Tarea 2.2 : Integrales

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① $\displaystyle\int \frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}\,dx$

Para integrar por partes:

$\displaystyle\int\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot\frac{1}{\sqrt{x}}\,dx$

$U=\sqrt[5]{x^3}+\sqrt[6]{x}$ $\quad du=\left(\frac{3}{5}x^{-\frac{2}{5}}+\frac{1}{6}x^{-\frac{5}{6}}\right)dx$

$dv=\frac{1}{\sqrt{x}}dx$ $\quad V=2\sqrt{x}$

Sostituyendo:

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)(2\sqrt{x})-\int 2\sqrt{x}\left(\frac{3}{5}x^{-\frac{2}{5}}+\frac{1}{6}x^{-\frac{5}{6}}\right)dx$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-2\int\sqrt{x}\left(\frac{3}{5}x^{-\frac{2}{5}}+\frac{1}{6}x^{-\frac{5}{6}}\right)dx$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-2\int x^{\frac{1}{2}}\left(\frac{3}{5}\left(\frac{1}{x^{\frac{2}{5}}}\right)+\frac{1}{6}\left(\frac{1}{x^{\frac{5}{6}}}\right)\right)dx$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-2\int x^{\frac{1}{2}}\left(\frac{3}{5x^{\frac{2}{5}}}+\frac{1}{6x^{\frac{5}{6}}}\right)dx$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-2\int x^{\frac{1}{2}}\left(\frac{6x^{5/6}(3)+5x^{\frac{2}{5}}}{30x^{\frac{37}{30}}}\right)$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-\frac{1}{15}\int\frac{18x^{5/6}+5x^{2/5}}{x^{11/15}}\,dx$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-\frac{1}{15}\int 18x^{1/10}+\frac{5}{x^{1/3}}\,dx \rightarrow$$

$$\begin{cases} 18\int x^{\frac{1}{10}}dx=18\left(\frac{10x\sqrt[10]{x}}{11}\right)\\ 5\int\frac{1}{x^{1/3}}dx=5\left(\frac{3\sqrt[3]{x^2}}{2}\right)\end{cases}$$

$$\left(\sqrt[5]{x^3}+\sqrt[6]{x}\right)\cdot 2\sqrt{x}-\frac{1}{15}\left(\frac{180x\sqrt[10]{x}}{11}+\frac{15\sqrt[3]{x^2}}{2}\right)$$

$$2\sqrt{x}\sqrt[5]{x^3}+2\sqrt{x}\sqrt[6]{x}-\frac{12x\sqrt[10]{x}}{11}-\frac{\sqrt[3]{x^2}}{2}\longrightarrow$$

$$\begin{cases} 2\sqrt[10]{x^5}\sqrt[10]{x^6}+2\sqrt[6]{x}\sqrt[6]{x^3}\\ 2\sqrt[10]{x^5x^6}+2\sqrt[6]{x\,x^3}\\ 2\sqrt[10]{x^{11}}+2\sqrt[6]{x^4}=2x\sqrt[10]{x}+2\sqrt[3]{x}\end{cases}$$

$$\boxed{\frac{10x\sqrt[10]{x}}{11}+\frac{3}{2}\sqrt[3]{x^2}+C}$$

Derivando

$$\frac{10}{11}x^{11/10}+\frac{3}{2}x^{2/3} = \frac{10}{11}\left(\frac{11}{10}x^{1/10}\right)+\frac{3}{2}\left(\frac{2}{3}x^{-1/3}\right) = \frac{110}{110}x^{1/10}+\frac{6}{6}x^{-1/3}$$

$$\left(x^{1/10}+x^{-1/3}\right)\frac{x^{1/2}}{x^{1/2}} = \frac{x^{3/5}+x^{1/6}}{x^{1/2}} = \boxed{\frac{\sqrt[5]{x^3}+\sqrt[6]{x}}{\sqrt{x}}}$$

(2) $$\int \frac{dx}{\sqrt{x-1}+\sqrt{x+1}} \qquad \frac{1}{\sqrt{x-1}+\sqrt{x+1}}\left(\frac{\sqrt{x-1}-\sqrt{x+1}}{\sqrt{x-1}-\sqrt{x+1}}\right) = \frac{\sqrt{x-1}-\sqrt{x+1}}{\left(\sqrt{x-1}+\sqrt{x+1}\right)\left(\sqrt{x-1}-\sqrt{x+1}\right)}$$

$$= \frac{\sqrt{x-1}-\sqrt{x+1}}{x-1-x-1}$$

$$\int \frac{\sqrt{x-1}-\sqrt{x+1}}{-2}\,dx = \frac{1}{2}\int \sqrt{x-1}-\sqrt{x+1}\,dx = \frac{1}{2}\left(\int\sqrt{x+1}\,dx-\int\sqrt{x-1}\,dx\right)$$

$$= \frac{1}{2}\left(\frac{2(x+1)\sqrt{x+1}}{3}-\frac{2(x-1)\sqrt{x-1}}{3}\right) = \frac{\sqrt{x+1}\,(x+1)-\sqrt{x-1}\,(x-1)}{3}$$

$$= \boxed{\frac{\sqrt{x+1}\,(x+1)-x\sqrt{x-1}+\sqrt{x-1}}{3}+c}$$

Derivando

$$\frac{d}{dx}\,\frac{\sqrt{x+1}\,(x+1)-x\sqrt{x-1}+\sqrt{x-1}}{3} = \frac{1}{3}\left(\sqrt{x+1}\,(x+1)-x\sqrt{x-1}+\sqrt{x-1}\right)\frac{d}{dx}$$

$$\frac{d}{dx}\sqrt{x+1}\,(x+1) = \frac{d}{dx}\left(\sqrt{x+1}\right)(x+1)+\sqrt{x+1}\left(\frac{d}{dx}(x+1)\right) = \frac{(x+1)}{2\sqrt{x+1}}+\sqrt{x+1}$$

$$\frac{d}{dx}-x\sqrt{x-1} = \frac{d}{dx}(-x)\left(\sqrt{x-1}\right)-x\left(\frac{d}{dx}\left(\sqrt{x-1}\right)\right) = -1\sqrt{x-1}-\frac{x}{2\sqrt{x-1}}$$

$$\frac{d}{dx}\sqrt{x-1} = \frac{1}{2\sqrt{x-1}}$$

$$\frac{1}{3}\left(\frac{x+1}{2\sqrt{x+1}}+\sqrt{x+1}-\sqrt{x-1}-\frac{x}{2\sqrt{x-1}}+\frac{1}{2\sqrt{x-1}}\right)$$

$$\frac{1}{3}\left(\frac{x+1}{2\sqrt{x+1}}+\sqrt{x+1}-\sqrt{x-1}+\frac{-x+1}{2\sqrt{x-1}}\right)=\frac{x+1}{6\sqrt{x+1}}+\frac{\sqrt{x+1}}{3}-\frac{\sqrt{x-1}}{3}+\frac{-x+1}{6\sqrt{x-1}}$$

$$\frac{\sqrt{x+1}}{6}+\frac{1}{3}\sqrt{x+1}-\frac{1}{3}\sqrt{x-1}+\frac{-\sqrt{x-1}}{6}=\frac{\sqrt{x+1}}{2}-\frac{\sqrt{x+1}}{2}$$

$$=\frac{1}{\sqrt{x-1}+\sqrt{x+1}}$$

③ $\int \frac{x+4}{x^2+1}\,dx$

$$\int \frac{x}{x^2+1}+\frac{4}{x^2+1}\,dx=\int\frac{x}{x^2+1}\,dx+\int\frac{4}{x^2+1}\,dx$$

Sustituyendo

$t=x^2+1$

$$\int\frac{x}{x^2+1}\,dx=\int\frac{1}{2t}\,dt=\frac{1}{2}\int\frac{1}{t}\,dt=\frac{1}{2}\ln(|t|)=\frac{1}{2}\ln(|x^2+1|)=\frac{1}{2}\ln(x^2$$

$$\int\frac{4}{x^2+1}\,dx=4\int\frac{1}{x^2+1}\,dx=4\arctan(x)$$

$$\boxed{\frac{1}{2}\ln(x^2+1)+4\arctan(x)+C}$$

Derivando

$$\frac{d}{dx}\left(\frac{1}{2}\ln(x^2+1)\right)+\frac{d}{dx}\left(4\operatorname{atan}(x)\right)$$

$$\frac{1}{2}\left(\frac{d}{dx}\right)\left(\ln(x^2+1)\right)\to g=x^2+1\to\frac{1}{2}\left(\frac{d}{dg}\ln(g)\right)\left(\frac{d}{dx}(x^2+1)\right)=\frac{1}{2}\left(\frac{1}{g}\right)(2x)$$

$$\frac{2x}{2(x^2+1)}=\frac{x}{x^2+1}$$

$$\frac{d}{dx}4\operatorname{atan}(x)=4\frac{d}{dx}\operatorname{atan}(x)=\frac{4}{1-x^2}$$

$$\boxed{\frac{x}{x^2+1}+\frac{4}{x^2+1}}$$

④ $\int \frac{1+\cos(x)}{\operatorname{sen}(x)^2}\,dx$ $\quad \operatorname{sen}(*)^2 = 1-\cos(*)^2 \quad \cos(x) = \frac{1-\tan\left(\frac{x}{2}\right)^2}{1+\tan\left(\frac{x}{2}\right)^2}$

$$\int \frac{1+\cos(x)}{1-\cos(x)^2}\,dx = \int \frac{1+\cos(x)}{(1-\cos(x))(1+\cos(x))} = \int \frac{1}{1-\cos(x)}\,dx$$

$$= \int \frac{1}{1-\frac{1-\tan(x/2)^2}{1+\tan(x/2)^2}}\,dx \rightarrow \quad dx = \frac{1}{t'}\,dt \qquad t' = \frac{1}{2}\left(1+\tan(x/2)^2\right)$$

$$t = \tan(x/2)$$

$$= \int \frac{1}{1-\frac{1-\tan(x/2)^2}{1+\tan(x/2)^2}}\left(\frac{1}{\frac{1}{2}\left(1+\tan(x/2)^2\right)}\right)dt = \int \frac{1}{1-\frac{1-t^2}{1+t^2}}\left(\frac{1}{\frac{1}{2}(1+t^2)}\right)dt$$

$$= \int \frac{1}{\frac{1+t^2-(1-t^2)}{1+t^2}}\left(\frac{1}{\frac{1}{2}+\frac{t^2}{2}}\right)dt = \int \frac{1}{\frac{1+t^2-1+t^2}{1+t^2}}\left(\frac{1}{\frac{1+t^2}{2}}\right)dt$$

$$= \int \frac{1}{\frac{2t^2}{1+t^2}}\left(\frac{2}{1+t^2}\right)dt = \int \frac{1+t^2}{2t^2}\left(\frac{2}{1+t^2}\right)dt = \int \frac{1}{t^2}\,dt$$

$$= \frac{1}{(2-1)t^{2-1}} = -\frac{1}{t} = -\frac{1}{\tan\left(\frac{x}{2}\right)} = \boxed{-\frac{1}{\tan\left(\frac{x}{2}\right)} + C}$$

Derivando

$$\frac{d}{dx}\left(-\frac{1}{\tan\left(\frac{x}{2}\right)}\right) = \frac{1}{\frac{1-\cos(x)}{\operatorname{sen}(x)}} = \frac{\operatorname{sen}(x)}{1-\cos(x)}$$

$u = \operatorname{sen}(x) \quad du = \cos(x)$

$v = 1-\cos(x) \quad dv = \operatorname{sen}(x)$

$$= \frac{[(1-\cos(x))(\cos(x))] - [\operatorname{sen}(x)\cdot\operatorname{sen}(x)]}{(1-\cos(x))^2}$$

$$= \frac{\cos(x) - \cos^2(x) - \text{sen}^2(x)}{(1-\cos(x))(1-\cos(x))} = \frac{\cos(x) - 1}{(1-\cos(x)) - (1-\cos(x))} = \frac{1}{1-\cos(x)}$$

$$\frac{1}{1-\cos(x)}\left(\frac{1+\cos(x)}{1+\cos(x)}\right) = \boxed{\frac{1+\cos(x)}{\text{sen}^2(x)}}$$

5) $\int x^2 e^x \, dx$

$u = x^2 \qquad du = 2x\,dx$

$dv = e^x dx \qquad v = e^x$

$\rightarrow x^2 e^x - \int e^x \, 2x\,dx = x^2 e^x - 2 \int e^x x\,dx$

$= x^2 e^x - 2\int x e^x dx \rightarrow \begin{matrix} u = x & du = 1 \\ dv = e^x dx & v = e^x \end{matrix} \rightarrow x^2 e^x - 2\left(x e^x - \int e^x dx\right)$

$= x^2 e^x - 2\left(x e^x - e^x\right) = x^2 e^x - 2x e^x + 2e^x = \boxed{e^x\left(x^2 - 2x + 2\right) + c}$

Derivando

$u = x^2 - 2x + 2 \qquad v = e^x$

$\frac{d}{dx} e^x\left(x^2 - 2x + 2\right)$

$du = 2x - 2 \qquad dv = e^x$

$= (x^2 - 2x + 2)e^x + (2x - 2)e^x$

$= x^2 e^x \cancel{- 2xe^x} \cancel{+ 2e^x} \cancel{+ 2xe^x} \cancel{- 2e^x}$

$= \boxed{x^2 e^x}$

⑥ $\int \frac{8x^2+6x+4}{x+1}\,dx$

$$\int \frac{8x^2}{x+1}\,dx + \int \frac{6x}{x+1}\,dx + \int \frac{4}{x+1}\,dx$$

$t = x+1$

$$\int \frac{8x^2}{x+1}\,dx = \int \frac{8t^2-16t+8}{t} = \int 8t - 16 + \frac{8}{t}\,dt = 4t^2 - 16t + 8\ln(|t|)$$

$$= 4(x+1)^2 - 16(x+1) + 8\ln(|x+1|) = \underline{4x^2 - 8x - 12 + 8\ln(|x+1|)}$$

$$\int \frac{6x}{x+1}\,dx = \int \frac{6t-6}{t}\,dt = \int 6 - \frac{6}{t}\,dx = 6t - 6\ln(|t|)$$

$$= 6(x+1) - 6\ln(|x+1|) = \underline{6x + 6 - 6\ln(|x+1|)}$$

$$\int \frac{4}{x+1}\,dx = \int \frac{4}{t}\,dt = 4\ln(|t|) = \underline{4\ln(|x+1|)}$$

$$= 4x^2 - 8x - 12 + 8\ln(|x+1|) + 6x + 6 - 6\ln(|x+1|) + 4\ln(|x+1|)$$

$$= \boxed{4x^2 - 2x + 6\ln(x+1) + C}$$

Derivando

$$4x^2 - 2x + 6\ln(x+1)$$

$$8x - 2 + \frac{6}{x+1} = \frac{(8x-2)(x+1)+6}{x+1} = \boxed{\frac{8x^2+6x+4}{x+1}}$$

⑦ $\int \frac{4^x+1}{2^2+1}\,dx$

$= \frac{1}{5}\int 4^x+1\,dx = \frac{1}{5}\left(\int 4^x dx + \int 1\,dx\right) = \frac{1}{5}\left(\frac{2^{2x-1}}{\ln(2)} + x\right)$

$= \boxed{\frac{2^{2x-1}}{5\ln(2)} + \frac{1}{5}x + C}$

Derivando

$\frac{d}{dx}\left(\frac{2^{2x-1}}{5\ln(2)} + \frac{1}{5}x\right) = \frac{d}{dx}\left(\frac{2^{2x-1}}{5\ln(2)}\right) + \frac{d}{dx}\left(\frac{1}{5}x\right)$

$\frac{1}{5\ln(2)}\frac{d}{dx}\left(2^{2x-1}\right) \to g=2x+1 \to \frac{1}{5\ln(2)}\frac{d}{dx}\left(2^g\right)\frac{d}{dx}(2x-1)$

$= \frac{1}{5\ln(2)}(\ln(2))(2^g)(2) = \frac{1}{5\ln(2)}\cdot\ln(2)\cdot 2^{2x-1}\cdot 2 + \frac{1}{2}$

$= \frac{1}{5}2^{2x-1}\cdot 2 + \frac{1}{2} = \frac{2^{2x}}{5} + \frac{1}{2} = \frac{4^x}{5} + \frac{1}{2} = \boxed{\frac{4^x+1}{2^2+1}}$ ▨

⑧ $\int e^{\sqrt{x}}\,dx \qquad t=\sqrt{x}$

$$\int 2te^{t}dt = 2\int te^{t}dt \longrightarrow \begin{array}{ll} u=t & du=dt \\ dv=e^{t}dt & v=e^{t} \end{array}$$

$$= 2\left(te^{t}-\int e^{t}dt\right) = 2\left(te^{t}-e^{t}\right) = 2\left(\sqrt{x}e^{\sqrt{x}}-e^{\sqrt{x}}\right)$$

$$= \boxed{2\sqrt{x}e^{\sqrt{x}}-2e^{\sqrt{x}}+C}$$

Derivando

$$\frac{d}{dx}\left(2e^{\sqrt{x}}\sqrt{x}-2e^{\sqrt{x}}\right) = \frac{d}{dx}\left(2e^{\sqrt{x}}\sqrt{x}\right)+\frac{d}{dx}\left(-2e^{\sqrt{x}}\right)$$

$$\frac{d}{dx}\left(2e^{\sqrt{x}}\sqrt{x}\right) = \left(\frac{d}{dx}2e^{\sqrt{x}}\right)(x)+2e^{\sqrt{x}}\left(\frac{d}{dx}\sqrt{x}\right) = 2e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)x+2e^{\sqrt{x}}+\frac{1}{2}$$

$$g=\sqrt{x}$$

$$\frac{d}{dx}\left(-2e^{\sqrt{x}}\right) = -2\left(\frac{d}{dx}e^{\sqrt{x}}\right) = -2\left(\frac{d}{dg}e^{g}\right)\left(\frac{d}{dx}\sqrt{x}\right) = -2e^{g}\left(\frac{1}{2\sqrt{x}}\right) = -2e^{\sqrt{x}}\left(\frac{1}{2\sqrt{x}}\right)$$

$$= \boxed{e^{\sqrt{x}}} \quad \blacksquare$$

⑨ $\int x^2 \operatorname{sen}(x)\,dx$ $\quad u = x^2 \quad du = 2x\,dx$

$dv = \operatorname{sen}(x)\,dx \quad v = -\cos(x)$

$x^2(-\cos(x)) - \int -\cos(x)\cdot 2x\,dx = x^2(-\cos(x)) - 1(-2)\int \cos(x)\,x\,dx$

$u = x \quad du = dx$

$x^2(-\cos(x)) + 2\int x\cos(x)\,dx \longrightarrow dv = \cos(x)\,dx \quad v = \sin(x)$

$x^2(-\cos(x)) + 2\left(x\operatorname{sen}(x) - \int \operatorname{sen}(x)dx\right) = x^2(-\cos(x)) + 2(x\operatorname{sen}(x) - (-\cos(x)))$

$= \boxed{-x^2\cos(x) + 2x\operatorname{sen}(x) + 2\cos(x) + C}$

Derivando

$\frac{d}{dx}\left(-x^2\cos(x) + 2x\operatorname{sen}(x) + 2\cos(x)\right)$

$\left(\frac{d}{dx} - x^2\cos(x)\right) + \left(\frac{d}{dx} 2x\operatorname{sen}(x)\right) + \left(\frac{d}{dx} 2\cos(x)\right)$

$\left(\frac{d}{dx} - x^2\right)\cos(x) - x^2\left(\frac{d}{dx}\cos(x)\right) = -2x\cos(x) - 2x^2(-\operatorname{sen}(x))$

$\left(\frac{d}{dx} 2x\right)\operatorname{sen}(x) + 2x\left(\frac{d}{dx}\operatorname{sen}(x)\right) = 2\operatorname{sen}(x) + 2x\cos(x)$

$\left(\frac{d}{dx} 2\cos(x)\right) = 2\left(\frac{d}{dx}\cos(x)\right) = 2(-\operatorname{sen}(x)) \quad = \boxed{x^2\operatorname{sen}(x)}$

(10) $\int \frac{e^x}{e^{2x}+2e^x+1}\,dx \qquad t=e^x$

$$\int \frac{1}{t^2+2t+1} = \int \frac{1}{(t+1)^2}\,dt \longrightarrow \begin{array}{l} dt = \frac{1}{u'}\,du \\ u = t+1 \end{array} \qquad u'=1$$

$$\int \frac{1}{(t+1)^2}\,du = \int \frac{1}{u^2}\,du = -\frac{1}{u} = -\frac{1}{t+1} = -\frac{1}{e^x+1}$$

$$= \boxed{\frac{-1}{e^x+1} + C}$$

Derivando

$$\frac{d}{dx}\left(-\frac{1}{e^x+1}\right) = \frac{\frac{d}{dx}(e^x+1)}{(e^x+1)^2} = \frac{\left(\frac{d}{dx}e^x\right)+\left(\frac{d}{dx}1\right)}{(e^x+1)^2}$$

$$= \frac{e^x}{(e^x+1)^2} = \boxed{\frac{e^x}{e^{2x}+2e^x+1}}$$