

**Test 1 – Answers**

Consider two events  $A$  and  $B$  of an outcome space. Determine the maximum and minimum possible values for  $P(A \cap B)$ , and the conditions under which each of these values is achieved, when  $P(A) = 0.5$  and  $P(B) = 0.6$ .

**Remark:**

For any event  $E$ , we have  $0 \leq P(E) \leq 1$ . This implies that 0 is the trivial lower bound of  $P(E)$  and 1 is the trivial upper bound of  $P(E)$ . But can we tighten these bounds? In other words, can we determine real numbers  $b_1$  and  $b_2$  such that  $b_1 > 0$ ,  $b_2 < 1$ , and  $b_1 \leq P(E) \leq b_2$ ? Can we also give some sufficient conditions under which the lower bound  $b_1$  is achieved, i.e.  $P(E) = b_1$ ? Similarly, can we give some sufficient conditions under which the upper bound  $b_2$  is achieved, i.e.  $P(E) = b_2$ ?

**An alternative formulation of the question:**

Consider two events  $A$  and  $B$  of an outcome space. Suppose that  $P(A) = 0.5$  and  $P(B) = 0.6$ .

- (a) Determine a real number  $b_1 > 0$  such that  $P(A \cap B) \geq b_1$  and give one sufficient condition under which  $P(A \cap B) = b_1$ .
- (b) Determine a real number  $b_2 < 1$  such that  $P(A \cap B) \leq b_2$  and give one sufficient condition under which  $P(A \cap B) = b_2$ .

**Solution:**

My proof of the inequalities  $P(A \cap B) \geq 0.1$  and  $P(A \cap B) \leq 0.5$  is based upon the approach taken by Zexi Tian.

Using the formula  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ , we get

$$P(A \cap B) = 0.5 + 0.6 - P(A \cup B) = 1.1 - P(A \cup B). \quad [1]$$

Hence, the problem of finding ‘good’ lower and upper bounds for  $P(A \cap B)$  reduces to the problem of finding ‘good’ lower and upper bounds for  $P(A \cup B)$ .

We will use the fact that  $P(E_1) \leq P(E_2)$  for any events  $E_1$  and  $E_2$  with the property  $E_1 \subseteq E_2$ .

First,  $P(A) \leq P(A \cup B)$ , since  $A \subseteq A \cup B$ ;

Second,  $P(B) \leq P(A \cup B)$ , since  $B \subseteq A \cup B$ .

It follows that  $\max\{P(A), P(B)\} \leq P(A \cup B)$ , i.e.  $0.6 \leq P(A \cup B)$ . Combining this with the fact that  $P(E) \leq 1$  for any event  $E$ , we obtain the following bounds for  $P(A \cup B)$ :

$$0.6 \leq P(A \cup B) \leq 1.$$

Next, we put this into [Equation \[1\]](#) and get

$$0.1 \leq P(A \cap B) \leq 0.5.$$

We have obtained non-trivial bounds for  $P(A \cap B)$  (in other words, a larger-than-0 lower bound and also a smaller-than-1 upper bound). It remains to investigate the question whether these bounds can be achieved.

🔖 A sufficient condition may be different for each bound.

For this purpose, we need to answer the following separate questions:

Question 1. When does equality hold in  $P(A \cap B) \geq 0.1$ ?

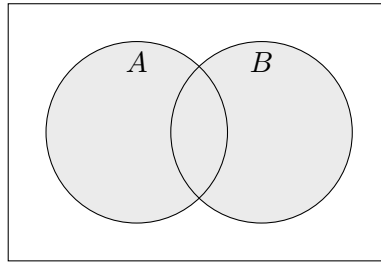
Question 2. When does equality hold in  $P(A \cap B) \leq 0.5$ ?

This is equivalent to finding answers to these two separate questions:

[Question 1★](#). When does equality hold in  $P(A \cup B) \geq 0.6$ ?

[Question 2★](#). When does equality hold in  $P(A \cup B) \leq 1$ ?

We can use Venn diagrams as a visual aid.



One answer to [Question 1★](#): If  $B = A \cup B$  then  $0.6 = P(B) = P(A \cup B)$ .

One answer to [Question 2★](#): If  $A \cup B$  forms the whole sample space then  $P(A \cup B) = 1$ .

🔗 Note that  $B = A \cup B \Leftrightarrow A \subseteq B$  is always true.

Here  $A \subsetneq B$ , since  $P(A) \neq P(B)$ .

One answer to Question 1: If  $A \subset B$  then  $P(A \cap B) = 0.5$ .

One answer to Question 2: If  $A \cup B$  forms the whole sample space then  $P(A \cap B) = 0.1$ .

**An alternative proof of the inequality  $P(A \cap B) \leq 0.5$ :**

First,  $P(A \cap B) \leq P(A)$ , since  $A \cap B \subseteq A$ ;

Second,  $P(A \cap B) \leq P(B)$ , since  $A \cap B \subseteq B$ .

It follows that  $P(A \cap B) \leq \min\{P(A), P(B)\} = 0.5$ .

Next, in order to answer the question “When does equality hold in  $P(A \cap B) \leq 0.5$ ?”, consider the case  $A \subseteq B$ . It is easy to see that in this case we get  $A = A \cap B$  and, hence,  $P(A) = P(A \cap B)$ . This proves that  $P(A \cap B) = 0.5$  whenever  $A \subset B$ .

**We cannot use the fact that  $A \cap B \neq \emptyset$  to conclude that  $P(A \cap B) \neq 0$ !**

**Proof of  $A \cap B \neq \emptyset$ :**

To obtain a contradiction, we suppose that  $A \cap B = \emptyset$ . Then, since  $A$  and  $B$  are mutually exclusive events, Axiom 3 implies that  $P(A \cup B) = P(A) + P(B) = 0.5 + 0.6 = 1.1$ . We thus get  $P(A \cup B) > 1$ , which is a contradiction of the fact that  $P(E) \leq 1$  for any event  $E$ .