Probability, Random Processes and Inference

Solución de la serie de ejercicios 2

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1. You just rented a large house and the realtor gave you 6 keys, one for each of the 6 doors of the house. Unfortunately, all keys look identical, so to open the front door, you try them at random. Find the PMF of the number of trials you will need to open the door under the assumption that after an unsuccessful trial, you mark the corresponding key, so you never try it again.

1

$$P(X=1)=\frac{1}{6}$$

$$P(X=2)=\frac{5}{6}\frac{1}{5}=\frac{1}{6}$$

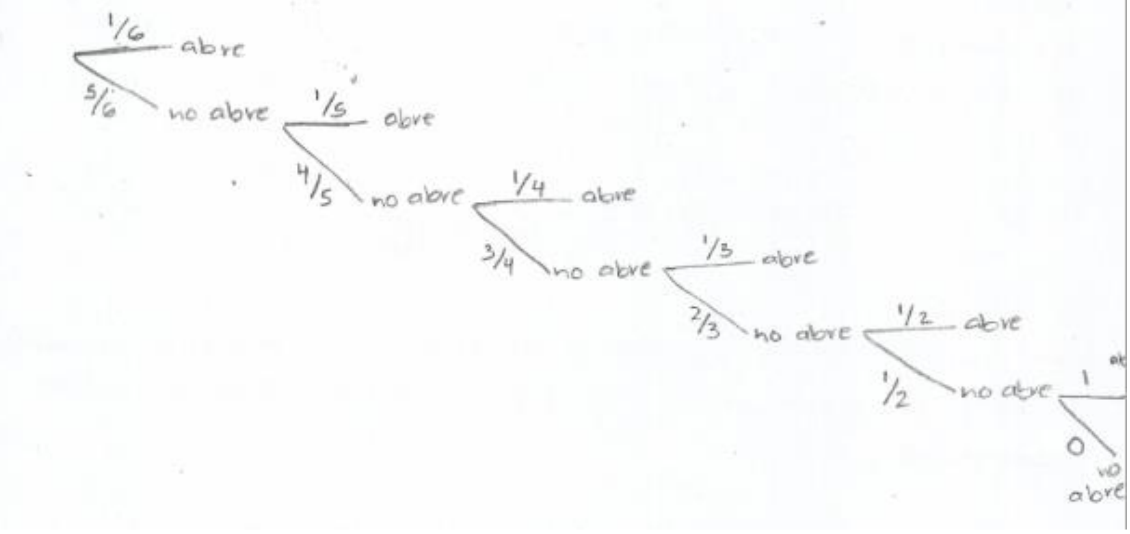
$$P(X=3)=\frac{5}{6}\frac{4}{5}\frac{1}{4}=\frac{1}{6}$$

$$P(X=4)=\frac{5}{6}\frac{4}{5}\frac{3}{4}\frac{1}{3}=\frac{1}{6}$$

$$P(X=5)=\frac{5}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}=\frac{1}{6}$$

$$P(X=6)=\frac{5}{6}\frac{4}{5}\frac{3}{4}\frac{2}{3}\frac{1}{2}\frac{1}{1}=\frac{1}{6}$$

$$P_X(x)=\begin{cases}\frac{1}{6} & \text{para } X=\{1,2,3,4,5,6\}\\ 0 & \text{cualquier otro caso}\end{cases}$$



2. Accidentally, two depleted batteries got into a set of five batteries. To remove the two depleted batteries, the batteries are tested one by one in a random order. Let the random variable X denote the number of batteries that must be tested to find the two depleted batteries. What is the probability mass function of X?

② Primero etiquetaremos las pilas del 1 al 5

Si pensamos que la forma en la que colocaremos las pilas es una permutación aleatoria de los números definidos anteriormente (1, 2, 3, 4, 5). Entonces, nuestro espacio muestral tendrá 5! resultados con la misma probabilidad.

Para encontrar las dos baterías que están agotadas necesitamos al menos 2 pruebas pero no mas de 4.

Necesitaremos 2 pruebas si las primeras 2 baterías se encuentran vacías.
El número de resultados para que las dos baterías esten en las posiciones 1 y 2 es $2(1)(3!)$

$$P(X=2) = \frac{2 \times 1 \times 3!}{5!} = \frac{12}{120} = \frac{1}{10}$$

Ahora, necesitaremos 3 prueba si las primeras 3 baterías no están vacías o si encontramos una segunda batería vacía en nuestra tercera prueba

$$P(X=3) = \frac{(3)(2)(1)(2!) + (2)(3)(1)(2!) + (3)(2)(1)(2!)}{5!}$$

$$P(X=3) = \frac{12 + 12 + 12}{120} = \frac{36}{120} = \frac{3}{10}$$

$P(X=4)$ la podemos deducir $P(X=4) = 1 - P(X=2) - P(X=3)$, y esto nos da de resultado $6/10$.

La función de masa de probabilidad de X podemos obtenerla con las probabilidades condicionales

$$P(X=0) = \left(\frac{2}{5}\right)\left(\frac{1}{4}\right) = \frac{2}{20} = \boxed{\frac{1}{10}}$$

$$P(X=2) = \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{5}\right)\left(\frac{3}{4}\right)\left(\frac{1}{3}\right) + \left(\frac{3}{5}\right)\left(\frac{2}{4}\right)\left(\frac{1}{3}\right)$$

$$= \frac{6}{60} + \frac{6}{60} + \frac{6}{60} = \frac{18}{60} = \boxed{\frac{3}{10}}$$

$$P_X(x) = \begin{cases} \frac{1}{10} & \text{para } x=2 \\ \frac{3}{10} & \text{para } x=3 \\ \frac{6}{10} & \text{, para } x=4 \\ 0 & \text{cualquer otro caso} \end{cases}$$

3. Two urns contain different proportions of red and black balls. Urn 1 has a proportion p1 of red balls and a proportion 1 – p1 of black balls whereas urn 2 has proportions of p2 and 1 - p2 of red balls and black balls, respectively. A compound experiment is performed in which an urn is chosen at random, followed by the selection of a ball. What is the probability that a red ball is selected?

(3) Definamos

$R = \{\text{bola roja seleccionada}\}$

$U_1 = \{\text{urna 1 elegida}\}$

$U_2 = \{\text{urna 2 elegida}\}$

Entonces, tenemos que:

$$P[R] = P[R|U_1]P[U_1] + P[R|U_2]P[U_2]$$

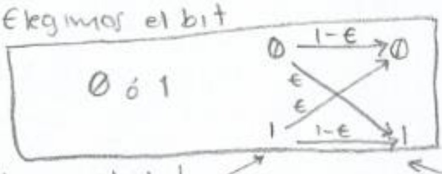
$$= p_1\left(\tfrac{1}{2}\right) + p_2\left(\tfrac{1}{2}\right) = \left(\tfrac{1}{2}\right)(p_1+p_2)$$

4. In a digital communication system a "0" or "1" is transmitted to a receiver. Typically, either bit is equally likely to occur so that a prior probability of 1/2 is assumed. At the receiver a decoding error can be made due to channel noise, so that a 0 may be mistaken for a 1 and vice versa. Defining the probability of decoding a 1 when a 0 is transmitted as ϵ and a 0 when a 1 is transmitted also as ϵ, what is the overall probability of an error?

(4) Tenemos que aunque conozcamos que bit se transmitió no conocemos el bit que nos llegó.

$P[0] = P[1] = 1/2$

Elegimos el bit



columna de bit transmitido

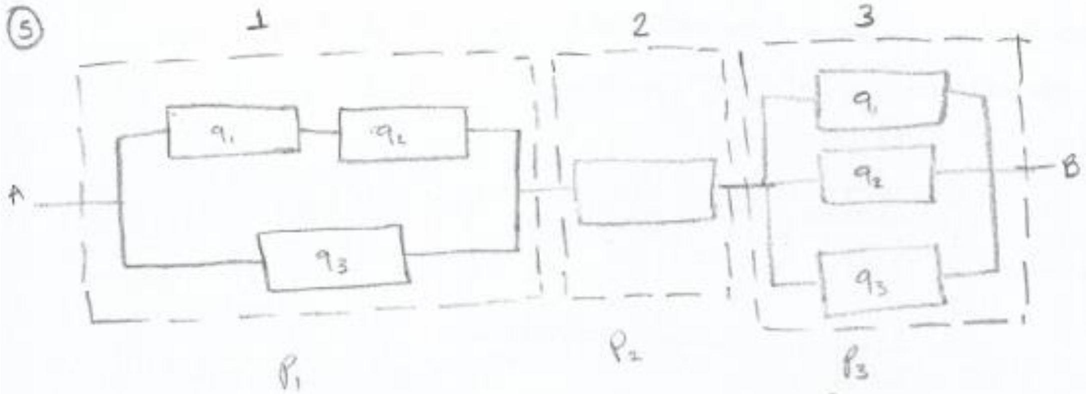
columna de bit recibido.

Por lo tanto, nuestra probabilidad de error es:

$$P[\text{error}] = P[\text{error} \mid \text{transmitimos } 0]\, P[\text{transmitimos } 0] + P[\text{error} \mid \text{transmitimos } 1]\, P[\text{transmitimos } 1]$$

$$= \epsilon \tfrac{1}{2} + \epsilon \tfrac{1}{2} = \boxed{\epsilon}$$

5. An electrical system consists of identical components, each of which is operational with probability p, independent of other components. The components are connected in three subsystems, as shown in the figure below. The system is operational if there is a path that starts at point A, ends at point B, and consists of operational components. What is the probability of this happening?



La probabilidad de que el sistema sea operable es:

$$P_1P_2P_3 \qquad q_1=q_2=q_3=p$$

$$P_2 = p \qquad P_3 = 1-(1-q_1)(1-q_2)(1-q_3)$$

$$= 1-(1-p)^3$$

$$q_1q_2 = p^2$$

$$P_1 = 1-(1-p^2)(1-p)$$

$$\boxed{P_1P_2P_3 = [1-(1-p^2)(1-p)][p][1-(1-p)^3]}$$

6. Let k be the number of active (nonsilent) speakers in a group of eight noninteracting (i.e., independent) speakers. Suppose that a speaker is active with probability 1/3. Find the probability that the number of active speakers is greater than six.

(6) Para $i = 1, 2, 3, 4, 5, 6, 7, 8$ definamos:

A_i = el i-ésimo speaker esté activo

El número de speakers activos entonces es el número de aciertos en ocho ensayos Bernoulli con $p = \frac{1}{3}$. Por lo tanto, la probabilidad de que haya seis speakers activos es:

$$P[K=7 \cup K=8] = P_8(7) + P_8(8)$$

$$P[K=7 \cup K=8] = \binom{8}{7}(p)^7(1-p)^1 + \binom{8}{8}(p)^8(1-p)^0$$

$$P[K=7 \cup K=8] = 8\left(\frac{1}{3}\right)^7\left(\frac{2}{3}\right) + 1\left(\frac{1}{3}\right)^8 = \frac{17}{3^8} = \frac{17}{6561}$$

$$= \boxed{0.002591}$$

7. A stock market trader buys 100 shares of stock A and 200 shares of stock B. Let X and Y be the price changes of A and B, respectively, over a certain time period, and assume that the joint PMF of X and Y is uniform over the set of integers x and y satisfying: −2 ⍰ X ⍰ 4, −1 ⍰ y − x ⍰ 1.

(a) Find the marginal PMFs and the means of X and Y.

(b) Find the mean of the trader’s profit.

⑦ Para cada x hay 3 y

Por lo tanto hay 21 posibilidades

Por lo tanto la distribución conjunta es:

$$P_{xy}(xy) = \begin{array}{c c} y & \\ \begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{array} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{21} \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{1}{21} \\ 0 & 0 & 0 & 0 & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} \\ 0 & 0 & 0 & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & 0 \\ 0 & 0 & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & 0 & 0 \\ 0 & \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & 0 & 0 & 0 \\ \frac{1}{21} & \frac{1}{21} & \frac{1}{21} & 0 & 0 & 0 & 0 \\ \frac{1}{21} & \frac{1}{21} & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{21} & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ & \begin{array}{ccccccc} -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array} \\ & x \end{array} \qquad P_y(y) = \begin{array}{c c} \begin{array}{c} 5 \\ 4 \\ 3 \\ 2 \\ 1 \\ 0 \\ -1 \\ -2 \\ -3 \end{array} & \begin{pmatrix} \frac{1}{21} \\ \frac{2}{21} \\ \frac{3}{21} \\ \frac{3}{21} \\ \frac{3}{21} \\ \frac{3}{21} \\ \frac{3}{21} \\ \frac{2}{21} \\ \frac{1}{21} \end{pmatrix} \end{array}$$

$$P_x(x) = \begin{pmatrix} \frac{3}{21} & \frac{3}{21} & \frac{3}{21} & \frac{3}{21} & \frac{3}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$$
$$\begin{array}{ccccccc} -2 & -1 & 0 & 1 & 2 & 3 & 4 \end{array}$$

$$E[x] = -2\left(\frac{3}{21}\right) - 1\left(\frac{3}{21}\right) + 0\left(\frac{3}{21}\right) + 1\left(\frac{3}{21}\right) + 2\left(\frac{3}{21}\right) + 3\left(\frac{3}{21}\right) + 4\left(\frac{3}{21}\right)$$

$$E[x] = \frac{-6-3+0+3+6+9+12}{21} = \frac{21}{21} = \boxed{1}$$

$$E[y] = 5\left(\frac{1}{21}\right) + 4\left(\frac{2}{21}\right) + 3\left(\frac{3}{21}\right) + 2\left(\frac{3}{21}\right) + 1\left(\frac{3}{21}\right) + 0\left(\frac{3}{21}\right) - 1\left(\frac{3}{21}\right) - 2\left(\frac{2}{21}\right) - 3\left(\frac{1}{21}\right)$$

$$E[y] = \frac{5+8+9+6+3-3-4-3}{21} = \frac{21}{21} = \boxed{1}$$

El beneficio es:

$$E[100x + 200y] = 100\,E[x] + 200\,E[y] = 300 \text{ units}$$

8. You roll a fair dice twice. Let the random variable X be the product of the outcomes of the two rolls. What is the probability mass function of X? What are the expected value and the standard deviation of X?

(8) $X = \{\text{producto de tirar 2 veces un dado}\}$

6	6	12	18	24	30	36
5	5	10	15	20	25	30
4	4	8	12	16	20	24
3	3	6	9	12	15	18
2	2	4	6	8	10	12
1	1	2	3	4	5	6
	1	2	3	4	5	6

$P(x=1) = \frac{1}{36}$

$P(x=2) = \frac{2}{36}$

$P(x=3) = \frac{2}{36}$

$P(x=4) = \frac{3}{36}$

$P(x=5) = \frac{2}{36}$

$P(x=6) = \frac{4}{36}$

$P(x=8) = \frac{2}{36}$

$P(x=9) = \frac{1}{36}$

$P(x=10) = \frac{2}{36}$

$P(x=12) = \frac{4}{36}$

$P(x=15) = \frac{2}{36}$

$P(x=16) = \frac{1}{36}$

$P(x=18) = \frac{2}{36}$

$P(x=20) = \frac{2}{36}$

$P(x=24) = \frac{2}{36}$

$P(x=25) = \frac{1}{36}$

$P(x=30) = \frac{2}{36}$

$P(x=36) = \frac{1}{36}$

$$P_X(x) = \begin{cases} \frac{1}{36} & \text{para } x = \{1,9,16,25,36\} \\ \frac{2}{36} & \text{para } x = \{2,3,5,8,10,15,18,20,24,30\} \\ \frac{3}{36} & \text{para } x = \{4\} \\ \frac{4}{36} & \text{para } x = \{6,12\} \\ 0 & \text{cualquier otro caso} \end{cases}$$

$$E(x) = \frac{1}{36}\sum_{i=1}^{6}\sum_{j=1}^{6} i \times j = \left(\frac{1}{6}\sum_{i=1}^{6} i\right)\left(\frac{1}{6}\sum_{j=1}^{6} j\right) = \left(\frac{1}{6}(21)\right)\left(\frac{1}{6}(21)\right) = \left(\frac{7}{2}\right)\left(\frac{7}{2}\right) = \underline{\underline{\frac{49}{4}}}$$

$$E(x^2) = \left(\frac{1}{6}\sum_{i=1}^{6} i^2\right)\left(\frac{1}{6}\sum_{j=1}^{6} j^2\right) = \left(\frac{1}{6}(91)\right)\left(\frac{1}{6}(91)\right) = \left(\frac{91}{6}\right)\left(\frac{91}{6}\right) = \underline{\underline{\frac{8281}{36}}}$$

$$Var(x) = \sigma_x^2 = E[(x-\mu(x))^2] = E[x^2] - E[x]^2 = \sqrt{\frac{8281}{36} - \left(\frac{49}{4}\right)^2} = \underline{\underline{8.94}}$$

9. Bobo, the amoeba, currently lives alone in a pond. After one minute Bobo will either die, split into two amoebas, or stay the same, with equal probability. Find the expectation and variance for the number of amoebas in the pond after one minute.

⑨ Sea X el número de amoebas al cabo de un minuto, entonces por definición de valor esperado:

$$E[x] = (0)(P(X=0)) + (1)(P(x=1)) + (2)(P(x=2))$$

$$= 0 + \frac{1}{3} + \frac{2}{3} = \boxed{1}$$

Para calcular la varianza, primero hay que evaluar el segundo momento $E[x^2]$:

$$E[x^2] = E[X] = (0)(P(x=0)) + (1)(P(x=1)) + (4)(P(x=2))$$

$$= 0 + \frac{1}{3} + \frac{4}{3} = \boxed{\frac{5}{3}}$$

Por lo tanto:

$$Var[x] = E[x^2] - (E[x])^2 = \frac{5}{3} - 1 = \boxed{\frac{2}{3}}$$

10. Let X be a random variable with PMF:

$$p_X(x) = \begin{cases} x^2/a, & \text{if } x = -3, -2, -1, 0, 1, 2, 3, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find a and E[X].

(b) What is the PMF of the random variable Z = (X - E[X])?

(c) Using the result from part (b), find the variance of X.

(d) Find the variance of X using the formula var(X) = ∑ x$(x - E[X])^{2}p$x(x).

(10)

a) El escalar debe satisfacer que:

$$1 = \sum_{x} px(x) = \frac{1}{a} \sum_{x=-3}^{3} x^2$$

entonces:

$$a = \sum_{x=-3}^{3} x^2 = (-3)^2 + (-2)^2 + (-1)^2 + 1^2 + 2^2 + 3^2 = \underline{28}$$

b) Si $z \in \{1,4,9\}$, entonces:

$$pz(z) = px(\sqrt{z}) + px(-\sqrt{z}) = \frac{z}{28} + \frac{z}{28} = \underline{\frac{z}{14}}$$

De lo contrario tenemos que:

$$pz(z) = 0$$

c) $$\mathrm{var}(X) = E[Z] = \sum_{z} z\,pz(z) = \sum_{z \in \{1,4,9\}} \frac{z^2}{14} = \underline{7}$$

d) Tenemos que

$$\begin{aligned}
\mathrm{var}(x) &= \sum_{x} (x - E[x])^2 px(x) \\
&= (1^2)(px(-1) + px(1)) + (2^2)(px(-2) + px(2)) + (3^2)(px(-3) + px(3)) \\
&= (2)\left(\tfrac{1}{28}\right) + (8)\left(\tfrac{4}{28}\right) + (18)\left(\tfrac{9}{28}\right) \\
&= \underline{7}
\end{aligned}$$