Computational and Variational Methods for Inverse Problems - Homework 1 Solution

Mohammad Afzal Shadab (ms82697)

mashadab@utexas.edu

Due Monday, September 20, 2021

---

1 Problem

Considering the following unconstrained optimization problem in which the length constraint is enforced approximately using a quadratic penalty method

$$\min_u \quad \Phi(u) = E(u) + \alpha(L(u) - L_0)^2 \tag{1.1}$$

$$\text{such that} \quad u(0) = u(1) = 0 \tag{1.2}$$

here the penalty parameter $\alpha > 0$.

(a) Length of chain

We need to find the stationary points for the minimum potential energy, for that purpose need to perform functional differentiation using Calculus of variation.

There can be multiple ways of proving it: such as using curvilinear coordinates transformation or just considering an infinitesimal chain element ds on the caternary. We will use the latter as it is much simpler conceptually.

So, considering an infinitesimal length of chain ds, using pythagorean theorem the length of the arc is

$$ds^2 = dx^2 + du^2 \tag{1.3}$$

where x is the horizontal distance and u is the vertical displacement of the chain. Taking dx^2 common from RHS and the square root of both sides gives

$$ds = \sqrt{1 + \left(\frac{du}{dx}\right)^2} dx \tag{1.4}$$

Then integrating the chain from $x = 0$ to $x = 1$ gives

$$\boxed{\int_0^{L(u)} ds = L(u) = \int_0^1 \sqrt{1 + \left(\frac{du}{dx}\right)^2} dx} \tag{1.5}$$

(b) Gravitational potential energy of the hanging chain

Similar to part (a), consider the gravitational potential energy of an element of chain

$$de = \text{vertical height * acceleration due to gravity * element mass} \tag{1.6}$$

$$= u * g * (\rho\, ds) \tag{1.7}$$

where ρ is the mass per using length of the chain. Using (1.4)

$$de = \rho g u \sqrt{1 + \left(\frac{du}{dx}\right)^2} dx \tag{1.8}$$

Now, integrating again from $x = 0$ to $x = 1$, we get

$$\boxed{\int_0^{E(u)} de = E(u) = \int_0^1 \rho g u \sqrt{1 + \left(\frac{du}{dx}\right)^2} dx} \tag{1.9}$$