Tutorial 8 – Solutions

Question 4:

Consider a sequence of Bernoulli trials, where the probability of success (S) on each trial is p. Let X denote the completed run-length that begins on the first trial. (For example $X = 3$ when get either SSSF or FFFS). Find the probability distribution of X, and thus, or otherwise, find $E[X]$.

Take an integer $x \in \{1, 2, 3, \ldots\}$. We want to find an explicit formula for $P[X = x]$.

It is easy to see that the probability of getting a run of x consecutive successes is equal to $p^x(1-p)$. Moreover, probability of getting a run of x consecutive failures is equal to $(1-p)^x p$.

Next, by the Probability Axiom 3, the probability of getting a run of x consecutive successes or failures is the sum of the above probabilities, namely $p^x(1-p) + (1-p)^x p$. We get

$$P[X = x] = p^x(1-p) + (1-p)^x p.$$

In addition, let us calculate the probability of getting a run of consecutive successes:

$$\sum_{x=1}^{\infty}(p^x(1-p)) = (1-p)\sum_{x=1}^{\infty} p^x = (1-p)\left(\sum_{x=0}^{\infty} p^x - 1\right) = (1-p)\left(\frac{1}{1-p} - 1\right) = p.$$

Here we have used the geometric series. Similarly, the probability of getting a run of consecutive failures is

$$\sum_{x=1}^{\infty}((1-p)^x p) = p\sum_{x=1}^{\infty}(1-p)^x = p\left(\sum_{x=0}^{\infty}(1-p)^x - 1\right) = p\left(\frac{1}{p} - 1\right) = 1-p.$$

Consider the following probabilistic program – written in R – that takes p as an input parameter:

```
create_patterns = function(p) {
  x = rbinom(1, 1, p)
  repeat {
    old_x = x
    x = rbinom(1, 1, p)
    if (x != old_x){
      break
    }
  }
}
```

This program generates either of two patterns of successes (S) and failures (F):

(1) a consecutive string of S followed by one F,

(2) a consecutive string of F followed by one S.

We claim that the probability of generating one of these patterns (sooner or later) is 1 – thus, the program terminates with probability 1.

We will prove that the set of all its terminating runs has probability one. Using the notation of regular expressions, we will prove the following statements:

1. $P(\text{S}\{x\}\ \text{F}) = p^x(1-p)$,
2. $P(\text{F}\{x\}\ \text{S}) = (1-p)^x p$,
3. $P(\text{S+ F}) = p$,
4. $P(\text{F+ S}) = 1-p$,
5. $P(\text{S+}) = 0$,

6. $P(\mathrm{F}+) = 0$.