Functional:

$$\min_{p,\theta,\lambda_1,\lambda_2} L = k_0 \int_X p(x,t_0)\log\frac{p(x,t_0)}{p_0(x)}dx + k_1\int_X p(x,t_1)\log\frac{p(x,t_1)}{p_1(x)}dx +$$

$$+\int_{t_0}^{t_1}\int_X \lambda_1(x,t)\left[\frac{\partial p(x,t)}{\partial t} + \frac{\partial}{\partial x}[f(x,\theta)p(x,t)]\right]dx\,dt +$$

$$+\int_{t_0}^{t_1}\lambda_2(t)\left[\int_X p(x,t)dx - 1\right]dt + \lambda_3(\dot{x} - f(x,\theta))$$

note that: x is $X(\theta(t))$

Variation by λ_1:

$$\delta\lambda_1: \int_X\int_t (\lambda_1+\delta\lambda_1)\left[\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[fp]\right]dx\,dt - \int_X\int_t \lambda_1\left[\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[fp]\right]dx\,dt = \int_X\int_t \delta\lambda_1\left[\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[fp]\right]dx\,dt$$

$$\delta\lambda_1\left(\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[fp]\right) = 0$$

Variation by λ_2:

$$\delta\lambda_2: \int_t(\lambda_2+\delta\lambda_2)\left[\int_X p\,dx - 1\right]dt - \int_t \lambda_2\left[\int_X p\,dx - 1\right]dt =$$

$$= \int_t \delta\lambda_1\left[\int_X p\,dx - 1\right]dt$$

$$\int_X p(x,t)dx = 1$$

Variation by λ_3:

$$\delta\lambda_3: \int_X\int_t(\lambda_3 - \delta\lambda_3)\left[\frac{dx}{dt} - f(x,\theta)\right] - \lambda_3\left[\frac{dx}{dt} - f(x,\theta)\right]dx\,dt =$$

$$= \int_X\int_t \delta\lambda_3\left[\frac{dx}{dt} - f\right]dx\,dt \qquad \frac{dx}{dt} - f(x,\theta) = 0$$

Variation by x:

We will use 1^{st} way to vary:

$$x \to x + \delta x$$

$$\frac{dy}{d(x+\delta x)} = \lim_{\Delta(x+\delta x)\to 0}\frac{y(x+\delta x|_{x=x_0} + \Delta(x+\delta x)) - y(x|_{x=x_0}+\delta x)}{\Delta x + \Delta\delta x} =$$

$$= \lim_{\to 0}\left[y(x+\delta x) + \frac{dy}{d(x+\delta x)}\Delta(x+\delta x) - y(x+\delta x)\right] =$$

$$= \lim_{\to 0} \left[y(x) + \frac{dy}{dx}\left(\delta x\Big|_{x=x_0} + \Delta x + \Delta \delta x\right) - y(x) - \frac{dy}{dx}\delta x \right] =$$

$$= \frac{dy}{dx}\Big|_{x=x_0}$$

Variation by x for Perron-Frobenius

$$\iint_{x\,t} \left(\lambda_1 + \frac{\partial \lambda_1}{\partial x}\delta x\right)\left[\frac{\partial P}{\partial t} + \frac{\partial^2 P}{\partial t \partial x}\delta x + \frac{\partial P}{\partial x}\delta_t x + \frac{\partial}{\partial x}\left(pf + p'f\delta x +\right.\right.$$

$$+ pf'\delta x + p'f'(\delta x)^2 \, dx\,dt = \iint_{x\,t} \left(\lambda_1 + \frac{\partial \lambda_1}{\partial x}\delta x\right)\left[\frac{\partial P}{\partial t} + \frac{\partial^2 P}{\partial t \partial x}\delta x + \frac{\partial P}{\partial x}\delta_t x +\right.$$

$$+ \frac{\partial P}{\partial x} f + \frac{\partial f}{\partial x} p + \frac{\partial^2 p}{\partial x^2} f \delta x + \frac{\partial P}{\partial x}\left(\frac{\partial t}{\partial x}\delta x + f\delta_x X\right) + \frac{\partial P}{\partial x}\frac{\partial t}{\partial x}\delta x +$$

$$+ p\frac{\partial^2 f}{\partial x^2}\delta x + p\frac{\partial t}{\partial x}\delta_x X\Big] dx\,dt = \iint_{x\,t}\left(\lambda_1 + \frac{\partial \lambda_1}{\partial x}\delta x\right)\left[\cancel{\frac{\partial P}{\partial t}} + \frac{\partial^2 P}{\partial t \partial x}\delta x +\right.$$

$$+ \frac{\partial P}{\partial x}\delta_t x + \cancel{\frac{\partial P}{\partial x}} f + \cancel{\frac{\partial t}{\partial x}} p + \frac{\partial^2 P}{\partial x^2} f \delta x + \frac{\partial P}{\partial x}\frac{\partial t}{\partial x}\delta x + \frac{\partial P}{\partial x}\delta_x X +$$

$$+ \frac{\partial P}{\partial x}\frac{\partial f}{\partial x}\delta x + p\frac{\partial^2 f}{\partial x^2}\delta x + p\frac{\partial t}{\partial x}\delta_x X\Big] - \lambda_1\left[\cancel{\frac{\partial P}{\partial t}} + \cancel{\frac{\partial t}{\partial x}} p + \cancel{\frac{\partial P}{\partial x}} f\right] dt\,dx =$$

$$\circledcirc$$

$$\circledast \int_t \lambda_1 \frac{\partial P}{\partial x}\delta_t x\,dt = \underbrace{\delta x \lambda_1 \frac{\partial P}{\partial x}\Big|_{t_0}^{t_1}}_{} - \int_t \delta x\left(\frac{\partial \lambda_1}{\partial t}\frac{\partial P}{\partial x} + \lambda_1 \frac{\partial^2 P}{\partial t \partial x}\right) dt$$

$$\circledast \int_x \lambda_1 \frac{\partial P}{\partial x} f \delta_x X\,dx = \delta x \lambda_1 \frac{\partial P}{\partial x} f\Big|_{X_{min}}^{X_{max}} - \int_t \delta x\left(\frac{\partial \lambda_1}{\partial x}\frac{\partial P}{\partial x} f + \lambda_1 \cdot \right.$$

$$\left.\cdot \left(\frac{\partial^2 P}{\partial x^2} f + \frac{\partial P}{\partial x}\cdot\frac{\partial f}{\partial x}\right)\right) dx$$

$$\circledast \int_x \lambda_1 p \frac{\partial f}{\partial x}\delta_x X\,dx = \delta_x \lambda_1 p \frac{\partial f}{\partial x}\Big|_{X_{min}}^{X_{max}} - \int_t \delta x\left(\frac{\partial \lambda_1}{\partial x}\frac{\partial f}{\partial x} p + \lambda_1 \cdot \right.$$

$$\left.\cdot\left(\frac{\partial^2 f}{\partial x^2} p + \frac{\partial f}{\partial x}\frac{\partial P}{\partial x}\right)\right) dx$$

$$\circledcirc \iint_{x\,t} \lambda_1 \cancel{\frac{\partial^2 P}{\partial t \partial x}}\delta x - \frac{\partial \lambda_1}{\partial t}\frac{\partial P}{\partial x}\delta x - \lambda_1 \cancel{\frac{\partial^2 P}{\partial t \partial x}}\delta x + \lambda_1 \cancel{\frac{\partial P}{\partial x^2}} f \delta x +$$

$$+ \lambda_1 \cancel{\frac{\partial P}{\partial x}\frac{\partial f}{\partial x}}\delta x - \frac{\partial \lambda_1}{\partial x}\cancel{\frac{\partial P}{\partial x}} f \delta x - \lambda_1 \cancel{\frac{\partial^2 P}{\partial x^2}} f \delta x - \lambda_1 \cancel{\frac{\partial P}{\partial x}}\cancel{\frac{\partial f}{\partial x}}\delta x +$$

$$+ \lambda_1 \cancel{\frac{\partial p}{\partial x} \frac{\partial f}{\partial x}} \delta x + \lambda_1 p \cancel{\frac{\partial^2 f}{\partial x^2}} \delta x - \lambda_1 \cancel{\frac{\partial^2 f}{\partial x^2}} p \, \delta x - \lambda_1 \cancel{\frac{\partial f}{\partial x} \frac{\partial p}{\partial x}} \delta x -$$

$$- \frac{\partial \lambda_1}{\partial x} \cancel{\frac{\partial f}{\partial x}} p \, \delta x + \frac{\partial \lambda_1}{\partial x} \frac{\partial p}{\partial t} \delta x + \frac{\partial \lambda_1}{\partial x} \cancel{\frac{\partial p}{\partial x}} f \, \delta x + \cancel{\frac{\partial \lambda_1}{\partial x} \frac{\partial f}{\partial x}} p \, \delta x \, dx \, dt =$$

$$= \iint \left(\frac{\partial \lambda_1}{\partial x} \frac{\partial p}{\partial t} - \frac{\partial \lambda_1}{\partial t} \frac{\partial p}{\partial x} \right) \delta x \cdot dx \, dt$$

Variation by x for kullback-leibler divergence:

$$\int_x k_0\, p(x+\delta x, t) \ln \frac{p(x+\delta x, t)}{\hat{p}_0(x+\delta x)} = \int_x k_0 p(\ln p + p'\delta x - \ln \hat{p}_0 + \hat{p}_0'\delta x) +$$

$$+ k_0 p'\delta x (\ln p + p'\delta x - \ln \hat{p}_0 - \hat{p}_0 \delta x) =$$

$$* \quad \ln(p + p'\delta x) = \ln\left(p\left(1 + \frac{p'\delta x}{p}\right)\right) = \ln p + \ln\left(1 + \frac{p'}{p}\delta x\right) = \ln p + \frac{p'}{p}\delta x$$

Similarly:

$$\ln(\hat{p}_0 + \hat{p}_0'\delta x) = \ln \hat{p}_0 + \frac{\hat{p}_0'}{\hat{p}_0}\delta x$$

$$\ominus \int_x k_0 p \left(\ln p + \frac{p'}{p}\delta x - \ln \hat{p}_0 - \frac{\hat{p}_0'}{\hat{p}_0}\delta x\right) + k_0 p'\delta x \left(\ln p + \frac{p'}{p}\delta x - \ln \hat{p}_0 - \frac{\hat{p}_0'}{\hat{p}_0}\delta x\right) = \int_x \delta x \underbrace{\left(k_0 p \frac{p'}{p} - k_0 \frac{\hat{p}_0'}{\hat{p}_0} p + k_0 p' \ln p - k_0 p' \ln \hat{p}_0\right)}$$

Similarly for term wit k_1 and p_1

Variation for differential equation:

$$\iint_{x\,t} \lambda_3 \left(\frac{d(x+\delta x)}{dt} - f(x+\delta x, \theta) - \frac{dx}{dt} + f(x,\theta)\right) dx\,dt = \iint_{x\,t} \frac{dx}{dt} + \delta_t x - f(x,\theta) - f'\delta x - \frac{dx}{dt} + f(x,\theta)\,\lambda_3\,dx\,dt = \iint_{x\,t} \lambda_3(t)(\delta_t x - f'\delta x)\,dx\,dt \ominus$$

$$\frac{dx+\delta x}{dt} = \frac{dx}{dt} + \frac{d\delta x}{dt} = \delta_t x$$

$$\circledast \int_t \lambda_3(t)\,\delta_t x\,dt = \lambda_3 \delta x \Big|_{t_0}^{t_1} - \int_t \frac{\partial \lambda_3}{\partial t} \delta x\,dt$$

$$\ominus \iint_{t\,x} \lambda_3(t)\left(-\frac{\partial \lambda_3}{\partial t}\delta x - \frac{\partial f}{\partial t}\delta x\right) dx\,dt + \lambda_3 \delta x \Big|_{t_0}^{t_1}$$

$$-\underbrace{\delta x \left(\frac{\partial \lambda_3}{\partial t} + \frac{\partial f}{\partial t}\right) = 0}$$

$$\underbrace{\lambda_3(t_0) = 0}$$

$$\underbrace{\lambda_3(t_1) = 0}$$

Variation for term with λ_2:

$$\int_t \lambda_2(t) \int_x [p(x+\delta x, t)\,dx - 1]\,dt - \int_t \lambda_2(t) \int_x [p(x,t)\,dx - 1]\,dt =$$

$$= \int_t \lambda_2(t) \int_x [p(x+\delta x, t) - p(x,t)]\,dx - 1 + 1\,dt = \int_t \lambda_2(t) \int_x [p(x,t) + p_x'\delta x - p(x,t)]\,dx\,dt = \int_t \lambda_2(t) \int_x p_x' \delta x\,dx\,dt \qquad \underbrace{\lambda_2 \frac{\partial p}{\partial x} = 0}$$

$$\delta x: \underbrace{\frac{\partial \lambda_1}{\partial x}\frac{\partial P}{\partial t} - \frac{\partial \lambda_1}{\partial t}\frac{\partial P}{\partial x}}_{\text{Perron-Frabenius}} + \underbrace{\frac{\partial \lambda_3}{\partial t} + \frac{\partial t}{\partial t}}_{\text{differential equation}} + \underbrace{\frac{\partial P}{\partial x}\lambda_2}_{\text{integr.}} = 0$$

$$\delta x(t_0): \lambda_3(t_0) - \lambda_1(t_0)\frac{\partial P}{\partial x}\Big|_{t_0} + k_0 \frac{\partial P}{\partial x}\Big|_{t_0} \; k_0 \frac{\partial \hat{P}_0}{\partial x}\frac{P}{\hat{P}_0}\Big|_{t_0} +$$

$$+ k_0 \frac{\partial P}{\partial x} \ln p\Big|_{t_0} - k_0 \frac{\partial P}{\partial x} \ln \hat{p}_0\Big|_{t_0} = 0.$$

$$\delta x(t_1): \lambda_3(t_1) + \lambda_1(t_1)\frac{\partial P}{\partial x}\Big|_{t_1} + k_1 \frac{\partial P}{\partial x}\Big|_{t_1} - k_1 \frac{\partial \hat{P}_1}{\partial x}\frac{P}{\hat{P}_1}\Big|_{t_1} +$$

$$+ k_1 \frac{\partial P}{\partial x} \ln p\Big|_{t_1} - k_1 \frac{\partial P}{\partial x} \ln \hat{p}_1\Big|_{t_1} = 0$$

$$\delta X_{max}: \lambda_1 \frac{\partial P}{\partial x} f + \lambda_1 p \frac{\partial f}{\partial x}\Big|_{X_{max}} = 0$$

$$\delta X_{min}: \lambda_1 \frac{\partial P}{\partial x} f + \lambda_1 p \frac{\partial f}{\partial x}\Big|_{X_{min}} = 0$$