrix whose elements below the leading diagonal are all zeros is known as the *upper triangular matrix.*

Illustration 3.12 The upper triangular matrix

$$\boldsymbol{U} = \begin{bmatrix} 2 & 3 & 7 \\ 0 & 5 & 12 \\ 0 & 0 & 10 \end{bmatrix}.$$

Lower Triangular Matrix A matrix whose elements above the leading diagonal are all zeros is known as the *lower triangular matrix.*

Illustration 3.13 The lower triangular matrix

$$\boldsymbol{L} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & 0 \\ 7 & 12 & 10 \end{bmatrix}$$

A generalized structure of an upper and a lower triangular matrix is shown in Fig. 3.2, where $\boldsymbol{O}$ indicates that all the elements there are zeros.

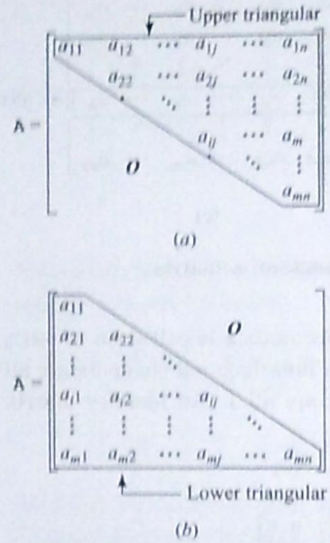

Fig. 3.2 Components of a triangular matrix.

Complex Matrix If at least one element of a matrix $\boldsymbol{A}$ is a complex number, then $\boldsymbol{A}$ is called a *complex matrix.*

Illustration 3.14 A complex matrix is

$$\boldsymbol{A} = \begin{pmatrix} 2+i & 0+i\sqrt{3} \\ -1.1 & 5.2 \end{pmatrix}$$

Sparse Matrix A matrix $\boldsymbol{A}$ is called a *sparse matrix* when the value of a large number of elements is zero. Also, the size of the matrix is reasonably large.

Illustration 3.15 A sparse matrix of size 7×10 is given below.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 100 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 18 & 0 \\ 11 & 0 & 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 27 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Boolean Matrix A matrix $\boldsymbol{B}$ called a *Boolean matrix* when each element of $\boldsymbol{B}$ is a logical quantity (i.e., either true/false or logical 0/1) or logical expression.

These matrices have a large number of uses in graph theory, automata theory, programming languages, compiler construction, etc. Here the element-wise operations are logical operations such as AND (i.e., $\bigwedge$) or OR (i.e., $\bigvee$) operations. In general, $A = [a_{ij}]_{n\times m}$, where $a_{ij} \in \{0, 1\}$ or {True, False}.

Illustration 3.16 A Boolean matrix A is described as follows:

$$A = \begin{bmatrix} \text{True} & \text{False} & \text{False} & \text{False} \\ \text{False} & \text{True} & \text{False} & \text{False} \\ \text{False} & \text{False} & \text{True} & \text{False} \\ \text{False} & \text{False} & \text{False} & \text{True} \end{bmatrix} \text{ or } A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

☞ ***Note*** Boolean matrices have many uses in the field of computer science and electronics.

3.2.3 Matrix Operations

In this section, different operations on matrices such as equality of two matrices and addition, subtraction, and multiplication of two matrices shall be discussed.

3.2.3.1 Equality of Matrices

Equality Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be *equal* if they are of the same order and each element of A is equal to the corresponding element of B, i.e., $a_{ij} = b_{ij}$ for all i and j.

Illustration 3.17 If

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix}, B = \begin{bmatrix} 1+1 & 1+3 \\ 5-2 & 10-3 \end{bmatrix}$$

then $a_{21} = 3$, $b_{21} = 5 - 2$, and so on. Therefore, $A = B$.

3.2.3.2 Addition/Subtraction of Matrices

Two matrices are said to be conformable for addition (or subtraction) if they are of same order.

Addition/Subtraction The *addition* (or *subtraction*) of two matrices of the same order is a matrix in which each element is the sum (or difference) of the corresponding elements in the given matrices.

If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$, then their sum, $A + B$ (or difference, $A - B$) is $C = [c_{ij}]_{m\times n}$, where

$$c_{ij} = \begin{cases} a_{ij} + b_{ij} \text{ (sum)} & \text{for } i = 1, 2, \ldots, m \\ a_{ij} - b_{ij} \text{ (difference)} & \text{for } j = 1, 2, \ldots, n \end{cases}$$

Illustration 3.18 If

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 5 & 7 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} -2 & 1 & -5 \\ -3 & 2 & -3 \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} 2 & 4 & -3 \\ 2 & 9 & 0 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} 6 & 2 & 7 \\ 8 & 5 & 6 \end{bmatrix}$$

Properties of Matrix Addition

(i) **Commutativity** If A and B are matrices of the same order, then $A + B = B + A$.

(ii) **Associativity** If A,B and C are matrices of same order, then $A + (B + C) = (A + B) + C$.

(iii) **Additive Identity** Given a matrix A, the zero matrix O of the same order is called its additive identity as $A + O = O + A = A$.

(iv) **Additive Inverse** Given a matrix A, the matrix $(-A)$ is called its additive inverse as $A + (-A) = O = (-A) + A$.

(v) **Cancellation Law** Given matrices A, B, and C, we have

$A + B = A + C \Rightarrow B = C$ [left cancellation]

$A + C = B + C \Rightarrow B = C$ [right cancellation]

Illustration 3.19 **[AND and OR Operations on Boolean Matrices]** Consider two Boolean matrices A and B, where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, the Boolean OR (i.e., $\bigvee$) operation on these matrices can be described as

$$A \bigvee B = \begin{bmatrix} 1\bigvee 0 & 1\bigvee 1 & 0\bigvee 1 \\ 0\bigvee 0 & 1\bigvee 1 & 1\bigvee 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Also, the Boolean AND (i.e., $\bigwedge$) operation on these matrices can be described as

$$A \bigwedge B = \begin{bmatrix} 1\bigwedge 0 & 1\bigwedge 1 & 0\bigwedge 1 \\ 0\bigwedge 0 & 1\bigwedge 1 & 1\bigwedge 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

3.2.3.3 Scalar Multiplication of Matrices

Scalar Multiplication If k is a scalar and A is a matrix, then kA or Ak is defined as the matrix obtained from the matrix A by multiplying each of its elements by k.

Illustration 3.20 Let k be a scalar quantity and

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Then

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Properties of Scalar Multiplication

(i) **Distributive Law** (*a*) $k(A + B) = kA + kB$

(*b*) $(k + l)A = kA + lA$

(ii) $(kl)A = k(lA) = l(kA)$

where A and B are two matrices; k and l are scalar quantities.

3.2.3.4 Matrix Multiplication

Multiplication Multiplication of two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined (we say, matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ are conformable for multiplication) if the number of columns of matrix $\boldsymbol{A}$ is equal to the number of rows of matrix $\boldsymbol{B}$.

If $\boldsymbol{A} = [a_{ij}]_{mxn}$ and $\boldsymbol{B} = [b_{ij}]_{nxp}$, then $\boldsymbol{AB} = \mathbf{C} = [c_{ij}]_{mxp}$ where $c_{ij} = \sum_{n=1}^{n} a_{ik}b_{kj}$ for all i, j such that $1 \le i \le m$, $1 \le j \le p$.
The product $\boldsymbol{AB}$ is referred to as $\boldsymbol{B}$ pre-multiplied by $\boldsymbol{A}$ or $\boldsymbol{A}$ post-multiplied by $\boldsymbol{B}$. Matrix $\boldsymbol{A}$ is called the *pre-factor* and matrix $\boldsymbol{B}$ is called the *post-factor*.
Post-multiplication In matrix multiplication $\boldsymbol{AB}$, the matrix $\boldsymbol{A}$ is said to be post-multiplied by the matrix $\boldsymbol{B}$.
Pre-multiplication In matrix multiplication $\boldsymbol{BA}$, the matrix $\boldsymbol{A}$ is said to be pre-multiplied by the matrix $\boldsymbol{B}$.

Illustration 3.21 Consider two matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ as defined below:
If

$$\boldsymbol{A} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

Then matrix multiplication operation $\boldsymbol{AB}$ can be written as

$$\boldsymbol{AB} = \begin{bmatrix} 3 & 1 & \frac{3}{2} \\ -\frac{5}{4} & -\frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & -\frac{1}{6} & \frac{5}{6} \end{bmatrix}$$

$$= \begin{bmatrix} 3 \times \frac{1}{3} + 1 \times -\frac{2}{3} + \frac{3}{2} \times \frac{1}{3} & 3 \times \frac{1}{3} + 1 \times \frac{1}{3} + \frac{3}{2} \times -\frac{1}{6} & \cdots \\ -\frac{5}{4} \times \frac{1}{3} - \frac{1}{4} \times -\frac{2}{3} - \frac{3}{4} \times \frac{1}{3} & -\frac{5}{4} \times \frac{1}{3} - \frac{1}{4} \times \frac{1}{3} - \frac{3}{4} \times -\frac{1}{6} & \cdots \\ -\frac{1}{4} \times \frac{1}{3} - \frac{1}{4} \times -\frac{2}{3} - \frac{1}{4} \times \frac{1}{3} & -\frac{1}{4} \times \frac{1}{3} - \frac{1}{4} \times \frac{1}{3} - \frac{1}{4} \times -\frac{1}{6} & \cdots \end{bmatrix}$$

$$\begin{matrix} 3 \times -\frac{2}{3} + 1 \times -\frac{2}{3} + \frac{3}{2} \times \frac{5}{6} \\ -\frac{5}{4} \times -\frac{2}{3} - \frac{1}{4} \times -\frac{2}{3} - \frac{3}{4} \times \frac{5}{6} \\ -\frac{1}{4} \times -\frac{2}{3} - \frac{1}{4} \times -\frac{2}{3} - \frac{1}{4} \times \frac{5}{6} \end{matrix} \Bigg]$$

$$= \begin{bmatrix} 1 - \frac{2}{3} + \frac{1}{2} & 1 + \frac{1}{3} - \frac{1}{4} & -2 - \frac{2}{3} + \frac{5}{4} \\ -\frac{5}{12} + \frac{1}{6} - \frac{1}{4} & -\frac{5}{12} - \frac{1}{12} + \frac{1}{8} & \frac{5}{6} + \frac{1}{6} - \frac{5}{8} \\ -\frac{1}{12} + \frac{1}{6} - \frac{1}{12} & -\frac{1}{12} - \frac{1}{12} + \frac{1}{24} & \frac{1}{6} + \frac{1}{6} - \frac{5}{24} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{6} & \frac{13}{12} & -\frac{17}{12} \\ -\frac{1}{2} & -\frac{3}{8} & \frac{3}{8} \\ 0 & -\frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

Illustration 3.22 **[Boolean Matrix Multiplication Operation]** Consider two Boolean matrices $\boldsymbol{A}$ and $\boldsymbol{B}$, where

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$

The Boolean matrix multiplication operation is denoted by $\odot$ and defined as matrix multiplication, where multiplication operations on matrix elements are replaced by $\wedge$ and addition operations on matrix elements are replaced by $\vee$, i.e.,

$$\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \odot \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \wedge 1 \vee 1 \wedge 0 \vee 0 \wedge 0 & 1 \wedge 0 \vee 1 \wedge 1 \vee 0 \wedge 1 \\ 0 \wedge 1 \vee 1 \wedge 0 \vee 1 \wedge 0 & 0 \wedge 0 \vee 1 \wedge 1 \vee 1 \wedge 1 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Properties of Matrix Multiplication

(i) **Commutativity** $AB \neq BA$ in general.
(ii) **Associativity** For three matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, if multiplication is defined, then $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$.
(iii) **Distributive Law** Multiplication of matrices is distributive over the addition.
   (a) $\boldsymbol{A}(\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{AB} + \boldsymbol{AC}$
   (b) $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{C} = \boldsymbol{AC} + \boldsymbol{BC}$
   where the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$ are conformable for addition and multiplication.
(iv) **Multiplicative Identity** For a matrix $\boldsymbol{A}_{m\times n}$, a unit matrix $\boldsymbol{I}_m$, i.e., $\boldsymbol{I}_{m\times m}$ is multiplicative identity if it is pre-multiplied as $\boldsymbol{I}_m \times \boldsymbol{A}_{m\times n}$ and $\boldsymbol{I}_n$ is multiplicative identity, if it is post-multiplied as $\boldsymbol{A}_{m\times n} \times \boldsymbol{I}_n$.
   For a square matrix $\boldsymbol{A}_{m\times m}$, there is only one multiplicative identity $\boldsymbol{I}_m$.
(v) **Multiplicative Inverse** A matrix $\boldsymbol{B}$ is called inverse of a matrix $\boldsymbol{A}$ if $\boldsymbol{AB} = \boldsymbol{BA} = \boldsymbol{I}$. Inverse of a matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^{-1}$. Note that only a square matrix may have a multiplicative inverse.

☞ Notes

1. If $\boldsymbol{AB} = \boldsymbol{O}$, then it is not necessary that either $\boldsymbol{A} = \boldsymbol{O}$ or $\boldsymbol{B} = \boldsymbol{O}$, or $\boldsymbol{A} = \boldsymbol{B} = \boldsymbol{O}$, where $\boldsymbol{O}$ is zero or null matrix.
2. Failure of the cancellation law of scalar algebra $\boldsymbol{AB} = \boldsymbol{AC}$, where $\boldsymbol{A} \neq \boldsymbol{O}$, does not imply that $\boldsymbol{B} = \boldsymbol{C}$.
3. A matrix $[a_{ij}]_{m\times n}$ is comfortable for multiplication with itself if $m = n$, i.e., $\boldsymbol{A}$ is a square matrix.

Illustration 3.23 Consider two matrices A and B, where

$$A = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } BA = \begin{bmatrix} 4 & 4 & -4 \\ -20 & -20 & 20 \\ -16 & -16 & 16 \end{bmatrix}$$

Here, $A \neq O$; $B \neq O$ but $AB = O$. Also $AB \neq BA$

3.2.4 Power of a Matrix

The product AA is written as A^2. Also, $A(AA) = (AA)A = AAA = A^3$.

In general, $AAA \cdots A$ (r times) $= A^r$ and A^r is the rth power of matrix A.

Illustration 3.24 Consider

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 9 \\ 6 & 10 \end{bmatrix}$$

☞ ***Note*** $I = I^2 = I^3 = \cdots = I^r$, where I is an identity matrix.

Nilpotent Matrix For a square non-zero matrix A, if there exists a natural number n such that $A^n = O$ but $A^{n-1} \neq O$, then the matrix A is called a *nilpotent matrix*. The natural number n is called *index* of the matrix A.

Illustration 3.25 If

$$A = \begin{bmatrix} 0 & 2 & 5 & 2 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then

$$A^3 = \begin{bmatrix} 0 & 0 & 0 & -48 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \neq O$$

But $A^4 = 0$. Therefore, A is a nilpotent matrix and its index is 4.

Idempotent Matrix A square matrix A is called an *idempotent matrix* if $A^2 = A$.

Illustration 3.26 Consider a matrix

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

Then

$$A^2 = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A$$

Hence, A is an idempotent matrix.

☞ ***Note*** Trivially, an identity matrix is an idempotent matrix.

Involutory Matrix A square matrix A is called an *involutory matrix* if $A^2 = I$.

Illustration 3.27 Consider a matrix

$$A = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix},$$

Then

$$A^2 = \begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} -5 & -8 & 0 \\ 3 & 5 & 0 \\ 1 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

☞ ***Note*** $I^2 = I$, so, I is also an involutory matrix.

Periodic Matrix A square matrix A is called a *periodic matrix* if $A^{m+1} = A$, where m is an integer. Here, m is called the *period*.

Illustration 3.28 Consider a matrix

$$A = \begin{bmatrix} 1 & -2 & -6 \\ -3 & 2 & 9 \\ 2 & 0 & -3 \end{bmatrix}$$

Then $A^3 = A$.
Therefore, A is a periodic matrix of period $= 2$.

3.2.5 Transpose of a Matrix

Transpose of a Matrix If the rows and columns of a matrix are interchanged, the resultant matrix is called the *transpose* of the original matrix.

The transpose of a matrix A is usually denoted by a A', A^T, or A^t. If the order of a matrix A is $m \times n$, then the order of its transpose is $n \times m$.

Properties

The transpose of a matrix has the following properties:
(i) $(A^T)^T = A$
(ii) $(kA)^T = kA^T$, where k is a scalar.

(iii) $(A+B)^T = A^T + B^T$
(iv) $(AB)^T = B^T A^T$
Consider two matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$

These properties are verified in Illustrations 3.29–3.31.

Illustration 3.29 The transpose of A is

$$A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

Again, the transpose of A^T is

$$(A^T)^T = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = A$$

Suppose k is a constant.

$$kA = k\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$$

Therefore,

$$(kA)^T = \begin{bmatrix} ka_{11} & ka_{21} & ka_{31} \\ ka_{12} & ka_{22} & ka_{32} \\ ka_{13} & ka_{23} & ka_{33} \end{bmatrix}$$

gain,

$$kA^T = k\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

i.e.,

$$kA^T = \begin{bmatrix} ka_{11} & ka_{21} & ka_{31} \\ ka_{12} & ka_{22} & ka_{32} \\ ka_{13} & ka_{23} & ka_{33} \end{bmatrix}$$

Hence, $(kA)^T = kA^T$

Illustration 3.30 The sum of A and B is

$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & a_{13}+b_{13} \\ a_{21}+b_{21} & a_{22}+b_{22} & a_{23}+b_{23} \\ a_{31}+b_{31} & a_{32}+b_{32} & a_{33}+b_{33} \end{bmatrix}$$

Now, the transpose of $A+B$ is

$$(A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & a_{31}+b_{31} \\ a_{12}+b_{12} & a_{22}+b_{22} & a_{32}+b_{32} \\ a_{13}+b_{13} & a_{23}+b_{23} & a_{33}+b_{33} \end{bmatrix}$$

Again

$$A^T + B^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} + \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & a_{31}+b_{31} \\ a_{12}+b_{12} & a_{22}+b_{22} & a_{32}+b_{32} \\ a_{13}+b_{13} & a_{23}+b_{23} & a_{33}+b_{33} \end{bmatrix}$$

Hence, $(A+B)^T = A^T + B^T$

Illustration 3.31 The product of A and B is

$$AB = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$$
$$= \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} \\ a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} \\ a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

Therefore

$$(AB)^T = \begin{bmatrix} a_{11}b_{11}+a_{12}b_{21}+a_{13}b_{31} & a_{21}b_{11}+a_{22}b_{21}+a_{23}b_{31} & a_{31}b_{11}+a_{32}b_{21}+a_{33}b_{31} \\ a_{11}b_{12}+a_{12}b_{22}+a_{13}b_{32} & a_{21}b_{12}+a_{22}b_{22}+a_{23}b_{32} & a_{31}b_{12}+a_{32}b_{22}+a_{33}b_{32} \\ a_{11}b_{13}+a_{12}b_{23}+a_{13}b_{33} & a_{21}b_{13}+a_{22}b_{23}+a_{23}b_{33} & a_{31}b_{13}+a_{32}b_{23}+a_{33}b_{33} \end{bmatrix}$$

The product of B^T and A^T is

$$B^T A^T = \begin{bmatrix} b_{11} & b_{21} & b_{31} \\ b_{12} & b_{22} & b_{32} \\ b_{13} & b_{23} & b_{33} \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$
$$= \begin{bmatrix} b_{11}a_{11}+b_{21}a_{12}+b_{31}a_{13} & b_{11}a_{21}+b_{21}a_{22}+b_{31}a_{23} & b_{11}a_{31}+b_{21}a_{32}+b_{31}a_{33} \\ b_{12}a_{11}+b_{22}a_{12}+b_{32}a_{13} & b_{12}a_{21}+b_{22}a_{22}+b_{32}a_{23} & b_{12}a_{31}+b_{22}a_{32}+b_{32}a_{33} \\ b_{13}a_{11}+b_{23}a_{12}+b_{33}a_{13} & b_{13}a_{21}+b_{23}a_{22}+b_{33}a_{23} & b_{13}a_{31}+b_{23}a_{32}+b_{33}a_{33} \end{bmatrix}$$

Hence, $(AB)^T = B^T A^T$

3.2.6 Determinant of a Square Matrix

Determinant of a Matrix If A is a square matrix $[a_{ij}]$ of size n, then the determinant of A is denoted by $|A|$ or $\det A$ or $\det(A)$ or $|a_{ij}|_{n\times n}$ is defined by

$$|A| = \sum (-1)^{j_1+j_2+\cdots+j_n-n} a_{1_{j_1}} a_{2_{j_2}} \cdots a_{n_{j_n}}$$

where the summation is extended over all possible permutations $j_1, j_2, \ldots, J_n$ of $1, 2, \ldots, n$.

Illustration 3.32 Consider a matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

Then the determinant of A is

$$|A| = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{vmatrix} = 2\times\begin{vmatrix} 5 & 0 \\ 2 & -1 \end{vmatrix} - 4\times\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} + 3\times\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix}$$
$$= 2\{5\times(-1) - 0\times 1\} - 4\{1\times(-1) - 0\times 3\} + 3(1\times 2 - 5\times 3)$$
$$= 2(-5-0) - 4(-1-0) + 3(2-15) = -10+4-39 = -45.$$

Properties of the Determinant of a Matrix

The determinant $|A|$ of a matrix $A_{n\times n}$ posseses the following properties:

(i) $|A|$ is a unique real number (> 0, $= 0$, or < 0).
(ii) If any two rows/columns of A are interchanged, then the value of the determinant $= -|A|$.

(iii) If two or more rows or columns of A are proportional (or linearly dependent), then $|A| = 0$.

(iv) If a row or column of A is multiplied by a scalar k, then determinant $= k\,|A|$.

(v) If every element of A is multiplied by a scalar k, then determinant $= k^n\,|A|$, i.e., $|kA| = k^n\,|A|$.

(vi) If a scalar multiple of any row (or column) of A is added to any other row (or column), then the value of $|A|$ remains unchanged.

(vii) $|A_{n\times n} \times B_{n\times n}| = |A_{n\times n}| \times |B_{n\times n}|$

(viii) Determinant of a square matrix A is given by

$$\det(A) = \begin{cases} \prod_{i=1}^{n} a_{ii}, & \text{when } A \text{ is a triangular matrix} \\ 1, & \text{when } A = I, \text{ an identity matrix} \\ \pm 1, & \text{when } A \text{ is an orthogonal matrix} \\ |A^{\mathrm{T}}|, & \text{when } A^{\mathrm{T}} \text{ is the transpose of } A \end{cases}$$

(ix) $|A^T A| = |I| \Rightarrow |A^T|\,|A| = |A|\,|A| = |A|^2 = 1$ or $|A| = \pm 1$.

3.2.7 Trace of a Square Matrix

Trace of a Square Matrix The trace of square matrix A is denoted by $Tr\,A$, or $trace(A)$, or $spur(A)$ and defined by a quantity which is the sum of diagonal elements of the matrix A, i.e.

$$\mathrm{Tr}A = \sum_{i=1}^{n} a_{ii}$$

where a_{ii} are the diagonal elements of the square matrix A of order n.

Illustration 3.33 Suppose

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

Then $trace(A) = 2 + 5 - 1 = 6$

Properties of trace of a square matrix

(i) $\mathrm{trace}(A \pm B) = \mathrm{trace}(A) \pm \mathrm{trace}(B)$

(ii) $\mathrm{trace}(AB) = \mathrm{trace}(BA)$

(iii) $\mathrm{trace}(A^T) = \mathrm{trace}(A)$

(iv) $\mathrm{trace}(A^*) = (\mathrm{trace}(A))^*$, where A^* is the complex conjugate of A.

☞ ***Note*** Here * denotes the complex conjugate of a complex matrix. These are defined in Section 3.22.

3.2.8 Singular and Non-singular Matrices

Singular or Non-singular Matrix A square matrix A is called a *singular* or *non-singular matrix* based on the following rule:

$$A = \begin{cases} \text{Singular matrix} & \text{if } |A| = 0 \\ \text{Non-singular matrix} & \text{if } |A| \neq 0 \end{cases}$$

Illustration 3.34 Consider a matrix

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 2 & 0 \\ 3 & 6 & -1 \end{bmatrix}$$

Then the determinant of A is

$$|A| = \begin{vmatrix} 2 & 4 & 3 \\ 1 & 2 & 0 \\ 3 & 6 & -1 \end{vmatrix} = 2 \times \begin{vmatrix} 2 & 0 \\ 6 & -1 \end{vmatrix} - 4 \times \begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} + 3 \times \begin{vmatrix} 1 & 2 \\ 3 & 6 \end{vmatrix}$$

$$= 2(2 \times -1 - 0 \times 6) - 4(1 \times -1 - 0 \times 3) + 3(1 \times 6 - 2 \times 3)$$

$$= 2(-2 - 0) - 4(-1 - 0) + 3(6 - 6) = -4 + 4 + 0 = 0$$

Therefore, matrix A is a singular matrix.

Illustration 3.35 Consider

$$A = \begin{bmatrix} 2 & 4 & 3 \\ 1 & 5 & 0 \\ 3 & 2 & -1 \end{bmatrix}$$

Then the determinant of A, i.e., $|A| = -45 \neq 0$. Therefore, A is a non-singular matrix.

3.2.9 Submatrices and Partitioning

In this section, submatrices and partitioning of matrices are discussed.

Submatrix A submatrix of a given matrix A is any matrix formed by deleting or removing some of its rows, or columns, or both from A.

Leading Submatrix of Order r A submatrix consisting of the elements of the first r rows and first r columns is called a leading submatrix of order r.

Illustration 3.36 Consider

$$A_{4\times 5} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

Then

$$\begin{bmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}, \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \text{ and } \begin{bmatrix} a_{13} & a_{15} \\ a_{23} & a_{25} \\ a_{43} & a_{45} \end{bmatrix}$$

are the submatrices of $A_{4\times 5}$.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \text{ and } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

are the leading submatrices of $A_{4\times 5}$.

Partition Matrix A matrix is called *partitioned* if it is divided into submatrices by horizontal and vertical lines between rows and columns.

Illustration 3.37 Consider

$$A=\left[\begin{array}{cc|cc} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ \hline 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{array}\right] \quad B=\left[\begin{array}{cc|cc} 1 & 5 & 9 & 13 \\ 2 & 6 & 10 & 14 \\ \hline 3 & 7 & 11 & 15 \\ 4 & 8 & 12 & 16 \end{array}\right]$$

If we consider

$$C=\begin{bmatrix} 1 & 5 \\ 2 & 6 \end{bmatrix}, D=\begin{bmatrix} 9 & 13 \\ 10 & 14 \end{bmatrix}, E=\begin{bmatrix} 3 & 7 \\ 4 & 8 \end{bmatrix}, \text{ and } F=\begin{bmatrix} 11 & 15 \\ 12 & 16 \end{bmatrix}$$

Then A can be written as

$$A=\left[\begin{array}{c|c} C & D \\ \hline E & F \end{array}\right]$$

3.2.9.1 Operations on Partition Matrices

If A and B are two partition matrices such that

$$A=\begin{bmatrix} C & D \\ E & F \end{bmatrix} \text{ and } B=\begin{bmatrix} P & Q \\ R & S \end{bmatrix},$$

where $C, D, E, F, P, Q, R,$ and S are the partitions of A and B. Then we can define the following operations:

Operation	Matrix notations
Addition	$A+B=\left[\begin{array}{c\|c} C+P & D+Q \\ \hline E+R & F+S \end{array}\right]$
Subtraction	$A-B=\left[\begin{array}{c\|c} C-P & D-Q \\ \hline E-R & F-S \end{array}\right]$
Multiplication	$AB=\left[\begin{array}{c\|c} CP+DR & CQ+DS \\ \hline EP+FR & EQ+FS \end{array}\right]$

3.2.10 Minors of a Matrix

Minors of a Matrix The minor of a matrix $A_{m\times n}$ of order r ($\le \min(m,n)$, i.e., minimum of m and n) is the determinant of a square matrix of order r. This square matrix of order r is obtained after deleting all elements of $A_{m\times n}$ except those in certain r rows and r columns.

Illustration 3.38 The determinant

$$\begin{vmatrix} a_{22} & a_{23} & a_{25} \\ a_{32} & a_{33} & a_{35} \\ a_{42} & a_{43} & a_{45} \end{vmatrix}$$

is a minor of order 3,

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \end{bmatrix}$$

Principal Minors In a square matrix A, those minors which have elements situated symmetrically with respect to the principal diagonal are called *principal minors.*

Illustration 3.39 The determinant

$$\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

is a principal minor of

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$$

3.2.11 Adjoint or Adjugate Matrix

In relation to a determinant, let us study the minor of an element and the co-factor of an element.

Minors of an Element The minor of an element a_{ij} of the determinant $|A|$ is denoted by M_{ij}. It is also a determinant obtained by deleting the ith row and jth column from the determinant $|A|$.

Co-factor The co-factor of an element a_{ij} of the determinant $|A|$ is denoted by A_{ij} and it is defined as $A_{ij}=(-1)^{i+j}M_{ij}$.

Illustration 3.40 Consider a matrix

$$A=\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and its determinant

$$|A|=\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}.$$

The minor of a_{12} is M_{12} and is computed by deleting the first row and second column of the determinant $|A|$, i.e.

$$M_{12}=\begin{vmatrix} [] & [] & [] \\ a_{21} & [] & a_{23} \\ a_{31} & [] & a_{33} \end{vmatrix}=\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

Similarly

$$M_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

etc.

The co-factor of a_{12} is A_{12} and is computed as $A_{12}=(-1)^{1+2}M_{12}$, i.e.

$$A_{12}=-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}.$$

Similarly

$$A_{11}=\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

etc.

Adjoint or Adjugate Matrix If $A=[a_{ij}]$ be a square matrix of order n, A_{ij} be the co-factor of a_{ij} in $|A|=|a_{ij}|$, and $B=[A_{ij}]$ is the matrix formed by the co-factors of the matrix A, then the transpose of the matrix B i.e., B', which is called the *adjoint* or *adjugate of the matrix A.*

The adjoint of the matrix A is denoted by adj A or adj (A).

Illustration 3.41 Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

and we get

$$B = \begin{bmatrix} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ -\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \\ \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} & -\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} & \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \end{bmatrix}$$

or

$$B = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

$$\left[\text{since } A_{21} = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \text{etc.}\right]$$

Therefore

$$\text{adj } A = B' = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

Properties of an Adjoint Matrix

The adjoint of a matrix $A_{n\times n}$, adj A, posseses the following properties:

(i) If A is a square matrix of order n, then A (adj A) $= |A|I_n =$ (adj A)A.
(ii) If A is a square matrix of order n, then (adj A^T) $=$ (adj A)^T.
(iii) If A and B are two square matrices of the same order, then adj $(AB) =$ adj(A).adj(B)
(iv) adj(adj A) $= |A|^{n-2}A$, where A is a non-singular matrix of order n.
(v) $|\text{adj}(\text{adj} A)| = |A|^{(n-1)^2}$, where A is a non-singular matrix of order n.
(vi) Adjoint of a diagonal matrix is a diagonal matrix.

3.2.12 Elementary Transformations on Matrices

Following elementary row (or column) operations or transformations can be performed on a matrix. Assume R_i and c_j are ith row and jth column of a given matrix.

1. **Interchange any two rows (or columns)** Interchange of rows i and j is denoted by $R_i \leftrightarrow R_j$. Similarly, $C_i \leftrightarrow C_j$ denotes the interchange of columns i and j.
2. **The multiplication of any row (or column) by a non-zero number (scalar)** The multiplication of elements of the ith row by a non-zero number k is denoted by kR_i. Similarly, kC_j denotes the multiplication of jth column by a non-zero scalar number k.
3. **Addition (or subtraction) of a constant multiple to the elements of any row (or column) to the corresponding elements of any other row (or column)** The addition of elements of the ith row and the k multiple to the corresponding element of the jth row is denoted by $R_i + kR_j$. Similarly, the addition of elements of the ith column and the k multiple to the corresponding element of the jth column is denoted by $C_i + kC_j$.

Illustration 3.42 Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

then by elementary transformations, we have

Interchange of operations

1. If we interchange row 1 (i.e., R_1) and row 2 (i.e., R_2), then the transformed matrix A is

$$\begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ [by } R_1 \leftrightarrow R_2]$$

2. Similarly, if we interchange column 2 (i.e., C_2) and column 3 (i.e., C_3), then the transformed matrix A is

$$\begin{bmatrix} a_{11} & a_{13} & a_{12} \\ a_{21} & a_{23} & a_{22} \\ a_{31} & a_{33} & a_{32} \end{bmatrix} \text{ [by } C_1 \leftrightarrow C_2]$$

Multiplication of row and column by a constant

1. If we multiply a scalar k with the row 2 (i.e., R_2) of A and replace the row 2 (i.e., R_2) of A, then the transformed matrix obtained from A is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ [by } R_2 \leftarrow kR_2]$$

2. Similarly, if we multiply a scalar p with the column 3 (i.e., C_3) of A and replace the column 3 (i.e., C_3) of A, then the transformed matrix obtained from A is

$$\begin{bmatrix} a_{11} & a_{12} & pa_{13} \\ a_{21} & a_{22} & pa_{23} \\ a_{31} & a_{32} & pa_{33} \end{bmatrix} \text{ [by } C_1 \leftrightarrow C_2]$$

☞ ***Note*** $R_2 \leftarrow kR_2$ of A means each elements in row 2 of A is multiplied by the constant k and R_2 replaced by kR_2.

Other combined operations

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} + ka_{11} & a_{22} + ka_{12} & a_{23} + ka_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ [by } R_2 \leftarrow R_2 + kR_1]$$

etc.

3.2.13 Equivalent Matrices

Equivalent Matrix Two matrices A and B are called equivalent and are expressed by $A \sim B$, if matrix B can be obtained by applying elementary row (or column) operation or operations on the matrix A, i.e., by a sequence of elementary transformations.

Illustration 3.43 If

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \text{ and } B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21}+a_{11} & a_{22}+a_{12} & a_{23}+a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then $A \sim B$, where B is obtained by performing $R_2 + R_1$ elementary row operation on A.

3.2.14 Elementary Matrices

Elementary Matrix A matrix is called an elementary matrix if it is obtained from a unit matrix by elementary transformations.

Illustration 3.44 Elementary matrices are obtained from a unit matrix of size 3, i.e.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

by the following operations:

(i) Either by interchanging R_1 and R_2 of I_3, i.e., $R_1 \leftrightarrow R_2$ or by interchanging C_1 and C_2 of I_3, i.e.

$$C_1 \leftrightarrow C_2 \text{ as } \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(ii) Multiplying a constant k with the column 3 or row 3 of I_3 by k, then we get

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & k \end{bmatrix}$$

(iii) By the operation $C_1 \leftarrow C_2 + kC_1$ on I_3, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ k & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Result 1 Elementary row (or column) transformations of a matrix A can be obtained by pre- (or post-) multiplication with the corresponding elementary matrices.

Illustration 3.45 Consider a matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Perform $R_1 \leftrightarrow R_2$ using elementary matrix multiplication. The elementary matrix obtained after performing $R_1 \leftrightarrow R_2$ is

$$R_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now

$$R_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Result 2 Suppose T_i is an elementary matrix corresponding to the given matrix A. If we apply a sequence of elementary operations $T_1, T_2, \ldots, T_k$ on A, then it becomes an elementary matrix I corresponding to A (where A and I are of the same size). This can be written as $T_k, T_{k-1}, \ldots, T_1 A = I$.

Section Review 3.1

1. Find A^8 when

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

2. If A and D are square matrices and $\det(A) \neq 0$, show that

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(A).\det(D - CA^{-1}B)$$
$$= \det(AD - ACA^{-1}B).$$

What does this reduce to if A and C commute? If A and B commute?

3. Compute the transpose of the following matrices:

(a) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

(b) $B = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}$

(c) $C = [5]$, (d) $\mathbf{D} = \begin{bmatrix} 3 & 5+6i \\ 2 & 3i \\ 2+3i & 2-i \end{bmatrix}$

3.3 Inverse of a Square Matrix

In this section, inverse of a matrix and its computational procedure and properties are defined.

Inverse of the Matrix If A is a square matrix of order n and a matrix B of order n exists such that $AB = BA = I_n$, Tel.: 2791266, 2791267

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