Variation by p:

$$\int_x \left(k_0\left[(p+\delta p)\ln\frac{p+\delta p}{\hat{p}_0(x)} - p\ln\frac{p}{\hat{p}_0}\right]\Big|_{(x,t_0)} + k_1\left[(p+\delta p)\ln\frac{p+\delta p}{\hat{p}_1} - p\ln\frac{p}{\hat{p}_1}\right]\right)dx + \iint_{x\,t}\lambda_1\left[\frac{\partial (p+\delta p)}{\partial t} + \frac{\partial}{\partial x}\left[f(p+\delta p)\right]\right] - \frac{\partial p}{\partial t} - \frac{\partial}{\partial x}[fp]\Big)\,dxdt$$

$$+ \int_t \lambda_2\left[\int_x (p+\delta p)dx - 1 - \int_x p\,dx + 1\right]dt =$$

$$= \int_x \left(k_0\left[p\left(\ln\frac{p+\delta p}{\hat{p}_0} - \ln\frac{p}{\hat{p}_0}\right) + \delta p\ln\frac{p+\delta p}{\hat{p}_0}\right]\Big|_{(x,t_0)} + (\text{the same with } k_1 \text{ and } p_1)\right.$$

$$+ \iint_{x\,t}\lambda_1\left(\cancel{\frac{\partial p}{\partial t}} + \delta_t p + \cancel{\frac{\partial f}{\partial x}p} + \frac{\partial f}{\partial x}\delta p + \cancel{\frac{\partial p}{\partial x}f} + \delta_x p f - \cancel{\frac{\partial p}{\partial t}} - \cancel{\frac{\partial f}{\partial x}p} - \cancel{\frac{\partial p}{\partial x}f}\right)dxdt + \int_t \lambda_2\left[\int_x (p-p+\delta p)dx\right]dt =$$

$$= \int_x \left(k_0\left[p\ln\frac{p+\delta p}{p} + \delta p\ln\frac{p+\delta p}{\hat{p}_0}\right]\Big|_{(x,t_0)} + k_1[\dots]\Big|_{x,t_1}\right)dx +$$

$$+ \iint_{x\,t}\lambda_1(x,t)\left(\frac{\partial f}{\partial x}\delta p + \delta_t p + \delta_x p f\right)dxdt + \iint_{t\,x}\lambda_2(t)\,\delta p\,dxdt \;=$$

$$\ln(p+\delta p) - \ln(\hat{p}) = \ln p + \frac{1}{p}\delta p - \ln(\hat{p})$$

$$\iint_{x\,t}\lambda_1\delta_t p_t\,dxdt = \int_x\left[\lambda_1\delta p\Big|_{t_0}^{t_1} - \int_{t_0}^{t_1}\frac{\partial\lambda_1}{\partial t}\delta p\,dt\right]dx$$

$$\iint_{t\,x}\lambda_1 f\delta_x p_x\,dxdt = \int_t\left[\lambda_1 f\delta p\Big|_{x_{min}}^{x_{max}} - \int_x\left(\frac{\partial\lambda_1}{\partial x}f\delta p + \lambda_1\frac{\partial f}{\partial x}\delta p\right)\right]dxdt$$

$$= \int_x\left(k_0\left[\delta p\left(1 + \ln p + \frac{\delta p}{p} - \ln(\hat{p}_0)\right)\right]\Big|_{(x,t_0)} + k_1[\dots]\Big|_{(x,t_1)}\right)dx +$$

$$+ \iint_{x\,t}\lambda_1\frac{\partial f}{\partial x}\delta p\,dxdt + \int_x\left[\lambda_1(x,t)\delta p\Big|_{t_0}^{t_1} - \int_t\frac{\partial\lambda_1}{\partial t}\delta p\,dt\right]dx + \int_t\left[\lambda_1 f\delta p\Big|_{x_{min}}^{x_{max}}\right.$$

$$\left. - \int_x\left(\frac{\partial\lambda_1}{\partial x}f\delta p + \lambda_1\frac{\partial f}{\partial x}\delta p\right)\right]dxdt + \iint_{x\,t}\lambda_2(t)\,\delta p\,dxdt$$

Results:

$$\delta p(x,t_0)\left[k_0\left(1+\ln p-\ln(p_0^{\top})\right)-\lambda_1(x,t_0)\right]=0$$

$$\delta p(x,t_1)\left[k_1\left(1+\ln p-\ln(p_1^{\top})\right)-\lambda_1(x,t_1)\right]=0$$

$$\lambda_1(x_{max},t)f(x_{max},\theta)\,\delta p(x_{max},t)=0$$

$$\lambda_1(x_{min},t)f(x_{min},\theta)\,\delta p(x_{min},t)=0$$

$$\delta p\left[\lambda_1(x,t)\frac{\partial t}{\partial x}-\frac{\partial\lambda_1}{\partial t}-\frac{\partial\lambda_1}{\partial x}f(x,\theta)-\lambda_1\frac{\partial t}{\partial x}+\lambda_2\right]=0$$

Variation by θ

KL:

$$\int_x k_0 p(x(\theta)+\delta\theta) \ln \frac{p((x(\theta)+\delta\theta), t_0)}{\hat{p}_0(x(\theta)+\delta\theta)} dx = \int_x k_0 (p(x(\theta)+x'\delta\theta)) \ln \frac{p(x(\theta)+x'\delta\theta, t_0)}{\hat{p}_0(x(\theta)+x'\delta\theta)} dx = \int_x k_0 (p(x(\theta)) + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta) \cdot$$

$$\cdot \ln \frac{(p(x(\theta), t_0) + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta)}{\hat{p}_0(x(\theta)) + \frac{d\hat{p}_0}{dx}\frac{dx}{d\theta}\delta\theta} dx = \int_x k_0 (p(x(\theta), t_0) + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta)$$

$$\left(\ln(p(x(\theta), t_0) + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta) - \ln(\hat{p}_0(x(\theta)) + \frac{d\hat{p}_0}{dx}\frac{dx}{d\theta}\delta\theta)\right) dx$$

$$\ln\left(p + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta\right) = \ln\left(p\left(1 + \frac{\frac{dp}{dx}\frac{dx}{d\theta}\delta\theta}{p}\right)\right) = \ln p + \ln\left(1 + \frac{dp}{dx}\frac{dx}{d\theta}\frac{\delta\theta}{p}\right)$$

$$= \ln p + \frac{dp}{dx}\frac{dx}{d\theta}\frac{1}{p}\delta\theta$$

$\Downarrow$ (the same)

$$\ln\left(\hat{p}_0 + \frac{d\hat{p}_0}{dx}\cdot\frac{dx}{d\theta}\delta\theta\right) = \ln(\hat{p}_0) + \frac{d\hat{p}_0}{dx}\cdot\frac{dx}{d\theta}\cdot\frac{1}{\hat{p}_0}\delta\theta$$

$$\int_x k_0 \left(p(x) + \frac{dp}{dx}\frac{dx}{d\theta}\delta\theta\right)\left[\left(\ln p + \frac{dp}{dx}\frac{dx}{d\theta}\frac{\delta\theta}{p}\right) - \left(\ln\hat{p}_0 + \frac{d\hat{p}_0}{dx}\frac{dx}{d\theta}\delta\theta\frac{1}{\hat{p}_0}\right)\right] dx =$$

$$= \int_x k_0 p \ln p + k_0 p \frac{dp}{dx}\frac{dx}{d\theta}\frac{\delta\theta}{p} - k_0 p \ln\hat{p}_0 - k_0 p \frac{d\hat{p}_0}{dx}\cdot\frac{dx}{d\theta}\frac{\delta\theta}{p} +$$

$$+ k_0 \frac{dp}{dx}\cdot\frac{dx}{d\theta}\delta\theta \ln p - k_0 \frac{dp}{dx}\cdot\frac{dx}{d\theta}\delta\theta\ln\hat{p}_0 - k_0 \ln p(\ln p - \ln\hat{p}_0) =$$

$$= \int_x \left(k_0 \frac{dp}{dx}\cdot\frac{dx}{d\theta} - k_0 p \frac{d\hat{p}_0}{dx}\frac{dx}{d\theta}\cdot\frac{1}{\hat{p}_0} + k_0 \frac{dp}{dx}\frac{dx}{d\theta}\ln p - k_0 \frac{dp}{dx}\frac{dx}{d\theta}\cdot\ln\hat{p}_0\right)$$

$$\delta\theta dx = \int_x k_0 \frac{dp}{dx}\frac{dx}{d\theta}\left(1 - \frac{p}{\hat{p}_0} + \frac{\ln p}{\hat{p}_0}\right)\delta\theta dx \Rightarrow$$

$$k_0 \frac{dp}{dx}\frac{dx}{d\theta}\left(1 - \frac{p}{\hat{p}_0} + \ln\frac{p}{\hat{p}_0}\right)\Big|_{t=t_0} = 0 \qquad \delta\theta(t_0)$$

(the same with $k_1, P_1, \hat{P}_1, t_1$)

$\int:$

$$\int_t \lambda_2(t)\int_x p(x(\theta+\delta\theta),t)dx-1]dt-\int_t\lambda_2\int_x[p(x(\theta),t)dx-1]dt=$$

$$=\int_t\lambda_2(t)\left[\int_x\left(p(x(\theta),t)+\frac{dp}{dx}\frac{dx}{d\theta}\delta\theta\right)dx-1\right]dt-\int\ldots dt=$$

$$=\int_t\lambda_2(t)\left[\int_x\frac{dp}{dx}\frac{dx}{d\theta}\delta\theta dx\right]dt \Rightarrow$$

$$\delta\theta\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\lambda_2=0$$

Differential: NODE

$$\iint_{xt}\lambda_3(t)\left(\frac{dx}{dt}-f(x(\theta+\delta\theta),t)\right)dxdt-\iint\ldots dxdt=$$

$$=\iint_{xt}\lambda_3\left(\frac{d(x(\theta)+x'\delta\theta)}{dt}-\left(f(x(\theta),t)+\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta\right)\right)dxdt-$$

$$-\iint\ldots dtdx=\iint_{xt}\lambda_3\left(\frac{d(x(\theta))}{dt}+\frac{d^2x}{dt}\delta\theta+\frac{dx}{d\theta}\delta_t\theta-\cancel{f}-\frac{\partial f}{dx}\frac{\partial x}{\partial\theta}\delta\theta\right)dxdt$$

$$-\cancel{\iint}\ldots=\iint_{xt}\lambda_3\left(\frac{dt\,dx}{dt\,d\theta}\right)\delta\theta+\frac{dx}{d\theta}\delta_t\theta-\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta\Big)dxdt=$$

(by parts)

$$=\iint_{xt}\lambda_3\left(\left(\frac{d^2x}{d\theta^2}\frac{d\theta}{dt}\delta\theta-\delta\theta\frac{d^2x}{d\theta^2}\frac{d\theta}{dt}-\frac{\partial t}{\partial t}\frac{dx}{\partial\theta}\delta\theta\right)-\delta\theta\frac{\partial\lambda_3}{dt}\frac{\partial x}{d\theta}\right)dxdt$$

$$\Rightarrow\quad -\delta\theta:\ \left(\lambda_3\frac{\partial f}{\partial x}\frac{\partial x}{d\theta}+\frac{\partial\lambda_3}{dt}\frac{dx}{d\theta}\right)=0,\quad \delta\theta(t_1):\lambda_3\frac{dx}{d\theta}=0$$

$$\delta\theta(t_0):-\lambda_3\frac{dx}{d\theta}=0$$

$$PF:\ \iint_{xt}\lambda_1(x(\theta+\delta\theta),t)\Big[\frac{\partial p(x(\theta),t)+\delta\theta}{dt}+$$

$$+\frac{\partial}{\partial x}\left[f(x(\theta+\delta\theta),t)p(x(\theta+\delta\theta,t))\right]\Big]dxdt-\int\ldots=$$

$$=\iint_{xt}\left(\lambda_1(x(\theta),t)+\frac{d\lambda_1}{dx}\frac{dx}{d\theta}\delta\theta\right)\Big[\frac{\partial\left(p+\frac{dp}{dx}\frac{dx}{d\theta}\delta\theta\right)}{dt}+\frac{\partial f}{dx}\Big[\left(f+\frac{\partial f}{dx}\frac{dx}{d\theta}\delta\theta\right)\cdot$$

$$\cdot\left(p+\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta\right)\Big]\Big]-\int\ldots dxdt=\iint_{xt}\left(\lambda_1+\frac{\partial\lambda_1}{dx}\frac{dx}{\partial\theta}\delta\theta\right).$$

$$\left[\frac{\partial p}{\partial t} + \frac{\partial^2 p}{\partial t\, dx}\frac{\partial x}{\partial \theta}\delta\theta + \frac{\partial x}{\partial\theta}\cdot\frac{\partial p}{\partial x}\delta_t\theta + \dots\right] dx\,dt - \int =$$

$$= \iint_{xt} \left(\lambda_1 + \frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta\right)\left[\frac{\partial^2 p}{\partial t\,\partial x}\frac{dx}{d\theta}\delta\theta + \frac{\partial x}{\partial\theta}\frac{\partial p}{\partial x}\delta_t\theta + 2\frac{\partial p}{\partial x}\frac{\partial t}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta + \right.$$

$$\left. + f\frac{\partial^2 p}{\partial x^2}\frac{\partial x}{\partial\theta} + p\frac{\partial^2 t}{\partial x^2}\frac{dx}{d\theta}\delta\theta\right] dx\,dt =$$

(by parts)

$$\int_t \lambda_1 \frac{\partial p}{\partial x}\frac{dx}{\partial\theta}\delta_t\theta\, dt = \left(\lambda_1\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta\Big|_{t_0}^{t_1}\right) - \int_x \dots$$

$$= \cancel{\iint \lambda_1 \frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\delta_t\theta\, dt} \quad \iint \lambda_1\frac{\partial p}{\partial t} + \lambda_1\frac{\partial^2 p}{\partial t\,dx}\frac{\partial x}{\partial\theta}\delta\theta + \lambda_1\frac{\partial p}{\partial x}\frac{\partial^2 x}{\partial\theta^2}\cdot$$

$$\cdot\frac{\partial\theta}{\partial t}\delta\theta - \frac{\partial\lambda_1}{\partial x}\frac{\partial p}{\partial x}\left(\frac{dx}{\partial\theta}\right)^2\frac{\partial\theta}{dt}\delta\theta - \frac{\partial\lambda_1}{\partial t}\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\delta\theta -$$

$$-\lambda_1\frac{\partial^2 p}{\partial x^2}\left(\frac{\partial x^2}{\partial\theta}\right)^2\frac{\partial\theta}{dt}\delta\theta - \lambda_1\frac{\partial^2 p}{\partial t\,dx}\frac{\partial x}{\partial\theta}\delta\theta - \lambda_1\frac{\partial p}{\partial x}\frac{\partial^2 x}{\partial\theta^2}\frac{\partial\theta}{dt}\delta\theta +$$

$$+\frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\frac{\partial p}{dt}\delta\theta + \frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial t}\frac{\partial p}{\partial x}\delta\theta + p\frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\frac{dp}{\partial x}\delta\theta =$$

$$= \iint\left(\frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\frac{\partial p}{\partial t} - \frac{\partial\lambda_1}{dt}\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta} - \lambda_1\frac{\partial^2 p}{\partial x^2}\left(\frac{dx}{\partial\theta}\right)^2\frac{\partial\theta}{dt} + p\frac{\partial\lambda_1}{\partial x}\frac{dx}{\partial\theta}\frac{\partial f}{dx}\right)$$

Results:

$$\delta\theta:\quad \frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\lambda_1 - \lambda_3\frac{\partial f}{\partial x}\frac{\partial x}{\partial\theta} - \frac{\partial\lambda_3}{\partial t}\frac{\partial x}{\partial\theta} + \frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\frac{\partial p}{\partial t} - \frac{\partial\lambda_1}{\partial t}\frac{\partial p}{\partial x}\cdot$$

$$\cdot\frac{\partial x}{dt} - \lambda_1\frac{\partial^2 p}{\partial x^2}\left(\frac{\partial x}{\partial\theta}\right)^2\frac{\partial\theta}{\partial t} + p\frac{\partial\lambda_1}{\partial x}\frac{\partial x}{\partial\theta}\frac{\partial f}{\partial x} = 0$$

$$\delta\theta(t_0):\quad -\lambda_1\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta} - \lambda_3\frac{\partial x}{\partial\theta} + k_0\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\left(1 - \frac{p}{\hat{p}_0} + \ln\frac{p}{\hat{p}_0}\right)\Bigg|_{t_0} = 0$$

$$\delta\theta(t_1):\quad \lambda_1\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta} - \lambda_3\frac{\partial x}{\partial\theta} + k_1\frac{\partial p}{\partial x}\frac{\partial x}{\partial\theta}\left(1 - \frac{p}{\hat{p}_1} + \ln\frac{p}{\hat{p}_1}\right)\Bigg|_{t_1} = 0$$