Functionale:

$$\min_{p,\theta,\lambda_1,\lambda_2} L = k_0 \int_x p(x,t_0)\log\frac{p(x,t_0)}{p_0(x)}dx + k_1\int_x p(x,t_1)\log\frac{p(x,t_1)}{p_1(x)}dx +$$

$$+\int_{t_0}^{t_1}\int_x \lambda_1(x,t)\left[\frac{\partial p(x,t)}{\partial t} + \frac{\partial}{\partial x}[f(x,\theta)p(x,t)]\right]dx\,dt +$$

$$+\int_{t_0}^{t_1}\lambda_2(t)\left[\int_x p(x,t)dx - 1\right]dt + \lambda_3(\dot{x} - f(x,\theta))$$

Variation by λ_1: (using second way to vary by differentiating by parameter α)

$$\delta\lambda_1\left(\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[fp]\right) = 0$$

Variation by λ_2: (2nd way to vary)

$$\int_x p(x,t)\,dx = 1$$

Variation by λ_3: (2nd way to vary)

$$\frac{dx}{dt} - f(x,\theta) = 0$$

Variation by x:

We will use 1^{st} way to vary:

$x \to x + \delta x$

$$\frac{dy}{d(x+\delta x)} = \lim_{\Delta(x+\delta x)\to 0}\frac{y(x+\delta x|_{x=x_0} + \Delta(x+\delta x)) - y(x|_{x=x_0} + \delta x)}{\Delta x + \Delta\delta x} =$$

$$= \lim_{\to 0}\left[y(x+\delta x) + \frac{dy}{d(x+\delta x)}\Delta(x+\delta x) - y(x+\delta x)\right] =$$

$$= \lim_{\to 0} \left[y(x) + \frac{dy}{dx}\left(\delta x\big|_{x=x_0} + \Delta x + \Delta \delta x\right) - y(x) - \frac{dy}{dx}\delta x \right] =$$

$$= \frac{dy}{dx}\Big|_{x=x_0}$$

Variation by x for Perron-Frabeneus

$$\iint_{x\,t} \left(\lambda_1 + \frac{\partial \lambda_1}{\partial x}\delta x\right)\Big[\frac{\partial P}{\partial t} + \frac{\partial^2 P}{\partial t \partial x}\delta x + \frac{\partial P}{\partial x}\delta_t x + \frac{\partial}{\partial x}(pf + p'f\delta x +$$

$$+ pf'\delta x + p'f'(\delta x)^2\, dx\, dt = \iint_{x\,t}\left(\lambda_1 + \frac{\partial\lambda_1}{\partial x}\delta x\right)\Big[\frac{\partial P}{\partial t} + \frac{\partial^2 P}{\partial t\partial x}\delta x + \frac{\partial P}{\partial x}\delta_t x +$$

$$+ \frac{\partial P}{\partial x} f + \frac{\partial f}{\partial x} p + \frac{\partial^2 P}{\partial x^2} f\delta x + \frac{\partial P}{\partial x}\left(\frac{\partial f}{\partial x}\delta x + f\delta_x x\right) + \frac{\partial P}{\partial x}\frac{\partial f}{\partial x}\delta x +$$

$$+ p\frac{\partial^2 f}{\partial x^2}\delta x + p\frac{\partial f}{\partial x}\delta_x x\Big]dx\, dt = \iint_{x\,t}\left(\lambda_1 + \frac{\partial \lambda_1}{\partial x}\delta x\right)\Big[\cancel{\frac{\partial P}{\partial t}} + \frac{\partial^2 P}{\partial t \partial x}\delta x +$$

$$+ \frac{\partial P}{\partial x}\delta_t x + \cancel{\frac{\partial P}{\partial x}} f + \cancel{\frac{\partial f}{\partial x}} p + \frac{\partial^2 P}{\partial x^2} f\delta x + \frac{\partial P}{\partial x}\frac{\partial f}{\partial x}\delta x + \frac{\partial P}{\partial x}\delta_x x +$$

$$+ \frac{\partial P}{\partial x}\frac{\partial f}{\partial x}\delta x + p\frac{\partial^2 f}{\partial x^2}\delta x + p\frac{\partial f}{\partial x}\delta_x x\Big] - \lambda_1\Big[\cancel{\frac{\partial P}{\partial t}} + \cancel{\frac{\partial f}{\partial x}} p + \cancel{\frac{\partial P}{\partial x}} f\Big]dt\, dx =$$

$$=$$

$$(*)\quad \int_t \lambda_1 \frac{\partial P}{\partial x}\delta_t x\, dt = \underbrace{\delta x\, \lambda_1 \frac{\partial P}{\partial x}\Big|_{t_0}^{t_1}} - \int_t \delta x\left(\frac{\partial\lambda_1}{\partial t}\frac{\partial P}{\partial x} + \lambda_1\frac{\partial^2 P}{\partial t\partial x}\right)dt$$

$$(*)\quad \int_x \lambda_1 \frac{\partial P}{\partial x} f\delta_x x\, dx = \delta x\, \lambda_1 \frac{\partial P}{\partial x} f\Big|_{x_{min}}^{x_{max}} - \int_t \delta x\left(\frac{\partial \lambda_1}{\partial x}\frac{\partial P}{\partial x} f + \lambda_1 \cdot \left(\frac{\partial^2 P}{\partial x^2} f + \frac{\partial P}{\partial x}\cdot\frac{\partial f}{\partial x}\right)\right)dx$$

$$(*)\quad \int_x \lambda_1 p\frac{\partial f}{\partial x}\delta_x x\, dx = \delta_x \lambda_1 p \frac{\partial f}{\partial x}\Big|_{x_{min}}^{x_{max}} - \int_t \delta x\left(\frac{\partial\lambda_1}{\partial x}\frac{\partial f}{\partial x} p + \lambda_1 \cdot \left(\frac{\partial^2 f}{\partial x^2} p + \frac{\partial f}{\partial x}\frac{\partial P}{\partial x}\right)\right)dx$$

$$= \iint_{x\,t} \lambda_1 \cancel{\frac{\partial^2 P}{\partial t\partial x}}\delta x - \frac{\partial \lambda_1}{\partial t}\frac{\partial P}{\partial x}\delta x - \lambda_1\cancel{\frac{\partial^2 P}{\partial t\partial x}}\delta x + \lambda_1\cancel{\frac{\partial^2 P}{\partial x^2} f}\delta x +$$

$$+ \lambda_1 \cancel{\frac{\partial P}{\partial x}\frac{\partial f}{\partial x}}\delta x - \frac{\partial\lambda_1}{\partial x}\cancel{\frac{\partial P}{\partial x}} f\delta x - \lambda_1\cancel{\frac{\partial^2 P}{\partial x^2} f}\delta x - \lambda_1\frac{\partial P}{\partial x}\cancel{\frac{\partial f}{\partial x}}\delta x +$$

$$+\lambda_1 \cancel{\frac{\partial p}{\partial x}\frac{\partial f}{\partial x}}\delta x + \lambda_1 \cancel{p\frac{\partial^2 f}{\partial x^2}}\delta x - \lambda_1 \cancel{\frac{\partial^2 f}{\partial x^2} p}\,\delta x - \lambda_1 \cancel{\frac{\partial f}{\partial x}\frac{\partial p}{\partial x}}\delta x -$$

$$-\frac{\partial \lambda_1}{\partial x}\cancel{\frac{\partial f}{\partial x}} p\,\delta x + \frac{\partial \lambda_1}{\partial x}\frac{\partial p}{\partial t}\delta x + \frac{\partial \lambda_1}{\partial x}\cancel{\frac{\partial p}{\partial x}} f\,\delta x + \cancel{\frac{\partial \lambda_1}{\partial x}\frac{\partial f}{\partial x}} p\,\delta x\, dx\, dt =$$

$$= \iint \left(\frac{\partial \lambda_1}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial \lambda_1}{\partial t}\frac{\partial p}{\partial x}\right)\delta x\, dx\, dt$$

$$\delta x: \underbrace{\frac{\partial \lambda_1}{\partial x}\frac{\partial p}{\partial t} - \frac{\partial \lambda_1}{\partial t}\frac{\partial p}{\partial x}}_{\text{Perron-Frabenius}} + \underbrace{\frac{\partial \lambda_3}{\partial t} + \frac{\partial f}{\partial t}}_{\text{differential equation}} + \underbrace{\frac{\partial p}{\partial x}\lambda_2}_{\text{Integr.}} = 0$$

$$\delta x(t_0): \lambda_3(t_0) - \lambda_1(t_0)\frac{\partial p}{\partial x}\Big|_{t_0} + k_0 \frac{\partial p}{\partial x}\Big|_{t_0} k_0 \frac{\partial \hat{p}_0}{\partial x}\frac{p}{\hat{p}_0}\Big|_{t_0} +$$

$$+ k_0 \frac{\partial p}{\partial x}\ln p\Big|_{t_0} - k_0 \frac{\partial p}{\partial x}\ln \hat{p}_0\Big|_{t_0} = 0.$$

$$\delta x(t_1): \lambda_3(t_1) + \lambda_1(t)\frac{\partial p}{\partial x}\Big|_{t_1} + k_1 \frac{\partial p}{\partial x}\Big|_{t_1} - k_1 \frac{\partial \hat{p}_1}{\partial x}\frac{p}{\hat{p}_1}\Big|_{t_1} +$$

$$+ k_1 \frac{\partial p}{\partial x}\ln p\Big|_{t_1} - k_1 \frac{\partial p}{\partial x}\ln \hat{p}_1\Big|_{t_1} = 0$$

$$\delta x_{max}: \lambda_1 \frac{\partial p}{\partial x} f + \lambda_1 p \frac{\partial f}{\partial x}\Big|_{x_{max}} = 0$$

$$\delta x_{min}: \lambda_1 \frac{\partial p}{\partial x} f + \lambda_1 p \frac{\partial f}{\partial x}\Big|_{x_{min}} = 0$$