Test 4 − Answers

Let X denote the concentration of a certain substance in one trial of an experiment and Y the concentration of the substance in a second trial of the experiment. Suppose that the joint pdf is given by

$$f(x,y) = \begin{cases} k(2x^2 + \frac{2}{3}xy) & \text{if } 0 < x \leq 1, 0 < y \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the value of k.

Solution: We begin by computing the integral of $f(x,y)$ over the region $(0,1] \times (0,1]$. We get

$$\int_0^1 \int_0^1 k(2x^2 + \tfrac{2}{3}xy)dxdy = \tfrac{5}{6}k.$$

So, if we take $k = \frac{6}{5}$ then $f(x,y)$ satisfies the following properties:

1. $f(x,y) \geq 0$ for all $0 < x \leq 1$ and $0 < y \leq 1$.

2. $\int_0^1 \int_0^1 f(x,y)dxdy = 1$.

From now on, we will assume that

$$f(x,y) = \begin{cases} \frac{12}{5}x^2 + \frac{4}{5}xy & \text{if } 0 < x < 1, 0 < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) Find the marginal pdf $f_X(x)$ of X.

Solution: $f_X(x) = \int_0^1 (\frac{12}{5}x^2 + \frac{4}{5}xy)dy = \frac{12}{5}x^2 + \frac{2}{5}x$.

(iii) Find the conditional pdf $f_{Y|X}(y|x)$ of Y given $X = x$.

Solution: $f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{\frac{12}{5}x^2+\frac{4}{5}xy}{\frac{12}{5}x^2+\frac{2}{5}x} = \frac{6x^2+2xy}{6x^2+x}$.

(iv) Find $E[X]$ and $E[Y|X=0.5]$.

Solution: First, we get

$$E[X] = \int_0^1 x f_X(x)dx = \int_0^1 x(\tfrac{12}{5}x^2 + \tfrac{2}{5}x)dx = \tfrac{11}{15}.$$

Second, the conditional pdf of Y given $X = 0.5$ is the function $\frac{6\cdot 0.5^2+(2\cdot 0.5)y}{6\cdot 0.5^2+0.5} = \frac{1}{2}y + \frac{3}{4}$. Hence,

$$E[Y|X=0.5] = \int_0^1 y(\tfrac{1}{2}y + \tfrac{3}{4})dy = \tfrac{13}{24}.$$

(v) Provide a formula for $P[X \le 0.5|X > 0.2]$ – a solution in terms of $f_X(x)$ is expected.

Solution: By definition of the conditional probability,

$$P[X \le 0.5|X > 0.2] = \frac{P[0.2 < X \le 0.5]}{P[X > 0.2]} = \frac{\int_{0.2}^{0.5} f_X(x)dx}{\int_{0.2}^{1} f_X(x)dx}.$$

If we want to evaluate this quotient, we proceed as follows:

$$\begin{aligned} P[0.2 < X \le 0.5] &= \int_{0.2}^{0.5} f_X(x)dx = \int_{0.2}^{0.5} (\tfrac{12}{5}x^2 + \tfrac{2}{5}x)dx \\ &= (\tfrac{4}{5}\cdot 0.5^3 + \tfrac{1}{5}\cdot 0.5^2) - (\tfrac{4}{5}\cdot 0.2^3 + \tfrac{1}{5}\cdot 0.2^2) = \tfrac{3}{20} - \tfrac{9}{625} \\ &= \tfrac{339}{2500} \end{aligned}$$

and

$$\begin{aligned}
P[X > 0.2] &= \int_{0.2}^{1} f_X(x)dx = \int_{0.2}^{1} (\tfrac{12}{5}x^2 + \tfrac{2}{5}x)dx \\
&= (\tfrac{4}{5} + \tfrac{1}{5}) - (\tfrac{4}{5} \cdot 0.2^3 + \tfrac{1}{5} \cdot 0.2^2) = 1 - \tfrac{9}{625} \\
&= \tfrac{616}{625}.
\end{aligned}$$

Hence, $P[X \leq 0.5 | X > 0.2] = \frac{339}{2500} \cdot \frac{625}{616} = \frac{339}{2464}$.

(vi) Find the probability that for both trials of the experiment, the average concentration is less than 0.5, i.e. find $P[(\frac{1}{2}X + \frac{1}{2}Y) < 0.5]$.

Problem: Find $P[X + Y < 1]$.

Approach: It is convenient to view the pair of two random variables X, Y – associated with the same experiment – as a 2-dimensional random vector (X, Y). In other words, if X and Y are both maps from the sample space Ω to the interval $(0, 1] \subset \mathbb{R}$ then the random vector (X, Y) is a map from Ω to the Cartesian product $(0, 1] \times (0, 1] \subset \mathbb{R}^2$.

The event "$X + Y < 1$" can be represented as the set $\{\omega \in \Omega : X(\omega) + Y(\omega) < 1\}$. It follows that

$$\begin{aligned}
P[X + Y < 1] &= P[\{\omega \in \Omega : X(\omega) + Y(\omega) < 1\}] \\
&= P[\{\omega \in \Omega : (X(\omega), Y(\omega)) \in B\}], \\
&\quad \text{where } B = \{(x, y) \in (0, 1] \times (0, 1] : x + y < 1\}.
\end{aligned}$$

In other words, the event "$X + Y < 1$" can be represented as the pre-image of the set B under the random vector (X, Y), in symbols $(X, Y)^{-1}(B)$.

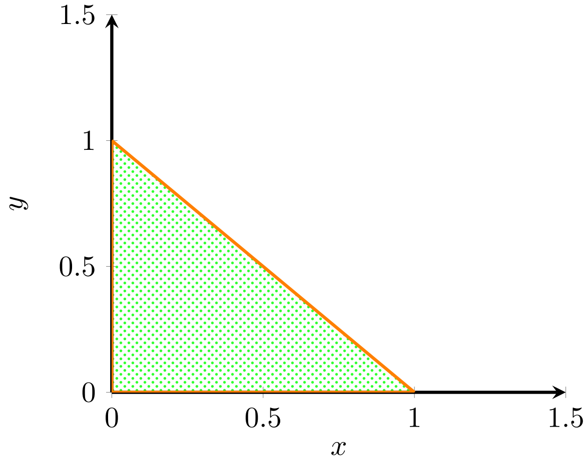

Figure 1: Region B corresponding to the event "$X + Y < 1$"

This makes it possible to determine the 'joint probability' by integrating the joint pdf over the region B, namely

$$
\begin{aligned}
P[X+Y<1] = P[(X,Y) \in B] &= \iint_B f(x,y)dxdy \\
&= \iint_{\substack{x\in(0,1] \\ y\in(0,1-x)}} f(x,y)dxdy = \int_0^1 \int_0^{1-x} f(x,y)dydx \\
&= \int_0^1 \left(-\tfrac{10}{5}x^3 + \tfrac{8}{5}x^2 + \tfrac{2}{5}x\right) dx = \tfrac{7}{30}.
\end{aligned}
$$

An alternative way to calculate this 'joint probability' is to evaluate the following integral:

$$
\begin{aligned}
P[X+Y<1] &= \iint_{\substack{y\in(0,1] \\ x\in(0,1-y)}} f(x,y)dxdy = \int_0^1 \int_0^{1-y} f(x,y)dxdy \\
&= \int_0^1 \left(-\tfrac{2}{5}y^3 + \tfrac{8}{5}y^2 - \tfrac{10}{5}y + \tfrac{4}{5}\right) dy = \tfrac{7}{30}.
\end{aligned}
$$