Test 1 − Answers

(i) Consider two events A and B of an outcome space. Determine the maximum and minimum possible values for $P(A \cap B)$, and the conditions under which each of these values is achieved, when $P(A) = 0.5$ and $P(B) = 0.6$.

Remark:

For any event E, we have $0 \leq P(E) \leq 1$. This implies that 0 is the trivial lower bound of $P(E)$ and 1 is the trivial upper bound of $P(E)$. But can we tighten these bounds? In other words, can we determine real numbers b_1 and b_2 such that $b_1 > 0$, $b_2 < 1$, and $b_1 \leq P(E) \leq b_2$? Can we also give some sufficient conditions under which the lower bound b_1 is achieved, i.e. $P(E) = b_1$? Similarly, can we give some sufficient conditions under which the upper bound b_2 is achieved, i.e. $P(E) = b_2$?

Alternative formulation of (i):

Consider two events A and B of an outcome space. Suppose that $P(A) = 0.5$ and $P(B) = 0.6$.

(a) Determine a real number $b_1 > 0$ such that $P(A \cap B) \geq b_1$ and give one sufficient condition under which $P(A \cap B) = b_1$.

(b) Determine a real number $b_2 < 1$ such that $P(A \cap B) \leq b_2$ and give one sufficient condition under which $P(A \cap B) = b_2$.

Solution:

My proof of the inequalities $P(A \cap B) \geq 0.1$ and $P(A \cap B) \leq 0.5$ is based upon the approach taken by Zexi Tian.

Using the formula $P(A \cup B) = P(A) + P(B) - P(A \cap B)$, we get

$$P(A \cap B) = 0.5 + 0.6 - P(A \cup B) = 1.1 - P(A \cup B). \qquad [1]$$

Hence, the problem of finding 'good' lower and upper bounds for $P(A \cap B)$ reduces to the problem of finding 'good' lower and upper bounds for $P(A \cup B)$.

We will use the fact that $P(E_1) \leq P(E_2)$ for any events E_1 and E_2 with the property $E_1 \subseteq E_2$.

First, $P(A) \leq P(A \cup B)$, since $A \subseteq A \cup B$;

Second, $P(B) \leq P(A \cup B)$, since $B \subseteq A \cup B$.

It follows that $\max\{P(A), P(B)\} \leq P(A \cup B)$, i.e. $0.6 \leq P(A \cup B)$. Combining this with the fact that $P(E) \leq 1$ for any event E, we obtain the following bounds for $P(A \cup B)$:

$$0.6 \leq P(A \cup B) \leq 1.$$

Next, we put this into Equation [1] and get

$$0.1 \leq P(A \cap B) \leq 0.5.$$

We have obtained non-trivial bounds for $P(A \cap B)$ (in other words, a larger-than-0 lower bound and also a smaller-than-1 upper bound). It remains to investigate the question whether these bounds can be achieved.

☞ A sufficient condition may be different for each bound.

For this purpose, we need to answer the following separate questions:

Question 1. When does equality hold in $P(A \cap B) \geq 0.1$?

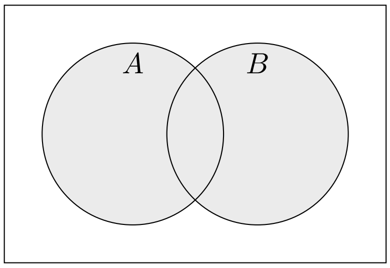
Question 2. When does equality hold in $P(A \cap B) \leq 0.5$?

This is equivalent to finding answers to these two separate questions:

Question 1✷. When does equality hold in $P(A \cup B) \geq 0.6$?

Question 2✷. When does equality hold in $P(A \cup B) \leq 1$?

We can use Venn diagrams as a visual aid.



One answer to Question 1✷: If $B = A \cup B$ then $0.6 = P(B) = P(A \cup B)$.

One answer to Question 2✷: If $A \cup B$ forms the whole sample space then $P(A \cup B) = 1$.

☞ Note that $B = A \cup B \Leftrightarrow A \subseteq B$ is always true.

Here $A \subsetneq B$, since $P(A) \neq P(B)$.

One answer to Question 1: If $A \subset B$ then $P(A \cap B) = 0.5$.

One answer to Question 2: If $A \cup B$ forms the whole sample space then $P(A \cap B) = 0.1$.

An alternative proof of the inequality $P(A \cap B) \leq 0.5$:

First, $P(A \cap B) \leq P(A)$, since $A \cap B \subseteq A$;

Second, $P(A \cap B) \leq P(B)$, since $A \cap B \subseteq B$.

It follows that $P(A \cap B) \leq \min\{P(A), P(B)\} = 0.5$.

Next, in order to answer the question "When does equality hold in $P(A \cap B) \leq 0.5$?", consider the case $A \subseteq B$. It is easy to see that in this case we get $A = A \cap B$ and, hence, $P(A) = P(A \cap B)$. This proves that $P(A \cap B) = 0.5$ whenever $A \subset B$.

We cannot use the fact that $A \cap B \neq \varnothing$ to conclude that $P(A \cap B) \neq 0$!

Proof of $A \cap B \neq \varnothing$:

To obtain a contradiction, we suppose that $A \cap B = \varnothing$. Then, since A and B are mutually exclusive events, Axiom 3 implies that

$$P(A \cup B) = P(A) + P(B) = 0.5 + 0.6 = 1.1.$$

We thus get $P(A \cup B) > 1$, which is a contradiction of the fact that $P(E) \leq 1$ for any event E.