Test 1

The following questions are taken from the exam in the year 2009.

(i) Consider two events A and B of an outcome space. Determine the maximum and minimum possible values for $P(A \cap B)$, and the conditions under which each of these values is achieved, when $P(A) = 0.5$ and $P(B) = 0.6$.

Remark:

For any event E, we have $0 \leq P(E) \leq 1$. This implies that 0 is the trivial lower bound of $P(E)$ and 1 is the trivial upper bound of $P(E)$. But can we tighten these bounds? In other words, can we determine real numbers b_1 and b_2 such that $b_1 > 0$, $b_2 < 1$, and $b_1 \leq P(E) \leq b_2$? Can we also give some sufficient conditions under which the lower bound b_1 is achieved, i.e. $P(E) = b_1$? Similarly, can we give some sufficient conditions under which the upper bound b_2 is achieved, i.e. $P(E) = b_2$?

An alternative formulation of (i):

Consider two events A and B of an outcome space. Suppose that $P(A) = 0.5$ and $P(B) = 0.6$.

(a) Determine a real number $b_1 > 0$ such that $P(A \cap B) \geq b_1$ and give one sufficient condition under which $P(A \cap B) = b_1$.

Show your steps of how you got your answer.

(b) Determine a real number $b_2 < 1$ such that $P(A \cap B) \leq b_2$ and give one sufficient condition under which $P(A \cap B) = b_2$.

Show your steps of how you got your answer.

(ii) Show from first principles that for two events A and B,

$$P(\text{Exactly one of the events } A \text{ or } B \text{ occurs}) = P(A) + P(B) - 2P(A \cap B)$$

The Axioms of Probability:

For a given experiment, let S denote the sample space. Let P be a set function that associates a real value $P(E)$ with each event $E \subseteq S$. If P satisfies the following properties then it is called a probability set function and $P(E)$ is called the probability of E:

Axiom 1 $P(E) \geq 0$ for every event E;

Axiom 2 $P(S) = 1$;

Axiom 3 $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$ where $E_1, E_2, \ldots$ are pairwise mutually exclusive events.

An alternative formulation of (ii):

Let A and B be two events. Let E be the event of exactly one of the events A or B occurring, i.e. E is the event of either A occurring or B occurring, but not both. Use the Axioms of Probability to show that $P(E) = P(A) + P(B) - 2P(A \cap B)$.

(iii) We consider 12 teams of two people (couples) at a particular time, including two particular teams designated as *team* A and *team* B. Suppose that at a later time six people (out of this group of 24) have been promoted, and it may be assumed that these promotions have occurred randomly among the 24 people.

(a) Find the probability that both people in *team* A have been promoted at the later time.

(b) Find also the probability that the four people in *teams* A and B have been promoted at the later time.

An alternative formulation of (iii):

Let X be a set consisting of 24 people, say $X = \{x_1, x_2, x_3, \ldots, x_{24}\}$. Let $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$. Suppose that 6 people from X have been promoted, and these promotions have occurred randomly among all members of X.

(a) Let E_1 be the event that all members of A have been promoted, i.e. E_1 is the event that x_1 and x_2 have been promoted. Find $P(E_1)$.
Show your steps of how you got your answer.

(b) Let E_2 be the event that all members of A and B have been promoted, i.e. E_2 is the event that x_1, x_2, x_3, and x_4 have been promoted. Find $P(E_2)$.
Show your steps of how you got your answer.

A combinatorial question similar to (iii):

Suppose $X = \{x_1, x_2, x_3, \ldots, x_{24}\}$ is a set of 24 distinct poker cards. For any integer n in the range $1 \leq n \leq 24$, an n-card hand is defined as the event of drawing n cards at random from a deck of these 24 cards, without replacement.

(a) Let $A = \{x_1, x_2\}$. Let E_1 be the event of getting a 6-card hand that contains x_1 and x_2. That is, E_1 is the event of drawing – without replacement – 4 cards at random from X, while avoiding the cards x_1 and x_2. Find $P(E_1)$.

(b) Let $A = \{x_1, x_2\}$ and $B = \{x_3, x_4\}$. Let E_2 be the event of getting a 6-card hand that contains x_1, x_2, x_3, and x_4. That is, E_2 is the event of drawing – without replacement – 2 cards at random from X, while avoiding the cards x_1, x_2, x_3, and x_4. Find $P(E_2)$.