Probability, Random Processes and Inference

Solución de la serie de ejercicios 1

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1. A drunkard removes two randomly chosen letters of the message HAPPY HOUR that is attached on a billboard in a pub. His drunk friend puts the two letters back in a random order. What is the probability that HAPPY HOUR appears again?

① Para este problema podemos tener los siguientes casos:

$C = \{$las palabras aparezcan en el mismo órden$\}$

$M_1 = \{$en el caso de la letra p, quitar ambas$\}$

$M_2 = \{$en el caso de la letra h, quitar ambas$\}$

$M_3 = \{$quitar dos letras cualquiera, excluyendo la p y la h$\}$

1 2
HAPPY HOUR
1 2

H=2
P=2

Entonces, para obtener el caso donde el cartel queda con el mismo órd "HAPPY HOUR", tenemos que calcular la probabilidad de C

$$P(C) = P(M_1)P(C|M_1) + P(M_2)P(C|M_2) + P(M_3)P(C|M_3)$$

Para los casos donde tomamos P o H, tenemos que

$P(C|M_1) = P(C|M_2) = 1$ ← esto porque no importa el órden de las letras, las palabras quedarán igual

$P(C|M_3) = \frac{1}{2}$ ← hay dos posibles formas de ordenar las letras

Por lo tanto:

$$P(C) = 1\left(\frac{1}{\binom{9}{2}}\right) + 1\left(\frac{1}{\binom{9}{2}}\right) + \frac{1}{2}\left(1 - \frac{1}{\binom{9}{2}} - \frac{1}{\binom{9}{2}}\right)$$

$$P(C) = 1\left(\frac{1}{\frac{9!}{2!(9-2)!}}\right) + 1\left(\frac{1}{\frac{9!}{2!(9-2)!}}\right) + \frac{1}{2}\left(1 - \frac{1}{\frac{9!}{2!(9-2)!}} - \frac{1}{\frac{9!}{2!(9-2)!}}\right)$$

$$\boxed{P(C) = \frac{19}{36}}$$

2. A producer has nine different engines stored, two of which were supplied by a particular supplier. Motors must be distributed in three production lines, with three motors in each line. If the assignment of the motors is made at random, calculate the probability that the two motors of the provider are assigned to the same line.

(2) Posibilidades para hacer 3 grupos con 3 motores cada uno:

$$\frac{9!}{3!3!3!} = 1680$$

Si dos motores están asignados a una línea, las posibilidades de acomodar los motores restantes (7) entre las 3 líneas, quedaría:

$$\frac{7!}{3!3!1!} = 140$$

Sea A = los dos motores asignados a la misma línea,

Entonces: $P(A) = \frac{140}{1680} = \boxed{\frac{1}{12}}$

3. A college has 10 (non-overlapping) time slots for its courses, and blithely assigns courses to time slots randomly and independently. A student randomly chooses 3 of the courses to enrol in. What is the probability that there is a conflict in the student's schedule?

(3) Cada curso de la universidad tiene 10 posibles horarios, esto nos dice que:

$$|S| = 10^3 = 1000$$

Para no causar conflicto con ninguno, tenemos que

$C = 10 \cdot 9 \cdot 8$

Por lo tanto:

$$P(C) = \frac{C}{|S|} = \frac{10 \cdot 9 \cdot 8}{1000} = \frac{720}{1000} = 0.72$$

Entonces, para que exista algun conflicto, tenemos que calcular $P(C^c)$:

$$P(C^c) = 1 - P(C) = 1 - 0.72 = \boxed{0.28}$$

4. A parking lot has 10 parking spaces arranged in a row. There are 7 cars parked. Assume that each car owner has picked at a random a parking place among the spaces available. Specify an appropriate sample space and determine the probability that the three empty places are adjacent to each other.

(4) $SS = \frac{10!}{3!(10-3)!} = \frac{10!}{3!7!} = \frac{720}{6} = \boxed{120}$

$\boxed{V = \text{vacío}}$

$P = \frac{8}{120} = \frac{1}{15} = \boxed{0.066}$

	1	2	3	4	5	6	7	8	9	10
1 -	V	V	V							
2 -		V	V	V						
3 -			V	V	V					
4 -				V	V	V				
5 -					V	V	V			
6 -						V	V	V		
7 -							V	V	V	
8 -								V	V	V

5. You are dealt a hand of four cards from a well-shuffled deck of 52 cards. Specify an appropriate sample space and determine the probability that you receive the four cards J, Q, K, A in any order, with suit irrelevant.

(5)

$|S| = \frac{52!}{4!(52-4)!} = 270{,}725$

$P(J, Q, K, A) = \frac{4 \times 4 \times 4 \times 4}{270725} = \frac{256}{270725} = \boxed{9.4561 \times 10^{-4}}$

6. (a) How many ways are there to split a dozen people into 3 teams, where one team has 2 people, and the other two teams have 5 people each?

(b) How many ways are there to split a dozen people into 3 teams, where each team has 4 people?

⑥

a) $\frac{12!}{2!(12-2)!} = \frac{12!}{2!10!} = 66$ $\qquad \frac{10!}{5!(10-5)!} = \frac{10!}{5!5!} = 252 \left(\frac{1}{2}\right)$

$\frac{5!}{5!(5-5)!}$ $\qquad 66 \cdot \frac{252}{2} = \boxed{8316}$

b) $\frac{12!}{4!4!4!}$ para saber cual equipo es cual, necesitamos dividirlo sobre $3!$, entonces:

$\frac{12!}{4!4!4!3!} = \boxed{5775}$

7. We deal from a well-shuffled 52-card deck. Calculate the probability that the 13 card is the first king to be dealt.

⑦ Probabilidad de obtener un rey después de 12 cartas $\left(\frac{4}{40}\right)$

$\frac{48}{52}\left(\frac{47}{51}\right)\left(\frac{46}{50}\right)\left(\frac{45}{49}\right)\left(\frac{44}{48}\right)\left(\frac{43}{47}\right)\left(\frac{42}{46}\right)\left(\frac{41}{45}\right)\left(\frac{40}{44}\right)\left(\frac{39}{43}\right)\left(\frac{38}{42}\right)\left(\frac{4}{40}\right)$

$= \boxed{0.033}$

8. A six-sided die is rolled three times independently. Which is more likely: a sum of 11 or a sum of 12? (This problem was posed by the French nobleman Méré to his friend Pascal in the 17th century).

(8) Permutaciones para obtener 11 con 3 dados:

(6,4,1), (5,5,1), (5,3,3), (4,5,2), (3,5,3), (2,6,3), (1,4,6)
(6,1,4), (5,1,5), (4,4,3), (4,6,1), (3,4,4), (2,3,6), (1,6,4)
(6,3,2), (5,4,2), (4,3,4), (4,1,6), (3,2,6), (2,4,5), (1,5,5)
(6,2,3), (5,2,4), (4,2,5), (3,3,5), (3,6,2), (2,5,4)

Permutaciones para obtener 12 con 3 dados

(6,5,1), (6,3,3), (5,3,4), (4,6,2), (3,5,4), (2,5,5),
(6,1,5), (5,5,2), (4,4,4), (4,2,6), (3,4,5), (1,6,5),
(6,4,2), (5,2,5), (4,3,5), (3,3,6), (2,4,6), (1,5,6),
(6,2,4), (5,4,3), (4,5,3), (3,6,3), (2,6,4), (5,1,6),
(5,6,1)

Espacio muestral: $6^3 = 216$ elementos

$P_r(11) = \frac{27}{216} = \boxed{0.125}$ $\quad P_r(12) = \frac{25}{216} = \boxed{0.1157}$

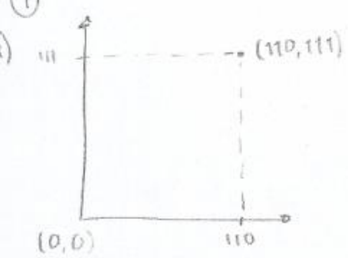
9. In a Cartesian coordinate system:

(a) How many paths are there from the point (0, 0) to the point (110, 111) in the plane such that each step either consists of going one unit up or one unit to the right?

(b) How many paths are there from (0, 0) to (210, 211), where each step consists of going one unit up or one unit to the right, and the path has to go through (110, 111)?

(9)

a)



si queremos llegar al punto (110,111) desde el origen (0,0), de cualquier forma, tenemos que dar 111 pasos hacia arriba y 110 a la derecha, por lo tanto, es mejor calcular sus combinaciones.

Para $x = 110$

$$P = \frac{211!}{110!\,(211-111)!} = \frac{211!}{110!\,11!}$$

Para $y = 111$

$$P = \frac{211!}{111!\,(211-111)!} = \frac{211!}{111!\,110!}$$

Por lo tanto, tenemos $\frac{211!}{110!\,111!}$ posibles rutas

b) Para este punto queremos llegar ahora al punto $(210, 211)$, pero ya tenemos que para el punto $(110, 111) = \frac{211!}{110!\,111!}$

Entonces $(210 - 110, 211 - 111) = (100, 100)$

$$\frac{200!}{100!\,100!}$$

Para obtener el total de pasos hasta el punto $(210, 211)$ desde el punto $(0, 0)$ solo necesitaríamos multiplicar ambos resultados

Tal que:

$$\left(\frac{211!}{110!\,111!}\right)\left(\frac{200!}{100!\,100!}\right)$$

10. A batch of 50 items contains 10 defective items. Suppose 10 items are selected at random and tested. What is the probability that exactly 5 of the items tested are defective?

10

Número total de salidas:

$$n(s) = \binom{50}{10} = \frac{50!}{10!\,(50-10)!}$$

Número de salidas que podemos obtener si 5 de 10 items son defectuosos

$$n(A) = \binom{10}{5}\binom{40}{5}$$

$$n(A) = \left(\frac{10!}{5!\,5!}\right)\left(\frac{40!}{5!\,35!}\right)$$

$$P(A) = \frac{\left(\frac{10!}{5!\,5!}\right)\left(\frac{40!}{5!\,35!}\right)}{\left(\frac{50!}{10!\,40!}\right)} = \boxed{0.01614}$$

11. To battle against spam, Bob installs two anti-spam programs. An email arrives, which is either legitimate (event L) or spam (event Lth) for j ∈ {1; 2}. Assume that 10% of Bob's email is legitimate and that the two programs are each "90% accurate" in c), and which program j marks as legitimate (event Mj) or marks as spam (event Mjc the sense that P(Mj | L) = P(Mj) = 9/10. Also assume that given whether an email is spam, the two programs' outputs are conditionally independent.

(a) Find the probability that the email is legitimate, given that the 1st program marks it as legitimate (simplify).

(b) Find the probability that the email is legitimate, given that both programs mark it as legitimate (simplify).

(c) Bob runs the 1st program and M1 occurs. He updates his probabilities and then runs the 2nd program. Let $\tilde{P}$(A) = P(A | M1) be the updated probability function after running the 1st program. Explain briefly in words whether or not $\tilde{P}$(L | M2) = P(L | M1 ∩ M2): is conditioning on M1 ∩ M2 in one step equivalent to first conditioning on M1, then updating probabilities, and then conditioning on M2?

11 el correo
L = no es spam
L^c = es spam

el programa marca que
M_j = no es spam
M_j^c = es spam

$P(M_j|L) = P(M_j^c|L^c) = \frac{9}{10}$

a) el correo no es spam y el programa 1 marca que no es spam

$$P(L|M_1) = \frac{P(L)P(M_1|L)}{P(M_1)} = \frac{P(L)P(M_1|L)}{P(L)P(M_1|L) + P(L^c)\cdot P(M_1|L^c)} = \frac{\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)}{\left(\frac{1}{10}\right)\left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)} = \boxed{\frac{1}{2}}$$

b) el correo no es spam y ambos programas marcan que no es spam

$$P(L|M_1,M_2) = \frac{P(L)P(M_1,M_2|L)}{P(M_1,M_2)} = \frac{P(L)P(M_1|L)P(M_2|L)}{P(L)P(M_1|L)P(M_2|L) + P(L^c)P(M_1|L^c)P(M_2|L^c)}$$

$$= \frac{\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right)}{\left(\frac{1}{10}\right)\left(\frac{9}{10}\right)\left(\frac{9}{10}\right) + \left(\frac{9}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{10}\right)} = \boxed{\frac{9}{10}}$$

c) Si, esto porque el teorema de Bayes nos dice que

$$P(B) = P(A_1 \cap B) + \dots + P(A_n \cap B)$$
$$= P(A_1)P(B|A_1) + \dots + P(A_n)P(B|A_n)$$

entonces cumple que $M_1 \cap M_2$ es un paso equivalente a condicionar primero M_1 y después condicionar sobre M_2

12. A bag contains one marble which is either green or blue, with equal probabilities. A green marble is put in the bag (so there are 2 marbles now), and then a random marble is taken out. The marble taken out is green. What is the probability that the remaining marble is also green?

12

V = {la primer canica es verde}

S = {la canica sacada es verde}

Q = {la canica que quedo es verde}

$P(V) = \frac{1}{2}$ = probabilidad de que la primera sea verde

$P(V^c) = \frac{1}{2}$ = probabilidad de que la primera sea azul

$P(S|V) = 1$ = probabilidad de sacar una verde si la primera es verde

$P(S|V^c) = 1/2$ = probabilidad de sacar una verde si la primera es verde.

Necesitamos encontrar que:

$$P(V|S) = \frac{P(S|V)\,P(V)}{P(S)} = \frac{P(S|V)\,P(V)}{P(S|V)P(V) + P(S|V^c)\,P(V)}$$

$$P(V|S) = \frac{1/2\,(1)}{(1/2)(1) + (1/2)(1/2)} = \boxed{\frac{2}{3}}$$

13. A spam filter is designed by looking at commonly occurring phrases in spam. Suppose that 80% of email is spam. In 10% of the spam emails, the phrase “free money” is used, whereas this phrase is only used in 1% of non-spam emails. A new email has just arrived, which does mention “free money”. What is the probability that it is spam?

13

$P(S)$ = spam = 0.8 = $\frac{8}{10}$

$P(S^c)$ = no spam = 0.2 = $\frac{2}{10}$

$P(F|S)$ = "free money" | spam = $\frac{1}{10}$

$P(F|S^c)$ = "freemoney" | no spam = $\frac{1}{100}$

$$P(S|F) = \frac{P(S)\,P(F|S)}{P(S)\,P(F|S) + P(S^c)\,P(F|S^c)} = \frac{\left(\frac{8}{10}\right)\left(\frac{1}{10}\right)}{\left(\frac{8}{10}\right)\left(\frac{1}{10}\right) + \left(\frac{2}{10}\right)\left(\frac{1}{100}\right)}$$

$$= \boxed{0.975}$$

14. A crime is committed by one of two suspects, A and B. Initially, there is equal evidence against both of them. In further investigation at the crime scene, it is found that the guilt party had a blood type found in 10% of the population. Suspect A does match this blood type, whereas the blood type of Suspect B is unknown.

(a) Given this new information, what is the probability that A is the guilty party?

(b) Given this new information, what is the probability that B's blood type matches that found at the crime scene?

(14) A = A es culpable
B = B es culpable
C = el tipo de sangre de A encaja
D = el tipo de sangre de B encaja

Por lo tanto, tenemos que:

$P(A) = 1/2$ $\quad$ $P(C|B) = 1/10$ ← esto porque si B es el culpable entonces la probabilidad de que el tipo de sangre de A coincida sería el mismo que el de la población.

$P(B) = 1/2$ $\quad$ $P(C|A) = 1$

a)

$$P(A|C) = \frac{P(A)P(C|A)}{P(A)P(C|A) + P(B)P(C|B)}$$

$$= \frac{(1/2)(1)}{(1/2)(1) + (1/2)(1/10)} = \frac{10}{11} = \boxed{0.90}$$

15. At a gas station, 40% of customers use regular gasoline (A1), 35% use gasoline plus (A2) and 25% use premium (A3). Of customers who use regular gasoline, only 30% fill their tank (event B). Of customers who use plus, 60% fill their tank, while those who use premium, 50% fill their tank.

(a) What is the probability that the next customer asks for gas plus and fills the tank?

(b) What is the probability that the next customer will fill the tank?

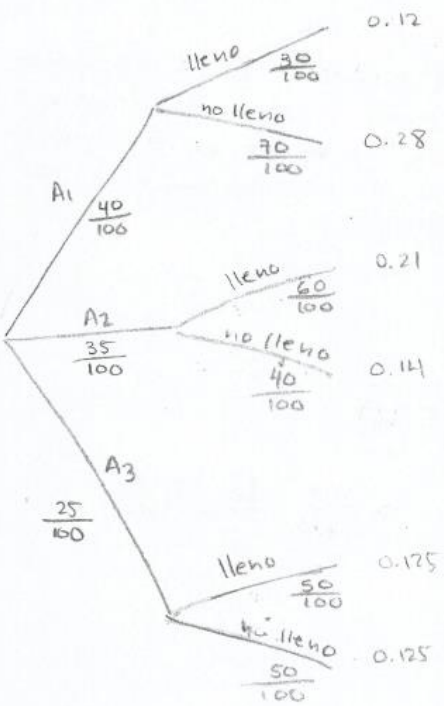
(c) If the next customer fills the tank, what is the probability of ordering regular gasoline? And what to ask for plus? And what to ask for premium?

(15)

A1 = 40% = regular —— 30% llena el tanque

A2 = 35% = plus —— 60% llena el tanque

A3 = 25% = premium —— 50% llena el tanque



a) $\left(\frac{35}{100}\right)\left(\frac{60}{100}\right) = \boxed{0.21}$

b) $\left(\frac{40}{100}\right)\left(\frac{30}{100}\right) + \left(\frac{35}{100}\right)\left(\frac{60}{100}\right) + \left(\frac{25}{100}\right)\left(\frac{50}{100}\right) = \boxed{0.455}$

c) $\frac{0.12}{0.455} = \boxed{0.264}$

$\frac{0.21}{0.455} = \boxed{0.462}$

$\frac{0.125}{0.455} = \boxed{0.275}$