Computational and Variational Methods for Inverse Problems - Homework 4 Solution

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1 Inverse problem for Burgers' equation

Consider the inverse problem for the viscosity field $m(x)$ in the one-dimensional Burgers' equation

$$u_t + uu_x - (mu_x)_x = f \quad \text{in } (0,L)\times(0,T) \tag{1.1}$$
$$u(0,t) = u(L,t) = 0 \quad \text{for all } t \in [0,T] \tag{1.2}$$
$$u(x,0) = 0 \quad \text{for all } x \in [0,L] \tag{1.3}$$

Given observations $d = d(x,t)$ for $t \in [T_1, T]$, where $T_1 > 0$, we invert for the viscosity field $m = m(x)$ by solving the minimisation problem

$$\min_{m\in\mathcal{M}} \mathcal{F}(m) := \frac{1}{2}\int_{T_1}^{T}\int_0^L (u-d)^2 \, dx \, dt + \frac{\beta}{2}\int_0^L \left(\frac{\mathrm{d}m}{\mathrm{d}x}\right)^2 dx \tag{1.4}$$

where the space $\mathcal{M}$ is defined as $\mathcal{M} = H^1(0,L)$, regularization parameter $\beta > 0$.

1.1 Weak form of the forward problem

The weak form of the forward problem can be found by integrating with a test function $p(x,t) : [0,L]\times[0,T] \to \mathbb{R}$. Therefore, the solution space $\mathcal{U}$ considered is

$$\mathcal{U} = \{u \in H^1(0,L)\times L^2(0,T) : u(0,t) = u(L,t) = 0 \text{ and } u(x,0) = 0\} \tag{1.5}$$

Similarly, the space for the adjoint variable is $\mathcal{P}$ given by

$$\mathcal{P} = \{p \in H^1[0,L]\times L^2[0,T] : p(0,t) = p(L,t) = 0\} \tag{1.6}$$

The homogeneous Dirichlet boundary conditions gave been enforced in the weak forms. Integrating the PDE against p yields

$$\int_0^T\int_0^L p\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(m\frac{\partial u}{\partial x}\right) - f\right) dx\, dt = 0 \tag{1.7}$$

Now, performing the integration by parts of the diffusion term, the "weak form" of the forward problem can be yielded as follows:

$$\boxed{\text{Find } u\in\mathcal{U} \text{ such that: } \int_0^T\int_0^L p\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f\right) + \frac{\partial p}{\partial x}\left(m\frac{\partial u}{\partial x}\right) dx\, dt = 0 \quad \forall p\in\mathcal{P}} \tag{1.8}$$

with homogeneous Dirichlet boundary conditions $p(0,t) = p(L,t) = 0$

1.2 Adjoint and gradient of the Lagrangian $\mathcal{L}$

Let's first form the lagrangian by clubbing the weak form of the forward problem (1.8) and the objective function (1.4) together.

$$\mathcal{L}(u,p,m) = \frac{1}{2}\int_{T_1}^{T}\int_0^L (u-d)^2\, dx\, dt + \frac{\beta}{2}\int_0^L\left(\frac{\mathrm{d}m}{\mathrm{d}x}\right)^2 dx + \int_0^T\int_0^L p\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - f\right) + \frac{\partial p}{\partial x}\left(m\frac{\partial u}{\partial x}\right) dx\, dt \tag{1.9}$$

(a) Adjoint equation, $\delta_u \mathcal{L} = 0 \; \forall \hat{u} \in \mathcal{U}$

The weak form of adjoint equation can be found by taking the Frechet derivative with respect to u.

$$\boxed{\text{Find } p \in \mathcal{P} \text{ such that: } \delta_u \mathcal{L} = \int_{T_1}^{T}\int_0^L \hat{u}(u-d)\, dx\, dt + \int_0^T\int_0^L p\frac{\partial \hat{u}}{\partial t} + p\hat{u}\frac{\partial u}{\partial x} + pu\frac{\partial \hat{u}}{\partial x} + \frac{\partial p}{\partial x} m \frac{\partial \hat{u}}{\partial x}\, dx\, dt = 0 \quad \forall \hat{u} \in \mathcal{U}} \tag{1.10}$$

The strong form of the adjoint equation is derived by using integration by parts

$$\int_0^T\int_0^L \mathbb{1}_{[T_1,T]}(t)\hat{u}(u-d)\, dx\, dt + \int_0^T\int_0^L \hat{u}\left[-\frac{\partial p}{\partial t} + p\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(pu) - \frac{\partial}{\partial x}\left(m\frac{\partial p}{\partial x}\right)\right] dx\, dt$$
$$+ \int_0^L [p\hat{u}]_0^T\, dx + \int_0^T [pu\hat{u}]_0^L\, dt + \int_0^T \left[m\frac{\partial p}{\partial x}\hat{u}\right]_0^L dt = 0$$

where $\mathbb{1}_{[T_1,T]}(t)$ is a function defined as

$$\mathbb{1}_{[T_1,T]}(t) = \begin{cases} 1, & t \in [T_1, T] \\ 0, & \text{else} \end{cases} \tag{1.11}$$

In this case, all the boundary term disappear due to the homogeneous Dirichlet boundary conditions. The initial conditions are also $u(x,0) = 0$. Furthermore, using

$$p\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}(pu) = p\frac{\partial u}{\partial x} - u\frac{\partial p}{\partial x} - p\frac{\partial u}{\partial x} = -u\frac{\partial p}{\partial x} \tag{1.12}$$

and the arbitrariness of $\hat{u}$ in the spatio-temporal domain, the strong form of the adjoint equation can finally be written as follows

$$\boxed{\begin{aligned} -\frac{\partial p}{\partial t} - \frac{\partial p}{\partial x}u - \frac{\partial}{\partial x}\left(m\frac{\partial p}{\partial x}\right) &= -\mathbb{1}_{[T_1,T]}(t)(u-d) && \text{in } (0,L)\times(0,T) \\ p(0,t) = p(L,t) &= 0 && \text{for all } t \in [0,T] \\ p(x,T) &= 0 && \text{for all } t \in [0,T] \end{aligned}}$$

(b) Gradient w.r.t. m, $\delta_m \mathcal{L} = \langle \mathcal{G}(m), \hat{m} \rangle \; \forall \hat{m} \in \mathcal{M}$

Taking the Frechet derivative of the Lagrangian (1.9) w.r.t. m

$$\boxed{\langle \mathcal{G}(m), \hat{m} \rangle = \beta \int_0^L \frac{\mathrm{d}m}{\mathrm{d}x}\frac{\mathrm{d}\hat{m}}{\mathrm{d}x}\, dx + \int_0^T\int_0^L \hat{m}\frac{\partial p}{\partial x}\frac{\partial u}{\partial x}\, dx\, dt \quad \forall \hat{m} \in \mathcal{M}} \tag{1.13}$$

To obtain the strong form, let's perform integration by parts on the first term on RHS,

$$\langle \mathcal{G}(m), \hat{m} \rangle = \int_0^L \hat{m}\left(-\beta\frac{\mathrm{d}^2 m}{\mathrm{d}x^2} + \int_0^T \frac{\partial p}{\partial x}\frac{\partial u}{\partial x}\, dt\right) dx + \left[\beta\frac{\mathrm{d}m}{\mathrm{d}x}\hat{m}\right]_0^L$$

By using the arbitrariness of $\hat{m}$ in the spatio-temporal domain, we finally get the strong form of the gradient

$$\boxed{\mathcal{G}(m) = \begin{cases} -\beta\dfrac{\mathrm{d}^2 m}{\mathrm{d}x^2} + \displaystyle\int_0^T \frac{\partial p}{\partial x}\frac{\partial u}{\partial x}\, dt & (x,t) \in (0,L)\times(0,T) \\ -\beta\left.\dfrac{\mathrm{d}m}{\mathrm{d}x}\right|_{x=0} & x = 0 \\ \beta\left.\dfrac{\mathrm{d}m}{\mathrm{d}x}\right|_{x=L} & x = L \end{cases}}$$

2 Inverse advection-diffusion inverse problem

Here we consider an inverse problem for advection-diffusion-reaction equation, on the domain $\Omega = [0,1] \times [0,1]$:

$$\min_{m \in H^1(\Omega)} \mathcal{F}(m) := \frac{1}{2} \int_\Omega (u(m) - d)^2 \, dx \, + \frac{\beta}{2} \int_\Omega |\nabla m|^2 \, dx \tag{2.1}$$

subject to the PDE constraint

$$-\nabla \cdot (k \nabla u) + \boldsymbol{v} \cdot \nabla u + 100 \exp(m) u^3 = f \quad \text{in } \Omega \tag{2.2}$$

$$u = 0 \quad \text{on } \Omega \tag{2.3}$$

We take $f = \max\{0.5, \exp(-25(x-0.7)^2 - 25(y-0.7)^2)\}$, $k = 1$ and $\boldsymbol{v} = (1,0)^T$. The "true" reaction coefficient field m is defined as

$$m(x,y) = \begin{cases} 4 & \text{if } (x-0.5)^2 + (y-0.5)^2 < 0.2^2 \\ 8 & \text{otherwise} \end{cases} \tag{2.4}$$

Data is generated by adding noise of standard deivation of 0.01 in true values of the parameters.

Let's first define the Lagrangian

$$\mathcal{L}(u,p,m) = \frac{1}{2} \int_\Omega (u(m) - d)^2 \, dx \, + \frac{\beta}{2} \int_\Omega |\nabla m|^2 \, dx \, + \int_\Omega k \nabla u \cdot \nabla p + p \boldsymbol{v} \cdot \nabla u + 100 \exp(m) u^3 p - f p \, dx$$

where the weak form of the PDE has been derived from integration by parts against p, and the boundary terms are nil due to the homogeneous boundary condition. We then proceed to derive the gradient as follows

1. Forward problem:

$$\text{Find } u \in H_0^1(\Omega) : \, \delta_p \mathcal{L} = \int_\Omega k \nabla u \cdot \nabla \hat{p} + \hat{p} \boldsymbol{v} \cdot \nabla u + 100 \exp(m) u^3 \hat{p} - f \hat{p} \, dx \, = 0 \quad \forall \hat{p} \in H_0^1(\Omega) \tag{2.5}$$

2. Adjoint problem:

$$\text{Find } p \in H_0^1(\Omega) : \, \delta_u \mathcal{L} = \int_\Omega \hat{u}(u-d) \, dx \, + \int_\Omega k \nabla \hat{u} \cdot \nabla p + p v \cdot \nabla \hat{u} + 300 \exp(m) u^2 p \hat{u} \, dx \, = 0 \quad \forall \hat{u} \in H_0^1(\Omega) \tag{2.6}$$

3. Gradient evaluation w.r.t. m:

$$\mathcal{G}(m; \hat{m}) = \delta_m \mathcal{L} = \beta \int_\Omega \nabla m \cdot \nabla \hat{m} + \int_\Omega 100 \exp(m) \hat{m} u^3 p \, dx \tag{2.7}$$

For deriving the Hessian action in direction $\tilde{m}$, let's define a new Lagrangian to enforce the forward and adjoint equations

$$\mathcal{L}^H(u,p,m,\tilde{u},\tilde{p},\tilde{m}) = \delta_m \mathcal{L}(\tilde{m}) + \delta_p \mathcal{L}(\tilde{p}) + \delta_u \mathcal{L}(\tilde{u}) \tag{2.8}$$

By substituting the gradient expressions from (2.5), (2.6) and (2.7) in (2.8), we get

$$\begin{aligned} \mathcal{L}^H(u,p,m,\tilde{u},\tilde{p},\tilde{m}) = \beta \int_\Omega \nabla m \cdot \nabla \tilde{m} + \int_\Omega 100 \exp(m) \tilde{m} u^3 p \, dx \, + \int_\Omega k \nabla u \cdot \nabla \tilde{p} + \tilde{p} \boldsymbol{v} \cdot \nabla u + 100 \exp(m) u^3 \tilde{p} - f \tilde{p} \, dx \\ + \int_\Omega \tilde{u}(u-d) \, dx \, + \int_\Omega k \nabla \tilde{u} \cdot \nabla p + p v \cdot \nabla \tilde{u} + 300 \exp(m) u^2 p \tilde{u} \, dx \end{aligned} \tag{2.9}$$

For the second variations, using the calculus of variations $\mathcal{L}^H$ for variable $\alpha \to \alpha + \epsilon \hat{\alpha}$ and performing $d/d\epsilon$ with limit $\epsilon \to 0$:

1. **Incremental forward problem**: $\delta_p \mathcal{L}^H = 0 \, \forall \tilde{p}$

$$\boxed{\begin{aligned}\text{Find } \tilde{u} \in H_0^1(\Omega) : \delta_p \mathcal{L}^H = \int_\Omega & 100 \exp(m) \tilde{m} u^3 \hat{p} \, dx \; + k \nabla \tilde{u} \cdot \nabla \hat{p} \\ & + \hat{p} v \cdot \nabla \tilde{u} + 300 \exp(m) u^2 \hat{p} \tilde{u} \, dx \; = 0 \quad \forall \hat{p} \in H_0^1(\Omega)\end{aligned}} \tag{2.10}$$

For strong form, performing integration by parts on the second term on RHS to get $\hat{p}$ out and using the arbitrariness of $\hat{p}$ on Ω, we get the strong form

$$100 \exp(m) \tilde{m} u^3 - \int_\Omega k \Delta \tilde{u} + v \cdot \nabla \tilde{u} + 300 \exp(m) u^2 \tilde{u} = 0 \quad \text{in } \Omega \tag{2.11}$$

$$\tilde{u} = 0 \quad \text{on } \partial\Omega \tag{2.12}$$

2. **Incremental adjoint problem**: $\delta_u \mathcal{L}^H = 0 \, \forall \tilde{u}$

$$\boxed{\begin{aligned}\text{Find } \tilde{p} \in H_0^1(\Omega) : \delta_u \mathcal{L}^H = & \int_\Omega 300 \exp(m) \tilde{m} u^2 \hat{u} p \, dx \; + \int_\Omega k \nabla \hat{u} \cdot \nabla \tilde{p} \, dx \; + \int_\Omega \tilde{p} \boldsymbol{v} \cdot \nabla \hat{u} \, dx \\ & + \int_\Omega 300 \exp(m) u^2 \hat{u} \tilde{p} \, dx \; + \int_\Omega \tilde{u} \hat{u} \, dx \; + \int_\Omega 600 \exp(m) u p \tilde{u} \hat{u} \, dx \; = 0 \\ & \forall \hat{u} \in H_0^1(\Omega)\end{aligned}}$$

$$\tag{2.13}$$

Again, integration by parts of the second and third terms and arbitrariness of $\hat{u}$ on Ω gives the strong form

$$300 \exp(m) \tilde{m} u^2 p - k \Delta \tilde{p} - \nabla \cdot (\tilde{p} \boldsymbol{v}) + 300 \exp(m) u^2 \tilde{p} + \tilde{u} + 600 \exp(m) u p \tilde{u} = 0 \quad \text{in } \Omega \tag{2.14}$$

$$\tilde{p} = 0 \quad \text{on } \partial\Omega \tag{2.15}$$

3. **Hessian action**: We wish to derive the action of the Hessian in direction $\tilde{m}$. Using the calculus of variations the gradient is $m \to m + \epsilon \hat{m}$ and performing $\mathrm{d}/\mathrm{d}\epsilon$ with $\epsilon \to 0$, we get the weak form of the Hessian action $\mathcal{H}(\tilde{m})\hat{m} = \delta_m \mathcal{L}^H$

$$\boxed{\begin{aligned}\mathcal{H}(\tilde{m})\hat{m} = \int_\Omega & \beta \nabla \tilde{m} \cdot \nabla \hat{m} + 100 \exp(m) \hat{m} \tilde{m} u^3 p + 300 \exp(m) \hat{m} \tilde{u} u^2 p \\ & + 100 \exp(m) \hat{m} u^3 \tilde{p} \;\; dx \quad \forall \hat{m} \in H^1(\Omega)\end{aligned}} \tag{2.16}$$

Finally, one last integration by parts gives

$$\begin{aligned}\mathcal{H}(\tilde{m})\hat{m} = \int_\Omega & -\beta \hat{m} \Delta \tilde{m} + 100 \exp(m) \hat{m} \tilde{m} u^3 p + 300 \exp(m) \hat{m} \tilde{u} u^2 p \\ & + 100 \exp(m) \hat{m} u^3 \tilde{p} \;\; dx + \int_\Omega \hat{m} \beta \nabla \tilde{m} \cdot n \, \mathrm{d}s\end{aligned} \tag{2.17}$$

From the arbitrariness of $\hat{m}$ on Ω gives the strong form

$$\boxed{\mathcal{H}(\tilde{m}) = \begin{cases} -\beta \Delta \tilde{m} + 100 \exp(m) \tilde{m} u^3 p + 300 \exp(m) \tilde{u} u^2 p + 100 \exp(m) u^3 \tilde{p} & \text{in } \Omega \\ \beta \nabla \tilde{m} \cdot n & \text{on } \partial\Omega \end{cases}}$$

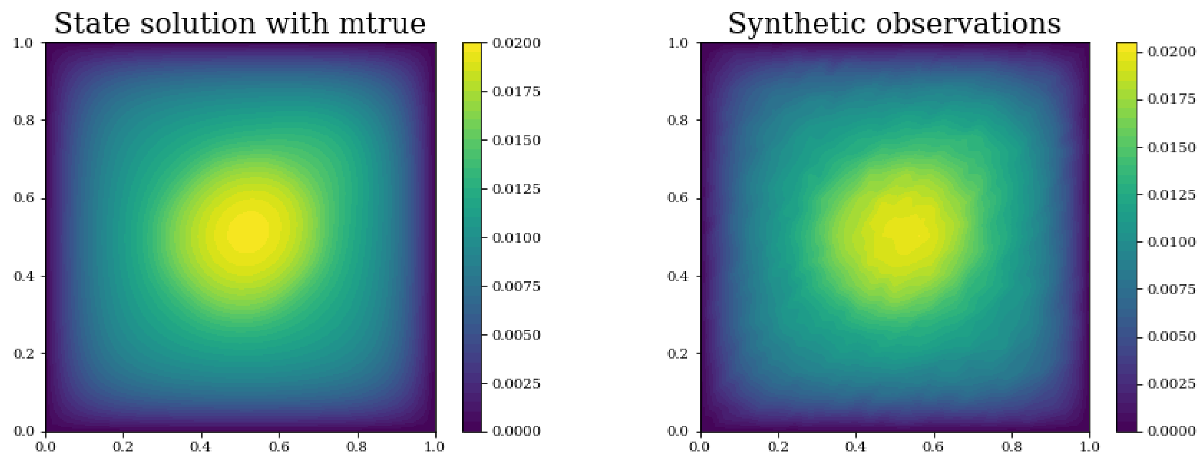

Figure 1: The true solution and noisy solution after adding 0.01% noise.

The jupyter notebook '`HW5_Q2.ipynb`' is provided for the solution codes.