12

Textbook

MIND ACTION SERIES
MATHEMATICS

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Continuously updated to comply with the National Curriculum and Assessment Policy Statement (NCAPS)

Written/compiled by top Educators

Creative, interactive, concise approach

MIND
ACTION
SERIES

Mathematics Textbook Grade 12 NCAPS

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MATHEMATICS GRADE 12 NCAPS TEXTBOOK INFORMATION

The concept, purpose and benefits

When the latest Curriculum and Assessment Policy Statement (CAPS) was made available, most educators and authors were still in the dark as to the exact extent of the content to be covered.

There was also a level of uncertainty as to how the content should be presented. Textbooks had to be produced based on limited information. As a result, an increasing number of educators found themselves having to supplement textbooks with additional notes.

After carefully observing examination trends and teaching the curriculum over the last number of years, the authoring team of *Mind Action Series Mathematics* have started on a project of revising our textbooks to alleviate the problems previously faced by Mathematics Educators.

Content has been better structured, more up-to-date examples and exercises have been provided, chapters dealing with difficult topics have been improved and a broader range of typical exam questions has been included. The language has been kept accessible, yet Mathematically correct.

We recommend our new series to educators and learners across the country in the belief that:

- it will help educators get through the curriculum more effectively
- it will enable educators to teach traditionally awkward topics (such as Probability theory, Statistics, Functions and Euclidean Geometry) with greater ease
- it will greatly reduce the need for additional notes
- it will be easier for learners to follow
- it will prepare learners better for examinations

We proudly present our New *Mind Action Series Mathematics* textbooks to each learner and educator in South Africa, certain that it will be of great benefit to all users.

Contents

CHAPTER ONE

Sequences and Series

REVISION OF QUADRATIC NUMBER PATTERNS

In a **quadratic** number pattern:

First term $= a+b+c$

First of the first differences $= 3a+b$

Second difference $= 2a$

General term: $T_n = an^2+bn+c$

EXAMPLE 1

Consider the quadratic number pattern 3; 11; 21; 33; …

(a) Determine a formula for the general term T_n.

(b) Calculate the value of T_{12}.

(c) Which term in this pattern is equal to 497?

Solution

(a)

$$\begin{array}{rccccccc}
a+b+c= & 3 & & 11 & & 21 & & 33 \\
3a+b= & & 8 & & 10 & & 12 & \\
2a= & & & 2 & & 2 & &
\end{array}$$

$2a = 2 \qquad \therefore a = 1$

$3a+b = 8$

$\therefore 3(1)+b = 8 \qquad \therefore b = 5$

$a+b+c = 3$

$\therefore 1+5+c = 3 \qquad \therefore c = -3$

$T_n = an^2+bn+c$

$\therefore T_n = n^2+5n-3$

(b) $T_{12} = (12)^2+5(12)-3 = 201$

(c) $n^2+5n-3 = 497$

$\therefore n^2+5n-500 = 0$

$\therefore (n-20)(n+25) = 0$

$\therefore n = 20$ or $n = -25$

N.A.

$\therefore n = 20$

$\therefore$ The 20^{th} term is -497.

In a **quadratic** number pattern:

- the second difference is constant.
- the first differences form a linear number pattern.

The fact that a quadratic number pattern has a constant second difference can be used to build equations to solve for unknown terms:

EXAMPLE 2

If $-2;\ x;\ 8;\ 19;\ y;\ \ldots$ is a quadratic number pattern, determine the values of x and y.

Solution

$$\begin{array}{ccccccccc} -2 & & x & & 8 & & 19 & & y \\ & x+2 & & 8-x & & 11 & & y-19 & \\ & & 6-2x & & 3+x & & y-30 & & \end{array}$$

$$\begin{array}{ll} 6-2x=3+x & 3+x=y-30 \\ \therefore 3x=3 & \therefore 3+1=y-30 \\ \therefore x=1 & \therefore y=34 \end{array}$$

EXERCISE 1

(a) Determine the general term of each of the following quadratic number patterns:

(1) $8;\ 13;\ 20;\ 29;\ \ldots$ (2) $-2;\ -9;\ -22;\ -41;\ \ldots$

(3) $0;\ \frac{5}{2};\ 6;\ \frac{21}{2};\ \ldots$ (4) $-1;\ -4\frac{1}{2};\ -10;\ -17\frac{1}{2};\ \ldots$

(b) Consider the quadratic number pattern $-1;\ 1;\ 7;\ 17;\ \ldots$

(1) Calculate the value of the 50^{th} term in this pattern.

(2) Determine n if $T_n = 6\,961$.

(c) Consider the quadratic number pattern $0;\ -2;\ -5;\ -9;\ldots$

(1) Calculate the value of T_{25}.

(2) Which term in this pattern is equal to -1034?

(d) Solve for x in the following quadratic number patterns:

(1) $2;\ 8;\ x;\ 26;\ \ldots$ (2) $6;\ 3x-1;\ x;\ -3;\ldots$

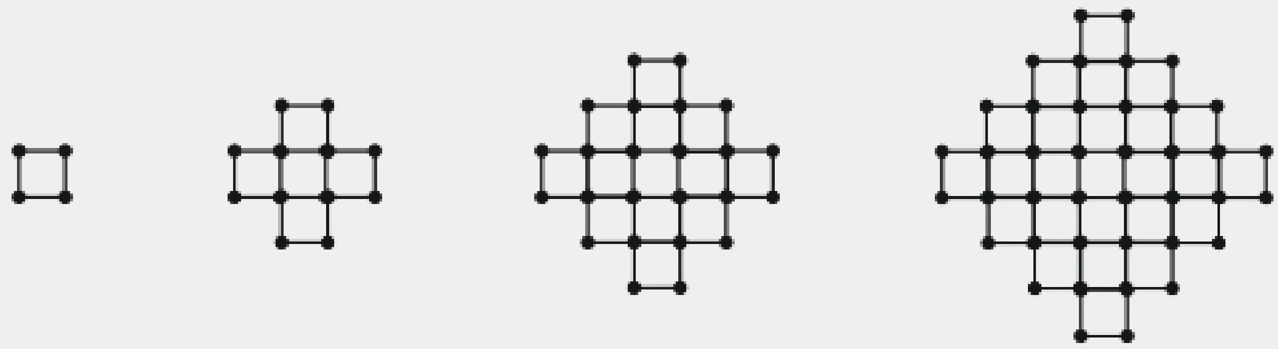
(e) If $3;\ x;\ 11;\ 21;\ y;\ \ldots$ is a quadratic number pattern, determine the values of x and y.

(f) The general term of a quadratic number pattern is given by $T_n = -(n-50)^2+20$.

(1) What is the constant second difference of the number pattern?

(2) Which term has the highest value and what is the value of this term?

(g) Consider the following pattern:



Pattern 1 Pattern 2 Pattern 3 Pattern 4

(1) Determine the number of squares in the n^{th} pattern.
(2) Determine the number of dots in the 75^{th} pattern.

(h)* A quadratic number pattern has a general term $T_n = an^2 + bn + c$. $T_2 - T_1 = 4$ and $T_3 - T_2 = -2$. The third term of the sequence is 7. Determine the values of a, b and c.

(i)* A certain number pattern has a constant second difference of 4. The first two terms are equal and the sum of the first 3 terms is 16. Determine the general term of the sequence.

SEQUENCES

A number pattern is also called a *sequence*.

ARITHMETIC SEQUENCES

An *arithmetic sequence* is a sequence where the **difference** between consecutive terms remains constant i.e. $T_2 - T_1 = T_3 - T_2 = T_4 - T_3 = ...$

In an **arithmetic** sequence, we define:

$$a = \text{the first term} = T_1$$
$$d = \text{constant difference} = T_2 - T_1 = T_3 - T_2 = ...$$

Consider the sequence 2; 5; 8; 11; 14; ...

We see that $a = 2$ and $d = 3$.

$$T_1 = 2 = a$$
$$T_2 = 5 = 2 + 3 = a + d$$
$$T_3 = 8 = 2 + 2(3) = a + 2d$$
$$T_4 = 11 = 2 + 3(3) = a + 3d$$

$$T_n = 2 + (n-1)(3) = a + (n-1)d$$

The **general term** of an **arithmetic sequence** is given by:

$$\boxed{T_n = a + (n-1)d}$$

This formula can be used to determine the value of any specific term of an arithmetic sequence.

EXAMPLE 3

Consider the arithmetic sequence $4; 9; 14; \ldots$

(a) Determine the general term of this sequence.

(b) Calculate the value of the 50^{th} term.

Solution

(a) $a = 4$ and $d = 9 - 4 = 5$

$T_n = a + (n-1)d$

$\therefore T_n = 4 + (n-1)(5)$

$\therefore T_n = 5n - 1$

(b) $T_{50} = 5(50) - 1 = 249$

EXAMPLE 4

Calculate the value of T_{36} in the arithmetic sequence $1; \ -2; \ -5; \ldots$

Solution

$a = 1$ and $d = -2 - 1 = -3$

$T_n = a + (n-1)d$

$\therefore T_n = 1 + (n-1)(-3)$

$\therefore T_n = -3n + 4$

$\therefore T_{36} = -3(36) + 4 = -104$

To determine the **position** of a term, we equate the general term (T_n) to the given term's value and solve for n:

EXAMPLE 5

Which term in the arithmetic sequence $1; 9; 17; \ldots$ is equal to 193?

Solution

$T_n = 1 + (n-1)(8)$

$\therefore T_n = 8n - 7$

$8n - 7 = 193$

$\therefore 8n = 200$

$\therefore n = 25$

$\therefore T_{25} = 193$

We can determine the number of terms in a sequence by equating T_n to the last term's value and solving for n:

EXAMPLE 6

Find the number of terms in the arithmetic sequence $-2;\ -6;\ -10;\ \dots;\ -150$.

Solution

$$T_n = -2 + (n-1)(-4)$$
$$\therefore T_n = -4n + 2$$

$$-4n + 2 = -150$$
$$\therefore -4n = -152$$
$$\therefore n = 38$$

$\therefore$ There are 38 terms in this sequence.

We can solve for unknowns in an arithmetic sequence by remembering that $T_2 - T_1 = T_3 - T_2 = \dots$

EXAMPLE 7

$x-1;\ 2x+1;\ 5x-1$ are the first three terms of an arithmetic sequence.

(a) Determine the value of x. (b) Determine T_n.

Solution

(a)
$$T_2 - T_1 = T_3 - T_2$$
$$\therefore 2x+1-(x-1) = 5x-1-(2x+1)$$
$$\therefore 2x+1-x+1 = 5x-1-2x-1$$
$$\therefore x+2 = 3x-2$$
$$\therefore 2x = 4$$
$$\therefore x = 2$$

(b) $T_1 = x-1 = 2-1 = 1$ $T_2 = 2x+1 = 2(2)+1 = 5$ $T_3 = 5x-1 = 5(2)-1 = 9$

$\therefore 1;\ 5;\ 9;\ \dots$ is the sequence.

$$T_n = a + (n-1)d$$
$$\therefore T_n = 1 + (n-1)(4)$$
$$\therefore T_n = 4n - 3$$

EXERCISE 2

(a) Determine the general term of each of the following arithmetic sequences:

(1) $2;\ 8;\ 14;\ \dots$ (2) $10;\ 7;\ 4;\ \dots$

(3) $-3;\ -1;\ 1;\ \dots$ (4) $-2;\ -4;\ -6;\ \dots$

(5) $\frac{1}{2};\ \frac{5}{2};\ \frac{9}{2};\ \dots$ (6) $3;\ \frac{7}{3};\ \frac{5}{3};\ \dots$

(7) $-1;\ -\frac{1}{4};\ \frac{1}{2};\ \dots$ (8) $\frac{1}{5};\ -\frac{9}{5};\ -\frac{19}{5};\ \dots$

(b) Consider the arithmetic sequence $3; 10; 17; ...$

(1) Determine the general term of this sequence.
(2) Calculate the value of the 35^{th} term in this sequence.
(3) Which term in this sequence is equal to 192?

(c) Consider the arithmetic sequence $\frac{1}{2}; -1; -\frac{5}{2}; ...$

(1) Determine the general term of this sequence.
(2) Calculate the value of the 100^{th} term in this sequence.
(3) Which term in this sequence is equal to -22?

(d) Consider the arithmetic sequence $12; 8; 4; ...$

(1) Calculate T_{23}.
(2) Determine the value of k if $T_k = -168$.

(e) Find the number of terms in the arithmetic sequence $-5; -11; -17; ... ; -491$.

(f) How many terms are there in the arithmetic sequence $-2; \frac{1}{2}; 3; ... ; \frac{361}{2}$?

(g) $x-8; x; 2x-5$ are the first three terms of an arithmetic sequence.

(1) Determine the value of x. (2) Determine the general term.
(3) Determine the value of the 115^{th} term.

(h) $x+1; 2x; 5x+5; ...$ is an arithmetic sequence.

(1) Determine the value of x. (2) Determine the general term.

(i) $2x-1; x-3; 1-3x$ are the first three terms of an arithmetic sequence.

(1) Determine the value of x. (2) Determine the general term.

(j)* $x; y; 8; ...$ and $1; 4x; 3y; ...$ are arithmetic sequences. Determine the values of x and y.

GEOMETRIC SEQUENCES

A *geometric sequence* is a sequence where the **ratio** between consecutive terms remains constant i.e. $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \frac{T_4}{T_3} = ...$

In a **geometric** sequence, we define:

$$a = \text{first term} = T_1$$
$$r = \text{constant ratio} = \frac{T_2}{T_1} = \frac{T_3}{T_2} = ...$$

Consider the sequence $3; 6; 12; 24; 48; ...$

We see that $a = 3$ and $r = 2$.

$$T_1 = 3 = a$$
$$T_2 = 6 = 3 \times 2 = a \times r$$
$$T_3 = 12 = 3 \times 2^2 = a \times r^2$$
$$T_4 = 24 = 3 \times 2^3 = a \times r^3$$

$$T_n = 3 \times 2^{n-1} = a \times r^{n-1} = ar^{n-1}$$

The **general term** of a **geometric sequence** is given by:

$$\boxed{T_n = ar^{n-1}}$$

This formula can be used to determine the value of any specific term of a geometric sequence.

We can also use this formula to determine the **position** of any term in a sequence by equating T_n to the given term's value and solving for n.

Note: To solve for an unknown in an exponential equation we can use the following:

$$\boxed{\text{If } b^x = c, \text{ then } x = \log_b c.}$$

For example, if $2^x = 5$, then $x = \log_2 5 = 2{,}32$.

EXAMPLE 8

Consider the geometric sequence $2;\ 6;\ 18;\ \ldots$

(a) Determine the general term of this sequence.
(b) Calculate the value of the 8th term.
(c) Which term in this sequence is equal to 1 458?

Solution

(a) $a = 2$ and $r = \dfrac{T_2}{T_1} = \dfrac{6}{2} = 3$

$T_n = ar^{n-1}$

$\therefore T_n = 2(3)^{n-1}$

(b) $T_8 = 2(3)^{8-1} = 4\ 374$

(c) $2(3)^{n-1} = 1\ 458$

$\therefore (3)^{n-1} = \dfrac{1\ 458}{2}$

$\therefore (3)^{n-1} = 729$

$\therefore (3)^{n-1} = 3^6$ $\quad$ ***Note:** $\log_3 729 = 6$

$\therefore n - 1 = 6$

$\therefore n = 7$

EXAMPLE 9

Consider the geometric sequence $12;\ -6;\ 3;\ \ldots;\ -\dfrac{3}{2\ 048}$.

(a) Determine the general term of this sequence.

(b) Calculate the value of the 6th term.

(c) Which term in this sequence is equal to $\dfrac{3}{256}$?

(d) How many terms are there in this sequence?

Solution

(a) $a = 12$ and $r = \frac{T_2}{T_1} = \frac{12}{-6} = -\frac{1}{2}$

$T_n = ar^{n-1}$

$\therefore T_n = 12\left(-\frac{1}{2}\right)^{n-1}$

(b) $T_6 = 12\left(-\frac{1}{2}\right)^{6-1} = -\frac{3}{8}$

(c) $12\left(-\frac{1}{2}\right)^{n-1} = \frac{3}{256}$

$\therefore \left(-\frac{1}{2}\right)^{n-1} = \frac{1}{1\,024}$

$\therefore \left(-\frac{1}{2}\right)^{n-1} = \left(-\frac{1}{2}\right)^{10}$ $\quad$ $*$**Note:** $\log_{\frac{1}{2}} \frac{1}{1\,024} = 10$

$\therefore n - 1 = 10$

$\therefore n = 11$

(d) $12\left(-\frac{1}{2}\right)^{n-1} = -\frac{3}{2\,048}$

$\therefore \left(-\frac{1}{2}\right)^{n-1} = -\frac{1}{8\,192}$

$\therefore \left(-\frac{1}{2}\right)^{n-1} = \left(-\frac{1}{2}\right)^{13}$ $\quad$ $*$**Note:** $\log_{\frac{1}{2}} \frac{1}{8\,192} = 13$

$\therefore n - 1 = 13$

$\therefore n = 14$

We can solve for unknowns in a geometric sequence by remembering that $\frac{T_2}{T_1} = \frac{T_3}{T_2} = \ldots$

EXAMPLE 10

$x+2;\ 3x+1;\ 7x-1$ are the first three terms of a geometric sequence. Determine the value(s) of x.

Solution

$\frac{T_2}{T_1} = \frac{T_3}{T_2}$

$\therefore \frac{3x+1}{x+2} = \frac{7x-1}{3x+1}$

$\therefore (3x+1)(3x+1) = (7x-1)(x+2)$

$\therefore 9x^2 + 6x + 1 = 7x^2 + 13x - 2$

$\therefore 2x^2 - 7x + 3 = 0$

$\therefore (2x-1)(x-3) = 0$

$\therefore x = \frac{1}{2}$ or $x = 3$

EXERCISE 3

(a) Determine the general term of each of the following geometric sequences:

(1)	$5;\ 10;\ 20;\ ...$	(2)	$4;\ -12;\ 36;\ ...$
(3)	$-3;\ -6;\ -12;\ ...$	(4)	$-2;\ 10;\ -50;\ ...$
(5)	$18;\ 6;\ 2;\ ...$	(6)	$20;\ -10;\ 5;\ ...$
(7)	$\frac{1}{2};\ \frac{3}{2};\ \frac{9}{2};\ ...$	(8)	$\frac{2}{3};\ -\frac{1}{3};\ \frac{1}{6};\ ...$

(b) Consider the geometric sequence $3;\ 6;\ 12;\ ...\ ;\ 1\ 536$.

(1) Determine the general term.
(2) Calculate the value of the sixth term.
(3) Which term in the sequence is equal to 384?
(4) How many terms are there in this sequence?

(c) Consider the geometric sequence $\frac{2}{9};\ -\frac{2}{3};\ 2;\ ...\ ;\ 1\ 062\ 882$.

(1) Determine the general term.
(2) Calculate the value of the 10^{th} term.
(3) Which term in the sequence is equal to -486?
(4) How many terms are there in this sequence?

(d) Consider the geometric sequence $\frac{27}{4};\ \frac{9}{4};\ \frac{3}{4};\ ...\ ;\ \frac{1}{2\ 916}$.

(1) Calculate T_6.
(2) Determine the value of k if $T_k = \frac{1}{972}$.
(3) How many terms are there in this sequence?

(e) Consider the geometric sequence $-320;\ 160;\ -80;\ ...\ ;\ -\frac{5}{64}$.

(1) Calculate T_8.
(2) Determine the value of k if $T_k = -\frac{5}{16}$.
(3) How many terms are there in this sequence?

(f) $x-5;\ 2x-4;\ 6x$ are the first three terms of a geometric sequence. Determine the possible value(s) of x.

(g) $x+1;\ x-1;\ x-2;\ ...$ is a geometric sequence.

(1) Determine the value of x.
(2) Determine the general term of the sequence.

(h)* $x;\ y;\ 8;\ ...$ is a geometric sequence and $1;\ x+1;\ y+1;\ ...$ is an arithmetic sequence. Determine the values of x and y.

SERIES

A *series* is created by adding the terms of a sequence. For example, the series associated with the sequence 1; 4; 9; ... ; 100 is $1+4+9+...+100$.

We use the symbol S_n to represent the **sum** of the first n terms of a series:

$$\boxed{S_n = T_1 + T_2 + T_3 + ... + T_n}$$

In the series $1+4+9+...+100$:

$$S_1 = T_1 = 1$$
$$S_2 = T_1 + T_2 = 1+4 = 5$$
$$S_3 = T_1 + T_2 + T_3 = 1+4+9 = 14$$

$$S_{10} = T_1 + T_2 + T_3 + ... + T_{10} = 1+4+9+...+100 = 385$$

ARITHMETIC SERIES

An *arithmetic series* is formed by adding the terms of an arithmetic sequence. For example, 2; 6; 10; ...; 62 is an arithmetic sequence and $2+6+10+...+62$ is an arithmetic series.

We can calculate the **sum of the first *n* terms** of an **arithmetic series** by using the following formula:

$$\boxed{S_n = \frac{n}{2}[2a+(n-1)d]}$$

S_n is the sum of the first n terms.
n is the number of terms.
a is the first term.
d is the constant difference.

Proof

Let the first term of an arithmetic series be a and the constant difference d.
$\therefore S_n = a + (a+d) + (a+2d) + ... + T_n$, where $T_n = a+(n-1)d$.

$$\begin{array}{lcccccccccccccc}
S_n & = & a & + & (a+d) & + & (a+2d) & + & \dots & + & (T_n-2d) & + & (T_n-d) & + & T_n \\
S_n & = & T_n & + & (T_n-d) & + & (T_n-2d) & + & \dots & + & (a+2d) & + & (a+d) & + & a \\
\hline
\therefore 2S_n & = & (a+T_n) & + & (a+T_n) & + & (a+T_n) & + & \dots & + & (a+T_n) & + & (a+T_n) & + & (a+T_n)
\end{array}$$

$\therefore 2S_n = n(a+T_n)$

$\therefore S_n = \frac{n}{2}(a+T_n)$

But $T_n = a+(n-1)d$

$\therefore S_n = \frac{n}{2}[a+a+(n-1)d]$

$\therefore S_n = \frac{n}{2}[2a+(n-1)d]$

EXAMPLE 11

Calculate the sum of the arithmetic series $3+7+11+...$ (to 30 terms).

Solution

$S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore S_{30} = \frac{30}{2}[2(3)+(30-1)(4)] = 1\,830$

When the last term (l) in the series is given, the following formula can be used:

$$\boxed{S_n = \frac{n}{2}[a+l]}$$

EXAMPLE 12

Consider the arithmetic series $-1-\frac{3}{2}-2-...-16$.

(a) Determine the number of terms in this series.

(b) Calculate the sum of the series.

Solution

(a) $T_n = a+(n-1)d$

$\therefore T_n = -1+(n-1)\left(-\frac{1}{2}\right)$

$\therefore -\frac{1}{2}n-\frac{1}{2} = -16$

$\therefore n = 31$

(b) $S_n = \frac{n}{2}[a+l]$

$\therefore S_{31} = \frac{31}{2}[-1+(-16)]$

$\therefore S_{31} = -\frac{527}{2}$

We can determine the number of terms that will add up to a specific value by equating S_n to the given value and solving for n:

EXAMPLE 13

How many terms of the arithmetic series $8+13+18+...$ will add up to 1 700?

Solution

$S_n = \frac{n}{2}[2a+(n-1)d]$

$\therefore \frac{n}{2}[2(8)+(n-1)(5)] = 1\,700$

$\therefore n[16+5n-5] = 3\,400$

$\therefore 5n^2+11n-3\,400 = 0$

$\therefore n = \frac{-(11)\pm\sqrt{(11)^2-4(5)(-3\,400)}}{2(5)}$

$\therefore n = 25$ or $n = \frac{-136}{5}$

N.A. (n is a natural number)

$\therefore$ The sum of the first 25 terms is 1 700.

To determine the least number of terms for which the sum of a series will be greater than a given value, or the greatest number of terms for which the sum will be less than a given value, equate S_n to the given value and solve for n. Round the answer according to the requirements of the question:

EXAMPLE 14

Consider the arithmetic series $-4-1+2+\dots$

(a) Calculate the smallest value of n for which $S_n > 300$.

(b) Determine the greatest number of terms that can be added together if the answer must be less than 100.

Solution

(a)
$$\frac{n}{2}[2(-4)+(n-1)(3)]=300$$
$$\therefore n[3n-11]=600$$
$$\therefore 3n^2-11n-600=0$$
$$\therefore n=\frac{-(-11)\pm\sqrt{(-11)^2-4(3)(-600)}}{2(3)}$$
$$\therefore n=16{,}09 \quad \text{or} \quad n=-12{,}43$$
N.A.

$\therefore$ The smallest possible value of n is 17.

(b)
$$\frac{n}{2}[2(-4)+(n-1)(3)]=100$$
$$\therefore n[3n-11]=200$$
$$\therefore 3n^2-11n-200=0$$
$$\therefore n=\frac{-(-11)\pm\sqrt{(-11)^2-4(3)(-200)}}{2(3)}$$
$$\therefore n=10{,}20 \quad \text{or} \quad n=-6{,}53$$
N.A.

$\therefore$ The greatest number of terms is 10.

EXERCISE 4

(a) Calculate the sum of each of the following arithmetic series:

(1) $3+8+13+\dots$ (to 20 terms)
(2) $20+17+14+\dots$ (to 35 terms)
(3) $-12-7-2+\dots$ (to 15 terms)
(4) $-3-7-11-\dots$ (to 30 terms)
(5) $(x)+(x+1)+(x+2)+\dots$ (to 10 terms)
(6) $2q+6q+10q+\dots$ (to 20 terms)

(b) Consider the arithmetic series $2+9+16+\dots+121$.

(1) Determine the number of terms in this series.
(2) Calculate the sum of the series.

(c) Consider the arithmetic series $10+6+2-\dots-66$.

(1) Determine the number of terms in the series.
(2) Calculate the sum of the series.

(d) Calculate the sum of each of the following arithmetic series:

(1) $7+9+11+\dots+105$
(2) $50+43+36+\dots-643$
(3) $2-3-8-\dots-368$
(4) $-15-12-9-\dots+432$
(5) $2+\frac{13}{4}+\frac{9}{2}+\dots+82$
(6) $\frac{11}{2}+\frac{7}{2}+\frac{3}{2}+\dots-\frac{65}{2}$
(7) $-2p+2p+6p+\dots+102p$
(8) $x-2x-5x-\dots-56x$

(e) How many terms of each the following arithmetic series must be added to obtain the given sum?

(1) $5+9+13+\dots=860$
(2) $10+7+4+\dots=-238$
(3) $3-2-7-\dots=-24450$
(4) $-1-3-5-\dots=-2500$

(f) Consider the arithmetic sequence $6; 10; 14; ...$

(1) Calculate the value of the 25th term in this sequence.
(2) Calculate the sum of the first 25 terms of this sequence.
(3) How many terms of the sequence will add up to 1 248?
(4) Calculate the least number of terms that must be added, if the answer must be more than 2 000.

(g) Consider the arithmetic series $4-2-8-...-260$.

(1) Calculate the sum of the series.
(2) Determine m if $4-2-8-...$ (to m terms) $=-1700$.
(3) Calculate the greatest value of k for which $S_k > -800$.

(h) Consider the arithmetic series $\frac{3}{2}+2+\frac{5}{2}+...$

(1) Calculate S_{12}.
(2) Calculate the value of k if $S_k = 231$.
(3) Calculate the greatest number of terms that can be added, if the answer must be less than 300.

(i) Consider the arithmetic sequence $-\frac{1}{5}; -\frac{11}{5}; -\frac{21}{5}; ...$

(1) Calculate the sum of the first 30 terms of this sequence.
(2) Calculate the least number of terms that must be added, if the answer must be less than -100.

(j)* Consider the arithmetic sequence $2x+1; 4x+1; 6x+1; ...$

(1) Determine the general term of this sequence in terms of x.
(2) Which term in the sequence is equal to $150x+1$?
(3) If the sum of the first 21 terms of the sequence is equal to 1 407, determine the value of x.

GEOMETRIC SERIES

A *geometric series* is formed by adding the terms of a geometric sequence. For example, $2; 6; 18; ...; 4372$ is a geometric sequence and $2+6+18+...+4374$ is a geometric series.

We can calculate the **sum of the first *n* terms** of a **geometric series** by using the following formulae:

$$\boxed{S_n = \frac{a(r^n-1)}{r-1};\ r \neq 1} \qquad \text{or} \qquad \boxed{S_n = \frac{a(1-r^n)}{1-r};\ r \neq 1}$$

S_n is the sum of the first n terms.
n is the number of terms.
a is the first term.
r is the constant ratio.

Note: The second formula is normally easier to use when $r < 1$.

Proof

Let the first term of a geometric series be a and the constant ratio r.

$\therefore S_n = a + ar + ar^2 + ... + ar^{n-1}$

$\therefore rS_n = ar + ar^2 + ... + ar^{n-1} + ar^n$

$$\begin{array}{rcccccccccccccc} rS_n & = & & & ar & + & ar^2 & + & \dots & + & ar^{n-2} & + & ar^{n-1} & + & ar^n \\ S_n & = & a & + & ar & + & ar^2 & + & \dots & + & ar^{n-2} & + & ar^{n-1} & & \\ \hline \therefore rS_n - S_n & = & -a & + & 0 & + & 0 & + & \dots & + & 0 & + & 0 & + & ar^n \end{array}$$

$\therefore rS_n - S_n = ar^n - a$

$\therefore S_n(r-1) = a(r^n - 1)$

$\therefore S_n = \dfrac{a(r^n - 1)}{r-1}$

Note: The alternative form $S_n = \dfrac{a(1-r^n)}{1-r}$ can easily be derived from this by multiplying the numerator and the denominator of the formula by -1:

$S_n = \dfrac{a(r^n - 1)}{r-1} \times \dfrac{-1}{-1}$

$\therefore S_n = \dfrac{a(1-r^n)}{1-r}$

EXAMPLE 15

Calculate the sum of each of the following geometric series:

(a) $2+6+18+...$ (to 10 terms)

(b) $10-5+\dfrac{5}{2}-...$ (to 15 terms)

Solution

(a) $S_n = \dfrac{a(r^n - 1)}{r-1}$

$\therefore S_{10} = \dfrac{2(3^{10}-1)}{3-1}$

$\therefore S_{10} = 59\ 048$

(b) $S_n = \dfrac{a(1-r^n)}{1-r}$

$\therefore S_{15} = \dfrac{10\left(1-\left(-\frac{1}{2}\right)^{15}\right)}{\left(1-\left(-\frac{1}{2}\right)\right)}$

$\therefore S_{15} = 6{,}67$

In some cases, we have to determine the **number of terms** in a series before we can calculate the sum of the series. To determine the number of terms in a series, we equate S_n to the last term in the series and solve for n:

EXAMPLE 16

Calculate the sum of each of the following geometric series:

(a) $27+9+3+...+\dfrac{1}{2\ 187}$

(b) $3-6+12-...-1\ 536$

Solution

(a) $T_n = ar^{n-1}$

$\therefore 27\left(\frac{1}{3}\right)^{n-1} = \frac{1}{2\ 187}$

$\therefore \left(\frac{1}{3}\right)^{n-1} = \frac{1}{59\ 049}$

$\therefore \left(\frac{1}{3}\right)^{n-1} = \left(\frac{1}{3}\right)^{10}$

$\therefore n-1=10$

$\therefore n=11$

$S_n = \frac{a(1-r^n)}{1-r}$

$\therefore S_{11} = \frac{27\left(1-\left(\frac{1}{3}\right)^{11}\right)}{1-\frac{1}{3}}$

$\therefore S_{11} = 40{,}50$

(b) $T_n = ar^{n-1}$

$\therefore 3(-2)^{n-1} = -1\ 536$

$\therefore (-2)^{n-1} = -512$

$\therefore (-2)^{n-1} = (-2)^9$

$\therefore n-1=9$

$\therefore n=10$

$S_n = \frac{a(r^n-1)}{(r-1)}$

$\therefore S_{10} = \frac{3((-2)^{10}-1)}{-2-1}$

$\therefore S_{10} = -1\ 023$

We can determine the number of terms that will add up to a specific value by equating S_n to the given value and solving for n:

EXAMPLE 17

Consider the geometric series $5+10+20+...$

(a) How many terms of the series will add up to 20 475?

(b) Determine the least number of terms for which the sum of the series will be greater than 5 000.

Solution

(a) $S_n = \frac{a\left(r^n-1\right)}{r-1}$

$\therefore \frac{5(2^n-1)}{2-1} = 20\ 475$

$\therefore 5(2^n-1) = 20\ 475$

$\therefore 2^n-1 = 4\ 095$

$\therefore 2^n = 4\ 096$

$\therefore 2^n = 2^{12}$

$\therefore n=12$

(b) $S_n = 10\ 000$

$\therefore \frac{5(2^n-1)}{2-1} = 5\ 000$

$\therefore 2^n-1 = 1\ 000$

$\therefore 2^n = 1\ 001$

$\therefore n = \log_2 1\ 001$

$\therefore n = 9{,}97$

$\therefore$ At least 10 terms.

EXAMPLE 18

Consider the geometric series $324+108+36+...$

(a) How many terms of the series will add up to $\frac{13\,120}{27}$?

(b) Determine the least number of terms for which the sum of the series will be greater than 485.

Solution

(a) $S_n=\frac{a(1-r^n)}{1-r}$

$$\therefore \frac{324\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}=\frac{13\,120}{27}$$

$$\therefore 1-\left(\frac{1}{3}\right)^n=\frac{6\,560}{6\,561}$$

$$\therefore \left(\frac{1}{3}\right)^n=\frac{1}{6\,561}$$

$$\therefore \left(\frac{1}{3}\right)^n=\left(\frac{1}{3}\right)^8$$

$$\therefore n=8$$

(b) $$\frac{324\left(1-\left(\frac{1}{3}\right)^n\right)}{1-\frac{1}{3}}=485$$

$$\therefore 1-\left(\frac{1}{3}\right)^n=\frac{485}{486}$$

$$\therefore \left(\frac{1}{3}\right)^n=\frac{1}{486}$$

$$\therefore n=\log_{\frac{1}{3}}\frac{1}{486}$$

$$\therefore n=5{,}63$$

$\therefore$ At least 6 terms.

EXERCISE 5

(a) Calculate the sum of each of the following geometric series:

(1) $3+12+48+...$ (to 10 terms)

(2) $4-8+16-...$ (to 15 terms)

(3) $200+100+50+...$ (to 8 terms)

(4) $-27+9-3+...$ (to 10 terms)

(5) $x+x^2+x^3+...$ (to 20 terms)

(6) $xy^{10}+xy^9+xy^8+...$ (to 8 terms)

(b) Calculate the sum of each of the following geometric series:

(1) $6+12+24+...+6\,144$

(2) $4-8+16-...+16\,384$

(3) $-5+15-45+...-3\,645$

(4) $100-10+1-...+\frac{1}{10\,000}$

(5) $200(1+(1{,}01)+(1{,}01)^2+...+(1{,}01)^{28})$

(6) $-12-6-3-...-\frac{3}{256}$

(7) $xy^2+xy+x+...+\frac{x}{y^{12}}$

(8) $x^5-x^3+x-...+\frac{1}{x^{11}}$

(c) How many terms of each the following geometric series must be added to obtain the given sum?

(1) $2+12+72+...=18\,662$

(2) $8+\frac{8}{3}+\frac{8}{9}+...=\frac{8\,744}{729}$

(3) $-4+16-64+...=52\,428$

(4) $\frac{3}{4}-\frac{3}{8}+\frac{3}{16}-...=\frac{513}{1\,024}$

(d) Consider the geometric series $3+9+27+...$

(1) Calculate S_8.

(2) Determine the value of k if $S_k = 88\ 572$.

(3) Determine the least number of terms for which the sum will be greater than 20 000.

(e) Consider the geometric sequence $\frac{3}{4}; \frac{3}{2}; 3; ...$

(1) Calculate the sum of the first 10 terms of the sequence.

(2) How many terms in this sequence will add up to $\frac{12\ 285}{4}$?

(3) Determine the smallest value of k for which $S_k > 1000$.

(f) Consider the geometric series $50+25+12\frac{1}{2}+...+\frac{25}{8\ 192}$.

(1) Calculate the sum of the series.

(2) Determine m if $50+25+12\frac{1}{2}+...(\text{to } m \text{ terms}) = \frac{6\ 375}{64}$.

(3) Determine the least number of terms for which the sum will be greater than 95.

(g)* Prove that $(1-x)(1+x+x^2+...+x^{99}) = 1-x^{100}$.

THE SUM TO INFINITY OF A GEOMETRIC SERIES

Consider the following two geometric series:

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}...$$

$S_1 = 1$ $\quad\therefore S_1 = 1$

$S_2 = 1+\frac{1}{2} = 1\frac{1}{2}$ $\quad\therefore S_2 = 1{,}5$

$S_3 = 1+\frac{1}{2}+\frac{1}{4} = 1\frac{3}{4}$ $\quad\therefore S_3 = 1{,}75$

$S_4 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8} = 1\frac{7}{8}$ $\quad\therefore S_4 = 1{,}875$

$S_5 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16} = 1\frac{15}{16}$ $\quad\therefore S_5 = 1{,}9375$

$S_6 = 1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32} = 1\frac{31}{32}$ $\quad\therefore S_6 = 1{,}984375$

As n increases, S_n approaches 2.
Mathematically we say that:
if $n \to \infty$ then $S_n \to 2$.
$\therefore$ This series **converges** to 2.

$$1+2+4+8+16+32+...$$

$S_1 = 1$ $\quad\therefore S_1 = 1$

$S_2 = 1+2 = 3$ $\quad\therefore S_2 = 3$

$S_3 = 1+2+4 = 7$ $\quad\therefore S_3 = 7$

$S_4 = 1+2+4+8 = 15$ $\quad\therefore S_4 = 15$

$S_5 = 1+2+4+8+16 = 31$ $\quad\therefore S_5 = 31$

$S_6 = 1+2+4+8+16+32 = 63$ $\quad\therefore S_6 = 63$

As n increases, S_n becomes very large.
Mathematically we say that:
if $n \to \infty$ then $S_n \to \infty$.
$\therefore$ This series **diverges**.

A geometric series will converge (the sum will approach a specific value), if the constant ratio is a number between -1 and 1.

Convergent geometric series:
$-1 < r < 1$

The value approached by a convergent geometric series is called the **sum to infinity** (S_∞) of the geometric series. We can calculate the sum to infinity of a convergent geometric series by using the following formula:

$$\boxed{S_\infty = \frac{a}{1-r} \quad \text{where } -1 < r < 1}$$

Where does this formula come from?

For r-values between -1 and 1:

If $n \to \infty$ then $r^n \to 0$

$$\therefore S_n = \frac{a(1-r^n)}{1-r} \to \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

EXAMPLE 19

Calculate the sum to infinity of each of the following geometric series:

(a) $1+\frac{1}{2}+\frac{1}{4}+...$ (b) $2-\frac{4}{3}+\frac{8}{9}-...$

Solution

(a)

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{1}{1-\frac{1}{2}}$$

$$\therefore S_\infty = 2$$

(b)

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{2}{1-\left(-\frac{2}{3}\right)}$$

$$\therefore S_\infty = \frac{6}{5}$$

We can use the sum to infinity formula to convert recurring decimals into common fractions. To do this we write the recurring decimal as an infinite series. For example, $0{,}\dot{2} = 0{,}2 + 0{,}02 + 0{,}002 + ...$

EXAMPLE 20

Convert the following each of the following recurring decimals into a common fraction, by first writing it as an infinite series:

(a) $0{,}\dot{3}$ (b) $0{,}\dot{1}\dot{6}$

Solution

(a) $0{,}\dot{3} = 0{,}3 + 0{,}03 + 0{,}003 + ...$

$a = 0{,}3$ and $r = \frac{T_2}{T_1} = \frac{0{,}03}{0{,}3} = 0{,}1$

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{0{,}3}{1-0{,}1}$$

$$\therefore S_\infty = \frac{1}{3} \qquad \therefore 0{,}\dot{3} = \frac{1}{3}$$

(b) $0{,}\dot{1}\dot{6} = 0{,}16 + 0{,}0016 + 0{,}000016 + ...$

$a = 0{,}16$ and $r = \frac{T_2}{T_1} = \frac{0{,}0016}{0{,}16} = 0{,}01$

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{0{,}16}{1-0{,}01}$$

$$\therefore S_\infty = \frac{16}{99} \qquad \therefore 0{,}\dot{1}\dot{6} = \frac{16}{99}$$

We can use the fact that geometric series are only convergent if $-1 < r < 1$, to determine when a given series converges:

EXAMPLE 21

Consider the series $3x + 3x(x-2) + 3x(x-2)^2 + ...$

(a) For which values of x will the series converge?

(b) If x is a value for which the series converges, calculate the sum to infinity of the series in terms of x.

Solution

(a) $r = \dfrac{T_2}{T_1} = \dfrac{3x(x-2)}{3x} = x - 2$

$-1 < r < 1$

$\therefore -1 < x - 2 < 1$

$\therefore 1 < x < 3$

(b) $S_\infty = \dfrac{a}{1-r}$

$\therefore S_\infty = \dfrac{3x}{1-(x-2)}$

$\therefore S_\infty = \dfrac{3x}{3-x}$

EXERCISE 6

(a) Calculate the sum to infinity of each of the following geometric series:

(1) $16 + 8 + 4 + ...$ (2) $15 + 5 + \dfrac{5}{3} + ...$

(3) $48 - 12 + 3 - ...$ (4) $-72 + 12 - 2 + ...$

(5) $\dfrac{9}{4} + \dfrac{3}{2} + 1 + ...$ (6) $\dfrac{25}{16} - \dfrac{5}{8} + \dfrac{1}{4} - ...$

(b) Convert the each of the following recurring decimals into a common fraction, by first writing it as an infinite series:

(1) $0{,}\dot{4}$ (2) $0{,}\dot{7}$ (3) $0{,}\dot{2}\dot{5}$

(4) $0{,}\dot{4}\dot{2}$ (5) $-2{,}\dot{6}$ (6) $1{,}2\dot{3}$

(c) Determine the values of x for which each of the following geometric series converges:

(1) $x + 2x^2 + 4x^3 + ...$ (2) $1 - x + x^2 - ...$

(3) $(x+1) + (x+1)^2 + (x+1)^3 + ...$ (4) $2(1-x) + 2(1-x)^2 + 2(1-x)^3 + ...$

(5) $(2x-5) + (2x-5)^2 + (2x-5)^3 + ...$ (6) $x(7-2x)^4 - x(7-2x)^5 + x(7-2x)^6 - ...$

(7) $(3x-2)^3 - (3x-2)^4 + (3x-2)^5 - ...$ (8) $(2-4x) + 2(2x-1)^2 - 2(2x-1)^3 + ...$

(d) The first two terms of a geometric series are $x + 4$ and $x^2 - 16$.

(1) For which values of x will the series converge?

(2) Calculate the sum to infinity of the series in terms of x.

(3) If the sum to infinity of the series is 5, determine the value of x.

(e) Michael calculated the sum to infinity of the geometric series $3 + 9 + 27 + ...$ as follows:

$$S_\infty = \frac{a}{1-r}$$

$$\therefore S_\infty = \frac{3}{1-3} = -\frac{3}{2}$$

Explain why Michael's calculation is incorrect.

SIGMA NOTATION

Writing out the terms of a series can be tedious and time consuming. In this section we will introduce a new notation, called sigma notation, which is quicker and easier to write down. We use the Greek letter $\sum$ (Sigma) to indicate the sum of a series:

$$\boxed{\sum_{k=1}^{n} T_k = T_1 + T_2 + T_3 + \ldots + T_n}$$

$\sum_{k=1}^{n} T_k$ is read as "the sum of T_k for k from 1 to n."

APPLYING SIGMA NOTATION

A series written in sigma notation takes the following form:

$$\sum_{k=\text{first value}}^{\text{last value}} (\text{expression in terms of } k)$$

Any variable may be used (not necessarily k).

The series represented by this notation can be found by assigning consecutive integer values to the variable, starting with the number below the sigma sign (first value) and ending with the number above the sigma sign (last value). Each of these integer values is substituted into the expression and each result obtained represents a term of the series. To calculate the value of the sum represented by the sigma, we add all the terms together, for example:

$$\sum_{k=1}^{5} 2k = 2(1) + 2(2) + 2(3) + 2(4) + 2(5) = 30$$

$$\sum_{k=3}^{7} 2^{k-1} = 2^{3-1} + 2^{4-1} + 2^{5-1} + 2^{6-1} + 2^{7-1} = 124$$

$$\sum_{i=3}^{4} i^2 = 3^2 + 4^2 = 25$$

$$\sum_{r=1}^{20} 3 = 3 + 3 + 3 + \ldots + 3 \text{ (20 terms)} = 3 \times 20 = 60$$

CALCULATING SIGMA VALUES BY USING THE SUM FORMULAE

It is important to note that:

$$\boxed{\sum_{k=1}^{n} T_k = S_n}$$

We can calculate the sum of a series written in sigma notation by first expanding the series and then using the appropriate sum formula:

EXAMPLE 22

Evaluate the following:

(a) $\sum_{k=1}^{10} (2k - 6)$ (b) $\sum_{i=1}^{8} 5(-2)^i$ (c) $\sum_{r=1}^{\infty} 2(4)^{1-r}$

Solution

(a) $\sum_{k=1}^{10}(2k-6)=(2(1)-6)+(2(2)-6)+(2(3)-6)+...$

$$=-4-2+0+...$$

This is an **arithmetic** series: $a=-4$ and $d=2$

There are 10 terms: $n=10$

$$S_n=\frac{n}{2}[2a+(n-1)d]$$

$$\therefore S_{10}=\frac{10}{2}[2(-4)+(10-1)(2)]$$

$$\therefore S_{10}=50$$

(b) $\sum_{i=1}^{8}5(-2)^i=5(-2)^1+5(-2)^2+5(-2)^3+...$

$$=-10+20-40+...$$

This is a **geometric** series: $a=-10$ and $r=-2$

There are 8 terms: $n=8$

$$S_n=\frac{a(r^n-1)}{r-1}$$

$$\therefore S_8=\frac{-10((-2)^8-1)}{-2-1}$$

$$\therefore S_8=850$$

(c) $\sum_{r=1}^{\infty}2(4)^{1-r}=2(4)^{1-1}+2(4)^{1-2}+2(4)^{1-3}+...$

$$=2+\frac{2}{4}+\frac{2}{16}+...$$

This is an **infinite geometric** series: $a=2$ and $r=\frac{1}{4}$

$$S_\infty=\frac{a}{1-r}$$

$$\therefore S_\infty=\frac{2}{1-\frac{1}{4}}$$

$$\therefore S_\infty=\frac{8}{3}$$

If the first value we assign to the variable (the number below the sigma sign) is 1, the number of terms is simply the value above the sigma sign. However, this is not always the case. In general, to calculate the number of terms, we use the following simple calculation:

$$\sum_{k=\text{bottom}}^{\text{top}} \qquad \boxed{n=\text{top}-\text{bottom}+1}$$

EXAMPLE 23

Evaluate the following:

(a) $\sum_{k=3}^{10}(5k+4)$ (b) $\sum_{k=-2}^{6}2(3)^{k+2}$ (c) $\sum_{i=5}^{\infty}(3)^{6-i}$

Solution

(a) $\sum_{i=3}^{10}(5k+4)=19+24+29+...$

$n=10-3+1=8$

$S_n=\frac{n}{2}[2a+(n-1)d]$

$\therefore S_8=\frac{8}{2}[2(19)+(8-1)(5)]=292$

(b) $\sum_{k=-2}^{6}2(3)^{k+2}=2+6+18+...$

$n=6-(-2)+1=9$

$S_n=\frac{a(r^n-1)}{(r-1)}$

$\therefore S_9=\frac{2(3^9-1)}{3-1}=19\ 682$

(c) $\sum_{i=5}^{\infty}(3)^{6-i}=3+1+\frac{1}{3}+...$

$S_\infty=\frac{a}{1-r}$

$\therefore S_\infty=\frac{3}{1-\left(\frac{1}{3}\right)}=\frac{9}{2}$

When the sum is given, we can determine the top value in sigma notation by first solving for n and then using the formula $n=\text{top}-\text{bottom}+1$:

EXAMPLE 24

Determine the value of m if

(a) $\sum_{r=4}^{m}(3r-2)=1\ 150$ (b) $\sum_{k=2}^{m}4(2)^{k+1}<34\ 000$

Solution

(a) $\sum_{r=4}^{m}(3r-2)=10+13+16+...$

$\frac{n}{2}[2(10)+(n-1)(3)]=1\ 150$

$\therefore n[3n+17]=2300$

$\therefore 3n^2+17n-2300=0$

$(n-25)(3n+92)=0$

$\therefore n=25$ or $n=-\frac{-92}{3}$
N.A.

$25=m-4+1$

$\therefore m=28$

(b) $\sum_{k=2}^{m}4(2)^{k+1}=32+64+128+...$

$\frac{32(2^n-1)}{2-1}=34\ 000$

$\therefore 2^n=\frac{2\ 127}{2}$

$\therefore n=\log_2\frac{2\ 127}{2}=10{,}05$

$\therefore n=10$

$10=m-2+1$

$\therefore \therefore m<11; m\in\mathbb{N}$

To write a series in sigma notation, we have to determine the **general term** (T_n) of the associated sequence and the number of terms in the series:

EXAMPLE 25

Write the following series in sigma notation:

(a) $-3-7-11-15-...-123$

(b) $-8+\frac{8}{3}-\frac{8}{9}+\frac{8}{27}-...$ (to infinity)

(c) $100+20+4+\frac{4}{5}+...+\frac{4}{3\,125}$

Solution

(a) $T_n=-3+(n-1)(-4)=-4n+1$

$-4n+1=-123$

$\therefore n=31$

$\therefore -3-7-11-...-123=\sum_{k=1}^{31}(-4k+1)$

(b) $T_n=-8\left(-\frac{1}{3}\right)^{n-1}$

$\therefore -8+\frac{8}{3}-\frac{8}{9}+\frac{8}{27}-...=\sum_{k=1}^{\infty}-8\left(-\frac{1}{3}\right)^{k-1}$

(c) $T_n=100\left(\frac{1}{5}\right)^{n-1}$

$100\left(\frac{1}{5}\right)^{n-1}=\frac{4}{3\,125}$

$\therefore \left(\frac{1}{5}\right)^{n-1}=\frac{1}{7\,8125}$

$\therefore \left(\frac{1}{5}\right)^{n-1}=\left(\frac{1}{5}\right)^{7}$

$\therefore n-1=7$

$\therefore n=8$

$100+20+4+...+\frac{4}{3125}=\sum_{k=1}^{8}100\left(\frac{1}{5}\right)^{k-1}$

EXERCISE 7

(a) Evaluate the following:

(1) $\sum_{k=1}^{50}(3k+1)$ (2) $\sum_{k=1}^{\infty}10(2)^{3-k}$ (3) $\sum_{i=1}^{10}6(2)^{i}$

(4) $\sum_{r=2}^{7}\frac{30}{3^{r-3}}$ (5) $\sum_{i=5}^{45}(20-5i)$ (6) $\sum_{k=1}^{8}\left(\frac{1}{2}\right)^{k}$

(7) $\sum_{k=2}^{\infty}\frac{8}{4^{k}}$ (8) $\sum_{r=2}^{20}2\,000(1{,}01)^{4-r}$ (9) $\sum_{t=6}^{40}5$

(10) $\sum_{t=-2}^{30}(4t+6)$ (11) $\sum_{p=0}^{\infty}3^{4-p}$ (12) $\sum_{i=12}^{20}(-5)$

(b) Determine the value of m in each of the following:

(1) $\sum_{k=1}^{m}(k+1)=1\ 595$ (2) $\sum_{r=1}^{m}5^r=19\ 530$ (3) $\sum_{p=3}^{m}(4p-5)=7\ 500$

(4) $\sum_{p=5}^{m}(2-3p)=-7\ 085$ (5) $\sum_{i=4}^{m}8\left(\frac{2}{3}\right)^{i-5}=\frac{2\ 660}{81}$ (6) $\sum_{k=-2}^{m}\frac{1}{2}(3)^{k-1}=\frac{1\ 640}{27}$

(7) $\sum_{k=1}^{m}(6k-8)<2\ 600$ (8) $\sum_{r=2}^{m}2(3-r)>-800$ (9) $\sum_{p=0}^{m}3(2)^{2p-3}<8\ 000$

(c) Write the following series in sigma notation:

(1) $-3+4+11+18+...+137$ (2) $2+2+2+2+...+2$ (15 terms)

(3) $6+2+\frac{2}{3}+\frac{2}{9}+...$ (to infinity) (4) $10+6+2-2-...-210$

(5) $-2+14-98+686-...+33\ 614$ (6) $3+3+5+9+...+555$

(7) $x+(x+3)+(x+6)+(x+9)+...+(x+120)$ (8) $4+4x+4x^2+4x^3+...+4x^{65}$

(d) For which values of x will the geometric series $\sum_{i=1}^{\infty}4(3-x)^i$ converge?

(e) Calculate the value of p if $\sum_{k=0}^{\infty}27p^{k+1}=\sum_{r=3}^{14}(30-3r)$.

PROBLEMS REQUIRING THE USE OF SIMULTANEOUS EQUATIONS

When we are only given some terms or sums and not the first three terms of the sequence, we will have to use simultaneous equations to determine the values of a and d or r:

EXAMPLE 26

(a) In an arithmetic sequence the third term is 13 and the eighth term is 38. Determine the first three terms of the sequence.

(b) If the seventh term of an arithmetic series is 16 and the sum of the first 10 terms is 115, determine the first term of the series and the constant difference.

Solution

(a) $\boxed{T_n=a+(n-1)d}$

$T_3=a+2d=13$ and $T_8=a+7d=38$

$a+7d=38$ .... A

$a+2d=13$ .... B

$5d=25$ .... A − B

$\therefore d=5$

Substitute $d=5$ into B:

$a+2(5)=13$

$\therefore a=3$

$\therefore$ The first three terms of the sequence are 3; 8; 13.

(b) $S_n = \frac{n}{2}[2a+(n-1)d]$

$T_7 = a+6d = 16$ ······· A and $S_{10} = \frac{10}{2}[2a+9d] = 115$

$\therefore 5[2a+9d] = 115$

$\therefore 2a+9d = 23$ ······· B

From A: $a = 16-6d$ substitute into B:

$2(16-6d)+9d = 23$

$\therefore 32-12d+9d = 23$

$\therefore -3d = -9$

$\therefore d = 3$

Substitute $d = 3$ into $a = 16-6d$:

$a = 16-6(3) = -2$

The first term is -2 and the constant difference is 3.

EXAMPLE 27

In a geometric sequence the sixth term is 8 and the twelfth term is 512. Determine the sequence.

Solution

$T_n = ar^{n-1}$

$T_6 = ar^5 = 8$ and $T_{12} = ar^{11} = 512$

$ar^{11} = 512$ ······· A

$ar^5 = 8$ ······· B

$r^6 = 64$ ······· A ÷ B

$\therefore r = \pm\sqrt[6]{64}$

$\therefore r = \pm 2$

Substitute $r = 2$ and $r = -2$ into B:

$a(2)^5 = 8$ or $a(-2)^5 = 8$

$\therefore 32a = 8$ $\therefore -32a = 8$

$\therefore a = \frac{1}{4}$ $\therefore a = -\frac{1}{4}$

The sequence is $\frac{1}{4}; \frac{1}{2}; 1;...$ or $-\frac{1}{4}; \frac{1}{2}; -1;...$

EXAMPLE 28

(a) The sum of the first three terms of a convergent geometric series is 21 and the sum to infinity is $\frac{64}{3}$. Determine the first three terms of the series.

(b) The sum of the first three terms of a geometric series is 52 and the sum of the next 3 terms is 1 404. Determine the first term and the constant ratio of the series.

Solution

(a) $\boxed{S_n = \frac{a(1-r^n)}{1-r}}$ and $\boxed{S_\infty = \frac{a}{1-r}}$

$S_3 = \frac{a(1-r^3)}{1-r} = 21$ and $S_\infty = \frac{a}{1-r} = \frac{64}{3}$ .... B

$\therefore \frac{a}{1-r} = \frac{21}{(1-r^3)}$ .... A

Let A = B:

$$\frac{21}{1-r^3} = \frac{64}{3}$$

$$\therefore 63 = 64 - 64r^3$$

$$\therefore 64r^3 = 1$$

$$\therefore r^3 = \frac{1}{64}$$

$$\therefore r = \sqrt[3]{\frac{1}{64}} = \frac{1}{4}$$

Substitute $r = \frac{1}{4}$ into B:

$$\frac{a}{1-\frac{1}{4}} = \frac{64}{3}$$

$$\therefore \frac{a}{\frac{3}{4}} = \frac{64}{3}$$

$$\therefore a = \frac{64}{3} \times \frac{3}{4}$$

$$\therefore a = 16$$

The first three terms of the series are 16; 4; 1.

(b) $\boxed{S_n = \dfrac{a(r^n - 1)}{r-1}}$

$S_3 = \dfrac{a(r^3-1)}{r-1} = 52$ and $S_6 = \dfrac{a(r^6-1)}{r-1} = 1\,404 + 52 = 1\,456$

$\dfrac{a}{r-1} = \dfrac{52}{r^3-1}$ .... A $\qquad \dfrac{a}{r-1} = \dfrac{1456}{r^6-1}$ .... B

Let A = B:

$$\frac{52}{r^3-1} = \frac{1\,456}{r^6-1}$$

$$\therefore \frac{r^6-1}{r^3-1} = \frac{1\,456}{52}$$

$$\therefore \frac{(r^3-1)(r^3+1)}{(r^3-1)} = 28$$

$$\therefore r^3 + 1 = 28$$

$$\therefore r^3 = 27$$

$$\therefore r = 3$$

Substitute $r = 3$ into S_3:

$$\frac{a(3^3-1)}{3-1} = 52$$

$$\therefore 13a = 52$$

$$\therefore a = 4$$

EXERCISE 8

(a) The first term of an arithmetic sequence is 2 and the fifth term is 26. Determine the constant difference.

(b) The first term of a geometric sequence is 5 and the fourth term is 40. Determine the constant ratio.

(c) The third term in an arithmetic sequence is 4 and the tenth term is -17. Determine the value of the first term and the constant difference.

(d) The seventh term of a geometric sequence is 192 and the tenth term is $-1\,536$. Determine the value of the first term and the constant ratio.

(e) The eighth term of an arithmetic sequence is 57 and the sum of the first 5 terms is 85. Determine the general term of this sequence.

(f) The second term of a geometric sequence is -12 and the sixth term is -972. Determine the first three terms of this sequence.

(g) The sum of the fourth term and the eight term of an arithmetic series is -54 and the sum of the first ten terms is -200. Determine the series.

(h) The sum of the first four terms of a geometric series is equal to 45 and the sum of the first eight terms is 765. Determine the first 3 terms of the series.

(i) The sum of the first three terms of an arithmetic series is 24 and the sum of the next three terms is -12. Calculate the sum of the first twenty terms of this series.

(j) The sum of the first three terms of a geometric series is equal to 15 and the sum of the next three terms is equal to $-\frac{15}{8}$. Calculate the sum of the first 8 terms of this series.

(k) The second term of a convergent geometric series is -12 and the sum to infinity is $\frac{192}{5}$. Determine the first term and the constant ratio.

(l) In a convergent geometric series, the sum of the first four terms is $\frac{80}{3}$ and the sum to infinity of the series is 27. Determine the general term.

(m) The sum of the first 5 terms of a convergent geometric series is 62 and the sum to infinity of the series is 64. Determine the sum of the first 10 terms.

(n)* The ratio between the sum of the fifth, sixth and seventh terms of a geometric series and the sum of the first three terms of the same series is 81 : 16. Determine the sum of the first ten terms, if the first term is 1024.

(o)* The sum of the first three terms of an arithmetic sequence is 15 and their product is 80. Determine the first three terms.

MORE ADVANCED QUESTIONS ON SERIES

MIXED SERIES

Some series are made up of two different series that alternate. In such cases we have to calculate the sum of each series separately and then add the answers together:

EXAMPLE 29

(a) Calculate the sum of the first 20 terms of the series $1+\frac{1}{2}+5+\frac{1}{4}+9+\frac{1}{8}+13+\frac{1}{16}+...$ correct to four decimal places.

(b) Calculate the sum of the series $3+2+9+2+15+2+21+2+...+597$.

Solution

(a) The two series that were combined are: $1+5+9+13+...$ and $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$

Note: The sum of 20 terms $\rightarrow$ 10 terms from each of these 2 series must be added.

$S_{10}=\frac{10}{2}[2(1)+(10-1)(4)]=190$ and $S_{10}=\frac{\frac{1}{2}\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}=\frac{1\,023}{1\,024}$

$\therefore S_{20}=190+\frac{1\,023}{1\,024}=190{,}9990$

(b) Consider the series $3+9+15+21+...+597$:

$T_n = 3+(n-1)(6)$
$\therefore T_n = 6n-3$

$6n-3=597$
$\therefore n=100$

$S_{100} = \frac{100}{2}[2(3)+(100-1)(6)] = 30\ 000$

Consider the series $2+2+2+2+...+2$ **Note**: there are 99 terms in this series.

$S_{99} = 2 \times 99 = 198$

$\therefore 3+2+9+2+15+...+597 = 30\ 000+198 = 30\ 198$

DETERMINING TERMS FROM A SUM FORMULA

A given sum formula can be used to determine the terms of the sequence. In general:

$$\boxed{T_n = S_n - S_{n-1}}$$

EXAMPLE 30

The sum of the first n terms of a series is given by $S_n = 2n^2+4n$.
(a) Calculate the sum of the first 200 terms of this series.
(b) Calculate the value of the 100th term in the series.
(c) How many terms must be added for the sum to be 4230?

Solution

(a) $S_{200} = 2(200)^2 + 4(200)$
$\therefore S_{200} = 80\ 800$

(b) $T_{100} = S_{100} - S_{99}$
$\therefore T_{100} = \left[2(100)^2+4(100)\right]-\left[2(99)^2+4(99)\right]$
$\therefore T_{100} = 402$

(c) $2n^2+4n = 4\ 230$
$\therefore n^2+2n-2\ 115=0$
$\therefore n = \frac{-2\pm\sqrt{2^2-4(1)(-2\ 115)}}{2}$
$\therefore n = 45$ or $n=-47$
N.A.

WRITING MIXED PRODUCTS AND QUOTIENTS IN SIGMA NOTATION

A series can be built by multiplying or dividing the terms of two different series. To write such a series in sigma notation, we determine the general term of each series separately and use the product or quotient of the two general terms:

EXAMPLE 31

Write each of the following series in sigma notation:

(a) $2\cdot5+4\cdot7+6\cdot9+8\cdot11+...+40\cdot43$

(b) $\frac{1}{3}+\frac{4}{9}+\frac{7}{27}+\frac{10}{81}+...+\frac{22}{6\ 561}$

Solution

(a) For the series $2+4+6+8+...+40$:

$T_n=2+(n-1)(2)=2n$

$2n=40$

$\therefore n=20$

For the series $5+7+9+11+...+43$:

$T_n=5+(n-1)(2)=2n+3$

$\therefore 2\cdot5+4\cdot7+6\cdot9+...+40\cdot43$

$=\sum_{k=1}^{20}2k(2k+3)$

(b) For the series $1+4+7+10+...+22$:

$T_n=1+(n-1)(3)=3n-2$

$3n-2=22$

$\therefore n=8$

For the series $3+9+27+...+6\ 561$:

$T_n=3(3)^{n-1}=3^n$

$\therefore \frac{1}{3}+\frac{4}{9}+\frac{7}{27}+...+\frac{22}{6\ 561}$

$=\sum_{k=1}^{8}\frac{3k-2}{3^k}$

EXERCISE 9

(a) Calculate the sum of each of the following series:

(1) $2+\frac{1}{2}+4+\frac{1}{4}+8+\frac{1}{8}+16+\frac{1}{16}+...$ (to 16 terms)

(2) $2+3+9+5+16+7+23+9+...$ (to 25 terms)

(3) $4+8+4+6+4+4+4+2+...$ (to 30 terms)

(4) $\frac{2}{3}-3+2-3+6-3+18-3+...$ (to 19 terms)

(5) $3+2+11+8+19+14+27+20+...+155+116$

(6) $5+3+10+5+20+7+40+9+...+17+1280$

(7) $8+5+12+5+18+5+27+5+...+\frac{19\ 683}{64}$

(8) $-1-3-1-5-1-7-1-9-...-73$

(b) For a certain series, it is given that $\sum_{i=1}^{n}T_i=5n^2-2n$.

(1) Calculate the sum of the first 50 terms of the series.

(2) Determine T_{20}.

(3) How many terms of the series will add up to 6 055?

(c) The sum to n terms of a series is given by $S_n=\frac{n}{2}[8n-7]$.

(1) How many terms must be added to get a sum of 9 825?

(2) Determine the sixth term of the series.

(3) What is the sum of the eighth term and the tenth term?

(4) What is the general term of this series?

(d) For a certain series $S_n = 64 - 64\left(\frac{1}{2}\right)^n$.

(1) How many terms must be added for the sum to be equal to $\frac{255}{4}$?

(2) Determine the value of T_4.

(3) Show that $T_n = 2^{6-n}$.

(4)* If $2^n = p$, determine the value of $S_{6-n} - S_{6+n}$ in terms of p.

(e) Write each of the following series in sigma notation:

(1) $5\cdot 7 + 11\cdot 13 + 17\cdot 19 + 23\cdot 25 + ... + 179\cdot 181$

(2) $256\cdot 3 + 128\cdot 9 + 64\cdot 27 + 32\cdot 81 + ... + 2\cdot 6561$

(3) $\frac{2}{120} + \frac{6}{117} + \frac{18}{114} + \frac{54}{111} + ... + \frac{13\,122}{96}$

(4) $2 + \frac{7}{(1{,}01)} + \frac{12}{(1{,}01)^2} + \frac{17}{(1{,}01)^3} + ... + \frac{102}{(1{,}01)^{20}}$

(5) $\frac{x+1}{x^8} + \frac{x+2}{x^9} + \frac{x+3}{x^{10}} + \frac{x+4}{x^{11}} + ... + \frac{x+20}{x^{27}}$

(f) Consider the series $1 + 2 + 3 + 4 + ... + 500$.

(1) Calculate the sum of the series.

(2) Calculate the sum of the multiples of 2 that are less than or equal to 500.

(3) Determine the sum of the series formed after the multiples of 2 is removed from the series $1 + 2 + 3 + 4 + ... + 500$.

(g) If the powers of 5 are removed from the sequence $1;\ 2;\ 3;\ 4;\ ... ;\ 5000$, calculate the sum of the remaining terms.

APPLICATIONS

In this section we will apply our knowledge of sequences and series to various scenarios. In each case, we will first rewrite the information given as a sequence or series:

EXAMPLE 32

Max trains to run the Comrades marathon. He runs 3 km in the first week and increases his distance by 2 km each week after that.

(a) What distance did he run during the 11th week?

(b) During which week did he run 17 km?

(c) What is the total distance he ran in the first 15 weeks?

(d) It is said that an athlete should run at least 600 km in preparation for the Comrades marathon. How many weeks should Max train before he will meet this goal?

Solution

(a) Sequence: $3;\ 5;\ 7;..$

$\therefore a = 3$ and $d = 2$

$T_n = 3 + (n-1)(2) = 2n + 1$

$T_{11} = 2(11) + 1 = 23$

$\therefore$ He ran 23 km in week 11.

(b) $2n + 1 = 17$

$\therefore 2n = 16$

$\therefore n = 8$

$\therefore$ He ran 17 km in week 8.

(c) $S_{15} = \frac{15}{2}[2(3)+(15-1)(2)]$

$\therefore S_{15} = 255$

$\therefore$ He ran 255 km in the first 15 weeks.

(d) $\frac{n}{2}[2(3)+(n-1)(2)] = 600$

$\therefore n[2n+4] = 1\ 200$

$\therefore 2n^2 + 4n - 1\ 200 = 0$

$\therefore n^2 + 2n - 600 = 0$

$\therefore n = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-600)}}{2(1)}$

$\therefore n = 23{,}52$ or $n = -25{,}52$

$\therefore$ He will have to train for 24 weeks.

EXAMPLE 33

Tracy saves R1 on the first of a month. Each day after that, she saves twice as much as the day before.

(a) How long will she have to save before she has saved R1 000 000?

(b) How much money will she have to save on the 20th day?

Solution

(a)

Sequence: 1; 2; 4;..
$\therefore a = 1$ and $r = 2$

$S_n = \frac{(1)(2^n - 1)}{2-1} = 1\ 000\ 000$

$\therefore 2^n - 1 = 1\ 000\ 000$

$\therefore 2^n = 1\ 000\ 001$

$\therefore n = \log_2 1\ 000\ 001$

$\therefore n = 19{,}93$

$\therefore$ It will take her 20 days.

(b) $T_n = (1)(2)^{n-1} = 2^{n-1}$

$T_{19} = 2^{20-1} = 524\ 288$

$\therefore$ She has to save R524 288 on day 20.

EXAMPLE 34

A tree with an initial height of 1,5 m is planted. In the first year it grows 1 m, in the second year it grows 0,5 m, in the third year it grows 0,25 m and so forth. Each year the tree grows 50% of the previous years' growth. What is the maximum height that the tree can reach?

Solution

For the growth of the tree:

Series: $1 + 0{,}5 + 0{,}25 + ..$
$\therefore a = 1$ and $r = 0{,}5$

$S_\infty = \frac{1}{1 - \frac{1}{2}} = 2$

$\therefore$ Maximum height $= 1{,}5 + 2 = 3{,}5$ m

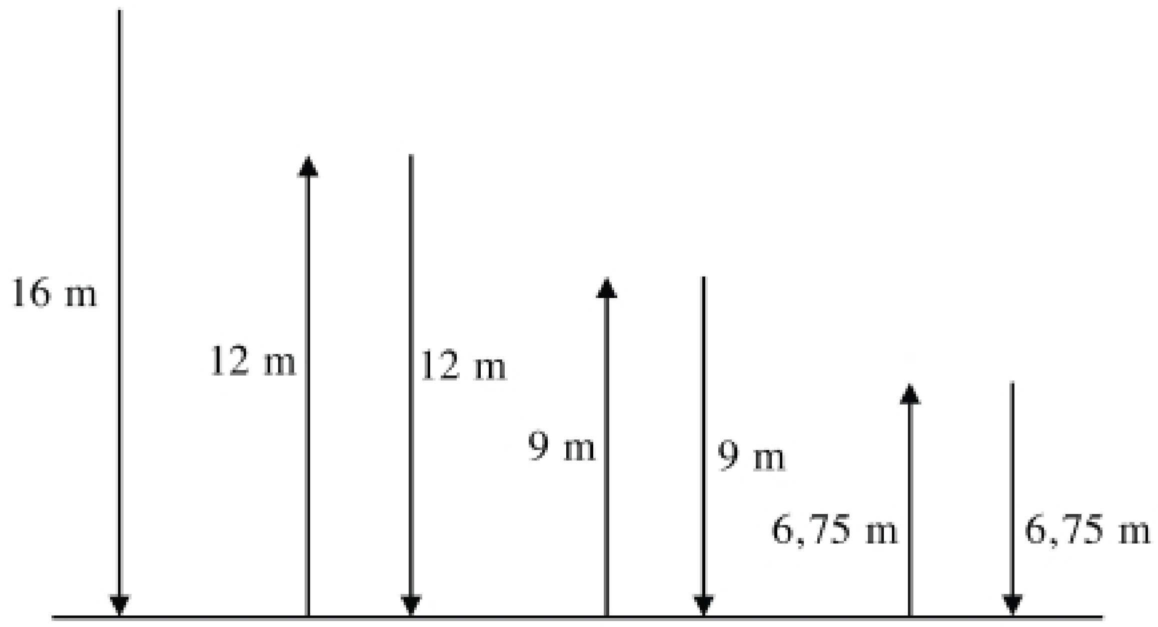
Note: There is no limit to how many years growth could continue, hence we use an **infinite series**.

EXAMPLE 35

A ball is dropped from a height of 16 m and bounces continuously. With each successive bounce, the ball reaches a height that is 75% of the previous height. If this motion continues indefinitely, what is the total vertical distance travelled by the ball over its entire journey?

Solution

The picture below represent this scenario:



From the sketch we write the series as:

$$16+12+12+9+9+6,75+6,75+...$$
$$=16+2(12+9+6,75+...)$$
$$=16+2\left(\frac{12}{1-0,75}\right)$$

*__Note__: Calculate $S_\infty = \frac{a}{1-r}$ for series $12+9+6,75+...$

$$=16+2(48)$$
$$=112$$

$\therefore$ The total vertical distance is 112 m

Note: There is no limit to how many times the ball can bounce, hence we use an **infinite series**.

EXERCISE 10

(a) Carol trains for the Cape Argus cycle race. She cycles a distance of 50 km in the first week and increases her distance by 15 km each week after that.
  (1) What distance did she cycle during the 15th week?
  (2) During which week did she cycle 185 km?
  (3) What is the total distance she cycled over a 20 week period?
  (4) Carol first started cycling 15 weeks before the end of the year. Her goal is to cycle 3 000 km before the end of the year. Will she be able to reach her goal?

(b) A company offers to pay you R0,01 for your first day of employment, R0,02 for your second day, R0,04 for your third day, R0,08 for your fourth day, and so forth. If this pattern continues, calculate your total earnings in the first month. (Assume 30 days in the month.)

(c) Joe dreams of being a striker for *Bafana Bafana* some day. Every day, after school, he takes 100 practice goal kicks from various angles and distances. On the first day he only managed to kick 12 goals. On the second next day he kicked 14 goals, on the third day 16 and on the fourth day 18. If the pattern continues, how long will he have to practice before he will kick 100 goals?

(d) Mary and Chris both start working after school. They each start with a salary of R100 000 per year. Mary's salary will increase by 5% per year and Chris' by R7 500 per year.

(1) How much will Mary earn per year after 5 years?

(2) What will Chris earn in total over a period of 10 years.

(3) They both stay at their jobs for 20 years. Who will be earning more at that time?

(4) After 20 years, who will have the greatest total earnings?

(e) A tree with an initial height of 2 m is planted. In the first year it grows 1,5 m. In the second year it grows 1,2 m, in the third year 0,96 m, in the fourth year 0,768 m, and so forth. What is the maximum height that the tree can reach?

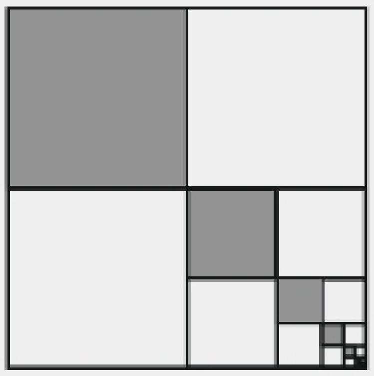
(f) A ball is dropped from a height of 10 m and bounces continuously. With each successive bounce, the ball reaches a height that is 50% of the previous height. If this motion continues indefinitely, what is the total vertical distance travelled by the ball over its entire journey?

(g) The diameter of the first circle in an infinite sequence of circles is 12 cm. Each subsequent circle's diameter is two thirds of the diameter of the previous circle. Calculate

(1) the total circumference of all the circles.

(2) the total area of the all the circles.

(h) The following diagram shows a *fractal* (self similar shape):
A square is divided into quarters and the top left quarter is shaded. The bottom right quarter is then divided into quarters and its top left quarter shaded. This pattern continues to infinity. What is the total shade area if the original square has a length of 2 m?



SUMMARY

	ARITHMETIC $a;\ a+d;\ a+2d;\ ...$	**GEOMETRIC** $a;\ ar;\ ar^2;\ ...$
Definition	Constant difference: $T_2 - T_1 = T_3 - T_2$	Constant ratio: $\frac{T_2}{T_1} = \frac{T_3}{T_2}$
General Term	$T_n = a + (n-1)d$	$T_n = ar^{n-1}$
Sum	$S_n = \frac{n}{2}[2a + (n-1)d]$ $S_n = \frac{n}{2}[a + l]$	$S_n = \frac{a(r^n - 1)}{r-1}$ or $S_n = \frac{a(1-r^n)}{1-r}$
Sum to infinity	Not applicable	$S_\infty = \frac{a}{1-r}$ when $-1 < r < 1$

QUADRATIC

Constant 2nd difference

$T_n = an^2 + bn + c$

- $a + b + c = T_1$
- $3a + b = T_2 - T_1$
- $2a$ = second difference

CONSOLIDATION AND EXTENSION EXERCISE

(a) Determine the general term of each of the following sequences:

(1) $-2;\ 7;\ 24;\ 49;\ ...$ (2) $\frac{5}{3};\ 5;\ 15;\ 45;\ ...$

(3) $\frac{13}{2};\ 3;\ -\frac{5}{2};\ -10;\ ...$ (4) $\frac{64}{5};\ -\frac{32}{5};\ \frac{16}{5};\ -\frac{8}{5};\ ...$

(5) $x;\ 2x+5;\ 3x+10;\ 4x+15;\ ...$ (6) $x^{-1};\ x^{-2};\ x^{-3};\ x^{-4};\ ...$

(b) Evaluate the sum of each of the following series:

(1) $7+\frac{15}{2}+8+\frac{17}{2}+...$ (to 15 terms) (2) $2-6+18-54+...-39\ 366$

(3) $100+50+25+\frac{25}{2}+...$ (to infinity) (4) $6+2-2-6-...-30$

(5) $2-1+7-5+12-9+17-13+...+72-57$ (6) $3x^2+9x^3+27x^4+...+6561x^9$

(c) Evaluate the following:

(1) $\sum_{k=1}^{8} 2(5)^{k-3}$ (2) $\sum_{r=3}^{20}(3r-8)$ (3) $\sum_{p=4}^{\infty} 2(3)^{4-p}$

(d) Write each the following series in sigma notation:

(1) $-2-\frac{3}{2}-1-\frac{1}{2}-...+8$ (2) $9+18+36+72+...+9\ 216$

(3) $10+13+12+7+...-617$ (4) $5+\frac{5}{3}+\frac{5}{9}+\frac{5}{27}+...$ (to infinity)

(5) $5\cdot 2+7\cdot 4+9\cdot 8+11\cdot 16+...$ (to 20 terms) (6) $-\frac{3}{5}-\frac{8}{15}-\frac{13}{45}-\frac{18}{135}-...-\frac{38}{10935}$

(e) Determine the value of m in each of the following:

(1) $\sum_{k=3}^{m}\left(\frac{1}{2}+\frac{3}{2}k\right)=1\ 067\frac{1}{2}$ (2) $\sum_{r=0}^{m} 72\left(\frac{1}{3}\right)^{r-1}=\frac{2\ 912}{9}$

(3) $\sum_{i=2}^{m} 32(2)^{4-i}<250$ (4) $\sum_{i=4}^{m}(2m+8)>10\ 500$

(f) Consider the sequence $\frac{3}{4};\ 3;\ 12;\ 48;\ ...$

(1) Determine the general term of this sequence.

(2) Calculate T_7.

(3) Determine the value of k if $T_k = 49\ 152$.

(4) Calculate the sum of the first 8 terms of the sequence.

(5) Determine the value of p if $\frac{3}{4}+3+12+...(\text{to } p \text{ terms})=\frac{1\ 023}{4}$.

(6) Determine the least number of terms for which the sum of this sequence will be greater than 10 000.

(g) Consider the sequence $-2;\ -7;\ -12;\ -17;...$

(1) Determine the general term of this sequence.

(2) What is the value of the 28^{th} term in the sequence?

(3) Which term in this sequence is equal to -87?

(4) Calculate the value of S_{10}.

(5) How many terms in the sequence add up to -1550?

(6) What is the largest possible value of k for which $S_k > -1000$?

(h) Consider the sequence 159; 120; 83; 48; ...

(1) Determine the general term of this sequence.

(2) Which term in this sequence will have the lowest value?

(3) Determine the general term of the sequence formed by the first differences of the given sequence.

(4) What is the difference between the 24^{th} and 25^{th} term of the original sequence?

(i) Consider the series $(2x+2)+(3x+3)+(4x+4)+(5x+5)+...+(30x+30)$.

(1) Determine the number of terms in the series.

(2) Calculate the sum of the series in terms of x.

(3) For which value of n is $S_n = 2\,555x + 2\,555$?

(j) Consider the series $\sum_{k=0}^{\infty} x^2(x-3)^k$.

(1) For which values of x will this series converge?

(2) If $x = \frac{5}{2}$, calculate the sum to infinity of the series.

(3) If the sum to infinity of the series is $\frac{49}{2}$, determine the value of x.

(k) Determine the value(s) of x in each of the following cases:

(1) $1;\ x;\ -19;\ 6x-2;\ ...$ is a quadratic number pattern.

(2) $2x;\ x+1;\ 6-x$ are the first three terms of an arithmetic sequence.

(3) $x+2;\ 4x-2;\ 6x+2$ are the first three terms of a geometric sequence.

(l) The sum of the fifth term and the eighth term of an arithmetic sequence is 25. The sum of the first 8 terms is 52. Determine the first three terms of the sequence.

(m) The sum to infinity of a convergent geometric series is $364\frac{1}{2}$. The sum of the first four terms is 360. Determine the first term and the constant ratio of this series.

(n) The sum of the first three terms of an arithmetic sequence is 36 and their product is 1140. Determine the three terms.

(o) A certain sequence has a constant second difference of 8. The sixth term of the sequence is 172 and the tenth term is 452. Determine the general term.

(p) The sum of the first n terms of a series is given by $S_n = \frac{n}{4}(7-2n)$.

(1) Calculate the sum of the first 30 terms of this series.

(2) How many terms must be added for the sum to equal $-3\,060$?

(3) Determine the value of the 30^{th} term.

(4) Determine the general term of the series.

(q) The arithmetic sequence $2; 5; 8; ... ; 209$ is given. The even numbers are removed from the sequence. Calculate the sum of the remaining terms.

(r) A restaurant has square tables which seat four people. When two tables are placed together, six people can be seated and when three tables are pushed together eight people can be seated.

(1) How many people can be seated if 10 tables are pushed together?

(2) How many tables will have to be pushed together, if a group of 50 has to be accommodated at one long table for a birthday party?

(s) Gold is extracted from old mine heaps using a chemical process. When 1 000 tons of gravel is processed for the first time, 30 kg of gold is recovered. The next time the same gravel is processed, 24 kg of gold is recovered. On the third attempt 19,2 kg of gold is recovered ,on the fourth 15,36 kg and so on.

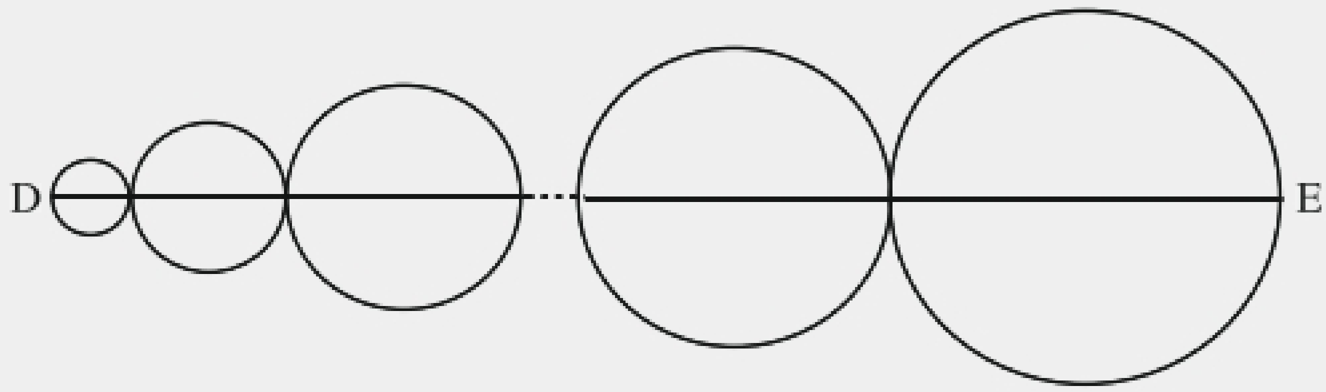
(1) How much gold will be recovered on the eighth attempt?

(2) Calculate the total amount of gold recovered after eight attempts.

(3) What is the maximum amount of gold that can be recovered from 1 000 tons of gravel?

(t) James has 20 containers. The first has a capacity of 50 *l*, the second of 25 *l*, the third 12,5 *l*, the fourth 6,25 *l*, and so forth. If the biggest container is empty and the other 19 are full of milk, will it be possible for James to pour all the milk into the biggest container?

(u) The diagram below shows a number of circles touching each other:



The area of the first circle is $16\pi\ \text{cm}^2$ and each circle has an area $\frac{9}{4}$ times the area of the previous circle. If the length of DE is $\frac{6305}{16}$ cm, determine the number of circles.

(v) $-2;\ 6;\ 22$ are the first 3 terms of a sequence. If the same number is added to each of these terms, the result is a geometric sequence. Determine this number.

(w)* Three numbers are in the ratio $1 : 3 : 10$. If 20 is subtracted from the largest number, the numbers form a geometric sequence. Determine the three numbers.

(x)* In a certain geometric series, the constant ratio is 3 the first term is 1 and $T_n = t$. Express S_n in terms of t.

(y)* Calculate the following sums:

(1) $\left(\frac{1}{4}+\frac{3}{4}\right)+\left(\frac{1}{6}+\frac{3}{6}+\frac{5}{6}\right)+\left(\frac{1}{8}+\frac{3}{8}+\frac{5}{8}+\frac{7}{8}\right)+...+\left(\frac{1}{50}+\frac{3}{50}+\frac{5}{50}+...+\frac{49}{50}\right)$

(2) $100^2-99^2+98^2-97^2+96^2-95^2+...+2^2-1^2$

(3) $1+\sin^2 2°+3+\sin^2 4°+5+\sin^2 6°+...+85+\sin^2 86°+87+\sin^2 88°$

CHAPTER TWO

Functions and Inverses

A function can be briefly described as a rule (or formula) linking two variables (usually x and y). It takes an input (x-value) and produces an output (y-value) by applying a certain mathematical procedure. For a rule to be called a function, it must produce **only one output** (y-value) for any given input (x-value). At the end of this chapter, we will discuss the theory of functions in more detail.

THE INVERSE OF A FUNCTION

The *inverse* of a function is a rule that follows the **opposite procedure** to the function itself.

Take, for example, the function $y = 2x + 1$. In this function, the input is first multiplied by 2 and then 1 is added. This can be illustrated by a flow diagram:

$$x \longrightarrow \boxed{\times 2} \longrightarrow \boxed{+1} \longrightarrow y$$

The inverse of this function does the exact opposite (the **opposite operations** in the **reverse order**):

$$y \longleftarrow \boxed{\div 2} \longleftarrow \boxed{-1} \longleftarrow x$$

1 is subtracted first and then the result is divided by 2. Thus the inverse of $y = 2x + 1$ is $y = \dfrac{x-1}{2}$.

Look at the following flow diagrams:

Notice that the inputs of the second flow diagram are the outputs of the first flow diagram and *vice versa*. This is always the case for a function and its inverse.

When using function notation, the inverse of a function f is denoted by f^{-1}:

$$f(x) = 2x + 1 \text{ and } f^{-1}(x) = \frac{x-1}{2}$$

DETERMINING THE INVERSE OF A FUNCTION

To determine the inverse of a function, we simply **swop x and y** in the defining equation and then make the new y the subject of the formula.

To find the inverse of a function, swop x and y.

If the function is written in the form $f(x) = ...$, first rewrite it in the form $y = ...$

The final answer for the inverse is written in the form $f^{-1}(x) = ...$

EXAMPLE 1

Determine f^{-1} if $f(x) = 2x + 1$.

Solution

For f: $\quad y = 2x + 1$

For f^{-1}: $\quad x = 2y + 1$

$$\therefore 2y = x - 1$$

$$\therefore y = \frac{x-1}{2}$$

$$\therefore f^{-1}(x) = \frac{x-1}{2}$$

EXAMPLE 2

Determine the equation of the inverse of $y = 3x^2$.

Solution

Original: $\quad y = 3x^2$

Inverse: $\quad x = 3y^2$

$$\therefore y^2 = \frac{x}{3}$$

$$\therefore y = \pm\sqrt{\frac{x}{3}}$$

Note: This inverse gives two outputs for each input. This means that the inverse is **not a function**. More about this later.

EXPONENTS AND LOGARITHMS

Logarithms (or *logs*) were designed to solve for **unknown exponents**.

For example: If $2^a = 8$, then $a = \log_2 8 = 3$.

In order to find the inverses of exponential and logarithmic functions, you will have to know the following conversions:

$\boxed{\text{If } x = a^y\text{, then } y = \log_a x}$ and $\boxed{\text{if } x = \log_a y\text{, then } y = a^x.}$

EXAMPLE 3

Determine the equation of the inverse of

(a) $f(x) = 5^x$

(b)* $g(x) = 3^{-x}$

(c)* $h(x) = -4^x$

Solution

(a) For f: $y = 5^x$

For f^{-1}: $x = 5^y$

$\therefore y = \log_5 x$

$\therefore f^{-1}(x) = \log_5 x$

(b)* For g: $y = 3^{-x}$

For g^{-1}: $x = 3^{-y}$

$\therefore -y = \log_3 x$

$\therefore y = -\log_3 x$

$\therefore g^{-1}(x) = -\log_3 x$

(c)* For h: $y = -4^x$

For h^{-1}: $x = -4^y$

$\therefore 4^y = -x$

$\therefore y = \log_4(-x)$

$\therefore h^{-1}(x) = \log_4(-x)$

EXAMPLE 4

Determine the equation of the inverse of

(a) $f(x) = \log_5 x$

(b)* $g(x) = \log_3(-x)$

(c)* $h(x) = -\log_2 x$

Solution

(a) For f: $y = \log_5 x$

For f^{-1}: $x = \log_5 y$

$\therefore y = 5^x$

$\therefore f^{-1}(x) = 5^x$

(b)* For g: $y = \log_3(-x)$

For g^{-1}: $x = \log_3(-y)$

$\therefore -y = 3^x$

$\therefore y = -3^x$

$\therefore g^{-1}(x) = -3^x$

(c)* For h: $y = -\log_2 x$

For h^{-1}: $x = -\log_2 y$

$\therefore -x = \log_2 y$

$\therefore y = 2^{-x}$

$\therefore h^{-1}(x) = 2^{-x}$

EXERCISE 1

(a) Determine the equation of the inverse of each of the following functions:

(1) $f(x) = x + 1$ (2) $y = x^2$ (3) $h(x) = 3x$

(4) $y = -x^2$ (5) $h(x) = 3x - 5$ (6) $y = 4x^2$

(7) $h(x) = \dfrac{x-2}{3}$ (8) $y = -\dfrac{x^2}{3}$ (9) $g(x) = -\dfrac{3}{2}x + 4$

(b) Determine the equation of the inverse of each of the following functions:

(1) $f(x) = 3^x$ (2) $g(x) = \left(\dfrac{1}{2}\right)^x$ (3)* $h(x) = 6^{-x}$

(4)* $g(x) = -5^x$ (5)* $f(x) = -7^{-x}$

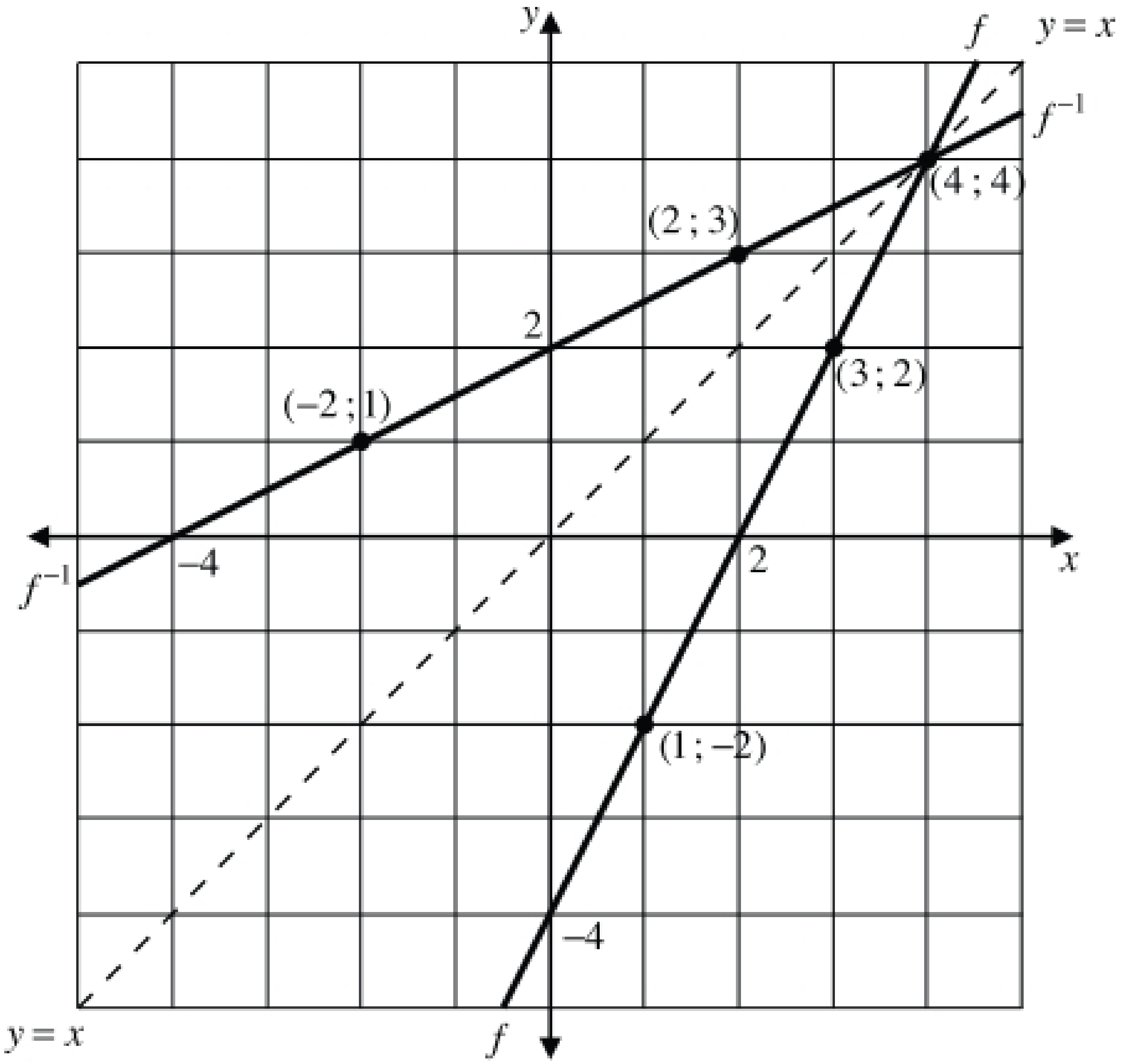
(c) Determine the equation of the inverse of each of the following functions:

(1) $f(x)=\log_7 x$ (2) $g(x)=\log_{\frac{1}{3}} x$ (3)* $h(x)=-\log_5 x$

(4)* $g(x)=\log_4(-x)$ (5)* $f(x)=-\log_{\frac{1}{2}}(-x)$

THE GRAPHS OF A FUNCTION AND ITS INVERSE

The following sketch shows the graphs of $f(x)=2x-4$ and its inverse $f^{-1}(x)=\dfrac{x+4}{2}$:



Take note of the following:

- The graphs of f and f^{-1} are symmetrical to each other in the line $y=x$.
- For every point $(a\,;b)$ on f, there is a point $(b\,;a)$ on f^{-1}.
- The x-intercept of f and the y-intercept of f^{-1} have the same value.
- The y-intercept of f and the x-intercept of f^{-1} have the same value.
- The graphs of f, f^{-1} and $y=x$ all intersect at the same point.

These principles apply to the graphs of any function and its inverse.

EXAMPLE 5

Given the function $f(x)=-3x+3$.

(a) Determine the equation of f^{-1}.

(b) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry.

(c) Determine the coordinates of the point of intersection between f and f^{-1}.

Solution

(a) For f: $y = -3x + 3$

For f^{-1}: $x = -3y + 3$

$\therefore 3y = -x + 3$

$\therefore y = \dfrac{-x+3}{3}$ $\quad\quad \therefore f^{-1}(x) = \dfrac{-x+3}{3}$

(b) For f:

x-intercept: $0 = -3x + 3$

$\therefore x = 1$

y-intercept: $y = 3$

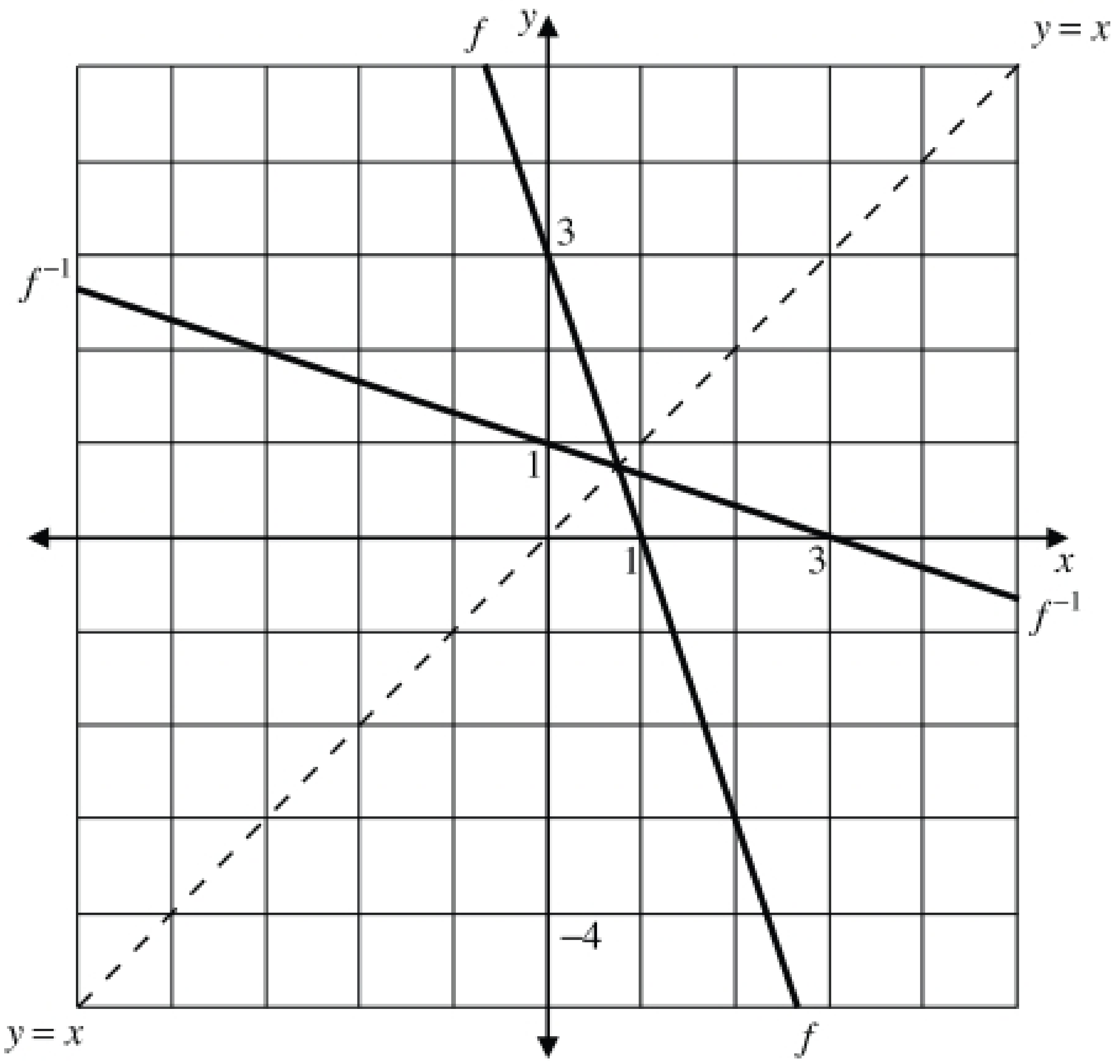
For f^{-1}:

x-intercept of f^{-1} = y-intercept of f

$\therefore x = 3$

y-intercept of f^{-1} = x-intercept of f

$\therefore y = 1$



(c) Solve f and $y = x$ simultaneously:

$-3x + 3 = x$

$\therefore -4x = -3$

$\therefore x = \dfrac{3}{4}$

$\therefore y = \dfrac{3}{4}$

Point of intersection: $\left(\dfrac{3}{4}; \dfrac{3}{4}\right)$

It is usually easier to solve a function and $y = x$ simultaneously than the function and its inverse.

Remember :

- a function, its inverse and the line $y = x$ all intersect at the same point.
- the x and y-coordinates of this point of intersection are always the same value.

EXAMPLE 6

Given the function $y = x^2$.

(a) Determine the equation of the inverse of this function.

(b) Sketch the graphs of $y = x^2$ and its inverse on the same set of axes and show the line of symmetry.

Solution

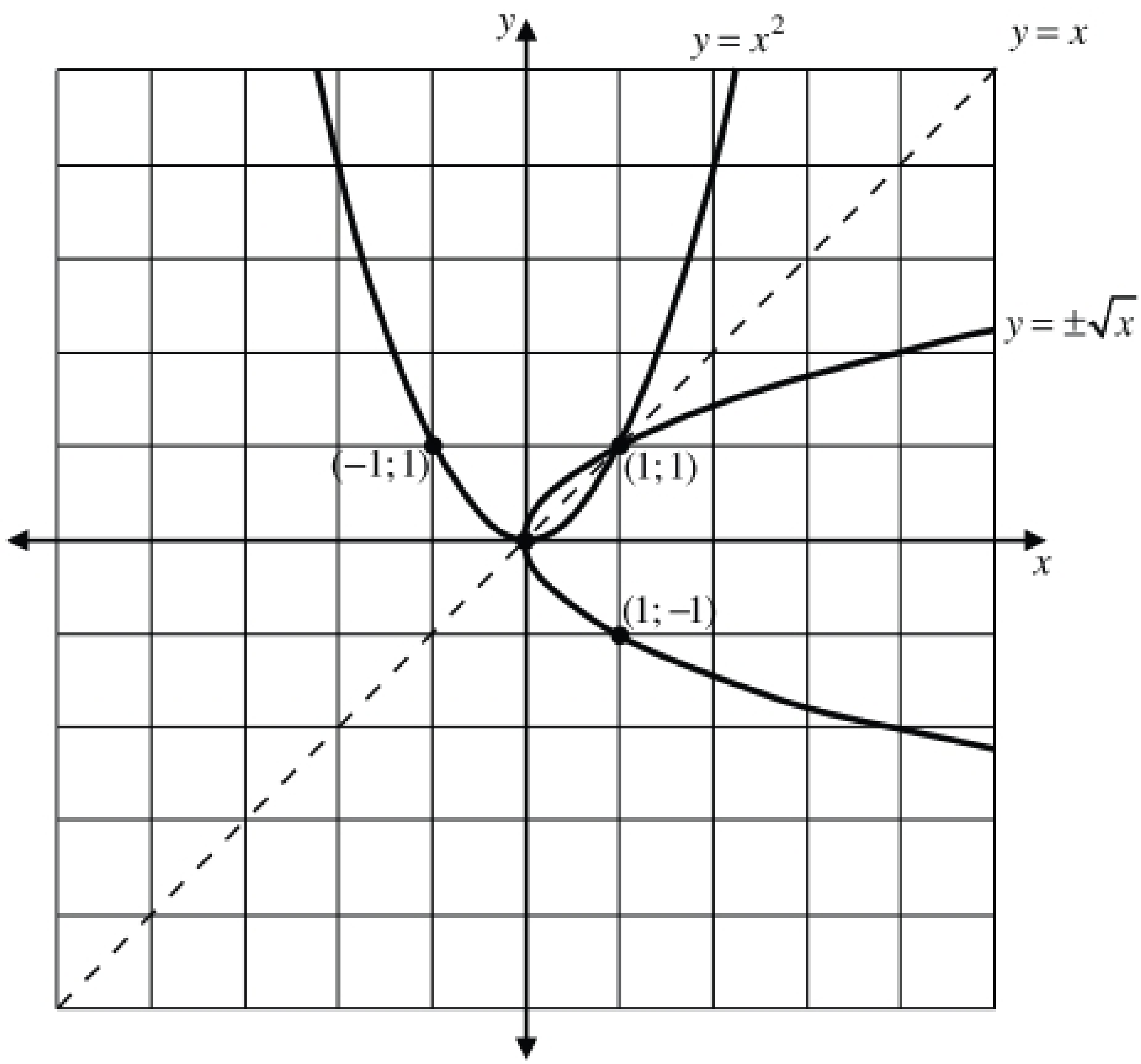
(a) Original: $y = x^2$

Inverse: $x = y^2$

$\therefore y = \pm\sqrt{x}$

(b) $y = x^2$ is the mother parabola.

Three points on $y = x^2$:	**Invert** coordinates for $y = \pm\sqrt{x}$:
$(-1;1)$	$(1;-1)$
$(0;0)$	$(0;0)$
$(1;1)$	$(1;1)$



EXAMPLE 7

Given the function $y=-2x^2$.

(a) Determine the equation of the inverse of this function.

(b) Sketch the graphs of $y=-2x^2$ and its inverse on the same set of axes and show the line of symmetry.

(c) Determine the coordinates of the points of intersection between $y=-2x^2$ and its inverse.

Solution

(a) Original : $y=-2x^2$

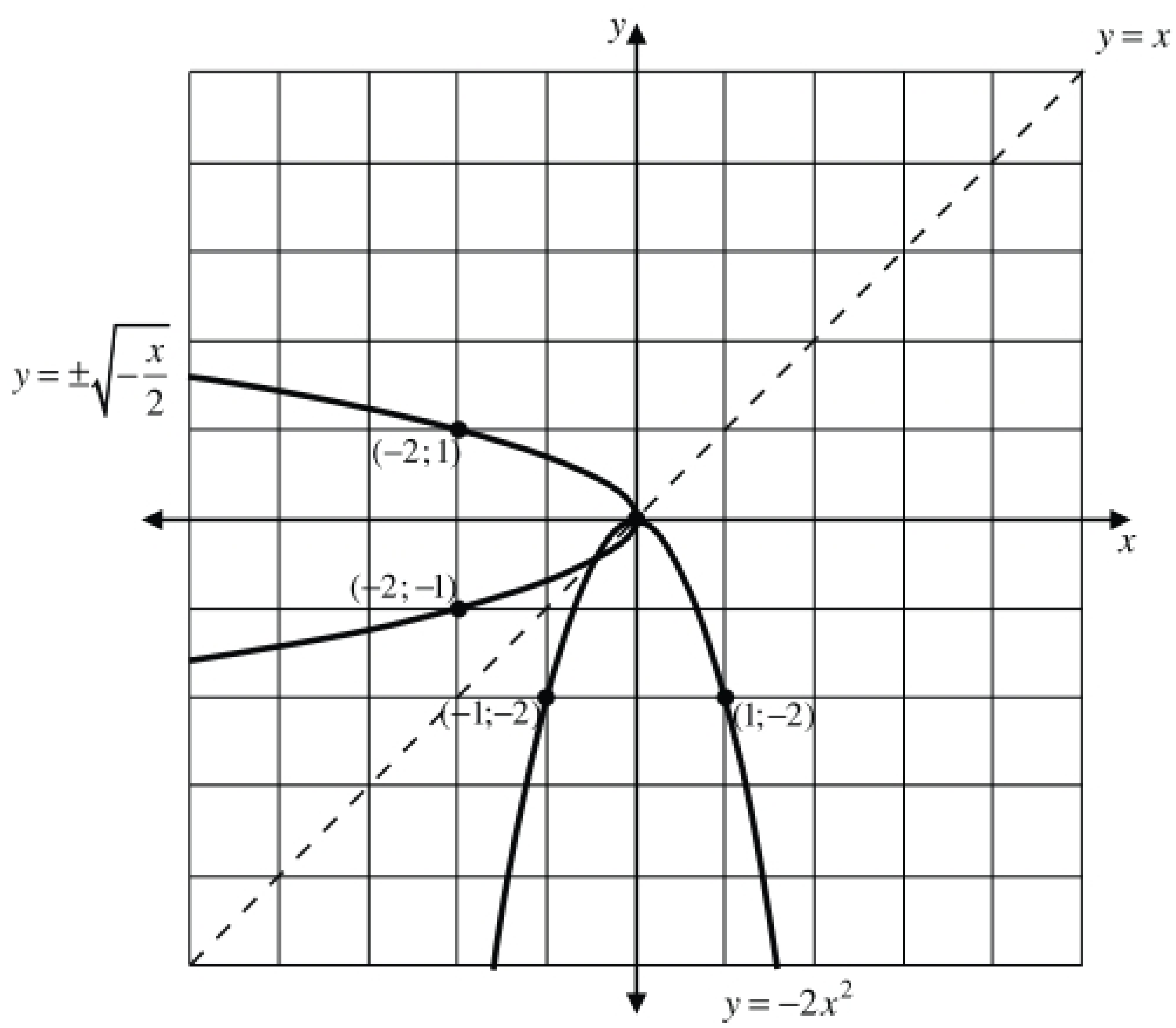
Inverse : $x=-2y^2$

$\therefore y^2=\dfrac{x}{-2}$

$\therefore y=\pm\sqrt{-\dfrac{x}{2}}$

(b) $y=-2x^2$ is a parabola with a negative orientation.

Three points on $y=-2x^2$:	**Invert** coordinates for $y=\pm\sqrt{-\dfrac{x}{2}}$:
$(-1\,;-2)$	$(-2\,;-1)$
$(0\,;0)$	$(0\,;0)$
$(1\,;-2)$	$(-2\,;1)$



(c) Solve $y=-2x^2$ and $y=x$ simultaneously:

$-2x^2=x$

$\therefore 2x^2+x=0$

$\therefore x(2x+1)=0$

$\therefore x=0$ or $x=-\dfrac{1}{2}$

Points of intersection: $(0\,;0)$ and $\left(-\dfrac{1}{2};-\dfrac{1}{2}\right)$

EXAMPLE 8

Determine the equation of the inverse of each of the following functions and then sketch the function and its inverse on the same set of axes, showing the line of symmetry:

(a) $f(x)=2^x$ (b) $g(x)=\left(\dfrac{1}{2}\right)^x$ (c) $h(x)=\log_3 x$

Solution

(a) For f: $y=2^x$

For f^{-1}: $x=2^y$

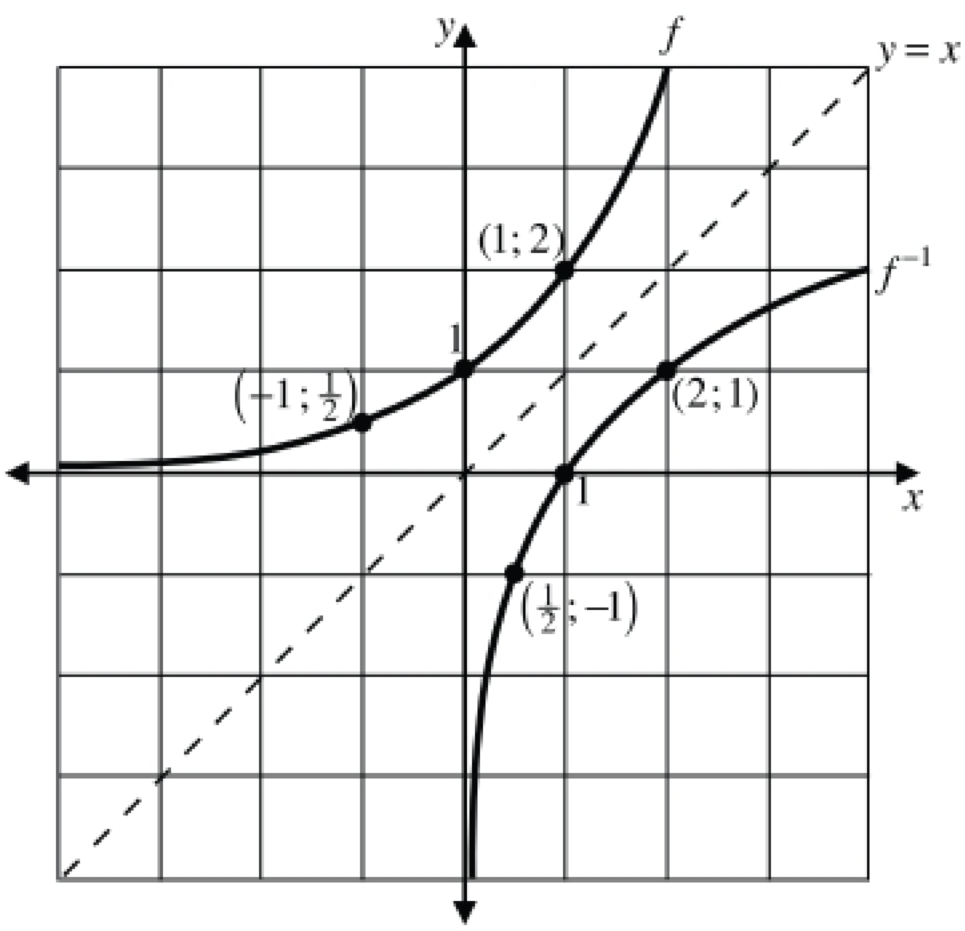
$\therefore y=\log_2 x$

$\therefore f^{-1}(x)=\log_2 x$

f is an exponential function.

The asymptote of f is $y=0$ (the x-axis). $\therefore$ The asymptote of f^{-1} is $x=0$ (the y-axis).

Three points on f:	**Invert** coordinates for f^{-1}:
$\left(-1;\dfrac{1}{2}\right)$	$\left(\dfrac{1}{2};-1\right)$
$(0\,;1)$	$(1\,;0)$
$(1\,;2)$	$(2\,;1)$



(b) For g: $y=\left(\frac{1}{2}\right)^x$

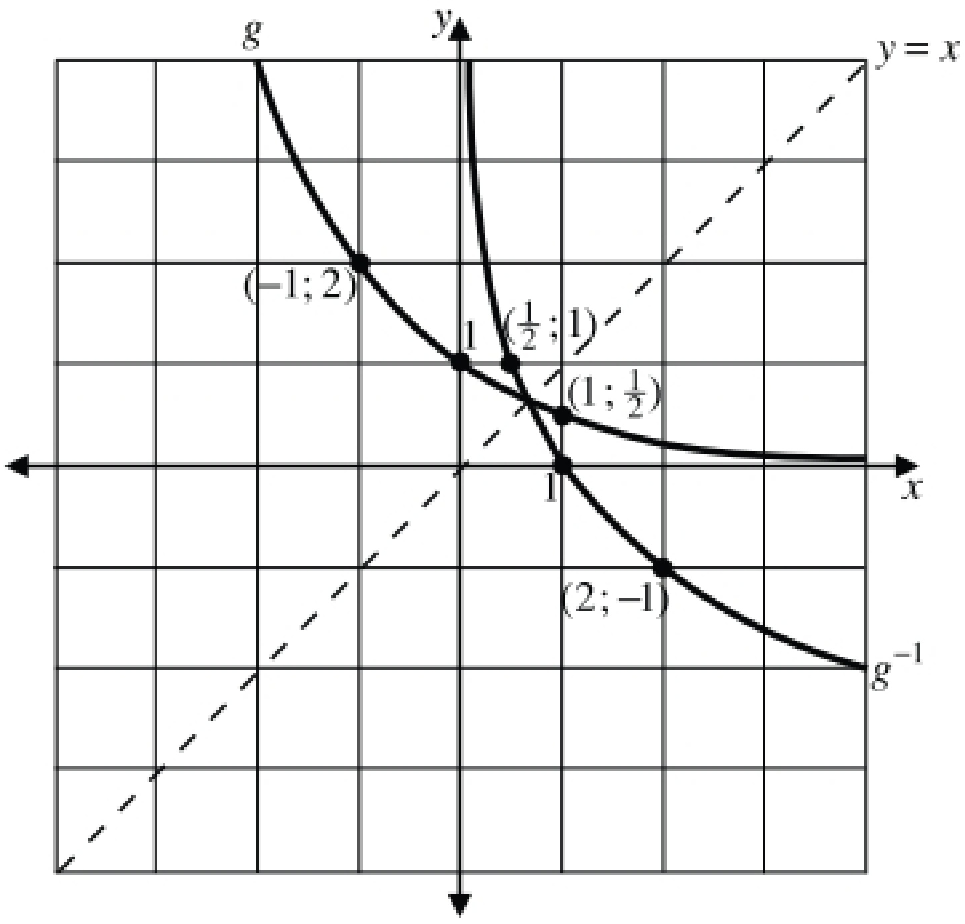
For g^{-1}: $x=\left(\frac{1}{2}\right)^y$

$\therefore y=\log_{\frac{1}{2}} x \qquad \therefore g^{-1}(x)=\log_{\frac{1}{2}} x$

g is an exponential function.

The asymptote of g is $y=0$ (the x-axis). $\therefore$ The asymptote of g^{-1} is $x=0$ (the y-axis).

Three points on g:	**Invert** coordinates for g^{-1}:
$(-1;2)$	$(2;-1)$
$(0;1)$	$(1;0)$
$\left(1;\frac{1}{2}\right)$	$\left(\frac{1}{2};1\right)$



(c) For h: $y=\log_3 x$

For h^{-1}: $x=\log_3 y$

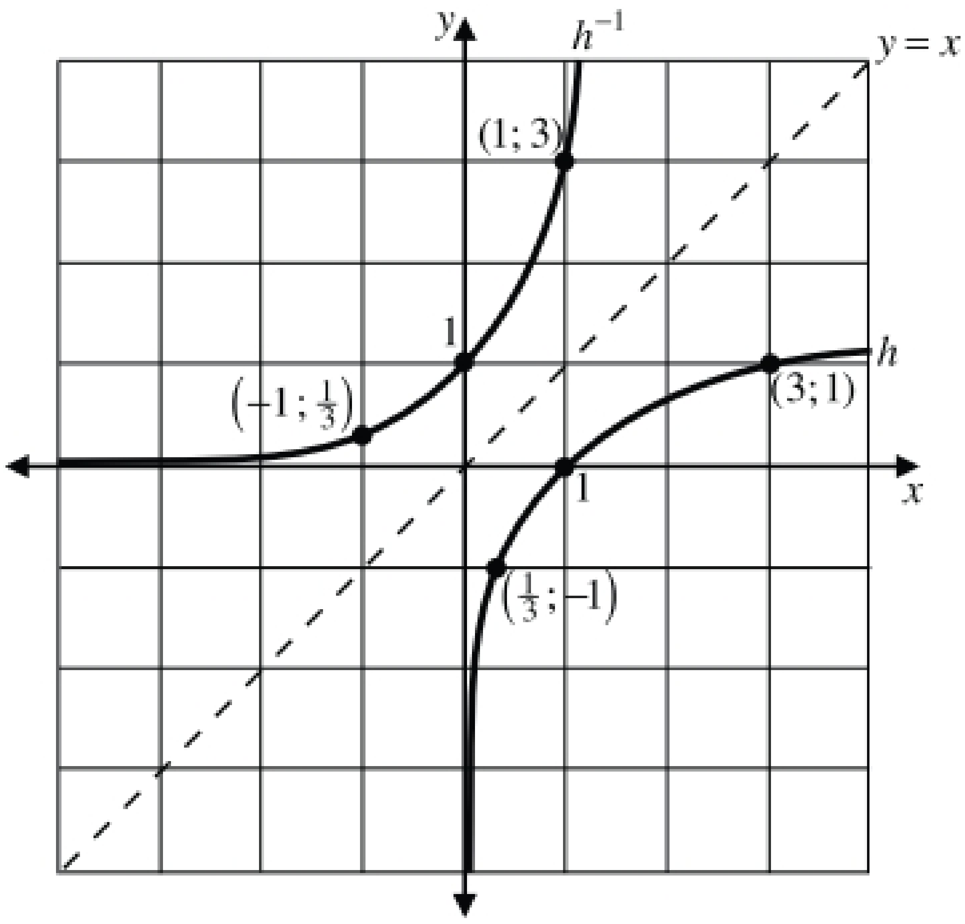
$\therefore y=3^x \qquad \therefore h^{-1}(x)=3^x$

h^{-1} is an exponential function.

In this case it is easier to start with h^{-1}.

The asymptote of h^{-1} is $y=0$ (the x-axis). $\therefore$ The asymptote of h is $x=0$ (the y-axis).

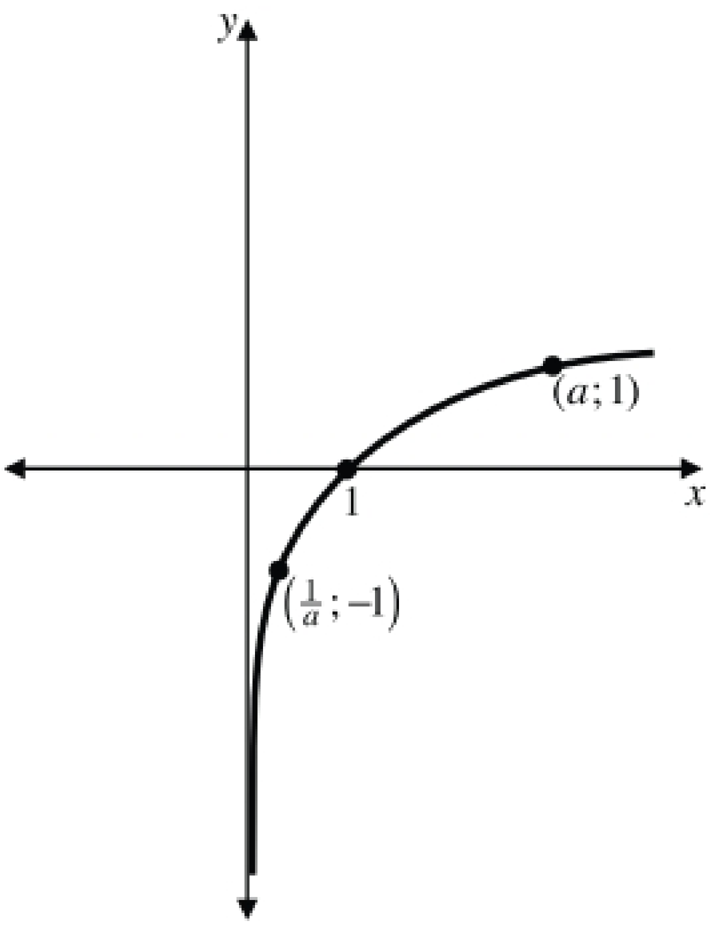
Three points on h^{-1}:	**Invert** coordinates for h:
$\left(-1;\frac{1}{3}\right)$	$\left(\frac{1}{3};-1\right)$
$(0;1)$	$(1;0)$
$(1;3)$	$(3;1)$



THE GRAPH OF A LOGARITHMIC FUNCTION

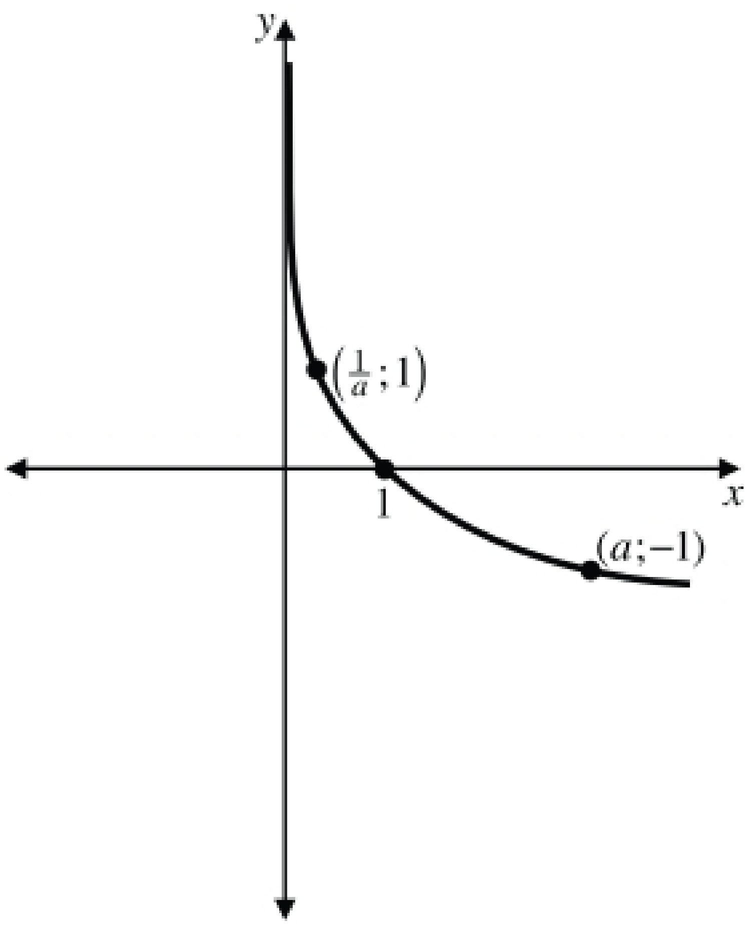
The graph of a logarithmic function can easily be drawn by considering it as the inverse of an exponential function (see Example 8 (c) above). It is, however, worthwhile to know the shapes of the two types of logarithmic functions:

$y = \log_a x \ \ ; \ a > 1$



- **Increasing** function
- x-intercept at $(1\,;0)$
- Asymptote: negative y-axis $(x = 0)$

$y = \log_a x \ \ ; \ 0 < a < 1$



- **Decreasing** function
- x-intercept at $(1\,;0)$
- Asymptote: positive y-axis $(x = 0)$

EXERCISE 2

(a) Determine the equation of the inverse of each of the following functions and then sketch the function and its inverse on the same set of axes, showing the line of symmetry:

(1)	$f(x)=3x-6$	(2)	$y=2x^2$	(3)	$g(x)=3^x$
(4)	$y=-x^2$	(5)	$h(x)=\left(\dfrac{1}{3}\right)^x$	(6)	$g(x)=\log_2 x$
(7)	$y=-\dfrac{1}{2}x^2$	(8)	$f(x)=4^{-x}$	(9)	$h(x)=\log_{\frac{1}{2}} x$
(10)*	$f(x)=-3^x$	(11)*	$h(x)=-\log_3 x$	(12)*	$g(x)=\log_2(-x)$

(b) Given the function $f(x)=4x+4$.

(1) Determine the equation of f^{-1}.

(2) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry.

(3) Calculate the coordinates of the point of intersection between f and f^{-1}.

(c) Given the function $y=3x^2$.

(1) Determine the equation of the inverse of this function.

(2) Sketch the graphs of $y=3x^2$ and its inverse on the same set of axes and show the line of symmetry.

(3) Calculate the coordinates of the points of intersection between $y=3x^2$ and its inverse.

REFLECTIONS

The following rules can be used to obtain the equations of three important reflections of any graph:

Reflection in the x-axis	**Reflection in the y-axis**	**Reflection in the line $y = x$**
Replace y with $-y$.	Replace x with $-x$.	Swop x and y (Find **inverse**)
$\boxed{y\rightarrow -y}$	$\boxed{x\rightarrow -x}$	$\boxed{x\leftrightarrow y}$

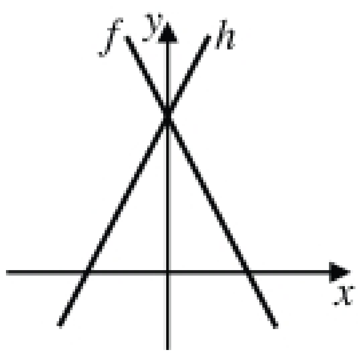
EXAMPLE 9

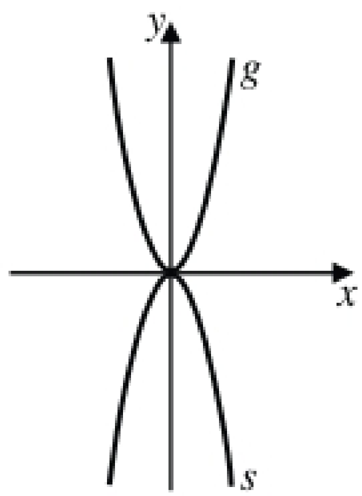
Given the functions $f(x)=-2x+8$ and $g(x)=3x^2$.

Determine the equation of

(a) h, the reflection of f in the y-axis.

(b) s, the reflection of g in the x-axis.





Solution

(a) For f: $y=-2x+8$

For h: $y=-2(-x)+8$ $\quad \boxed{x\rightarrow -x}$

$\therefore y=2x+8$

$\therefore h(x)=2x+8$

(b) For g: $y=3x^2$

For s: $-y=3x^2$ $\quad \boxed{y\rightarrow -y}$

$\therefore y=-3x^2$

$\therefore s(x)=-3x^2$

EXAMPLE 10

The graph of $f(x) = a^x$ passes through the point $P(-2\ ;\ 25)$.
g is the reflection of f in the line $x = 0$.

(a) Determine the value of a.

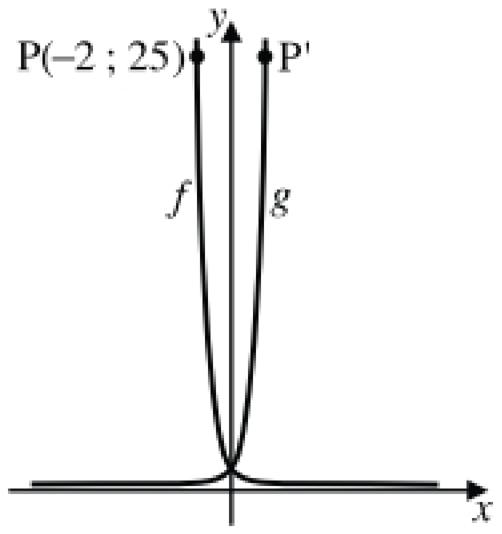
(b) If $g(x) = b^x$, determine the value of b.

(c) Determine the coordinates of P', the image of P, on g.

(d) If h is the reflection of f in the line $y = x$, determine

  (1) the equation of h.

  (2) the coordinates of P'', the image of P, on h.



Solution

(a) Substitute $P(-2\ ;\ 25)$ into $y = a^x$:

$$25 = a^{-2}$$

$$\therefore 25 = \frac{1}{a^2}$$

$$\therefore a^2 = \frac{1}{25}$$

$$\therefore a = \pm\sqrt{\frac{1}{25}} = \pm\frac{1}{5}$$

But the base of an exponential function is always positive.

$$\therefore a = \frac{1}{5}$$

Calculator shortcut :

$$a^{-2} = 25$$

$$\therefore a = \sqrt[-2]{25} = \frac{1}{5}$$

(b) For f: $y = \left(\frac{1}{5}\right)^x$

For g: $y = \left(\frac{1}{5}\right)^{-x}$ $\boxed{x \to -x}$

$$\therefore y = 5^x$$

$$\therefore b = 5$$

(c) Reflection in the y-axis.
$\therefore$ Change the sign of the x-coordinate: $P(-2\ ;\ 25) \to P'(2\ ;\ 25)$
$\therefore$ The coordinates of P' are $(2\ ;\ 25)$.

(d) (1) For f: $y = \left(\frac{1}{5}\right)^x$

For h: $x = \left(\frac{1}{5}\right)^y$ $\boxed{x \leftrightarrow y}$

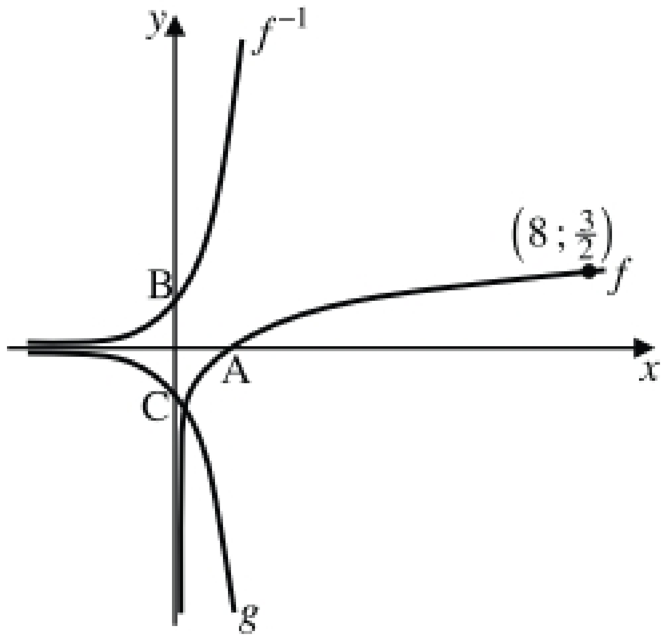
$$\therefore y = \log_{\frac{1}{5}} x$$

$$\therefore h(x) = \log_{\frac{1}{5}} x$$

(2) Reflection in the line $y = x$.
$\therefore$ Swop the x- and y-coordinates: $P(-2\ ;\ 25) \to P''(25\ ;\ -2)$
$\therefore$ The coordinates of P'' are $(25\ ;\ -2)$.

EXAMPLE 11

The sketch alongside shows the graphs of $f(x)=\log_b x$, f^{-1} and g, the reflection of f^{-1} in the x-axis. $\left(8\,;\frac{3}{2}\right)$ is a point on f.



(a) Determine the value of b.
(b) Determine the equation of f^{-1}.
(c) Determine the equation of g.
(d) Write down the domain of f.
(e) Write down the range of
 (1) f^{-1} (2) g.
(f) Write down the coordinates of A, B and C.
(g) For which values of x is
 (1) $f(x)<0$? (2) $g(x)\geq -1$?
 (3) $f^{-1}(x)<8$?
(h) If the graph of $y=f(x+t)$ passes through the point $\left(11\,;\frac{3}{2}\right)$, write down the value of t.

Solution

(a) Substitute $\left(8\,;\frac{3}{2}\right)$ into $y=\log_b x$:

$$\frac{3}{2}=\log_b 8$$
$$\therefore b^{\frac{3}{2}}=8$$
$$\therefore b=8^{\frac{2}{3}}$$
$$\therefore b=(2^3)^{\frac{2}{3}}$$
$$\therefore b=2^2=4$$

Calculator shortcut :
$b^{\frac{3}{2}}=8$
$\therefore b=\sqrt[\frac{3}{2}]{8}=4$

(b) For f: $y=\log_4 x$
For f^{-1}: $x=\log_4 y$ $\boxed{x\leftrightarrow y}$
$\therefore y=4^x$
$\therefore f^{-1}(x)=4^x$

(c) For f^{-1}: $y=4^x$
For g: $-y=4^x$ $\boxed{y\to -y}$
$\therefore y=-4^x$
$\therefore g(x)=-4^x$

(d) $x>0$ or $x\in(0\,;\infty)$

(e) (1) $y>0$ or $y\in(0\,;\infty)$
(2) $y<0$ or $y\in(-\infty\,;0)$

(f) $A(1\,;0)$, $B(0\,;1)$ and $C(0\,;-1)$

(g) (1) $0<x<1$
(2) $x\leq 0$
(3) $x<\frac{3}{2}$

(h) The graph of f must be shifted 3 units right.
$\therefore t=-3$

EXERCISE 3

(a) Given the function $f(x)=4x-2$. Determine the equation of

(1) g, the reflection of f in the x-axis.

(2) h, the reflection of f in the y-axis.

(b) Given the function defined by $y=-\frac{2}{3}x^2$. Determine the equation of the reflection of this function in

(1) the line $y=0$.

(2) the line $y=x$.

(c) Given the functions $f(x)=7^x$ and $g(x)=\log_{\frac{1}{6}} x$.

(1) Determine the equation of the reflection of f in

(i) the x-axis.

(ii) the line $x=0$.

(iii) the line $y=x$.

(2) Determine the equation of the reflection of g in

(i) the y-axis.

(ii) the line $y=0$.

(iii) the line $y=x$.

(d) The graph of $f(x)=b^x$ passes through the point $A\left(\frac{1}{2}\,;2\right)$. g is the reflection of f in the line $y=x$ and h is the reflection of f in the line $x=0$.

(1) Determine the value of b.

(2) Determine the equation of g.

(3) Determine the equation of h in the form $h(x)=a^x$.

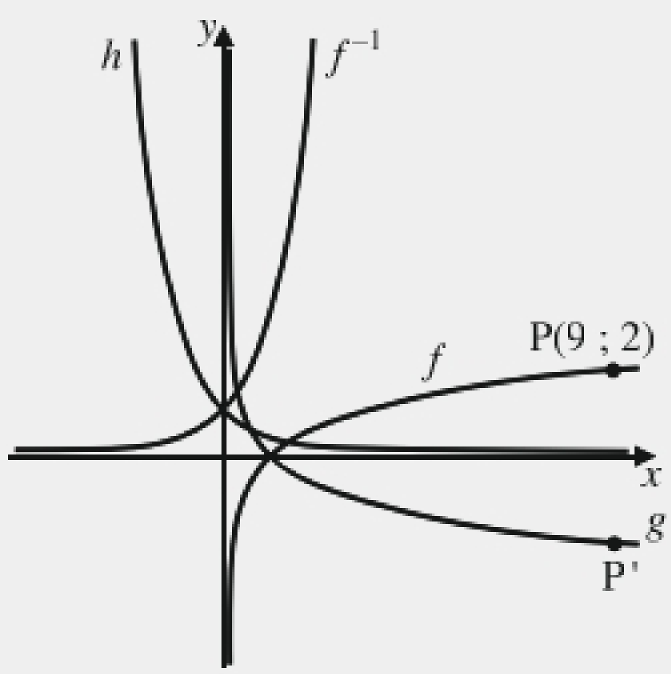
(4) Determine the coordinates of

(i) A', the image of A, on g.

(ii) A'', the image of A, on h.

(5) Sketch the graphs of f, g and h on the same set of axes, showing all relevant details.

(e) The graph of $f(x)=\log_a x$ passes through the point $P(9\,;2)$. g is the reflection of f in the x-axis and h is the reflection of f^{-1} in the y-axis.



(1) Determine the value of a.

(2) Determine the equation of f^{-1}.

(3) Determine the equation of h.

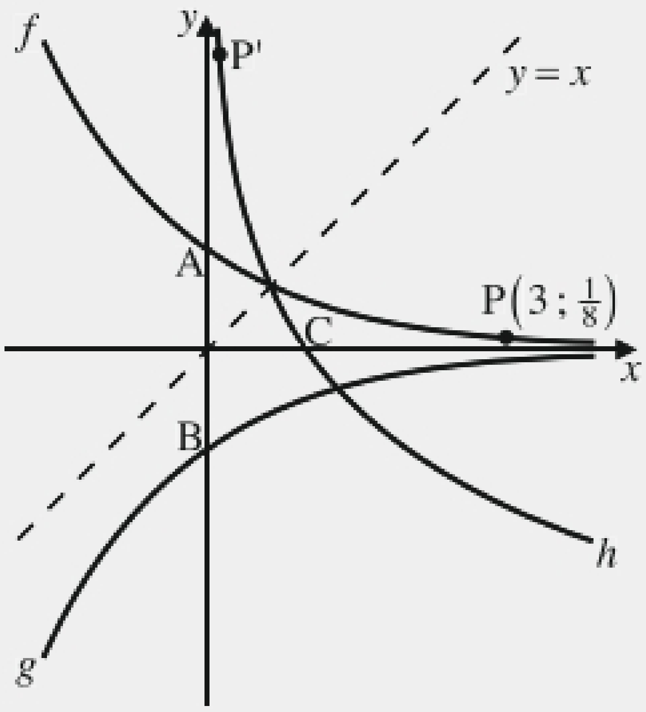
(4) Write down the coordinates of P', the image of P, on g.

(5) Determine the equation of g in the form $g(x)=\log_b x$.

(6) Show that $g(x)=h^{-1}(x)$.

(f) The sketch alongside shows the graph of $f(x) = a^x$, passing through the point $P\left(3\,;\,\frac{1}{8}\right)$.

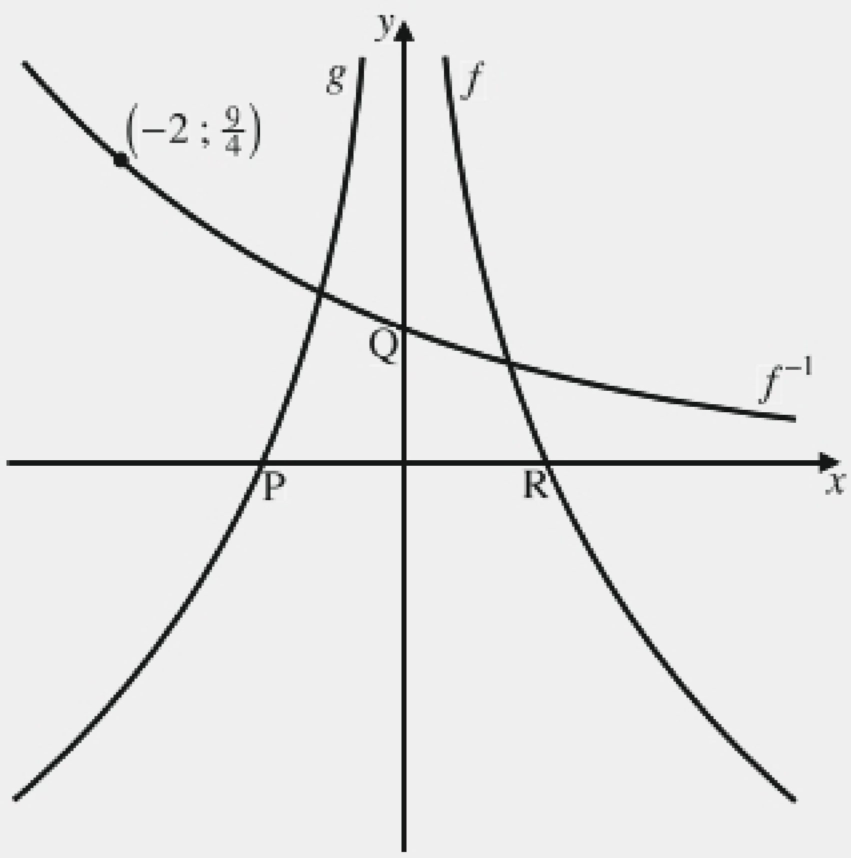
g is the reflection of f in the x-axis and h is the reflection of f in the line $y = x$.



(1) Determine the value of a.
(2) Determine the equation of g.
(3) Write down the range of
  (i) f
  (ii) g
(4) Determine the equation of h.
(5) Write down the domain of h.
(6) Write down the coordinates of A, B and C.
(7) Determine the coordinates of P', the image of P, on h.
(8) For which values of x is
  (i) $f(x) < \frac{1}{8}$?
  (ii) $-1 < g(x) < 0$?
  (iii) $\log_{\frac{1}{2}} x \geq 3$?

(g)* The sketch alongside shows the graphs of $f(x) = \log_b x$, f^{-1} and g - the reflection of f in the y-axis.

f^{-1} passes through the point $\left(-2\,;\,\frac{9}{4}\right)$.



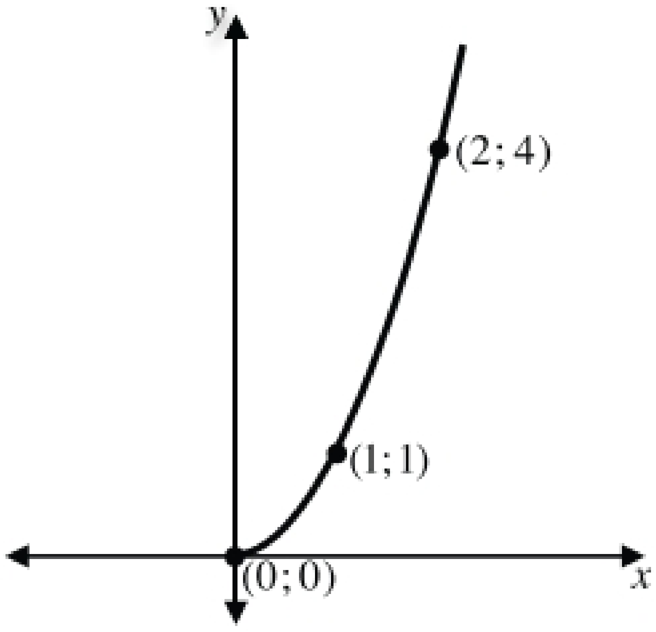
(1) Determine the value of b.
(2) Determine the equation of g.
(3) Write down the coordinates of P, Q and R.
(4) Write down the range of f^{-1}.
(5) Write down the domain of
  (i) f
  (ii) g
(6) For which values of x is
  (i) $f(x) < 0$?
  (ii) $f(x) \geq -2$?
  (iii) $f^{-1}(x) \cdot g(x) > 0$?
(7) If the graph of $y = f(x+k)$ passes through point P, determine the value of k.
(8) How many units must the graph of f^{-1} be shifted down to pass through the point P?

RESTRICTIONS ON FUNCTIONS

Certain functions are given with restrictions on their domains. Consider, for example, the following function:

$$f(x) = x^2 \ ; \ x \geq 0.$$

Since the restriction $x \geq 0$ is given, the graph of this function will not include any points with negative x-coordinates:



When applying a transformation (e.g. inverse or reflection) to a restricted function, **always** write down the **domain** as well as the **range** of the function. Any transformation must be applied to the **function formula** as well as to the **domain** and the **range**. This will be easier to do if you write both the domain and the range as **inequalities**:

EXAMPLE 12

Given the function $f(x) = x^2 \ ; x \geq 0$.

(a) Write down the range of f.

(b) Determine the equation of f^{-1}.

(c) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry.

(d) g is the reflection of f in the y-axis.

  (1) Determine the equation of g.

  (2) Determine the equation of g^{-1}.

  (3) Sketch the graph of g and g^{-1} together on a new set of axes and show the line of symmetry.

Solution

(a) x^2 can never be negative. $\therefore y \geq 0$

(b) For f: $y = x^2$ $x \geq 0$ $y \geq 0$

For f^{-1}: $x = y^2$ $y \geq 0$ $x \geq 0$ $\boxed{x \leftrightarrow y \text{ everywhere}}$

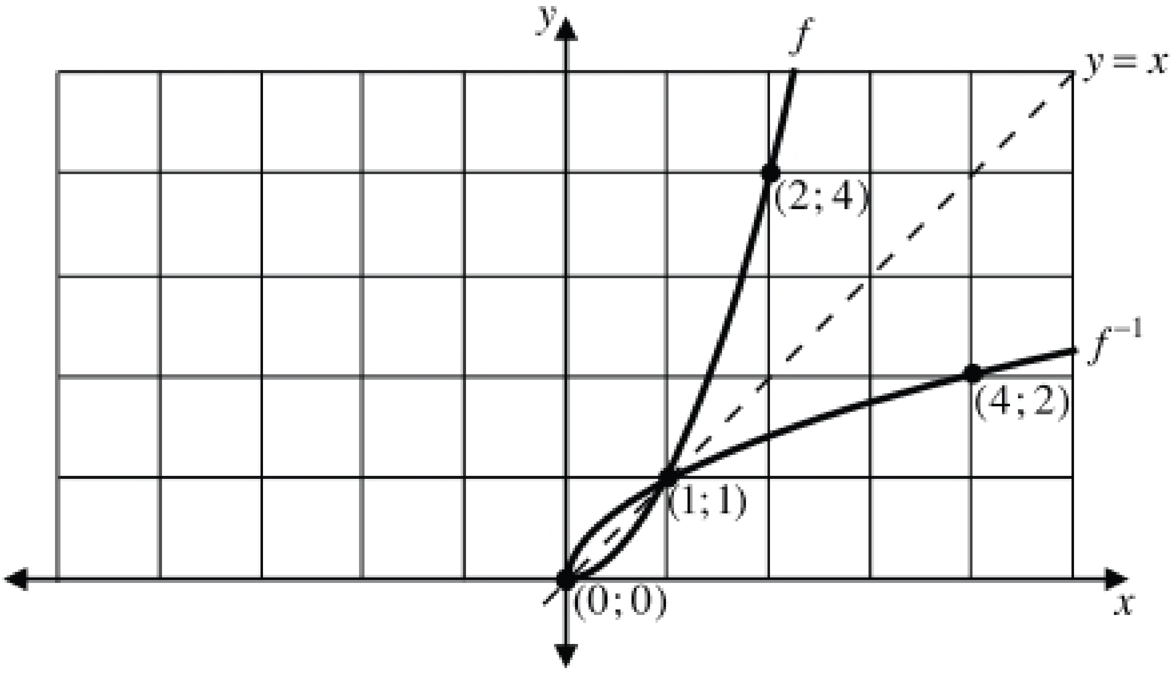
$\therefore y = \pm\sqrt{x}$

But since $y \geq 0$, $y = +\sqrt{x}$

$\therefore f^{-1}(x) = \sqrt{x}$

(**Note :** It is not necessary to write $x \geq 0$ here, as this is implied by the $\sqrt{\ }$ sign.)

(c)



(d) (1) For f: $y = x^2$ $x \geq 0$ $y \geq 0$

For g: $y = (-x)^2$ $-x \geq 0$ $y \geq 0$ $\boxed{x \to -x \text{ everywhere}}$

$\therefore y = x^2$ $\therefore x \leq 0$

$\therefore g(x) = x^2 \;\; ; \; x \leq 0$

(**Note:** It is important to write $x \leq 0$ here to show that the domain of g is restricted.)

(2) For g: $y = x^2$ $x \leq 0$ $y \geq 0$

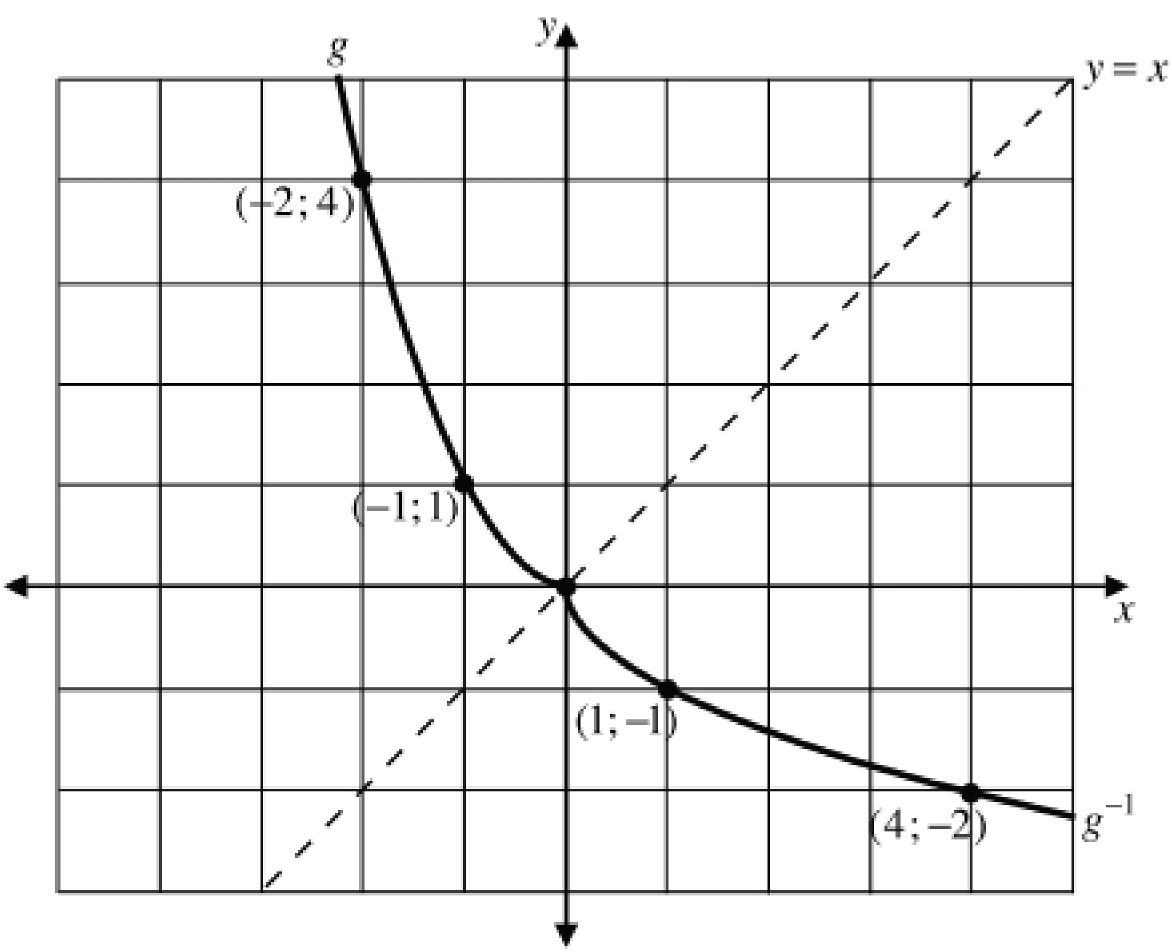
For g^{-1}: $x = y^2$ $y \leq 0$ $x \geq 0$ $\boxed{x \leftrightarrow y \text{ everywhere}}$

$\therefore y = \pm\sqrt{x}$

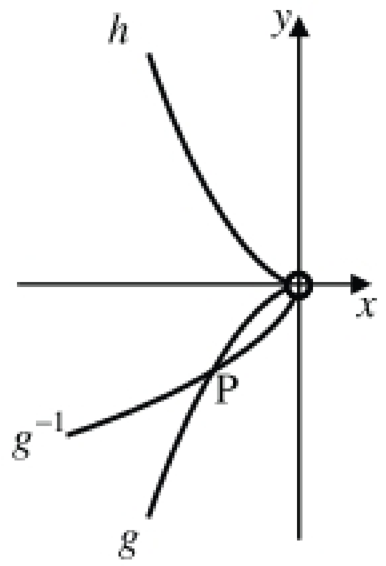
But since $y \leq 0$, $y = -\sqrt{x}$

$\therefore g^{-1}(x) = -\sqrt{x}$

(3)



EXAMPLE 13



The sketch alongside shows the graphs of g, g^{-1} and h.

$g(x) = -\frac{1}{2}x^2$; $x < 0$. h is the reflection of g in the x-axis.

(a) Write down the range of g.

(b) Determine the equation of h.

(c) Determine the equation of g^{-1}.

(d) Calculate the coordinates of P.

(e) For which values of x is $g(x) \geq g^{-1}(x)$?

Solution

(a) $y < 0$ (Since $x = 0$ is excluded, $y = 0$ is also excluded.)

(b) For g: $y = -\frac{1}{2}x^2$ $\quad x < 0 \quad$ $y < 0$

For h: $-y = -\frac{1}{2}x^2$ $\quad x < 0 \quad$ $-y < 0$ $\quad$ $\boxed{y \to -y \text{ everywhere}}$

$\therefore y = \frac{1}{2}x^2$ $\quad$ $\therefore y > 0$

$\therefore h(x) = \frac{1}{2}x^2$; $x < 0$

(c) For g: $y = -\frac{1}{2}x^2$ $\quad x < 0 \quad$ $y < 0$

For g^{-1}: $x = -\frac{1}{2}y^2$ $\quad y < 0 \quad$ $x < 0$ $\quad$ $\boxed{x \leftrightarrow y \text{ everywhere}}$

$\therefore y^2 = -2x$

$\therefore y = \pm\sqrt{-2x}$

But since $y < 0$, $y = -\sqrt{-2x}$

$\therefore g^{-1}(x) = -\sqrt{-2x}$; $x \neq 0$

(**Note:** It is important to write $x \neq 0$ here to show that 0 is excluded from the domain.)

(d) Solve g and $y = x$ simultaneously:

$-\frac{1}{2}x^2 = x$

$\therefore x^2 = -2x$

$\therefore x^2 + 2x = 0$

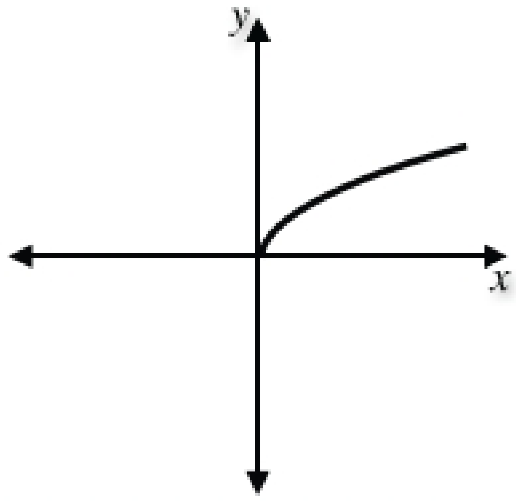
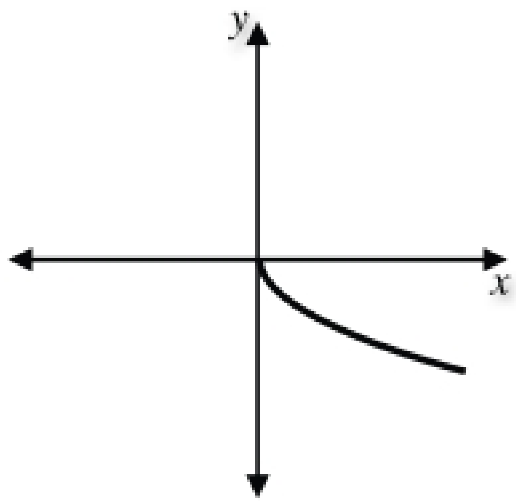
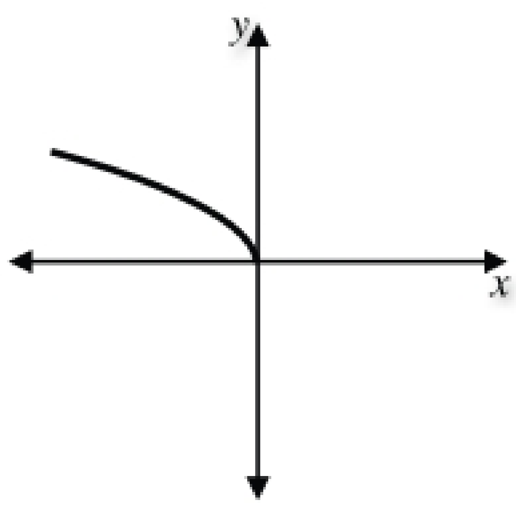
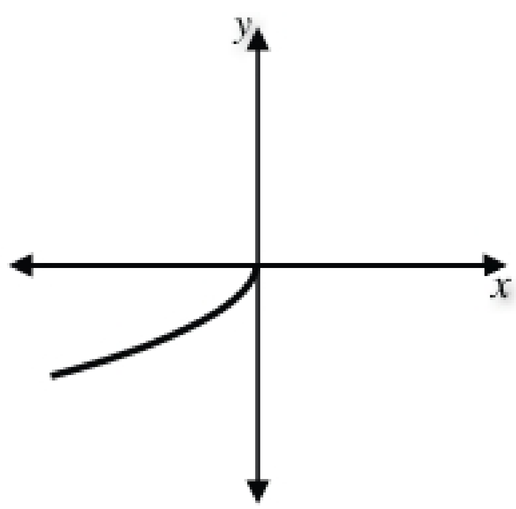
$\therefore x(x+2) = 0$

$\therefore x = 0$ or $x = -2$ $\rightarrow$ P$(-2\,;-2)$

(e) $-2 \leq x < 0$

SQUARE ROOT FUNCTIONS

The square root of a negative number is non-real. Furthermore, the expression $\sqrt{x}$ can never be equal to a negative number (e.g. $\sqrt{9} = +3$). This implies that any function containing a square root has restrictions on its domain and range, even if no restrictions are specifically stated:

Function	Domain	Range	Graph
$y = \sqrt{x}$	$x \geq 0$	$y \geq 0$	
$y = -\sqrt{x}$	$x \geq 0$	$y \leq 0$	
$y = \sqrt{-x}$	$x \leq 0$	$y \geq 0$	
$y = -\sqrt{-x}$	$x \leq 0$	$y \leq 0$	

EXAMPLE 14

Given the function $f(x)=\sqrt{-x}$.

(a) Write down the domain and range of f.

(b) Determine the equation of f^{-1}.

(c) Sketch the graphs of f and f^{-1} on the same set of axes and show the line of symmetry.

Solution

(a) Domain: $x \le 0$ Range: $y \ge 0$

(b) For f: $y=\sqrt{-x}$ $x \le 0$ $y \ge 0$

For f^{-1}: $x=\sqrt{-y}$ $y \le 0$ $x \ge 0$

$x \leftrightarrow y$ everywhere

$\therefore x^2=-y$

$\therefore y=-x^2$

$\therefore f^{-1}(x)=-x^2 \;;\; x \ge 0$

(c) Three points on $f^{-1}(x)=-x^2 \;;\; x \ge 0$:

$(0;0)$

$(1;-1)$

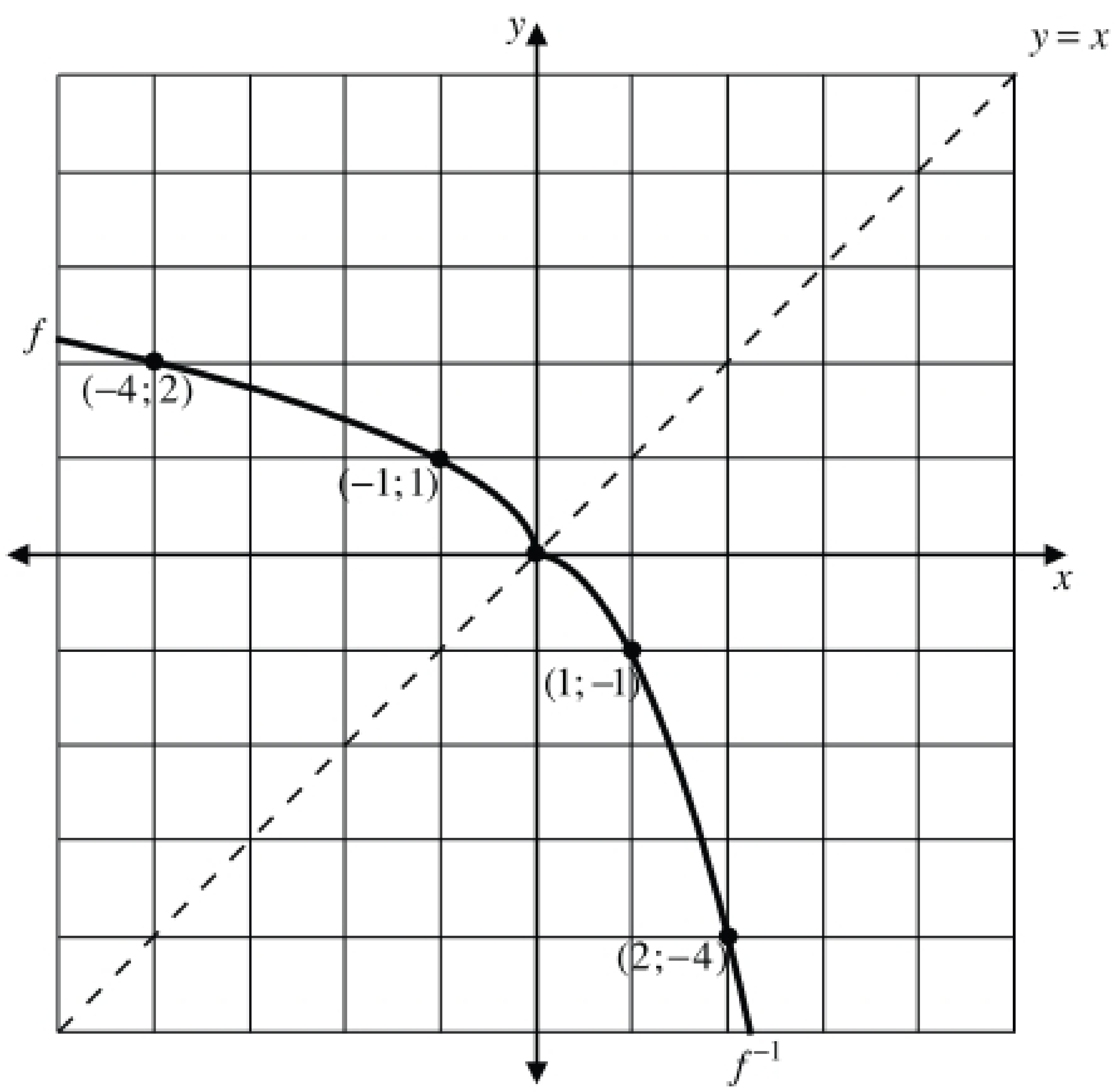
$(2;-4)$

Invert coordinates for $f(x)=\sqrt{-x}$:

$(0;0)$

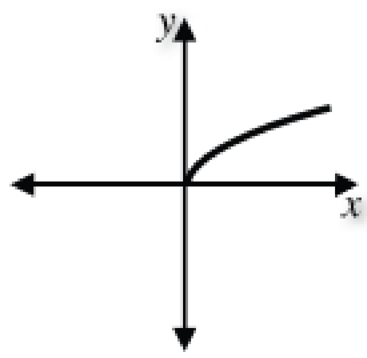
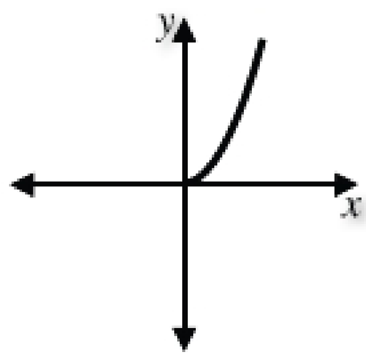
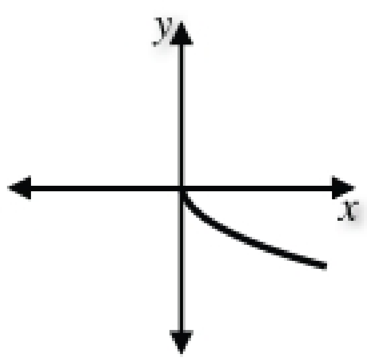
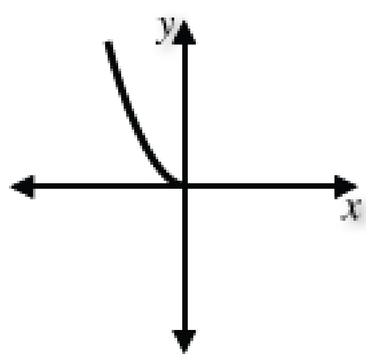
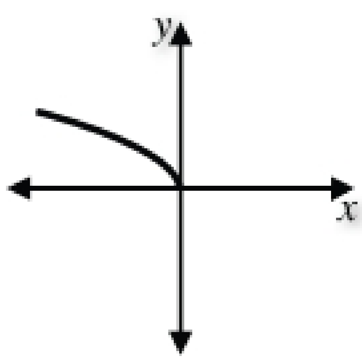
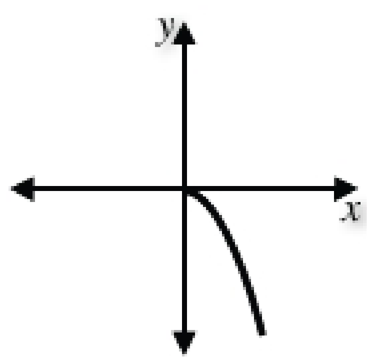
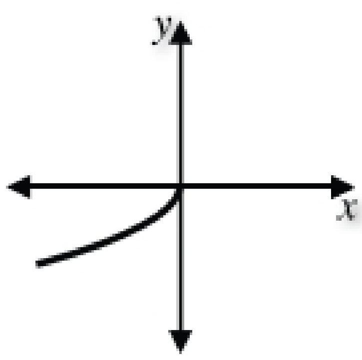
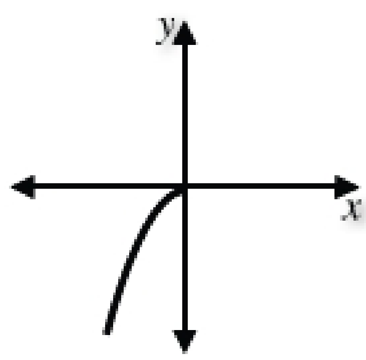
$(-1;1)$

$(-4;2)$



SUMMARY

SQUARE ROOT FUNCTIONS AND THEIR INVERSES

Function		Inverse	
$\boxed{y=\sqrt{x}}$ 	$x \geq 0$ $y \geq 0$	$\boxed{y=x^2 \; ; \; x \geq 0}$ 	$y \geq 0$ $x \geq 0$
$\boxed{y=-\sqrt{x}}$ 	$x \geq 0$ $y \leq 0$	$\boxed{y=x^2 \; ; \; x \leq 0}$ 	$y \geq 0$ $x \leq 0$
$\boxed{y=\sqrt{-x}}$ 	$x \leq 0$ $y \geq 0$	$\boxed{y=-x^2 \; ; \; x \geq 0}$ 	$y \leq 0$ $x \geq 0$
$\boxed{y=-\sqrt{-x}}$ 	$x \leq 0$ $y \leq 0$	$\boxed{y=-x^2 \; ; \; x \leq 0}$ 	$y \leq 0$ $x \leq 0$

EXERCISE 4

(a) Given $f(x) = -x^2$; $x \geq 0$. Let g be the reflection of f in the x-axis.

(1) Write down the range of f.

(2) Determine the equation of f^{-1}.

(3) Determine the equation of g.

(4) Sketch the graphs of f, f^{-1} and g on the same set of axes.

(b) Given $g(x) = \frac{1}{3}x^2$; $x < 0$. Let h be the reflection of g in the y-axis.

(1) Determine the equation of g^{-1}.

(2) Sketch the graphs of g and g^{-1} on the same set of axes.

(3) Determine the equation of h.

(4) Sketch the graphs of h and h^{-1} together on a new set of axes.

(c) Given $f(x) = -\sqrt{x}$.

(1) Write down the domain and range of f.

(2) Write down the equation of f^{-1}.

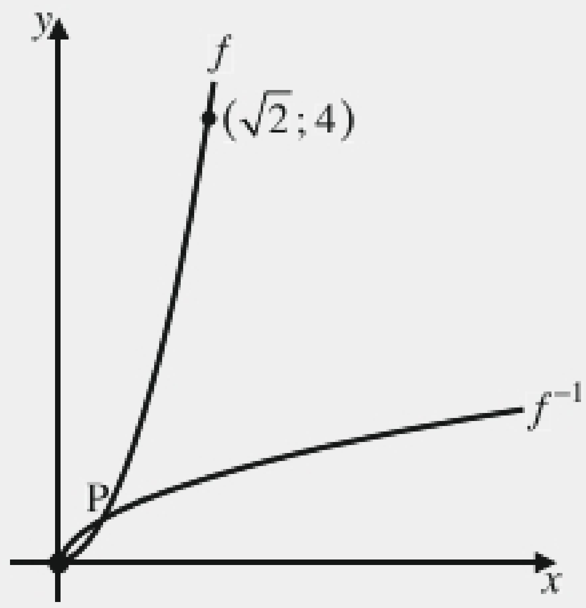
(3) Sketch the graphs of f and f^{-1} on the same set of axes.

(4) Determine the equation of

(i) g, the reflection of f in the x-axis.

(ii) h, the reflection of f in the y-axis.

(d) The sketch alongside shows the graph of $f(x) = ax^2$, with a restriction on its domain, and the graph of f^{-1}. f passes through the point $(\sqrt{2}; 4)$. P is the point of intersection of f and f^{-1}.



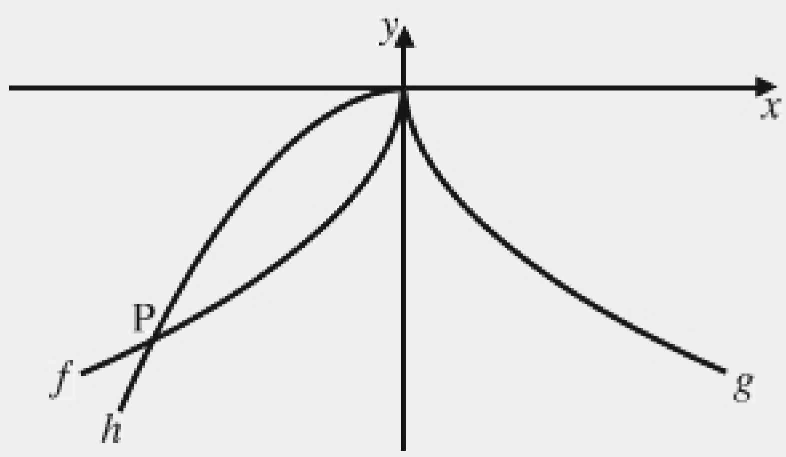
(1) Write down the domain of f.

(2) Determine the value of a.

(3) Calculate the coordinates of P.

(4) For which values of x is $f(x) > f^{-1}(x)$?

(e) The sketch alongside shows the graphs of $f(x) = -\sqrt{-2x}$, g and h. g is the reflection of f in the y-axis and h is the reflection of f in the line $y = x$.



(1) Determine the equation of g.

(2) Determine the equation of h.

(3) Explain why the x-coordinate of P can be calculated by solving the equation $-\frac{1}{2}x^2 = x$.

(4) Calculate the coordinates of P.

(5) For which values of x is $f(x) \geq h(x)$?

THE THEORY OF FUNCTIONS

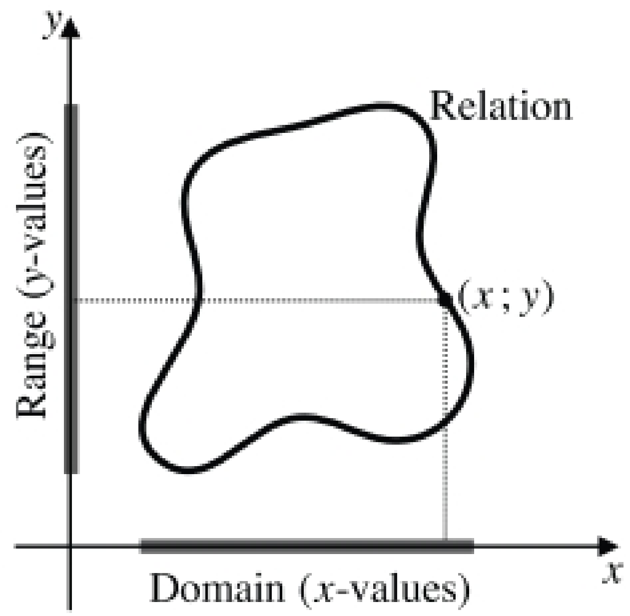
So far, we have thought of a function as a rule (or formula/equation) which can be represented by a graph. The proper definition of a function is more specific. In this section we will introduce the concept of a *relation*, define the concept of a function and distinguish between different types of functions.

RELATIONS

Any graph can be seen as a relationship between two sets of values:

- a set of ***x*-values** (the **domain**)
- a set of ***y*-values** (the **range**)

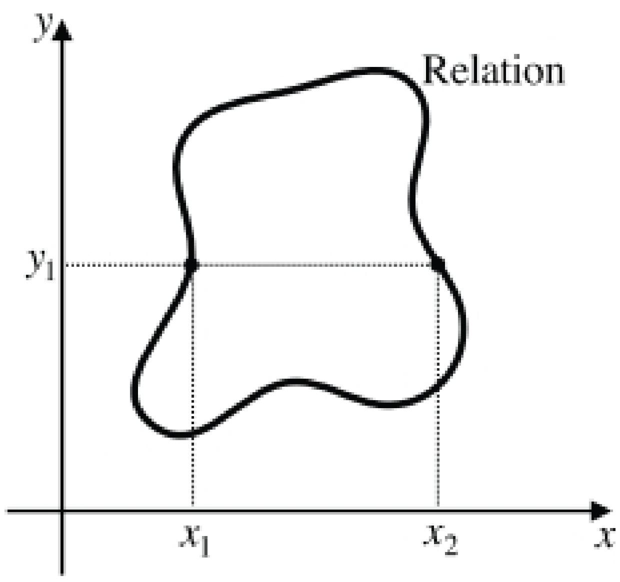
Values in the domain (x-values) are paired with values in the range (y-values). The result is a set of **ordered pairs** of the form $(x\ ;\ y)$. This set of ordered pairs is called a *relation.*



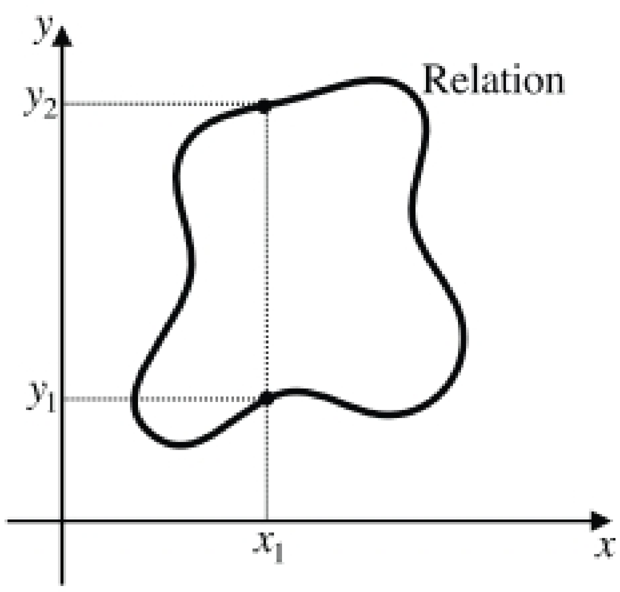
A *relation* is a set of ordered pairs. (A **graph** of **any shape**).

Note:

- It is possible that **more than one *x*-value** can be associated with **one *y*-value**.



- It is also possible that **one *x*-value** can be associated with **more than one *y*-value**.

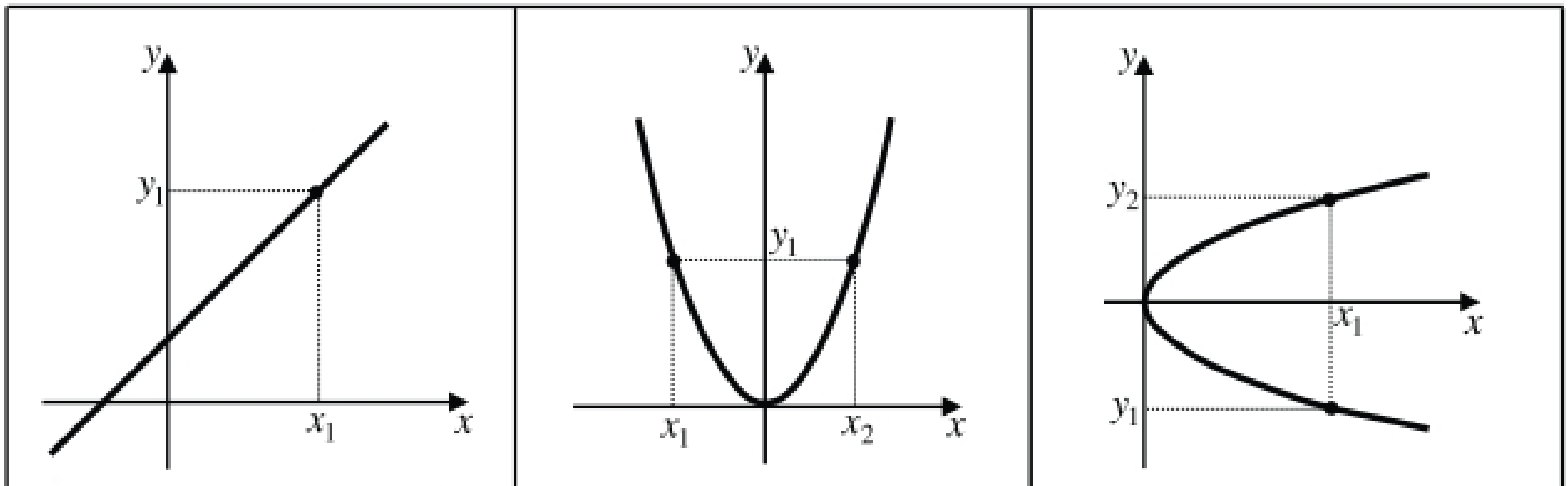


FUNCTIONS

A *function* is a special type of relation. In a function, x-values are thought of as *inputs* and y-values as *outputs*. For each input, the output must be **unique,** meaning that no input can produce more than one output. For this reason, a function is defined as follows:

> A *function* is a relation for which **every x-value** in the domain is associated with **only one y-value** in the range.

Although no x-value may be associated with more than one y-value, different x-values may be associated with the same y-value.

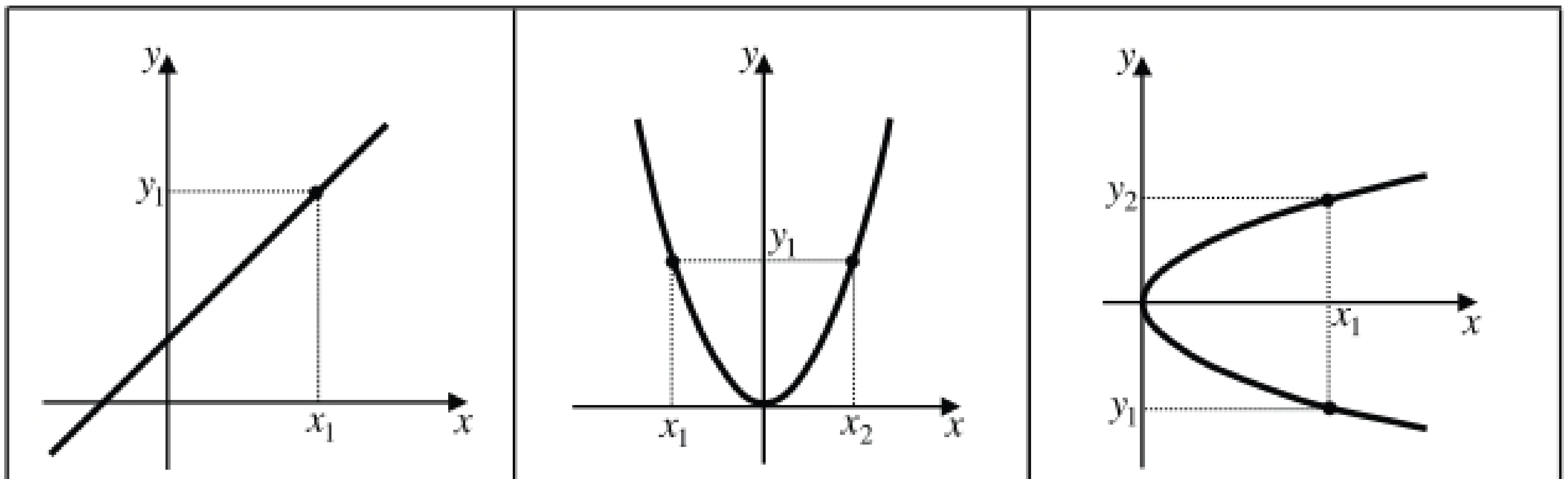


One x → One y	More than one x → One y	One x → More than one y
✔ This is a function.	✔ This is a function.	✘ This is **not** a function.

ONE-TO-ONE FUNCTIONS

> In a *one-to-one* function **every x-value** in the domain is associated with **only one y-value** in the range AND **every y-value** in the range is associated with **only one x-value** in the domain.

A function that is **not** one-to-one is said to be *many-to-one.*

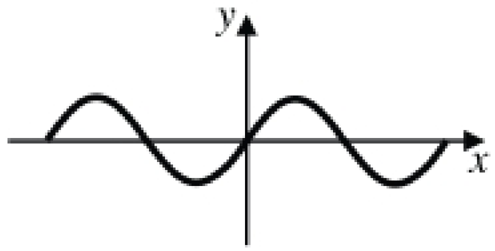


One x → One y	More than one x → One y	One x → More than one y
This function in **one-to-one**.	This function is **many-to-one**.	This is **not** a function.

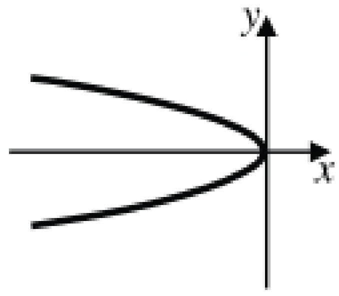
EXAMPLE 15

State whether the relation represented by each of the following graphs is a **one-to-one function**, a **many-to-one function** or **not a function**:

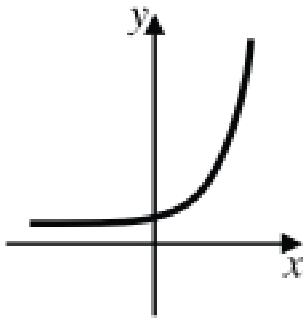
(a)



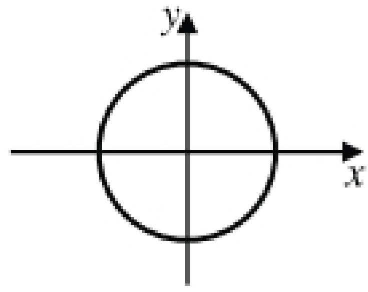
(b)



(c)



(d)



Solution

(a) Many-to-one function.

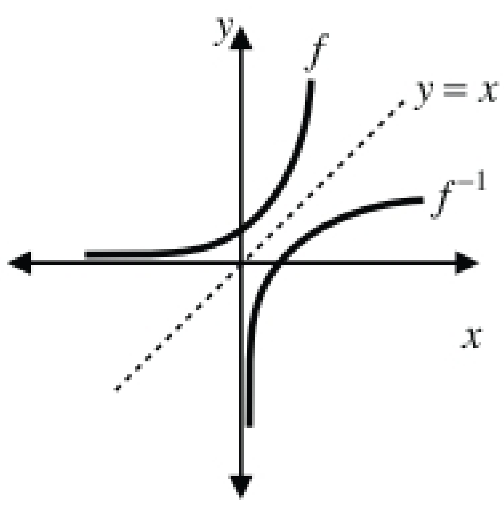
(b) Not a function.

(c) One-to-one function.

(d) Not a function.

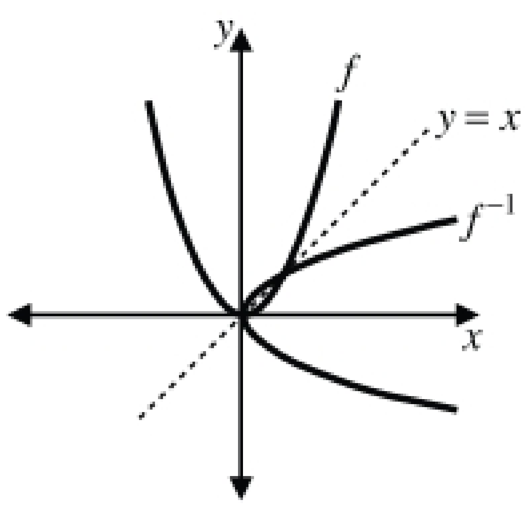
INVERSES OF ONE-TO-ONE AND MANY-TO-ONE FUNCTIONS

The **inverse** of a **one-to-one** function is a **function**:



$\boxed{f \text{ is a one-to-one function } \rightarrow f^{-1} \text{ is a function}}$

The **inverse** of a **many-to-one** function is **not a function**:



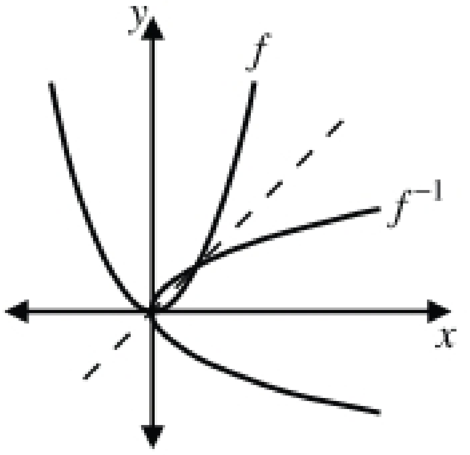
$\boxed{f \text{ is a many-to-one function } \rightarrow f^{-1} \text{ is } \textbf{not} \text{ a function}}$

RESTRICTING A FUNCTION SO THAT ITS INVERSE IS ALSO A FUNCTION

Sometimes mathematicians wish to **restrict the domain** of a many-to-one function so that the result is a one-to-one function. This is done to ensure that the **inverse will be a function**. It is usually done in such a way that the **range** of the original function **is preserved**:

EXAMPLE 16

The sketch alongside shows the graph $f(x) = x^2$ and its inverse.



(a) Write down the domain and range of f.

(b) Is f one-to-one or many-to-one?

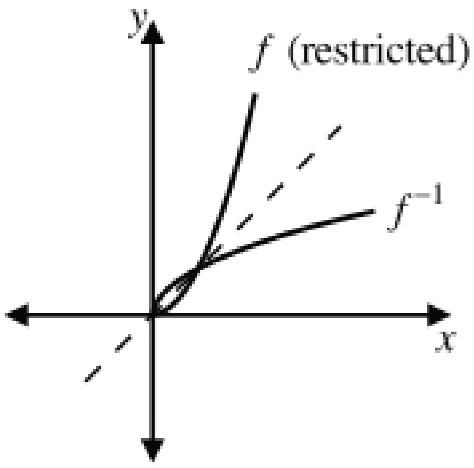
(c) Is f^{-1} a function?

(d) f is restricted with $x \geq 0$. The sketch alongside shows the graph of the restricted function f and its inverse.

(1) What is the range of the restricted function f?

(2) Is the restricted function f one-to-one or many-to-one?

(3) Is the inverse of the restricted function f a function?



(e) Write down another possible restriction on the domain of f, such that the inverse will be a function and draw a rough sketch of the graph of this restricted function f and its inverse on the same set of axes. (Your restriction may not cause the range of f to change.)

Solution

(a) Domain: $x \in \mathbb{R}$ Range: $y \geq 0$

(b) Many-to-one

(c) No

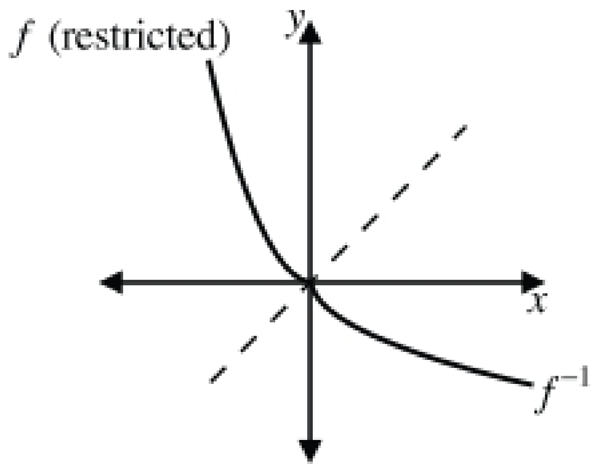
Notice that the range remained unchanged by the restriction.

(d) (1) $y \geq 0$

(2) One-to-one

(3) Yes

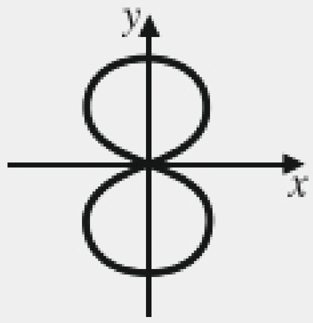
(e) $x \leq 0$



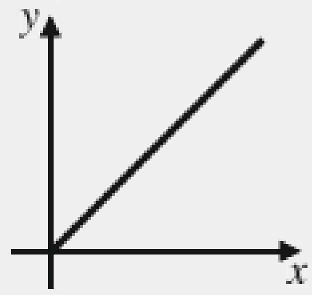
EXERCISE 5

(a) State whether the relation represented by each of the following graphs is a **one-to-one function**, a **many-to-one function** or **not a function**:

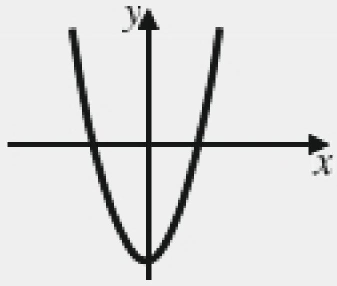
(1)



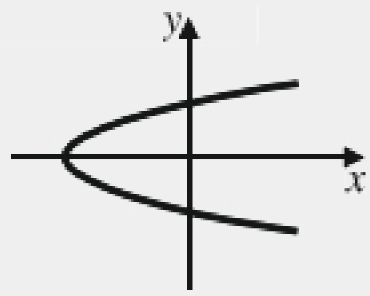
(2)



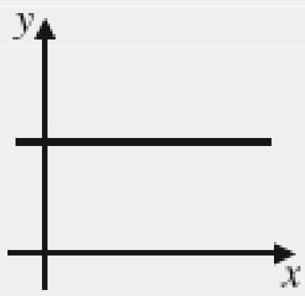
(3)



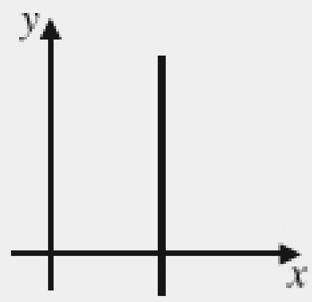
(4)



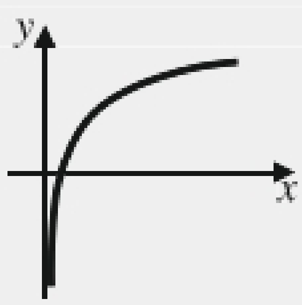
(5)



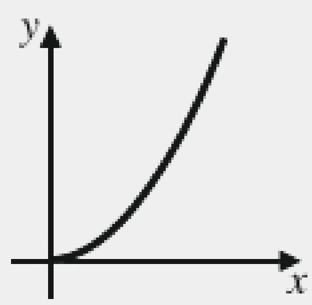
(6)



(7)



(8)



(b) State whether each of the following functions are **one-to-one** or **many-to-one**:

(1)	$y=-3x+1$	(2)	$y=3x^2$	(3)	$y=x^3$
(4)	$y=2^x$	(5)	$y=x^2\ ;\ x\leq 0$	(6)	$y=\cos 2x$
(7)	$y=\sqrt{x}$	(8)	$y=\log_3 x$	(9)	$y=\dfrac{4}{x}$
(10)*	$y=-x^2\ ;\ x\geq -2$	(11)*	$y=-2x^2\ ;\ x>1$	(12)	$y=\tan x$

(c) Given $f(x)=-2x^2$.

(1) Draw a rough graph of f and f^{-1} on the same set of axes.

(2) Is f one-to-one?

(3) Write down two possible restrictions on the domain of f, such that the inverse will be a function. (Your restrictions may not cause the range of f to change.)

(4) Draw rough sketches of the graphs of restricted function f and its inverse for each one of the two possible restrictions that you have listed in (3).

MORE ABOUT LOGARITHMS

The value of $\log_b x$ is the unique number c, such that $b^c = x$. (E.g. $\log_2 8 = 3$ because $2^3 = 8$.)

Some important facts about logarithms:

- In the expression $\log_b x$, b is called the *base* and x is called the *argument*.
- The expression $\log_b x$ is defined only if $x > 0$, $b > 0$ and $b \neq 1$.
- When the base of the logarithm is 10, this base (10) is traditionally omitted: $\log_{10} x = \log x$.

SOLVING EQUATIONS

Logarithms can be used to solve any exponential equation of the form $b^x = c$:

$$\boxed{\text{If } b^x = c, \text{ then } x = \log_b c}$$

EXAMPLE 17

Solve for x:
(Round to two decimals.)

(a) $3^x = 10$ (b) $5 \cdot 4^{3x-1} = 60$
(c) $2^{2x} - 7 \cdot 2^x + 12 = 0$

Solution

(a)
$$3^x = 10$$
$$\therefore x = \log_3 10$$
$$\therefore x = 2{,}10$$

(b)
$$5 \cdot 4^{3x-1} = 60$$
$$\therefore 4^{3x-1} = 12$$
$$\therefore 3x - 1 = \log_4 12$$
$$\therefore 3x = \log_4 12 + 1$$
$$\therefore x = \frac{\log_4 12 + 1}{3}$$
$$\therefore x = 0{,}93$$

(c)
$$2^{2x} - 7 \cdot 2^x + 12 = 0$$
$$\therefore (2^x - 4)(2^x - 3) = 0$$
$$\therefore 2^x = 4 \text{ or } 2^x = 3$$
$$\therefore x = 2 \text{ or } x = \log_2 3$$
$$\therefore x = 2 \text{ or } x = 1{,}58$$

Logarithmic equations are solved by converting to the exponential form:

$$\boxed{\text{If } \log_b x = p, \text{ then } x = b^p}$$

EXAMPLE 18

Solve for x:
(a) $\log_3 x = 2$ (b) $\log_x 20 = 3$ (Round to two decimals)
(c) $4 \log x = 12$

Solution

(a) $\log_3 x = 2$

$\therefore x = 3^2$

$\therefore x = 9$

(b) $\log_x 20 = 3$

$\therefore x^3 = 20$

$\therefore x = \sqrt[3]{20}$

$\therefore x = 2{,}71$

(c) $4\log x = 12$

$\therefore \log x = 3$

$\therefore x = 10^3$

$\therefore x = 1\,000$

APPLICATIONS

Logarithms are also used to solve problems about exponential growth:

EXAMPLE 19

The population of a city is given by the formula $P = ab^x$ where x is the number of years that have passed since 1 January 2000. On 1 January 2000, the population of the city was 450 000.
On 1 January 2002, the population of the city was 477 405.

(a) Determine the value of
(1) a (2) b

It is now given that $a = 450\ 000$ and $b = 1{,}03$.

(b) What was the population of the city (to the nearest whole number) on
(1) 1 January 2010? (2) 1 January 2020?

(c) In which year will the population of the city reach 1 000 000?

(d) The *population doubling time* of a city is the time it takes for the population of the city to double. What is the population doubling time of this city? (Round to two decimals.)

Solution

(a) (1) When $x = 0$, $P = 450\ 000$:

$450\ 000 = ab^0$

$\therefore a = 450\ 000$

(2) When $x = 2$, $P = 477\ 405$

$477\ 405 = 450\ 000b^2$

$\therefore b^2 = \dfrac{477\ 405}{450\ 000}$

$\therefore b = \sqrt{\dfrac{477\ 405}{450\ 000}}$

$\therefore b = 1{,}03$

(b) (1) $P = 450\ 000(1{,}03)^x$

$\therefore P = 450\ 000(1{,}03)^{10}$

$\therefore P = 604762$

(2) $P = 450\ 000(1{,}03)^x$

$\therefore P = 450\ 000(1{,}03)^{20}$

$\therefore P = 812\ 750$

(c) $P = 450\ 000(1{,}03)^x$

$\therefore 1\ 000\ 000 = 450\ 000(1{,}03)^x$

$\therefore \dfrac{1\ 000\ 000}{450\ 000} = 1{,}03^x$

$\therefore 1{,}03^x = \dfrac{20}{9}$

$\therefore x = \log_{1,03}\left(\dfrac{20}{9}\right)$

$\therefore x = 27$

(d) $P = 450\ 000(1{,}03)^x$

$\therefore 900\ 000 = 450\ 000(1{,}03)^x$

$\therefore 2 = 1{,}03^x$

$\therefore x = \log_{1,03} 2$

$\therefore x = 23{,}45$

The doubling time is 23,45 years.

THE LAWS OF LOGARITHMS **(Not for examination purposes.)**

The following laws of logarithms can be derived from the exponent laws and the definition of a logarithm:

- $\log_b x + \log_b y = \log_b(xy)$ (E.g. $\log_6 9 + \log_6 4 = \log_6 36 = 2$)
- $\log_b x - \log_b y = \log_b\left(\dfrac{x}{y}\right)$ (E.g. $\log 40 - \log 4 = \log 10 = 1$)
- $\log_b x^n = n\log_b x$ (E.g. $\log_5 8 = \log_5 2^3 = 3\log_5 2$)
- $\log_c x = \dfrac{\log_b x}{\log_b c}$ (E.g. $\log_4 8 = \dfrac{\log_2 8}{\log_2 4} = \dfrac{3}{2}$)

The following rules are also very useful:

- $\log_b 1 = 0$
- $\log_a a = 1$
- $\log_{\frac{1}{a}} a = -1$
- $\log_c b = \dfrac{1}{\log_b c}$

EXERCISE 6

(a) Solve for x: (Round to two decimals.)

(1) $2^x = 13$

(2) $2 \cdot \left(\dfrac{1}{3}\right)^x = 24$

(3) $5^{2x} = 0{,}1$

(4) $7^{2x-3} = 60$

(5) $2^{x+3} - 2^x = 42$

(6) $3^{2x} - 11 \cdot 3^x = -18$

(b) Solve for x, without the use of a calculator:

(1) $\log_2 x = 5$ (2) $\log(x+40) = 2$

(3) $\log_3 x = -1$ (4) $\log_x 8 = 3$

(5) $\log_x 9 = -2$ (6) $3\log_x 4 = 2$

(c) Solve for x: (Round to two decimals.)

(1) $\log_x 25 = 4$ (2) $2\log_7 3x = 1$

(d) For which values of x are the following expressions undefined?

(1) $\log_2(x-3)$ (2) $\log_{x+2} 5$

(e) The number of bacteria cells (N) in a culture, t hours after the start of a scientific experiment, is given by the formula $N = a \cdot 3^{kt}$. The number of bacteria in the culture triples every 4 hours. Exactly 12 hours after the experiment started, there are 81 000 bacteria cells in the culture.

(1) Determine the value of k.

(2) If $k = \frac{1}{4}$, determine the value of a.

It is now given that $a = 3000$ and $k = \frac{1}{4}$.

(3) How many bacteria cells were present in the culture at the beginning of the experiment?

(4) How many bacteria cells will be present in the culture 24 hours after the start of the experiment?

(5) How long after the start of the experiment will there be 10 000 000 bacteria cells in the culture?

(f) The pH of a solution is given by the formula $\text{pH} = -\log C$, where C is the concentration of H^+ ions in the solution (in moles per litre).

(1) In a certain soft drink, the concentration of H^+ ions is 0,000 004 moles per litre. What is the pH of the soft drink? (Round to one decimal.)

(2) The pH of white vinegar is 2,4. What is the concentration of H^+ ions in white vinegar? (Round to three decimals.)

CONSOLIDATION AND EXTENSION EXERCISE

(a) Given $f(x) = -2x + 4$. g is the reflection of f in the x-axis.

(1) Determine the equation of f^{-1}.

(2) Determine the equation of g.

(3) Sketch the graphs of f, f^{-1} and g on the same set of axes.

(4) Calculate the coordinates of the point of intersection between f and f^{-1}.

(b) Given the function $h(x) = \left(\frac{1}{4}\right)^x$. $f(x) = a^x$ is the reflection of h in the y-axis.

(1) Determine the equation of h^{-1}.

(2) Determine the value of a.

(3) Sketch the graphs of f, h and h^{-1} on the same set of axes.

(c) Given the function $g(x) = -2x^2$; $x \geq 0$. h is the reflection of g in the x-axis.

(1) Determine the equation of h.

(2) Determine the equation of g^{-1}.

(3) Sketch the graphs of g, h and g^{-1} on the same set of axes.

(4) f is the reflection of g in the y-axis

(i) Determine the equation of f.

(ii) Determine the equation of f^{-1}.

(iii) Sketch the graph of f and f^{-1} on the same set of axes.

(d) Given the function $f(x) = -\sqrt{3x}$.

(1) Write down the domain and range of f.

(2) Determine the equation of f^{-1}.

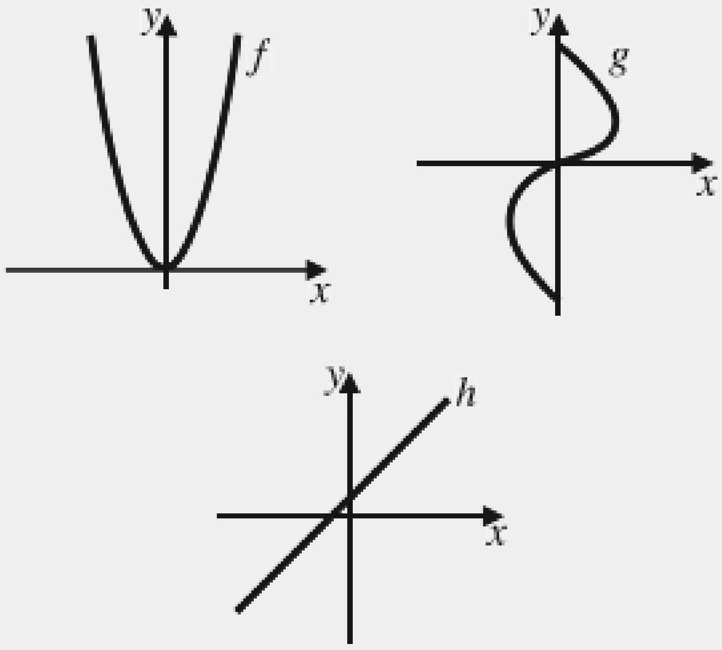
(3) Sketch the graphs of f and f^{-1} on the same set of axes.

(4) Write down the equation of

(i) g, the reflection of f in the line $x = 0$.

(ii) h, the reflection of f in the line $y = 0$.

(e) The sketch alongside shows the graphs of three relations: f, g and h.



(1) Which one of these three relations is

(i) a one-to-one function?

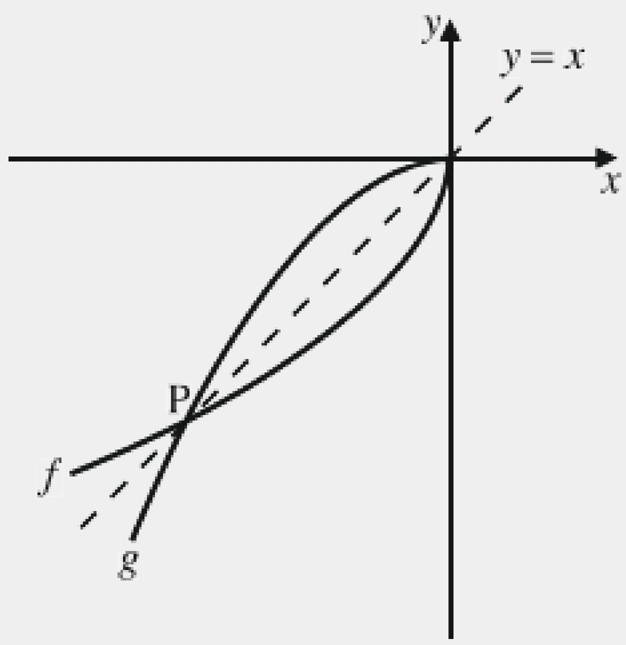
(ii) not a function?

(2) Write down the range of f.

(3) Is the inverse of g a function?

(4) State two possible ways of restricting f so that its inverse will be a function. (Your restriction may not cause the range of f to change.)

(f) The sketch alongside shows the graph of $f(x) = -2\sqrt{-x}$ and g, the reflection of f in the line $y = x$.



(1) Write down the domain and range of f.

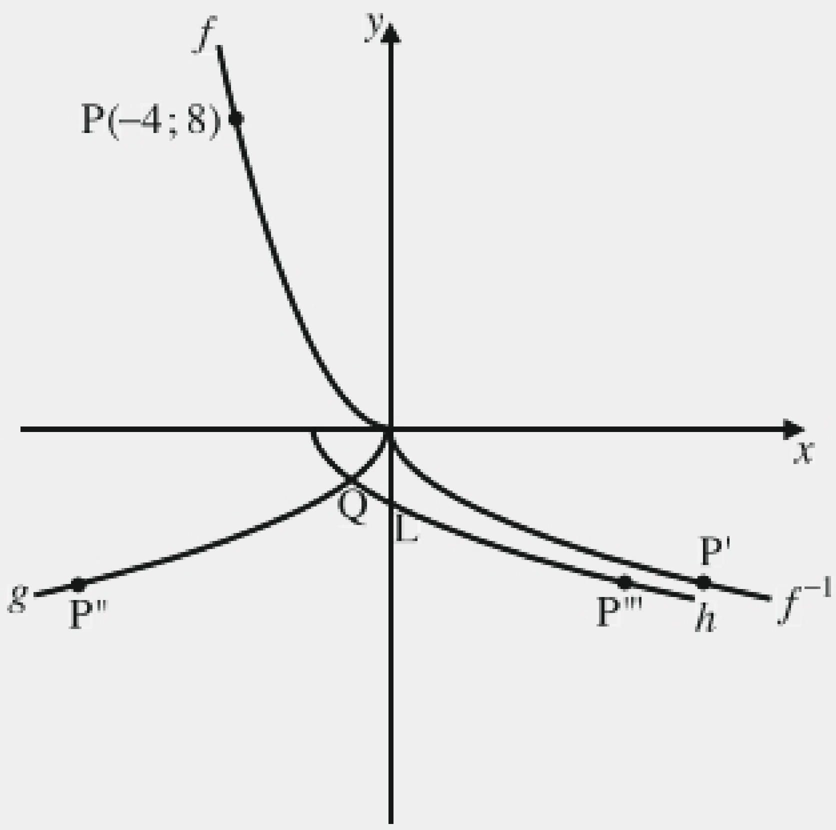
(2) Write down the equation of g.

(3) Calculate the coordinates of P.

(4) For which values of x is $f(x) \leq g(x)$?

(5) Determine the equation of h, the reflection of f in the x-axis.

(g) The sketch alongside shows the graphs of f, f^{-1}, g and h. $f(x) = ax^2$; $x \leq 0$. The graph of f passes through the point P(−4 ; 8).



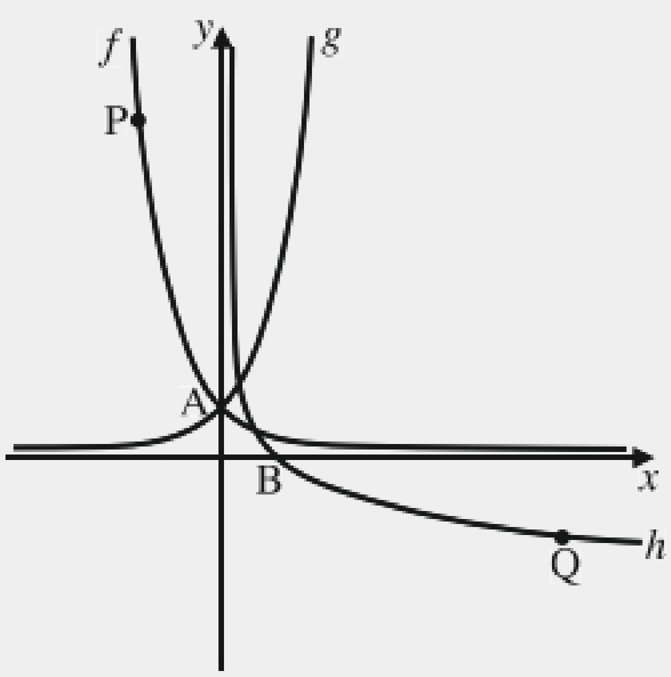
g is the reflection of f^{-1} in the y-axis and h is a translation of f^{-1}, 2 units to the left.

(1) Determine the value of a.

It is now given that $f(x) = \frac{1}{2}x^2$; $x \leq 0$.

(2) Determine the equation of
  (i) f^{-1}
  (ii) g
  (iii)* h
(3) Write down the coordinates of
  (i) P', the image of P, on f^{-1}.
  (ii) P'', the image of P, on g.
  (iii) P''', the image of P, on h.
(4)* Calculate the coordinates of
  (i) L
  (ii) Q

(h) The sketch alongside shows the graph of $f(x) = a^x$. The graph of f passes through point P.



$g(x) = b^x$ is the reflection of f in the line $x = 0$ and h is the reflection of f in the line $y = x$.

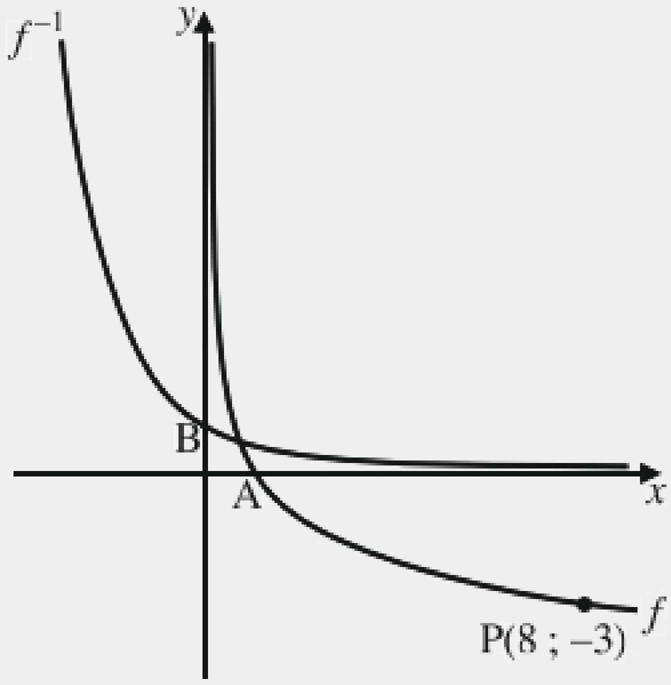
Q is the image of P on h.

(1) Express b in terms of a.
(2) Write down the
  (i) range of g.
  (ii) domain of h.
(3) Write down the coordinates of
  (i) A
  (ii) B

It is now given that the coordinates of P are $\left(-\frac{3}{2} ; \sqrt{27}\right)$.

(4) Determine the values of a and b.
(5) Determine the equation of h.
(6) Write down the coordinates of Q.

(i) The sketch alongside shows the graphs of $f(x) = \log_b x$ and f^{-1}. The graph of f passes through the point P(8 ; –3).



(1) Determine the value of b.

(2) Determine the equation of f^{-1}.

(3) Write down the coordinates of

(i) A (ii) B

(4) For which values of x is

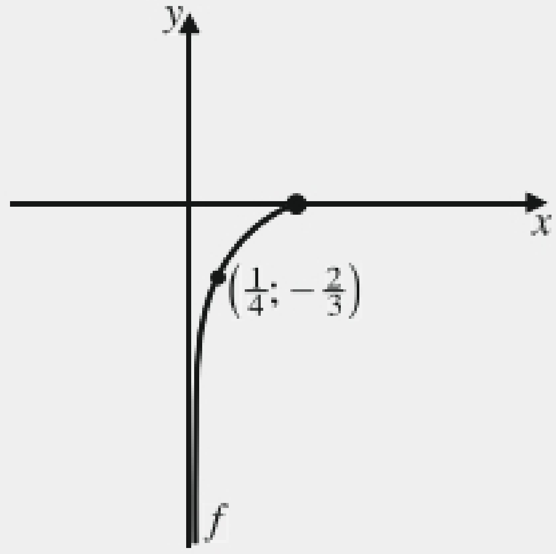
(i) $f(x) \geq 0$?

(ii) $0 < f^{-1}(x) \leq 1$?

(iii) $\log_{\frac{1}{2}} x < -3$?

(iv) $f^{-1}(x) > 8$?

(j)* The sketch shows the graph of $f(x) = \log_a x$, a logarithmic function with a restricted domain. The graph of f passes through the point $\left(\frac{1}{4} ; -\frac{2}{3}\right)$.



(1) Write down the domain of f.

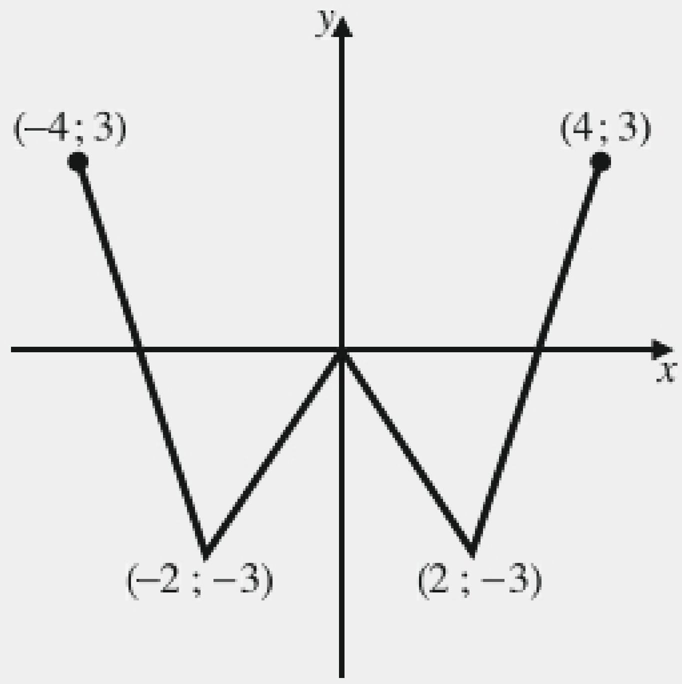
(2) Determine the value of a.

(3) Write down the value of $f^{-1}\left(-\frac{2}{3}\right)$.

(4) Determine the equation of f^{-1}, stating its domain.

(5) Write down the range of f^{-1}.

(k)* The graph of $y = \text{W}(x)$ is shown alongside.



(1) Write down the domain of W.

(2) Write down the range of W.

(3) Is W a function?

(4) Sketch the graph of

(i) $y = -\text{W}(x)$

(ii) $y = \text{W}(-x)$

(iii) $x = \text{W}(y)$

(5) State two possible ways of restricting W so that its inverse will be a function. (Your restriction may not cause the range of W to change.)

(l)* The straight line $f(x) = 3x + d$ and its inverse $f^{-1}(x) = mx + c$ intersect where $x = -\frac{9}{2}$.

Find the value(s) of

(1) d (2) m and c.

(m)* Given $f(x) = x + a$, $g(x) = x + b$ and $h(x) = mx + c$.

(1) If g is the inverse of f, express b in terms of a.

(2) If h is its own inverse and $c \neq 0$, determine the value of m.

(n)** Given $f(x)=x^2+bx$.

(1) The coordinates of the points of intersection of f and its inverse are $(0\ ;0)$ and $(p\ ;q)$. Express p and q in terms of b.

(2) Is f a one-to-one function?

(3) f is restricted, with $x \geq t$, so that its inverse is a function and its range remains unchanged. Express t in terms of b.

(o)** How would you restrict each of the following functions so that its inverse will be a function and its range remains unchanged?

(1) $f(x)=\sin x$ (2) $g(x)=\cos x$ (3) $h(x)=\tan x$

There are many possibilities. Give one possibility in each case.

CHAPTER THREE

Cubic Polynomials

A polynomial is an algebraic expression consisting of one or more terms. All the coefficients of a polynomial are real numbers and the exponents must be whole numbers (positive integers or 0).

The highest exponent of a polynomial determines its *degree*. For example, $2x^4 - 5x^3 - x^2 + \frac{1}{2}x - 1$ is a polynomial of the fourth degree.

In this chapter, we focus on polynomials of the third degree (also called *cubic polynomials*).

FACTORISING CUBIC POLYNOMIALS

You have already learnt how to factorise some cubic expressions:

- Cubic expressions without a constant term, with a **common factor**, e.g. $x^3 - 2x^2 - 3x$.
- Cubic expressions, with **four terms**, that can be factorised by **grouping** terms, e.g. $x^3 + 2x^2 - 9x - 18$.
- The **sum or difference of two cubes**:

$$x^3 \pm b^3 = (x \pm b)(x^2 \mp bx + b^2) \quad \text{and}$$
$$a^3x^3 \pm b^3 = (ax \pm b)(x^2 \mp abx + b^2)$$

Let us revise this briefly:

EXAMPLE 1

Factorise the following expressions fully:

(a) $x^3 - 2x^2 - 3x$ (b) $x^3 + 2x^2 - 9x - 18$

(c) $x^3 + 27$ (d) $125x^3 - 8$

Solution

(a)
$$x^3 - 2x^2 - 3x$$
$$= x(x^2 - 2x - 3)$$
$$= x(x-3)(x+1)$$

(b)
$$x^3 + 2x^2 - 9x - 18$$
$$= x^2(x+2) - 9(x+2)$$
$$= (x+2)(x^2 - 9)$$
$$= (x+2)(x+3)(x-3)$$

(c)
$$x^3 + 27$$
$$= (x+3)(x^2 - 3x + 9)$$

(d)
$$125x^3 - 8$$
$$= (5x-2)(25x^2 + 10x + 4)$$

EXERCISE 1

Factorise the following expressions fully:

(a)	$x^3 - x^2 - 12x$	(b)	$x^3 + x^2 - 4x - 4$
(c)	$8x^3 + 1$	(d)	$x^3 - 1000$
(e)	$x^3 - 3x^2 + 3x - 9$	(f)	$(x-1)^3 - 27$
(g)	$64x^3 + 125$	(h)	$2x^3 + 10x^2 + x + 5$
(i)	$16x^4 - 54x$	(j)	$4x^3 + 12x^2 - x - 3$
(k)	$6x^3 - 3x^2 - 4x + 2$	(l)	$16x^3 + 16$
(m)	$(x+2)^3 - (x+2)^2 - 6x - 12$	(n)	$x^4 - 2x^3 - 9x^2 + 18x$

In order to be able to factorise a greater variety of cubic expressions, there are certain principles concerning polynomial division that you have to know:

POLYNOMIAL DIVISION

When a number is divided by another number that is not a factor of the first number, there will always be a remainder. For example, when 29 is divided by 8, the remainder is 5:

$$\begin{array}{r|l} & 3 \text{ rem } 5 \\ \hline 8 & 29 \\ & \underline{24} \\ & \ \ 5 \end{array}$$

Similarly, when a polynomial is divided by another polynomial of a lower degree, there will also be a remainder. For example, when $x^2 + 6x + 9$ is divided by $x + 2$, the remainder is 1:

$$\begin{array}{r|rrrrr} & x & + & 4 & & \text{rem } 1 \\ \hline x + 2 & x^2 & + & 6x & + & 9 \\ & x^2 & + & 2x & & \\ \hline & & & 4x & + & 9 \\ & & & 4x & + & 8 \\ \hline & & & & & 1 \end{array}$$

THE REMAINDER THEOREM

There is an easy method to determine the remainder, called the *Remainder Theorem*, which states that:

> When a polynomial $f(x)$ is divided by $x + a$, the remainder is given by $f(-a)$.

For example, when dividing $f(x) = x^2 + 6x + 9$ by $x + 2$, the remainder is given by $f(-2)$:

$$\text{Rem} = f(-2) = (-2)^2 + 6(-2) + 9 = 1.$$

The Remainder Theorem can be generalised further, to state that:

> When a polynomial $f(x)$ is divided by $ax + b$, the remainder is given by $f\left(-\frac{b}{a}\right)$.

EXAMPLE 2

(a) Determine the remainder when $f(x) = x^3 + 5x^2 - 6x + 3$ is divided by $x+1$.

(b) Determine the remainder when $g(x) = 8x^3 - 4x^2 + 2x - 7$ is divided by $2x-1$.

Solution

(a) $\text{Rem} = f(-1) = (-1)^3 + 5(-1)^2 - 6(-1) + 3$ $\quad\therefore \text{Rem} = 13$

(b) $\text{Rem} = g\left(\frac{1}{2}\right) = 8\left(\frac{1}{2}\right)^3 - 4\left(\frac{1}{2}\right)^2 + 2\left(\frac{1}{2}\right) - 7$ $\quad\therefore \text{Rem} = -6$

THE FACTOR THEOREM

The Remainder Theorem states that when $f(x)$ is divided by $ax+b$, the remainder is $f\left(-\frac{b}{a}\right)$.

If $f\left(-\frac{b}{a}\right) = 0$, it is actually the remainder that is 0, which implies that $ax+b$ is a factor of $f(x)$.

This conclusion can be summarised in the *Factor Theorem*:

$$\boxed{\text{If } f\left(-\frac{b}{a}\right) = 0, \text{ then } ax+b \text{ is a factor of } f(x).}$$

EXAMPLE 3

(a) Show that $x-2$ is a factor of $f(x) = 4x^3 - 3x^2 - 2x - 16$.

(b) Show that $2x+1$ is a factor of $g(x) = 8x^3 + 4x^2 - 2x - 1$.

Solution

(a) $f(2) = 4(2)^3 - 3(2)^2 - 2(2) - 16 = 32 - 12 - 4 - 16 = 0$ $\quad\therefore x-2$ is a factor of $f(x)$.

(b) $g\left(-\frac{1}{2}\right) = 8\left(-\frac{1}{2}\right)^3 + 4\left(-\frac{1}{2}\right)^2 - 2\left(-\frac{1}{2}\right) - 1 = 0$ $\quad\therefore 2x+1$ is a factor of $g(x)$.

USING THE FACTOR THEOREM TO FACTORISE CUBIC POLYNOMIALS

Cubic polynomials can have up to three linear factors. We factorise a cubic polynomial by first constructing a pair of matching factors - one linear (short bracket) and one quadratic (long bracket).

To factorise a cubic expression of the form $x^3 + bx^2 + cx + d$, follow these steps:

- Look at the **constant term** of the cubic expression (d) and list its **factors**.
- Substitute the variable (x) with factors of d, until you find a value that gives a result of 0.
- Use the **factor theorem** to write down the first **linear factor** of the expression (short bracket).
- Use **inspection** to construct the matching **quadratic factor** (long bracket).
- **Factorise the quadratic factor** if possible.

Note that, when applying the factor theorem to find the first linear factor, we only use factors of the constant term (d). For example, when factorising $x^3 + 3x^2 - 9x + 5$, we would not use the values ± 2 or ± 3 to find our first factor. Rather, we would only test ± 1 and ± 5 (the factors of 5).

EXAMPLE 4

Factorise

(a) $x^3 + 5x^2 - 2x - 24$ (b) $x^3 - 3x - 2$

Solution

(a) Let $f(x) = x^3 + 5x^2 - 2x - 24$

Test factors of 24: $\pm 1 \quad \pm 2 \quad \pm 3 \quad \pm 4 \quad \pm 6 \quad \pm 8 \quad \pm 12 \quad \pm 24$

$f(1) = 1^3 + 5(1)^2 - 2(1) - 24 = -20$ $\quad \therefore x - 1$ is NOT a factor

$f(-1) = (-1)^3 + 5(-1)^2 - 2(-1) - 24 = -18$ $\quad \therefore x + 1$ is NOT a factor

$f(2) = 2^3 + 5(2)^2 - 2(2) - 24 = 0$ $\quad \therefore x - 2$ IS a factor

As soon as one linear factor is found, we can start factorising the expression. We use the linear factor $(x - 2)$ and construct a matching quadratic factor by inspection:

$$x^3$$

$$x^3 + 5x^2 - 2x \boxed{-24} = (x \quad -2)(x^2 \; \ldots\ldots \; +12)$$

$$\boxed{-24}$$

Match the **first** and **last** terms of the LHS to the **products** of the **first terms** and **last terms** in the brackets on the RHS.

$$x^3 + 5x^2 - 2x - 24 = (x \quad -2)(x^2 \quad +7x \quad +12)$$

$$-2x^2$$

$$+ \quad +7x^2$$

$$+5x^2$$

Match the x^2 term of the LHS to the **sum** of the **two products** that give x^2 terms on the RHS.

$x^3 + 5x^2 - 2x - 24$

$= (x - 2)(x^2 + 7x + 12)$

$= (x - 2)(x + 4)(x + 3)$

Note: We could have started with any of the factors as our first factor. Each possible first linear factor has its own unique, matching quadratic factor, but the final result is always the same:

$\therefore x^3 + 5x^2 - 2x - 24$

$= (x + 3)(x^2 + 2x - 8)$

$= (x + 3)(x - 2)(x + 4)$

or

$\therefore x^3 + 5x^2 - 2x - 24$

$= (x + 4)(x^2 + x - 6)$

$= (x + 4)(x - 2)(x + 3)$

(b) Let $f(x)=x^3-3x-2$

$f(-1)=(-1)^3-3(-1)-2=0 \quad \therefore\ x+1$ is a factor

$$x^3 + 0x^2 - 3x - 2 = (x+1)(x^2 - 1x - 2)$$

$$+1x^2$$

$$\underline{+\ -1x^2}$$

$$0x^2$$

Note that this expression does not have an x^2 term, and we have to add $0x^2$.

$$\begin{aligned}&x^3-3x-2\\&=(x+1)(x^2-x-2)\\&=(x+1)(x+1)(x-2)\end{aligned}$$

When factorising cubic expressions of the form ax^3+bx^2+cx+d, we use $\dfrac{\text{factors of } d}{\text{factors of } a}$ to find the first linear factor. For example, when factorising $2x^3+3x^2+19x+9$, we test the values:

$$\pm\frac{1}{1}\ ;\ \pm\frac{3}{1}\ ;\ \pm\frac{9}{1}\ ;\ \pm\frac{1}{2}\ ;\ \pm\frac{3}{2}\ ;\ \pm\frac{9}{2}$$

<u>EXAMPLE 5</u>

Factorise $2x^3+3x^2+4x-3$.

<u>Solution</u>

Let $f(x)=2x^3+3x^2+4x-3$

Test $\dfrac{\text{factors of } 3}{\text{factors of } 2}$: $\quad \pm\frac{1}{1};\pm\frac{3}{1};\pm\frac{1}{2};\pm\frac{3}{2}$

$f\left(\frac{1}{2}\right)=2\left(\frac{1}{2}\right)^3+3\left(\frac{1}{2}\right)^2+4\left(\frac{1}{2}\right)-3=0 \quad \therefore 2x-1$ is a factor

$$2x^3$$

$$2x^3 + 3x^2 + 4x \boxed{-3} = (2x - 1)(x^2 \ \ldots\ldots\ldots + 3)$$

$$\boxed{-3}$$

$$2x^3 + 3x^2 + 4x - 3 = (2x - 1)(x^2 + 2x + 3)$$

$$-1x^2$$

$$\underline{+\ +4x^2}$$

$$+3x^2$$

$2x^3+3x^2+4x-3=(2x-1)(x^2+2x+3)$ The expression cannot be factorised any further.

EXERCISE 2

(a) Determine the remainder when $f(x)=x^3+8x^2-15x+7$ is divided by $x-1$.

(b) $g(x)=27x^3+9x^2-3x+7$ is divided by $3x+1$. Determine the remainder.

(c) Let $f(x)=x^3+5x^2+2x-8$.

(1) Show that $x+4$ is a factor of $f(x)$.

(2) Hence factorise $f(x)$ fully.

(d) Show that $3x-2$ is a factor of $3x^3+10x^2-23x+10$ and factorise the expression fully.

(e) Factorise the following expressions fully:

(1) x^3+x^2-5x+3 (2) x^3-4x^2+x+6

(3) $x^3+x^2-8x-12$ (4) $x^3-13x-12$

(5) $x^3-9x^2+26x-24$ (6) $-x^3-4x^2+3x+18$

(7) $-x^3+12x-16$ (8) $x^3-x^2-22x+40$

(9) $2x^3-5x^2-23x-10$ (10) $3x^3+2x^2+2x-1$

(11) $12x^3-16x^2-7x+6$ (12)* $(x-4)^3-19(x-4)+30$

SOLVING CUBIC EQUATIONS

In order to solve a cubic equation:

- Transpose all terms to the LHS so that the $\text{RHS}=0$.
- **Factorise** the expression on the LHS.
- Apply the **zero factor law** (each factor $=0$).

Remember to always try the most basic techniques of factorising first, i.e. taking out a common factor or a common bracket, grouping or the sum or difference of cubes. If it is not possible to factorise by using any of these techniques, use the Factor Theorem.

EXAMPLE 6

Solve for x:

(a) $x^3+x^2-16x-16=0$ (b) $x^3-2x^2=5x-6$

Solution

(a) $x^3+x^2-16x-16=0$

$\therefore x^2(x+1)-16(x+1)=0$

$\therefore (x+1)(x^2-16)=0$

$\therefore (x+1)(x+4)(x-4)=0$

$\therefore x=-1$ or $x=-4$ or $x=4$

(b) $x^3-2x^2=5x-6$

$\therefore x^3-2x^2-5x+6=0$

$(1)^3-2(1)^2-5(1)+6=0$

$\therefore x-1$ is a factor

$\therefore (x-1)(x^2-x-6)=0$

$\therefore (x-1)(x-3)(x+2)=0$

$\therefore x=-1$ or $x=3$ or $x=-2$

Cubic equations with real coefficients can have a **maximum of three** real roots and a **minimum of one** real root. Real roots can be rational or irrational. The following example illustrates a variety of possibilities:

EXAMPLE 7

Solve for x in the following equations:

(a) $x^3 - 3x^2 + 4 = 0$ (b) $x^3 - 2x^2 - 8x - 5 = 0$

(c) $2x^3 - 16 = 0$

Solution

(a) $(-1)^3 - 3(-1)^2 + 4 = 0 \quad \therefore\ x+1$ is a factor

$(x+1)(x^2 - 4x + 4) = 0$

$\therefore (x+1)(x-2)^2 = 0$

$\therefore x = -1$ or $x = 2$

This equation has **two real** and **rational** roots.

(b) $f(x) = x^3 - 2x^2 - 8x - 5$

$(-1)^3 - 2(-1)^2 - 8(-1) - 5 = 0 \quad \therefore\ x+1$ is a factor

$(x+1)(x^2 - 3x - 5) = 0$

$$\therefore x = -1 \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{3 \pm \sqrt{(-3)^2 - 4(1)(-5)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{29}}{2}$$

$\therefore x = -1$ or $x = 4{,}19$ or $x = -1{,}19$

This equation has **three real roots**, **two** of which are **irrational**.

(c) $2x^3 - 16 = 0$

$\therefore x^3 - 8 = 0$

$\therefore (x-2)(x^2 + 2x + 4) = 0$

$$\therefore x = 2 \text{ or } x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-2 \pm \sqrt{2^2 - 4(1)(4)}}{2(1)}$$

$$= \frac{-2 \pm \sqrt{-12}}{2}$$

N/A

$\therefore x = 2$

This equation has **one real** root.

Alternative solution :

$2x^3 - 16 = 0$

$\therefore 2x^3 = 16$

$\therefore x^3 = 8$

$\therefore x = \sqrt[3]{8} = 2$

EXERCISE 3

Solve for x:

(a) $x^3 - 8x^2 = 0$ (b) $x^3 = x$

(c) $x^3 - 6x^2 + 11x - 6 = 0$ (d) $x^3 - 7x^2 = -8x - 16$

(e) $2x^3 - 2 = 0$ (f) $x^3 - 5x - 2x^2 + 6 = 0$

(g) $x^3 + 10 = x^2 + x$ (h) $2 + 3x - x^3 = 0$

(i) $x^2 - 3x + \dfrac{12}{x} = 4$ (j) $x - \dfrac{11}{x} = 4 - \dfrac{30}{x^2}$

(k) $4x^3 - 8x^2 = x - 2$ (l) $10x^3 + 5x^2 - 15x = 0$

(m) $2x^3 + 12x^2 + 22x + 12 = 0$ (n) $2x^3 - 3x^2 + 4x + 3 = 0$

(o) $2x^3 + 2x^2 = 7x - 3$ (p) $3x^3 + 2x^2 - 61x + 20 = 0$

(q) $7x^3 + 5x^2 - 3x = 1$ (r) $8x^4 + 7x^3 + 2x = 17x^2$

MORE ABOUT THE REMAINDER AND FACTOR THEOREM

In the following examples, we apply the remainder and/or factor theorem to polynomials with *unknown coefficients*:

EXAMPLE 8

(a) When the function $h(x) = 2x^3 + 5x^2 + ax + 12$ is divided by $x + 3$, the remainder is 15. Find the value of a.

(b) Find the value of p if division of $f(a) = 2a^3 - a^2 - 5a - p$ by $a + 2$ gives a remainder of 6.

(c) Given $f(x) = x^3 + kx^2 - 15x + 9$. Determine the value of k if $f(x)$ is divisible by $x + 3$.

Solution

(a) $h(-3) = 2(-3)^3 + 5(-3)^2 + a(-3) + 12 = 15$

$\therefore -54 + 45 - 3a + 12 = 15$

$\therefore -3a = 12$

$\therefore \quad a = -4$

(b) $f(-2) = 2(-2)^3 - (-2)^2 - 5(-2) - p = 6$

$\therefore\ -16 - 4 + 10 - p = 6$

$\therefore\ -p = 16$

$\therefore \quad p = -16$

(c) $f(-3) = (-3)^3 + k(-3)^2 - 15(-3) + 9 = 0$

$\therefore -27 + 9k + 45 + 9 = 0$

$\therefore\ 9k + 27 = 0$

$\therefore k = -3$

EXAMPLE 9

Prove that $x + 3a$ is a factor of $f(x) = x^3 - 2x + 3ax^2 - 6a$.

Solution

$f(-3a) = (-3a)^3 - 2(-3a) + 3a(-3a)^2 - 6a$

$= -27a^3 + 6a + 27a^3 - 6a$

$= 0$

$\therefore x + 3a$ is a factor of $f(x)$.

EXAMPLE 10

When $f(x) = 2x^3 - ax^2 + bx + 7$ is divided by $x - 3$, the remainder is 19. When $f(x)$ is divided by $x - 1$, the remainder is 11. Determine the values of a and b.

Solution

$f(3) = 2(3)^3 - a(3)^2 + b(3) + 7 = 19$
$\therefore 54 - 9a + 3b + 7 = 19$
$\therefore -9a + 3b = -42$
$\therefore 3a - b = 14 \quad \ldots 1$

$f(1) = 2(1)^3 - a(1)^2 + b(1) + 7 = 11$
$\therefore 2 - a + b + 7 = 11$
$\therefore -a + b = 2 \quad \ldots 2$

$1 + 2$: $2a = 16$
$\therefore a = 8$

Into 2: $-8 + b = 2$
$\therefore b = 10$

If a polynomial $g(x)$ is a factor of a polynomial $f(x)$, then any factor of $g(x)$ is also a factor of $f(x)$. We will use this fact in the following example:

EXAMPLE 11

If $x^2 - 2x - 3$ is a factor of $x^3 + ax^2 + bx + 6$, determine the values of a and b.

Solution

Let $f(x) = x^3 + ax^2 + bx + 6$.

$x^2 - 2x - 3 = (x - 3)(x + 1)$

If $x^2 - 2x - 3$ is a factor of $f(x)$, then both $x - 3$ and $x + 1$ are factors of $f(x)$:

$\therefore f(3) = 3^3 + a(3)^2 + b(3) + 6 = 0$ and $f(-1) = (-1)^3 + a(-1)^2 + b(-1) + 6 = 0$
$\therefore 27 + 9a + 3b + 6 = 0$ $\qquad$ $\therefore -1 + a - b + 6 = 0$
$\therefore 9a + 3b = -33$ $\qquad$ $\therefore a - b = -5 \quad \ldots 2$
$\therefore 3a + b = -11 \quad \ldots 1$

$1 + 2$: $4a = -16$
$\therefore a = -4$

Into 2: $-4 - b = -5$
$\therefore b = 1$

EXERCISE 4

(a) Find the value of k if $h(x)=x^3-5x^2+kx+12$ leaves a remainder of -8 when divided by $x+2$.

(b) Determine the value of a if $x-2a$ is a factor of $x^3-2ax^2+4x+16$.

(c) It is given that $f(x)=x^3-5x^2+px-9$. When $f(x)$ is divided by $x-3$, the remainder is $6q$. Find p in terms of q.

(d) If it is given that 2 is a root of $x^3-tx^2-7x-10=0$, determine the value of t, and hence determine the other two roots of the equation.

(e) When $f(x)=x^3+mx^2-nx+8$ is divided by $x-1$, the remainder is 12. When $f(x)$ is divided by $x+1$, the remainder is 14. Determine the values of m and n.

(f) The cubic polynomial $3x^3+rx^2-4x-s$ has a factor of $3x-1$, and leaves a remainder of 2 when it is divided b y $x-1$. Determine the values of r and s.

(g) If x^2-x-6 is a factor of ax^3-7x+b, determine the values of a and b.

(h)* If $f(x)=(x^2-2x-3)\cdot Q(x)+5$, where $f(x)$ and $Q(x)$ are both polynomials, determine

(1) the remainder when $f(x)$ is divided by $x-3$.

(2) $Q(x)$ if $f(x)=x^3-3x^2-x+8$.

CONSOLIDATION AND EXTENSION EXERCISE

(a) Find the remainder if $f(x)=2x^3-7x^2-4x+3$ is divided by $x+1$.

(b) Given that $g(x)=x^3+3x^2-2x-4$.

(1) Show that $x+1$ is a factor of $g(x)$.

(2) Factorise $g(x)$ fully.

(3) Solve x if $g(x)=0$.

(c) If $f(x)=2x^3-14x+12$:

(1) Calculate $f(2)$ and explain the implication of your answer.

(2) Factorise $f(x)$ fully.

(3) Solve for x if $f(x)=0$.

(d) Given that $f(x)=2x^3-7x^2-5x+4$.

(1) Show that $x=\dfrac{1}{2}$ is a root of $f(x)=0$.

(2) Factorise $f(x)$ fully.

(e) Factorise the following expressions fully:

(1) x^3-3x^2	(2) $2x^3+8x^2-10x$
(3) $-4x^3+25x$	(4) x^3-64
(5) $x^3-5x^2-9x+45$	(6) $x^3-7x^2+14x-6$
(7) x^3-4x^2+x+6	(8) $-x^3-3x^2+4$
(9) $12-13x+x^3$	(10) $2x^3+12x^2+24x+16$
(11) $2x^3-x^2-13x-6$	(12) $12x^3-16x^2-5x+3$

(f) Solve the following equations:

(1)	$x^3 = 81x$	(2)	$x^3 - 5x^2 + 8x - 4 = 0$
(3)	$x^3 + 4x + 12 = 7x^2$	(4)	$x^3 + 27 = 0$
(5)	$x^3 - 9x - x^2 + 9 = 0$	(6)	$-x^3 + 7x - 6 = 0$
(7)	$x^2 - x - 1 = \dfrac{15}{x}$	(8)	$x^3 - 5x^2 + 4x = 0$
(9)	$(x-2)^3 - 8 = 0$	(10)	$x + 3 = \dfrac{10}{x} + \dfrac{24}{x^2}$
(11)	$x^3 - 31x - 30 = 0$	(12)	$x^3 - 10x^2 + 31x - 30 = 0$
(13)	$9x^3 = 9x^2$	(14)	$2x^3 - 3x^2 - 23x + 12 = 0$
(15)	$2x^3 - x^2 - 13x - 6 = 0$	(16)	$3x^3 - 7x^2 + 4 = 0$
(17)	$4x^3 - 2x^2 + 10x - 5 = 0$	(18)	$7x^3 - 12x^2 = 5x - 2$
(19)	$5x^3 - 12x^2 - 13x + 12 = 0$	(20)*	$8(x+1)^3 + 7(x+1)^2 - 17(x+1) + 2 = 0$

(g) Show that $x^3 - 4x^2 + 3x + 8 = 0$ has only one real root.

(h) Determine the value of p if $8x^3 - 12x^2 - px + 7$ is divided by $2x - 3$ and leaves a remainder of -8.

(i) If $x - k$ is a factor of $2x^3 + k^2x + 24$, find the value of k.

(j) Given $g(x) = x^3 - 3ax^2 - 5x - b$. When $g(x)$ is divided by x the remainder is 15 and when $g(x)$ is divided by $x - 2$ the remainder is 7. Determine
(1) the values of a and b.
(2) the remainder when $g(x)$ is divided by $2x + 1$.

(k) The graph of $h(x) = 2x^3 + 3x^2 - 30x + p$ intersects the x-axis at 1.
Determine
(1) the value of p.
(2) the other x-intercept(s) of h.

(l) If $x^2 - 2x - 15$ is a factor of $f(x) = x^3 + mx + n$, determine the values of m and n.

(m)* Given $g(x) = 4x^3 + bx^2 + cx + d$. Determine the values of b, c and d, if
$g\left(-\dfrac{5}{2}\right) = g(3) = g(-2) = 0$.

(n)* In a cubic polynomial, the coefficient of x^2 is 7, and the constant term is -6.
The polynomial is divisible by $2x + 3$ as well as by $x - 1$. Determine the third factor of the polynomial.

(o)* Given $f(x) = (2x-1)(x+3)\cdot Q(x) - 6x + 1$, where $f(x)$ and $Q(x)$ are both polynomials.
(1) What will the remainder be when $f(x)$ is divided by $x + 3$?
(2) Given that $x - 2$ is a factor of $Q(x)$, determine $f(2)$.

(p)* Prove that $a - b - 5$ is a factor of $a^2 - 2ab + b^2 - 3a + 3b - 10$.

CHAPTER FOUR

Differential Calculus

Calculus is a branch of mathematics, dealing with problems concerning **gradients of tangents** to curves (*differential calculus*) and **areas enclosed by curves** (*integral calculus*). To solve these problems, mathematicians had to find effective methods to deal with quantities approaching the **infinitely small** and the **infinitely large**.

The most important breakthroughs in the field were made in the seventeenth century by the English mathematician and scientist Sir Isaac Newton, and German mathematician Gottfried Leibniz. These two mathematicians are regarded as the inventors of Calculus. Their methods were ingenious, but lacked mathematical precision. It was only in the nineteenth century that mathematicians such as Cauchy, Weierstrass and Riemann made the discipline precise by introducing the idea of a *limit* - a concept that you will briefly encounter in this chapter.

Calculus was originally developed to solve problems from the Physical Sciences (especially Mechanics), but today it is applied to a variety of subjects.

In this chapter, we will focus on only one of the two branches of Calculus, namely *Differential Calculus*.

THE AVERAGE GRADIENT BETWEEN TWO POINTS ON A CURVE

We can calculate the **average gradient** between any **two points** on a curve, by simply calculating the gradient of the line connecting those two points. The average gradient of a function f, between $x = x_1$ and $x = x_2$ is given by the formula:

$$\boxed{\text{Average gradient} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}}$$

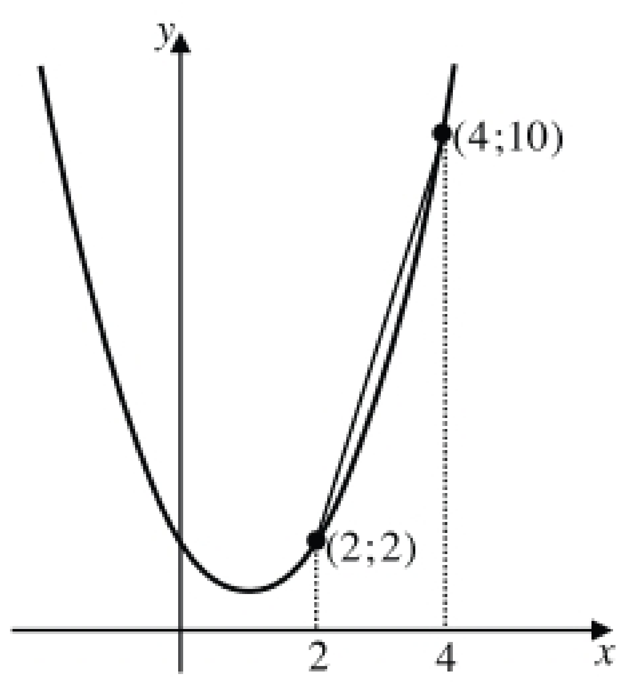
EXAMPLE 1

Calculate the average gradient of $f(x) = x^2 - 2x + 2$, between $x = 2$ and $x = 4$.

Solution

$$\begin{aligned} x_1 &= 2 \\ f(x_1) &= f(2) \\ &= 2^2 - 2(2) + 2 \\ &= 2 \end{aligned} \qquad \begin{aligned} x_2 &= 4 \\ f(x_2) &= f(4) \\ &= 4^2 - 2(4) + 2 \\ &= 10 \end{aligned}$$

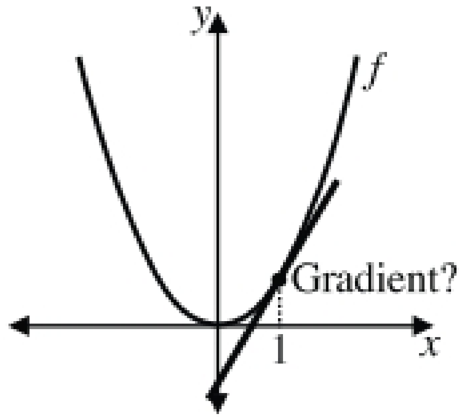
$$\begin{aligned} \text{Average gradient } &= \frac{f(x_2) - f(x_1)}{x_2 - x_1} \\ &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{10 - 2}{4 - 2} \\ &= 4 \end{aligned}$$



THE GRADIENT AT A SINGLE POINT ON A CURVE

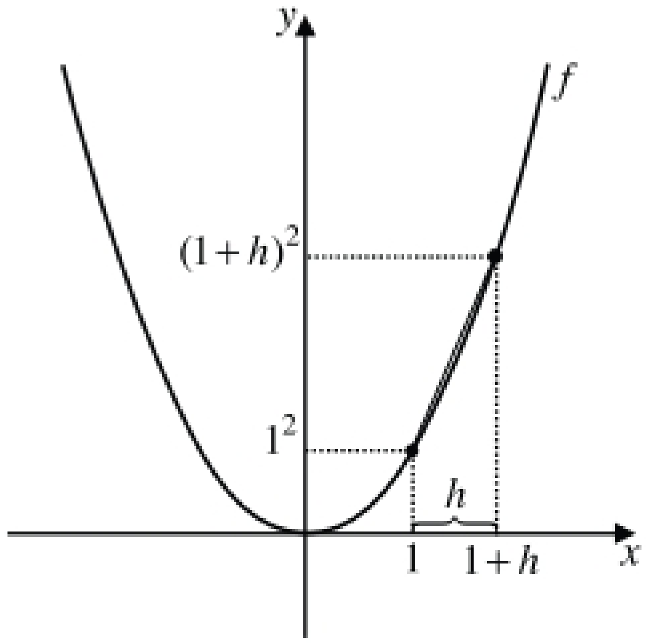
In differential calculus, we are interested in finding the gradient at **a single point** on a curve.

Consider, for example, the function $f(x) = x^2$. Suppose we wish to determine the gradient of the graph of f at the point where $x = 1$. In this case, what we are really looking for is the **gradient of the tangent** to f at the point where $x = 1$:



Notice that we only have **one point** on the tangent. We need **two points** to calculate the gradient of a straight line. To estimate the gradient of the tangent, we choose a second point **on the graph of *f***, close to the point where $x = 1$, and calculate the **average gradient** of f between these two points:

Let the x-value of the second point be $1 + h$. (h is the horizontal distance between the two points.)



$$\begin{aligned}
\text{Average gradient } &= \frac{f(1+h) - f(1)}{1+h-1} \\
&= \frac{(1+h)^2 - 1^2}{h} \\
&= \frac{1+2h+h^2-1}{h} \\
&= \frac{2h+h^2}{h} \\
&= \frac{h(2+h)}{h} \\
&= 2+h \qquad\qquad (h \neq 0)
\end{aligned}$$

How can we use this <u>average gradient</u> $(2 + h)$ to find the <u>gradient of the tangent</u> at $x = 1$?

The closer the distance h is to 0, the closer this average gradient will be to the gradient of the tangent at $x = 1$. h can never equal 0, since h is the denominator in the original expression $\dfrac{(1+h)^2 - 1^2}{h}$.

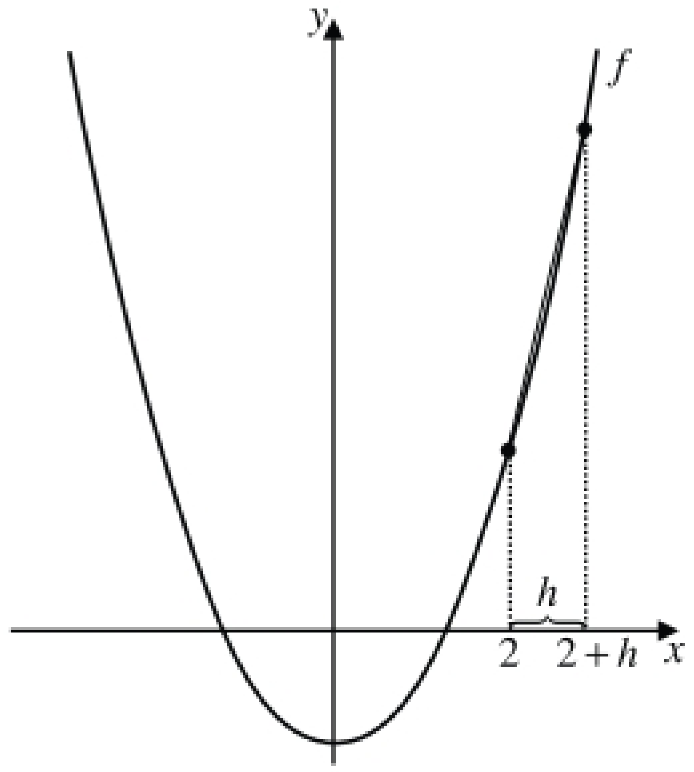
As h gets closer to 0, the value of our result $2 + h$ gets closer to 2, and thus the gradient of the tangent will be 2. To express this idea mathematically, we use the concept of a *limit* (abbreviated "lim"). We write:

$$\text{Gradient of the tangent} = \lim_{h \to 0}(2+h) = 2$$

- The symbol $\lim\limits_{h \to 0}$ indicates that h is **not equal** to 0, but **approaches** 0.
- By writing $\lim\limits_{h \to 0}(2+h) = 2$, we are saying that the expression $2 + h$ approaches 2, as h approaches 0.

EXAMPLE 2

Given the function $f(x)=2x^2-3$.



(a) Calculate the average gradient of f, between $x=2$ and $x=2+h$, in terms of h and simplify.

(b) What is the average gradient of f, between $x=2$ and $x=2+h$, when

(1) $h=1$?

(2) $h=0{,}1$?

(3) $h=0{,}01$?

(c) Determine the gradient of the tangent to f at $x=2$.

Solution

(a) $f(2)=2(2)^2-3$
$=5$

$$\begin{aligned}f(2+h)&=2(2+h)^2-3\\&=2(4+4h+h^2)-3\\&=8+8h+2h^2-3\\&=2h^2+8h+5\end{aligned}$$

$$\begin{aligned}\text{Average gradient}&=\frac{f(2+h)-f(2)}{2+h-2}\\&=\frac{2h^2+8h+5-5}{h}\\&=\frac{2h^2+8h}{h}\\&=\frac{h(2h+8)}{h}\\&=2h+8\end{aligned}$$

(b) (1) $\text{Average gradient}=2h+8$
$=2(1)+8$
$=10$

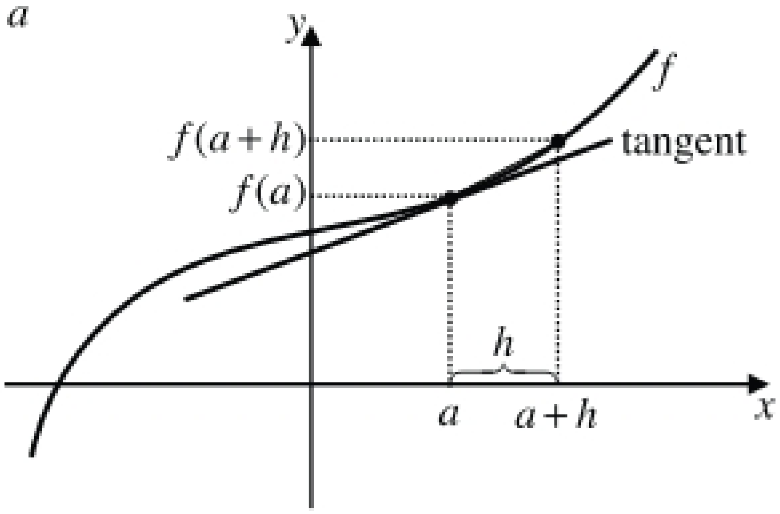
(2) $\text{Average gradient}=2(0{,}1)+8$
$=8{,}2$

(3) $\text{Average gradient}=2(0{,}01)+8$
$=8{,}02$

(c) $\text{Gradient of the tangent}=\lim\limits_{h\to 0}(2h+8)$
$=2(0)+8$
$=8$

The **gradient of the tangent** to **any function** f at $x = a$ is given by:

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

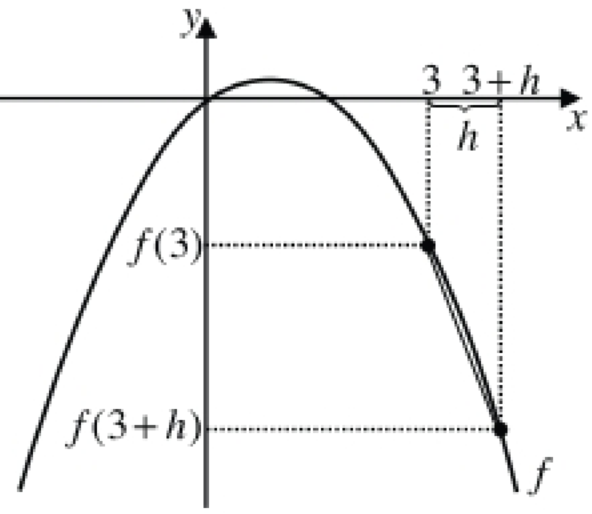


EXAMPLE 3

Given the function $f(x) = -x^2 + x$.

(a) Determine the value of $\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$.

(b) Explain what your answer in (a) means.



Solution

(a) $f(3+h) = -(3+h)^2 + (3+h)$

$= -(9 + 6h + h^2) + 3 + h$

$= -9 - 6h - h^2 + 3 + h$

$= -h^2 - 5h - 6$

$f(3) = -3^2 + 3$

$= -6$

$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h}$

$= \lim_{h \to 0} \frac{-h^2 - 5h - 6 - (-6)}{h}$

$= \lim_{h \to 0} \frac{-h^2 - 5h}{h}$

$= \lim_{h \to 0} \frac{h(-h-5)}{h}$

$= \lim_{h \to 0} (-h - 5)$

$= -0 - 5$

$= -5$

(b) The gradient of the tangent to f at $x = 3$ is -5.

EXERCISE 1

(a) Calculate the average gradient of

(1) $f(x)=x^2-1$, between $x=0$ and $x=5$.

(2) $f(x)=-x^3+x^2$, between $x=-1$ and $x=3$.

(b) Given the function $f(x)=-3x^2$.

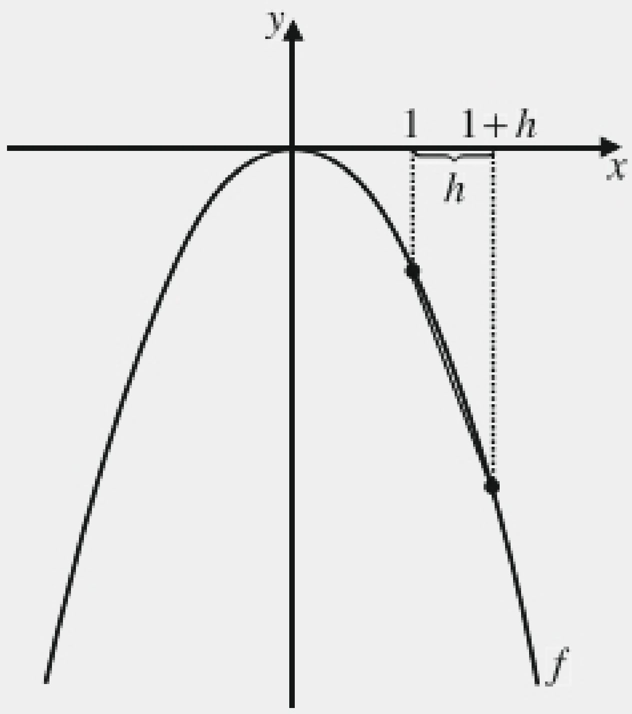
(1) Calculate the average gradient of f between $x=1$ and $x=1+h$ in terms of h and simplify.

(2) What is the average gradient of f between $x=1$ and $x=1+h$ when

(i) $h=1$? (ii) $h=0{,}1$?

(iii) $h=0{,}01$?

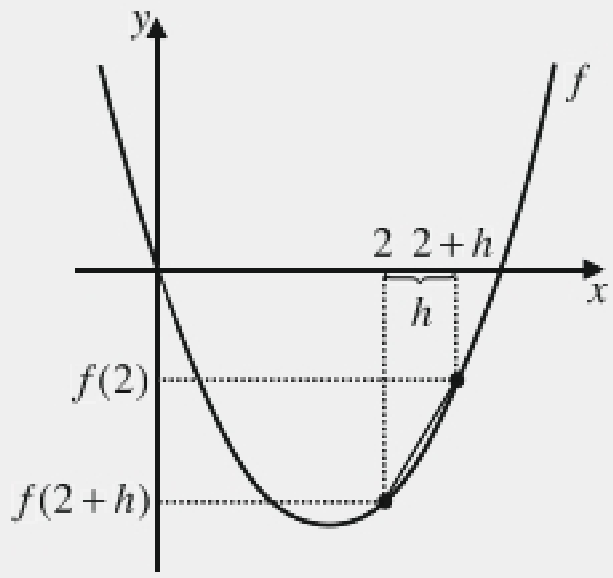
(3) Determine the gradient of the tangent to f at $x=1$.



(c) Given $f(x)=x^2-3x$.

(1) Determine the value of $\lim\limits_{h\to 0}\dfrac{f(2+h)-f(2)}{h}$.

(2) Explain what your answer in (1) means.



THE DERIVATIVE (FIRST PRINCIPLES)

The function that gives the **gradient of the tangent** at any point on a given function f is called the *derivative* (or *gradient function*) and is written as f'.

The **derivative** is defined by

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

The derivative is a function of x (i.e. it is expressed in terms of x).

When we use this formula to determine the derivative, we say that we are finding the derivative **from first principles**. There are easier ways to find the derivative, which will be discussed later in this chapter.

EXAMPLE 4

Determine the derivative of $f(x)=x^2+1$ from first principles.

Solution

$f(x)=x^2+1$

$$f(x+h)=(x+h)^2+1$$
$$=x^2+2hx+h^2+1$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0}\frac{x^2+2hx+h^2+1-(x^2+1)}{h} \\ &= \lim_{h\to 0}\frac{x^2+2hx+h^2+1-x^2-1}{h} \\ &= \lim_{h\to 0}\frac{2hx+h^2}{h} \\ &= \lim_{h\to 0}\frac{h(2x+h)}{h} \\ &= \lim_{h\to 0}(2x+h) \\ &= 2x+0 \\ &= 2x \end{aligned}$$

EXAMPLE 5

Given the function $g(x)=-x^2+3x-2$.

(a) Determine $g'(x)$ from first principles.

(b) Determine the gradient of the tangent to g at $x=-2$.

(c) At which value of x is the gradient of the tangent to g equal to -3?

Solution

(a) $g(x)=-x^2+3x-2$

$$\begin{aligned} g(x+h) &= -(x+h)^2+3(x+h)-2 \\ &= -(x^2+2hx+h^2)+3(x+h)-2 \\ &= -x^2-2hx-h^2+3x+3h-2 \end{aligned}$$

$$\begin{aligned} g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0}\frac{-x^2-2hx-h^2+3x+3h-2-(-x^2+3x-2)}{h} \\ &= \lim_{h\to 0}\frac{-x^2-2hx-h^2+3x+3h-2+x^2-3x+2}{h} \\ &= \lim_{h\to 0}\frac{-2hx-h^2+3h}{h} \\ &= \lim_{h\to 0}\frac{h(-2x-h+3)}{h} \\ &= \lim_{h\to 0}(-2x-h+3) \\ &= -2x+3 \end{aligned}$$

(b) $g'(-2) = -2(-2) + 3 = 7$

(c) $g'(x) = -3$

$\therefore -2x + 3 = -3$

$\therefore -2x = -6$

$\therefore x = 3$

EXAMPLE 6

Determine $f'(x)$ from first principles if

(a) $f(x) = x^3$

(b) $f(x) = \dfrac{2}{x}$

Solution

(a) $f(x) = x^3$

$$\begin{aligned} f(x+h) &= (x+h)^3 \\ &= (x+h)(x+h)^2 \\ &= (x+h)(x^2 + 2hx + h^2) \\ &= x^3 + 2hx^2 + h^2x + hx^2 + 2h^2x + h^3 \\ &= x^3 + 3hx^2 + 3h^2x + h^3 \end{aligned}$$

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{x^3 + 3hx^2 + 3h^2x + h^3 - x^3}{h} \\ &= \lim_{h \to 0} \frac{3hx^2 + 3h^2x + h^3}{h} \\ &= \lim_{h \to 0} \frac{h(3x^2 + 3hx + h^2)}{h} \\ &= \lim_{h \to 0} (3x^2 + 3hx + h^2) \\ &= 3x^2 + 3(0)x + (0)^2 \\ &= 3x^2 \end{aligned}$$

(b)

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{\frac{2}{x+h} - \frac{2}{x}}{h} \\ &= \lim_{h \to 0} \frac{2x - 2(x+h)}{x(x+h)} \times \frac{1}{h} \\ &= \lim_{h \to 0} \frac{-2h}{x(x+h)} \times \frac{1}{h} \\ &= \lim_{h \to 0} \frac{-2}{x(x+h)} \\ &= \frac{-2}{x(x+0)} \\ &= \frac{-2}{x^2} \end{aligned}$$

EXERCISE 2

(a) Determine the derivative of each of the following functions from first principles:

(1)	$f(x)=x^2-3$	(2)	$g(x)=3x^2+1$
(3)	$h(x)=2-x^2$	(4)	$f(x)=x^2-4x$
(5)	$g(x)=-x^2+3x$	(6)	$h(x)=x^2+2x-3$
(7)	$f(x)=2x^3$	(8)	$g(x)=-x^3$
(9)	$h(x)=\dfrac{3}{x}$	(10)	$f(x)=\dfrac{-4}{x}$

(b) Given the function $f(x)=-x^2-5x$.

(1) Determine the average gradient of f between $x=-1$ and $x=3$.
(2) Determine $f'(x)$ from first principles.
(3) Calculate the gradient of the tangent to f at $x=2$.

(c) Given the function $g(x)=\dfrac{8}{x}$.

(1) Determine $g'(x)$ from first principles.
(2) Calculate $g'(-1)$.
(3) Explain what your answer in (2) means.
(4) At which value(s) of x is the gradient of the tangent to g equal to -2?
(5)* State whether the following statement is true or false and motivate your answer: "The gradient of any tangent to g is negative."

(d) Determine $f'(x)$ from first principles if

(1) $f(x)=3x-1$ (2) $f(x)=5$

(e)* Determine the derivative of each of the following functions from first principles:

(1)	$f(x)=c$	(c is a constant.)
(2)	$f(x)=mx+c$	(m and c are constants.)
(3)	$f(x)=ax^2$	(a is a constant.)
(4)	$f(x)=ax^2+bx+c$	(a, b and c are constants.)

DIFFERENTIATION

When we "find the derivative of a function", we also say that we "*differentiate* the function". The process is called **differentiation** and the result is called the **derivative** (or the **gradient function**).

NOTATION

Different notations are used to indicate derivatives:

The derivative of a function $f(x)$: $f'(x)$

The derivative of y with respect to x: $\dfrac{dy}{dx}$

The derivative of an expression: $D_x[\text{Expression}]$ or $\dfrac{d}{dx}[\text{Expression}]$

THE RULES OF DIFFERENTIATION

To find the derivative of a function from first principles is long and tedious. There are rules that can be used to find derivatives quickly:

RULE 1 $\boxed{D_x\left[ax^n\right]=(na)x^{n-1}}$ Example: $D_x\left[3x^4\right]=12x^3$

Special cases $\boxed{D_x\left[ax\right]=a}$ Example: $D_x\left[2x\right]=2$

$\boxed{D_x\left[a\right]=0}$ Example: $D_x\left[5\right]=0$

EXAMPLE 7

Determine

(a) $D_x\left[2x^3\right]$ (b) $f'(x)$ if $f(x)=3x^2$ (c) $\dfrac{dy}{dx}$ if $y=x^4$

(d) $D_x\left[5x\right]$ (e) $\dfrac{d}{dx}[12]$ (f) $g'(x)$ if $g(x)=\dfrac{3x^5}{5}$

Solution

(a) $D_x\left[2x^3\right]=6x^2$

(b) $f(x)=3x^2$

$\therefore f'(x)=6x$

(c) $y=x^4$

$\therefore \dfrac{dy}{dx}=4x^3$

(d) $D_x\left[5x\right]=5$

(e) $\dfrac{d}{dx}\left[12\right]=0$

(f) $g(x)=\dfrac{3x^5}{5}$

$=\dfrac{3}{5}x^5$

$\therefore g'(x)=3x^4$

If the variable is in the **denominator** or under a **root sign**, we apply the **laws of exponents** to rewrite the expression to the form ax^n before applying RULE 1:

EXAMPLE 8

Determine $f'(x)$ if

(a) $f(x)=\dfrac{1}{x^2}$ (b) $f(x)=\dfrac{3}{x^4}$ (c) $f(x)=\dfrac{1}{2x^3}$

(d) $f(x)=\dfrac{2x^6}{x^3}$ (e) $f(x)=\dfrac{x^2}{3x^5}$ (f) $f(x)=\dfrac{5x^9}{4x^{11}}$

(g) $f(x)=\sqrt{x}$ (h) $f(x)=\sqrt[3]{x}$ (i) $f(x)=\dfrac{3}{\sqrt[3]{x^4}}$

Solution

(a) $f(x)=\dfrac{1}{x^2}$
$=x^{-2}$
$\therefore f'(x)=-2x^{-3}$

(b) $f(x)=\dfrac{3}{x^4}$
$=3x^{-4}$
$\therefore f'(x)=-12x^{-5}$

(c) $f(x)=\dfrac{1}{2x^3}$
$=\dfrac{1}{2}x^{-3}$
$\therefore f'(x)=-\dfrac{3}{2}x^{-4}$

(d) $f(x)=\dfrac{2x^6}{x^3}$
$=2x^3$
$\therefore f'(x)=6x^2$

(e) $f(x)=\dfrac{x^2}{3x^5}$
$=\dfrac{1}{3}x^{-3}$
$\therefore f'(x)=-x^{-4}$

(f) $f(x)=\dfrac{5x^9}{4x^{11}}$
$=\dfrac{5}{4}x^{-2}$
$\therefore f'(x)=-\dfrac{5}{2}x^{-3}$

(g) $f(x)=\sqrt{x}$
$=x^{\frac{1}{2}}$
$\therefore f'(x)=\dfrac{1}{2}x^{-\frac{1}{2}}$

(h) $f(x)=\sqrt[3]{x}$
$=x^{\frac{1}{3}}$
$\therefore f'(x)=\dfrac{1}{3}x^{-\frac{2}{3}}$

(i) $f(x)=\dfrac{3}{\sqrt[3]{x^4}}$
$=3x^{-\frac{4}{3}}$
$\therefore f'(x)=-4x^{-\frac{7}{3}}$

RULE 2

$$\boxed{D_x\left[f(x) \pm g(x) \pm ...\right]=D_x\left[f(x)\right] \pm D_x\left[g(x)\right] \pm ...}$$

Example: $D_x\left[2x^3+x^2-3x\right]=6x^2+2x-3$

EXAMPLE 9

Determine $\dfrac{dy}{dx}$ if

(a) $y=3x^2+2x-5$

(b) $y=x^3-\dfrac{5}{x^2}+\dfrac{2}{3\sqrt{x^3}}$

(c) $y=\pi^2x+\pi^3-\sqrt{5}$

Solution

(a) $y=3x^2+2x-5$
$\therefore \dfrac{dy}{dx}=6x+2$

(b) $y=x^3-\dfrac{5}{x^2}+\dfrac{2}{3\sqrt{x^3}}$
$\therefore y=x^3-5x^{-2}+\dfrac{2}{3}x^{-\frac{3}{2}}$
$\therefore \dfrac{dy}{dx}=3x^2+10x^{-3}-x^{-\frac{5}{2}}$

(c) $y=\pi^2x+\pi^3-\sqrt{5}$
$\therefore \dfrac{dy}{dx}=\pi^2+0-0=\pi^2$

EXERCISE 3

(a) Determine $f'(x)$ if

(1) $f(x)=x^3$	(2) $f(x)=2x^5$	(3) $f(x)=-6x$
(4) $f(x)=3$	(5) $f(x)=3x^{-2}$	(6) $f(x)=-2x^{\frac{3}{2}}$
(7) $f(x)=\frac{1}{4}x^2$	(8) $f(x)=\frac{x^3}{2}$	(9) $f(x)=\frac{3x^2}{4}$
(10) $f(x)=\frac{1}{x^3}$	(11) $f(x)=\frac{-6}{x}$	(12) $f(x)=\frac{5}{2x^2}$
(13) $f(x)=\frac{x^5}{x^3}$	(14) $f(x)=\frac{3x^4}{2x^7}$	(15) $f(x)=\sqrt[3]{x^2}$
(16) $f(x)=4\sqrt{x^3}$	(17) $f(x)=\frac{3}{\sqrt{x}}$	(18) $f(x)=\frac{-6x}{\sqrt[3]{x}}$

(b) Determine

(1) $f'(x)$ if $f(x)=-x^2+3x-1$	(2) $\frac{dy}{dx}$ if $y=4x^3-2x^2+5x+7$
(3) $D_x\left[x^{-2}-3x^{-1}+5\right]$	(4) $\frac{d}{dx}\left[x^{\frac{1}{2}}-5x^{-\frac{1}{2}}+3x^{\frac{2}{3}}\right]$
(5) $f'(u)$ if $f(u)=\frac{3}{u^2}+\sqrt{u}$	(6) $D_a\left[\sqrt[3]{a^5}+4\sqrt{a}-\frac{2}{a}\right]$
(7) $\frac{d}{dk}\left[\frac{4}{\sqrt{k}}-3k+\sqrt{5}\right]$	(8) $\frac{ds}{dt}$ if $s=\frac{3t^2}{4}+\frac{2}{3t}-\frac{9}{\sqrt[3]{t^2}}$
(9) $g'(x)$ if $g(x)=\sqrt{2x^3}+x^{\sqrt{2}}$	(10) $\frac{dy}{dx}$ if $y=\pi x^2+\pi^2x+\pi^3$

DIFFERENTIATING MORE COMPLEX EXPRESSIONS

Our two rules allow us to differentiate any expression of the form $ax^p \pm bx^q \pm cx^r \pm ...$ (i.e. only **sums** and/or **differences** of terms consisting of a **constant coefficient**, a variable and a **constant exponent**). More complex expressions must be simplified till they contain:

- NO Brackets
- NO Algebraic Fractions
- NO Roots of algebraic expressions

EXAMPLE 10

Determine $D_x\left[x(3x+2)(x-3)\right]$.

Solution

$$\begin{aligned}&D_x\left[x(3x+2)(x-3)\right]\\&=D_x\left[x(3x^2-7x-6)\right]\\&=D_x\left[3x^3-7x^2-6x\right]\\&=9x^2-14x-6\end{aligned}$$

EXAMPLE 11

Determine $f'(x)$ if

(a) $f(x)=\dfrac{x^4-3x}{x^3}$ (b) $f(x)=\dfrac{4x+\sqrt{x}}{x^2}$

(c) $f(x)=\dfrac{4x^2-9}{2x-3}$ (d) $f(x)=\dfrac{x^2-2x-3}{6-2x}$

Solution

(a)
$$\begin{aligned} f(x)&=\frac{x^4-3x}{x^3}\\ &=\frac{x^4}{x^3}-\frac{3x}{x^3}\\ &=x-3x^{-2}\\ \therefore f'(x)&=1+6x^{-3} \end{aligned}$$

(b)
$$\begin{aligned} f(x)&=\frac{4x+\sqrt{x}}{x^2}\\ &=\frac{4x}{x^2}+\frac{x^{\frac{1}{2}}}{x^2}\\ &=4x^{-1}+x^{-\frac{3}{2}}\\ \therefore f'(x)&=-4x^{-2}-\frac{3}{2}x^{-\frac{5}{2}} \end{aligned}$$

(c)
$$\begin{aligned} f(x)&=\frac{4x^2-9}{2x-3}\\ &=\frac{(2x-3)(2x+3)}{2x-3}\\ &=2x+3\\ \therefore f'(x)&=2 \end{aligned}$$

(d)
$$\begin{aligned} f(x)&=\frac{x^2-2x-3}{6-2x}\\ &=\frac{(x-3)(x+1)}{-2(x-3)}\\ &=\frac{x+1}{-2}\\ &=-\frac{1}{2}x-\frac{1}{2}\\ \therefore f'(x)&=-\frac{1}{2} \end{aligned}$$

LETTERS REPRESENTING CONSTANTS

When we find a derivative of an expression containing different letters, we differentiate **with respect to** the letter that represents the **variable**. This means that other letters are regarded as constants. To know which letter represents the variable that you have to differentiate with respect to, look for the letter that takes the place of $\boldsymbol{x}$ in the notation that is being used:

$$f'(\boldsymbol{x}) \qquad D_{\boldsymbol{x}} \qquad \frac{dy}{d\boldsymbol{x}}$$

EXAMPLE 12

Determine

(a) $f'(x)$ if $f(x)=ax^2+bx+c$ (a, b and c are constants.)

(b) $\dfrac{dv}{dt}$ if $v=u+at$ (u and a are constants.)

(c) $D_m\left[pm^3-q^2m^2+r^2\right]$ (p, q and r are constants.)

(d) $f'(n)$ if $f(n)=\sqrt{a}n^3-b\sqrt{n}$ (a and b are constants.)

Solution

(a) $f(x) = ax^2 + bx + c$

$\therefore f'(x) = 2ax + b$

(b) $v = u + at$

$\therefore \dfrac{dv}{dt} = 0 + a = a$

(c) $D_m\left[pm^3 - q^2m^2 + r^2\right]$

$= 3pm^2 - 2q^2m + 0$

$= 3pm^2 - 2q^2m$

(d) $f(n) = \sqrt{a}n^3 - b\sqrt{n}$

$= \sqrt{a}n^3 - bn^{\frac{1}{2}}$

$\therefore f'(x) = 3\sqrt{a}n^2 - \dfrac{1}{2}bn^{-\frac{1}{2}}$

EXERCISE 4

(a) Determine

(1) $D_x\left[(x-3)(x^2+1)\right]$

(2) $\dfrac{d}{dx}\left[-2x(x-1)^2\right]$

(3) $\dfrac{dy}{dx}$ if $y = \left(x + \dfrac{1}{x}\right)\left(x - \dfrac{1}{x}\right)$

(4) $f'(x)$ if $f(x) = \left(\sqrt{x} + \dfrac{1}{\sqrt{x}}\right)^2$

(b) Differentiate each of the following functions with respect to x:

(1) $f(x) = \dfrac{x^4 + x}{x^2}$

(2) $y = \dfrac{2x^3 - x}{3x^2}$

(3) $y = \dfrac{2x - \sqrt{x}}{\sqrt{x}}$

(4) $h(x) = \dfrac{3\sqrt{x} + \sqrt[3]{x}}{x}$

(5) $g(x) = \dfrac{\sqrt{x^3} - 3x}{2\sqrt{x}}$

(6) $f(x) = \dfrac{(2x+3)^2}{\sqrt[3]{x}}$

(7) $y = \dfrac{x^2 - 1}{x + 1}$

(8) $g(x) = \dfrac{x^3 - 4x}{2 - x}$

(9) $h(x) = \dfrac{-x^2 + 3x + 4}{2x + 2}$

(10) $y = \dfrac{x^3 + 8}{2 + x}$

(c) Determine

(1) $D_x\left[ax^3 + bx^2 + cx + d\right]$ (a, b, c and d are constants.)

(2) $\dfrac{d}{dx}\left[p\sqrt{x} + 5qx^2 + \dfrac{r}{x}\right]$ (p, q and r are constants.)

(3) $\dfrac{dy}{dx}$ if $y = a^2x^3 + 3bx - \dfrac{x}{c} + d^3$ (a, b, c and d are constants.)

(4) $D_k\left[mk^3 - n^3k + p^2\right]$ (m, n and p are constants.)

(5) $\dfrac{dA}{dr}$ if $A = 2\pi r^2 + 2\pi rh$ (h is a constant.)

(6) $s'(t)$ if $s = ut + \dfrac{1}{2}at^2$ (u and a are constants.)

(d)* Determine $\dfrac{dy}{dx}$ if

(1) $xy - x^3 = x^2$

(2) $3y - xy = x^4 - 9x^2$

(e)* If $y = \sqrt{x} + x$ and $x = t^3$, determine

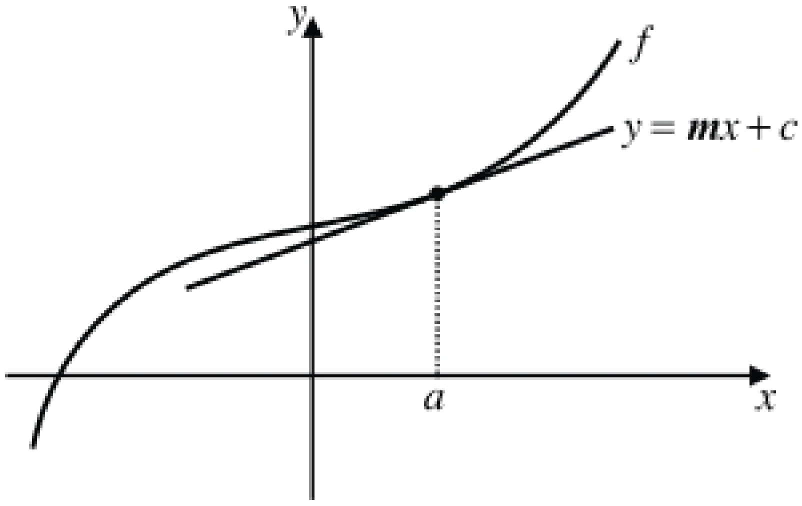
(1) $\dfrac{dy}{dx}$

(2) $\dfrac{dt}{dx}$

(3) $\dfrac{dy}{dt}$

TANGENTS

The **derivative** can be used to calculate the **gradient** of the tangent to a graph at any point on the graph:



$\boxed{m = f'(a)}$

EXAMPLE 13

Determine the gradient of the tangent to the graph of $f(x) = x^3 - 2x^2 + 3$ at $x = 2$.

Solution

$$\begin{aligned} f'(x) &= 3x^2 - 4x \\ m &= f'(2) \\ &= 3(2)^2 - 4(2) \\ &= 4 \end{aligned}$$

THE EQUATION OF A TANGENT

The equation of **any** straight line can be determined by using the following formula:

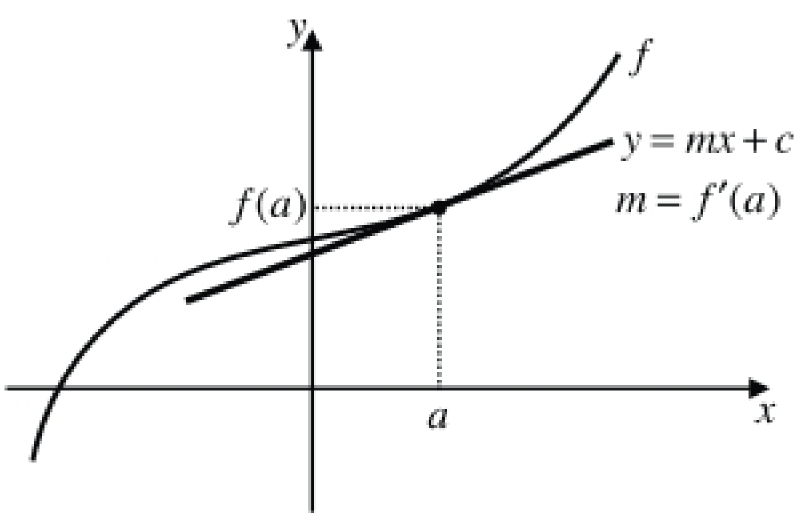
$$\boxed{y - y_1 = m(x - x_1)}$$

In this formula

- m is the gradient of the line.
- $(x_1\,;\,y_1)$ are the coordinates of any point on the line.

This formula can be used to determine the equation of the tangent to a graph at any point on the graph. If an x-value of a point on the graph is given, we can find

- the y-value, by using the original function $f(x)$: $y = f(x\text{-value})$
- the gradient of the tangent, by using the derivative $f'(x)$: $m = f'(x\text{-value})$



EXAMPLE 14

Determine the equation of the tangent to the graph of $f(x)=x^2-x$ at $x=3$.

Solution

- Calculate the y-value, by using the original function $f(x)$:

  $f(x)=x^2-x$

  $\therefore y=f(3)$

  $=3^2-3$

  $=6$

  $\therefore$ The coordinates of the point of contact is $(3; 6)$.

- Determine the gradient of the tangent, by using the derivative $f'(x)$:

  $f'(x)=2x-1$

  $m=f'(3)$

  $=2(3)-1$

  $=5$

- Determine the equation of the tangent, by using the formula $y-y_1=m(x-x_1)$:

  $y-6=5(x-3)$

  $\therefore y-6=5x-15$

  $\therefore y=5x-9$

FINDING THE POINT OF CONTACT

The point where a tangent touches a graph is called the *point of contact* (or *point of tangency*). If the gradient of a tangent to a graph is given, we can determine the x-value of the point of contact by **equating** the **derivative** to the given **gradient** and then solving for x:

EXAMPLE 15

The graph of $f(x)=x-2x^2$ has a tangent with a gradient of 9.

(a) Calculate the coordinates of the point of contact between the tangent and the graph.

(b) Determine the equation of the tangent.

Solution

(a) $f(x)=x-2x^2$

$\therefore f'(x)=1-4x$

$f'(x)=9$

$\therefore 1-4x=9$

$\therefore -4x=8$

$\therefore x=-2$

$y=f(-2)$

$=(-2)-2(-2)^2$

$=-10$

$\therefore$ The point of contact is $(-2;-10)$.

(b) $y - y_1 = m(x - x_1)$

$y + 10 = 9(x + 2)$

$\therefore y + 10 = 9x + 18$

$\therefore y = 9x + 8$

EXAMPLE 16

The graph of $f(x) = x^3 + 3x^2 - 4x$ has two tangents that are parallel to the line $y - 5x + 7 = 0$. Determine the equations of both these tangents.

Solution

The equation of the line $y - 5x + 7 = 0$ can be rewritten as $y = 5x - 7$.
$\therefore$ The gradient of the tangents must be 5.

$f'(x) = 5$

$\therefore 3x^2 + 6x - 4 = 5$

$\therefore 3x^2 + 6x - 9 = 0$

$\therefore x^2 + 2x - 3 = 0$

$\therefore (x + 3)(x - 1) = 0$

$\therefore x = -3$ or $x = 1$

For $x = -3$:

$y = f(-3)$

$= (-3)^3 + 3(-3)^2 - 4(-3)$

$= 12$

$\therefore$ The point of contact is $(-3; 12)$.

Equation of the tangent:

$y - 12 = 5(x + 3)$

$\therefore y - 12 = 5x + 15$

$\therefore y = 5x + 27$

For $x = 1$:

$y = f(1)$

$= (1)^3 + 3(1)^2 - 4(1)$

$= 0$

$\therefore$ The point of contact is $(1; 0)$.

Equation of the tangent:

$y - 0 = 5(x - 1)$

$\therefore y = 5x - 5$

The equations of the two tangents are $y = 5x + 27$ and $y = 5x - 5$.

EXERCISE 5

(a) (1) Determine the gradient of the tangent to the graph of $f(x) = x^2 + x$ at $x = 1$.

(2) Determine the gradient of the tangent to the graph of $y = 4x - x^3$ at $x = -1$.

(b) (1) Determine the equation of the tangent to the graph of $f(x) = -x^2 + 5x$ at $x = 3$.

(2) Determine the equation of the tangent to the graph of $g(x) = -\dfrac{12}{x}$ at $x = -2$.

(c) Calculate the coordinates of the point of contact between the graph of $f(x) = 2x^2 - 3x$ and its tangent that has a gradient of 5.

(d) Determine the equation of the tangent to the graph of $f(x)=x^2+2x$ that has a gradient of 4.

(e) The graph of $y=x^3-6x^2+7$ has two tangents with a gradient of -9. Determine the equations of both these tangents.

(f) (1) Determine the equation of the tangent to the graph of $f(x)=-2x^2+5$, parallel to the line $y=8x+7$.

(2) Determine the equation of the tangent to the graph of $h(x)=\dfrac{4}{x^2}$, perpendicular to the line $x+8y=12$.

(g)* The gradient of the tangent to the graph of $f(x)=x^3+ax^2$ at $x=2$ is 4. Determine the value of *a*.

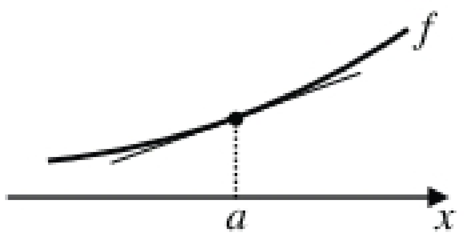
(h)* A line with a gradient of -3 is a tangent to the graph of $y=ax^2+bx+3$, at the point $(2;5)$. Determine the values of *a* and *b*.

(i)* The equation of the tangent to the parabola defined by $y=px^2+qx$ at $x=3$ is $y=15x-18$. Determine the values of *p* and *q*.

GRAPHICAL INTERPRETATION OF THE DERIVATIVE

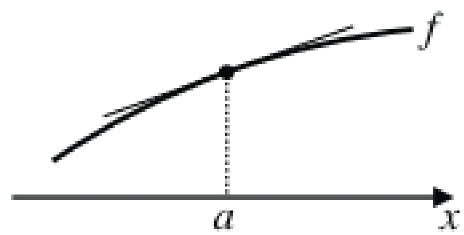
We can use the value of the derivative at a specific point on a function to establish whether the function is *increasing* or *decreasing* at that point:

- If $f'(a)>0$, then f is **increasing** at $x=a$:



$f'(a)>0$

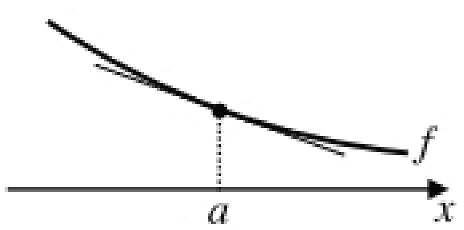
f is increasing at $x=a$



$f'(a)>0$

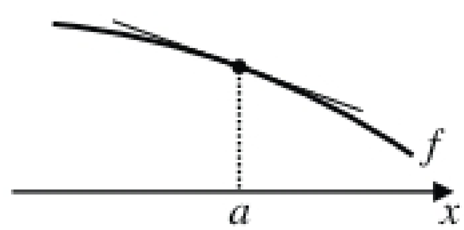
f is increasing at $x=a$

- If $f'(a)<0$, then f is **decreasing** at $x=a$:



$f'(a)<0$

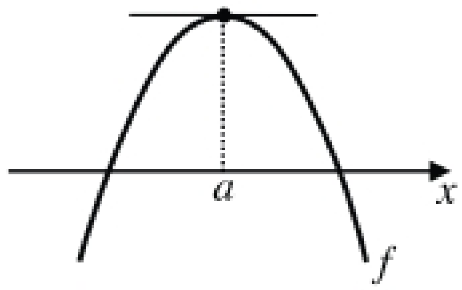
f is decreasing at $x=a$



$f'(a)<0$

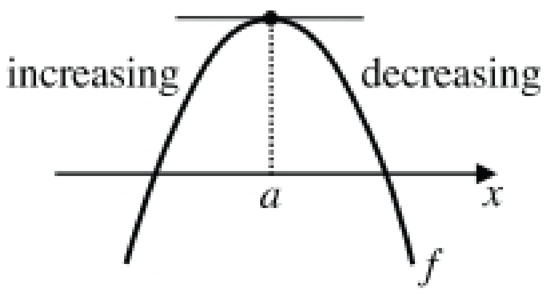
f is decreasing at $x=a$

- If $f'(a) = 0$, then f has a ***stationary point*** at $x = a$.

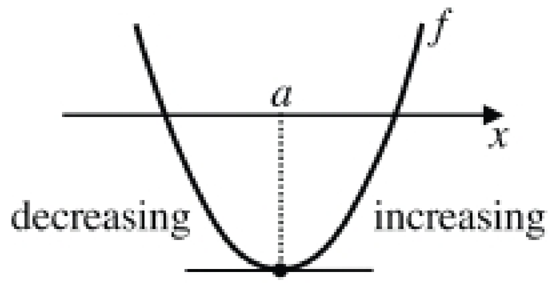


$f'(a) = 0$
f has a stationary point at $x = a$

- ✦ If the function is **increasing before** the stationary point and **decreasing after** the stationary point (or *vice versa*), then f is said to be **<u>both</u> increasing and decreasing** at the stationary point:

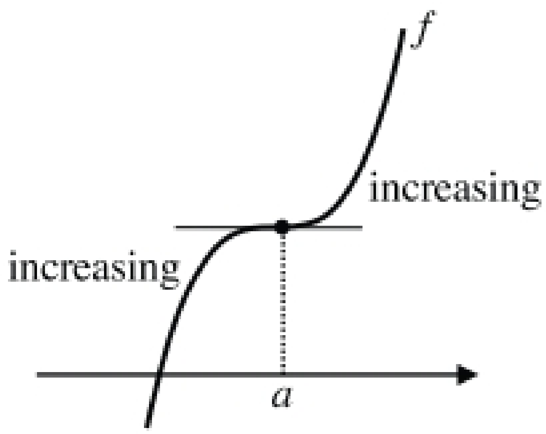


f is increasing and decreasing at $x = a$



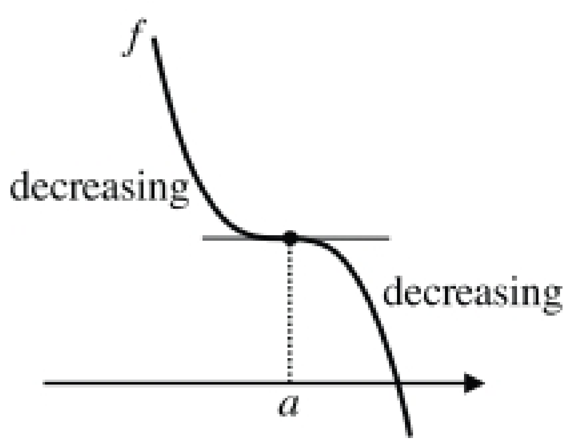
f is increasing and decreasing at $x = a$

- ✦ If the graph is **increasing before and after** the stationary point, then f is **increasing** at the stationary point:



f is increasing at $x = a$

- ✦ If the graph is **decreasing before and after** the stationary point, then f is **decreasing** at the stationary point:

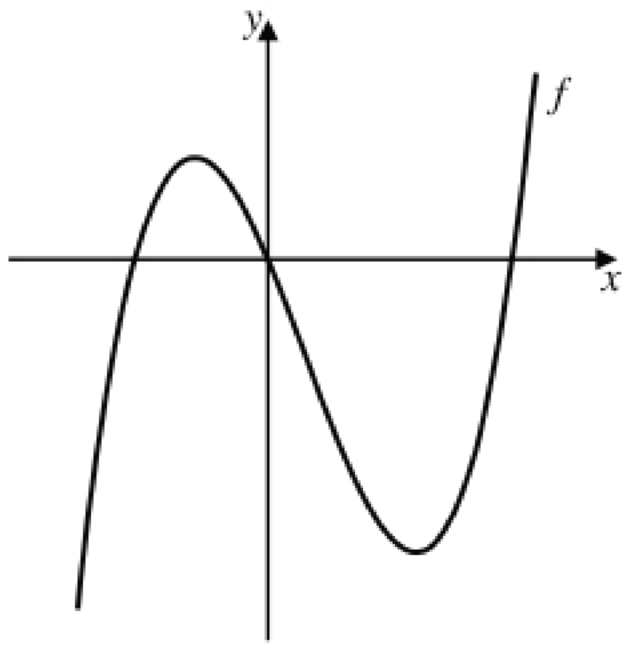


f is decreasing at $x = a$

EXAMPLE 17

The graph of $f(x)=2x^3-3x^2-12x$ is shown alongside.
Determine, by calculation, whether f is increasing or decreasing at

(a) $x=-2$
(b) $x=0$
(c) $x=-1$
(d) $x=3$
(e) $x=2$

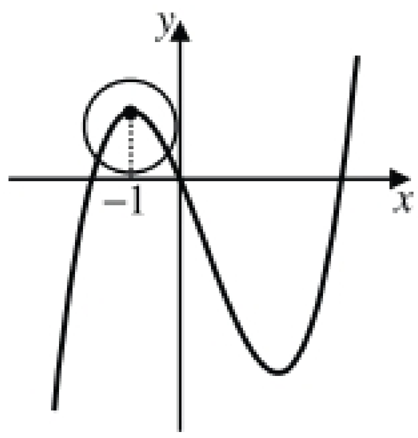


Solution

(a) $f'(x)=6x^2-6x-12$
$\therefore f'(-2)=6(-2)^2-6(-2)-12$
$=24>0$
$\therefore f$ is increasing at $x=-2$.

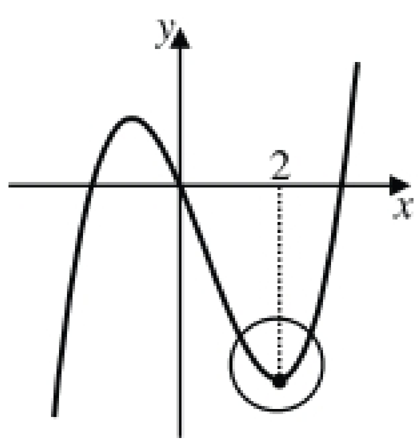
(b) $f'(0)=6(0)^2-6(0)-12$
$=-12<0$
$\therefore f$ is decreasing at $x=0$.

(c) $f'(-1)=6(-1)^2-6(-1)-12$
$=0$
$\therefore f$ has a stationary point at $x=-1$.
f is increasing before the stationary point and decreasing after the stationary point.
$\therefore f$ is both increasing and decreasing at $x=-1$.



(d) $f'(3)=6(3)^2-6(3)-12$
$=24>0$
$\therefore f$ is increasing at $x=3$.

(e) $f'(2)=6(2)^2-6(2)-12$
$=0$
$\therefore f$ has a stationary point at $x=2$.
f is decreasing before the stationary point and increasing after the stationary point.
$\therefore f$ is both increasing and decreasing at $x=2$.



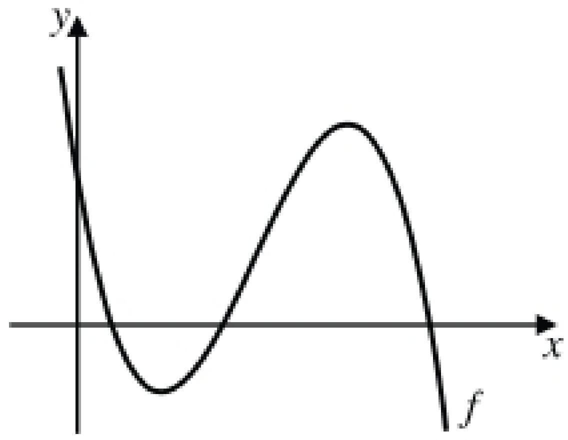
To calculate the coordinates of the **stationary point(s)** of a function f:

- Let $f'(x)=0$ and solve for x.
- Use the original function $f(x)$ to determine the y-value.

For stationary point(s): $f'(x)=0$

EXAMPLE 18

The sketch alongside shows the graph of $f(x) = -x^3 + 6x^2 - 9x + 3$.



(a) Calculate the coordinates of the stationary points of f.
(b) For which values of x is f
  (1) increasing?
  (2) decreasing?

Solution

(a) $f'(x) = 0$

$\therefore -3x^2 + 12x - 9 = 0$

$\therefore x^2 - 4x + 3 = 0$

$\therefore (x-1)(x-3) = 0$

$\therefore x = 1 \text{ or } x = 3$

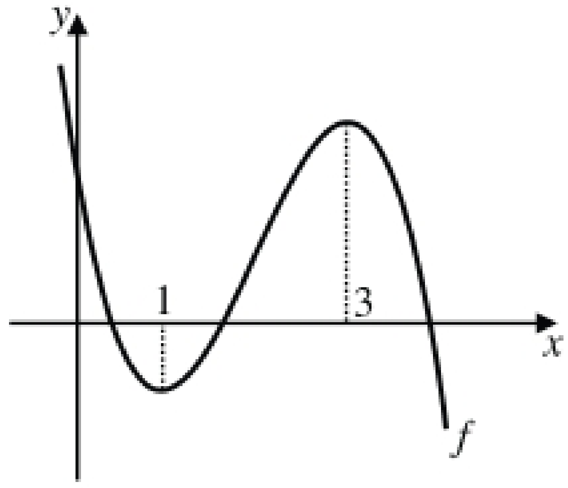
When $x = 1$, $y = f(1) = -1^3 + 6(1)^2 - 9(1) + 3 = -1 \rightarrow (1;-1)$

When $x = 3$, $y = f(1) = -3^3 + 6(3)^2 - 9(3) + 3 = -1 \rightarrow (3;3)$

$\therefore$ The stationary points of f are $(1;-1)$ and $(3;3)$.

(b) (1) $1 \leq x \leq 3$

(2) $x \leq 1$ or $x \geq 3$



Remember to include the stationary points in these intervals, since this function is both increasing and decreasing at its stationary points.

THE SECOND DERIVATIVE

The *second derivative* of a function f is written as f'' and is obtained by differentiating the function twice. (The second derivative is the derivative of the derivative.)

EXAMPLE 19

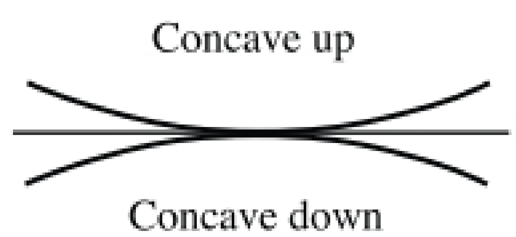
Find the second derivative of $f(x) = x^3 - 5x^2 + 3x$.

Solution

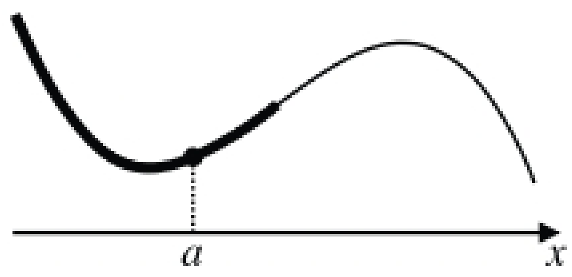
$f'(x) = 3x^2 - 10x + 3$
$f''(x) = 6x - 10$

GRAPHICAL INTERPRETATION OF THE SECOND DERIVATIVE

The second derivative is a measure of the *concavity* of a function. If the graph of a function curves upwards, the function is said to be *concave up*. If the graph of a function curves downwards, the function is said to be *concave down*.



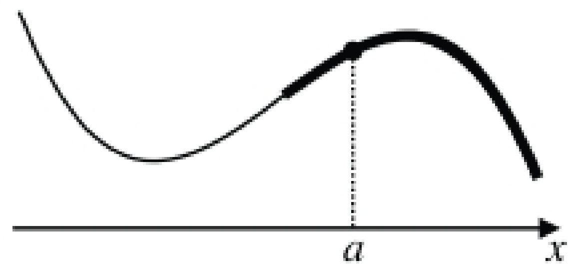
- If $f''(a) > 0$, then f is **concave up** at $x = a$:



$f''(a) > 0$

f is concave up at $x = a$

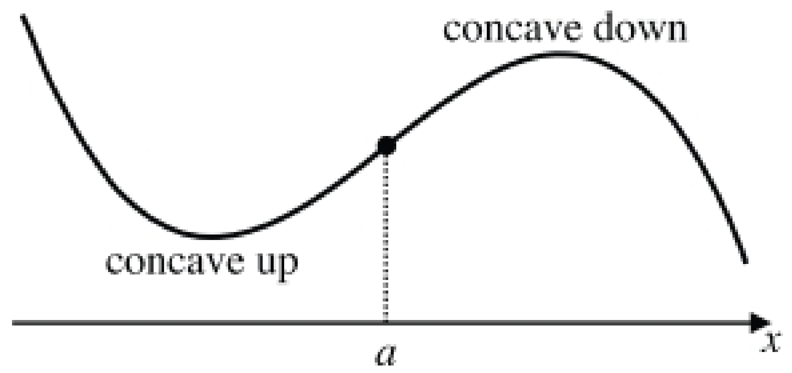
- If $f''(a) < 0$, then f is **concave down** at $x = a$:



$f''(a) < 0$

f is concave down at $x = a$

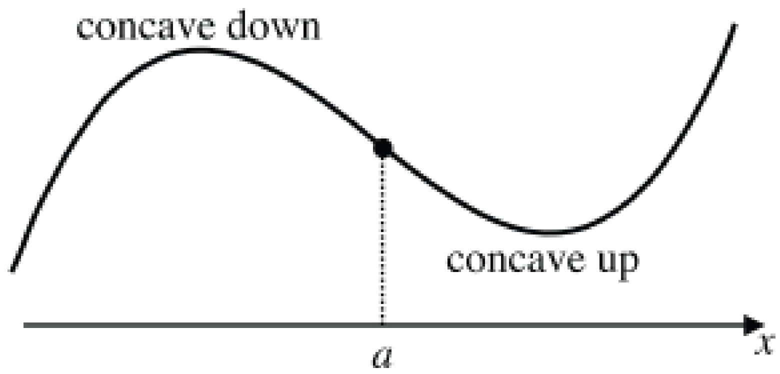
- A point where a function **changes from concave up to concave down** (or *vice versa*) is called a ***point of inflection***. If a function has a point of inflection at $x = a$, then $f''(a) = 0$.



f changes from concave up to concave down at $x = a$

$f''(a) = 0$

f has a point of inflection at $x = a$



f changes from concave down to concave up at $x = a$

$f''(a) = 0$

f has a point of inflection at $x = a$

Note: The fact that the second derivative equals 0 at a point doesn't always guarantee that the point is a point of inflection. To verify that a point is in fact a point of inflection of a function, we have to check that the function is concave up before the point and concave down after the point (or *vice versa*).

EXAMPLE 20

The graph of $f(x) = -x^3 - 6x^2 + x$ is shown alongside.

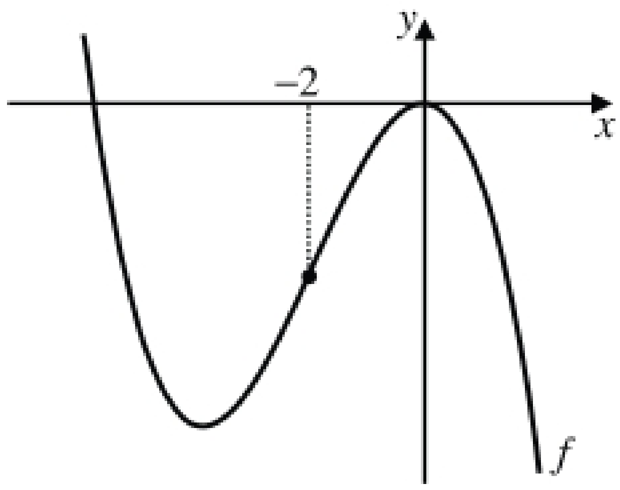
(a) Determine, by calculation, whether f is concave up or concave down at

(1) $x = -3$

(2) $x = -1$

(3) $x = 0$

(b) Show that f has a point of inflection at $x = -2$.



Solution

(a) (1)

$$f'(x) = -3x^2 - 12x + 1$$
$$f''(x) = -6x - 12$$
$$\therefore f''(-3) = -6(-3) - 12$$
$$= 6 > 0$$

$\therefore f$ is concave up at $x = -3$.

(2)

$$f''(-1) = -6(-1) - 12$$
$$= -6 < 0$$

$\therefore f$ is concave down at $x = -1$.

(3)

$$f''(0) = -6(0) - 12$$
$$= -12 < 0$$

$\therefore f$ is concave down at $x = 0$.

(b)

$$f''(0) = -6(-2) - 12$$
$$= 0$$

f is concave up before $x = -2$ and concave down after $x = -2$.

$\therefore f$ has a point of inflection at $x = -2$.

To calculate the coordinates of the **point(s) of inflection** of a function f:

- Let $f''(x) = 0$ and solve for x.
- Use the original function $f(x)$ to determine the y-value.

For point(s) of inflection:
$f''(x) = 0$

EXAMPLE 21

Determine the coordinates of the point of inflection of $f(x) = x^3 - 9x^2 + 3x - 5$.

Solution

$$f'(x) = 3x^2 - 18x + 3$$
$$f''(x) = 6x - 18$$

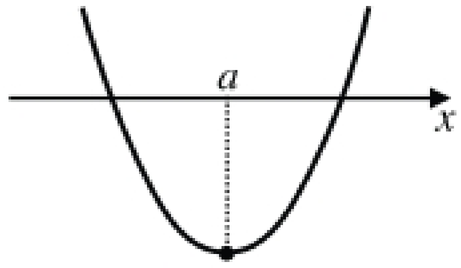
$$f''(x) = 0$$
$$\therefore 6x - 18 = 0$$
$$\therefore x = 3$$
$$\therefore y = f(3) = 3^3 - 9(3)^2 + 3(3) - 5 = -50 \rightarrow (3; -50)$$

$\therefore$ The point of inflection of f is $(3; -50)$.

CLASSIFICATION OF STATIONARY POINTS

Local minima and maxima

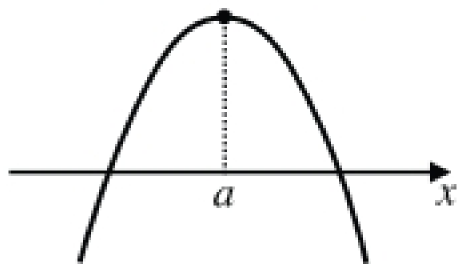
- If $f'(a)=0$ and $f''(a)>0$, then f has a ***local minimum*** at $x=a$.



Local Minimum

- If $f'(a)=0$ and $f''(a)<0$, then f has a ***local maximum*** at $x=a$.

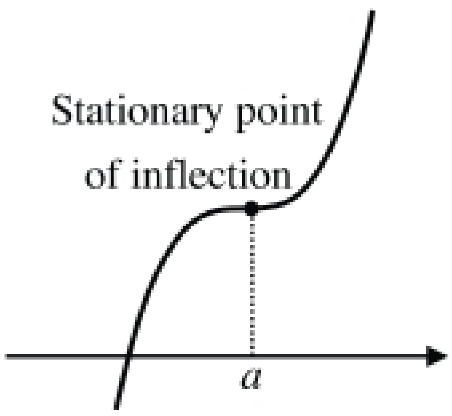
Local Maximum

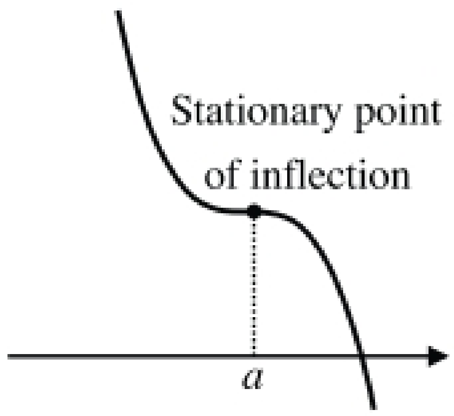


Stationary points of inflection

A stationary point that is also a point of inflection is neither a local minimum nor a local maximum. Such a stationary point is called a stationary point of inflection.

If a function has a stationary point of inflection at $x=a$, then $f'(a)=0$ and $f''(a)=0$.





EXAMPLE 22

The points $(1;-1)$ and $(3;3)$ are stationary points of the function $f(x)=-x^3+6x^2-9x+3$. Classify each of these stationary points as a local minimum, local maximum or stationary point of inflection.

Solution

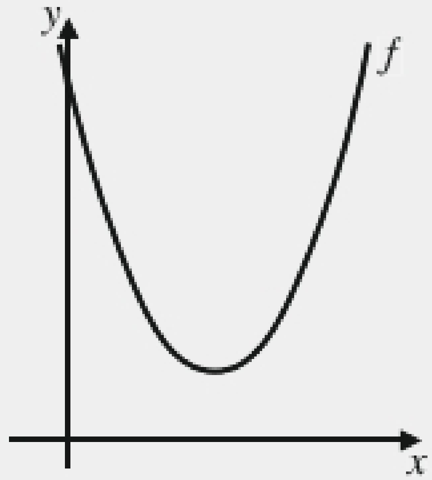
$f'(x)=-3x^2+12x-9$
$f''(x)=-6x+12$

$f''(1)=-6(1)+12=6>0$
$\therefore f$ has a local minimum at $x=1$.

$f''(3)=-6(3)+12=-6<0$
$\therefore f$ has a local maximum at $x=3$.

EXERCISE 6

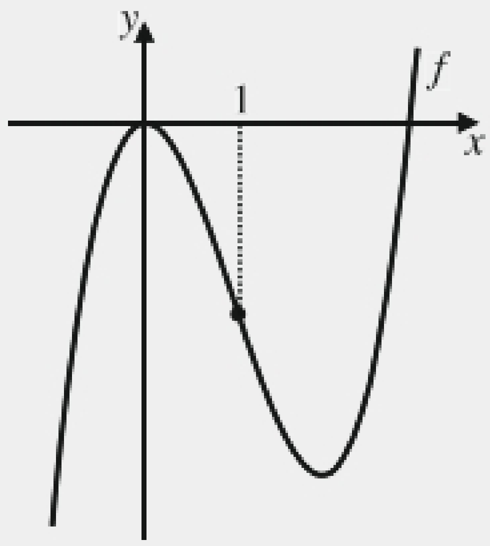
(a) The graph of $f(x) = x^2 - 4x + 5$ is shown alongside.



(1) Determine, by calculation, whether f is increasing or decreasing at
  (i) $x = 1$
  (ii) $x = 3$
  (iii) $x = 2$

(2) Determine $f''(x)$ and comment on the meaning of the result.

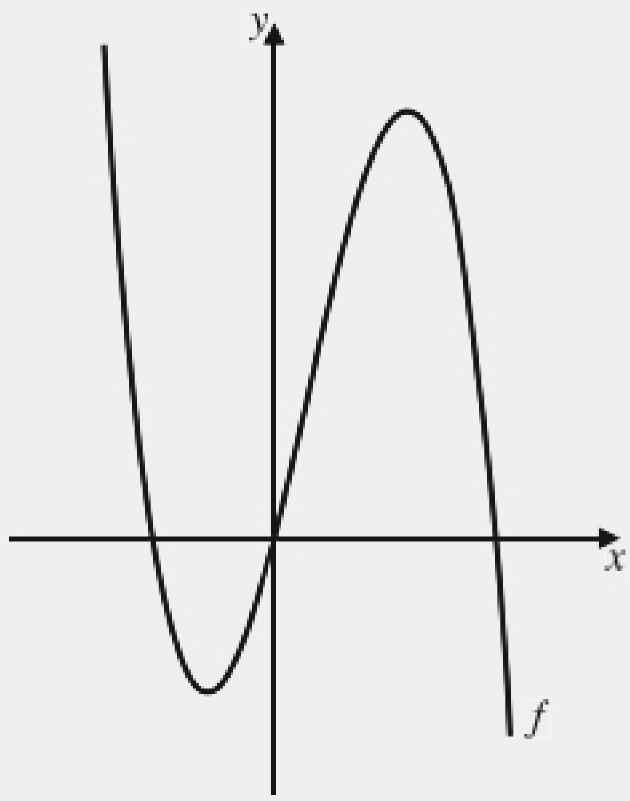
(b) The graph of $f(x) = x^3 - 3x^2$ is shown alongside.



(1) Determine, by calculation, whether f is concave up or concave down at
  (i) $x = -1$
  (ii) $x = \dfrac{1}{2}$
  (ii) $x = 2$

(2) Show that f has a point of inflection at $x = 1$.

(c) The sketch alongside shows the graph of $f(x) = -2x^3 + 3x^2 + 12x$.



(1) Calculate the coordinates of the stationary points of f.

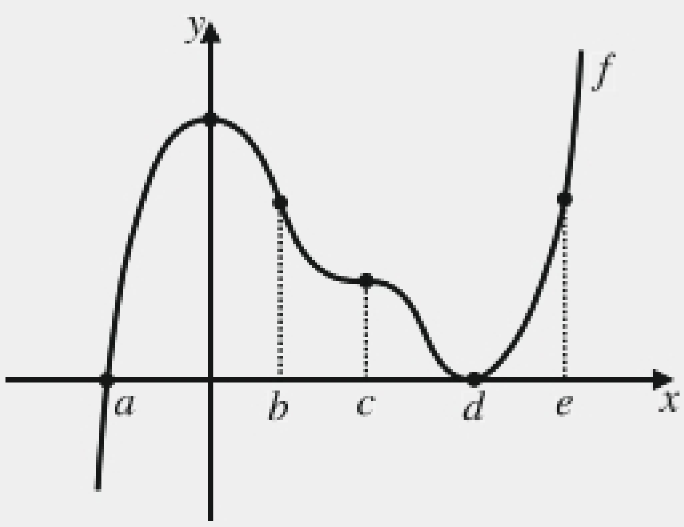
(2) Classify each of the stationary points that you have calculated in (1) as a local minimum, local maximum or stationary point of inflection.

(3) Calculate the coordinates of the point of inflection of f.

(4) For which values of x is f
  (i) increasing?
  (ii) decreasing?

(d) Find and classify the stationary point(s) of $y = x^3 + 3x^2 + 3x + 5$.

(e) The sketch alongside shows the graph of $y = f(x)$. State whether each of the following is positive, negative or zero:



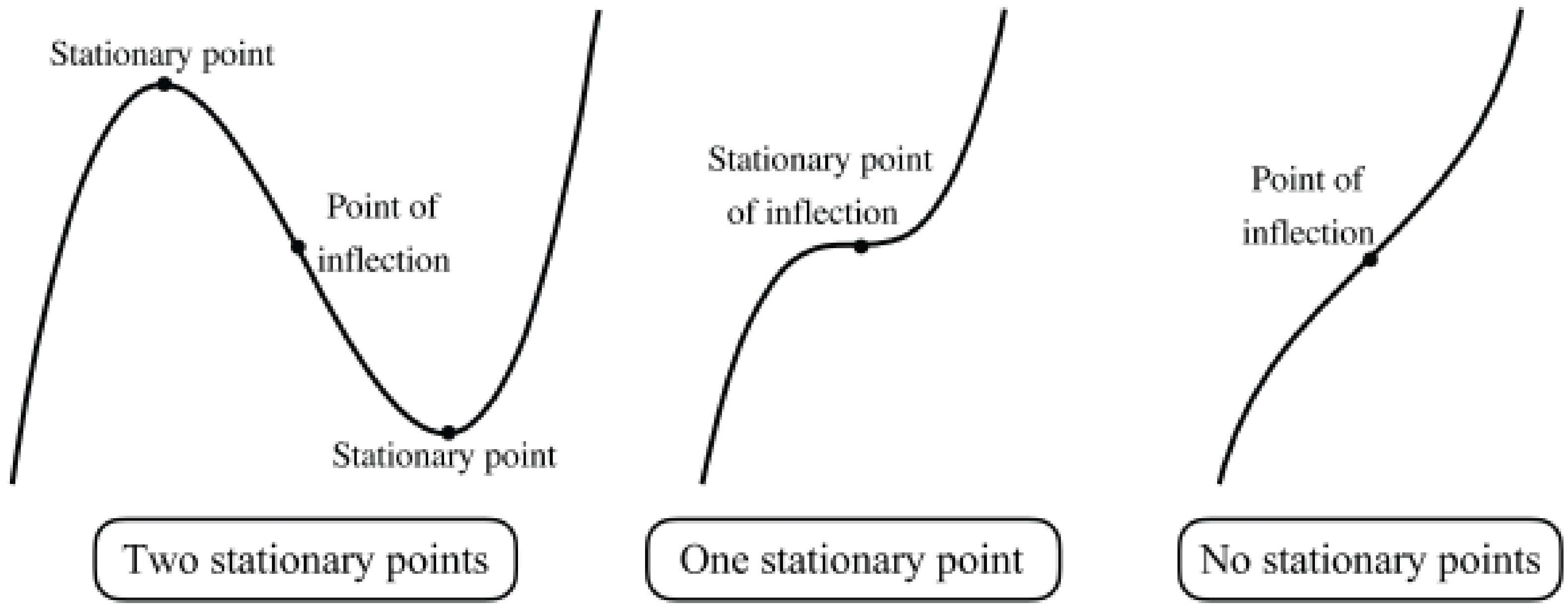
(1)	$f'(a)$	(2)	$f''(a)$
(3)	$f'(0)$	(4)	$f''(0)$
(5)	$f'(b)$	(6)	$f''(b)$
(7)	$f'(c)$	(8)	$f''(c)$
(9)	$f'(d)$	(10)	$f''(d)$
(11)	$f'(e)$	(12)	$f''(e)$

CUBIC FUNCTIONS

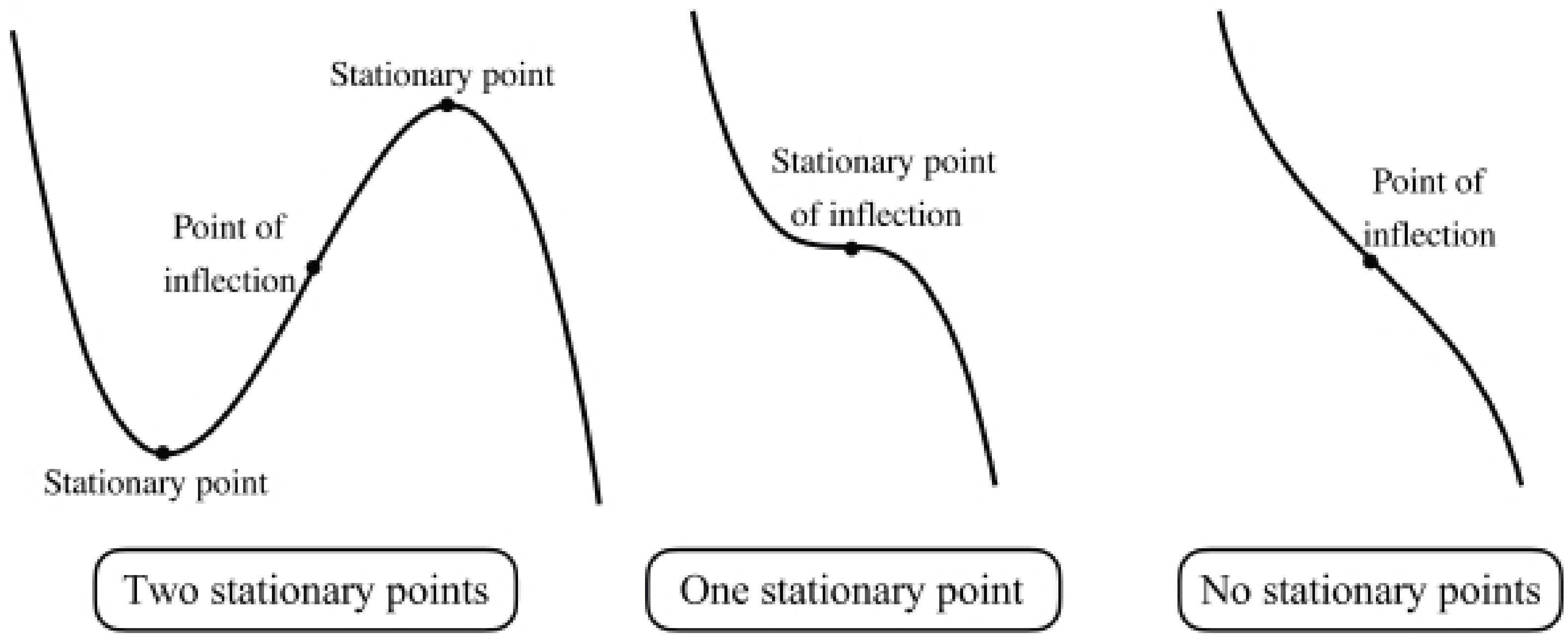
A cubic function is a function of the form $f(x) = ax^3 + bx^2 + cx + d$.

A cubic function can have **up to two stationary points**. All cubic graphs have **one point of inflection**.

- If $a > 0$, the possible shapes are :



- If $a < 0$, the possible shapes are :



SKETCHING THE GRAPH OF A CUBIC FUNCTION

When sketching the graph of a cubic function, the following details have to be shown on the graph:

- x-intercepts
- y-intercept
- stationary point(s)
- the point of inflection

Graphs don't have to be drawn to scale, but the **shape** must be clear. It is advisable to start by calculating the stationary points, since the number of stationary points determines the shape of the graph.

EXAMPLE 23

Sketch the graph of $f(x)=x^3-6x^2$.

Solution

$f(x)=x^3-6x^2 \qquad f'(x)=3x^2-12x \qquad f''(x)=6x-12$

Stationary points:

$f'(x)=0$

$\therefore 3x^2-12x=0$

$\therefore x^2-4x=0$

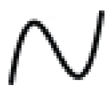
$\therefore x(x-4)=0$

$\therefore x=0$ or $x=4$

When $x=0$, $y=f(0)=0^3-6(0)^2=0 \rightarrow (0;0)$

When $x=4$, $y=f(4)=4^3-6(4)^2=-32 \rightarrow (4;-32)$

Shape:

Two stationary points and $a>0$: 

Point of inflection:

$f''(x)=0$

$\therefore 6x-12=0$

$\therefore x=2$

$\therefore y=f(2)=2^3-6(2)^2=-16 \rightarrow (2;-16)$

***x*-intercepts:**

$f(x)=0$

$\therefore x^3-6x^2=0$

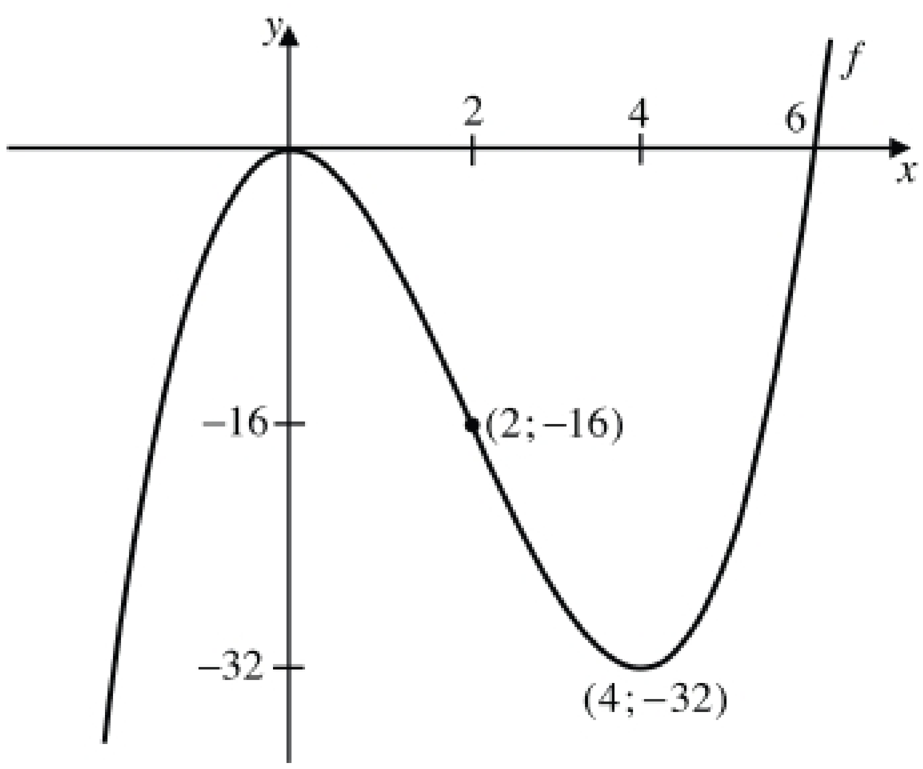
$\therefore x^2(x-6)=0$

$\therefore x=0$ or $x=6$

***y*-intercept:**

$x=0$

$y=f(0)=0^3-6(0)^2=0$



EXAMPLE 24

Sketch the graph of $y = -x^3 + x^2 + 5x + 3$.

Solution

$y = -x^3 + x^2 + 5x + 3 \qquad y' = -3x^2 + 2x + 5 \qquad y'' = -6x + 2$

Stationary points:

$y' = 0$

$\therefore -3x^2 + 2x + 5 = 0$

$\therefore 3x^2 - 2x - 5 = 0$

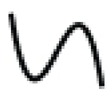
$\therefore (3x - 5)(x + 1) = 0$

$\therefore x = \dfrac{5}{3}$ or $x = -1$

When $x = \frac{5}{3}$, $y = -\left(\frac{5}{3}\right)^3 + \left(\frac{5}{3}\right)^2 + 5\left(\frac{5}{3}\right) + 3 = \frac{256}{27} \rightarrow \left(\frac{5}{3}; \frac{256}{27}\right)$

When $x = -1$, $y = -(-1)^3 + (-1)^2 + 5(-1) + 3 = 0 \rightarrow (-1; 0)$

Shape:

Two stationary points and $a < 0$: 

Point of inflection:

$y'' = 0$

$\therefore -6x + 2 = 0$

$\therefore x = \frac{1}{3}$

$\therefore y = -\left(\frac{1}{3}\right)^3 + \left(\frac{1}{3}\right)^2 + 5\left(\frac{1}{3}\right) + 3 = \frac{1}{3} \rightarrow \left(\frac{1}{3}; \frac{128}{27}\right)$

x-intercepts:

$y = 0$

$\therefore -x^3 + x^2 + 5x + 3 = 0$

$\therefore x^3 - x^2 - 5x - 3 = 0$

$\therefore (x + 1)(x^2 - 2x - 3) = 0$

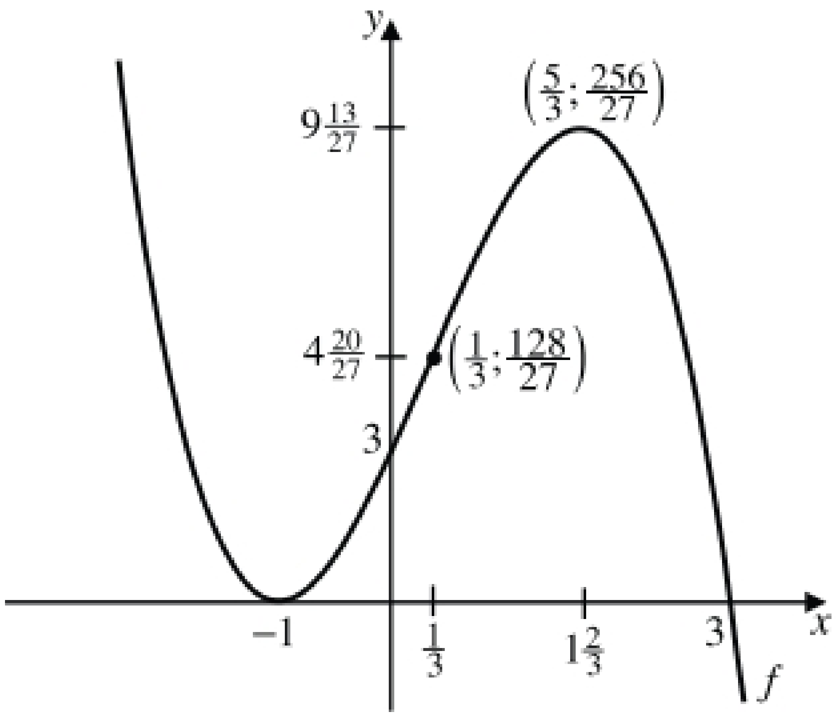
$\therefore (x + 1)(x + 1)(x - 3) = 0$

$\therefore x = -1$ or $x = 3$

y-intercept:

$x = 0$

$y = 0^3 - 0^2 + 5(0) + 3 = 3$



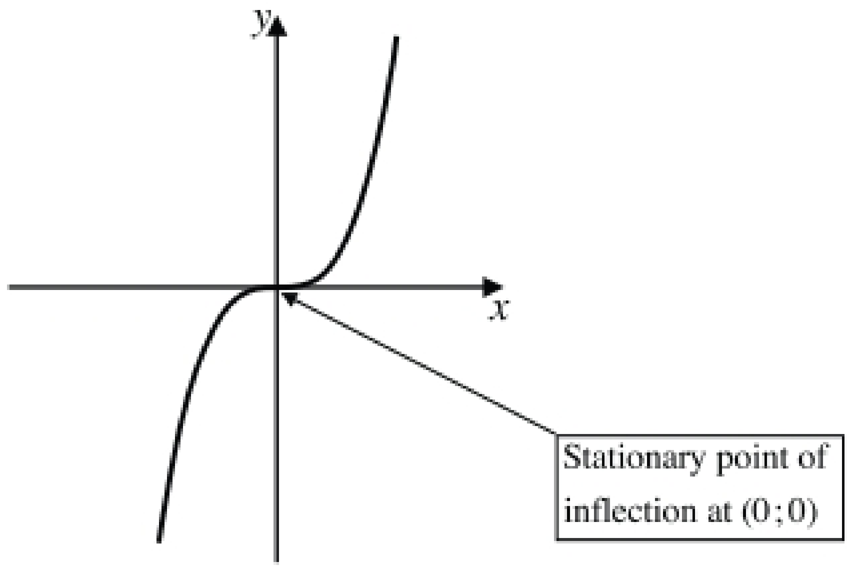
EXERCISE 7

Sketch the graph of each of the following functions, showing
- all intercepts with axes;
- stationary points;
- the point of inflection.

(a)	$f(x) = x^3 - 3x^2$	(b)	$g(x) = -x^3 + 6x^2 - 9x$
(c)	$h(x) = x^3 - 2x^2 - 15x$	(d)	$y = -x^3 + 3x^2 - 4$
(e)	$y = x^3 - 3x - 2$	(f)	$g(x) = -x^3 - 5x^2 - 3x + 9$
(g)	$f(x) = x^3 - 8x^2 + 5x + 14$	(h)	$h(x) = 2x^3 + 5x^2 - 4x - 3$

CUBIC FUNCTIONS WITH ONLY ONE STATIONARY POINT

The cubic function $y = x^3$ has only one stationary point at $(0;0)$, which is also a point of inflection:



Graphs of functions of the form $y = a(x+p)^3 + q$ can be drawn by applying the standard transformations to the graph of $y = x^3$:

EXAMPLE 25

Sketch the graph of $y = (x+2)^3 - 1$.

Solution

$p = 2$ and $q = -1$

Translate the graph of $y = x^3$, 2 units left and 1 unit down.

$\therefore$ The stationary point of inflection will be at $(-2;-1)$.

x-intercepts:

$y = 0$

$\therefore (x+2)^3 - 1 = 0$

$\therefore (x+2)^3 = 1$

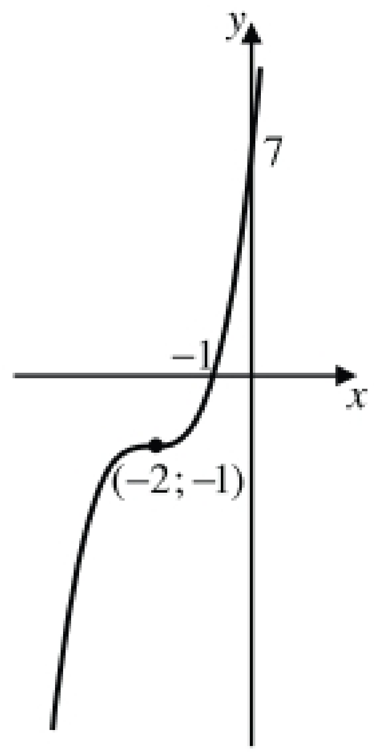
$\therefore x + 2 = 1$

$\therefore x = -1$

y-intercept:

$x = 0$

$y = (0+2)^3 - 1 = 7$



EXAMPLE 26

Sketch the graph of $f(x) = -x^3 + 3x^2 - 3x + 9$.

Solution

$f(x) = -x^3 + 3x^2 - 3x + 9 \qquad f'(x) = -3x^2 + 6x - 3 \qquad f''(x) = -6x + 6$

Stationary points:

$f'(x) = 0$

$\therefore -3x^2 + 6x - 3 = 0$

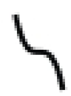
$\therefore x^2 - 2x + 1 = 0$

$\therefore (x-1)^2 = 0$

$\therefore x = 1$

When $x = 1$, $y = f(1) = -1^3 + 3(1)^2 - 3(1) + 9 = \; \rightarrow \; (1;8)$

Shape:

One stationary point and $a < 0$: 

Point of inflection:

$f''(x) = 0$

$\therefore -6x + 6 = 0$

$\therefore x = 1$

$\therefore y = 8 \rightarrow (1;8)$

***x*-intercepts:**

$f(x) = 0$

$\therefore -x^3 + 3x^2 - 3x + 9 = 0$

$\therefore x^3 - 3x^2 + 3x - 9 = 0$

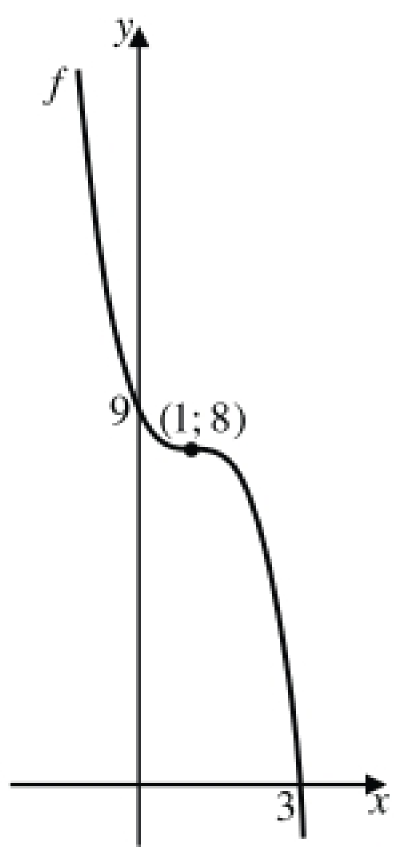
$\therefore (x-3)(x^2+3) = 0$

$\therefore x = 3$

***y*-intercept:**

$x = 0$

$y = f(0) = -0^3 + 3(0)^2 - 3(0) + 9 = 9$



CUBIC FUNCTIONS WITH NO STATIONARY POINTS

It is also possible for a cubic function to have no stationary points. An example of such a function is

EXAMPLE 27

Sketch the graph of $g(x) = -x^3 + 3x^2 - 9x$.

Solution

$g(x) = -x^3 + 3x^2 - 9x \qquad g'(x) = -3x^2 + 6x - 9 \qquad g''(x) = -6x + 6$

Stationary points:

$g'(x) = 0$

$\therefore -3x^2 + 6x - 9 = 0$

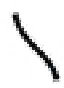
$\therefore x^2 - 2x + 3 = 0$

$$\therefore x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$\therefore x = \frac{2 \pm \sqrt{-8}}{2} \quad \text{(Non-real)}$$

$\therefore$ The function has no stationary points.

Shape:

No stationary points and $a < 0$: 

Point of inflection:

$g''(x) = 0$

$\therefore -6x + 6 = 0$

$\therefore x = 1 \qquad \therefore y = g(1) = -1^3 + 3(1)^2 - 9(1) = -7 \rightarrow (1; -7)$

***x*-intercepts:**

$g(x) = 0$

$\therefore -x^3 + 3x^2 - 9x = 0$

$\therefore -x(x^2 - 3x - 9) = 0$

$$\therefore x = 0 \text{ or } x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-9)}}{2(1)}$$

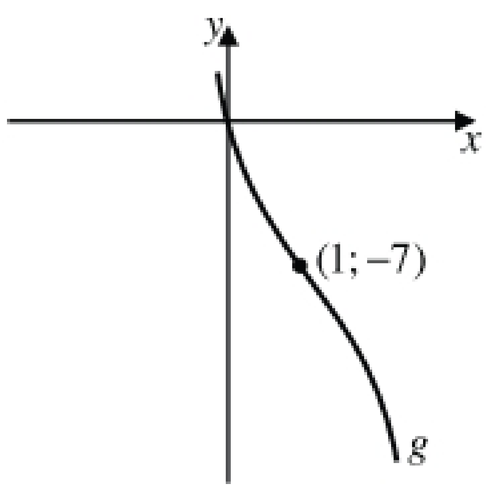
(Non-real)

$\therefore x = 0$

***y*-intercept:**

$x = 0$

$y = g(0) = -0^3 + 3(0)^2 - 9(0) = 0$



EXERCISE 8

(a) Sketch the graphs of the following functions, showing all relevant details:

(1) $y = 2x^3$ (2) $f(x) = -x^3 + 8$

(3) $g(x) = (x+1)^3$ (4) $y = (x-1)^3 - 8$

(5) $h(x) = -3(x+2)^3 - 3$

(b) Given the function $f(x) = -x^3 + 6x^2 - 12x + 7$

(1) Calculate the coordinates of the stationary point(s) of *f*.

(2) Write down the coordinates of the point of inflection of *f*.

(3) Sketch the graph of *f*, showing all relevant details.

(c) Given the function $g(x) = x^3 + 3x^2 + 6x + 8$

(1) Show that *g* has no stationary points.

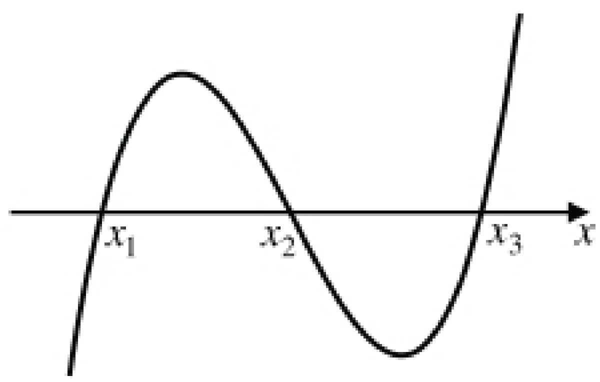
(2) Determine the coordinates of the point of inflection of *g*.

(3) Sketch the graph of *g*, showing all relevant details.

FINDING THE EQUATION OF A CUBIC FUNCTION

- If the *x*-intercepts of a cubic function are x_1, x_2 and x_3, then the equation of the function is given by

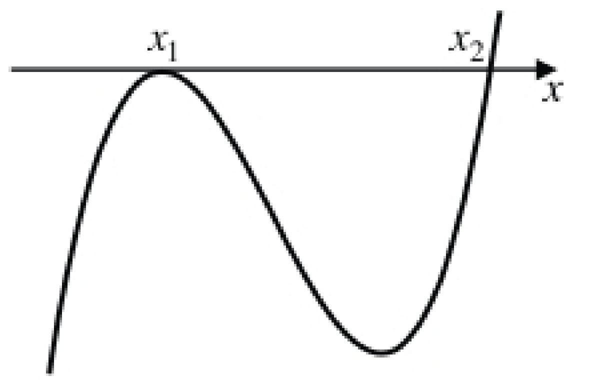
$$\boxed{y = a(x - x_1)(x - x_2)(x - x_3)}$$



- If one of the *x*-intercepts is also a turning point, this *x*-intercept can be used twice:

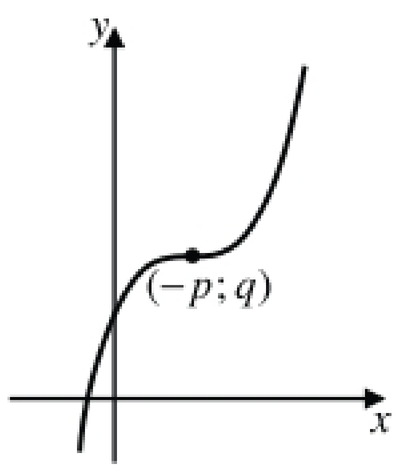
$$y = a(x - x_1)(x - x_1)(x - x_2)$$

$$\therefore \quad \boxed{y = a(x - x_1)^2(x - x_2)}$$



- If a cubic function has a stationary point of inflection at $(-p; q)$, then the equation of the function is given by:

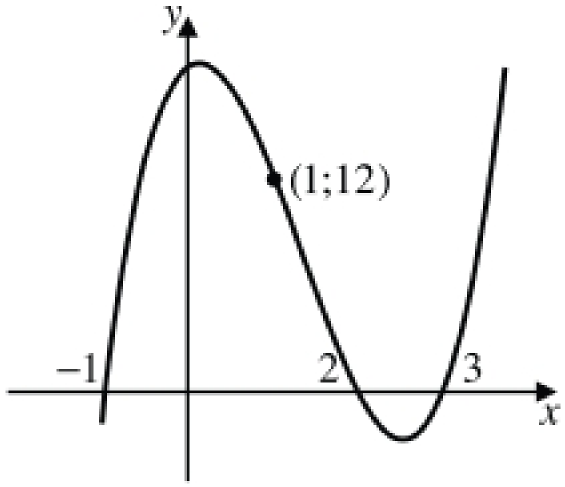
$$\boxed{y = a(x + p)^3 + q}$$



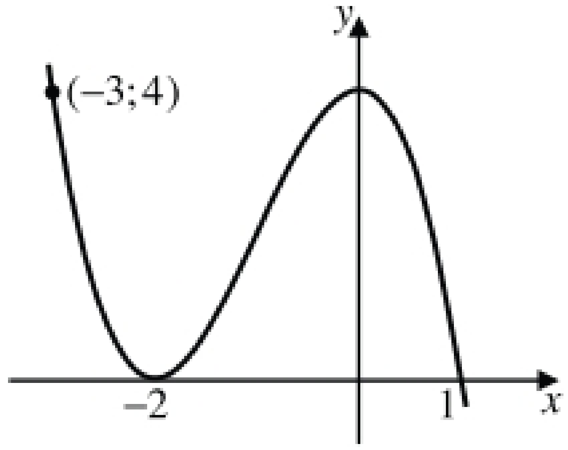
EXAMPLE 28

Determine the equation of each of the following cubic functions in the form $y = ax^3 + bx^2 + cx + d$:

(a)



(b)



Solution

(a) $y = a(x+1)(x-2)(x-3)$

Substitute the point $(1;12)$:

$12 = a(1+1)(1-2)(1-3)$

$\therefore 12 = 4a$

$\therefore a = 3$

$y = 3(x+1)(x-2)(x-3)$

$\therefore y = 3(x+1)(x^2 - 5x + 6)$

$\therefore y = 3(x^3 - 4x^2 + x + 6)$

$\therefore y = 3x^3 - 12x^2 + 3x + 18$

(b) $y = a(x+2)^2(x-1)$

Substitute the point $(-3;4)$:

$4 = a(-3+2)^2(-3-1)$

$\therefore 4 = -4a$

$\therefore a = -1$

$y = -(x+2)^2(x-1)$

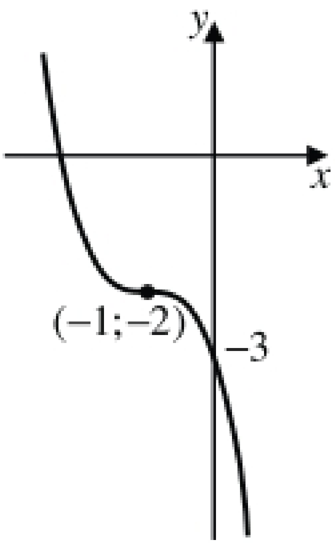
$\therefore y = -(x^2 + 4x + 4)(x-1)$

$\therefore y = -(x^3 + 3x^2 - 4)$

$\therefore y = -x^3 - 3x^2 + 4$

EXAMPLE 29

Determine the equation of the following cubic function in the form $y = a(x+p)^3 + q$:



Solution

$y = a(x+1)^3 - 2$

Substitute the point $(0;-3)$:

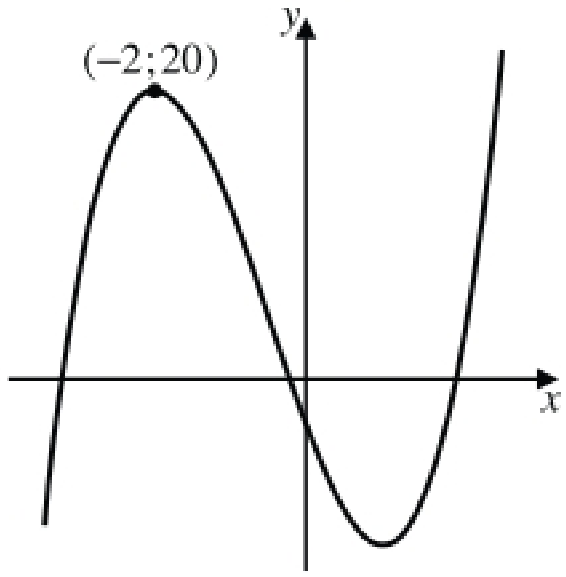
$-3 = a(0+1)^3 - 2$

$\therefore -1 = a$

$\therefore y = -(x+1)^3 - 2$

EXAMPLE 30

The sketch alongside shows the graph of
$f(x) = ax^3 + bx^2 - 12x$.
$(-2;20)$ is a stationary point of f.
Determine the values of a and b.



Solution

$f(-2) = 20$

$\therefore a(-2)^3 + b(-2)^2 - 12(-2) = 20$

$\therefore -8a + 4b + 24 = 20$

$\therefore -8a + 4b = -4$

$\therefore 2a - b = 1 \quad ...\ 1$

$(-2;20)$ is a stationary point.

$\therefore f'(-2) = 0 \qquad \left[f'(x) = 3ax^2 + 2bx - 12 \right]$

$\therefore 3a(-2)^2 + 2b(-2) - 12 = 0$

$\therefore 12a - 4b = 12$

$\therefore 3a - b = 3 \quad ...\ 2$

$2 - 1 : a = 2$

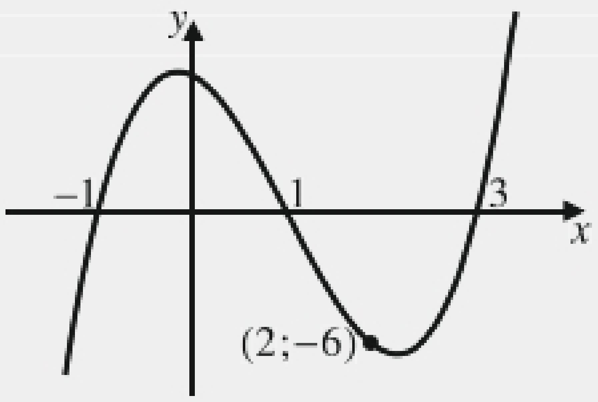
Into 1 : $2(2) - b = 1$

$\therefore b = 3$

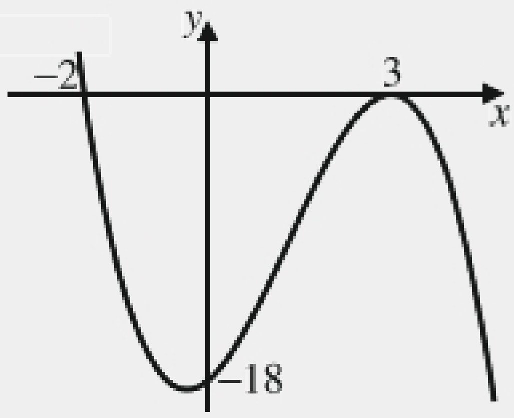
EXERCISE 9

(a) Determine the equations of the following cubic functions in the form $y = ax^3 + bx^2 + cx + d$:

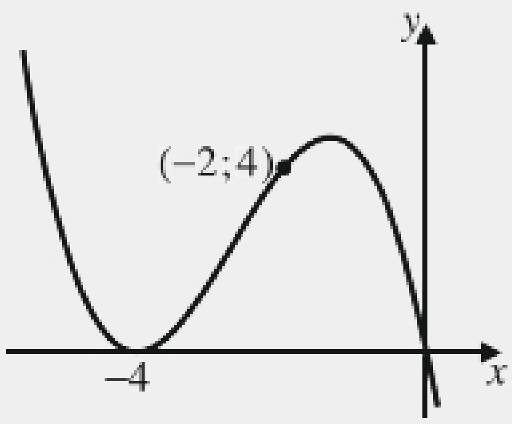
(1)



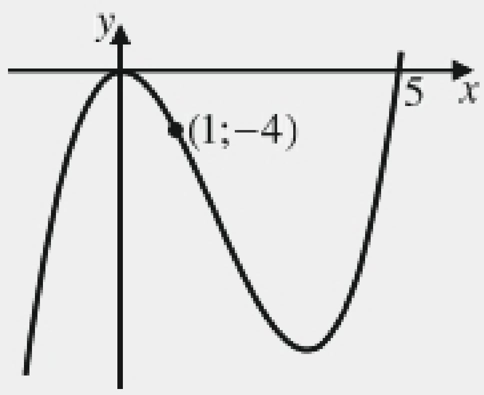
(2)



(3)

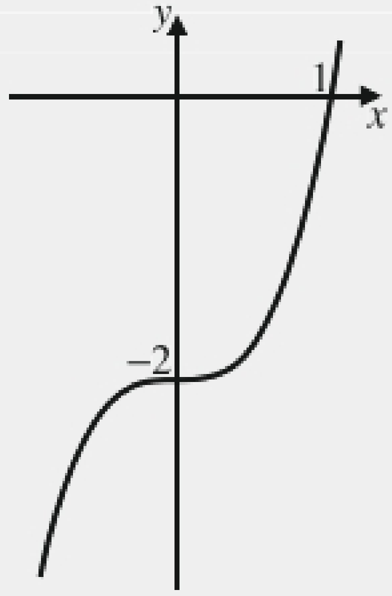


(4)

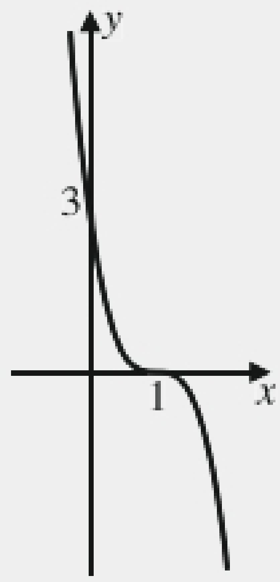


(b) Determine the equation of each of the following functions in the form $y = a(x+p)^3 + q$:

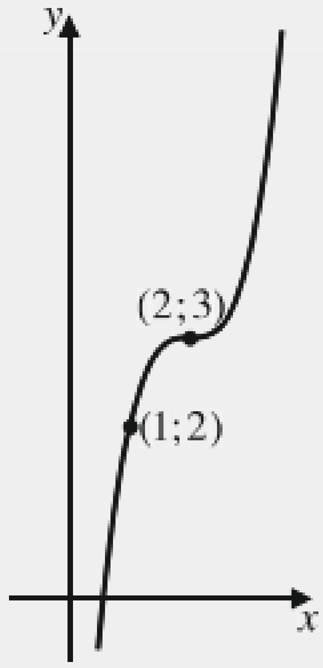
(1)



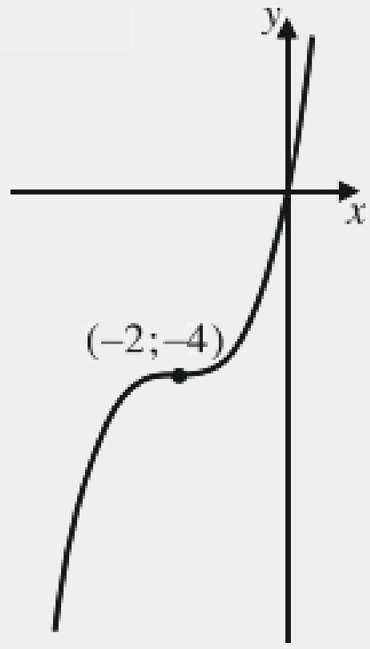
(2)



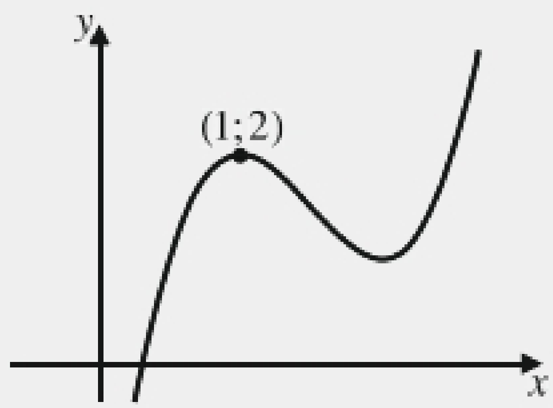
(3)



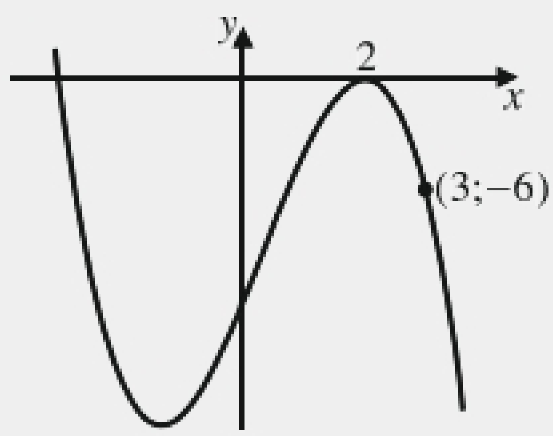
(4)



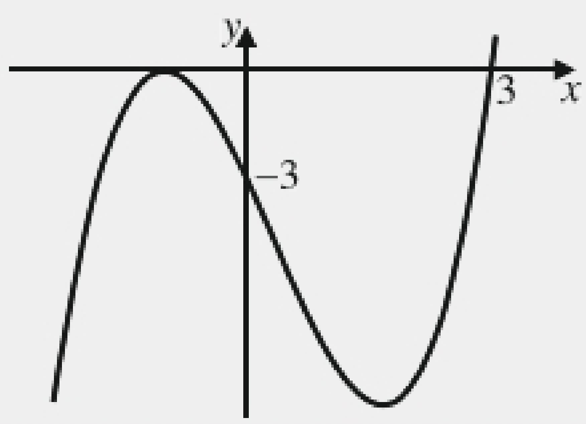
(c) The sketch alongside shows the graph of $f(x) = ax^3 - 9x^2 + bx - 3$.
$(1;\, 2)$ is a stationary point of f.
Determine the values of a and b.



(d) The sketch alongside shows the graph of $y = -x^3 + ax^2 + bx + c$. Determine the values of a, b and c.

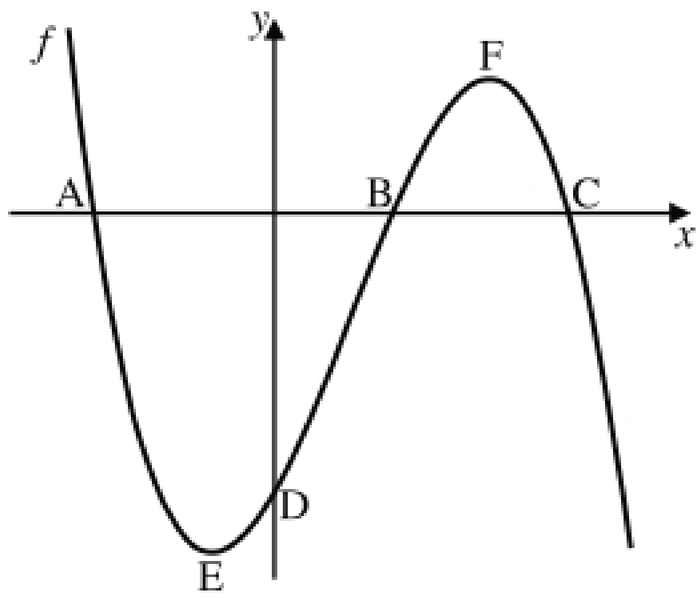


(e)* The sketch alongside shows the graph of $g(x) = x^3 + px^2 + qx + r$. Determine the values of p, q and r.



EXAMPLE 31

The sketch alongside shows the graph of $f(x)=-x^3+4x^2+11x-30$. E and F are stationary points of f.



(a) Calculate the length of AC.
(b) Calculate the coordinates of E and F.
(c) Calculate the x-coordinate of the point of inflection of f.
(d) Determine the equation of the tangent to f at D.
(e) The tangent to f at D also intersects the graph of f at another point. Calculate the x-coordinate of this point.
(f) For which values of x is
(1) f increasing? (2) $f(x)\geq 0$?
(3) $f'(x)<0$? (4) $f(x)\cdot f'(x)<0$?
(g) For which value(s) of k will $f(x)=k$ have
(1) two distinct real roots? (2) three distinct real roots?
(3) one real root?
(h) For which values of t will the graph of $y=f(x)+t$ have two negative x-intercepts and one positive x-intercept?

Solution

(a) Calculate x-intercepts:

$f(x)=0$

$\therefore -x^3+4x^2+11x-30=0$

$\therefore x^3-4x^2-11x+30=0$

$\therefore (x-2)(x^2-2x-15)=0$

$\therefore (x-2)(x+3)(x-5)=0$

$\therefore x=2$ or $x=-3$ or $x=5$

$\therefore \text{AC}=5-(-3)=8$ units

(b) Stationary points:

$f'(x)=0$

$\therefore -3x^2+8x+11=0$

$\therefore 3x^2-8x-11=0$

$\therefore (3x-11)(x+1)=0$

$\therefore x=\dfrac{11}{3}$ or $x=-1$

When $x=-1$, $y=f(-1)=-(-1)^3+4(-1)^2+11(-1)-30=-36 \rightarrow \text{E}(-1;-36)$

When $x=\frac{11}{3}$, $y=f\left(\frac{11}{3}\right)=-\left(\frac{11}{3}\right)^3+4\left(\frac{11}{3}\right)^2+11\left(\frac{11}{3}\right)-30=\frac{400}{27} \rightarrow \text{F}\left(\frac{11}{3};\frac{400}{27}\right)$

(c) $f''(x) = 0$

$\therefore -6x + 8 = 0$

$\therefore x = \dfrac{4}{3}$

(d) D$(0; -30)$

$m = f'(0)$

$= -3(0)^2 + 8(0) + 11$

$= 11$

$y + 30 = 11(x - 0)$

$\therefore y = 11x - 30$

(e) $-x^3 + 4x^2 + 11x - 30 = 11x - 30$

$\therefore -x^3 + 4x^2 = 0$

$\therefore -x^2(x - 4) = 0$

$\therefore x = 0$ or $x = 4$

N.A.

$\therefore x = 4$

(f) (1) $-1 \leq x \leq \frac{11}{3}$ (2) $x \leq -3$ or $2 \leq x \leq 5$

(3) $x < -1$ or $x > \frac{11}{3}$ (4) $x < -3$ or $-1 < x < 2$ or $\frac{11}{3} < x < 5$

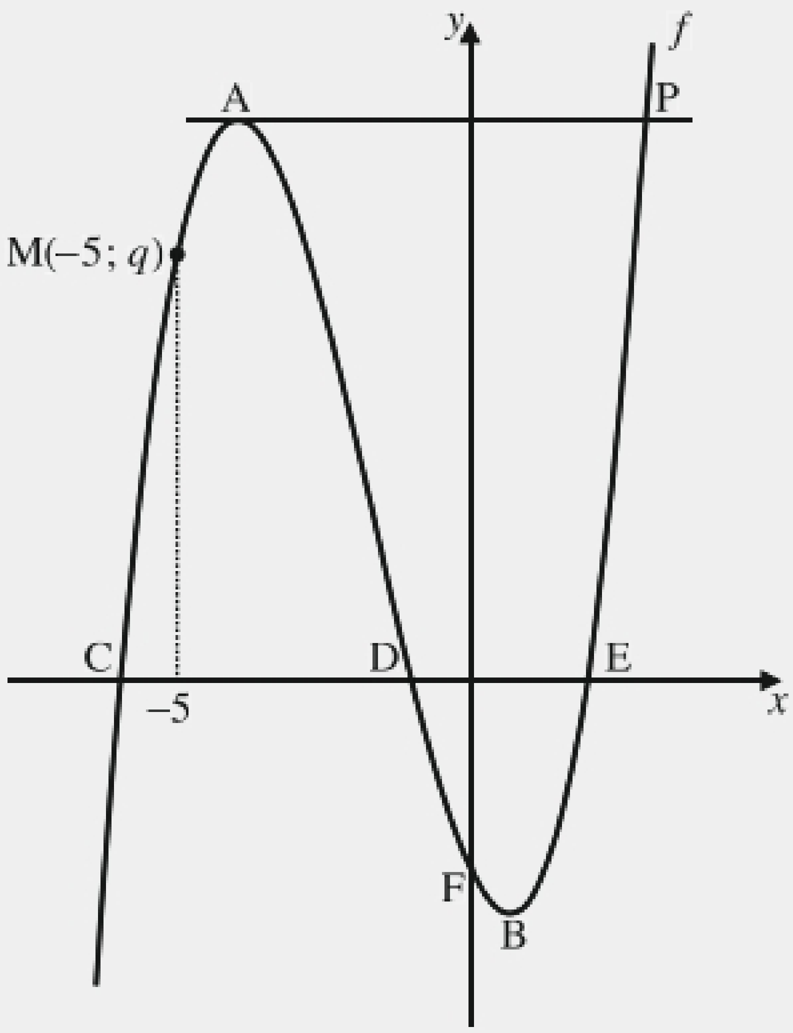
(g) (1) $k = -36$ or $k = \frac{400}{27}$ (2) $-36 < k < \frac{400}{27}$

(3) $k < -36$ or $k > \frac{400}{27}$

(h) $30 < t < 36$

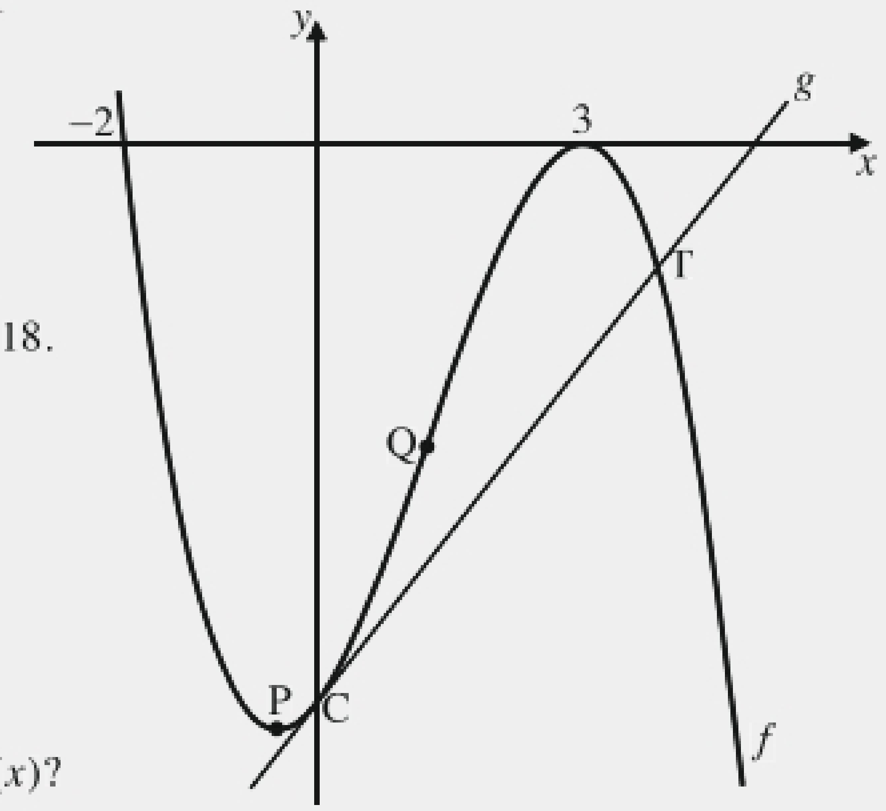
EXERCISE 10

(a) The sketch alongside shows the graph of $f(x) = x^3 + 5x^2 - 8x - 12$.
A and B are stationary points of f.
The tangent to the graph, at A, cuts the graph again at P. M$(-5; q)$ is a point on f.



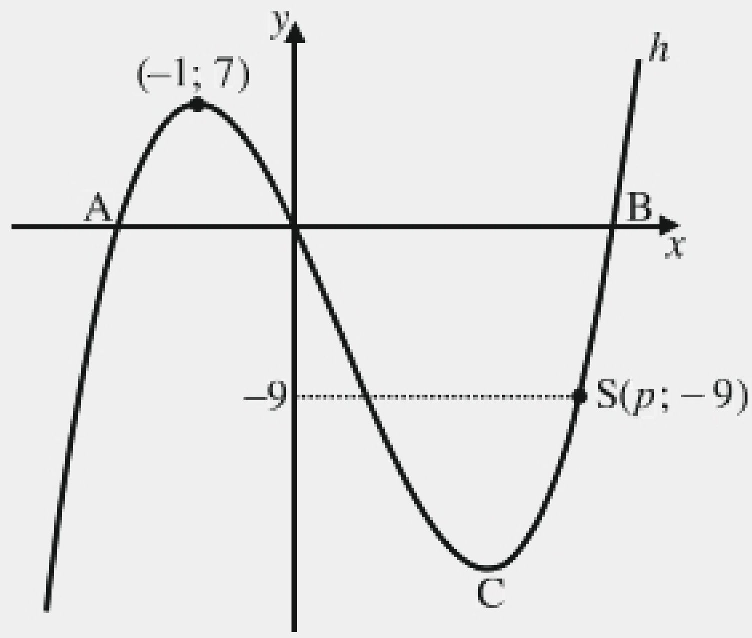
(1) Calculate the coordinates of C, D and E.
(2) Write down the coordinates of F.
(3) Calculate the coordinates of A and B.
(4) For which values of x is f decreasing?
(5) Calculate the x-coordinate of the point of inflection of f.
(6) Calculate the coordinates of P.
(7) Determine the equation of the tangent to f at M.
(8) For which values of x is
  (i) $f(x) \leq 0$?
  (ii) $f'(x) > 0$?
  (iii) $f(x) \cdot f'(x) \geq 0$?

(b) The sketch alongside shows the graphs of $f(x)=-x^3+bx^2+cx+d$ and g.
The graph of f cuts the y-axis at C.
g is the tangent to f at C.
P is a stationary point of f and Q is the point of inflection. f and g intersect at T.



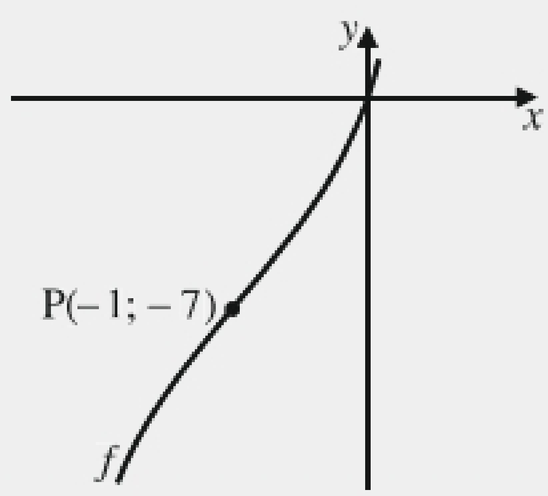
(1) Show that $b=4$, $c=3$ and $d=-18$.
(2) Calculate the coordinates of P.
(3) Calculate the x-coordinate of Q.
(4) For which values of x is f
 (i) increasing?
 (ii) concave down?
(5) Determine the equation of g.
(6) Calculate the coordinates of T.
(7) For which values of x is $f(x)<g(x)$?
(8) For which value(s) of k will the graph of $y=f(x)+k$ have three x-intercepts?
(9) For which value(s) of b will the equation $f(x+b)=0$ have two positive roots?

(c) The sketch alongside shows the graph of $h(x)=ax^3+bx^2-12x$.
$(-1;7)$ and C are stationary points of h.
$\mathrm{S}(p;-9)$ is a point on h.



(1) Show that $a=2$ and $b=-3$.
(2) Calculate the length of AB. (Round to two decimal places.)
(3) Determine the value of p.
(4) Determine the equation of the tangent to h at S.
(5) Calculate the coordinates of the point of inflection of h.
(6) If it is further given that the coordinates of C are $(2;-20)$, determine the value(s) of t for which $h(x)=t$ has
 (i) three distinct real roots.
 (ii) two distinct real roots.
 (iii) one real root.

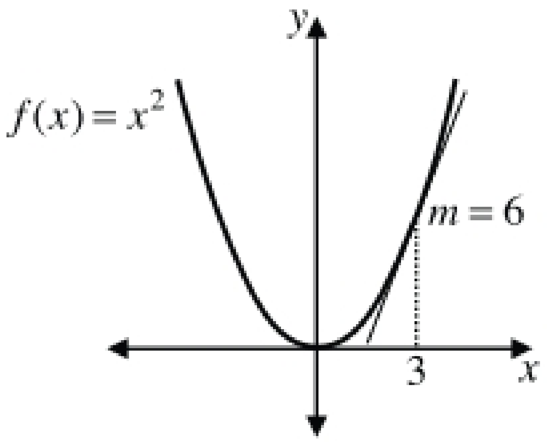
(d) The sketch shows the graph of $f(x)=x^3+mx^2+nx$.
The point $\mathrm{P}(-1;-7)$ is the point of inflection of f.

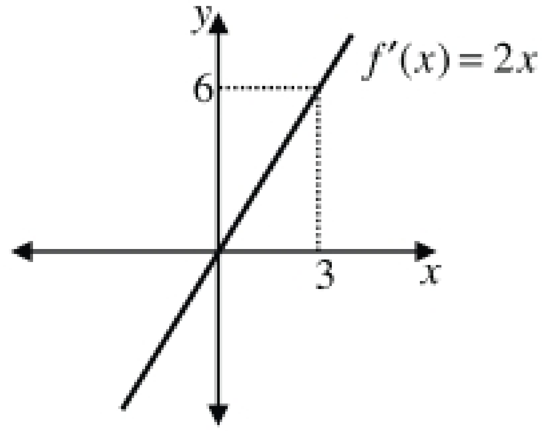


(1) Show that $m=3$ and $n=9$.
(2) Show that f has no stationary points.
(3) Determine the equation of the tangent to f at P.
(4) Determine the equations of the two tangents to f, parallel to the line $y-9x=10$.

THE GRAPH OF THE DERIVATIVE OF A FUNCTION

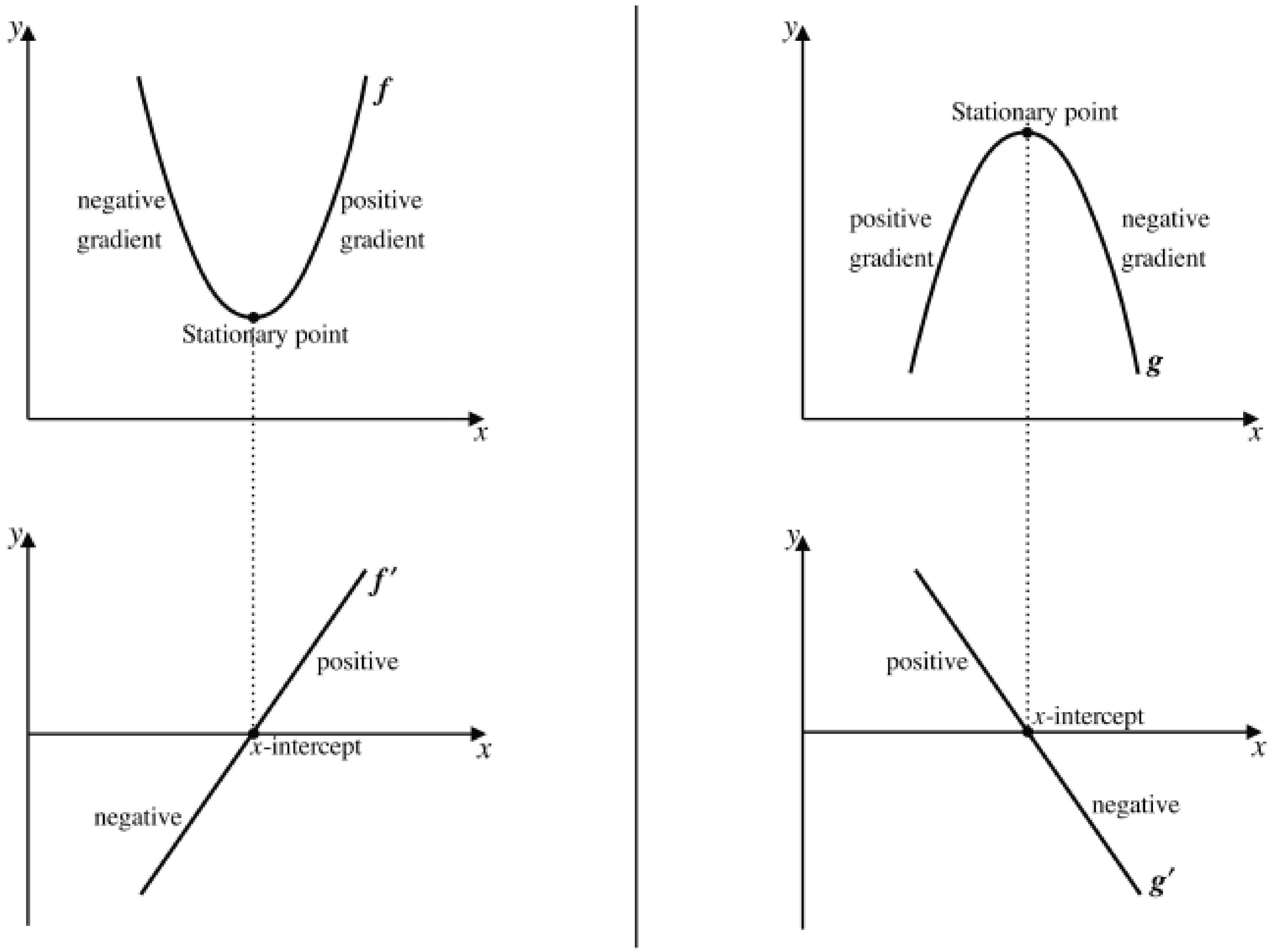
The derivative of a function is itself also a function. The graph of the derivative of a function can be drawn to show how the gradient of the original function changes from point to point. The ***y*-value** on the graph of the **derivative** is equal to the **gradient** of the **original function** at each x-value. Graphs of the function $f(x) = x^2$ and its derivative $f'(x) = 2x$ are shown as an example:





THE DERIVATIVE OF A QUADRATIC FUNCTION

The derivative of a **quadratic function** is a **linear function**:

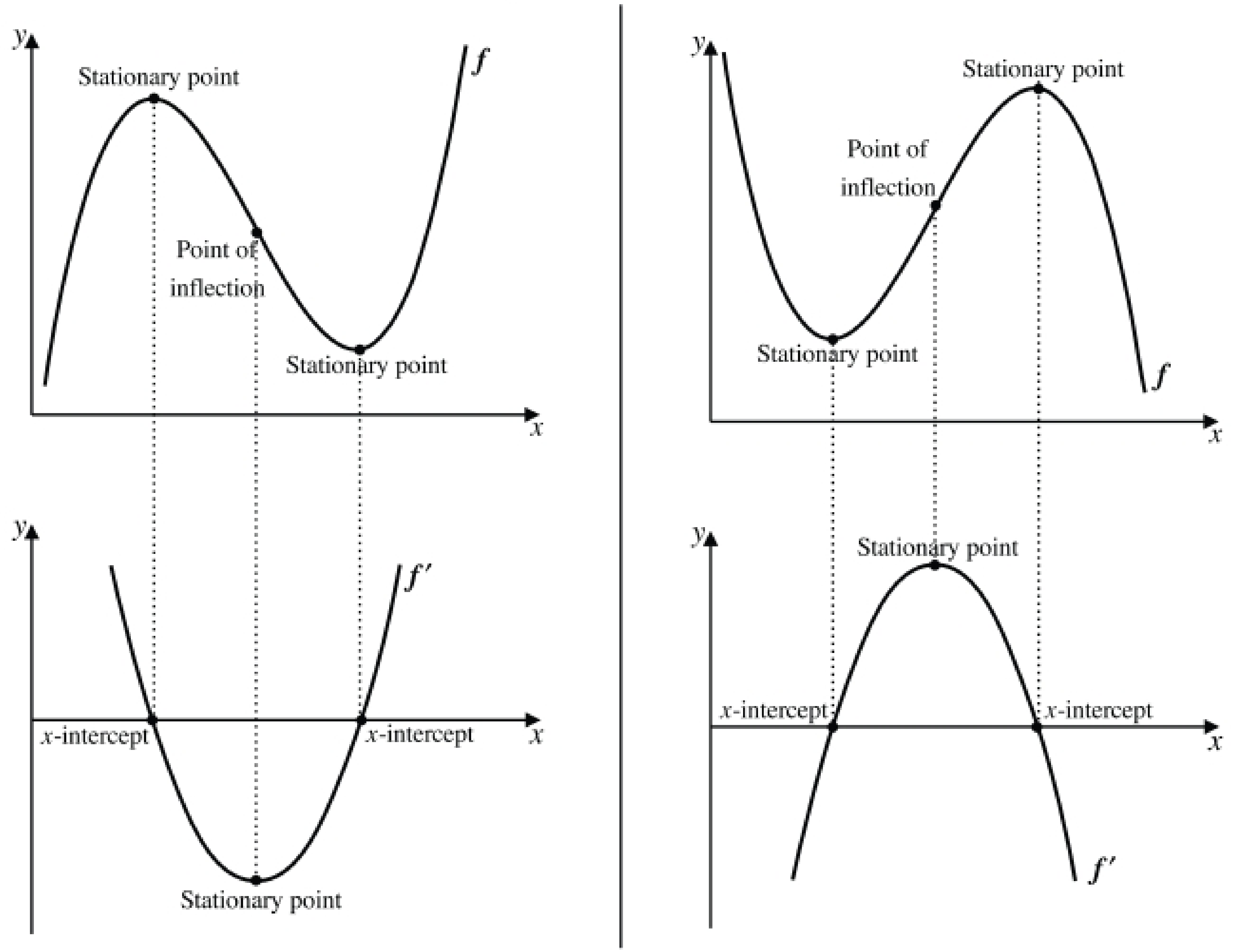


Note that the graph of the **derivative** has an ***x*-intercept** at the x-value where the **original function** has a **stationary point**.

$$\boxed{f' \text{ has an } x\text{-intercept where } f \text{ has a stationary point.}}$$

THE DERIVATIVE OF A CUBIC FUNCTION

The derivative of a **cubic function** is a **quadratic function**:



Note that

- the graph of the **derivative** has **x-intercepts** at the x-values where the **original function** has **stationary points**.
- the graph of the **derivative** has a **stationary point** at the x-value where the **original function** has a **point of inflection**.

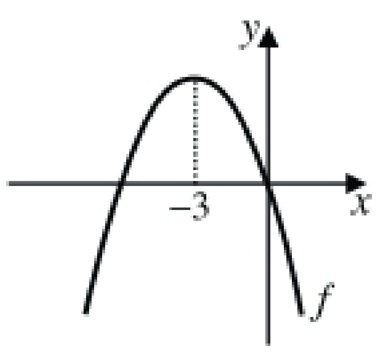
f' has x-intercepts where f has stationary points.

f' has a stationary point where f has a point of inflection.

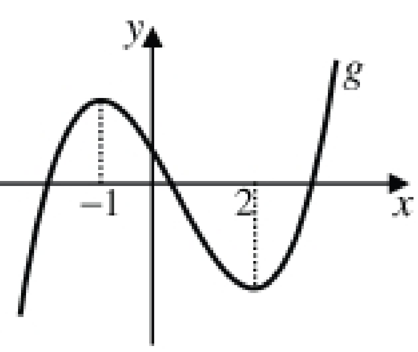
EXAMPLE 32

Draw a rough sketch of the graph of the derivative of each of the following functions:

(a)

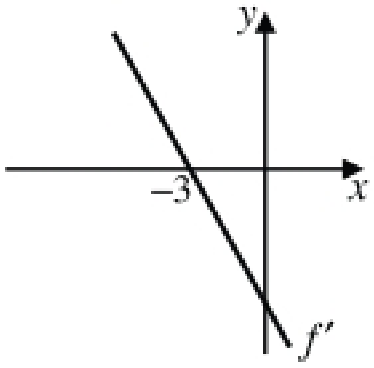


(b)

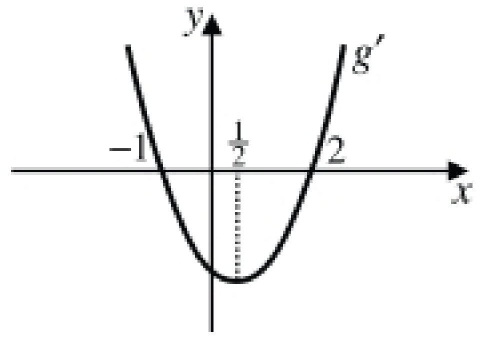


Solution

(a)

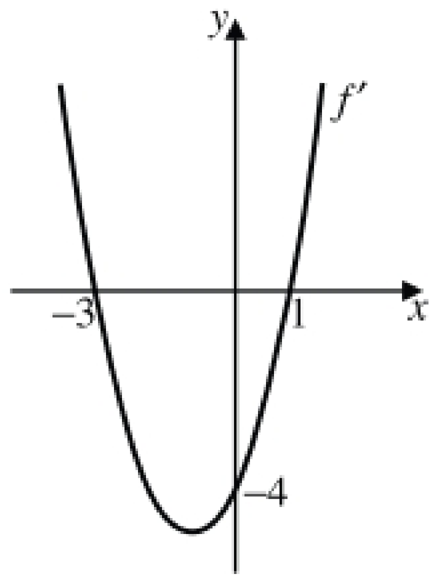


(b)



EXAMPLE 33

In the sketch alongside, the graph of $y = f'(x)$ is shown.



(a) What is the gradient of the tangent to f at $x = 0$?

(b) Write down the x-coordinate(s) of the stationary point(s) of f.

(c) What is the x-coordinate(s) of the point(s) of inflection of f?

(d) What is the gradient of the tangent to f at $x = -2$?

(e) For which values of x is f

(i) increasing? (ii) decreasing?

(f) Is f concave up or concave down at $x = 0$?

(g) Draw a rough sketch of the graph of $y = f(x)$ if $f(0) > 0$.

Solution

(a) -4

(b) $x = -3$; $x = 1$

(c) $x = \dfrac{-3+1}{2}$

$\therefore x = -1$

(d) By symmetry about the line $x = -1$,

$f'(-2) = -4$.

$\therefore$ The gradient of f at $x = -2$ is -4.

(e) (i) f is increasing when f' is positive. We also include the stationary points.

$\therefore x \leq -3$ or $x \geq 1$

(ii) f is decreasing when f' is negative. We also include the stationary points.

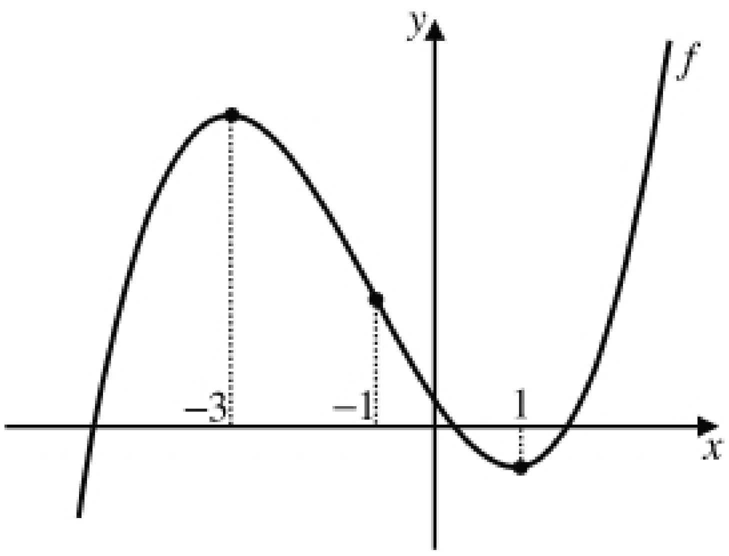
$\therefore -3 \leq x \leq 1$

(f) The gradient of f' is positive at $x = 0$.

$\therefore f''(0) > 0$

$\therefore f$ is concave up at $x = 0$.

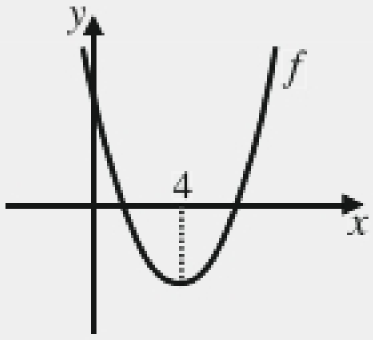
(g)



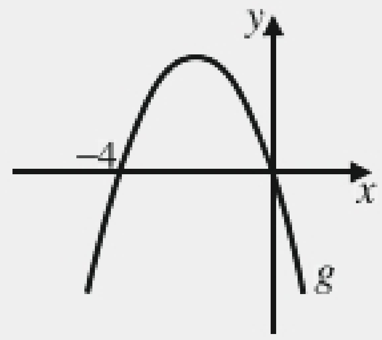
EXERCISE 11

(a) Draw a rough sketch of the graph of the derivative of each of the following functions:

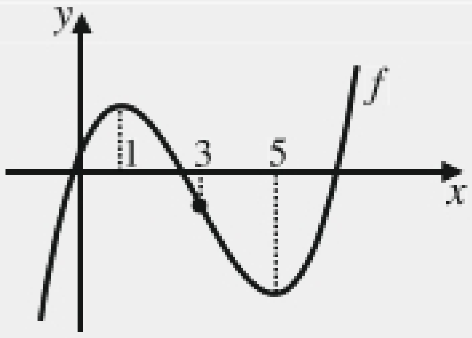
(1)



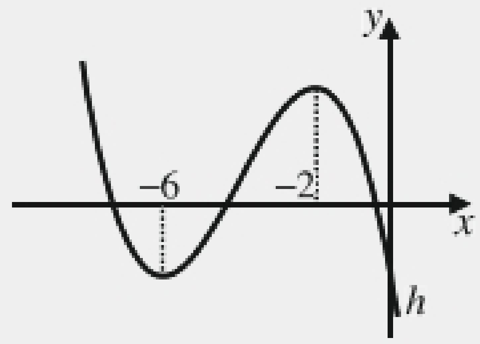
(2)



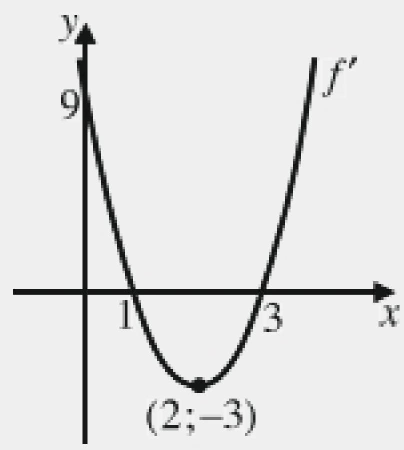
(3)



(4)

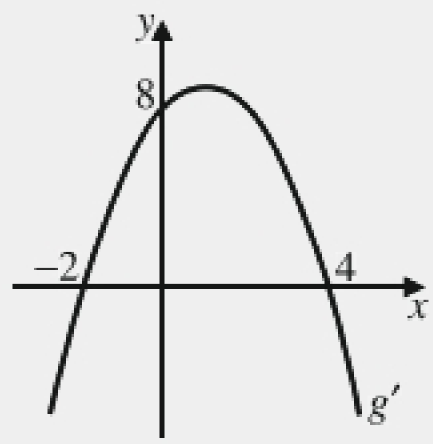


(b) In the sketch alongside, the graph of $y = f'(x)$ is shown.



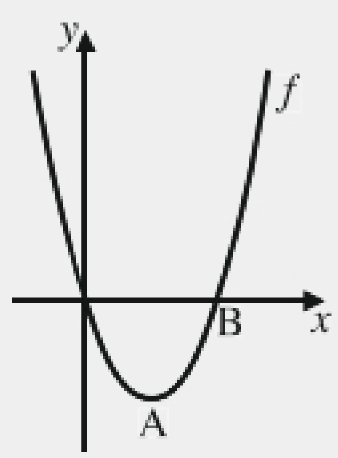
(1) What is the gradient of the tangent to f at $x = 0$?
(2) Write down the x-coordinate(s) of the stationary point(s) of f.
(3) Write down the x-coordinate of the point of inflection of f.
(4) What is the gradient of the tangent to f at its point of inflection?
(5) For which values of x is f
 (i) increasing? (ii) decreasing?
(6) Is f concave up or concave down at
 (i) $x = 1$? (ii) $x = 4$?
(7) Draw a rough sketch of the graph of $y = f(x)$ if $f(0) = 0$.

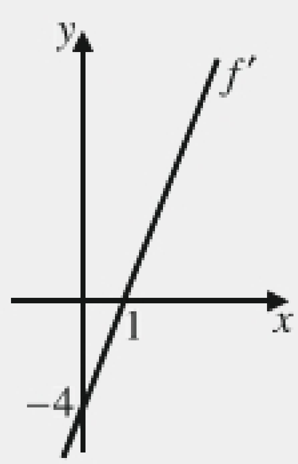
(c) The sketch alongside shows the graph of g', a quadratic function.



(1) What is the gradient of g at $x = 0$?
(2) Write down the x-coordinates of the stationary points of g?
(3) What is the x-coordinate of the point of inflection of g?
(4) What is the gradient of g at $x = 2$?
(5) For which values of x is g decreasing?
(6) Draw a rough sketch of the graph of g if $g(0) < 0$.
(7) Determine the equation of
 (i) g' (ii)** g if $g(0) = 2$.

(d) The graphs of $y = f(x)$ and $y = f'(x)$ are shown alongside.

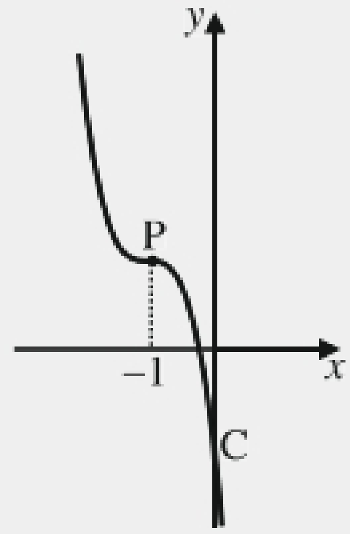




(1) What is the gradient of the tangent to f at $x = 0$?
(2) What is the x-coordinate of point A?
(3) What are the coordinates of B?
(4) Determine the equation of f.
(5) Sketch the graph of $y = f''(x)$.

(e)* The sketch alongside shows the graph of a cubic function h, with a single stationary point at P. The gradient of the tangent to h at C is -3.

(1) Draw a rough sketch of the graph of h'.
(2) Determine the equation of h'.
(3) Sketch the graph of h''.



APPLICATIONS OF THE DERIVATIVE

RATES OF CHANGE

The gradient of a function can be seen as the **rate at which the function changes**. In order to find the *rate of change* of a function at a specific moment in time, we calculate the **derivative** at that point.

The **rate of change** of a function $f(t)$, at $t = a$, is given by $f'(a)$.

- A **positive rate of change** indicates that the function value is **increasing**.
- A **negative rate of change** indicates that the function value is **decreasing**.

EXAMPLE 34

The volume of water in a tank changes according to the formula $V(t) = -t^2 + 24t + 6$, where t is the time in hours, from the moment the inlet and outlet of the tank was connected to a pump.
The volume is measured in litres.

(a) How much water is in the tank initially?
(b) At what rate does the volume of water in the tank change
 (1) after exactly 6 hours? (2) after exactly 15 hours?
(c) After how many hours will there be 50 litres of water in the tank?
(d) After how many hours will the volume of water decrease at a rate of 12 litres per hour?

Solution

(a) $V(t) = -t^2 + 24t + 6$

$\therefore V(0) = -0^2 + 24(0) + 6$
$= 6l$

"initially" means $t = 0$

(b) (i) $V'(t) = -2t + 24$
$= -2(6) + 24$
$= 12\ l/\text{hour}$

This question is about a **rate of change** so we use the **derivative**.

(ii) $V'(t) = -2(15) + 24$
$= -6\ l/\text{hour}$

The **negative** rate indicates that the volume is **decreasing**.

(c) $V(t) = 50$

$\therefore -t^2 + 24t + 6 = 50$

$\therefore t^2 - 24t + 44 = 0$

$\therefore (t-2)(t-22) = 0$

$\therefore t = 2 \text{ or } t = 22$

The volume reaches 50 litres after 2 hours and again after 22 hours.

(d) $V'(t) = -12$

The rate of change is **negative**, since the volume is **decreasing**.

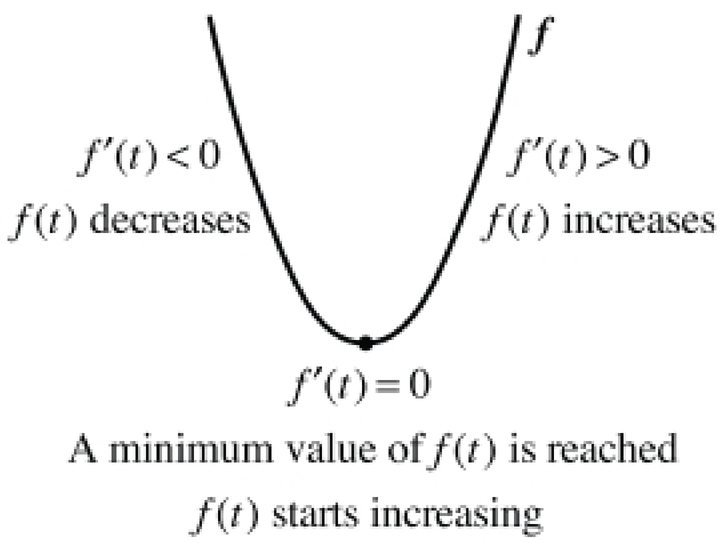
$\therefore -2t + 24 = -12$

$\therefore 2t = 36$

$\therefore t = 18$ hours

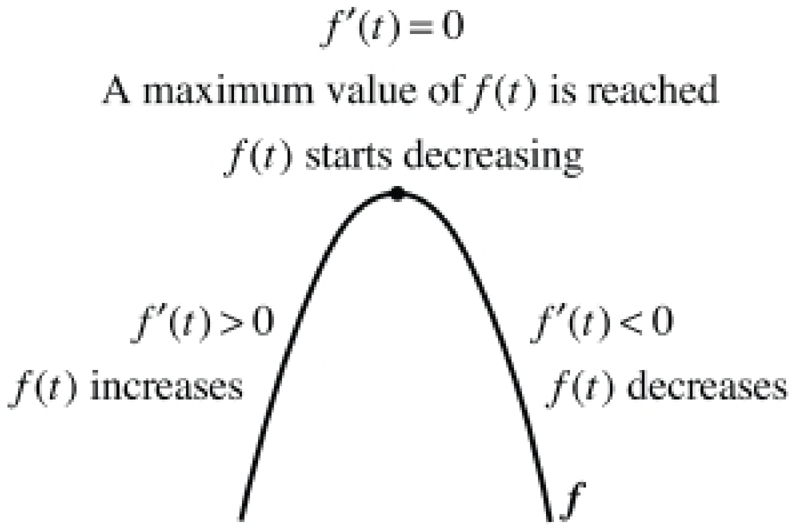
MINIMUM AND MAXIMUM VALUES

- A function reaches a local **minimum** value at any point where the gradient of the function changes from negative to positive. This happens where the value of the **derivative** is **zero**. At this point, the function value **starts increasing**:



When $f(t)$ reaches a minimum, $f'(t) = 0$.

- A function reaches a local **maximum** value at any point where the gradient of the function changes from positive to negative. This happens where the value of the **derivative** is **zero**. At this point, the function value **starts decreasing**:



When $f(t)$ reaches a maximum, $f'(t) = 0$.

EXAMPLE 35

Consider again the scenario in Example 34:

The volume of water in a tank changes according to the formula $V(t) = -t^2 + 24t + 6$, where t is the time in hours, from the moment the inlet and outlet of the tank was connected to a pump. The volume is measured in litres.

(a) After how many hours does the volume of water in the tank start decreasing?

(b) What is the maximum volume of water in the tank?

Solution

(a) $V'(t) = 0$

$\therefore -2t + 24 = 0$

$\therefore 2t = 24$

$\therefore t = 12$ hours

(b) $V(t) = -t^2 + 24t + 6$

$\therefore V(12) = -12^2 + 24(12) + 6$

$= 150\ l$

MOTION

When referring to moving objects

- the **rate of change of position**, with respect to time, is called ***velocity***.
- the **rate of change of velocity**, with respect to time, is called ***acceleration***.

Let $s(t)$ represent the position (displacement, distance, height etc.) of an object, after t seconds of motion.

- The velocity $v(t)$ of the object, at any specific time t, is given by $s'(t)$.

$$\boxed{v(t) = s'(t)}$$

- The acceleration of the object, at any specific time t, is given by $v'(t)$, which equals $s''(t)$.

$$\boxed{a(t) = v'(t) = s''(t)}$$

- At the time when the object reaches its **maximum/minimum displacement** from its initial position, and **changes its direction**, $s'(t) = 0$ i.e. $v(t) = 0$.

$$\boxed{\text{When } s(t) \text{ is a maximum/minimum, } s'(t) = 0.}$$

The velocity and acceleration can be **positive** or **negative**.

- The numerical value of the **velocity** tells us **how fast** an object is moving (i.e. the **speed** of the object) and the **sign** of the velocity indicates the **direction** in which the object is moving.
- The **acceleration** tells us how rapidly the **velocity** of the object **increases or decreases**.
  - If the acceleration has the **same sign as the velocity**, the object is **speeding up**.
  - If the acceleration has the **opposite sign to the velocity**, the object is **slowing down**.

EXAMPLE 36

The displacement of a particle, in metres, from a point P, after t seconds, is given by $s(t) = 3t^2 - 60t + 400$. The particle moves in a straight line from east to west and back. The particle is initially west of P.

(a) How far is the particle from point P initially?
(b) How fast and in what direction is the particle moving after exactly 4 seconds?
(c) Calculate the acceleration of the particle.
(d) What is the minimum distance between the particle and point P.
(e) Is the particle *speeding up* or *slowing down* after exactly
 (1) 6 seconds? (2) 12 seconds?

Solution

(a) $s(t) = 3t^2 - 60t + 400$

$\therefore s(0) = 3(0)^2 - 60(0) + 400 = 400 \text{ m}$

(b) $v(t) = s'(t) = 6t - 60$

$\therefore v(4) = 6(4) - 60 = -36 \text{ m/s}$

The particle is moving at 36 m/s EAST.

(c) $a(t) = v'(t) = 6$

The acceleration is 6 m/s^2.

(d) $s'(t) = 0$

$\therefore 6t - 60 = 0$

$\therefore t = 10 \text{ s}$

$s(10) = 3(10)^2 - 60(10) + 400 = 100$

The minimum distance is 100 m.

(e) (1) $v(6) = 6(6) - 60 = -24$

$a = +6$

a and v have opposite signs.

$\therefore$ The particle is *slowing down*.

(2) $v(12) = 6(12) - 60 = +12$

$a = +6$

a and v have the same sign.

$\therefore$ The particle is *speeding up*.

EXERCISE 12

(a) The volume of water (in m^3) in a reservoir is given by $V(t) = -8t^2 + 80t + 40$, where t is the time in hours after a pump was activated.
 (1) What is the initial volume of water in the reservoir?
 (2) At what rate is the volume of the water in the tank changing after exactly
  (i) 4 hours? (ii) 10 hours?
 (3) After how many hours does the volume of water in the reservoir
  (i) increase at a rate of 48 m^3 per hour?
  (ii) decrease at a rate of 32 m^3 per hour?
 (4) After how many hours does the volume of water in the reservoir reach a maximum?
 (5) What is the maximum volume of water in the reservoir?

(b) During an experiment, the temperature T (in degrees Celcius) in a container varies with time t (in minutes), according to the formula $T = 49 + 6t^2 - t^3$. The experiment lasts for 8 minutes.

(1) What is the temperature in the container at the start of the experiment?
(2) What is the temperature in the container at the end of the experiment?
(3) What is the rate of change of the temperature, with respect to time, exactly 3 minutes after the start of the experiment?
(4) At what rate is the temperature changing at the moment when a temperature of 49°C is reached for a second time?
(5) At what time does the temperature in the container start decreasing?
(6) What is the maximum temperature in the container during the experiment?
(7) During which time interval is the temperature in the container increasing?

(c) The height (in metres) of a stone, thrown upwards from the top of a building is given by $h = 35 + 30t - 5t^2$, where t is the time (in seconds) from the moment that the stone was thrown.

(1) From what height is the stone initially thrown?
(2) At what velocity is the stone initially thrown?
(3) What is the velocity of the stone after exactly 2 seconds?
(4) What is the acceleration of the stone?
(5) After how many seconds does the stone reach its maximum height?
(6) What is the maximum height reached by the stone?
(7) At what speed does the stone hit the ground?
(8) Was the stone *speeding up* or *slowing down* after exactly
(i) 2 seconds? (ii) 5 seconds?

(d) The displacement of a particle, in metres, from a point P, after t seconds, is given by $s(t) = -\frac{1}{10}t^3 + 3t^2$ where $0 \le t \le 30$. Initially, the particle is north of point P and moving northward.

(1) What is the average velocity of the particle over the first 5 seconds?
(2) What is the velocity of the particle after exactly 5 seconds?
(3) What is the acceleration of the particle after exactly 5 seconds?
(4) Is the particle *speeding up* or *slowing down* after exactly 5 seconds?
(5) At what time does the particle change direction (start moving southward)?
(6) What is the maximum northward displacement of the particle from point P?
(7) Is the particle *speeding up* or *slowing down* after exactly 25 seconds?
(8) At what velocity is the particle moving at the moment when it passes point P again?
(9) Determine the maximum speed reached by the particle during its northward motion.

MORE PROBLEMS ABOUT MINIMUM AND MAXIMUM VALUES

In order to determine a **minimum** or **maximum** value of any function $f(x)$, follow these steps:

- Let $f'(x) = 0$.
- Solve for x.
- Substitute the x-value(s) obtained into the original function $f(x)$ to obtain the minimum/maximum value.
(If more than one value is obtained, the context of the problem will usually indicate which one is applicable and gives the required minimum or maximum.)

EXAMPLE 37

(a) Determine the maximum value of $f(x) = -x^2 + 4x + 10$.

(b) Determine the minimum value of $A = x + \dfrac{16}{x} - 2$ where $x > 0$.

Solution

(a) $f(x) = -x^2 + 4x + 10$.

$\therefore f'(x) = -2x + 4$

For the maximum value of $f(x)$, let $f'(x) = 0$:

$-2x + 4 = 0$

$\therefore x = 2$

Maximum value $= f(2) = -2^2 + 4(2) + 10 = 14$

(b) $A = x + \dfrac{16}{x} - 2$

$= x + 16x^{-1} - 2$

$\therefore \dfrac{dA}{dx} = 1 - 16x^{-2}$

For the minimum value of A, let $\dfrac{dA}{dx} = 0$:

$1 - 16x^{-2} = 0$

$\therefore 1 = \dfrac{16}{x^2}$

$\therefore x^2 = 16$

$\therefore x = \pm 4$

But $x > 0$ $\therefore x = 4$

Minimum value of A $= 4 + \dfrac{16}{4} - 2 = 6$

We often have to determine an expression for the quantity that has to be maximised or minimised first:

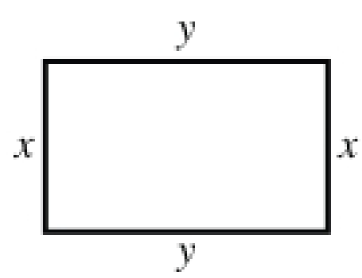
EXAMPLE 38

Determine the maximum area of a rectangle with a perimeter of 100 m.

Solution

The area of the rectangle depends on two variables, the length and the breadth. The first step is to express one of these variables in terms of the other:

Let the breadth of the rectangle be x and the length y.



Perimeter $= 100$

$\therefore 2x + 2y = 100$

$\therefore x + y = 50$

$\therefore y = 50 - x$

Next, we find an expression for the area of the rectangle:

Let the area of the rectangle be A.

$A = xy$

$\therefore A = x(50 - x)$

$\therefore A = 50x - x^2$

Now let $\frac{dA}{dx} = 0$:

$50 - 2x = 0$

$\therefore x = 25$

Substitute $x = 25$ into the formula for A:

Maximum value of $\text{A} = 50(25) - 25^2 = 625\ \text{m}^2$

Note: In this case the maximum area is obtained when $x = 25$ m and $y = 50 - 25 = 25$ m. This means that the rectangle will be a square and the area $25 \times 25 = 625\ \text{m}^2$. It is interesting that the maximum area of a rectangle with a given perimeter is always obtained when the rectangle is a square.

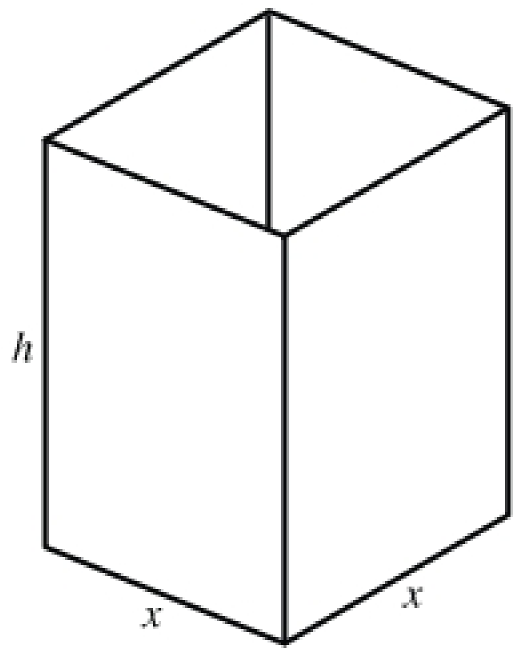
EXAMPLE 39

A box with a square base is to be made to contain a volume of $108\ \text{cm}^3$. The box must be open at the top. Let the base length be x and the height h as shown in the sketch alongside.

(a) Show that the total outer surface area of the box is given by $A(x) = x^2 + \frac{432}{x}$.

(b) Determine the dimensions of the box for which the outer surface area will be a minimum.

(c) Determine the minimum outer surface area.



Solution

(a) Volume $= L \times B \times H$

$= x \times x \times h$

$= x^2 h$

$\therefore x^2 h = 108$

$\therefore h = \frac{108}{x^2}$

$A(x) = x^2 + 4xh$ $\boxed{\textbf{1 square } (x^2) \text{ and } \textbf{4 rectangles } (x \times h)}$

$= x^2 + 4x\left(\frac{108}{x^2}\right)$

$= x^2 + \frac{432}{x}$

(b) $A(x) = x^2 + \dfrac{432}{x}$

$= x^2 + 432x^{-1}$

$\therefore A'(x) = 2x - 432x^{-2}$

For the minimum value of A, let $A'(x) = 0$:

$2x - 432x^{-2} = 0$

$\therefore 2x - \dfrac{432}{x^2} = 0$

$\therefore 2x^3 - 432 = 0$

$\therefore x^3 = 216$

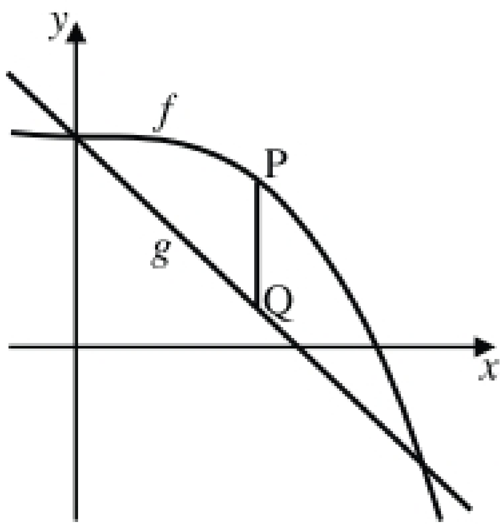
$\therefore x = 6$

$\therefore h = \dfrac{108}{6^2} = 3$

The base length is 6 cm and the height is 3 cm.

(c) Minimum surface area $= A(6) = 6^2 + \dfrac{432}{6} = 108 \text{ cm}^2$

EXAMPLE 40



The sketch alongside shows the graphs of $f(x) = -\dfrac{1}{3}x^3 + 9$ and $g(x) = -4x + 9$. PQ is a line segment, parallel to the y-axis and to the right of the y-axis, with P on f and Q on g. Determine the maximum length of PQ.

Solution

PQ = Top function − Bottom function

$= f(x) - g(x)$

$= -\dfrac{1}{3}x^3 + 9 - (-4x + 9)$

$= -\dfrac{1}{3}x^3 + 4x$

$\therefore \text{PQ}' = -x^2 + 4$

For the maximum value of PQ, let $\text{PQ}' = 0$:

$-x^2 + 4 = 0$

$\therefore x^2 = 4$

$\therefore x = \pm 2$

But PQ is to the right of the y-axis $\therefore x = 2$

$\therefore$ Maximum length of PQ $= -\dfrac{1}{3}(2)^3 + 4(2) = \dfrac{16}{3}$ units.

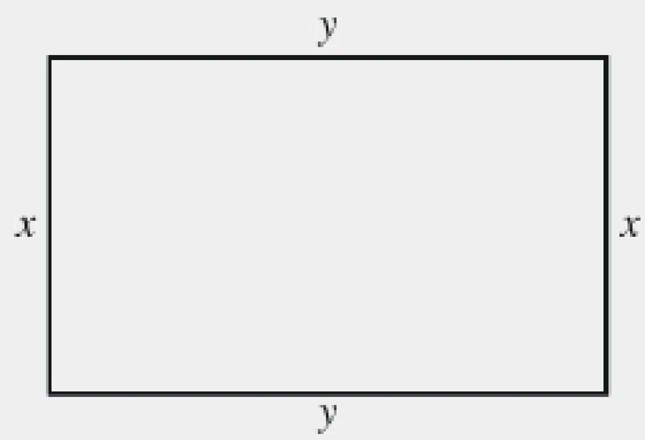
EXERCISE 13

(a) Determine the minimum value of $f(x)=x^2-6x+11$.

(b) Given $P=4-x^2-\dfrac{54}{x}$, determine

(1) the value of x for which P is a maximum. (2) the maximum value of P.

(c) The sketch alongside shows a rectangle with a perimeter of 80 m. Let the breadth and length of the rectangle be x and y respectively.

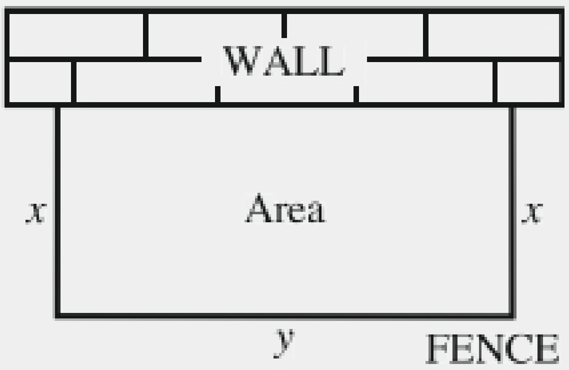


(1) Show that the area of the rectangle is given by $A(x)=40x-x^2$.

(2) Determine the dimensions of the rectangle for which the area will be a maximum.

(3) Calculate the maximum area of the rectangle.

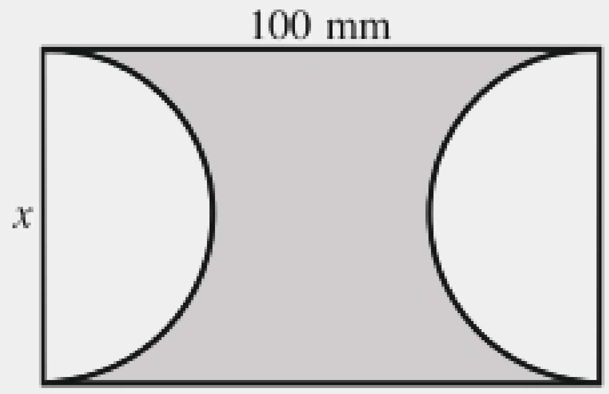
(d) A rectangular area is to be enclosed between a wall and a U-shaped fence, as shown in the sketch alongside.The total length of the fence (not including the wall) is 120 m.



(1) Show that the area between the wall and the fence is given by $A=-2x^2+120x$.

(2) Determine the maximum possible area that can be enclosed between the wall and the fence.

(e) Two semi circles are cut from a rectangle with length 100 mm and breadth x, as shown in the sketch.



(1) Express the remaining (shaded) area A in terms of x.

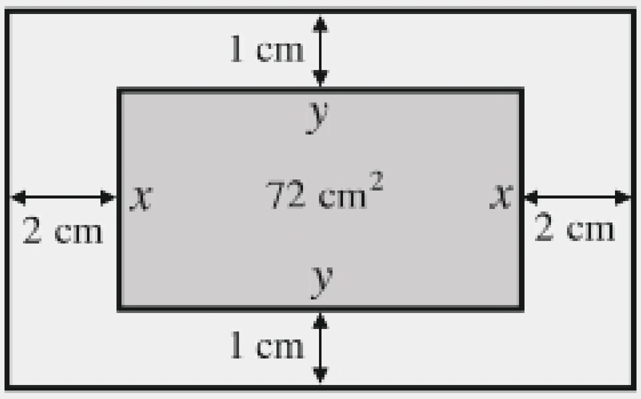
(2) Determine the maximum possible shaded area.

(f) A flyer must be designed to allow for a rectangular area of 72 cm^2 for printing.

The following margins are required:

- top an bottom margins of 1 cm each
- left and right margins of 2 cm each

Let x and y be the breadth and length of the area allocated for printing.

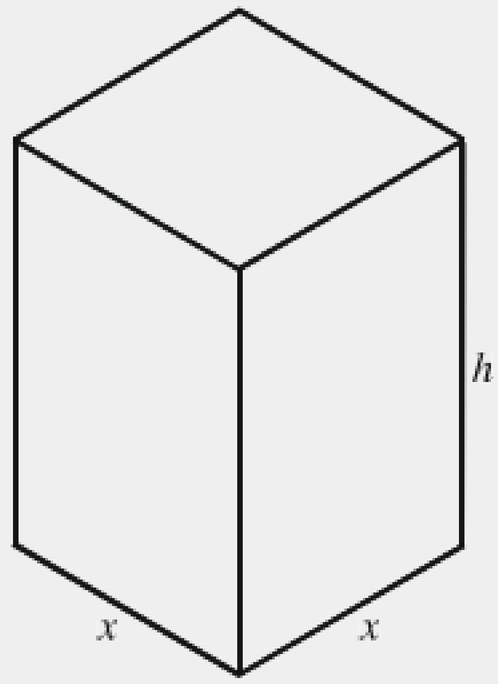


(1) Show that the total area of the flyer, including the margins, is given by

$$A(x)=4x+\frac{144}{x}+80.$$

(2) Determine the values of x and y for which the total area of the flyer is minimised.

(g) A rectangular prism, with a square base, has a volume of $125\ \text{cm}^3$. Let the base length be x and the height h as shown in the sketch alongside.

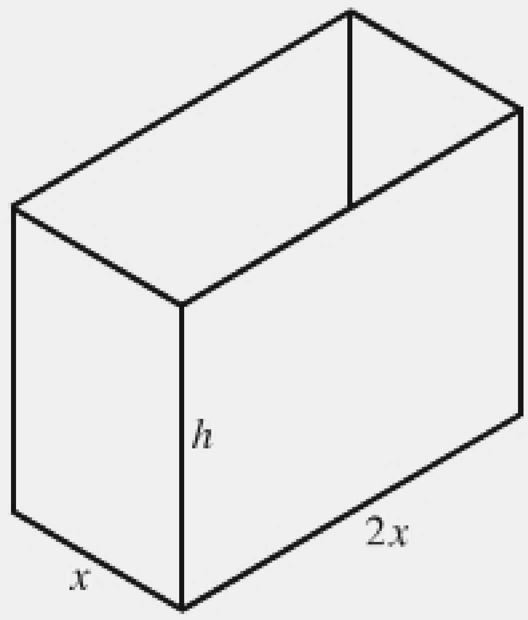


(1) Show that the surface area of the prism is given by $S(x) = 2x^2 + \dfrac{500}{x}$.

(2) For which value of x will the surface area be a minimum?

(3) Calculate the minimum surface area of the rectangular prism.

(h) The sketch alongside shows a box (open at the top) with length $2x$, breadth x and height h. The total outer surface area is $150\ \text{m}^2$.

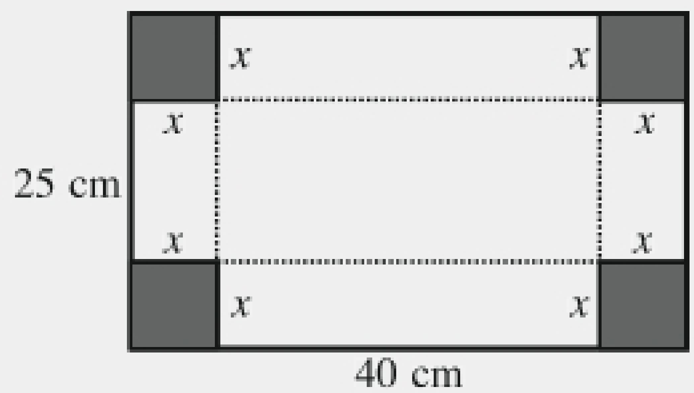


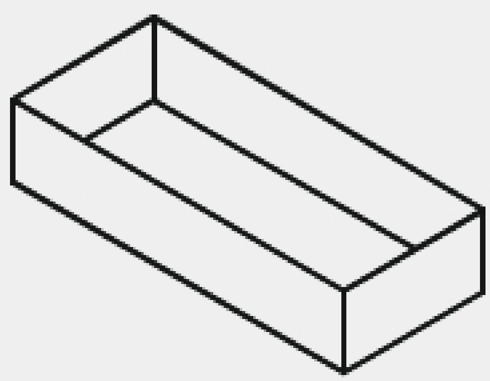
(1) Show that $h = \dfrac{150 - 2x^2}{6x}$.

(2) Express the volume V of the box in terms of x.

(3) Determine the maximum volume of the box.

(i) Four congruent squares must be cut from the four corners of a rectangular cardboard sheet with length 40 cm and breadth 25 cm as shown in the sketch alongside. The remaining cardboard must be folded along the dotted lines to form a box (open at the top).





(1) Show that the volume of the resulting box is given by $V(x) = 4x^3 - 130x^2 + 1000x$.

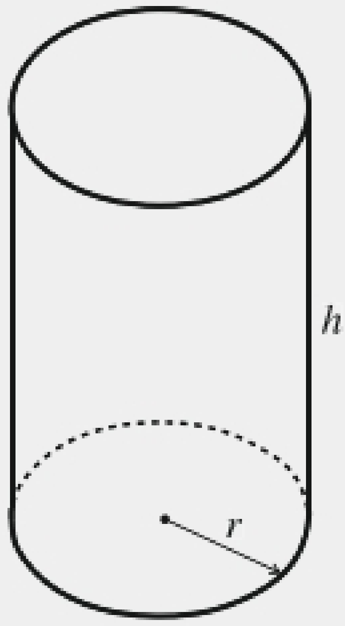
(2) Determine the value of x for which the volume is a maximum.

(3) Calculate the maximum possible volume of the box.

(j) A cylinder must have a volume of $128\pi \text{ cm}^3$. Let the radius of the base be r and the height h as shown in the sketch alongside.

(1) Show that the surface area of the cylinder is given by $A(r) = 2\pi r^2 + \dfrac{256\pi}{r}$.

(2) Determine the minimum surface area of the cylinder.

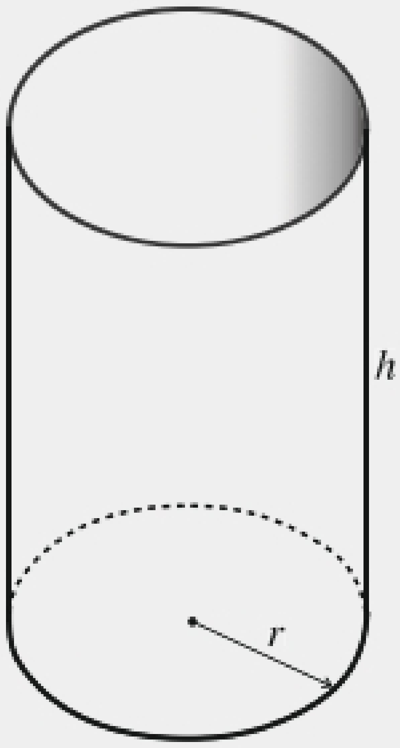


(k) A cylinder (open at the top) is to be made with a total outer surface area of $108\pi \text{ cm}^2$. Let the radius of the base be r and the height h as shown in the sketch alongside.

(1) Express h in terms of r.

(2) Show that the volume of the cylinder is given by $V = 54\pi r - \dfrac{\pi}{2}r^3$.

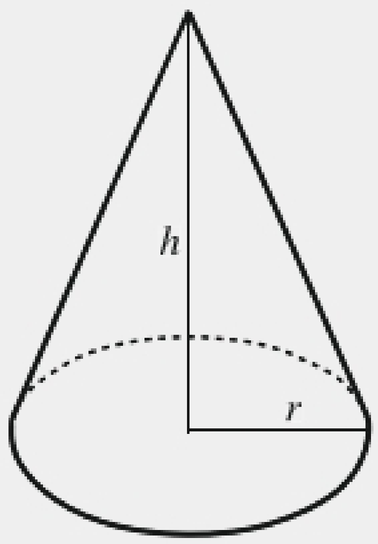
(3) Determine the maximum volume of the cylinder.



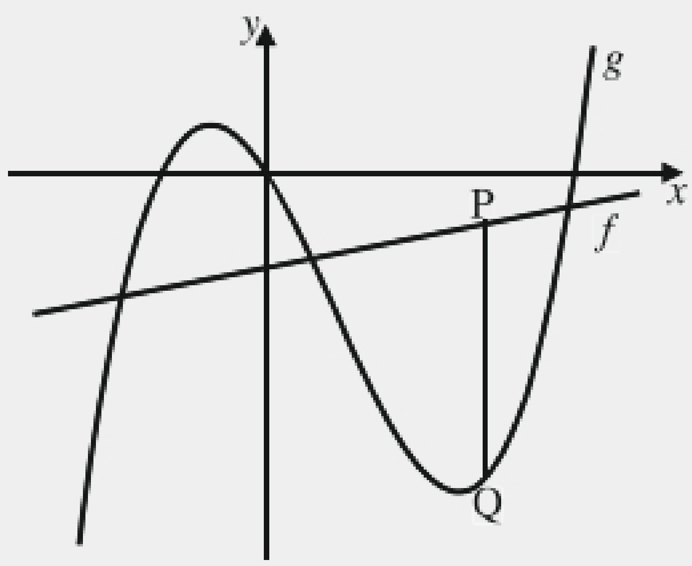
(l) A cone with radius r and height h is shown in the sketch alongside. The sum of the base radius and the height is 30 cm.

(1) Determine the volume V of the cone in terms of r.

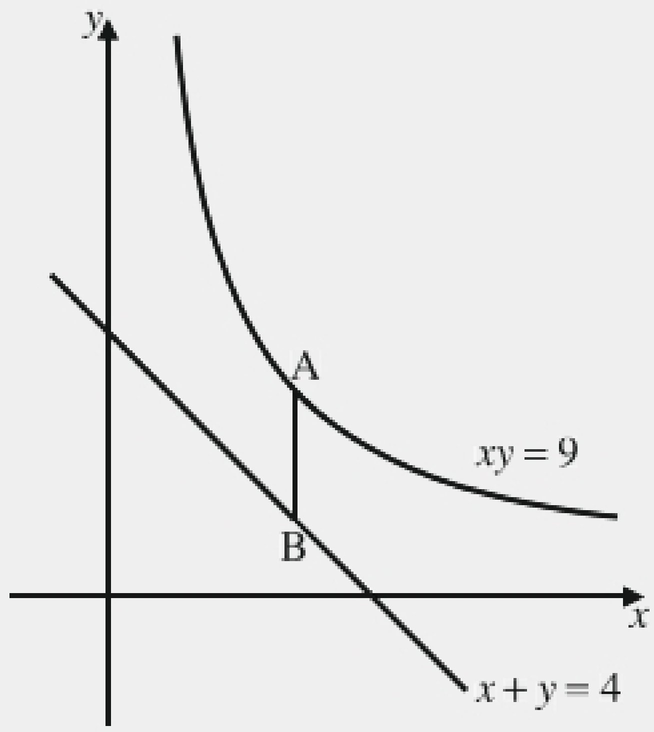
(2) What is the height of the cone when the cone has a maximum volume?



(m) The sketch alongside shows the graphs of $f(x) = x - 7$ and $g(x) = x^3 - 3x^2 - 8x$. PQ is a line segment, parallel to the y-axis, with P on f and Q on g. P is above Q. Determine the maximum length of PQ.



(n) The sketch alongside shows the graphs of $xy = 9$ and $x + y = 4$. AB is a line segment, parallel to the y-axis, with A on the graph of $xy = 9$ and B on the graph of $x + y = 4$.

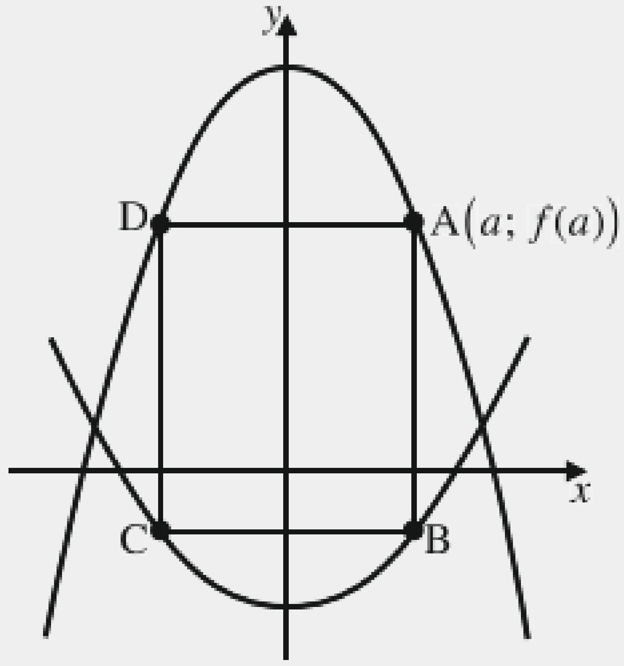


(1) Express the length of AB in terms of x.

(2) Determine the minimum length of AB.

(3) What are the coordinates of B when the length of AB is a minimum?

(o) In the sketch alongside, $f(x) = -2x^2 + 27$ and $g(x) = x^2 - 9$. ABCD is a rectangle with A and D on f, and B and C on g. AB is parallel to the y-axis. The coordinates of A are $(a;\, f(a))$.



(1) Express the area of rectangle ABCD in terms of a.

(2) For which value of a will the area of rectangle ABCD be a maximum?

(3) Calculate the maximum area of rectangle ABCD.

(p)* A cylindrical container must be made out of metal to contain exactly 500 ml of tomato juice. Determine the dimensions of the container if the least possible amount of metal must be used to make the container. (Give the dimensions in centimetres and round to two decimal places.)

(q)* At a price of R200 per T-shirt, 100 T-shirts are sold. For every R10 increase in the price of the T-shirts, 2 fewer T-shirts are sold. At what price must the T-shirts be sold to maximise the revenue?

CONSOLIDATION AND EXTENSION EXERCISE

(a) Given the function $f(x) = 2x^2 - 3x$.

(1) Determine the average gradient of f between $x = -2$ and $x = -1$.

(2) Determine the gradient of the tangent to f, at $x = 2$, from first principles.

(b) Determine the derivative of each of the following functions from first principles:

(1) $f(x) = -2x^3$ (2) $g(x) = \dfrac{5}{x}$

(c)* Given $\dfrac{f(x+h)-f(x)}{h}=\dfrac{2hx-3h+h^2}{h}$.

(1) Determine the average gradient of f between $x=2$ and $x=3$.

(2) Determine the gradient of the tangent to f at $x=4$.

(3)** If $f(0)=5$, determine the equation of f.

(d) Determine

(1) $f'(x)$ if $f(x)=4x^3-3x^2+5x-12$

(2) $D_x\left[\sqrt[3]{x}-\dfrac{5}{x}+x(x-1)\right]$

(3) $\dfrac{dy}{dx}$ if $y=\dfrac{2}{3x^3}-\dfrac{5x^2}{7}+\dfrac{3x}{\sqrt{x^3}}$

(4) $\dfrac{d}{dx}\left[-3x(x-2)^2\right]$

(5) $g'(x)$ if $g(x)=\left(\sqrt[3]{x}+\dfrac{1}{\sqrt{x}}\right)\left(\sqrt[3]{x}-\dfrac{1}{\sqrt{x}}\right)$

(6) $\dfrac{d}{dx}\left[\dfrac{2x^2-3x-1}{x^2}\right]$

(7) $\dfrac{dp}{dx}$ if $p=\dfrac{x^2+2x}{\sqrt{x}}$

(8) $f'(a)$ if $f(a)=\dfrac{\sqrt[3]{a^2}-3}{2a}$

(9) $h'(x)$ if $h(x)=\dfrac{4x^2-9}{3-2x}$

(10) $\dfrac{d}{dp}\left[\dfrac{p^2-3p+2}{p\sqrt{p}-\sqrt{p}}\right]$

(11) $D_x\left[\sqrt{2}\cdot x^{\sqrt{2}}+\sqrt[3]{2x}-\pi^3\right]$

(12) $\dfrac{d}{dx}\left[p^3x^2-qx+r^5\right]$ where p, q and r are constants.

(13) $\dfrac{dk}{dt}$ if $k=nt^n-tn^3$ where n is a constant.

(e)* Determine $\dfrac{dy}{dx}$ if

(1) $x=(y+1)^3$

(2) $xy+1=y+x^3$

(f) Determine the equation of the tangent to the graph of $y=\sqrt{x^3}$ at $x=4$.

(g) Determine the equations of the tangents to the graph of $f(x)=x^3-3x$ that are parallel to $y=9x$.

(h) The gradient of the tangent to the graph of $y=x^3+px^2-qx$ at the point $(1\,;4)$ is 6. Determine the values of p and q.

(i) The line $4y-3x=q$ is a tangent to the graph of $g(x)=p\sqrt{x}$ at $x=4$. Determine the values of p and q.

(j) The graphs of $f(x)=x^3-4x^2$ and $g(x)=ax^2-15x$ have equal gradients at $x=-1$. Determine

(1) the value of a.

(2) another value of x for which f and g will have equal gradients.

(k) Sketch the graphs of the following functions, showing all relevant details:

(1) $y = x^3 - 2x^2 - 7x - 4$ (2) $y = -x^3 + 5x^2 + 8x - 12$

(l) Given the function $f(x) = 2x^3 + 6x^2 + 6x + 4$.

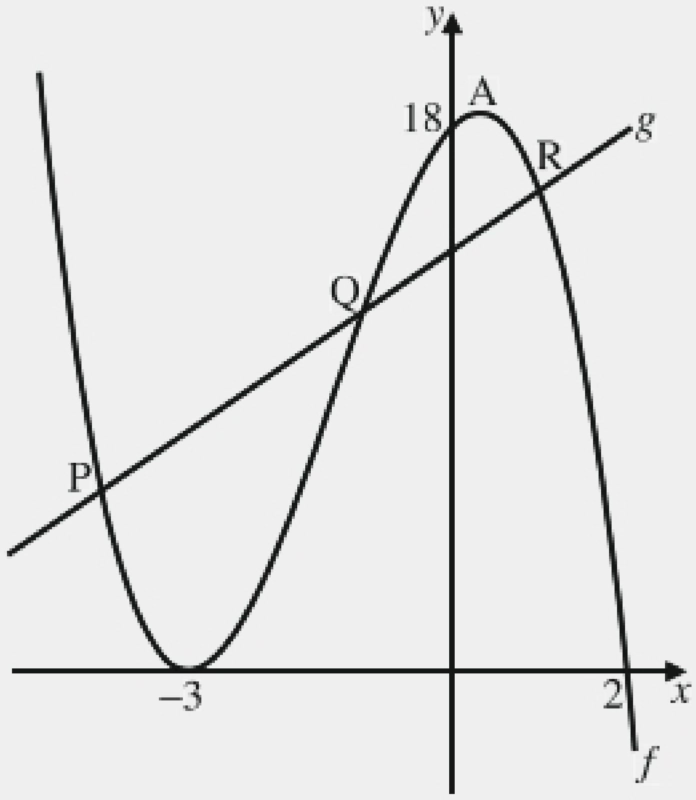
(1) Calculate the coordinates of the stationary point(s) of the function f.

(2) Sketch the graph of f showing all relevant details.

(3) Write $f(x)$ in the form $a(x+p)^3 + q$.

(m) A cubic function f is such that $f'(1) = 0$, $f'(3) = f(3) = 0$, $f(0) = 0$ and $f'(2) > 0$. Draw a rough sketch of f.

(n) The sketch alongside shows the graph of $f(x) = ax^3 + bx^2 + cx + d$ and $g(x) = 2x + 14$.



(1) Show that $a = -1$, $b = -4$, $c = 3$ and $d = 18$.

(2) Calculate the coordinates of A, a stationary point of f.

(3) Calculate the coordinates of the point of inflection of f.

(4) Calculate the x-coordinates of P, Q and R.

(5) Determine the equation of the tangent to f at $x = -2$.

(6) For which value(s) of x is

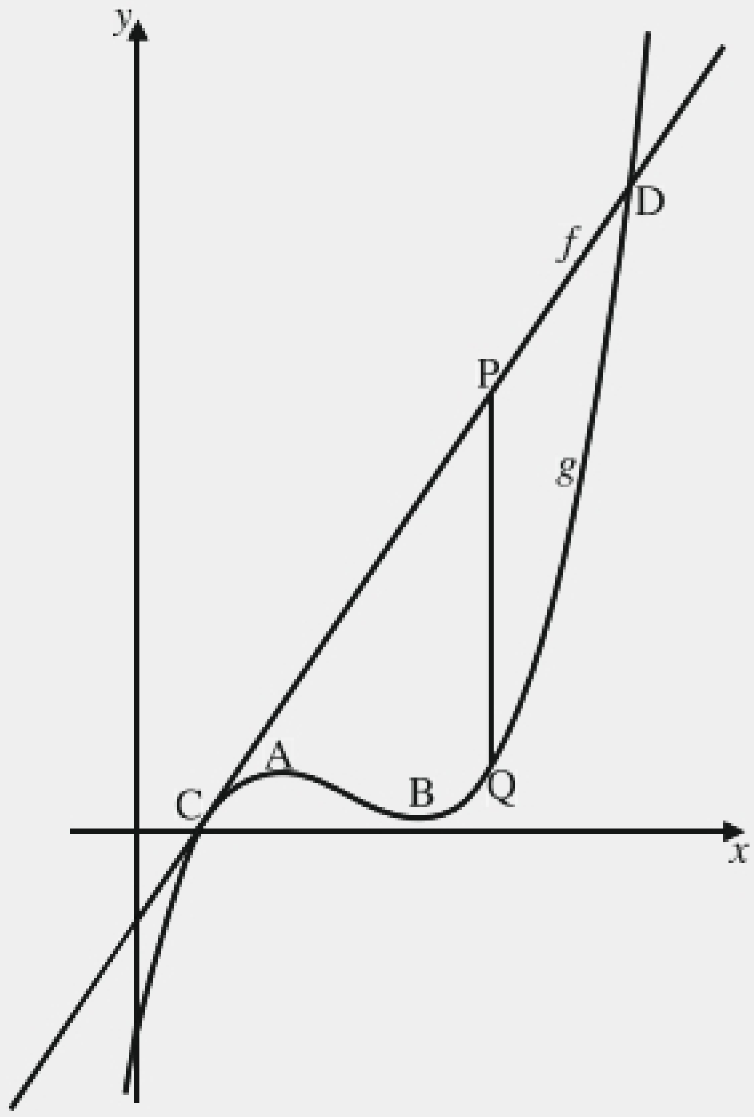
(i) f increasing?

(ii) $f(x) \geq g(x)$?

(iii) $x \cdot f'(x) < 0$?

(iv) $f(x) \cdot f'(x) \geq 0$?

(o) The sketch alongside shows the graphs of $f(x) = 9x + c$ and $g(x) = x^3 - ax^2 + bx - 15$. f is the tangent to g at $x = 1$. A and B are the stationary points of f. The coordinates of B are $(4; 1)$. PQ is a line segment, parallel to the y-axis with P on f and Q on g.



(1) Show that $a = 9$ and $b = 24$.

(2) Calculate the coordinates of A.

(3) Show that $c = -8$.

(4) Calculate the coordinates of D.

(5) For which values of x is g decreasing?

(6) Calculate the maximum length of PQ.

(7) For which values of k will

(i) $g(x) = k$ have two distinct real roots?

(ii) $x^3 - 9x^2 + 24x - 15 = k$ have three distinct real roots?

(iii) $y = g(x) + k$ have only one x-intercept?

(p) The sketch alongside shows the graph of f', a quadratic function.

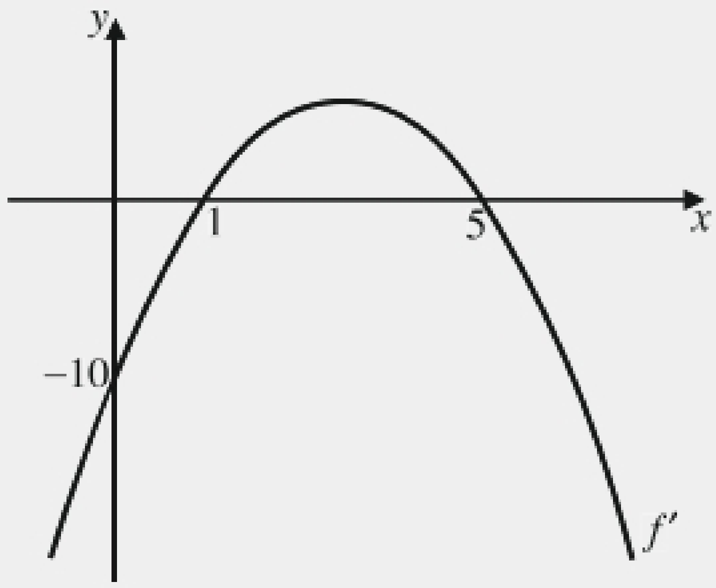
(1) What is the gradient of the tangent to f at
(i) $x = 0$? (ii) $x = 6$?

(2) Write down the x-coordinate(s) of the stationary point(s) of f.

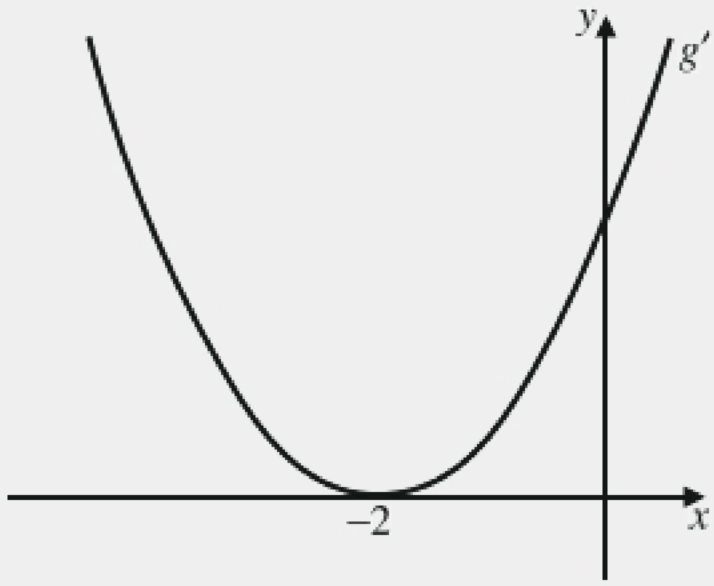
(3) Write down the x-coordinate of the point of inflection of f.

(4) For which values of x is f decreasing?

(5) Is f concave up or concave down at
(i) $x = 1$? (ii) $x = 4$?



(q) The sketch alongside shows the graph of $y = g'(x)$. g is a quadratic function.
$g(-2) = 1$, $g(0) = 9$ and $g(-3) = 0$.
Draw a rough sketch of g.



(r) During an experiment, the temperature T (in degrees Celcius) in an enclosed space is given by $T = -2t^3 + 27t^2 - 108t + 140$. t is the time in hours. The experiment lasts for 7 hours.

(1) What is the temperature in the enclosed space at the start of the experiment?

(2) What is the temperature in the enclosed space at the end of the experiment?

(3) At what rate is the temperature changing exactly 2 hours into the experiment?

(4) At what time does the temperature start increasing?

(5) What is the minimum temperature reached during the experiment?

(6) At what time does the temperature increase most rapidly?

(s) An object moves in a vertical line under the influence of gravity such that its height above ground level is given by $s(t) = -5t^2 + 15t + 20$, where $s(t)$ is measured in metres and t is the time in seconds after the moment stone was thrown.

(1) At what velocity was the stone thrown?

(2) At what velocity does the stone move exactly one second after it was thrown?

(3) Is the stone *speeding up* or *slowing down* after exactly
(i) 1 second? (ii) 3 seconds?

(4) After how many seconds does the stone reach its maximum height?

(5) After how many seconds does the stone reach ground level?

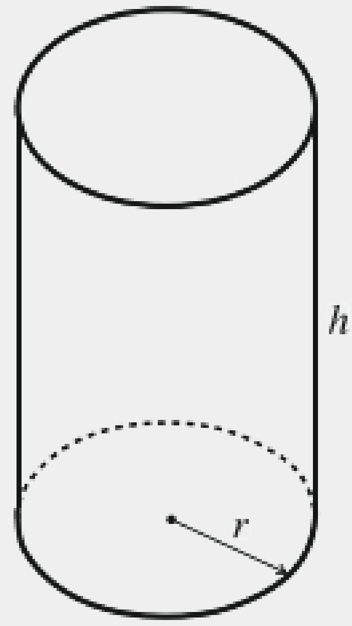
(6) At what speed does the stone hit the ground?

(t) Given the function $f(x) = 6x - 24\sqrt{x} + 30$.

(1) For which value of x does $f(x)$ have a minimum value?

(2) Determine the minimum value of $f(x)$.

(u) A cylindrical can is to be made so that the sum of its height and the circumference of its base is 15π cm. Let the radius of the base be r and the height h as shown in the sketch alongside.



(1) Show that the capacity (volume) of the can is given by $C = 15\pi^2 r^2 - 2\pi^2 r^3$

(2) Determine the maximum capacity of the can and give your answer correct to the nearest millilitre.

(v) A piece of wire with length 1 m is cut into two smaller pieces. One of the pieces is bent into a square and the other into a circle. Let the side length of the square be x.

(1) Show that the sum of the areas of the square and the circle is given by

$$S = \frac{\pi + 4}{\pi}x^2 - \frac{2}{\pi}x + \frac{1}{4\pi}$$

(2) If the piece of wire is cut so as to minimise the sum of the areas of the circle and the square, determine

(i) the value of x in centimetres (round to the nearest centimetre).

(ii) the lengths of the two pieces of wire in centimetres (round to the nearest centimetre).

(w)* The function $y = x^3 + bx^2 + cx + d$ has a stationary point of inflection at $(3; 5)$. Determine the values of b, c and d.

(x)* Let $f(x) = ax^3 + bx^2 + cx$

(1) Show that the x-coordinates of the stationary points of f are given by

$$x = \frac{-b \pm \sqrt{b^2 - 3ac}}{3a}.$$

(2) Use the formula in (1) to determine how many stationary points f will have if

(i) $a = 1,\ b = 3$ and $c = 3$

(ii) $a = b = c \neq 0$

(3) Express the x-coordinate of the point of inflection of f in terms of a and b.

(y)* For which values of k will the function $f(x) = x^3 + 3x^2 + kx - 1$ have both a local minimum and a local maximum?

(z)** Given $f(x) = 3x^4 - 4x^3$.

(1) Calculate all the x-intercepts of f.

(2) Calculate the coordinates of all the stationary points of f and classify each stationary point as a local minimum, a local maximum or a stationary point of inflection.

(3) Calculate the coordinates of all the points of inflection of f.

(4) Sketch the graph of f showing all relevant details.

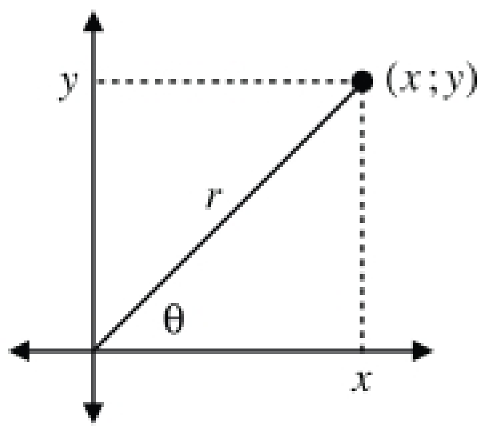
CHAPTER FIVE

General Trigonometry

This year you will learn some new trigonometric identities. You will have to be familiar with all the content you've learnt in Grade 11 as it will be used extensively throughout this chapter.

OVERVIEW OF BASIC CONCEPTS

DEFINITIONS AND PYTHAGORAS



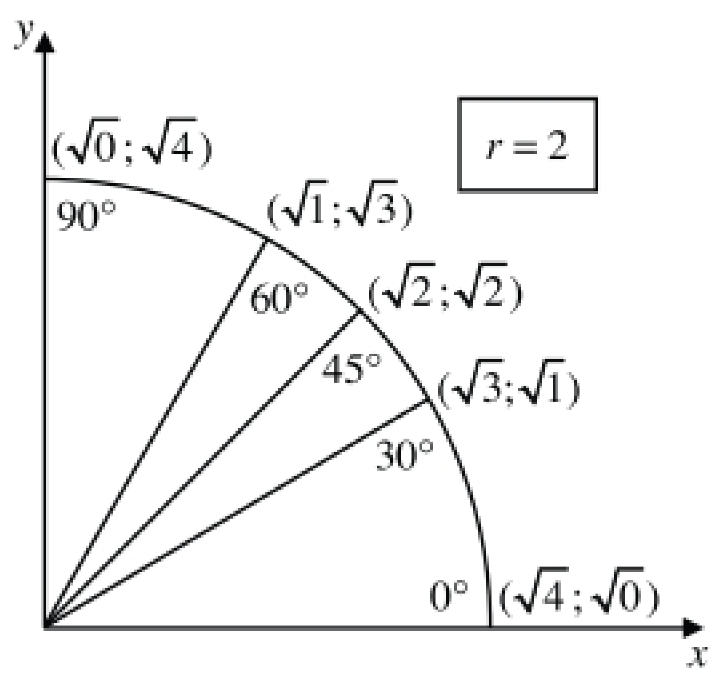
$$\sin\theta = \frac{y}{r}$$

$$\cos\theta = \frac{x}{r}$$

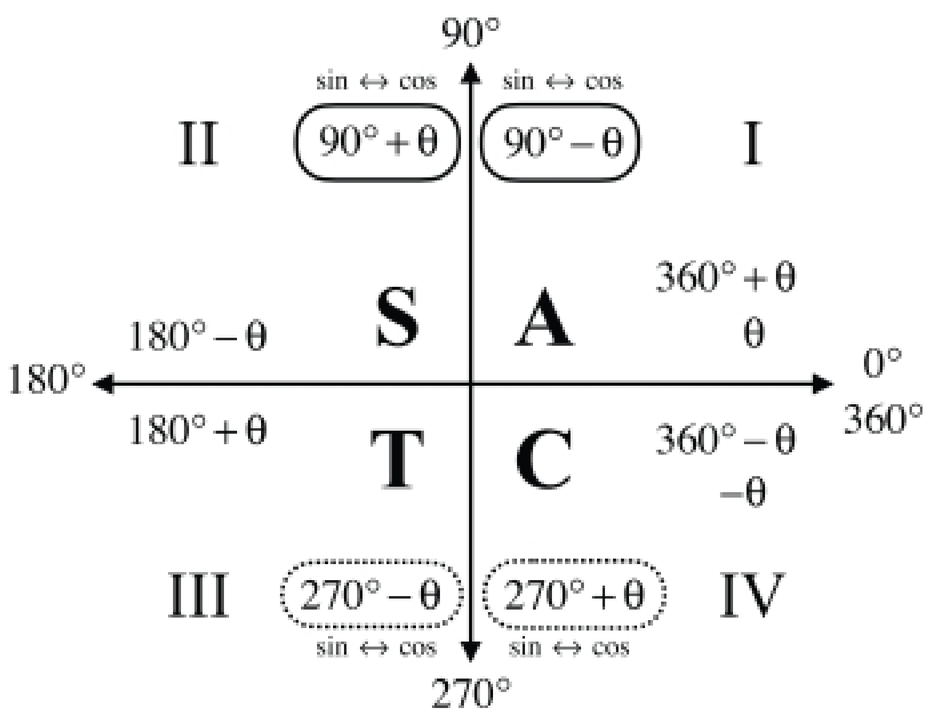
$$\tan\theta = \frac{y}{x}$$

$$x^2 + y^2 = r^2$$

SPECIAL ANGLES



REDUCTION



* Any multiple of 360° can be added to (or subtracted from) an angle if necessary.

IDENTITIES

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = 1 - \cos^2\theta \qquad \cos^2\theta = 1 - \sin^2\theta$$

COMPOUND ANGLES

A compound angle is an angle formed by adding or subtracting two angles, for example $\hat{A}+\hat{B}$, $\alpha-\beta$, $\theta+30°$, $x-60°$ etc. When dealing with compound angles, the following identities apply:

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$
$$\sin(A-B)=\sin A\cos B-\cos A\sin B$$

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$
$$\cos(A-B)=\cos A\cos B+\sin A\sin B$$

These formulae can be used to **expand** a trigonometric ratio of a compound angle into an expression consisting of trigonometric ratios of the individual angles:

EXAMPLE 1

Expand the following and simplify if possible:

(a) $\sin(x+y)$ (b) $\cos(\alpha+10°)$

(c) $\cos(x-60°)$ (d) $\sin(3x-45°)$

Solution

(a) $\sin(x+y)$
$=\sin x\cos y+\cos x\sin y$

(b) $\cos(\alpha+10°)$
$=\cos\alpha\cos 10°-\sin\alpha\sin 10°$

(c) $\cos(x-60°)$
$=\cos x\cos 60°+\sin x\sin 60°$
$=\cos x\left(\dfrac{1}{2}\right)+\sin x\left(\dfrac{\sqrt{3}}{2}\right)$
$=\dfrac{\cos x+\sqrt{3}\sin x}{2}$

(d) $\sin(3x-45°)$
$=\sin 3x\cos 45°-\cos 3x\sin 45°$
$=\sin 3x\left(\dfrac{\sqrt{2}}{2}\right)-\cos 3x\left(\dfrac{\sqrt{2}}{2}\right)$
$=\dfrac{\sqrt{2}\sin 3x-\sqrt{2}\cos 3x}{2}$

The compound angle formulae can also be used to **contract** certain trigonometric expressions into a **single trigonometric ratio**:

EXAMPLE 2

Write the following as a single trigonometric ratio:

(a) $\cos\alpha\cos\beta+\sin\alpha\sin\beta$ (b) $\sin x\cos 40°+\cos x\sin 40°$

(c) $\cos 2x\sin 3x-\sin 2x\cos 3x$ (d) $\cos 5A\cos A-\sin A\sin 5A$

Solution

(a) $\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$=\cos(\alpha-\beta)$

(b) $\sin x\cos 40°+\cos x\sin 40°$
$=\sin(x+40°)$

(c) $\cos 2x\sin 3x-\sin 2x\cos 3x$
$=\sin 3x\cos 2x-\cos 3x\sin 2x$
$=\sin(3x-2x)$
$=\sin x$

(d) $\cos 5A\cos A-\sin A\sin 5A$
$=\cos 5A\cos A-\sin 5A\sin A$
$=\cos(5A+A)$
$=\cos 6A$

EXAMPLE 3

Simplify: $\sin(x+30°)-\sin(x-30°)$

Solution

$$\begin{aligned}\sin(x+30°)-\sin(x-30°) &= \sin x\cos 30° + \cos x\sin 30° - (\sin x\cos 30° - \cos x\sin 30°)\\ &= 2\cos x\sin 30°\\ &= 2\cos x\left(\frac{1}{2}\right)\\ &= \cos x\end{aligned}$$

EXAMPLE 4

Calculate the values of the following without the use of a calculator:

(a) $\cos 75°$

(b) $\sin 70°\sin 10° + \cos 10°\cos 70°$

Solution

(a) $\cos 75°$

$$\begin{aligned}&= \cos(30°+45°)\\ &= \cos 30°\cos 45° - \sin 30°\sin 45°\\ &= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &= \frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

(b) $\sin 70°\sin 10° + \cos 10°\cos 70°$

$$\begin{aligned}&= \cos 70°\cos 10° + \sin 70°\sin 10°\\ &= \cos(70°-10°)\\ &= \cos 60°\\ &= \frac{1}{2}\end{aligned}$$

EXAMPLE 5*

Write $\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x$ as a single trigonometric ratio:

Solution

We know that $\frac{1}{2}=\sin 30°$ and $\frac{\sqrt{3}}{2}=\cos 30°$.

$$\begin{aligned}\therefore \frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\sin x &= \sin 30°\cos x + \cos 30°\sin x\\ &= \sin(30°+x)\end{aligned}$$

EXERCISE 1

(a) Expand the following and simplify if possible:

(1) $\cos(\alpha+\beta)$

(2) $\sin(x-25°)$

(3) $\sin(x+45°)$

(4) $\cos(30°-5x)$

(b) Write the following as a single trigonometric ratio:

(1) $\sin x\cos y - \cos x\sin y$

(2) $\cos\alpha\cos 12° + \sin\alpha\sin 12°$

(3) $\cos\theta\cos 4\theta - \sin\theta\sin 4\theta$

(4) $\sin 43°\cos 33° - \cos 43°\sin 33°$

(5) $\cos 14°\sin 6° + \cos 6°\sin 14°$

(6) $\sin 6x\sin 5x + \cos 6x\cos 5x$

(7) $\sin(x-20°)\cos 20° + \cos(x-20°)\sin 20°$

(c) Simplify the following:

(1) $\sin(\theta - 45°) + \sin(\theta + 45°)$ (2) $\cos(\alpha + 60°) - \cos(\alpha - 60°)$

(3) $\sin(\beta + 60°) - \cos(\beta - 30°)$ (4) $\cos(A + 60°) - \sin(A - 30°)$

(5) $\cos(x + 60°)\cos x + \sin(x + 60°)\sin x$ (6) $\sqrt{3}\cos(x - 30°) - \cos(x - 60°)$

(d) Calculate the following without the use of a calculator:

(1) $\sin 75°$ (2) $\cos 15°$

(3) $\sin 50° \cos 10° + \cos 50° \sin 10°$ (4) $\cos 70° \cos 40° + \sin 70° \sin 40°$

(5) $(\cos 20° \sin 65° - \sin 20° \cos 65°)^2$ (6) $-\sin 40° \cos 10° + \cos 40° \sin 10°$

(7) $-\cos 80° \cos 35° - \sin 80° \sin 35°$ (8) $\sin 55° \sin 5° - \cos 55° \cos 5°$

(e) Show that $\sqrt{3}\cos x + \sin x = 2\cos(x - 30°)$.

(f)* Write the following as a single trigonometric ratio:

(1) $\frac{1}{2}\cos 10° - \frac{\sqrt{3}}{2}\sin 10°$ (2) $\frac{\sqrt{2}(\cos\alpha + \sin\alpha)}{2}$

(g)* (1) Show that $\sin(\theta + x) + \sin(\theta - x) = 2\sin\theta\cos x$.

(2) Hence show that $\sin 43° + \sin 17° = \cos 13°$.

(h)* Calculate the value of $\sin 50° + \cos 50° \tan 20°$ without the use of a calculator.

DOUBLE ANGLES

A double angle is an angle formed by adding an angle to itself, for example $\hat{A} + \hat{A} = 2\hat{A}$, $\alpha + \alpha = 2\alpha$, $10° + 10° = 20°$ *et cetera*. Consider the following:

$$\begin{aligned} &\sin 2A \\ &\sin(A + A) \\ &= \sin A \cos A + \cos A \sin A \\ &= 2\sin A \cos A \end{aligned}$$

$$\boxed{\sin 2A = 2\sin A \cos A}$$

$$\begin{aligned} &\cos 2A \\ &= \cos(A + A) \\ &= \cos A \cos A - \sin A \sin A \\ &= \cos^2 A - \sin^2 A \end{aligned}$$

$$\boxed{\cos 2A = \cos^2 A - \sin^2 A}$$

By using the identity $\sin^2 A + \cos^2 A = 1$, we can rewrite the formula for $\cos 2A$ into two alternative forms:

$$\begin{aligned} \cos^2 A &= 1 - \sin^2 A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - \sin^2 A - \sin^2 A \\ &= 1 - 2\sin^2 A \end{aligned}$$

$$\boxed{\cos 2A = 1 - 2\sin^2 A}$$

$$\begin{aligned} \sin^2 A &= 1 - \cos^2 A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= \cos^2 A - (1 - \cos^2 A) \\ &= 2\cos^2 A - 1 \end{aligned}$$

$$\boxed{\cos 2A = 2\cos^2 A - 1}$$

The double angle formulae can be used to simplify trigonometric expressions:

EXAMPLE 6

Simplify the following:

(a) $\dfrac{\sin 2\theta}{\sin\theta}$ (b) $\dfrac{\cos 2\alpha}{\cos\alpha - \sin\alpha}$

(c) $\dfrac{\cos 2A + 1}{2\cos A}$ (d) $\cos 2x + 2\sin^2 x$

Solution

(a) $\dfrac{\sin 2\theta}{\sin\theta}$

$= \dfrac{2\sin\theta\cos\theta}{\sin\theta}$

$= 2\cos\theta$

(b) $\dfrac{\cos 2\alpha}{\cos\alpha - \sin\alpha}$

$= \dfrac{\cos^2\alpha - \sin^2\alpha}{\cos\alpha - \sin\alpha}$

$= \dfrac{(\cos\alpha - \sin\alpha)(\cos\alpha + \sin\alpha)}{\cos\alpha - \sin\alpha}$

$= \cos\alpha + \sin\alpha$

(c) $\dfrac{\cos 2A + 1}{2\cos A}$

$= \dfrac{2\cos^2 A - 1 + 1}{2\cos A}$

$= \dfrac{2\cos^2 A}{2\cos A}$

$= \cos A$

(d) $\cos 2x + 2\sin^2 x$

$= 1 - 2\sin^2 x + 2\sin^2 x$

$= 1$

The double angle formulae can also be used to **contract** certain trigonometric expressions into a **single trigonometric ratio**:

EXAMPLE 7

Write each of the following as a single trigonometric ratio:

(a) $2\sin\theta\cos\theta$ (b) $\cos^2 x - \sin^2 x$ (c) $2\cos^2 B - 1$

(d) $1 - 2\sin^2\alpha$ (e) $2\sin 3x\cos 3x$ (f) $2\cos^2\dfrac{x}{2} - 1$

(g) $\cos^2 10° - \sin^2 10°$ (h) $2\cos 33°\sin 33°$ (i) $1 - 2\sin^2 21°$

Solution

(a) $2\sin\theta\cos\theta$
$= \sin 2\theta$

(b) $\cos^2 x - \sin^2 x$
$= \cos 2x$

(c) $2\cos^2 B - 1$
$= \cos 2B$

(d) $1 - 2\sin^2\alpha$
$= \cos 2\alpha$

(e) $2\sin 3x\cos 3x$
$= \sin 6x$

(f) $2\cos^2\dfrac{x}{2} - 1$
$= \cos x$

(g) $\cos^2 10° - \sin^2 10°$
$= \cos 20°$

(h) $2\cos 33°\sin 33°$
$= \sin 66°$

(i) $1 - 2\sin^2 21°$
$= \cos 42°$

EXAMPLE 8

Calculate the following without the use of a calculator:

(a) $2\sin 15°\cos 15°$

(b) $\sin 22{,}5°\cos 22{,}5°$

(c) $\cos^2 15° - \sin^2 15°$

(d) $\cos^2 15° + \sin^2 15°$

(e) $\sin^2 22{,}5° - \cos^2 22{,}5°$

(f) $-\cos^2 22{,}5° - \sin^2 22{,}5°$

(g) $2\cos^2 15° - 1$

(h) $1 - 2\sin^2 22{,}5°$

(i) $2\sin^2 22{,}5° - 1$

(j) $2\cos^2 15°$

Solution

(a)
$$\begin{aligned}&2\sin 15°\cos 15°\\&=\sin(2\times 15°)\\&=\sin 30°\\&=\frac{1}{2}\end{aligned}$$

(b)
$$\begin{aligned}&\sin 22{,}5°\cos 22{,}5°\\&=\frac{1}{2}(2\sin 22{,}5°\cos 22{,}5°)\\&=\frac{1}{2}\sin(2\times 22{,}5°)\\&=\frac{1}{2}\sin(45°)\\&=\frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)\\&=\frac{\sqrt{2}}{4}\end{aligned}$$

(c)
$$\begin{aligned}&\cos^2 15° - \sin^2 15°\\&=\cos(2\times 15°)\\&=\cos 30°\\&=\frac{\sqrt{3}}{2}\end{aligned}$$

(d)
$$\begin{aligned}&\cos^2 15° + \sin^2 15°\\&=1\end{aligned}$$

(e)
$$\begin{aligned}&\sin^2 22{,}5° - \cos^2 22{,}5°\\&=-(\cos^2 22{,}5° - \sin^2 22{,}5°)\\&=-\cos(2\times 22{,}5°)\\&=-\cos 45°\\&=-\frac{\sqrt{2}}{2}\end{aligned}$$

(f)
$$\begin{aligned}&-\cos^2 22{,}5° - \sin^2 22{,}5°\\&=-(\cos^2 22{,}5° + \sin^2 22{,}5°)\\&=-1\end{aligned}$$

(g)
$$\begin{aligned}&2\cos^2 15° - 1\\&=\cos(2\times 15°)\\&=\cos 30°\\&=\frac{\sqrt{3}}{2}\end{aligned}$$

(h)
$$\begin{aligned}&1 - 2\sin^2 22{,}5°\\&=\cos(2\times 22{,}5°)\\&=\cos 45°\\&=\frac{\sqrt{2}}{2}\end{aligned}$$

(i) $2\sin^2 22{,}5° - 1$
$= -(1 - 2\sin^2 22{,}5°)$
$= -\cos(2 \times 22{,}5°)$
$= -\cos 45°$
$= -\dfrac{\sqrt{2}}{2}$

(j) $2\cos^2 15°$
$= 2\cos^2 15° - 1 + 1$
$= \cos(2 \times 15°) + 1$
$= \cos 30° + 1$
$= \dfrac{\sqrt{3}}{2} + 1$
$= \dfrac{\sqrt{3}+2}{2}$

EXERCISE 2

(a) Write each of the following as a single trigonometric ratio:

(1)	$\cos^2\alpha - \sin^2\alpha$	(2)	$2\sin x\cos x$	(3)	$1 - 2\sin^2\theta$
(4)	$2\cos^2 P - 1$	(5)	$2\sin\dfrac{\theta}{4}\cos\dfrac{\theta}{4}$	(6)	$1 - 2\sin^2 5\alpha$
(7)	$\cos^2 11° - \sin^2 11°$	(8)	$2\sin 5°\cos 5°$	(9)	$2\cos^2 35° - 1$

(b) Calculate the following without the use of a calculator:

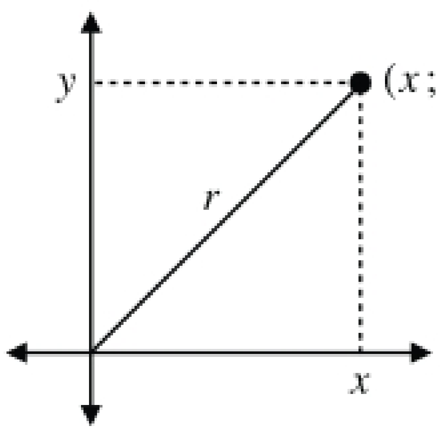
(1)	$2\sin 22{,}5°\cos 22{,}5°$	(2)	$\cos^2 22{,}5° - \sin^2 22{,}5°$
(3)	$\sin^2 15° - \cos^2 15°$	(4)	$\cos^2 22{,}5° + \sin^2 22{,}5°$
(5)	$-\cos^2 15° - \sin^2 15°$	(6)	$2\cos^2 22{,}5° - 1$
(7)	$1 - 2\sin^2 15°$	(8)	$1 - 2\cos^2 15°$
(9)	$2\sin^2 15° - 1$	(10)	$\sin 15°\cos 15°$
(11)	$2\cos^2 22{,}5°$	(12)	$\dfrac{1}{2} - \sin^2 15°$
(13)	$(\cos 15° + \sin 15°)^2$	(14)	$(\sin 15° - \cos 15°)(\sin 15° + \cos 15°)$

(c) Simplify the following:

(1)	$\dfrac{\sin 2x}{2\cos^2 x}$	(2)	$\dfrac{\cos 2\theta}{\sin\theta + \cos\theta}$	(3)	$\dfrac{1 - \cos 2A}{1 - \cos^2 A}$
(4)	$\dfrac{(\cos\alpha - \sin\alpha)^2}{\sin 2\alpha - 1}$	(5)	$\dfrac{\cos 2x}{\cos^4 x - \sin^4 x}$		

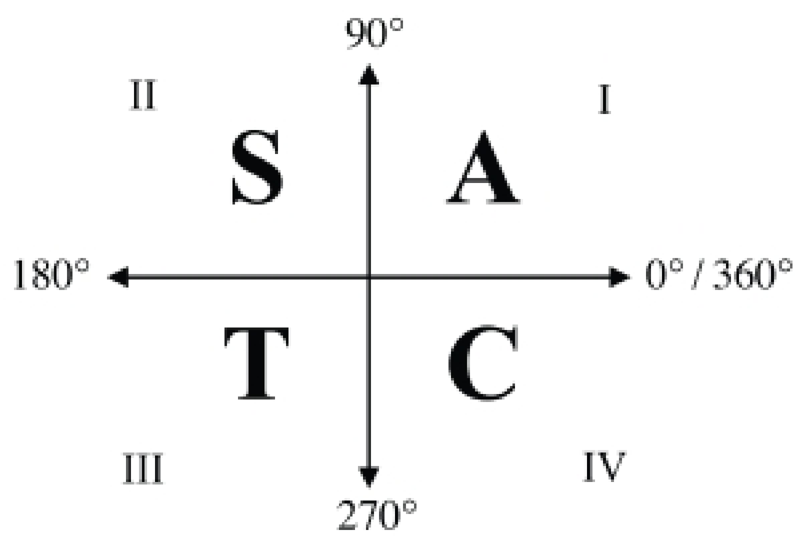
(d)* Calculate the following without the use of a calculator: $\dfrac{\sin 39°}{\sin 13°} - \dfrac{\cos 39°}{\cos 13°}$

APPLYING THE DEFINITIONS OF THE TRIGONOMETRIC RATIOS



$$x^2 + y^2 = r^2$$

$$\sin\theta = \frac{y}{r} \qquad \cos\theta = \frac{x}{r} \qquad \tan\theta = \frac{y}{x}$$



The definitions of the trigonometric ratios can be used together with the compound and/or double angle formulae to calculate the values of trigonometric expressions without the use of a calculator:

EXAMPLE 9

If $5\tan\alpha - 12 = 0$ with $\alpha \in (180°\,;\,360°)$ and $5\cos\beta = -3$ with $\sin\beta > 0$, determine the value of the following trigonometric ratios without the use of a calculator and with the aid of a diagram:

(a) $\sin(\alpha+\beta)$ (b) $\sin(\alpha-\beta)$ (c) $\cos(\alpha+\beta)$

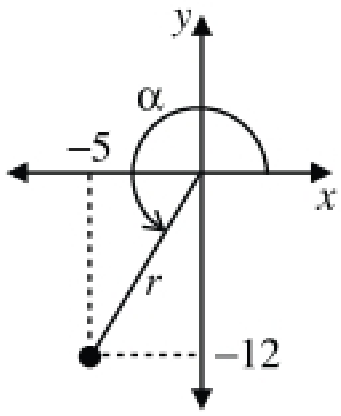
(d) $\sin 2\beta$ (e) $\cos 2\beta$

Solution

$\boxed{\alpha}$

$\tan\alpha = \boxed{+}\dfrac{12}{5}$ $\qquad$ $\alpha \in (180°\,;\,360°)$

α is a third quadrant angle:



$(-5)^2 + (-12)^2 = r^2$

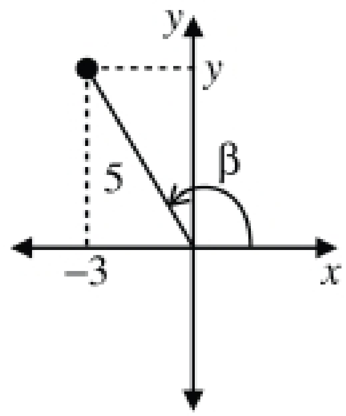
$\therefore r = 13$

$x = -5$ $\quad$ $y = -12$ $\quad$ $r = 13$

$\boxed{\beta}$

$\cos\beta = \boxed{-}\dfrac{3}{5}$ $\qquad$ $\sin\beta > 0$

β is a second quadrant angle:



$(-3)^2 + y^2 = 5^2$

$\therefore y = 4$

$x = -3$ $\quad$ $y = 4$ $\quad$ $r = 5$

(a) $\sin(\alpha+\beta)$

$= \sin\alpha\cos\beta + \cos\alpha\sin\beta$

$= \left(\dfrac{-12}{13}\right)\left(\dfrac{-3}{5}\right) + \left(\dfrac{-5}{13}\right)\left(\dfrac{4}{5}\right) = \dfrac{16}{65}$

(b) $\sin(\alpha-\beta)$

$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$

$= \left(\dfrac{-12}{13}\right)\left(\dfrac{-3}{5}\right) - \left(\dfrac{-5}{13}\right)\left(\dfrac{4}{5}\right) = \dfrac{56}{65}$

(c) $\cos(\alpha+\beta)$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$= \left(\dfrac{-5}{13}\right)\left(\dfrac{-3}{5}\right) - \left(\dfrac{-12}{13}\right)\left(\dfrac{4}{5}\right) = \dfrac{63}{65}$

(d) $\sin 2\beta$

$= 2\sin\beta\cos\beta$

$= 2\left(\dfrac{4}{5}\right)\left(\dfrac{-3}{5}\right) = -\dfrac{24}{25}$

(e) $\cos 2\beta$

$= 2\cos^2\beta - 1$

$= 2\left(-\dfrac{3}{5}\right)^2 - 1 = \dfrac{18}{25} - 1 = -\dfrac{7}{25}$

EXERCISE 3

(a) If $\sin A = \frac{4}{5}$ with $90° < \hat{A} < 360°$ and $\tan B = -\frac{3}{4}$ where $180° < \hat{B} < 360°$, determine the value of the following without the use of a calculator and with the aid of a diagram:

(1) $\cos(A - B)$ (2) $\sin(A + B)$ (3) $\cos 2A$

(4) $\sin 2B$

(b) If $5\cos\alpha + 4 = 0$ with $\alpha \in [0°;180°]$ and $13\sin\beta + 12 = 0$ with $\tan\beta > 0$, determine the value of the following without the use of a calculator and with the aid of a diagram:

(1) $\sin(\alpha - \beta)$ (2) $\cos(\alpha + \beta)$ (3) $\cos 2\alpha$

(4) $\tan 2\alpha$

(c) If $2\tan 2\theta = -\sqrt{5}$ with $0° < 2\theta < 180°$, determine the value of the following without the use of a calculator and with the aid of a diagram:

(1) $\sin 2\theta$ (2) $\sin\theta\cos\theta$ (3) $2\cos^2\theta$

(4) $\cos\theta$

(d) $6\sin 2P + \sqrt{11} = 0$ with $2\hat{P} \in [270°; 360°]$, determine the value of the following without the use of a calculator and with the aid of a diagram:

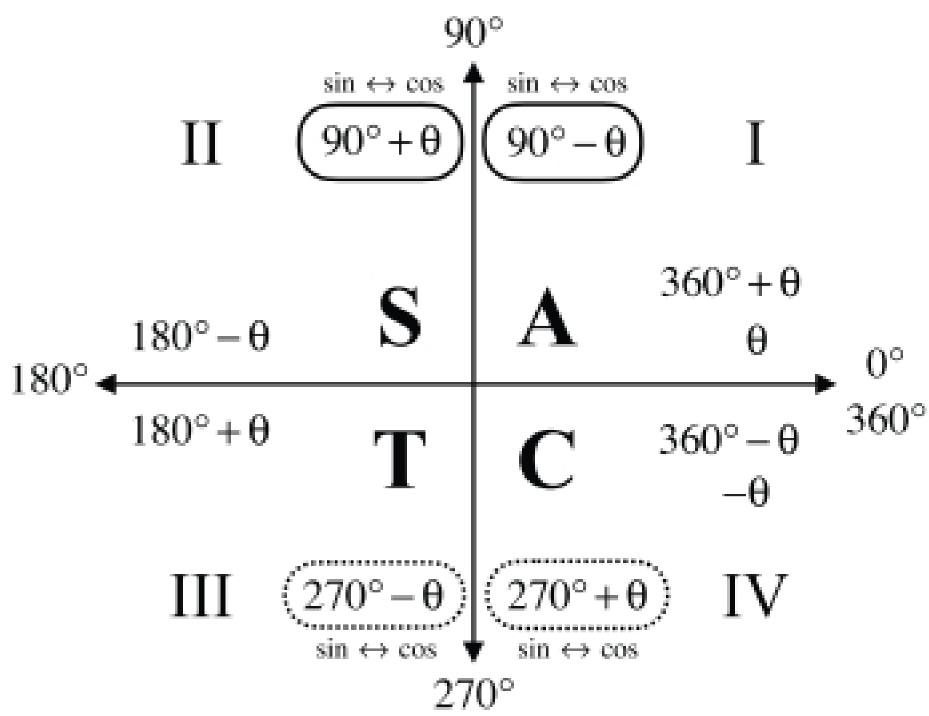
(1) $\cos 2P$ (2)* $\cos P$ (3)* $\sin P$

(e) If $\tan^2\theta = 7$ with $\theta \in (-90°; -180°)$, determine the value of the following without the use of a calculator and with the aid of a diagram:

(1) $\cos(\theta + 45°)$ (2) $\sin 2\theta$

REDUCTION

The following diagram (the CAST-diagram) summarises all the reductions you have learnt about in Grade 11:



* Any multiple of 360° can be added to (or subtracted from) an angle if necessary.

Reduction is often necessary before compound- or double angle patterns can be recognised:

EXAMPLE 10

Simplify the following expressions:

(a) $\sin(360°-\alpha)\cos(180°-\beta)+\cos(360°-\alpha)\sin(180°+\beta)$

(b) $\cos(-\theta)\cos(360°+\theta)+\cos(90°-\theta)\sin(540°+\theta)$

(c) $\sin(180°-x)\sin(90°-x)+\cos(x-180°)\sin(-x)$

Solution

(a) $\sin(360°-\alpha)\cos(180°-\beta)+\cos(360°-\alpha)\sin(180°+\beta)$
$=(-\sin\alpha)(-\cos\beta)+(\cos\alpha)(-\sin\beta)$
$=\sin\alpha\cos\beta-\cos\alpha\sin\beta$
$=\sin(\alpha-\beta)$

(b) $\cos(-\theta)\cos(360°+\theta)+\cos(90°-\theta)\sin(540°+\theta)$
$=\cos(-\theta)\cos(360°+\theta)+\cos(90°-\theta)\sin(180°+\theta)$ $*\,540°+\theta-360°=180°+\theta$
$=(\cos\theta)(\cos\theta)+(\sin\theta)(-\sin\theta)$
$=\cos^2\theta-\sin^2\theta$
$=\cos 2\theta$

(c) $\sin(180°-x)\sin(90°-x)+\cos(x-180°)\sin(-x)$
$=\sin(180°-x)\sin(90°-x)+\cos(180°+x)\sin(-x)$ $*\,x-180°+360°=180°+x$
$=(\sin x)(\cos x)+(-\cos x)(-\sin x)$
$=\sin x\cos x+\cos x\sin x$
$=2\sin x\cos x$
$=\sin 2x$

EXAMPLE 11

Simplify: $$\frac{\cos(720°-2\alpha)}{\sin^2(180°+\alpha)+\sin(90°+\alpha)\cos(\alpha+180°)}$$

Solution

$$\frac{\cos(720°-2\alpha)}{\sin^2(180°+\alpha)+\sin(90°+\alpha)\cos(\alpha+180°)}$$
$$\frac{\cos 2\alpha}{\sin^2\alpha+(\cos\alpha)(-\cos\alpha)}$$
$$=\frac{\cos 2\alpha}{\sin^2\alpha-\cos^2\alpha}$$
$$=\frac{\cos 2\alpha}{-(\cos^2\alpha-\sin^2\alpha)}$$
$$=\frac{\cos 2\alpha}{-\cos 2\alpha}$$
$$=-1$$

After reducing all angles, the basic identities ($\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$) are also often used, together with compound- or double angle formulae, to simplify expressions:

EXAMPLE 12

Simplify the following expressions:

(a) $\dfrac{\tan(-\theta)\cos^2(180° - \theta)}{\sin(360° - 2\theta)}$

(b) $\dfrac{\cos^2(-\theta) + \cos^2(90° + \theta) - \cos(180° + 2\alpha)}{\sin^2(90° + \alpha)}$

Solution

(a)
$$\begin{aligned}
&\frac{\tan(-\theta)\cos^2(180° - \theta)}{\sin(360° - 2\theta)} \\
&= \frac{(-\tan\theta)(\cos^2\theta)}{-\sin 2\theta} \\
&= \frac{\left(-\dfrac{\sin\theta}{\cos\theta}\right)\left(\dfrac{\cos^2\theta}{1}\right)}{-2\sin\theta\cos\theta} \\
&= \frac{-\sin\theta\cos\theta}{-2\sin\theta\cos\theta} \\
&= \frac{1}{2}
\end{aligned}$$

(b)
$$\begin{aligned}
&\frac{\cos^2(-\theta) + \cos^2(90° + \theta) - \cos(180° + 2\alpha)}{\sin^2(90° + \alpha)} \\
&= \frac{\cos^2\theta + \sin^2\theta - (-\cos 2\alpha)}{\cos^2\alpha} \\
&= \frac{1 + \cos 2\alpha}{\cos^2\alpha} \\
&= \frac{1 + 2\cos^2\alpha - 1}{\cos^2\alpha} \\
&= \frac{2\cos^2\alpha}{\cos^2\alpha} \\
&= 2
\end{aligned}$$

EXAMPLE 13

Simplify the following expressions:

(a) $\dfrac{2\cos(-x - 180°)\cos(90° + x)}{1 - 2\sin^2(-x)}$

(b) $\dfrac{1 - \sin^2(180° + 2x)}{1 + 2\cos(180° + x)\sin(90° - x)}$

Solution

(a)
$$\begin{aligned}
&\frac{2\cos(-x - 180°)\cos(90° + x)}{1 - 2\sin^2(-x)} \\
&= \frac{-2\cos(180° - x)\cos(90° + x)}{1 - 2\sin^2(-x)} \\
&= \frac{2(-\cos x)(-\sin x)}{1 - 2\sin^2 x} \\
&= \frac{2\sin x\cos x}{1 - 2\sin^2 x} \\
&= \frac{\sin 2x}{\cos 2x} \\
&= \tan 2x
\end{aligned}$$

(b)
$$\begin{aligned}
&\frac{1 - \sin^2(180° + 2x)}{1 + 2\cos(180° + x)\sin(90° - x)} \\
&= \frac{1 - \sin^2 2x}{1 + 2(-\cos x)(\cos x)} \\
&= \frac{1 - \sin^2 2x}{1 - 2\cos^2 x} \\
&= \frac{\cos^2 2x}{-(2\cos^2 x - 1)} \\
&= \frac{\cos^2 2x}{-\cos 2x} \\
&= -\cos 2x
\end{aligned}$$

EXERCISE 4

(a) Simplify the following expressions:

(1) $\sin(180°-\alpha)\cos(360°-\beta)+\cos(180°+\alpha)\sin(360°-\beta)$

(2) $\sin(90°-A)\cos(-B)+\sin(-A)\cos(90°-B)$

(3) $\sin(90°+x)\cos(360°+y)-\sin(180°-x)\cos(90°+y)$

(4) $\sin(-P)\cos(Q-180°)+\sin(180°+Q)\cos(360°-P)$

(5) $\cos^2(180°-\theta)+\sin(360°-\theta)\sin(540°-\theta)$

(6) $\cos(450°-x)\cos(360°-x)+\sin(-x)\cos(180°-x)$

(7) $\cos(-\alpha)\sin(90°+\alpha)+\sin^2(180°+\alpha)$

(8) $\sin(720°-\alpha)\cos(360°-\beta)+\cos(180°+\alpha)\sin(180°-\beta)$

(9) $\cos(90°-x)\sin(-x-180°)+\cos(180°-x)\cos(1080°+x)$

(10) $\sin(A-180°)\cos(-A)-\sin(90°+A)\sin(360°+A)$

(11) $\sin(180°-\theta)\cos(\theta-90°)+\cos(\theta+180°)\cos(360°+\theta)$

(b) Simplify the following expressions:

(1) $1-2\tan(180°-\theta)\sin(360°-\theta)\sin(90°+\theta)$

(2) $\dfrac{\sin(-\alpha)\cos(180°-\alpha)-\cos(360°+\alpha)\sin(180°+\alpha)}{\cos^2(180°-\alpha)+\cos^2(90°-\alpha)}$

(3) $\dfrac{\cos^2(180°+\theta)+\sin(\theta-180°)\sin(360°+\theta)}{\sin(90°-2\theta)}$

(4) $\dfrac{2\tan(x+180°)\cos(720°-x)}{\cos(450°+x)}+1$

(5) $\dfrac{\cos(90°-2x)}{2-2\sin(-x)\cos(90°+x)}$

(6) $\dfrac{\cos^2(180°-\theta)-\cos^2\alpha-\sin^2(\alpha-180°)}{\cos(360°-2\theta)-1}$

(c)* Given that $\cos(A-B)=\cos A\cos B+\sin A\sin B$,

(1) prove that $\cos(A+B)=\cos A\cos B-\sin A\sin B$.

(2) Hence prove that

(i) $\sin(A+B)=\sin A\cos B+\cos A\sin B$

(ii) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

EXPRESSIONS INVOLVING NUMERICAL ANGLES

When calculating trigonometric expressions involving numerical angles, without the use of a calculator, always **reduce all angles to acute angles** first. After doing this, you will have to be able to recognise compound- or double angle patterns.

Also look out for:

- co-functions of complementary angles.
- special angles.

EXAMPLE 14

Calculate the following without the use of a calculator:

(a) $\cos 290°\cos(-10°)+\cos 160°\sin 550°$

(b) $\sin 345°\sin 255°$

(c) $\dfrac{\cos 202,5°\cos 157,5°}{\sin(-210°)}-\tan 225°$

Solution

(a)
$$\begin{aligned}
&\cos 290°\cos(-10°)+\cos 160°\sin 550° \\
&=\cos 290°\cos(-10°)+\cos 160°\sin 190° \qquad *550°-360°=190° \\
&=\cos(360°-70°)\cos(-10°)+\cos(180°-20°)\sin(180°+10°) \\
&=(\cos 70°)(\cos 10°)+(-\cos 20°)(-\sin 10°) \\
&=\cos 70°\cos 10°+\cos 20°\sin 10° \\
&=\cos 70°\cos 10°+\sin 70°\sin 10° \\
&=\cos(70°-10°) \\
&=\cos 60° \\
&=\frac{1}{2}
\end{aligned}$$

(b)
$$\begin{aligned}
&\sin 345°\sin 255° \\
&=(-\sin 15°)(-\sin 75°) \\
&=\sin 15°\sin 75° \\
&=\sin 15°\cos 15° \\
&=\frac{1}{2}(2\sin 15°\cos 15°) \\
&=\frac{1}{2}\sin 30° \\
&=\frac{1}{2}\times\frac{1}{2} \\
&=\frac{1}{4}
\end{aligned}$$

(c)
$$\begin{aligned}
&\frac{\cos 202,5°\cos 157,5°}{\sin(-210°)}-\tan 225° \\
&=\frac{\cos 202,5°\cos 157,5°}{\sin 150°}-\tan 225° \\
&=\frac{(-\cos 22,5°)(-\cos 22,5°)}{\sin 30°}-\tan 45° \\
&=\frac{\cos^2 22,5°}{\left(\frac{1}{2}\right)}-1 \\
&=2\cos^2 22,5°-1 \\
&=\cos 45° \\
&=\frac{\sqrt{2}}{2}
\end{aligned}$$

The compound- and double angle formulae can often be used to manipulate an expression to a numerical value, without the use of a calculator, even if no special angles are present:

EXAMPLE 15

Calculate the following without the use of a calculator:

(a) $\dfrac{\sin 154°}{\sin 193°\cos 373°}$

(b) $\dfrac{\cos^2 173°-\sin^2 7°}{\sin(-66°)\cos 190°-\cos 66°\sin 350°}$

Solution

(a) $\dfrac{\sin 154°}{\sin 193° \cos 373°}$

$= \dfrac{\sin 26°}{(-\sin 13°)(\cos 13°)}$

$= \dfrac{2\sin 13° \cos 13°}{-\sin 13° \cos 13°} = -2$

(b) $\dfrac{\cos^2 173° - \sin^2 7°}{\sin(-66°)\cos 190° - \cos 66° \sin 350°}$

$\dfrac{\cos^2 7° - \sin^2 7°}{(-\sin 66°)(-\cos 10°) - (\cos 66°)(-\sin 10°)}$

$= \dfrac{\cos^2 7° - \sin^2 7°}{\sin 66° \cos 10° + \cos 66° \sin 10°}$

$= \dfrac{\cos 14°}{\sin 76°}$

$= \dfrac{\cos 14°}{\cos 14°} = 1$

The basic identities ($\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ and $\sin^2\theta + \cos^2\theta = 1$) are also often very useful:

EXAMPLE 16

Calculate without the use of calculator: $\dfrac{\tan(-12°)\cos 348°}{\sin^2 10° + \sin^2 80° - 2\sin^2 321°}$

Solution

$\dfrac{\tan(-12°)\cos 348°}{\sin^2 10° + \sin^2 80° - 2\sin^2 321°}$

$= \dfrac{(-\tan 12°)(\cos 12°)}{\sin^2 10° + \cos^2 10° - 2\sin^2 39°}$

$= \dfrac{-\dfrac{\sin 12°}{\cos 12°} \times \dfrac{\cos 12°}{1}}{1 - 2\sin^2 39°}$

$= \dfrac{-\sin 12°}{\cos 78°}$

$= \dfrac{-\sin 12°}{\sin 12°} = -1$

EXERCISE 5

(a) Calculate the following without the use of a calculator:

(1) $\sin 236° \cos 169° + \cos 371° \sin(-56°)$

(2) $\sin 25° \sin 175° + \cos(-25°)\sin 275°$

(3) $\cos^2 195° - \cos^2 285°$

(4) $-\cos^2 197° - \sin^2 343°$

(5) $\dfrac{\sin^2(-337{,}5°)}{\cos 300°} + \tan(-45°)$

(6) $\dfrac{\sin 157{,}5° \cos 562{,}5°}{\sin 330°}$

(7) $\sin(-165°)\cos 345°$

(8) $\cos 7{,}5° \cos 82{,}5° \cos 195°$

(b) Calculate the following without the use of a calculator:

(1) $\dfrac{\sin 134°}{\sin 337° \cos(-157°)}$

(2) $\dfrac{\cos^2 148° + \sin(-32°)\sin 392°}{\sin 206°}$

(3) $\dfrac{2\cos^2 204° - 2\sin 150°}{\cos 320° \cos(-8°) + \sin 220° \cos 82°}$

(4) $\dfrac{\sin 356° \cos 184°}{\sin^2(-41°) - \cos^2 139°}$

(c) Calculate the following without the use of a calculator:

(1) $\dfrac{\tan 190° \cos^2 170°}{\cos 290°}$ (2) $\dfrac{\sin^2 17° + \cos^2 17° - 2\sin^2 205°}{\sin(-140°)}$

(3) $\dfrac{\cos^2 165°}{\cos 60°} - \cos^2 97° - \cos^2 727°$ (4) $\dfrac{\cos 204° \tan 315° + 2\cos 156° \sin^2 348°}{\sin^2(-5°) + \sin^2 85° - \sin^2 24°}$

(5) $\sin 140° - \cos 220° \tan(-155°)$ (6) $\dfrac{\cos^4 190° - \sin^4(-10°)}{\cos 55° \cos 325°}$

(d) Simplify the following without the use of a calculator:

$$\frac{\cos(40° - x)\cos x - \sin(40° - x)\sin x}{\sin 205° \cos 155°}$$

(e)* Calculate the value of $\dfrac{\cos 2° + \sqrt{3}\sin(-2°)}{4\sin 346° \cos 194°}$ without the use of a calculator.

TRIGONOMETRIC VALUES REPRESENTED BY LETTERS

When a trigonometric value is represented by a letter, other trigonometric ratios or expressions can be written in terms of the given letter by using:

- reduction
- co-functions
- the definitions of the trigonometric ratios
- compound- or double angle formulae
- the basic identities

or a combination of these.

EXAMPLE 17

If $\sin 14° = k$, write the following in terms of k:

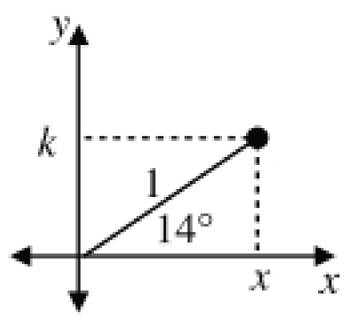
(a)	$\sin 194°$	(b)	$\cos 76°$	(c)	$\cos 14°$
(d)	$\sin 28°$	(e)	$\cos 28°$	(f)	$\tan 28°$
(g)	$\cos 44°$	(h)	$\sin 7° \cos 7°$	(i)	$1 - 2\sin^2 7°$

Solution

(a) $\sin 194°$
$= \sin(180 + 14°)$
$= -\sin 14°$
$= -k$

(b) $\cos 76°$
$= \cos(90° - 14°)$
$= \sin 14°$
$= k$

(c) $\sin 14° = \dfrac{k}{1} \left(\dfrac{y}{r}\right)$



$x^2 + y^2 = r^2$
$\therefore x^2 + k^2 = 1^2$
$\therefore x^2 = 1 - k^2$
$\therefore x = \sqrt{1 - k^2}$

$\boxed{x = \sqrt{1 - k^2}}$
$\boxed{y = k}$
$\boxed{r = 1}$

$$\cos 14° = \frac{x}{r} = \frac{\sqrt{1 - k^2}}{1} = \sqrt{1 - k^2}$$

(d) $\sin 28°$
$= \sin(2 \cdot 14°)$
$= 2\sin 14° \cos 14°$
$= 2k\sqrt{1-k^2}$

(e) $\cos 28°$
$= \cos(2 \cdot 14°)$
$= 1 - 2\sin^2 14°$
$= 1 - 2k^2$

(f) $\tan 28°$
$= \dfrac{\sin 28°}{\cos 28°}$
$= \dfrac{2k\sqrt{1-k^2}}{1-2k^2}$

(g) $\cos 44°$
$= \cos(30° + 14°)$
$= \cos 30° \cos 14° - \sin 30° \sin 14°$
$= \dfrac{\sqrt{3}}{2} \cdot \sqrt{1-k^2} - \dfrac{1}{2}k$
$= \dfrac{\sqrt{3-3k^2} - k}{2}$

(h) $\sin 7° \cos 7°$
$= \dfrac{1}{2}(2\sin 7° \cos 7°)$
$= \dfrac{1}{2}\sin 14°$
$= \dfrac{1}{2}k$

(i) $1 - 2\sin^2 7°$
$= \cos 14°$
$= \sqrt{1-k^2}$

EXAMPLE 18

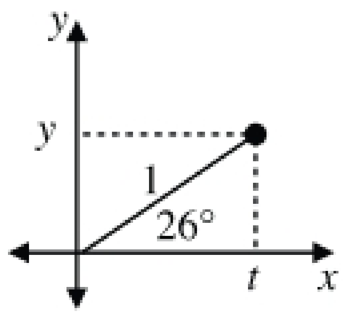
If $\cos 26° = t$, write the following in terms of t:

(a)	$\cos(-26°)$	(b)	$\sin(-26°)$	(c)	$\tan 206°$
(d)	$\sin 34°$	(e)	$\cos^2 167° - \sin^2 193°$	(f)	$(\cos 13° + \sin 13°)^2$
(g)	$2\cos^2 13°$	(h)	$\cos 13°$	(i)	$\sin 13°$

Solution

(a) $\cos(-26°) = \cos 26° = t$

(b) $\cos 26° = \dfrac{t}{1} \quad \left(\dfrac{x}{r}\right)$



$x^2 + y^2 = r^2$
$\therefore t^2 + y^2 = 1^2$
$\therefore y^2 = 1 - t^2$
$\therefore y = \sqrt{1-t^2}$

$\boxed{x = t}$
$\boxed{y = \sqrt{1-t^2}}$
$\boxed{r = 1}$

$\sin(-26°) = -\sin 26° = -\dfrac{\sqrt{1-t^2}}{1} = -\sqrt{1-t^2}$

(c) $\tan 206°$
$= \tan 26°$
$= \dfrac{\sqrt{1-t^2}}{t}$

(d) $\sin 34°$
$= \sin(60° - 26°)$
$= \sin 60° \cos 26° - \cos 60° \sin 26°$
$= \dfrac{\sqrt{3}}{2}t - \dfrac{1}{2}\sqrt{1-t^2} = \dfrac{\sqrt{3}t - \sqrt{1-t^2}}{2}$

(e) $\cos^2 167° - \sin^2 193°$
$= \cos^2 13° - \sin^2 13°$
$= \cos 26°$
$= t$

(f)* $(\cos 13° + \sin 13°)^2$
$= \cos^2 13° + 2\sin 13° \cos 13° + \sin^2 13°$
$= 1 + \sin 26°$
$= 1 + \sqrt{1 - t^2}$

(g) $2\cos^2 13°$
$= 2\cos^2 13° - 1 + 1$
$= \cos 26° + 1$
$= t + 1$

(h) $2\cos^2 13° = t + 1$
$\therefore \cos^2 13° = \dfrac{t+1}{2}$
$\therefore \cos 13° = \sqrt{\dfrac{t+1}{2}}$

(i) $\cos 26° = t$
$\therefore 1 - 2\sin^2 13° = t$
$\therefore 1 - t = 2\sin^2 13°$
$\therefore \dfrac{1-t}{2} = \sin^2 13°$
$\therefore \sin 13° = \sqrt{\dfrac{1-t}{2}}$

EXAMPLE 19

If $\sin 10° \cos 10° = m$, write the following in terms of m:

(a) $\cos 110°$ (b) $\sin^2 10° - \sin^2 80°$ (c) $\tan(-20°)$

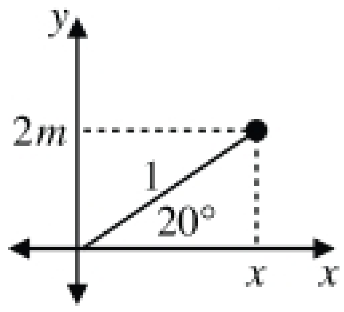
Solution

(a) $\sin 10° \cos 10° = m$
$\therefore 2\sin 10° \cos 10° = 2m$
$\therefore \sin 20° = 2m$

$\cos 110°$
$= \cos(90° + 20°)$
$= -\sin 20°$
$= -2m$

(b) $\sin^2 10° - \sin^2 80°$
$= \sin^2 10° - \cos^2 10°$
$= -(\cos^2 10° - \sin^2 10°)$
$= -\cos 20°$
$= \dfrac{-\sqrt{1-4m^2}}{1}$ *
$= -\sqrt{1-4m^2}$

$*\sin 20° = \dfrac{2m}{1} \quad \left(\dfrac{y}{r}\right)$



$x^2 + y^2 = r^2$
$\therefore x^2 + (2m)^2 = 1^2$
$\therefore x^2 = 1 - 4m^2$
$\therefore x = \sqrt{1-4m^2}$

$x = \sqrt{1-4m^2}$

$y = 2m$

$r = 1$

(c) $\tan(-20°) = -\tan 20° = -\dfrac{2m}{\sqrt{1-4m^2}}$

EXERCISE 6

(a) If $\sin 8° = k$, write the following in terms of k:

(1)	$\sin 172°$	(2)	$\cos 82°$	(3)	$\cos 8°$
(4)	$\sin 16°$	(5)	$\cos 16°$	(6)	$\sin 68°$
(7)	$\tan(-8°)$	(8)	$\tan 82°$	(9)	$\tan 196°$
(10)	$2\sin 4° \cos 4°$	(11)	$1-2\sin^2 4°$	(12)	$(\sin 4° - \cos 4°)^2$

(b) If $\cos 18° = p$, write the following in terms of p:

(1)	$\cos 198°$	(2)	$\sin 108°$	(3)	$\sin(-18°)$
(4)	$\tan 522°$	(5)	$\cos 36°$	(6)	$\sin 144°$
(7)	$\tan 216°$	(8)	$\cos 12°$	(9)	$\sin 78°$
(10)	$2\cos^2 9°$	(11)	$\sin^2 9°$	(12)	$\sin 9°$

(c) If $\tan 21° = t$, write the following in terms of t:

(1)	$\tan 339°$	(2)	$\sin 201°$	(3)	$\cos(-21°)$
(4)	$\sin 48°$	(5)	$\cos 132°$	(6)	$\cos 39°$

(d) If $\sin 12° = m$, write the following in terms of m:

(1)	$\tan 12°$	(2)	$\sin 114°$
(3)	$\sin 6° \cos 6°$	(4)	$\cos^2 6° - \sin^2 6°$
(5)	$2\sin^2 6° - 1$	(6)	$\cos 58° \cos 20° - \sin 58° \sin 20°$

(e) If $\cos 158° = k$, write the following in terms of k:

(1)	$\cos 22°$	(2)	$\cos 44°$	(3)	$\cos 134°$
(4)	$\sin(-11°)\cos 191°$	(5)	$\sin 11°$	(6)	$\cos 11°$

(f) If $\cos^2 23° = t$, determine the following in terms of t:

(1)	$\sin^2 337°$	(2)	$\sin 23°$	(3)	$\cos 46°$
(4)	$\sin 46°$	(5)	$\tan(-226°)$	(6)	$\sin 37°$

(g) If $\sin 17° \cos 17° = p$, determine the following in terms of p:

(1)	$\sin 34°$	(2)	$\sin 56°$	(3)	$\cos 68°$
(4)	$\tan 56°$	(5)	$\sin 274°$	(6)	$\sin^2(-17°) - \sin^2 107°$

(h)* If $\cos 10° = k$, express the following in terms of k:

(1) $\cos 70° + \sin 70° \tan 10°$ (2) $\dfrac{1}{\tan 5°} - \tan 5°$

(i)* If $\cos 13° + \sin 13° = m$, determine the following in terms of m:

(1)	$\sin 26°$	(2)	$\cos 26°$	(3)	$\sin 52°$
(4)	$\cos^2 13° - \sin^2 13°$	(5)	$\cos^4 13° - \sin^4 13°$	(6)	$\cos 13° - \sin 13°$

PROVING IDENTITIES

When two expressions give the same result for all values of the variable (for which they are both defined), the equality of these expressions is called an *identity*.

When proving identities, we always work with the two sides (LHS and RHS) **separately** and simplify or manipulate the two sides till they equal the same expression. To do this we use:

- **Trigonometric principles**

  ✓ Basic identities:

$$\boxed{\tan\theta = \frac{\sin\theta}{\cos\theta}}$$

$$\boxed{\begin{aligned}&\sin^2\theta + \cos^2\theta = 1\\ &\sin^2\theta = 1 - \cos^2\theta\\ &\cos^2\theta = 1 - \sin^2\theta\end{aligned}}$$

  ✓ Double angle formulae:

$$\boxed{\sin 2\theta = 2\sin\theta\cos\theta}$$

$$\boxed{\begin{aligned}&\cos 2\theta = \cos^2\theta - \sin^2\theta\\ &\cos 2\theta = 2\cos^2\theta - 1\\ &\cos 2\theta = 1 - 2\sin^2\theta\end{aligned}}$$

  ✓ Compound angle formulae:

$$\boxed{\begin{aligned}&\sin(A+B) = \sin A\cos B + \cos A\sin B\\ &\sin(A-B) = \sin A\cos B - \cos A\sin B\end{aligned}}$$

$$\boxed{\begin{aligned}&\cos(A+B) = \cos A\cos B - \sin A\sin B\\ &\cos(A-B) = \cos A\cos B + \sin A\sin B\end{aligned}}$$

- **Algebraic principles**

  ✓ Basic algebra:

  - Adding/subtracting like terms
  - Algebraic multiplication
  - Basic simplification of fractions

  ✓ Using the LCD to add/subtract fractions. (Always simplify both sides to a single fraction!)

  ✓ Factorisation:

  - Common factor
  - Difference of two squares
  - Quadratic trinomials

Hint: Start by replacing $\tan\theta$ by $\dfrac{\sin\theta}{\cos\theta}$ and $\sin 2\theta$ by $2\sin\theta\cos\theta$ wherever they occur.

IDENTITIES INVOLVING DOUBLE ANGLES

EXAMPLE 20

Prove that: $\tan\theta + \dfrac{\cos\theta}{\sin\theta} = \dfrac{2}{\sin 2\theta}$

Solution

$$\begin{aligned}\text{LHS} &= \tan\theta + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}\\ &= \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}\\ &= \frac{1}{\cos\theta\sin\theta}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \frac{2}{\sin 2\theta}\\ &= \frac{2}{2\sin\theta\cos\theta}\\ &= \frac{1}{\cos\theta\sin\theta}\end{aligned}$$

$\therefore \text{LHS} = \text{RHS}$

When $\cos 2\theta$ occurs in an expression, we replace it with one of the following:

$$\cos^2\theta - \sin^2\theta$$
$$2\cos^2\theta - 1$$
$$1 - 2\sin^2\theta$$

We base our choice on which one of these three will make it easiest to simplify the expression we are working with. It also helps to look at the other side of the identity to see what we are working towards:

EXAMPLE 21

Prove that:

(a) $\dfrac{\sin 2A}{1+\cos 2A} = \tan A$

(b) $\dfrac{1-\cos 2x}{\sin x} = \dfrac{\sin 2x}{\cos x}$

(c) $\dfrac{\cos 2\theta}{\cos\theta - \sin\theta} = \dfrac{\sin\theta}{\tan\theta} + \tan\theta\cos\theta$

Solution

(a)
$$\begin{aligned}\text{LHS} &= \frac{\sin 2A}{1+\cos 2A}\\ &= \frac{2\sin A\cos A}{1+2\cos^2 A - 1}\\ &= \frac{2\sin A\cos A}{2\cos^2 A}\\ &= \frac{\sin A}{\cos A}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \tan A\\ &= \frac{\sin A}{\cos A}\end{aligned}$$

(b) $\text{LHS} = \dfrac{1-\cos 2x}{\sin x}$

$= \dfrac{1-(1-2\sin^2 x)}{\sin x}$

$= \dfrac{2\sin^2 x}{\sin x}$

$= 2\sin x$

$\text{RHS} = \dfrac{\sin 2x}{\cos x}$

$= \dfrac{2\sin x\cos x}{\cos x}$

$= 2\sin x$

(c) $\text{LHS} = \dfrac{\cos 2\theta}{\cos\theta - \sin\theta}$

$= \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta - \sin\theta}$

$= \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta - \sin\theta}$

$= \cos\theta + \sin\theta$

$\text{RHS} = \dfrac{\sin\theta}{\tan\theta} + \tan\theta\cos\theta$

$= \dfrac{\sin\theta}{\left(\dfrac{\sin\theta}{\cos\theta}\right)} + \left(\dfrac{\sin\theta}{\cos\theta}\right)\cos\theta$

$= \dfrac{\sin\theta}{1} \times \dfrac{\cos\theta}{\sin\theta} + \dfrac{\sin\theta}{\cos\theta} \times \dfrac{\cos\theta}{1}$

$= \cos\theta + \sin\theta$

<u>EXAMPLE 22</u>

Prove that:

(a) $\dfrac{\cos 2\alpha + 2\sin\alpha - 1}{2\cos\alpha - \sin 2\alpha} = \tan\alpha$

(b) $\dfrac{\cos 2\theta + \cos\theta}{\sin^2\theta} = \dfrac{2\sin\theta - \tan\theta}{\tan\theta - \sin\theta}$

<u>Solution</u>

(a) $\text{LHS} = \dfrac{\cos 2\alpha + 2\sin\alpha - 1}{2\cos\alpha - \sin 2\alpha}$

$= \dfrac{1 - 2\sin^2\alpha + 2\sin\alpha - 1}{2\cos\alpha - 2\sin\alpha\cos\alpha}$

$= \dfrac{2\sin\alpha - 2\sin^2\alpha}{2\cos\alpha - 2\sin\alpha\cos\alpha}$

$= \dfrac{2\sin\alpha(1-\sin\alpha)}{2\cos\alpha(1-\sin\alpha)}$

$= \dfrac{\sin\alpha}{\cos\alpha}$

$\text{RHS} = \tan\alpha$

$= \dfrac{\sin\alpha}{\cos\alpha}$

(b) $\text{LHS} = \dfrac{\cos 2\theta + \cos\theta}{\sin^2\theta}$

$= \dfrac{2\cos^2\theta - 1 + \cos\theta}{1 - \cos^2\theta}$

$= \dfrac{2\cos^2\theta + \cos\theta - 1}{1 - \cos^2\theta}$

$= \dfrac{(2\cos\theta - 1)(\cos\theta + 1)}{(1-\cos\theta)(1+\cos\theta)}$

$= \dfrac{2\cos\theta - 1}{1 - \cos\theta}$

$\text{RHS} = \dfrac{2\sin\theta - \tan\theta}{\tan\theta - \sin\theta}$

$= \dfrac{2\sin\theta - \dfrac{\sin\theta}{\cos\theta}}{\dfrac{\sin\theta}{\cos\theta} - \sin\theta}$

$= \dfrac{2\sin\theta\cos\theta - \sin\theta}{\sin\theta - \sin\theta\cos\theta}$

$= \dfrac{\sin\theta(2\cos\theta - 1)}{\sin\theta(1-\cos\theta)}$

$= \dfrac{2\cos\theta - 1}{1 - \cos\theta}$

EXAMPLE 23

Prove that: $\dfrac{1-\sin 2x}{\cos x-\sin x}=\dfrac{\cos 2x}{\cos x+\sin x}$

Solution

$$\begin{aligned}\text{LHS}&=\frac{1-\sin 2x}{\cos x-\sin x}\\&=\frac{1-2\sin x\cos x}{\cos x-\sin x}\\&=\frac{\cos^2 x+\sin^2 x-2\sin x\cos x}{\cos x-\sin x}\\&=\frac{\cos^2 x-2\sin x\cos x+\sin^2 x}{\cos x-\sin x}\\&=\frac{(\cos x-\sin x)^2}{\cos x-\sin x}\\&=\cos x-\sin x\end{aligned}$$

$$\begin{aligned}\text{RHS}&=\frac{\cos 2x}{\cos x+\sin x}\\&=\frac{\cos^2 x-\sin^2 x}{\cos x+\sin x}\\&=\frac{(\cos x-\sin x)(\cos x+\sin x)}{\cos x+\sin x}\\&=\cos x-\sin x\end{aligned}$$

EXERCISE 7

(a) Prove the following identities:

(1) $\dfrac{\sin 2\theta}{\cos\theta}-\sin\theta=\tan\theta\cos\theta$

(2) $(\cos A+\sin A)^2=1+\sin 2A$

(3) $\dfrac{\sin 2\alpha}{1-\sin^2\alpha}=2\tan\alpha$

(4) $\dfrac{1}{1-\cos x}-\dfrac{1}{1+\cos x}=\dfrac{4\cos x}{\tan x\sin 2x}$

(5) $\dfrac{\cos x}{\cos x-\sin x}-\dfrac{\cos x}{\cos x+\sin x}=\tan 2x$

(6) $\dfrac{1-\cos 2A}{\sin 2A}=\tan A$

(7) $\dfrac{2\sin^2\theta}{1+\cos 2\theta}=\tan^2\theta$

(8) $\dfrac{\cos 2\alpha-\sin^2\alpha}{1-\cos^2\alpha}=\dfrac{1}{\tan^2\alpha}-2$

(9) $\dfrac{\cos 2x+1}{\cos x}=\dfrac{2\sin x}{\tan x}$

(10) $\dfrac{\cos x-\tan x\sin x}{\cos x+\tan x\sin x}=\cos 2x$

(b) Prove the following identities:

(1) $\cos x\sin x+\sin^2 x\tan x=\tan x$

(2) $\dfrac{\tan A\cdot\sin 2A}{1-\cos A}=2+2\cos A$

(3) $\dfrac{\cos\alpha+\sin\alpha}{\cos 2\alpha}=\dfrac{\cos^2\alpha+\sin^2\alpha}{\cos\alpha-\sin\alpha}$

(4) $\dfrac{\sin 2x-2\sin x}{\cos 2x-1}=\dfrac{1}{\sin x}-\dfrac{1}{\tan x}$

(5) $\dfrac{\sin\theta-\sin 2\theta}{\cos\theta-\cos 2\theta-1}=\tan\theta$

(6) $\dfrac{\cos 2\alpha+\cos\alpha}{\sin 2\alpha-\sin\alpha}=\dfrac{\cos\alpha+1}{\sin\alpha}$

(7) $\dfrac{\sin 2x-\cos x}{\sin x-\cos 2x}=\dfrac{\cos x}{1+\sin x}$

(8) $\dfrac{\cos\theta-\sin\theta}{1-\sin 2\theta}=\dfrac{\cos\theta+\sin\theta}{\cos 2\theta}$

(9) $\dfrac{\cos 2\alpha-\sin\alpha}{1+\sin\alpha+\cos^2\alpha}=\dfrac{2\sin\alpha-1}{\sin\alpha-2}$

(10) $\dfrac{\sin A-\tan A}{\sin A+\tan A}=\dfrac{\cos 2A-1}{2(\cos A+1)^2}$

(11) $\dfrac{1+\tan x}{1-\tan x}=\dfrac{1+\sin 2x}{\cos 2x}$

(12) $\dfrac{\cos 2A}{3-\sin 2A-4\sin^2 A}=\dfrac{\tan A+1}{\tan A+3}$

(c) Prove that

(1) $\sin 4x = 4\sin x\cos x - 8\sin^3 x\cos x$ (2) $\cos 4x = 8\cos^4 x - 8\cos^2 x + 1$

(d)* Prove the following identities:

(1) $\dfrac{2+2\cos x}{\sin 2x} = \dfrac{\tan x}{1-\cos x}$ (2) $\dfrac{\sin 2\theta}{2\sin\theta + \cos 2\theta - 1} = \dfrac{1+\sin\theta}{\cos\theta}$

(3) $\sqrt{\dfrac{1+\sin\alpha}{1-\sin\alpha}} = \dfrac{2\cos\alpha + \sin 2\alpha}{\cos 2\alpha + 1}$

IDENTITIES INVOLVING COMPOUND ANGLES

EXAMPLE 24

Prove that:

(a) $\sin(A+B) + \sin(A-B) = 2\sin A\cos B$ (b) $\cos 3\alpha = 4\cos^3\alpha - 3\cos\alpha$

Solution

(a)
$$\begin{aligned} \text{LHS} &= \sin(A+B) + \sin(A-B) \\ &= \sin A\cos B + \cos A\sin B + \sin A\cos B - \cos A\sin B \\ &= 2\sin A\cos B \end{aligned}$$

$\text{RHS} = 2\sin A\cos B$

(b)
$$\begin{aligned} \text{LHS} &= \cos 3\alpha \\ &= \cos(2\alpha + \alpha) \\ &= \cos 2\alpha\cos\alpha - \sin 2\alpha\sin\alpha \\ &= (2\cos^2\alpha - 1)\cos\alpha - (2\sin\alpha\cos\alpha)\sin\alpha \\ &= 2\cos^3\alpha - \cos\alpha - 2\sin^2\alpha\cos\alpha \\ &= 2\cos^3\alpha - \cos\alpha - 2(1-\cos^2\alpha)\cos\alpha \\ &= 2\cos^3\alpha - \cos\alpha - 2\cos\alpha + 2\cos^3\alpha \\ &= 4\cos^3\alpha - 3\cos\alpha \end{aligned}$$

$\text{RHS} = 4\cos^3\alpha - 3\cos\alpha$

EXAMPLE 25

Prove that: $\dfrac{\sin 3x\cos x - \cos 3x\sin x}{\cos^2 x - \sin^2 x} = \tan 2x$

Solution

$$\begin{aligned} \text{LHS} &= \frac{\sin(3x-x)}{\cos 2x} \\ &= \frac{\sin 2x}{\cos 2x} \end{aligned}$$

$$\begin{aligned} \text{RHS} &= \tan 2x \\ &= \frac{\sin 2x}{\cos 2x} \end{aligned}$$

EXAMPLE 26

Prove that: $\dfrac{\sin 3x+\sin 7x}{\cos 3x+\cos 7x}=\tan 5x$

Solution

Notice that the average of $3x$ and $7x$ is $5x$. $\left(\dfrac{3x+7x}{2}=\dfrac{10x}{2}=5x\right)$

We can write:

- $3x$ as $5x-2x$
- $7x$ as $5x+2x$

$$\begin{aligned}\text{LHS} &= \frac{\sin 3x+\sin 7x}{\cos 3x+\cos 7x}\\ &= \frac{\sin(5x-2x)+\sin(5x+2x)}{\cos(5x-2x)+\cos(5x+2x)}\\ &= \frac{\sin 5x\cos 2x-\cos 5x\sin 2x+\sin 5x\cos 2x+\cos 5x\sin 2x}{\cos 5x\cos 2x+\sin 5x\sin 2x+\cos 5x\cos 2x-\sin 5x\sin 2x}\\ &= \frac{2\sin 5x\cos 2x}{2\cos 5x\cos 2x}\\ &= \frac{\sin 5x}{\cos 5x}\end{aligned}$$

$$\begin{aligned}\text{RHS} &= \tan 5x\\ &= \frac{\sin 5x}{\cos 5x}\end{aligned}$$

EXERCISE 8

(a) Prove that:

(1) $\cos(A+B)+\cos(A-B)=2\cos A\cos B$

(2) $\cos\alpha\sin\beta=\dfrac{\sin(\alpha+\beta)-\sin(\alpha-\beta)}{2}$

(3) $\dfrac{\cos(x+y)-\sin(x-y)}{\cos 2x}=\dfrac{\cos y+\sin y}{\cos x+\sin x}$

(4) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(b) Prove that:

(1) $\sin 3x=3\sin x-4\sin^3 x$

(2) $\dfrac{\cos 3\theta}{\cos\theta}=1-4\sin^2\theta$

(c) Prove that:

(1) $\dfrac{\cos 5\alpha\cos 3\alpha+\sin 3\alpha\sin 5\alpha}{\cos\alpha-\sin\alpha}=\cos\alpha+\sin\alpha$

(2) $\dfrac{\sin\frac{3x}{2}\cos\frac{x}{2}+\cos\frac{3x}{2}\sin\frac{x}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}=4\cos x$

(d) Prove that

(1) $\cos 5x-\cos 15x=2\sin 10x\sin 5x$

(2) $\dfrac{\sin 3x+\sin x}{2}=\sin 2x\cos x$

(3) $\dfrac{\sin 4\theta-\sin 2\theta}{\cos 4\theta+\cos 2\theta}=\tan\theta$

(4) $\dfrac{\sin 9\alpha-\sin 5\alpha}{\cos 9\alpha-\cos 5\alpha}=-\dfrac{1}{\tan 7\alpha}$

TRIGONOMETRIC EQUATIONS

In Grade 11 you learned how to solve trigonometric equations. Let us revise the basic concepts:

THE GENERAL SOLUTION OF A TRIGONOMETRIC EQUATION

To find the **general solution** of a trigonometric equation of the form $\sin\theta = a$, $\cos\theta = a$ or $\tan\theta = a$:

- Calculate the reference angle by applying $\sin^{-1}$, $\cos^{-1}$ or $\tan^{-1}$ to the numerical value of a **ignoring its sign**.
- Look at the sign of a and the given trigonometric ratio to determine in which **two** quadrants the solution must be according to the CAST diagram. Use the following rules to determine the solutions in the applicable quadrants:

Quadrant I:	$\theta = \text{ref}\angle$	$+k\cdot 360°$
Quadrant II:	$\theta = 180° - \text{ref}\angle$	$+k\cdot 360°$
Quadrant III:	$\theta = 180° + \text{ref}\angle$	$+k\cdot 360°$
Quadrant IV:	$\theta = 360° - \text{ref}\angle$	$+k\cdot 360°$

Note: For $\sin\theta = a$ and $\cos\theta = a$, the value of a must be in the interval $[-1\,;1]$, else there will be no solution. For $\tan\theta = a$, the value of a can be any real number.

EXAMPLE 27

Solve for θ:

(a) $\sin\theta = 0{,}3$

(b) $\tan(3\theta - 15°) = -2{,}5$

Solution

(a) $\sin\theta = \boxed{+}0{,}3$

$\text{ref}\angle = \sin^{-1}(0{,}3) = 17{,}46°$

sin is **positive** in quadrants **I** and **II**:

I: $\theta = 17{,}46° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

II: $\theta = 180° - 17{,}46° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

$= 162{,}54° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

(b) $\tan(3\theta - 15°) = \boxed{-}2{,}5$

$\text{ref}\angle = \tan^{-1}(2{,}5) = 68{,}20°$

tan is **negative** in quadrants **II** and **IV**:

II:

$3\theta - 15° = 180° - 68{,}20° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

$\therefore 3\theta = 126{,}80° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

$\therefore \theta = 42{,}27° + k\cdot 120°\ ;\ k \in \mathbb{Z}$

IV:

$3\theta - 15° = 360° - 68{,}20° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

$\therefore 3\theta = 306{,}80° + k\cdot 360°\ ;\ k \in \mathbb{Z}$

$\therefore \theta = 102{,}27° + k\cdot 120°\ ;\ k \in \mathbb{Z}$

SOLUTIONS IN A GIVEN INTERVAL

To find solutions in a given interval, we find the general solution first and then substitute k with integer values to generate solutions in the interval specified:

EXAMPLE 28

Solve for x if $2\cos 2x = -1$ and $x \in (-180°\,;\,270°)$.

Solution

$2\cos 2x = -1$

$\therefore \cos 2x = \boxed{-}\frac{1}{2}$

$\text{ref}\,\angle = \cos^{-1}\left(\frac{1}{2}\right) = 60°$

cos is **negative** in quadrants **II** and **III**:

II: $2x = 180° - 60° + k \cdot 360° \;;\; k \in \mathbb{Z}$

$\therefore 2x = 120° + k \cdot 360° \;;\; k \in \mathbb{Z}$

$\therefore x = 60° + k \cdot 180° \;;\; k \in \mathbb{Z}$

$k = -1: \quad x = 60° + (-1)\cdot 180° = -120°$

$k = 0: \quad x = 60° + 0 \cdot 180° = 60°$

$k = 1: \quad x = 60° + 1 \cdot 180° = 240°$

III: $2x = 180° + 60° + k \cdot 360° \;;\; k \in \mathbb{Z}$

$\therefore 2x = 240° + k \cdot 360° \;;\; k \in \mathbb{Z}$

$\therefore x = 120° + k \cdot 180° \;;\; k \in \mathbb{Z}$

$k = -1: \quad x = 120° + (-1)\cdot 180° = -60°$

$k = 0: \quad x = 120° + 0 \cdot 180° = 120°$

$x \in \{-120°\,;\,-60°\,;\,60°\,;\,120°\,;\,240°\}$

USING THE BASIC IDENTITIES TO SOLVE EQUATIONS

Equations of the form $a\sin\theta = b\cos\theta$ are solved by dividing both sides by $\cos\theta$ and then using the identity $\frac{\sin\theta}{\cos\theta} = \tan\theta$.

Note: Generally, dividing by a variable expression in an equation is not allowed as this variable expression may equal 0. In equations of the form $a\sin\theta = b\cos\theta$ it can be shown that $\cos\theta$ cannot possibly equal 0 and hence division by $\cos\theta$ is allowed in this case.

EXAMPLE 29

Solve for θ: $\quad 3\cos\theta - 2\sin\theta = 0$

Solution

$3\cos\theta - 2\sin\theta = 0$

$\therefore 2\sin\theta = 3\cos\theta$

$\therefore \frac{2\sin\theta}{\cos\theta} = \frac{3\cos\theta}{\cos\theta}$

$\therefore 2\tan\theta = 3$

$\therefore \tan\theta = \frac{3}{2}$

$\therefore \theta = 56{,}31° + k \cdot 360° \;\text{ or }\; \theta = 236{,}31 + k \cdot 360° \;;\; k \in \mathbb{Z}$

The identity $\sin^2\theta+\cos^2\theta=1$ is used to solve certain equations. The goal is usually to make factorisation possible:

EXAMPLE 30

Solve for θ:

(a) $\cos\theta+1=3\sin^2\theta$ (b) $7\sin^2\theta+\sin\theta\cos\theta-6=0$

Solution

(a) $\cos\theta+1=3\sin^2\theta$

$\cos\theta$ in the equation $\rightarrow$ Replace $\sin^2\theta$ with $1-\cos^2\theta$:

$\therefore \cos\theta+1=3(1-\cos^2\theta)$

$\therefore \cos\theta+1=3-3\cos^2\theta$

$\therefore 3\cos^2\theta+\cos\theta-2=0$

$\therefore (3\cos\theta-2)(\cos\theta+1)=0$

$\therefore \cos\theta=\dfrac{2}{3}$ or $\cos\theta=-1$

- $\cos\theta=\dfrac{2}{3}$: $\theta=48{,}19°+k\cdot 360°$ or $\theta=311{,}81°+k\cdot 360°$; $k\in\mathbb{Z}$
- $\cos\theta=-1$: $\theta=180°+k\cdot 360°$; $k\in\mathbb{Z}$

(b) $7\sin^2\theta+\sin\theta\cos\theta-6=0$

$\sin\theta\cos\theta$ in the equation $\rightarrow$ Multiply the constant term by $\sin^2\theta+\cos^2\theta$:

$\therefore 7\sin^2\theta+\sin\theta\cos\theta-6\times(\sin^2\theta+\cos^2\theta)=0$

$\therefore 7\sin^2\theta+\sin\theta\cos\theta-6\sin^2\theta-6\cos^2\theta=0$

$\therefore \sin^2\theta+\sin\theta\cos\theta-6\cos^2\theta=0$

$\therefore (\sin\theta+3\cos\theta)(\sin\theta-2\cos\theta)=0$

$\therefore \sin\theta+3\cos\theta=0$ or $\sin\theta-2\cos\theta$

$\therefore \sin\theta=-3\cos\theta$ or $\sin\theta=2\cos\theta$

$\therefore \dfrac{\sin\theta}{\cos\theta}=-3$ or $\dfrac{\sin\theta}{\cos\theta}=2$

$\therefore \tan\theta=-3$ or $\tan\theta=2$

- $\tan\theta=-3$: $\theta=108{,}43+k\cdot 360°$ or $\theta=288{,}43°+k\cdot 360°$; $k\in\mathbb{Z}$
- $\tan\theta=2$: $\theta=63{,}43°+k\cdot 360°$ or $\theta=243{,}43°+k\cdot 360°$; $k\in\mathbb{Z}$

EQUATIONS OF THE FORM $\sin\alpha=\sin\beta$; $\cos\alpha=\cos\beta$ or $\tan\alpha=\tan\beta$

EXAMPLE 31

Solve for α:

(a) $\sin(\alpha+10°)=\sin(\alpha-30°)$ (b) $\tan(3\alpha-40°)=-\tan(\alpha+20°)$

(c) $\cos(\alpha+30°)=\sin 2\alpha$

Solution

(a) $\sin(\alpha+10^\circ)=\boxed{+}\sin(\alpha-30^\circ)$ ref $\angle=\alpha-30^\circ$

sin is **positive** in quadrants **I** and **II**:

I: $\alpha+10^\circ=\alpha-30^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 10^\circ=-30^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
No solution.

II: $\alpha+10^\circ=180^\circ-(\alpha-30^\circ)+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha+10^\circ=180^\circ-\alpha+30^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 2\alpha=200^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha=100^\circ+k\cdot 180^\circ\ ;\ k\in\mathbb{Z}$

(b) $\tan(3\alpha-40^\circ)=\boxed{-}\tan(\alpha+20^\circ)$ ref $\angle=\alpha+20^\circ$

tan is **negative** in quadrants **II** and **IV**:

II: $3\alpha-40^\circ=180^\circ-(\alpha+20^\circ)+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 3\alpha-40^\circ=180^\circ-\alpha-20^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 4\alpha=200^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha=50^\circ+k\cdot 90^\circ\ ;\ k\in\mathbb{Z}$

IV: $3\alpha-40^\circ=360^\circ-(\alpha+20^\circ)+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 3\alpha-40^\circ=360^\circ-\alpha-20^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 4\alpha=380^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha=95^\circ+k\cdot 90^\circ\ ;\ k\in\mathbb{Z}$

(c) $\cos(\alpha+30^\circ)=\sin 2\alpha$
$\cos(\alpha+30^\circ)=\boxed{+}\cos(90^\circ-2\alpha)$ ref $\angle=90^\circ-2\alpha$

cos is **positive** in quadrants **I** and **IV**:

I: $\alpha+30^\circ=90^\circ-2\alpha+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore 3\alpha=60^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha=20^\circ+k\cdot 120^\circ\ ;\ k\in\mathbb{Z}$

II: $\alpha+30^\circ=360^\circ-(90^\circ-2\alpha)+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha+30^\circ=360^\circ-90^\circ+2\alpha+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore -\alpha=240^\circ+k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$
$\therefore \alpha=-240^\circ-k\cdot 360^\circ\ ;\ k\in\mathbb{Z}$

EXERCISE 9

(a) Solve for x:

(1) $2\tan x+5=0$
(2) $\cos(x+10^\circ)=0{,}8$
(3) $\sin 3x=0{,}2$
(4) $\sin(2x+10^\circ)=0{,}3;\quad x\in[-90^\circ\,;270^\circ]$
(5) $3\cos 5x+2=0;\quad x\in(0^\circ\,;90^\circ)$
(6) $4\cos^2 2x-3=0$

(b) Solve for α:

(1) $3\sin\alpha-2\cos\alpha=0$
(2) $4\sin^2\alpha+3\cos\alpha=4;\quad \alpha\in[-360^\circ\,;360^\circ]$
(3) $\sin\alpha-2\cos^2\alpha+2=0$
(4) $2\sin\alpha=5\sin(90^\circ-\alpha)$
(5) $3\cos^2\alpha+4\cos\alpha-2=\sin^2\alpha$
(6) $\cos^2\alpha+7\sin\alpha\cos\alpha+2=0$
(7) $\tan\alpha=\cos\alpha$

(c) Solve for θ:

(1) $\tan(2\theta-30^\circ)=\tan(\theta+60^\circ)$ (2) $\cos(\theta+10^\circ)=-\cos\theta;\quad \theta\in(-180^\circ;180^\circ]$

(3) $\sin(2\theta-15^\circ)=\cos(\theta+60^\circ)$ (4) $\cos(\theta+20^\circ)=-\sin(\theta-45^\circ)$

(d) Solve for α: $3\sin\alpha\cos\alpha-2\sin\alpha-\cos\alpha-6\cos^2\alpha+6=0$

USING DOUBLE- AND COMPOUND ANGLE FORMULAE TO SOLVE EQUATIONS

When an angle and its double angle (e.g. θ and 2θ) both occur as angles in a trigonometric equation, we often apply the double angle formulae to rewrite the trigonometric ratios of double angles in terms of ratios of the single angle. The goal is usually to make factorisation possible:

EXAMPLE 32

Solve for θ:

(a) $2\sin 2\theta=3\cos\theta$ (b) $3\sin\theta-\cos 2\theta+2=0$

(c) $\cos 2\theta+2\sin 2\theta-4\sin^2\theta=0$

Solution

(a) $2\sin 2\theta=3\cos\theta$

Replace $\sin 2\theta$ with $2\sin\theta\cos\theta$:

$\therefore 2(2\sin\theta\cos\theta)=3\cos\theta$

$\therefore 4\sin\theta\cos\theta=3\cos\theta$

$\therefore 4\sin\theta\cos\theta-3\cos\theta=0$

$\therefore \cos\theta(4\sin\theta-3)=0$

$\therefore \cos\theta=0$ or $4\sin\theta-3=0$

$\therefore \cos\theta=0$ or $\sin\theta=\dfrac{3}{4}$

- $\cos\theta=0$: $\theta=90^\circ+k\cdot 360^\circ$ or $\theta=270^\circ+k\cdot 360^\circ$; $k\in\mathbb{Z}$
- $\sin\theta=\dfrac{3}{4}$: $\theta=48{,}59^\circ+k\cdot 360^\circ$ or $\theta=131{,}41^\circ+k\cdot 360^\circ$; $k\in\mathbb{Z}$

(b) $3\sin\theta-\cos 2\theta+2=0$

$\sin\theta$ in the equation $\rightarrow$ Replace $\cos 2\theta$ with $1-2\sin^2\theta$:

$3\sin\theta-(1-2\sin^2\theta)+2=0$

$\therefore 3\sin\theta-1+2\sin^2\theta+2=0$

$\therefore 2\sin^2\theta+3\sin\theta+1=0$

$\therefore (2\sin\theta+1)(\sin\theta+1)=0$

$\therefore \sin\theta=-\dfrac{1}{2}$ or $\sin\theta=-1$

- $\sin\theta=-\dfrac{1}{2}$: $\theta=210^\circ+k\cdot 360^\circ$ or $\theta=330^\circ+k\cdot 360^\circ$; $k\in\mathbb{Z}$
- $\sin\theta=-1$: $\theta=270^\circ+k\cdot 360^\circ$; $k\in\mathbb{Z}$

(c) $\cos 2\theta + 2\sin 2\theta - 4\sin^2\theta = 0$

Replace $\sin 2\theta$ with $2\sin\theta\cos\theta$:

$\cos 2\theta + 2(2\sin\theta\cos\theta) - 4\sin^2\theta = 0$

$\therefore \cos 2\theta + 4\sin\theta\cos\theta - 4\sin^2\theta = 0$

$\sin\theta\cos\theta$ in the equation $\rightarrow$ Replace $\cos 2\theta$ with $\cos^2\theta - \sin^2\theta$:

$\therefore \cos^2\theta - \sin^2\theta + 4\sin\theta\cos\theta - 4\sin^2\theta = 0$

$\therefore \cos^2\theta + 4\sin\theta\cos\theta - 5\sin^2\theta = 0$

$\therefore (\cos\theta - \sin\theta)(\cos\theta + 5\sin\theta) = 0$

$\therefore \cos\theta = \sin\theta$ or $\cos\theta = -5\sin\theta$

$\therefore \tan\theta = 1$ or $\tan\theta = -\dfrac{1}{5}$

- $\tan\theta = 1$: $\theta = 45° + k\cdot 360°$ or $\theta = 225° + k\cdot 360°$; $k \in \mathbb{Z}$
- $\tan\theta = -\dfrac{1}{5}$: $\theta = 168{,}69° + k\cdot 360°$ or $\theta = 348{,}69° + k\cdot 360°$; $k \in \mathbb{Z}$

In some cases, the compound- or double angle formulae are used to rewrite expressions as a single trigonometric ratio:

EXAMPLE 33

Solve for α:

(a) $\cos\alpha\cos 10° + \sin\alpha\sin 10° = 0{,}365$ (b) $\sin\alpha\cos\alpha = -0{,}4$

Solution

(a) $\cos\alpha\cos 10° + \sin\alpha\sin 10° = 0{,}365$

Do you recognise the compound angle pattern? $\cos\alpha\cos 10° + \sin\alpha\sin 10° = \cos(\alpha - 10°)$:

$\therefore \cos(\alpha - 10°) = 0{,}365$

$\therefore \alpha - 10° = 68{,}59° + k\cdot 360°$ or $\alpha - 10° = 291{,}41° + k\cdot 360°$; $k \in \mathbb{Z}$

$\therefore \alpha = 78{,}59° + k\cdot 360°$ or $\alpha = 301{,}41° + k\cdot 360°$; $k \in \mathbb{Z}$

(b) $\sin\alpha\cos\alpha = -0{,}4$

Multiply both sides by 2 to create $2\sin\alpha\cos\alpha$, which can be replaced with $\sin 2\alpha$:

$2\sin\alpha\cos\alpha = -0{,}8$

$\therefore \sin 2\alpha = -0{,}8$

$\therefore 2\alpha = 233{,}13° + k\cdot 360°$ or $2\alpha = 306{,}87° + k\cdot 360°$; $k \in \mathbb{Z}$

$\therefore \alpha = 116{,}57° + k\cdot 180°$ or $\alpha = 153{,}44° + k\cdot 180°$; $k \in \mathbb{Z}$

EXAMPLE 34*

Solve for x: $\cos x+\sqrt{3}\sin x=\sqrt{3}$

Solution

$\cos x+\sqrt{3}\sin x=\sqrt{3}$

Divide both sides of the equation by 2 to create the values $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$:

$\frac{1}{2}\cos x+\frac{\sqrt{3}}{2}\sin x=\frac{\sqrt{3}}{2}$

The values $\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ can be replaced with trigonometric ratios of special angles:

$\therefore \sin 30°\cos x+\cos 30°\sin x=\sin 60°$

$\therefore \sin(x+30°)=\sin 60°$

$\therefore x+30°=60°+k\cdot 360°$ or $x+30°=120°+k\cdot 360°$; $k\in\mathbb{Z}$

$\therefore x=30°+k\cdot 360°$ or $x=90°+k\cdot 360°$; $k\in\mathbb{Z}$

EXERCISE 10

Solve for x:

(a)	$3\sin 2x-2\sin x=0$	(b)	$\cos 2x+5\cos x-2=0$; $x\in[0°;360°]$
(c)	$3\sin x\cos x=1$	(d)	$3\cos 20°\sin x+3\sin 20°\cos x=2$; $x\in(-200°;160°)$
(e)	$12\sin^2 x=\sin 2x+4\cos^2 x$	(f)	$1-\sin 2x=9\cos^2 x$
(g)	$\cos 2x+\sin^2 x-\sin 2x=0$	(h)	$4\sin^2 x-\sin x=\cos 2x$; $x\in(-180°;180°]$
(i)	$\cos 2x+2\sin 2x+2=0$	(j)	$\sin x\cos 15°-\cos x\cos 75°=-\sin 2x$
(k)	$3\cos 2x+9=13\cos x$	(l)	$\cos x\cos 35°-\sin x\sin 35°-\sin x=0$
(m)	$\sin 2x=\tan x$	(n)	$\sqrt{2}\sin x-\sqrt{2}\cos x=1$
(o)	$\sin 2x=\sin^2 x$	(p)	$3\sin 2x-3\sin x=2-4\cos x$
(q)	$\sin 3x-\cos 2x\sin x=\cos x$	(r)	$\cos 60°\cos(2x+30°)=\sin x\cos x$

USING EQUATIONS TO DETERMINE WHERE IDENTITIES ARE UNDEFINED

An identity is undefined when:
- any denominator equals 0.
- $\tan\theta$ is undefined, i.e. when $\theta=90°+k\cdot 180°$; $k\in\mathbb{Z}$. (If $\tan\theta$ occurs in the identity.)

EXAMPLE 35

For which values of x is the identity $\frac{\cos x-\cos 2x-1}{\sin x-\sin 2x}=\frac{1}{\tan x}$ undefined?

Solution

This identity will be undefined if

- $\sin x - \sin 2x = 0$
- $\tan x = 0$ (Since $\tan x$ is a denominator in this identity.)
- $\tan x$ is undefined.

- $\sin x - \sin 2x = 0$

  $\therefore \sin 2x = \sin x$

  $\therefore 2x = x + k \cdot 360°$ or $2x = 180° - x + k \cdot 360°$; $k \in \mathbb{Z}$

  $\therefore x = k \cdot 180°$ or $3x = 180 + k \cdot 360°$; $k \in \mathbb{Z}$

  $\therefore x = k \cdot 180°$ or $x = 60° + k \cdot 120°$; $k \in \mathbb{Z}$

- $\tan x = 0$

  $\therefore x = k \cdot 180°$; $k \in \mathbb{Z}$

- $\tan x$ is undefined.

  $\therefore x = 90° + k \cdot 180°$; $k \in \mathbb{Z}$

$\therefore$ The identity is undefined when $x = k \cdot 180°$ or $x = 60° + k \cdot 120°$ or $x = 90° + k \cdot 180°$; $k \in \mathbb{Z}$.

EXERCISE 11

(a) For which values of θ are the following identities undefined?

(1) $\dfrac{1 - \cos 2\theta}{\sin 2\theta} = \tan\theta$

(2) $\dfrac{\sin 2\theta - \cos\theta}{\sin\theta - \cos 2\theta} = \dfrac{\cos\theta}{1 + \sin\theta}$

(3) $\dfrac{\cos 2\theta + 1}{\cos\theta} = \dfrac{2\sin\theta}{\tan\theta}$

(4) $\dfrac{\cos 2\theta + \cos\theta}{\sin^2\theta} = \dfrac{2\sin 2\theta - 2\sin\theta}{2\sin\theta - \sin 2\theta}$

(5) $\dfrac{1 + \tan\theta}{1 - \tan\theta} = \dfrac{1 + \sin 2\theta}{\cos 2\theta}$

(6) $\dfrac{\cos 3\theta + \cos 7\theta}{\sin 5\theta + \sin 7\theta} = \dfrac{1}{\tan 5\theta}$

(b) For which values of x in the interval $[0° ; 360°)$ are the following identities undefined?

(1) $\dfrac{1}{1 - \cos x} - \dfrac{1}{1 + \cos x} = \dfrac{4\cos x}{\tan x \sin 2x}$

(2) $\dfrac{\cos x}{\cos x - \sin x} - \dfrac{\cos x}{\cos x + \sin x} = \tan 2x$

(c) For which values of α in the interval $(-180° ; 180°]$ are the following identities undefined?

(1) $\dfrac{\cos 2\alpha - \sin^2\alpha}{1 - \cos^2\alpha} = \dfrac{1}{\tan^2\alpha} - 2$

(2) $\dfrac{2\sin^2\alpha}{1 + \cos 2\alpha} = \tan^2\alpha$

TRIGONOMETRIC GRAPHS

In Grade 11, you have learnt how to sketch and interpret the graphs of the following standard trigonometric functions:

- $y = a\sin k(x + p) + q$
- $y = a\cos k(x + p) + q$
- $y = a\tan k(x + p) + q$

Amplitude: For the sin- and cos graphs, the amplitude is given by the numerical value of a (ignoring the sign). The tan-graph has no amplitude.

Period: For the sin- and cos graphs, the period is $\dfrac{360°}{k}$. (The sign of k is ignored.)

For the tan graph, the period becomes $\dfrac{180°}{k}$. (The sign of k is ignored.)

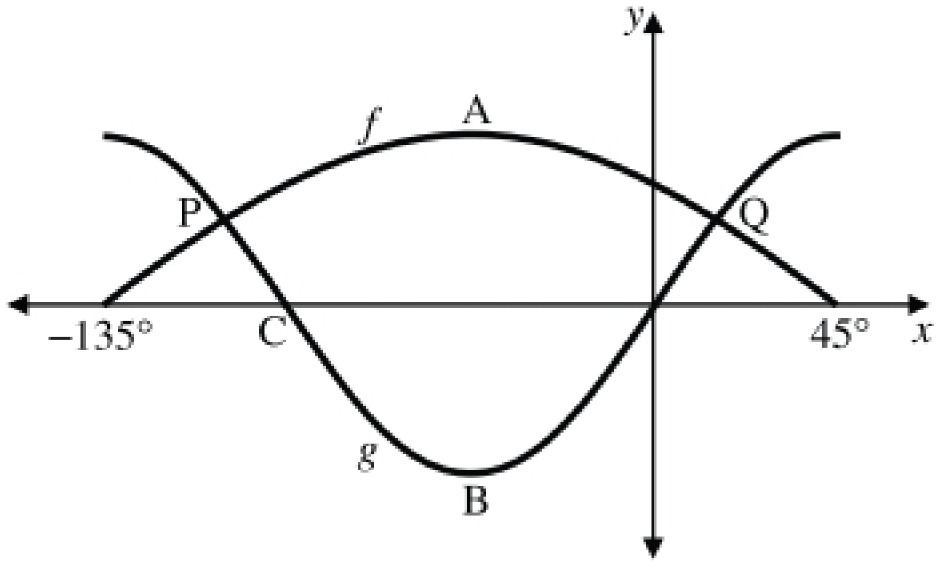
Shifts:

p indicates a **horizontal** shift.
- If p is positive the graph shifts left.
- If p is negative the graph shifts right.

q indicates a **vertical** shift.
- If q is positive the graph shifts up.
- If q is negative the graph shifts down.

EXAMPLE 36

The graphs of $f(x)=\cos(x+45°)$ and $g(x)=\sin 2x$, where $-135° \leq x \leq 45°$, are shown:



(a) Write down the

(1) amplitude of f. (2) period of g.

(b) Write down the the coordinates of

(1) A (2) B

(3) C

(c) Determine the coordinates of P and Q.

(d) For which values of x, in the interval $[-135°;45°]$, is

(1) $f(x)<g(x)$? (2) $f(x)\cdot g(x)\leq 0$?

(3) $f(x)-g(x)=2$?

(e) Use the given graphs, and your calculations above, to determine the value(s) of x, in the interval $[-135°;45°]$, for which

(1) $\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\sin x=1$ (2) $\sin x\cos x=-\frac{1}{2}$

(3) $\sqrt{2}\cos x-\sqrt{2}\sin x=4\sin x\cos x$

(f) Explain how the graph of f can be transformed to produce the graph of $h(x)=\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin x-1$.

Solution

(a) (1) 1 (2) $180°$

(b) (1) $\text{A}(-45°;1)$ (2) $\text{B}(-45°;-1)$ (3) $\text{C}(-90°;0)$

(c) $f(x) = g(x)$

$\therefore \cos(x+45^\circ) = \sin 2x$

$\therefore \cos(x+45^\circ) = \cos(90^\circ - 2x)$

$\therefore x+45^\circ = 90^\circ - 2x + k\cdot 360^\circ$ or $x+45^\circ = 360^\circ - (90^\circ - 2x) + k\cdot 360^\circ$

$\therefore 3x = 45^\circ + k\cdot 360^\circ$ or $-x = 225^\circ + k\cdot 360^\circ\ ;\ k \in \mathbb{Z}$

$\therefore x = 15^\circ + k\cdot 120^\circ$ or $x = -225^\circ - k\cdot 360^\circ\ ;\ k \in \mathbb{Z}$

$k = 0:\quad x = 15^\circ$ No solutions in the applicable interval.

$k = -1:\ x = -105^\circ$

P: $x = -105^\circ\quad y = \cos(-105^\circ + 45^\circ) = \frac{1}{2}\qquad \mathrm{P}\left(-105^\circ;\ \frac{1}{2}\right)$

Q: $x = 15^\circ\quad y = \cos(15^\circ + 45^\circ) = \frac{1}{2}\qquad \mathrm{Q}\left(15^\circ;\ \frac{1}{2}\right)$

(d) (1) $-135^\circ \le x < -105^\circ$ or $15^\circ < x \le 45^\circ$ (2) $-90^\circ \le x \le 0^\circ$

(3) $x = -45^\circ$

(e) (1) $\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = 1$

$\therefore \cos 45^\circ \cos x - \sin 45^\circ \sin x = 1$

$\therefore \cos(x+45^\circ) = 1$

$\therefore f(x) = 1$

This happens at A.

$\therefore x = -45^\circ$

(2) $\sin x \cos x = -\frac{1}{2}$

$\therefore 2\sin x\cos x = -1$

$\therefore \sin 2x = -1$

$\therefore g(x) = -1$

This happens at B.

$\therefore x = -45^\circ$

(3) $\sqrt{2}\cos x - \sqrt{2}\sin x = 4\sin x\cos x$

$\therefore \frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\sin x = \frac{4}{2}\sin x\cos x$

$\therefore \cos 45^\circ \cos x - \sin 45^\circ \sin x = 2\sin x\cos x$

$\therefore \cos(x+45^\circ) = \sin 2x$

$\therefore f(x) = g(x)$

This happens at P and Q.

$\therefore x = -105^\circ$ or $x = 15^\circ$

(f) $h(x) = \frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x - 1$

$= \cos 30^\circ \cos x + \sin 30^\circ \sin x - 1$

$= \cos(x - 30^\circ) - 1$

$\therefore f$ must be translated 75° right and 1 unit up.

MINIMUM- AND MAXIMUM VALUES

From the basic sin- and cos graphs we know that:

The minimum value of $\sin\theta$ is -1 and the maximum value of $\sin\theta$ is 1.

The minimum value of $\cos\theta$ is -1 and the maximum value of $\cos\theta$ is 1.

We can use these facts to determine the minimum- and maximum values of certain trigonometric expressions:

EXAMPLE 37

Determine the minimum- and maximum value of each of the following expressions:

(a) $\sin x \cos x$ (b) $\sqrt{3}\sin x + \cos x$

(c) $3\sin^2 x + 2$ (d) $(\cos x + 2)^2$

(e) $(\sin x - 3)^2$ (f) $(2\cos x + 1)^2$

Solution

(a) $\sin x \cos x$

$= \frac{1}{2}(2\sin x \cos x)$

$= \frac{1}{2}\sin 2x$

- The minimum value is obtained when $\sin 2x = -1$ $\therefore$ Minimum value $= \frac{1}{2}(-1) = -\frac{1}{2}$
- The maximum value is obtained when $\sin 2x = 1$ $\therefore$ Maximum value $= \frac{1}{2}(1) = \frac{1}{2}$

(b) $\sqrt{3}\sin x + \cos x$

$= 2\left(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x\right)$

$= 2(\sin 60^\circ \sin x + \cos 60^\circ \cos x)$

$= 2\sin(x + 60^\circ)$

- The minimum value is obtained when $\sin(x+60^\circ) = -1$ $\therefore$ Minimum value $= 2(-1) = -2$
- The maximum value is obtained when $\sin(x+60^\circ) = 1$ $\therefore$ Maximum value $= 2(1) = 2$

(c) $3\sin^2 x + 2$

The minimum value of a square is 0.

- The minimum value is obtained when $\sin x = 0$ $\therefore$ Minimum value $= 3(0)^2 + 2 = 2$
- The maximum value is obtained when $\sin x = 1$ $\therefore$ Maximum value $= 3(1)^2 + 2 = 5$

(d) $(\cos x + 2)^2$

- The minimum value is obtained when $\cos x = -1$ $\therefore$ Minimum value $= (-1+2)^2 = 1$
- The maximum value is obtained when $\cos x = 1$ $\therefore$ Maximum value $= (1+2)^2 = 9$

(e) $(\sin x - 3)^2$

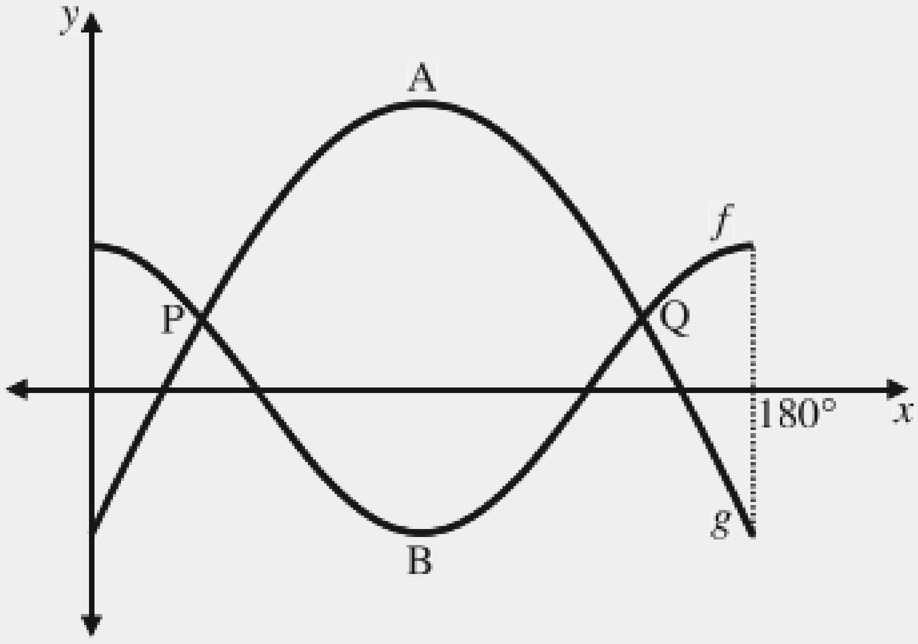
- The minimum value is obtained when $\sin x = 1$ $\therefore$ Minimum value $= (1-3)^2 = 4$
- The maximum value is obtained when $\sin x = -1$ $\therefore$ Maximum value $= (-1-3)^2 = 16$

(f) $(2\cos x+1)^2$

- The minimum value is obtained when $\cos x = -\dfrac{1}{2}$ $\therefore$ Minimum value $= \left[2\left(-\dfrac{1}{2}\right)+1\right]^2 = 0$
- The maximum value is obtained when $\cos x = 1$ $\therefore$ Maximum value $= [2(1)+1]^2 = 9$

EXERCISE 12

(a) The following sketch shows the graphs of $f(x)=\cos 2x$ and $g(x)=3\sin x-1$, where $x\in[0°;180°]$:



(1) Write down the

(i) period of *f*. (ii) amplitude of *g*.

(2) Write down the coordinates of

(i) A (ii) B

(3) Determine the coordinates of P and Q.

*For the questions that follow, you may assume that $x_P = 30°$ and $x_Q = 150°$.

(4) For which values of x, in the interval $[0°;180°]$, is

(i) $f(x)\le 0$? (ii) $f(x)>g(x)$?

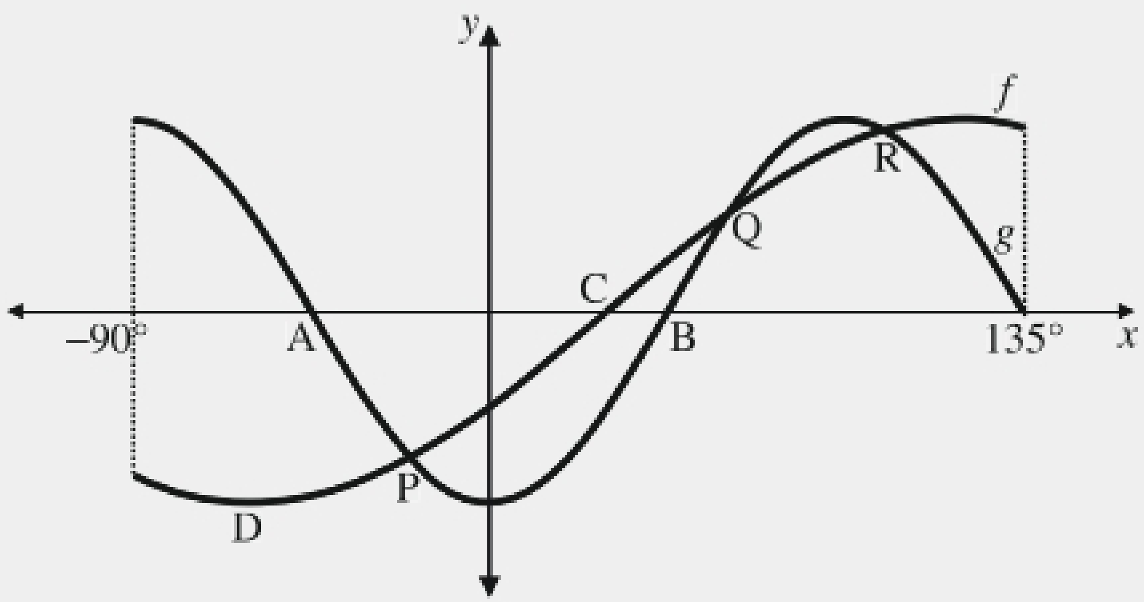
(5) Use the given graphs, and your calculations above, to determine the value(s) of x, in the interval $[0°;180°]$, for which

(i) $\sin^2 x-\cos^2 x=1$ (ii) $2\cos^2 x = 3\sin x$

(6) Explain how the graph of f can be transformed to produce the graph of

(i) $s(x)=2\cos^2 x$ (ii) $t(x)=2\sin^2 x-1$

(b) The following sketch shows the graphs of $f(x)=\sin(x-30°)$ and $g(x)=-\cos 2x$, where $-90° \le x \le 135°$:



(1) Write down the

(i) period of f. (ii) amplitude of g.

(2) Write down the coordinates of

(i) A (ii) B

(iii) C

(3) Determine the x-coordinates of P, Q and R.

*For the questions that follow, you may assume that $x_P=-20°$, $x_Q=60°$ and $x_R=100°$.

(4) For which values of x, in the interval $[-90°;135°]$, is

(i) $f(x)<g(x)$? (ii) $f(x)\cdot g(x)\ge 0$?

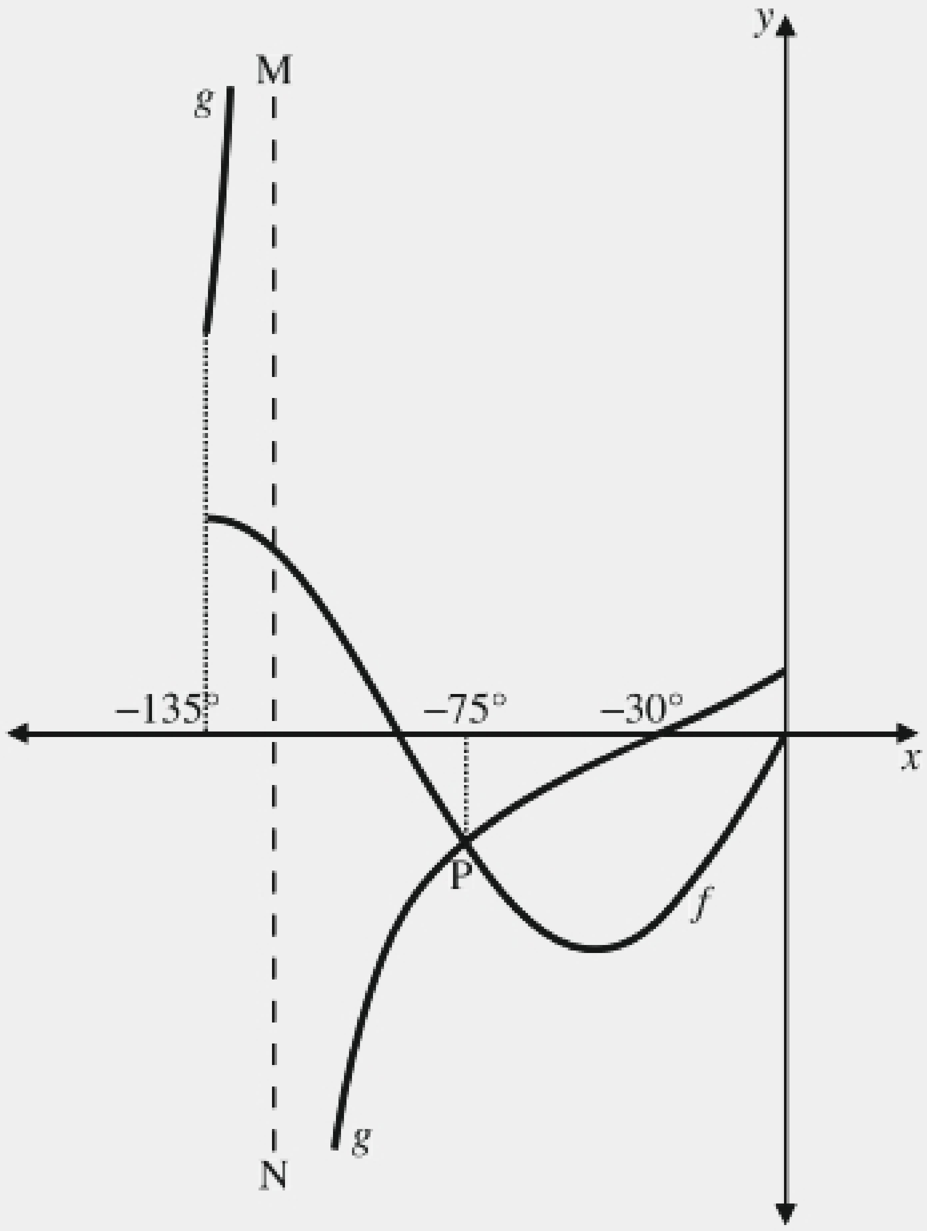
(5) Use the given graphs, and your calculations above, to determine the value(s) of x, in the interval $[-90°;135°]$, for which

(i) $\sqrt{3}\sin x-\cos x=-2$ (ii) $\cos x-\sqrt{3}\sin x=2\cos 2x$

(6) Explain how the graph of f must be transformed to produce the graph of $h(x)=\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x$.

(7) Explain how the graph of g must be transformed to produce the graph of $p(x)=2\sin^2 x$.

(c) The following sketch shows the graphs of $f(x)=\sin 2x$ and $g(x)=a\tan(x+b)$, where $x\in[-135°;0°]$. MN is an asymptote of g. f and g intersect at P, where $x=-75°$:



(1) Write down the value of b.

(2) Determine the value of a.

(3) Write down the equation of MN (the asymptote of g).

(4) For which values of x, in the interval $[-135°;0°]$, is

(i) $f(x)\geq g(x)$? (ii) $f(x)\cdot g(x)\leq 0$?

(5) Explain how the graph of f can be transformed to produce the graph of $h(x)=(\sin x+\cos x)^2$.

(6) Use the given graphs, and your calculations above, to determine the value(s) of x, in the interval $[-135°;0°]$, for which

(i) $4\sin x\cos x+1=0$ (There are **two** values!)

(ii) $\dfrac{\sqrt{3}\sin x+\cos x}{\sqrt{3}\cos x-\sin x}=2\sin 2x$

(d) Determine the minimum- and maximum value of each of the following expressions:

(1) $\sin 3x \cos 3x$ (2) $4\sin x \cos x + 1$

(3) $\cos^2 x - \sin^2 x$ (4) $(\sin x + \cos x)^2$

(5) $2\sin x \cos 10° + 2\cos x \sin 10°$ (6) $\cos x - \sqrt{3}\sin x$

(7) $\sqrt{8}\cos x - \sqrt{8}\sin x$ (8) $3\cos^2 x + 1$

(9) $5 - 2\sin^2 x$ (10) $(\sin x + 3)^2$

(11) $(\cos x - 2)^2 + 1$ (12) $(2\cos x + 1)^2$

(13) $(3\sin x - 1)^2 + 2$

(e) Rewrite each of the following functions into a standard trigonometric function and hence sketch its graph for $x \in [-180°;180°]$:

(1) $y = 2\cos^2 x$ (2) $y = -2\sin^2 x$

(3) $y = 4\sin x \cos x$ (4) $y = \sin 2x \cos 2x$

(5) $y = 4\sin x \cos x \cos 2x$ (6) $y = \cos 4x \cos x + \sin 4x \sin x$

(7) $y = \dfrac{\sin x \cos x}{\cos^2 x - \sin^2 x}$ (8) $y = \sin 2x \cos(x - 60°) + \cos 2x \sin(x - 60°)$

(9) $y = \dfrac{\sqrt{3}}{2}\cos x + \dfrac{1}{2}\sin x$ (10) $y = \sin x - \sqrt{3}\cos x$

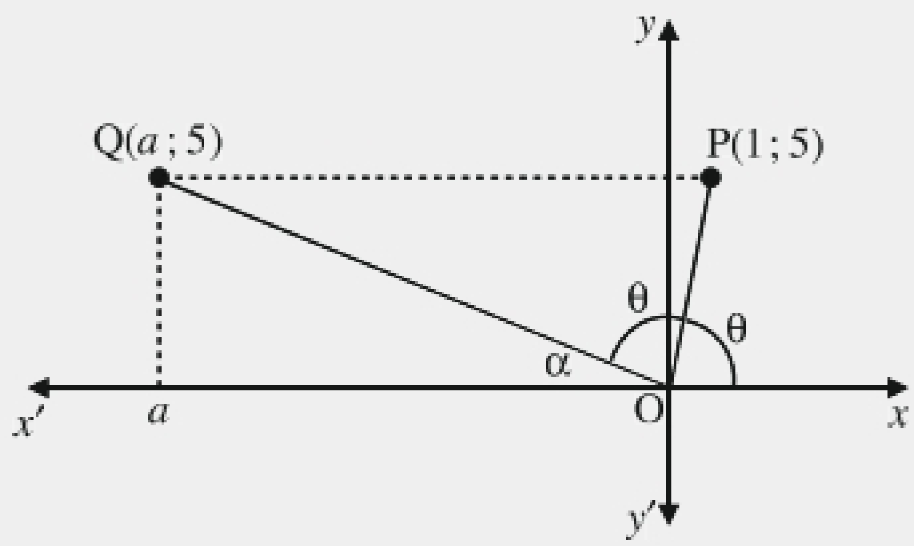
(11) $y = \sqrt{2}(\sin x + \cos x)$ (12)** $y = \dfrac{\cos x + \sin x}{\cos x - \sin x}$

CONSOLIDATION AND EXTENSION EXERCISE

(a) If $4\tan\alpha - 3 = 0$, with $\alpha \in (180°;360°)$ and $5\cos\beta + 3 = 0$, with $\sin\beta > 0$, determine the value of the following without the use of a calculator and with the aid of a diagram:

(1) $\sin(\alpha + \beta)$ (2) $\sin 2\alpha$ (3) $\cos 2\beta$

(b) In the following sketch the coordinates of P are $(1;5)$ and the coordinates of Q are $(a;5)$. $\hat{POx} = \theta$, $\hat{POQ} = \theta$ and $\hat{QOx'} = \alpha$.



Determine

(1) the length of OP. (2) the value of $\sin 2\theta$.

(3) the length of OQ. (4) the value of a.

(5) the value of $\tan\alpha$.

(c) If $\cos 2\theta = \frac{7}{9}$ and $180° < \theta < 360°$, determine the value of $\sin\theta$, without the use of a calculator.

(d) Simplify:

(1) $\dfrac{\tan(-\theta)\cos^2(180° - \theta)}{\sin(-2\theta)}$

(2) $\dfrac{\cos(360° - \theta)\cos(-\theta) + \sin(-\theta - 180°)\cos(90° + \theta)}{\tan 225° - 2\cos^2(180° + \theta)}$

(e) Calculate the values of the following without the use of a calculator:

(1) $\sin 165°$ (2) $\sin 202{,}5°\cos 157{,}5°$

(3) $\cos 10°\cos 410° + \cos 80°\sin(-50°)$ (4) $\cos 195°\cos 525° - \sin^2(-15°)$

(5) $\sin^2 192° + \sin^2 78° + \dfrac{\sin^2(-15°)}{\sin 210°}$ (6) $\dfrac{\cos 75°}{\sin 5°} - \dfrac{\sin 75°}{\cos 5°}$

(f) Given that $\cos\theta = t$, with $-90° < \theta < 0°$.

(1) What is the range of possible values of t ?

(2) Express the following in terms of t :

(i) $\tan(180° + \theta)$ (ii) $\sin 2\theta$ (iii) $\cos(60° + \theta)$

(g) If $\cos 16° = p$, write the following in terms of p:

(1) $\cos 334°$ (2) $\sin(-74°)$ (3) $\sin 16°$

(4) $\cos 32°$ (5) $\sin 32°$ (6) $\cos 76°$

(7) $\sin 8°\cos 8°$ (8) $\cos 8°$ (9) $\sin 8°$

(h) If $\sin 175°\cos 5° = t$, write the following in terms of t:

(1) $\sin 10°$ (2) $\cos 200°$ (3) $\sin 20°$

(i) If $\sin\theta = x$,

(1) write $\sin 3\theta$ in terms of x.

(2)* Hence show that $\sin 10°$ is a root of the equation $8x^3 - 6x + 1 = 0$.

(j) Prove that:

(1) $\dfrac{1 - \cos 2\theta}{1 - \cos\theta} = \dfrac{\sin 2\theta + 2\sin\theta}{\sin\theta}$ (2) $\dfrac{\tan x\cos x + \cos 2x - 1}{\cos x - \sin 2x} = \tan x$

(3) $\dfrac{\cos 2A + \cos A}{\sin^2 A} = \dfrac{2\sin 2A - 2\sin A}{2\sin A - \sin 2A}$ (4) $\dfrac{1}{\cos 2\alpha} + \tan 2\alpha = \dfrac{\cos\alpha + \sin\alpha}{\cos\alpha - \sin\alpha}$

(5) $\dfrac{\cos(x + 45°)}{\cos(x - 45°)} = \dfrac{\cos 2x}{1 + \sin 2x}$ (6) $\dfrac{2\cos\theta - \sin 2\theta}{\cos 2\theta + 1} = \dfrac{2\cos^2\theta}{\sin 2\theta + 2\cos\theta}$

(7) $\dfrac{\sin 13x + \sin 7x}{\cos 13x - \cos 7x} = -\dfrac{1}{\tan 3x}$ (8) $2\sin 5y\cos 4y - \sin 9y = \sin y$

(9) $8\sin\theta\cos\theta\cos 2\theta\cos 4\theta = \sin 8\theta$ (10)* $\dfrac{4\cos 2x - \sin 4x}{\cos x - \sin x} = 4(\sin^3 x + \cos^3 x)$

(k) For which values of x are the following identities undefined?

(1) $\dfrac{\cos(x+45^\circ)}{\cos(x-45^\circ)}=\dfrac{\cos 2x}{1+\sin 2x}$ (2) $\dfrac{\sin 13x+\sin 7x}{\cos 13x-\cos 7x}=-\dfrac{1}{\tan 3x}$

(l) Solve for x:

(1) $\tan(x+15^\circ)=-\tan(2x-35^\circ)$; $x\in[-180^\circ;180^\circ]$

(2) $8\sin 2x\cos 2x-3=0$; $x\in[-90^\circ;180^\circ]$

(m) Determine the general solution of each of the following equations:

(1) $\cos 2\theta-9\cos\theta=4$ (2) $5\sin 2\theta-\cos 2\theta=8\cos^2\theta-3$

(3) $\sin\theta-\sqrt{3}\cos\theta=1$ (4) $\sin 2\theta\cos 32^\circ-\cos 2\theta\sin 32^\circ=\cos 3\theta$

(n) (1) Prove that $\dfrac{\cos x}{\sin 2x}-\dfrac{\cos 2x}{2\sin x}=\sin x$.

(2) Hence solve for x if $1+2\cos 2x=\dfrac{\cos 2x}{2\sin x}-\dfrac{\cos x}{\sin 2x}$.

(o) (1) Show that $2\cos^2(45^\circ-x)=1+\sin 2x$.

(2) Hence determine the value of $\cos^2 15^\circ$ without the use of a calculator.

(p) (1) Prove that $\dfrac{2\tan\theta}{1+\tan^2\theta}=\sin 2\theta$.

(2) Hence determine

(i) the value of $\dfrac{2\tan 15^\circ}{1+\tan^2 15^\circ}$ without the use of a calculator.

(ii)* the minimum value of $\dfrac{(1-\tan\theta)^2}{1+\tan^2\theta}$.

(q)* Determine the general solution of $\sin x=\sin 43^\circ+\sin 17^\circ$, without the use of a calculator.

(r)* If $\cos 62^\circ=t$, express $\dfrac{\cos 16^\circ}{\cos 14^\circ}+\dfrac{\sin 16^\circ}{\sin 14^\circ}$ in terms of t.

(s)* It is given that $\cos 36^\circ$ is a root of the equation $4x^2-2x-1=0$.

(1) Use this fact to determine the value of $\cos 36^\circ$ in simplest surd form, without the use of a calculator.

(2) Show that $\cos 108^\circ$ is the other root of the equation, without the use of a calculator.

(t)* Consider the following pattern:

$$T_1=2\sin 2^\circ\cos 2^\circ$$
$$T_2=4\sin 2^\circ\cos 2^\circ\cos 4^\circ$$
$$T_3=8\sin 2^\circ\cos 2^\circ\cos 4^\circ\cos 8^\circ$$
$$\vdots$$
$$T_n=2^n\sin 2^\circ\cos 2^\circ\cos 4^\circ\cos 8^\circ\ldots\cos(2^n)^\circ$$

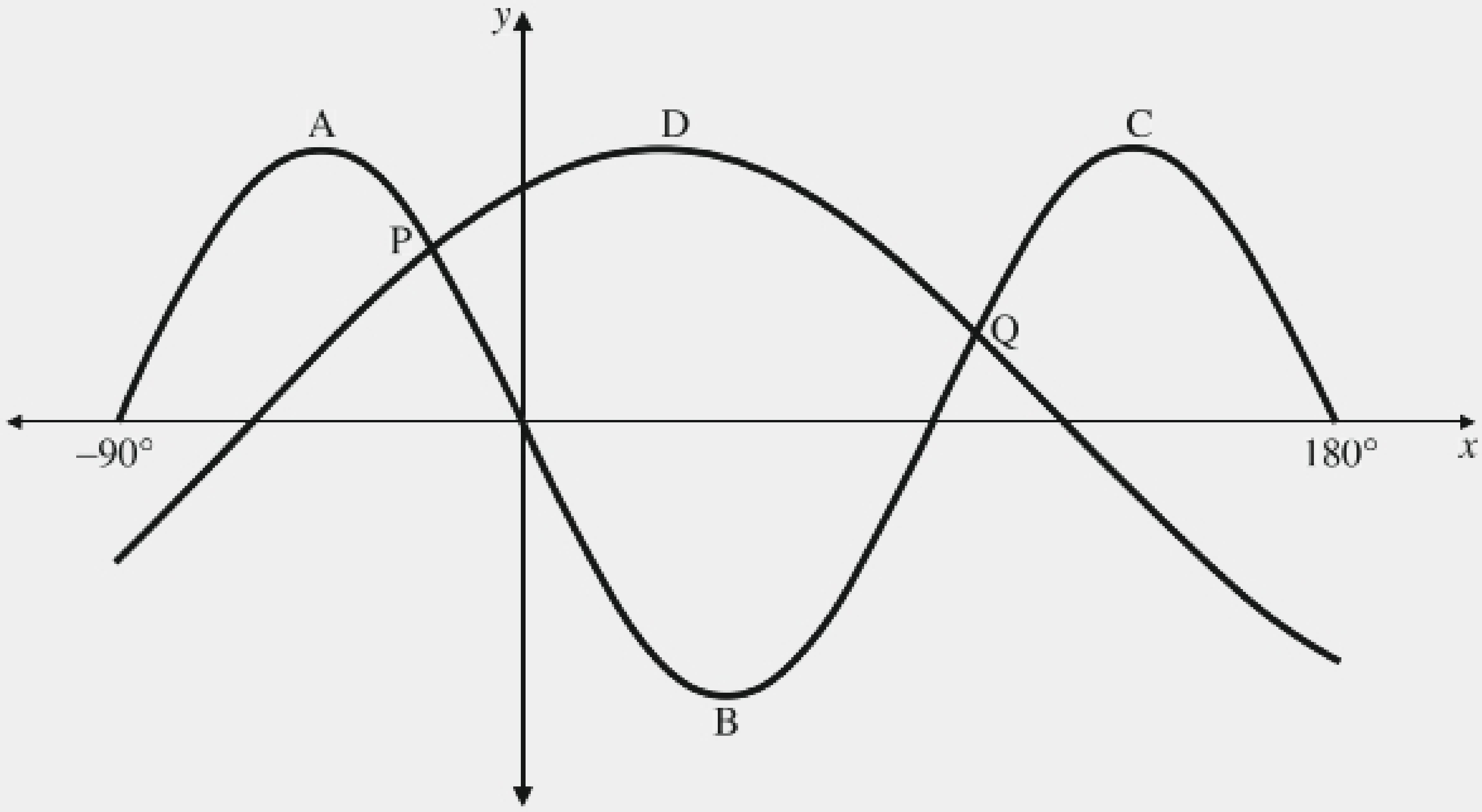
(1) Simplify each of T_1, T_2 and T_3 to a single trigonometric ratio.

(2) Write down an expression for T_n as a single trigonometric ratio.

(3) If $\sin 2^\circ=k$, express each of the following in terms of k:

(i) $\cos 2^\circ$ (ii) T_8 (iii) T_{12}

(u) The following sketch shows the graphs of $f(x)=-\sin ax$ and $g(x)=\cos(x-30°)$, where $x\in[-90°;180°]$:



(1) What is the period of f?

(2) Write down the value of a.

(3) Write down the coordinates of
(i) A (ii) B (iii) C (iv) D

(4) Determine the coordinates of P and Q.

(5) For which values of x is
(i) $f(x)\leq g(x)$? (ii) $f(x)\cdot g(x)<0$?

(6) Use the graphs to determine the value(s) of x for which
(i) $\sin x\cos x=-\frac{1}{2}$ (ii) $\sqrt{3}\cos x+\sin x=2$
(iii) $\sqrt{3}\cos x+\sin x+4\cos x\sin x=0$

(7) Explain how the graph of f can be transformed to produce the graph of $h(x)=(\sin x+\cos x)^2$.

(8) Explain how the graph of g can be transformed to produce the graph of $p(x)=\frac{\sqrt{2}(\sin x+\cos x)}{2}$.

(v)* Sound waves can be represented by sine and/or cosine functions. When two sound waves, defined by $f(x)=a_1\sin x$ and $g(x)=a_2\cos x$, are superimposed, the result is a wave defined by the function $h(x)=f(x)+g(x)$. This resulting function can always be written in the form $h(x)=a\sin(x+\alpha)$.

If the two sound waves defined by $f(x)=2\sin x$ and $g(x)=3\cos x$ are superimposed, determine the amplitude of the resulting wave function.

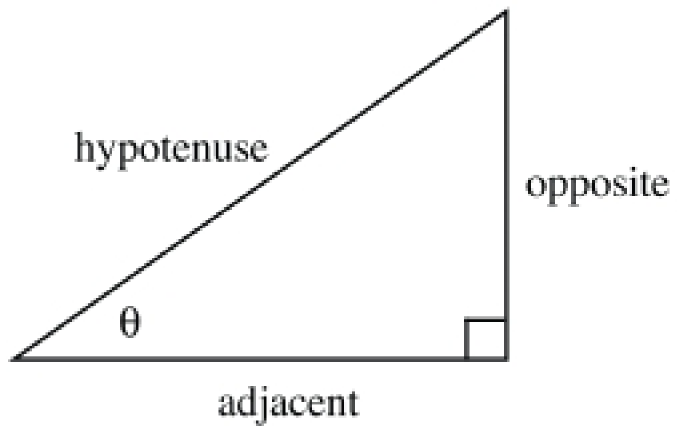
CHAPTER SIX

Trigonometry in Triangles

In this chapter, we will use the basic trigonometric ratios as well as the sine-, cosine- and area rules to solve problems in three dimensions.

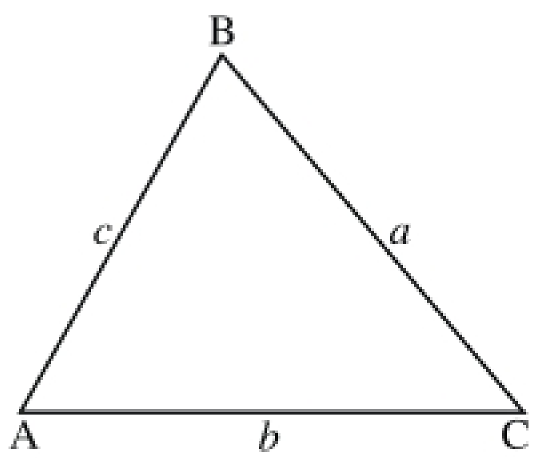
OVERVIEW OF CONCEPTS

RIGHT ANGLED TRIANGLES



$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan\theta = \frac{\text{opposite}}{\text{adjacent}}$$

ALL TRIANGLES



Sine rule:

$$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c} \quad \text{or} \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A \quad \text{or} \quad b^2 = a^2 + c^2 - 2ac\cos B \quad \text{or} \quad c^2 = a^2 + b^2 - 2ab\cos C$$

Area rule:

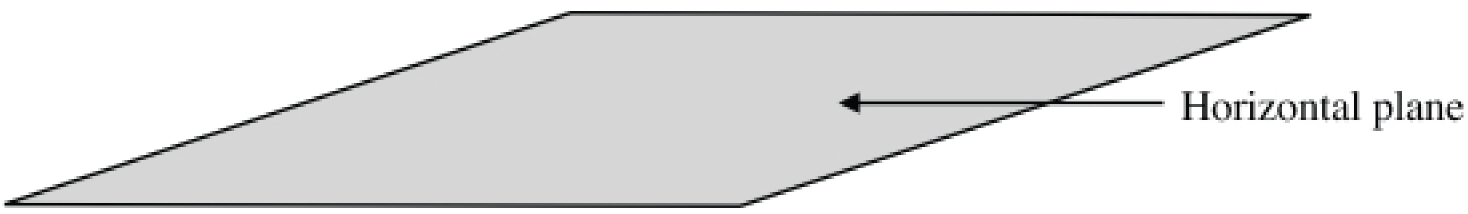
$$\text{Area of } \Delta ABC = \frac{1}{2}bc\sin A \quad \text{or} \quad \text{Area of } \Delta ABC = \frac{1}{2}ac\sin B \quad \text{or} \quad \text{Area of } \Delta ABC = \frac{1}{2}ab\sin C$$

WORKING IN THREE DIMENSIONS

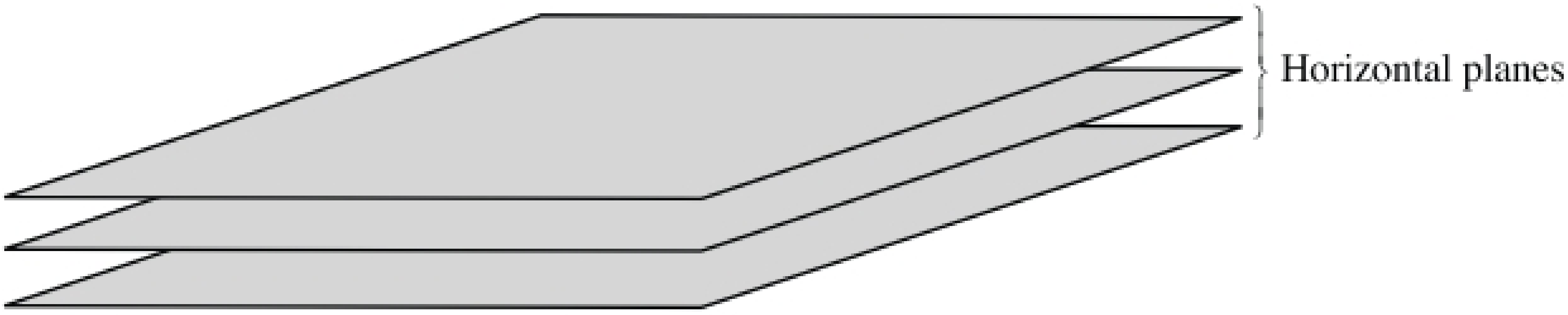
PLANES AND LINES

Horizontal planes

Imagine a "flat" rectangular area on the earth's surface. (Ignore the roundness of the earth.)
If we extend this flat surface indefinitely, this represents what we call a *horizontal plane*:

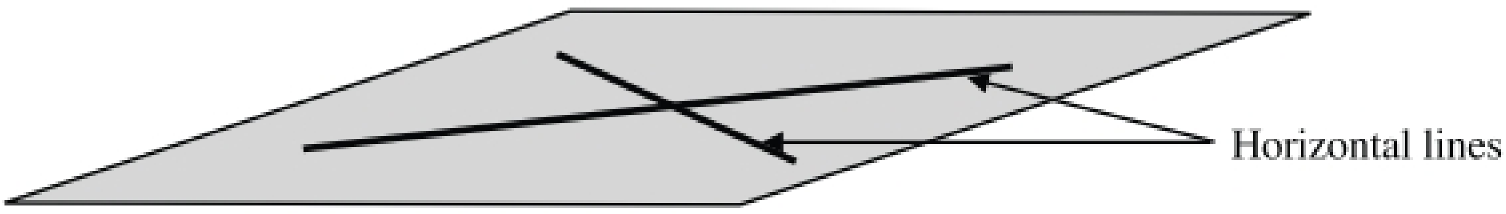


Any plane parallel to this plane is also a horizontal plane:



Horizontal lines

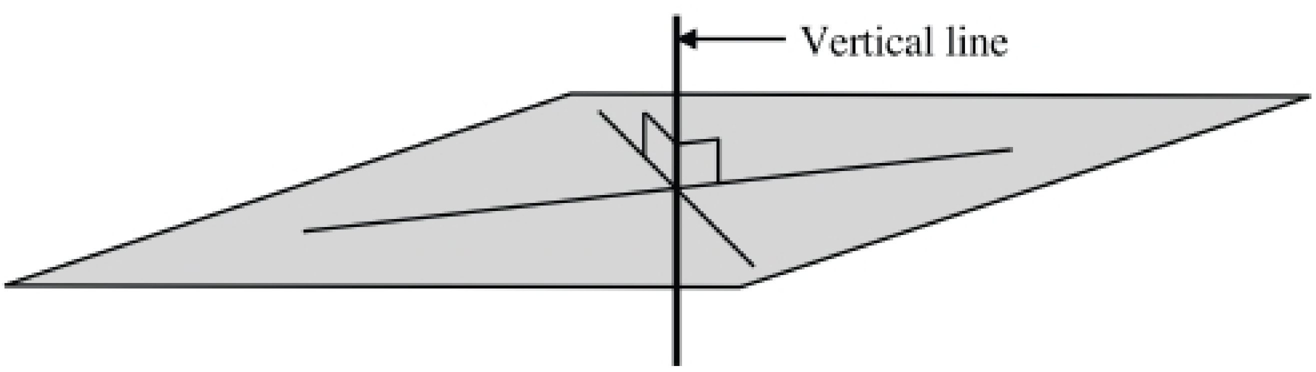
Any line in a horizontal plane is called a *horizontal line*:



Horizontal lines are not necessarily parallel to each other.

Vertical lines

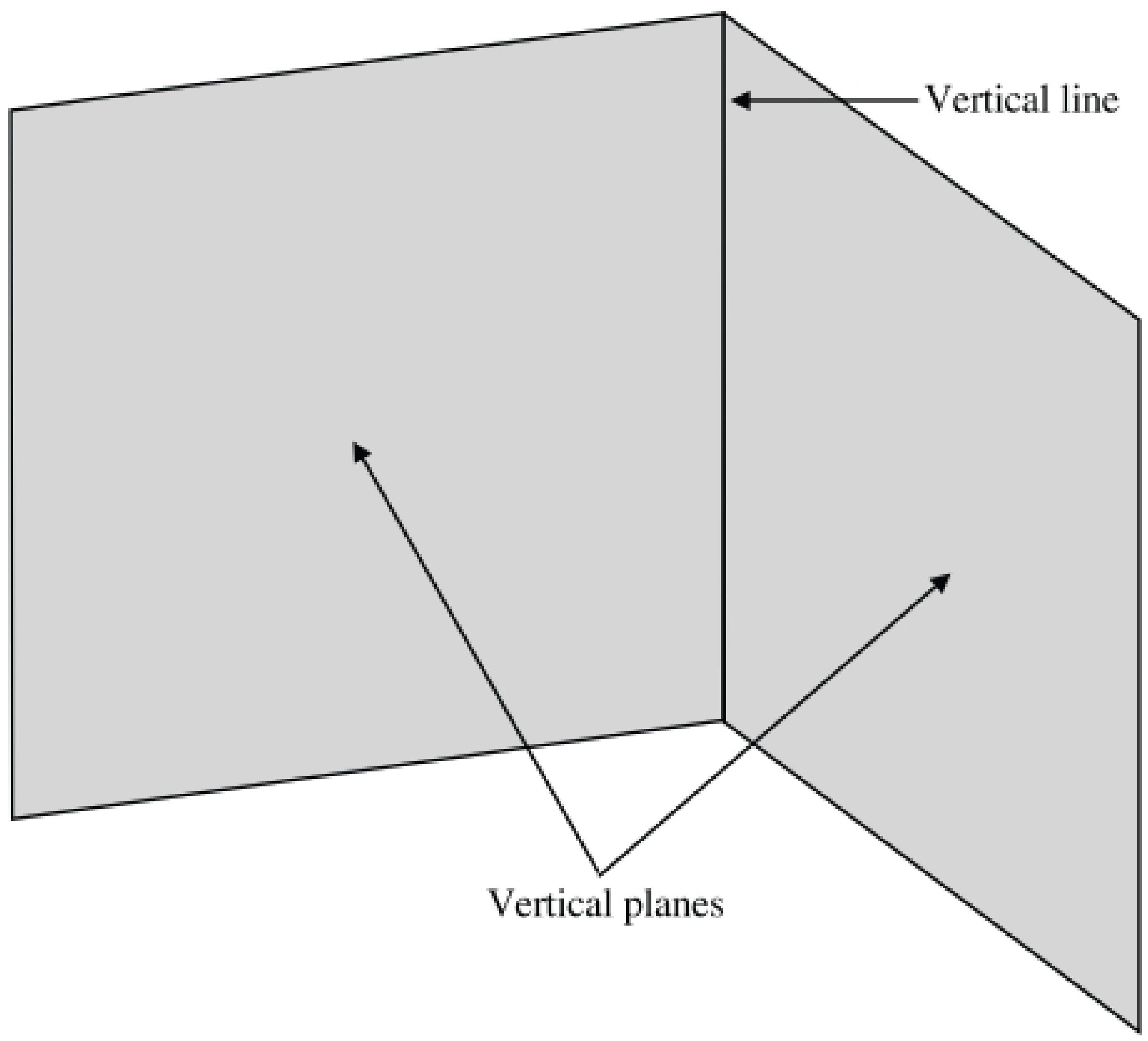
Vertical lines are lines that are **perpendicular to ALL horizontal lines**:



- There is only one vertical line through any given point.
- All vertical lines are parallel to each other.

Vertical planes

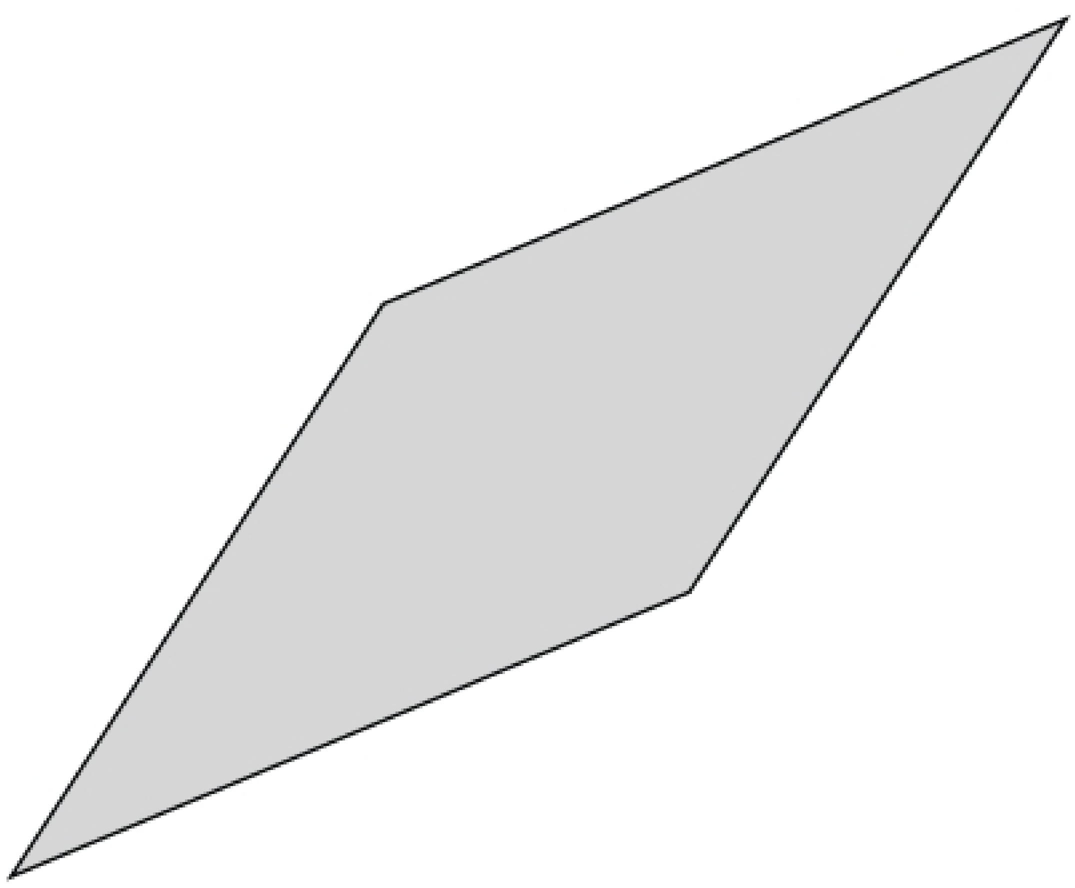
Any plane containing a vertical line is called a *vertical plane*:



- There are infinitely many vertical planes containing a given vertical line.
- Not all lines in a vertical plane are vertical lines!

Inclined planes

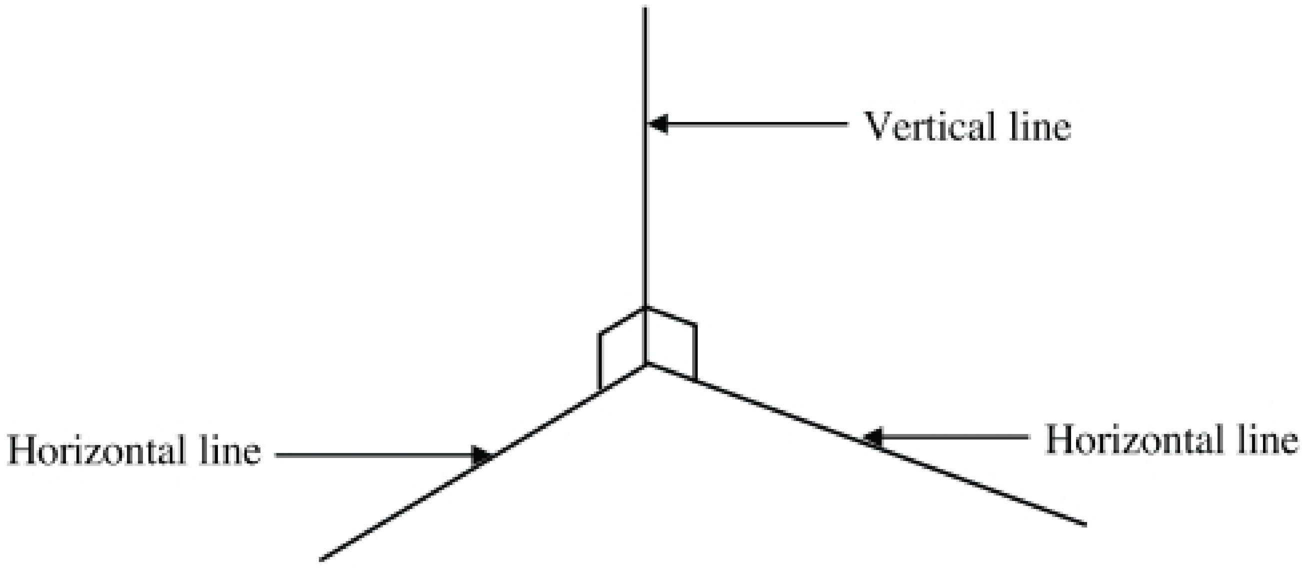
A plane that is **neither horizontal nor vertical** is called an *inclined plane*:



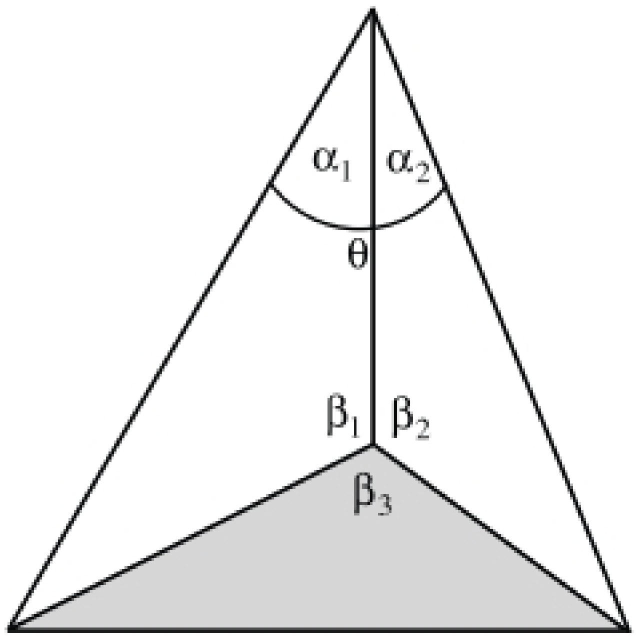
SOME IMPORTANT PRINCIPLES

When working with drawings of three dimensional objects, remember the following principles:

- Any vertical line is perpendicular to any horizontal line:



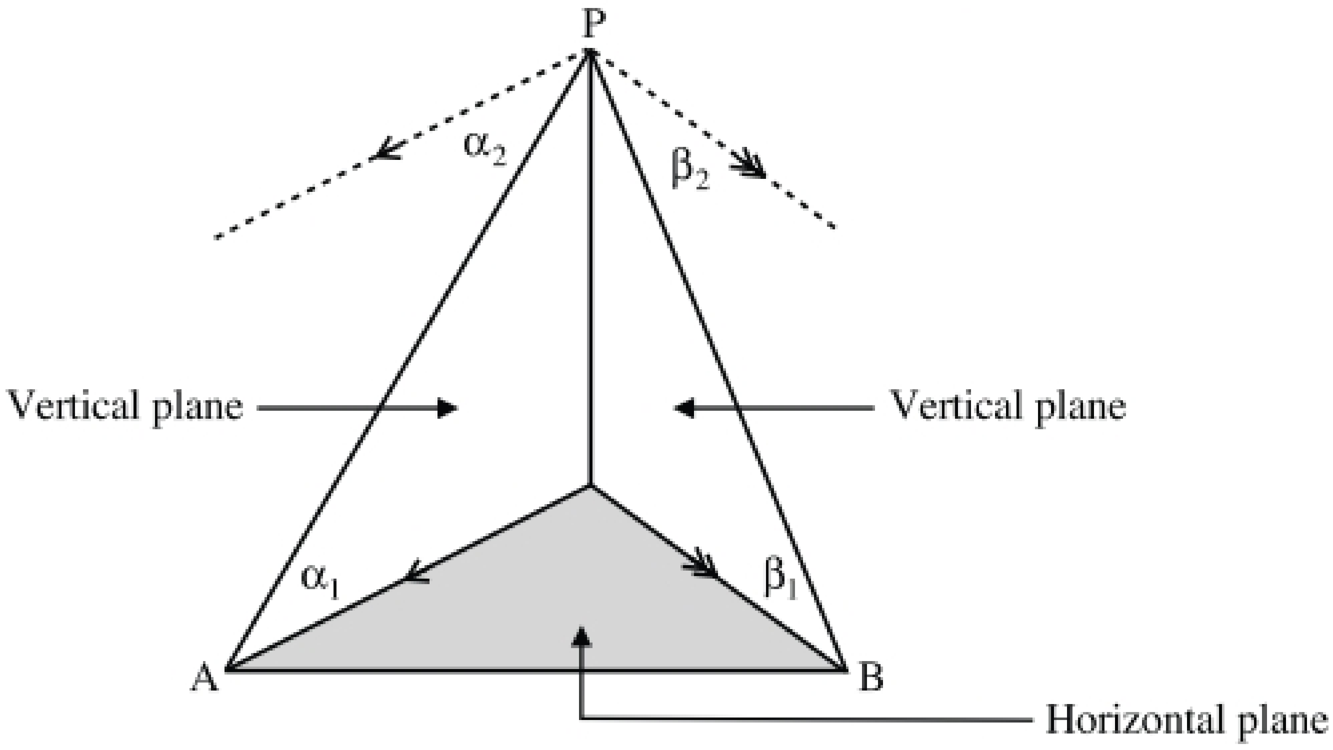
- Adding or subtracting angles in different planes is meaningless:



$\boxed{\alpha_1 + \alpha_2 \neq \theta}$

$\boxed{\beta_1 + \beta_2 + \beta_3 \neq 360°}$

- *Angles of elevation and depression* are always measured in vertical planes:



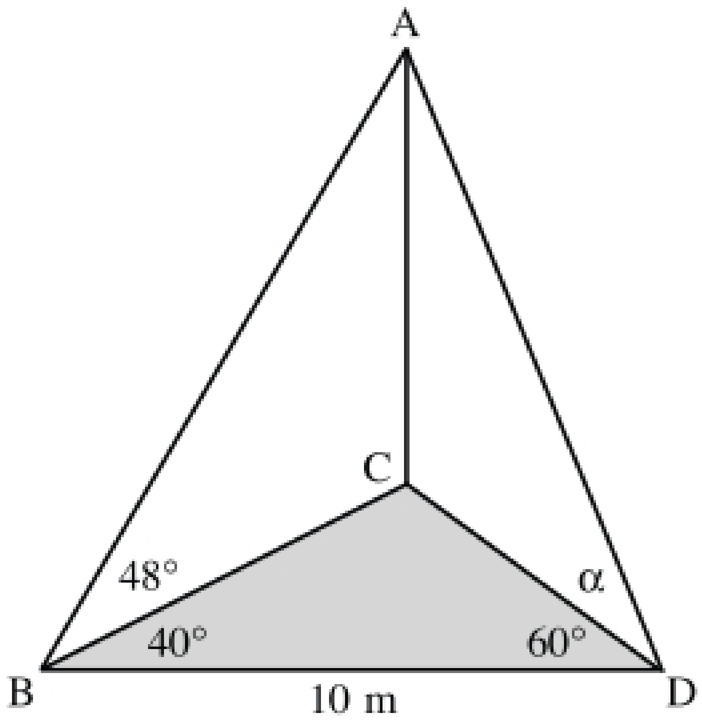
Angle of elevation of P from A $= \alpha_1$	Angle of elevation of P from B $= \beta_1$
Angle of depression of A from P $= \alpha_2$	Angle of depression of B from P $= \beta_2$
$\alpha_1 = \alpha_2$	$\beta_1 = \beta_2$

SOLVING PROBLEMS IN THREE DIMENSIONS

When solving problems involving triangles in different planes, always look for a **common side** between the triangle containing the side/angle you are required to calculate and another triangle containing information. Start by calculating the common side.

EXAMPLE 1

In the following sketch, AC is a vertical line. ΔBCD lies in a horizontal plane. $BD = 10$ m, $C\hat{B}D = 40°$ and $C\hat{D}B = 60°$. The angle of elevation of A from B is $48°$ and the angle of elevation of A from D is α.



(a) Calculate the length of
  (1) AB (2) AC (3) AD

(b) Calculate the size of α.

(c) Write down the angle of depression of B from A.

(d) Calculate the size of $A\hat{B}D$.

(e) Calculate the area of ΔABD.

Solution

(a) (1) In ΔBCD:

$B\hat{C}D = 80°$ ($\angle$s of Δ)

$$\frac{BC}{\sin 60°} = \frac{10}{\sin 80°}$$

$$\therefore BC = \frac{10 \sin 60°}{\sin 80°} = 8{,}79 \text{ m}$$

In ΔABC:

AC is vertical and BC horizontal $\therefore A\hat{C}B = 90°$

$$\therefore \cos 48° = \frac{8{,}79}{AB}$$

$$\therefore AB = \frac{8{,}79}{\cos 48°} = 13{,}14 \text{ m}$$

(2) $\underline{\text{In } \Delta ABC:}$

$$\tan 48^\circ = \frac{AC}{BC}$$

$$\therefore \tan 48^\circ = \frac{AC}{8,79}$$

$$\therefore AC = 8,79 \tan 48^\circ = 9,76 \text{ m}$$

(3) $\underline{\text{In } \Delta BCD:}$

$$\frac{CD}{\sin 40^\circ} = \frac{10}{\sin 80^\circ}$$

$$\therefore CD = \frac{10 \sin 40^\circ}{\sin 80^\circ}$$

$$= 6,52 \text{ m}$$

$\underline{\text{In } \Delta ACD:}$

AC is vertical and CD horizontal $\therefore A\hat{C}D = 90^\circ$

$$\therefore AD^2 = AC^2 + CD^2 \qquad \text{(Pythagoras)}$$

$$\therefore AD^2 = 9,76^2 + 6,52^2$$

$$\therefore AD^2 = 137,768$$

$$\therefore AD = 11,74 \text{ m}$$

(b) $$\tan \alpha = \frac{AC}{CD}$$

$$\therefore \tan \alpha = \frac{9,76}{6,52}$$

$$\therefore \alpha = 56,26^\circ$$

(c) Angle of depression of B from A = Angle of elevation of A from B $= 48^\circ$

(d) $\underline{\text{In } \Delta ABD:}$

$$AD^2 = AB^2 + BD^2 - 2AB \cdot BD \cdot \cos A\hat{B}D$$

$$\therefore 11,74^2 = 13,14^2 + 10^2 - 2(13,14)(10)\cos A\hat{B}D$$

$$\therefore 2(13,14)(10)\cos A\hat{B}D = 13,14^2 + 10^2 - 11,74^2$$

$$\therefore \cos A\hat{B}D = \frac{13,14^2 + 10^2 - 11,74^2}{2(13,14)(10)}$$

$$\therefore \cos A\hat{B}D = 0,5130593607$$

$$\therefore A\hat{B}D = 59,13^\circ$$

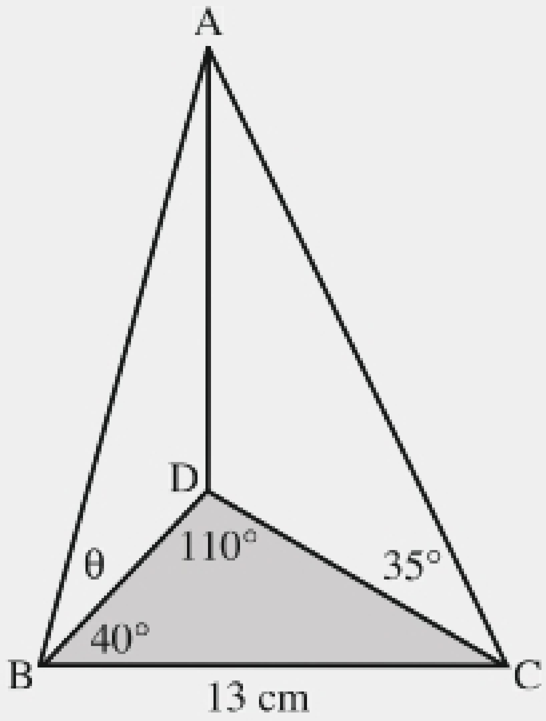
(e) $$\text{Area of } \Delta ABD = \frac{1}{2} AB \cdot BD \cdot \sin A\hat{B}D$$

$$= \frac{1}{2}(13,14)(10)\sin 59,13^\circ$$

$$= 56,39 \text{ m}^2$$

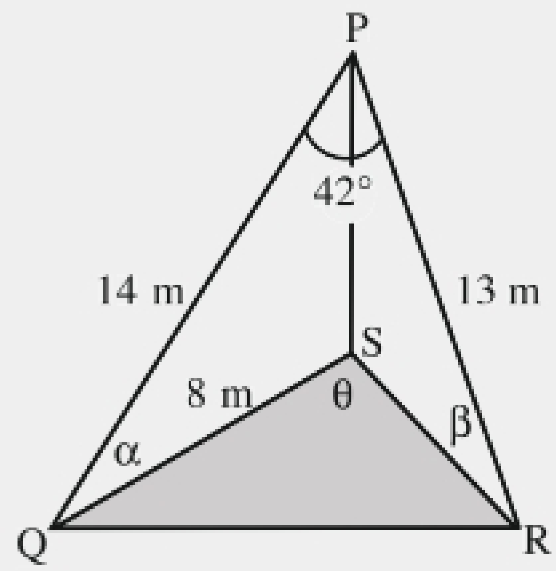
EXERCISE 1

(a) In the sketch alongside, AD is a vertical line. ΔBDC lies in a horizontal plane. $BC = 13$ cm, $D\hat{B}C = 40°$, $B\hat{D}C = 110°$. The angle of elevation of A from C is $35°$ and the angle of elevation of A from B is θ.



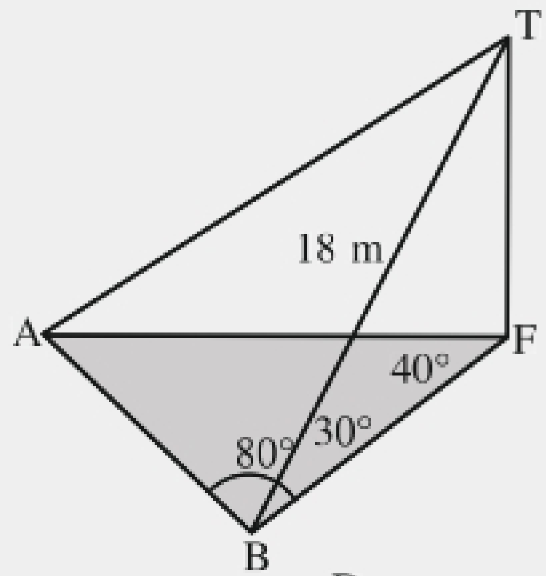
Calculate
(1) the length of AD.
(2) the length of AB.
(3) the magnitude of θ.
(4) the magnitude of $B\hat{A}C$.
(5) the area of ΔABC.

(b) In the sketch alongside, ΔQSR lies in a horizontal plane. PS is a vertical line. $PQ = 14$ m, $QS = 8$ m, $PR = 13$ m and $Q\hat{P}R = 42°$ and $Q\hat{S}R = \theta$. From Q, the angle of elevation to P is α. From R, the angle of elevation to P is β.



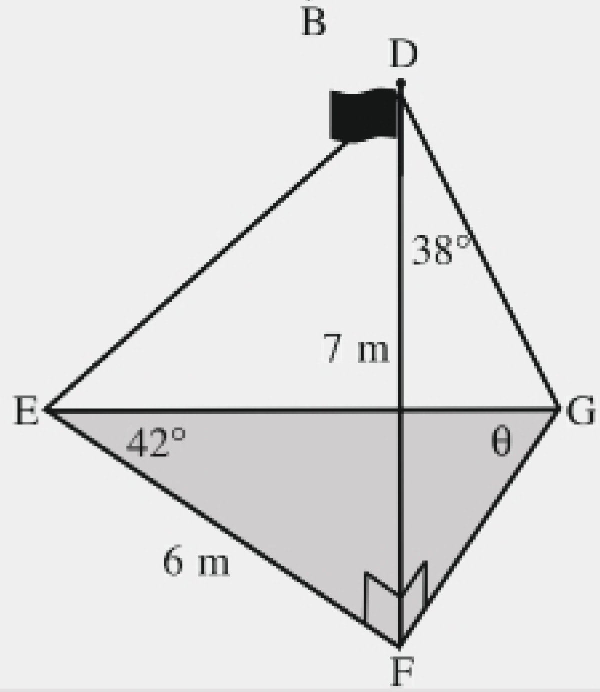
Calculate
(1) the length of RS.
(2) the size of β.
(3) the angle of depression of Q from P.
(4) the size of θ.

(c) In the sketch alongside, TF represents a vertical tower. A, B and F are three points on ground level. $TB = 18$ m, $A\hat{B}F = 80°$ and $A\hat{F}B = 40°$. The angle of elevation from B to the top of the tower (T) is $30°$.



Calculate
(1) the height of the tower TF.
(2) the distance between A and B.
(3) the angle of elevation of T from A.

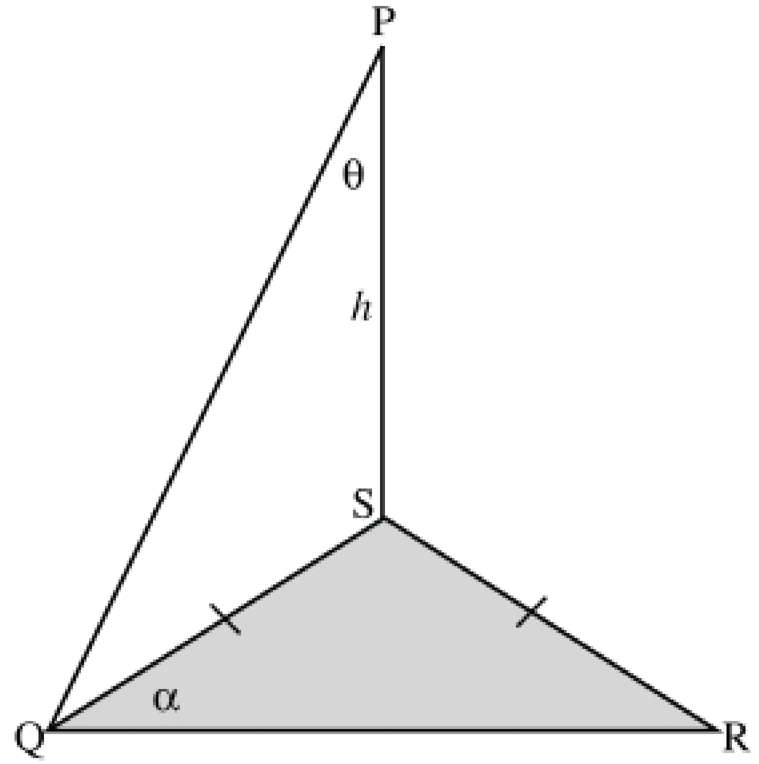
(d) In the sketch alongside, DF is a vertical flagpole and E, F and G are three points on ground level. $DF = 7$ m, $EF = 6$ m, $F\hat{D}G = 38°$ and $F\hat{E}G = 42°$. $E\hat{G}F = \theta$, where θ is an acute angle.



Calculate
(1) the magnitude of θ.
(2) the length of EG
(3) the magnitude of $E\hat{D}G$.
(4) the area of ΔEDG.
(5) the angle of elevation of D from E.

EXAMPLE 2

In the following sketch, PS is a vertical line. ΔSQR lies in a horizontal plane. $Q\hat{P}S = \theta$, $S\hat{Q}R = \alpha$ and $PS = h$. $SQ = SR$.



(a) Show that $QR = 2h\tan\theta\cos\alpha$.

(b) If $\theta = 45°$, $\alpha = 30°$ and $h = \sqrt{3}$ units, determine the length of QR, without the use of a calculator.

Solution

(a) In ΔPQS:

PS is vertical and QS is horizontal $\therefore P\hat{S}Q = 90°$

$$\therefore \tan\theta = \frac{QS}{PS}$$

$$\therefore \tan\theta = \frac{QS}{h}$$

$$\therefore QS = h\tan\theta$$

In ΔSQR:

$S\hat{R}Q = \alpha$ (∠s opp = sides)

$\therefore Q\hat{S}R = 180° - 2\alpha$ (∠s of Δ)

$$\therefore \frac{QR}{\sin(180° - 2\alpha)} = \frac{QS}{\sin\alpha}$$

$$\therefore \frac{QR}{\sin 2\alpha} = \frac{h\tan\theta}{\sin\alpha}$$

$$\therefore QR = \frac{h\tan\theta\sin 2\alpha}{\sin\alpha}$$

$$= \frac{h\tan\theta \cdot 2\sin\alpha\cos\alpha}{\sin\alpha}$$

$$= 2h\tan\theta\cos\alpha$$

(b) $QR = 2h\tan\theta\cos\alpha$

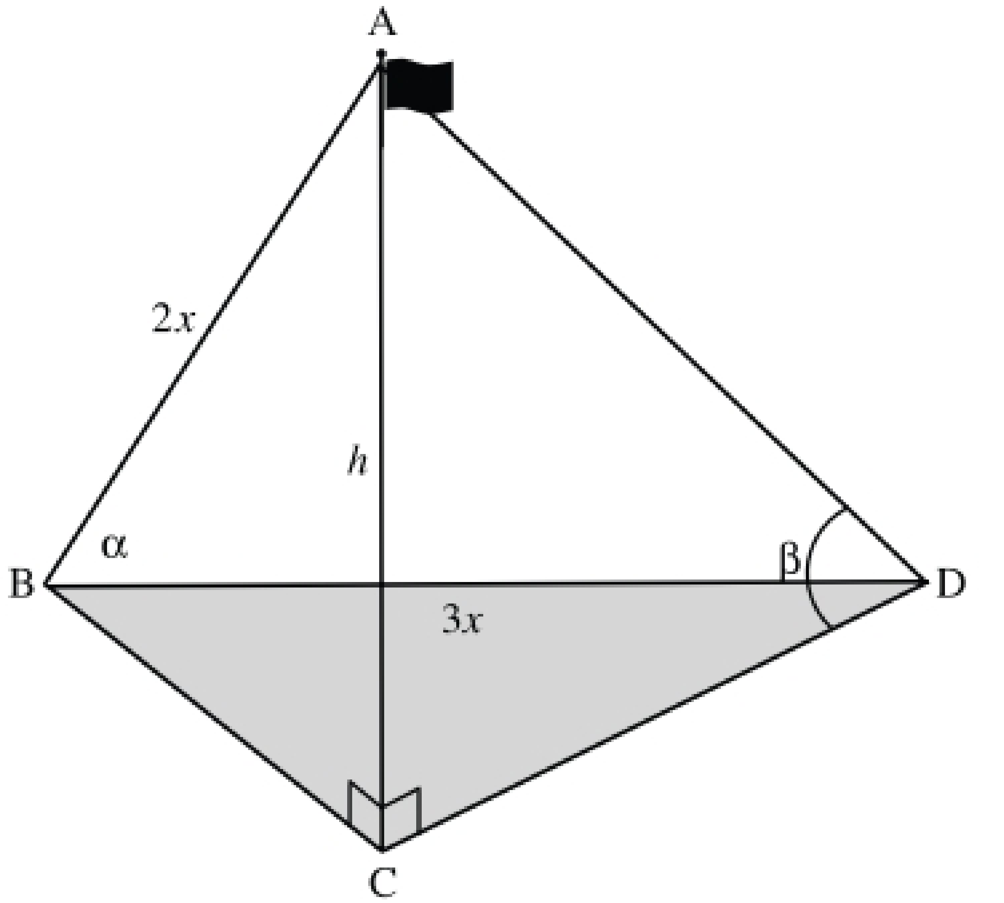
$$\therefore QR = 2\left(\sqrt{3}\right)\tan 45°\cos 30°$$

$$= 2 \times \sqrt{3} \times 1 \times \frac{\sqrt{3}}{2}$$

$$= 3 \text{ units}$$

EXAMPLE 3

The following sketch shows a vertical flagpole AC with height h. B, C and D are three points on ground level. $AB = 2x$, $BD = 3x$ and $A\hat{B}D = \alpha$. The angle of elevation of the top of the flagpole (A) from point D is β.



(a) Show that $h = x\sin\beta\sqrt{13-12\cos\alpha}$.

(b) Calculate the angle of elevation of the top of the flagpole (A) from point B if $\alpha = 48°$, $\beta = 56°$ and $x = 10$ m.

Solution

(a) In ΔABD:

$$AD^2 = (2x)^2 + (3x)^2 - 2(2x)(3x)\cos\alpha$$

$$\therefore AD^2 = 4x^2 + 9x^2 - 12x^2\cos\alpha$$

$$\therefore AD^2 = 13x^2 - 12x^2\cos\alpha$$

$$\therefore AD = \sqrt{13x^2 - 12x^2\cos\alpha} = \sqrt{x^2(13-12\cos\alpha)} = x\sqrt{13-12\cos\alpha}$$

In ΔACD:

$$\frac{h}{AD} = \sin\beta$$

$$\therefore h = AD\sin\beta$$

$$\therefore h = x\sqrt{13-12\cos\alpha}\sin\beta = x\sin\beta\sqrt{13-12\cos\alpha}$$

(b) The angle of elevation of A from B is $A\hat{B}C$.

$$\sin A\hat{B}C = \frac{h}{2x} = \frac{x\sin\beta\sqrt{13-12\cos\alpha}}{2x}$$

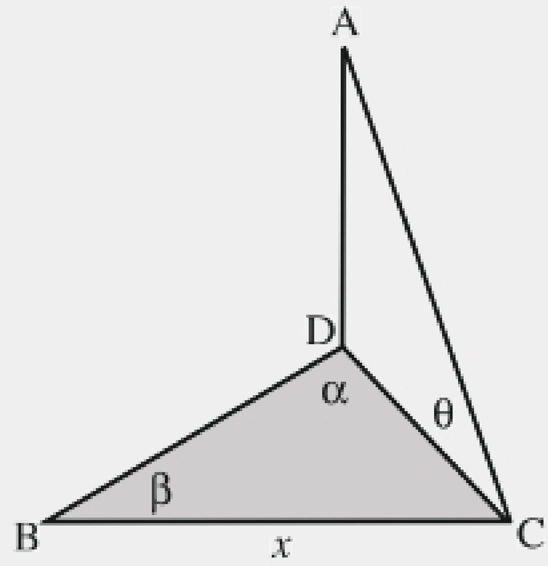
$$\therefore \sin A\hat{B}C = \frac{10\sin 56°\sqrt{13-12\cos 48°}}{2(10)}$$

$$\therefore \sin A\hat{B}C = 0{,}9241475528$$

$$\therefore A\hat{B}C = 67{,}54°$$

EXERCISE 2

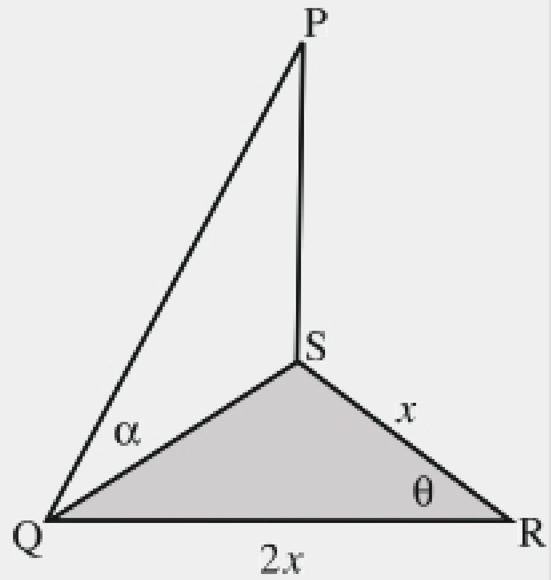
(a) In the sketch alongside, AD is a vertical line and ΔBDC lies in a horizontal plane. $B\hat{D}C = \alpha$, $D\hat{B}C = \beta$ and $A\hat{C}D = \theta$. $BC = x$.



(1) Show that $AD = \dfrac{x \sin\beta \tan\theta}{\sin\alpha}$.

(2) If $x = 5$ m, $\alpha = 120°$ and $\theta = \beta = 30°$, determine the length of AD, without the use of a calculator.

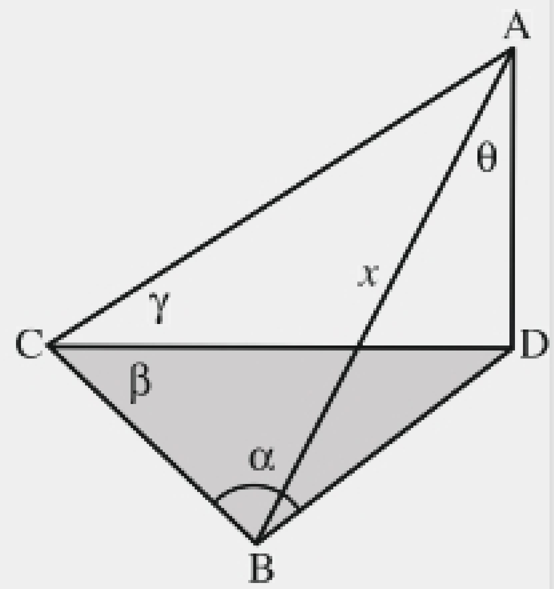
(b) In the sketch alongside, PS is a vertical line and ΔQSR lies in a horizontal plane. $P\hat{Q}S = \alpha$ and $Q\hat{R}S = \theta$. $SR = x$ and $QR = 2x$.



(1) Show that $PQ = \dfrac{x\sqrt{5 - 4\cos\theta}}{\cos\alpha}$.

(2) If $x = 12$ cm, $\alpha = 50°$ and $\theta = 35°$, calculate the length of PQ.

(c) In the sketch alongside, AD is a vertical line and ΔBCD lies in a horizontal plane. $C\hat{B}D = \alpha$, $B\hat{C}D = \beta$, $A\hat{C}D = \gamma$ and $B\hat{A}D = \theta$. $AB = x$.

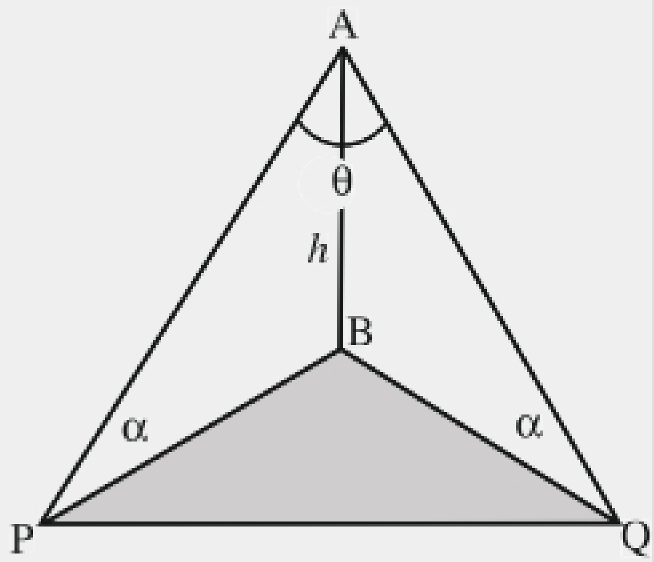


Show that

(1) $BC = \dfrac{x \sin\theta \sin(\alpha + \beta)}{\sin\beta}$

(2) $AC = \dfrac{x \sin\theta \sin\alpha}{\sin\beta \cos\gamma}$

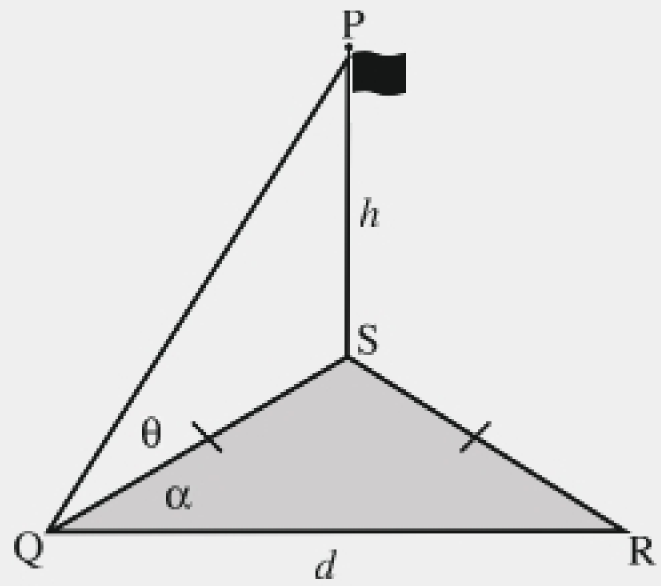
(d) In the sketch alongside, AB is a vertical line. B, P and Q are three points in the same horizontal plane. $A\hat{P}B = A\hat{Q}B = \alpha$ and $P\hat{A}Q = \theta$. $AB = h$.



(1) Show that $PQ = \dfrac{h\sqrt{2(1 - \cos\theta)}}{\sin\alpha}$.

(2) If $PQ = 5$ units, $h = 3$ units and $\theta = 65°$, calculate the size of α.

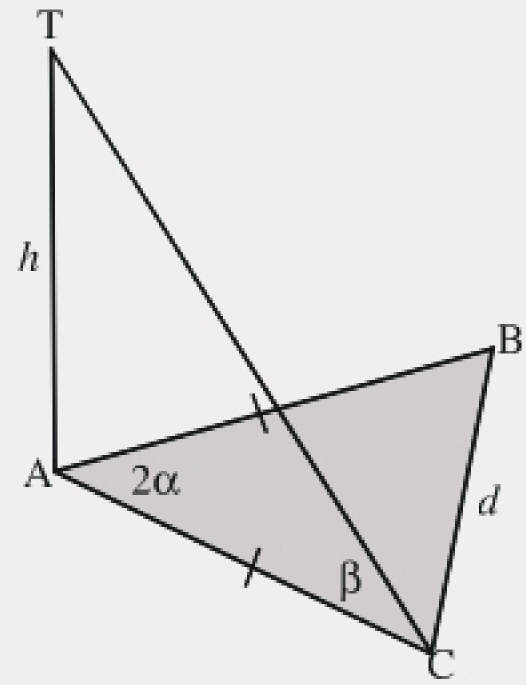
(e) In the sketch alongside, PS represents a vertical flagpole with height h. Q, S and R are points on ground level. The angle of elevation of the top of the flagpole (P) from Q is θ. $S\hat{Q}R = \alpha$ and $QR = d$. $SQ = SR$.



(1) Show that $h = \dfrac{d\tan\theta}{2\cos\alpha}$.

(2) Calculate the height of the flagpole if $d = 20$ m, $\theta = 58^\circ$ and $\alpha = 32^\circ$.

(f) A, B and C are three points on ground level. TA is a vertical tower with height h. From C, the angle of elevation to the top of the building (T) is β. $B\hat{A}C = 2\alpha$ and $TA = h$ and $BC = d$. $AB = AC$.

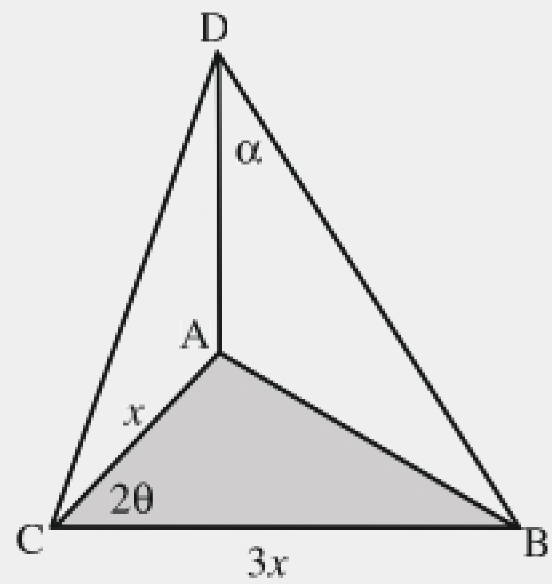


(1) Show that $h = \dfrac{d\tan\beta}{2\sin\alpha}$.

(2) Calculate the distance between B and C if $\alpha = 22^\circ$, $\beta = 47^\circ$ and the height of the tower is 25 m.

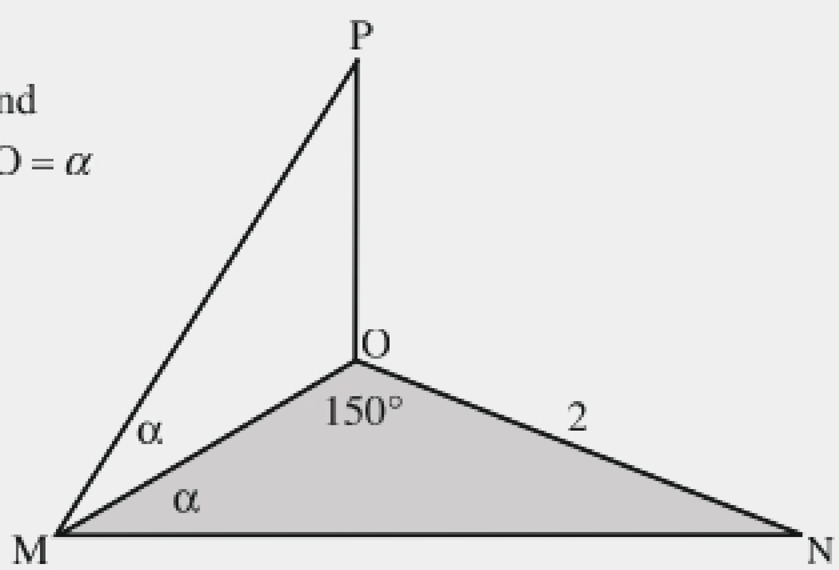
(g) In the sketch alongside, DA is a vertical line. A, B and C are three points in the same horizontal plane. $A\hat{D}B = \alpha$, $A\hat{C}B = 2\theta$, $AC = x$ and $CB = 3x$.

Show that $DB = \dfrac{2x\sqrt{3\sin^2\theta + 1}}{\sin\alpha}$.



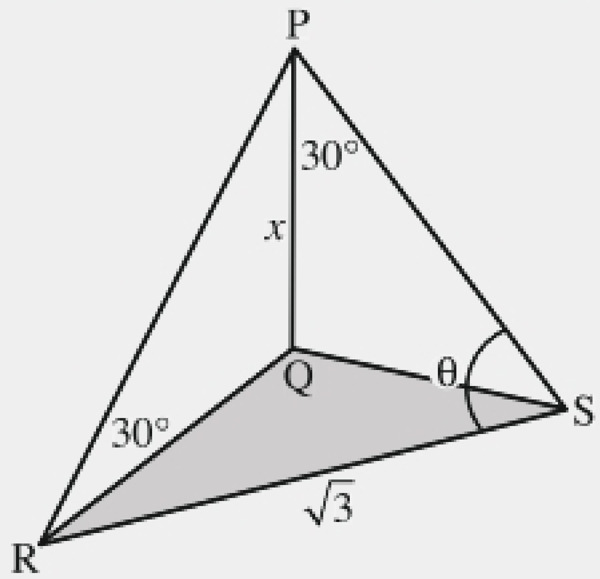
(h) In the sketch alongside, PO is a vertical line and ΔMON lies in a horizontal plane. $P\hat{M}O = N\hat{M}O = \alpha$ and $M\hat{O}N = 150^\circ$. $ON = 2$ units.

Show that $OP = 1 - \sqrt{3}\tan\alpha$.



(i) In the sketch alongside, PQ is a vertical line with length x. $RS = \sqrt{3}$ units. $P\hat{R}Q = Q\hat{P}S = 30°$ and $P\hat{S}R = \theta$.

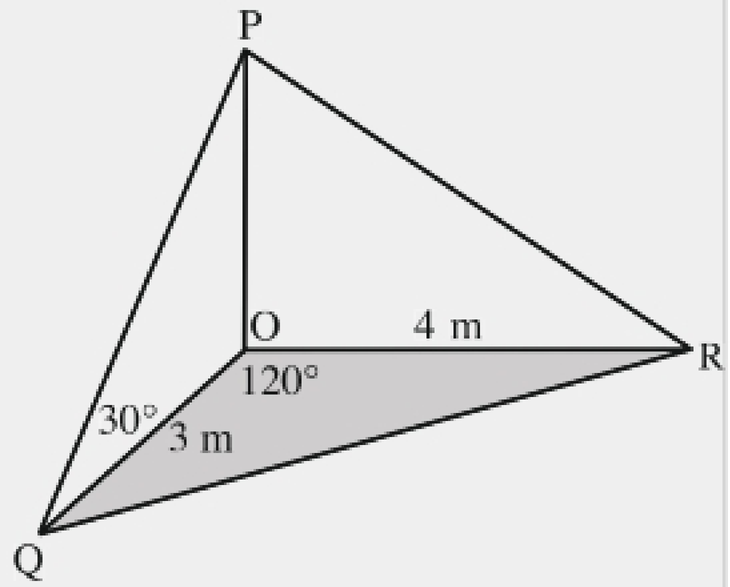
Show that $\cos\theta = \dfrac{9 - 8x^2}{12x}$



CONSOLIDATION AND EXTENSION EXERCISE

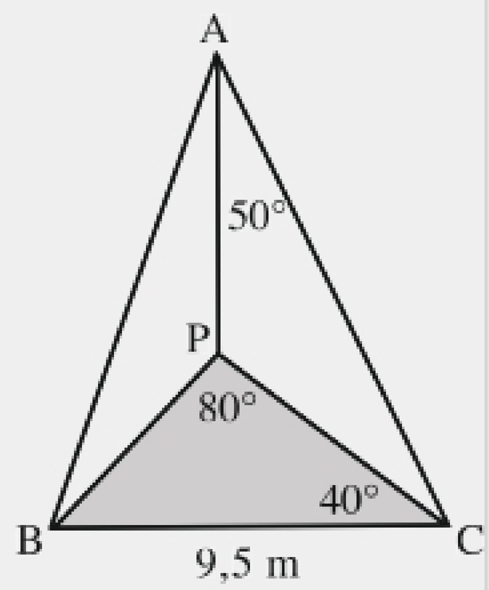
(a) In the sketch alongside, ΔOQR lies in a horizontal plane. PO is a vertical line segment. $P\hat{Q}O = 30°$, $Q\hat{O}R = 120°$, $QO = 3$ m and $OR = 4$ m.

Calculate
(1) the length of PR.
(2) the magnitude of $Q\hat{P}R$.
(3) the area of ΔPQR.
(4) the angle of elevation of P from R.



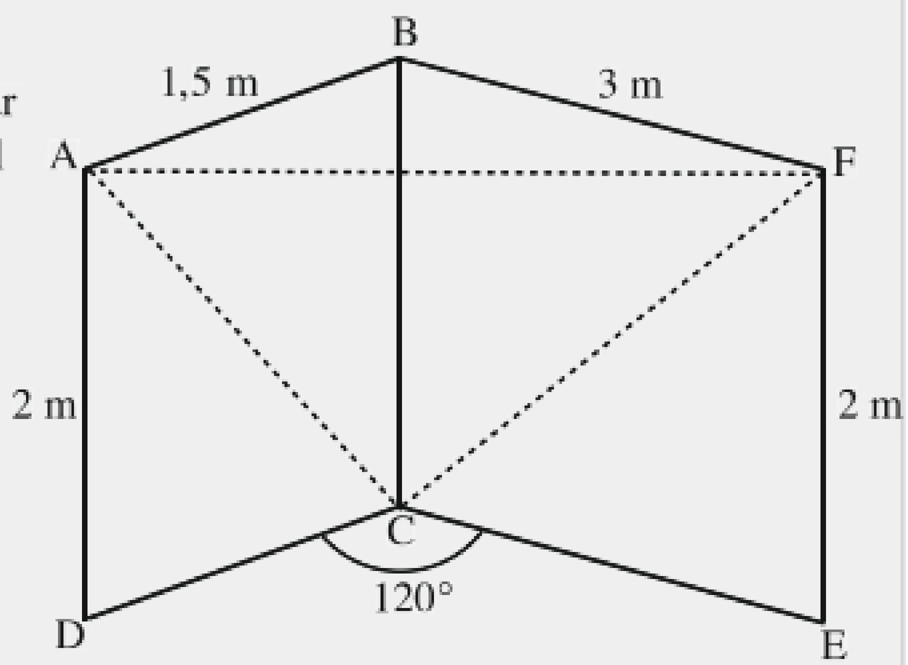
(b) In the sketch alongside, AP is a vertical line and ΔBPC lies in a horizontal plane. $P\hat{A}C = 50°$, $P\hat{C}B = 40°$ and $B\hat{P}C = 80°$. $BC = 9{,}5$ m.

Calculate
(1) the length of AC.
(2) the length of AB.
(3) the size of $A\hat{B}C$.
(4) the angle of depression of B from A.
(5) the total surface area of the shape (tetrahedron APBC).

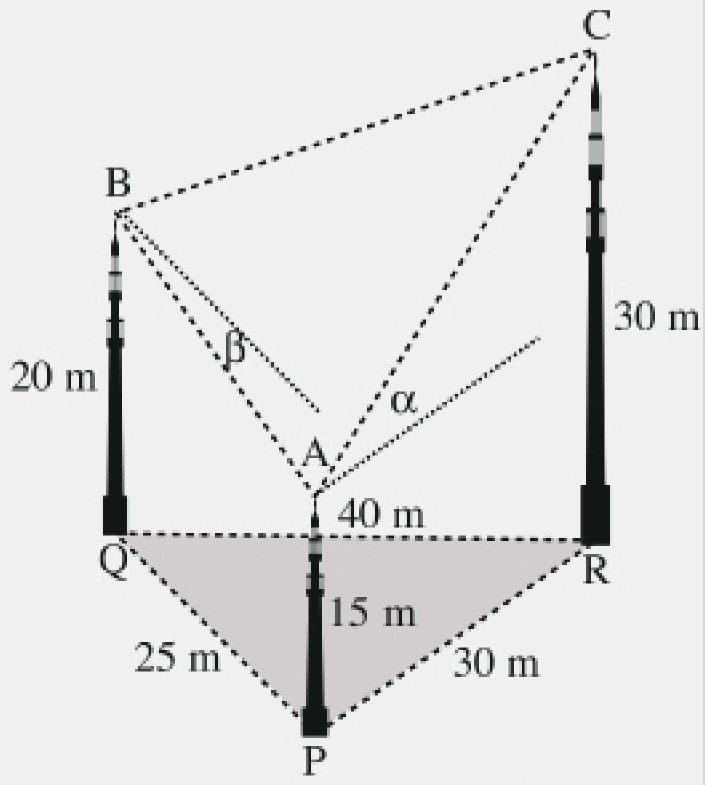


(c) The sketch alongside shows two rectangular advertising screens, each in its own vertical plane. The screens have dimensions $1{,}5 \text{ m} \times 2 \text{ m}$ and $3 \text{ m} \times 2 \text{ m}$. The angle between the screens is $120°$.

Calculate
(1) the distance between point A and point F.
(2) the magnitude of $A\hat{C}F$.



(d)* The following sketch shows three vertical towers, AP, BQ and CR, with heights 15 m, 20 m and 30 m respectively. The distances between the feet of the towers are:
$PQ = 25\text{ m},\ PR = 30\text{ m}$ and $QR = 40\text{ m}$.
The angle of elevation of C from A is α and the angle of depression of A from B is β.

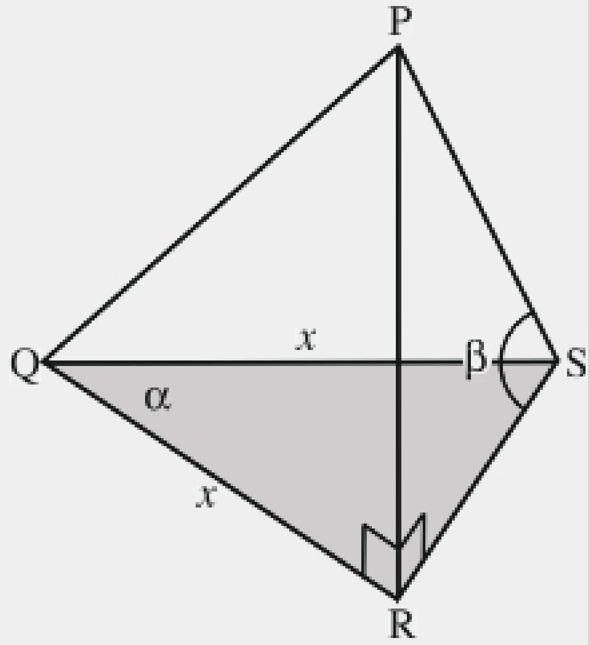


Calculate the magnitude of

(1) α (2) β

(3) $B\hat{A}C$

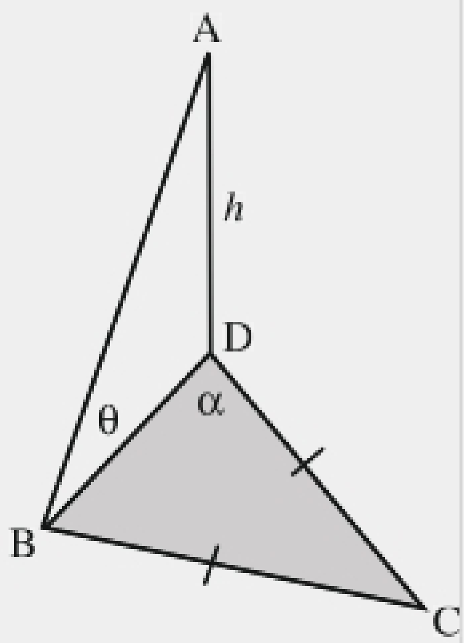
(e) In the sketch alongside Q, R and S represent three points on a flat surface at ground level. PR represents a vertical tower. The angle of elevation from S to the top of the tower (P) is β. $QR = QS = x$. $R\hat{Q}S = \alpha$.



(1) Show that $PR = x\tan\beta\sqrt{2 - 2\cos\alpha}$.

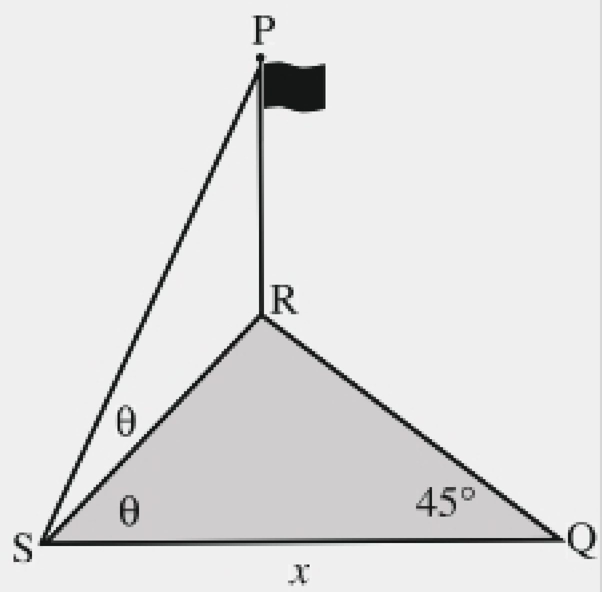
(2) Given that $\alpha = 34°$, the height of the tower is 30 m, and the distance from Q to the foot of the tower is 35 m, determine the size of β.

(f) In the sketch alongside, AD is a vertical line. B, C and D are three points in the same horizontal plane.
$A\hat{B}D = \theta$, $B\hat{D}C = \alpha$ and $AD = h$. $BC = DC$.



Show that $BC = \dfrac{h}{2\tan\theta\cos\alpha}$.

(g)* In the sketch alongside, PR represents a vertical flagpole. R, S and Q are three points in the same horizontal plane.
$SQ = x$ and $P\hat{S}R = R\hat{S}Q = \theta$. $S\hat{Q}R = 45°$.

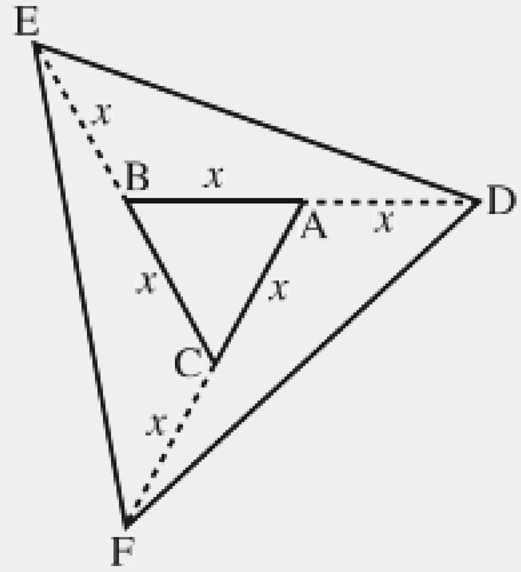


Show that $PR = \dfrac{2x\sin\theta}{\sin 2\theta + \cos 2\theta + 1}$.

(h) The sketch alongside shows ΔABC, an equilateral triangle with side length x. BA, CB and AC are extended by their own length to D, E and F respectively. ΔDEF is also an equilateral triangle.

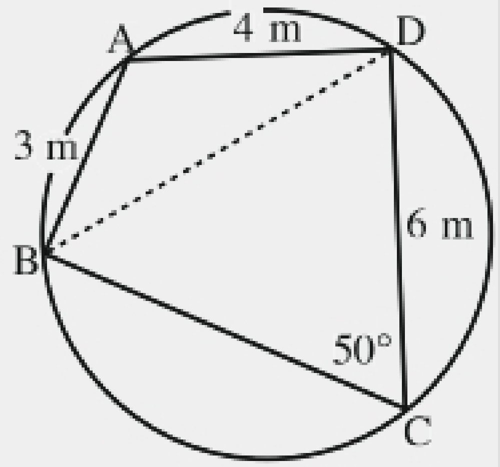
(1) Determine, in terms of x, and simplify
  (i) the length of ED.
  (ii) the area of ΔDEF.

(2) Calculate the size of $E\hat{D}B$.



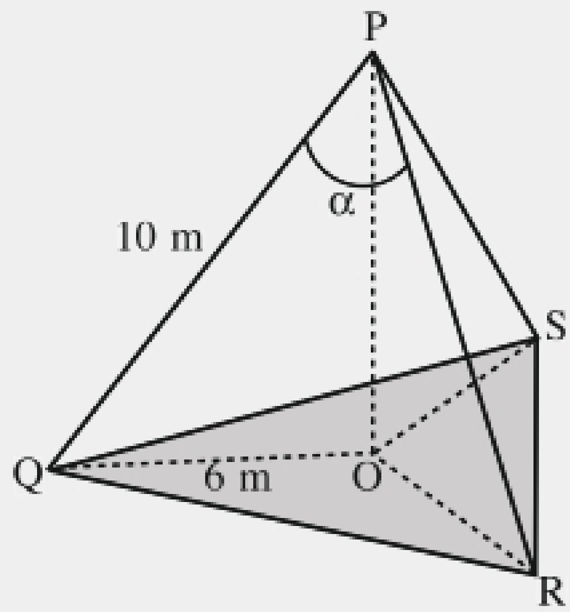
(i) The sketch alongside shows a circle passing through points A, B, C and D. $AB = 3$ m, $AD = 4$ m and $DC = 6$ m. $B\hat{C}D = 50°$.

Calculate
(1) the length of BD.
(2) the area of quadrilateral ABCD.
(3) the area of the circle.



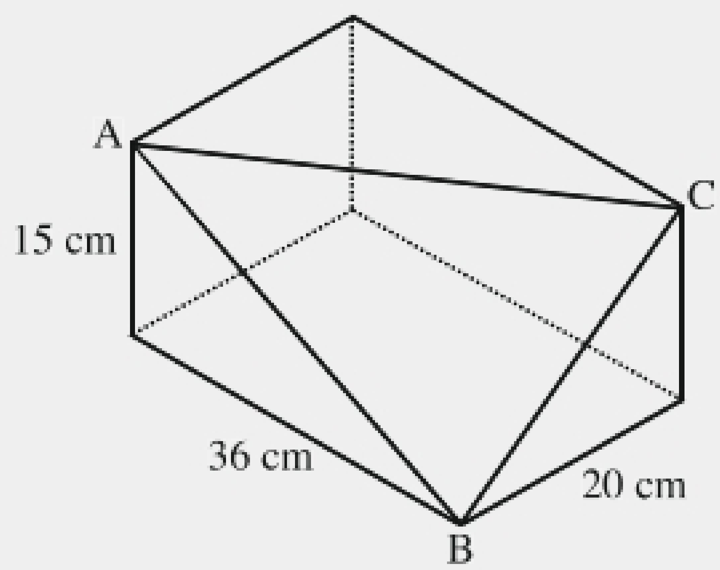
(j) The following sketch shows a right triangular pyramid, with base ΔQSR. PO is the height of the pyramid. $PQ = PR = PS = 10$ m and $OQ = OR = OS = 6$ m. $Q\hat{P}R = \alpha$.

Calculate
(1) the volume of the pyramid.
(2) the size of α.
(3) the surface area of the pyramid.



(k) The sketch alongside shows a metal block from which a part has been cut away. The block was cut along the plane containing vertices A, B and C, as shown in the sketch. The length, breadth and height of the original block was 36 cm, 20 cm and 15 cm respectively.

Calculate
(1) the size of $B\hat{A}C$.
(2) the surface area of the remaining part of the block.
(3)* the volume of the remaining part of the block.



Analytical Geometry

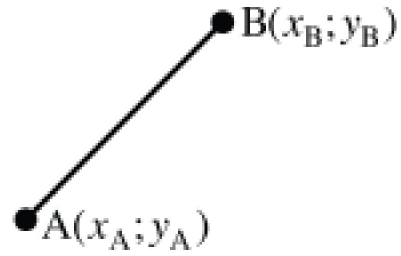
The purpose of Analytical Geometry is to translate geometrical objects into algebraic equations. We can then solve geometrical problems using algebraic methods. In Grade 10 and 11, we focussed on objects consisting of **straight lines**.

OVERVIEW OF BASIC CONCEPTS

DISTANCE/LENGTH

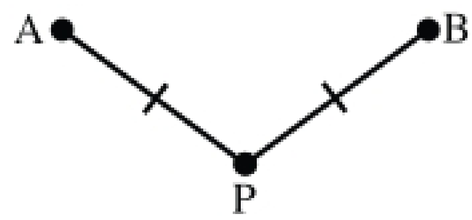
The distance between any two points $A(x_A; y_A)$ and $B(x_B; y_B)$ is given by

$$\boxed{AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2}}$$



Equidistance

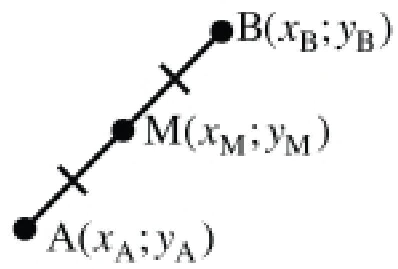
Point P is said to be equidistant from points A and B if $AP = PB$.



MIDPOINT

The coordinates of the midpoint $M(x_M; y_M)$ of a line segment AB, with $A(x_A; y_A)$ and $B(x_B; y_B)$, are given by

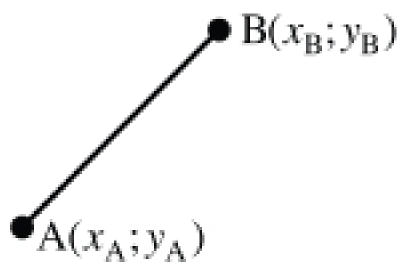
$$\boxed{x_M = \frac{x_A + x_B}{2}} \qquad \boxed{y_M = \frac{y_A + y_B}{2}}$$

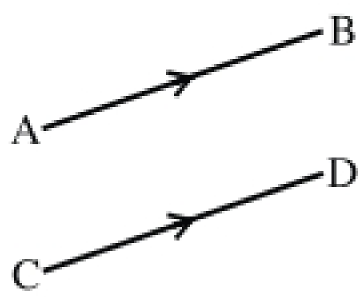
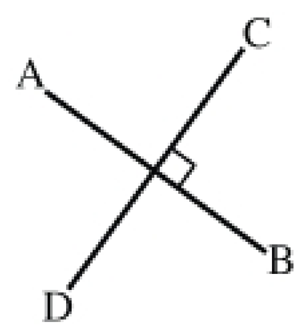
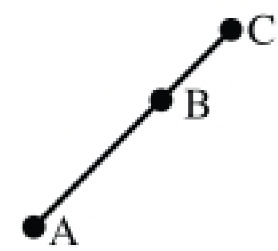


GRADIENT

For any two points $A(x_A; y_A)$ and $B(x_B; y_B)$, on a straight line AB, the gradient of the line is given by

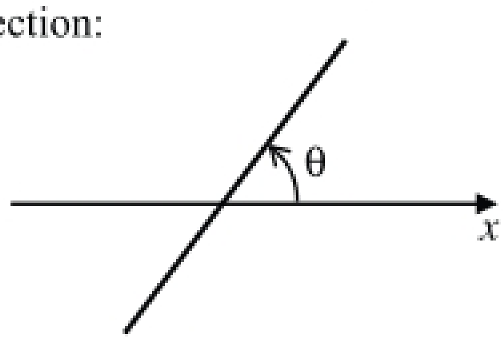
$$\boxed{m_{AB} = \frac{y_B - y_A}{x_B - x_A}}$$

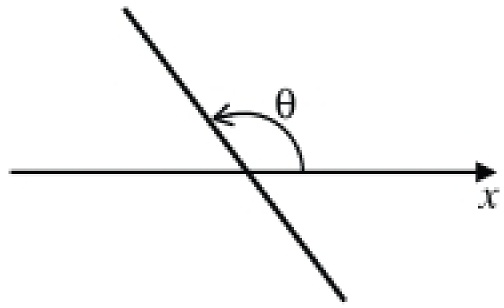


Parallel Lines	**Perpendicular Lines**	**Collinear Points**
		
$m_{AB} = m_{CD}$	$m_{AB} \times m_{CD} = -1$	$m_{AB} = m_{BC} = m_{AC}$

INCLINATION

The inclination θ of a line is the angle between the line and the positive x-axis, measured in an anti-clockwise direction:





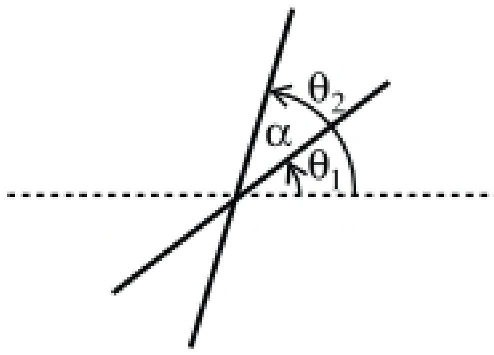
If the gradient of a straight line is m, the inclination is the unique angle θ, such that

$$\boxed{\tan\theta = m \text{ and } \theta \in [0°;180°)}$$

If $m \geq 0$: $\theta = \tan^{-1} m$ If $m < 0$: $\theta = \tan^{-1} m + 180°$

The angle between two lines

In order to calculate an angle between two lines, we draw a horizontal line through the point of intersection and then use the inclinations of the two lines to help us determine the angle between the lines:



$$\boxed{\alpha = \theta_2 - \theta_1}$$

THE EQUATION OF A STRAIGHT LINE

Standard form

The equation of a straight line has the form:

$$\boxed{y = mx + c}$$

In this form, m is the gradient and c is the y-intercept.

Horizontal and vertical lines

The equations of horizontal- and vertical lines have the forms:

$$\boxed{\textbf{Horizontal Line: } y = \text{constant}} \qquad \boxed{\textbf{Vertical Line: } x = \text{constant}}$$

The gradient of a horizontal line is 0. The gradient of a vertical line is undefined.

Gradient-point form

If given the gradient m and a point $(x_1; y_1)$ on a straight line, then the equation of the line is:

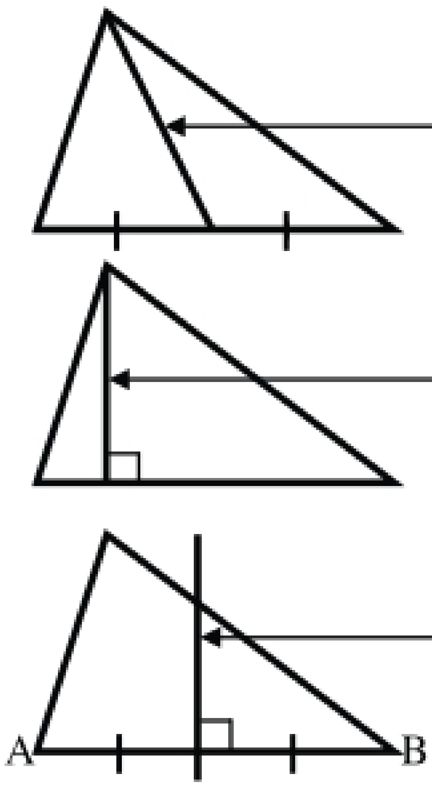
$$\boxed{y - y_1 = m(x - x_1)}$$

SPECIAL LINES IN TRIANGLES

Median: A median is a line drawn from a vertex of a triangle to the midpoint of the opposite side.

Altitude: An altitude is a line drawn from a vertex of a triangle, perpendicular to the opposite side.

Perpendicular Bisector: A perpendicular bisector of a line segment is a line that passes through the midpoint of the line segment and is perpendicular to the line segment.



	Vertex	Midpoint	Perpendicular
Median	✔	✔	
Altitude	✔		✔
Perpendicular Bisector of AB		✔	✔

PROVING DIFFERENT TYPES OF QUADRILATERALS

To prove that a quadrilateral is a …	show that…
trapezium	one pair of opposite sides are parallel
parallelogram	diagonals bisect each other
rhombus	it is a parallelogram AND diagonals are perpendicular to each other
rectangle	it is a parallelogram AND one angle is 90^o
square	it is a rhombus AND one angle is 90^o OR it is a rectangle AND diagonals are perpendicular to each other
kite	one diagonal is bisected by the other at 90^o

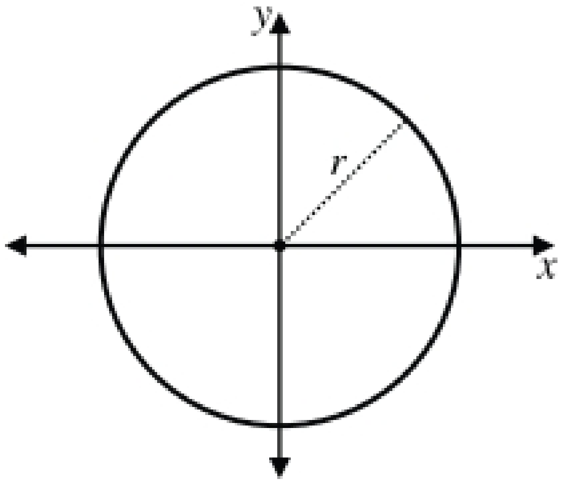
CIRCLES

This year, the focus of our study of Analytical Geometry will be on **circles** in the Cartesian plane.

CENTRE AT THE ORIGIN

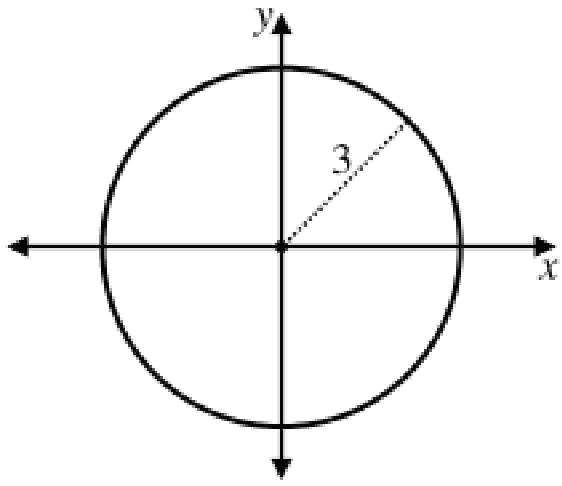
The equation of a circle with radius r and its centre at the origin is

$$\boxed{x^2 + y^2 = r^2}$$



EXAMPLE 1

Determine the equation of the circle with a radius of 3 units and centre $(0;0)$.



Solution

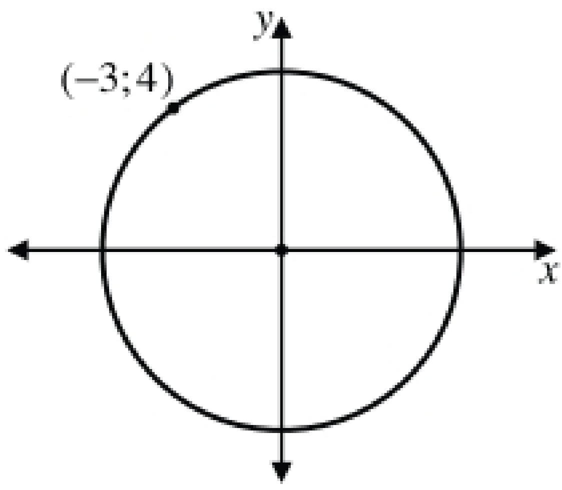
$x^2 + y^2 = r^2$

$\therefore x^2 + y^2 = 3^2$

$\therefore x^2 + y^2 = 9$

EXAMPLE 2

Determine the equation of the circle with centre $(0;0)$ and passing through the point $(-3;4)$.



Solution

$x^2 + y^2 = r^2$

Substitute the point $(-3;4)$ into the equation:

$(-3)^2 + 4^2 = r^2$

$\therefore 9 + 16 = r^2$

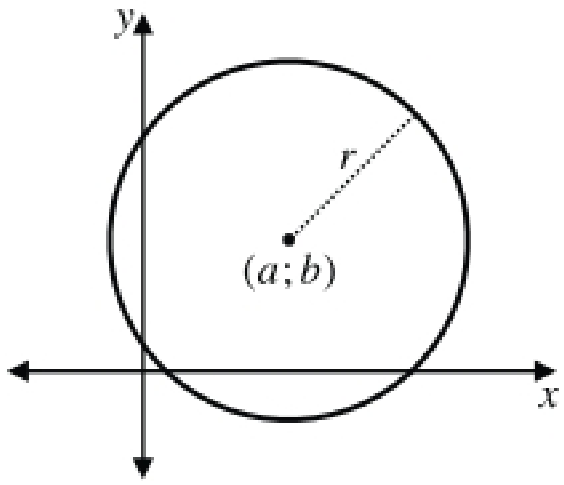
$\therefore r^2 = 25$

$\therefore x^2 + y^2 = 25$

CENTRE AT ANY POINT

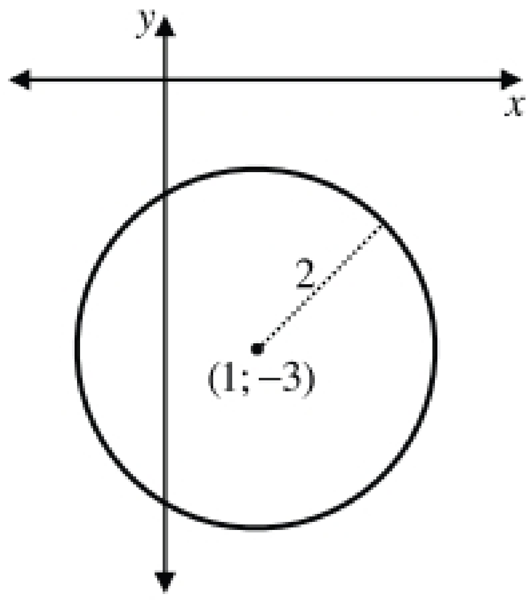
The equation of a circle with radius r and its centre at the point $(a;b)$ is

$$\boxed{(x-a)^2+(y-b)^2=r^2}$$



EXAMPLE 3

Write down the equation of the circle with centre $(1;-3)$ and radius 2 units.



Solution

$(x-a)^2+(y-b)^2=r^2$

$\therefore (x-1)^2+(y+3)^2=2^2$

$\therefore (x-1)^2+(y+3)^2=4$

EXAMPLE 4

Write down the radius and the coordinates of the centre of the circles with the following equations:

(a) $x^2+y^2=1$

(b) $(x+2)^2+(y-3)^2=25$

(c) $(x-1)^2+y^2=16$

(d) $x^2+(y+2)^2=10$

Solution

(a) Radius: 1 unit
Centre: $(0;0)$

(b) Radius: 5 units
Centre: $(-2;3)$

(c) Radius: 4 units
Centre: $(1;0)$

(d) Radius: $\sqrt{10}$ units
Centre: $(0;-2)$

EXAMPLE 5

Determine the equation of the circle

(a) with centre $(3;2)$ and passing through the point $(1;-2)$.

(b) with $(-3;-7)$ and $(3;1)$ the endpoints of one of its diameters.

Solution

(a) $(x-a)^2+(y-b)^2=r^2$

$\therefore (x-3)^2+(y-2)^2=r^2$

Substitute the point $(1;-2)$ into the equation:

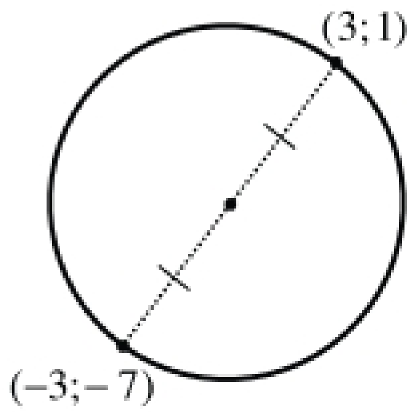
$(1-3)^2+(-2-2)^2=r^2$

$\therefore 4+16=r^2$

$\therefore r^2=20$

$\therefore (x-3)^2+(y-2)^2=20$

(b) The midpoint of the diameter is the centre of the circle:



$\text{Centre}=\left(\dfrac{-3+3}{2};\dfrac{-7+1}{2}\right)=(0;-3)$

$\therefore x^2+(y+3)^2=r^2$

Substitute the point $(3;1)$ into the equation:

$\therefore 3^2+(1+3)^2=r^2$

$\therefore 9+16=r^2$

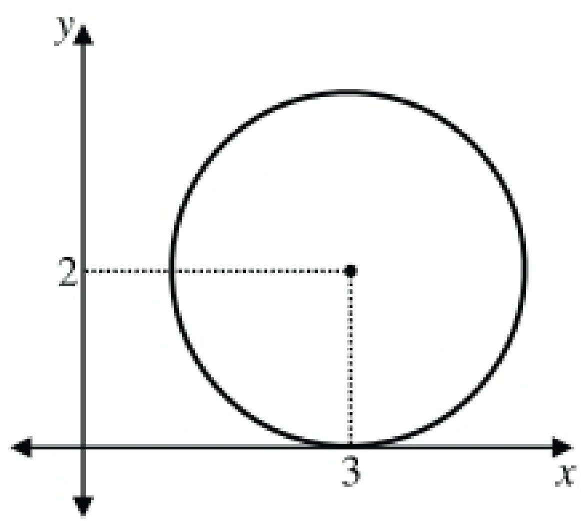
$\therefore 25=r^2$

$\therefore x^2+(y+3)^2=25$

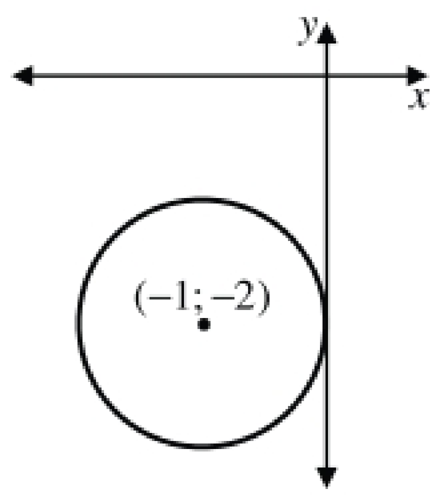
EXAMPLE 6

Determine the equation of the circle in each one of the following sketches:

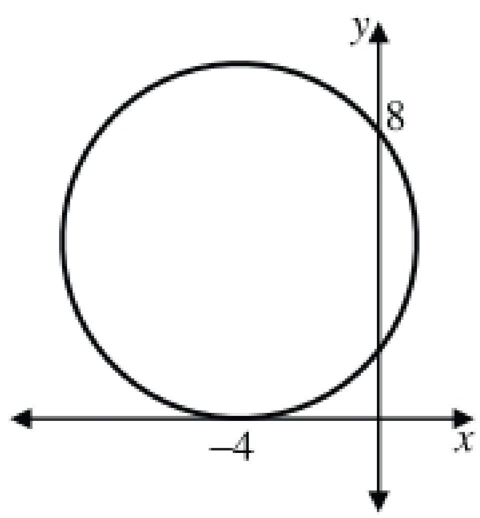
(a)



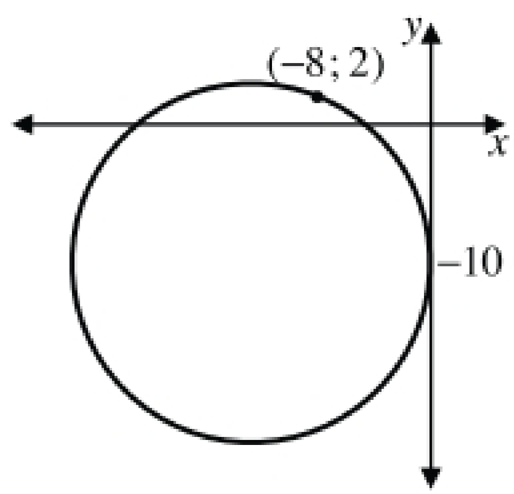
(b)



(c)*

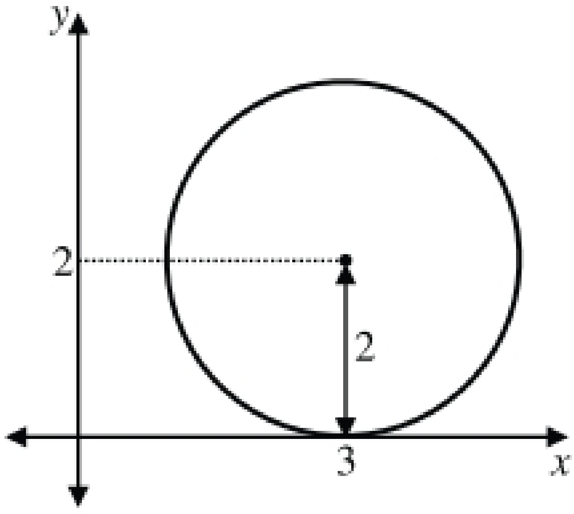


(d)*



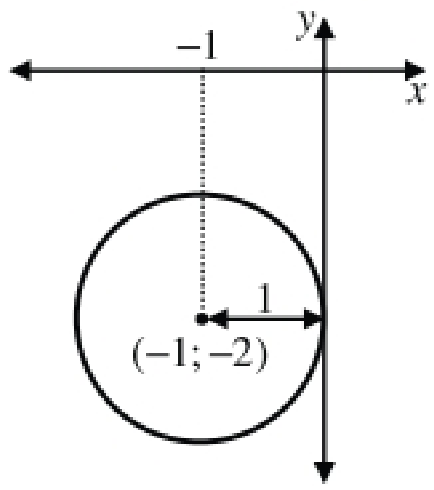
Solution

(a)



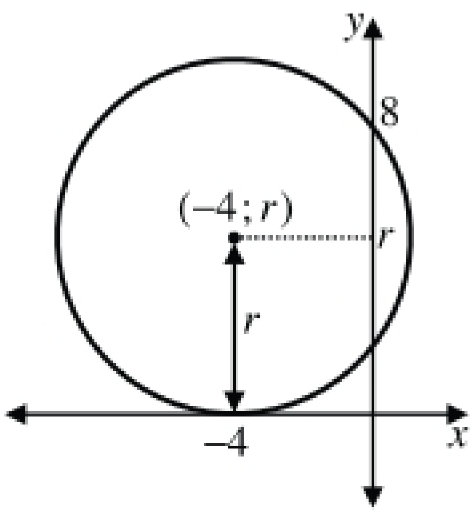
$(x-3)^2+(y-2)^2=4$

(b)



$(x+1)^2+(y+2)^2=1$

(c)



$(x+4)^2+(y-r)^2=r^2$

Substitute $(0;8)$ into the equation:

$(0+4)^2+(8-r)^2=r^2$

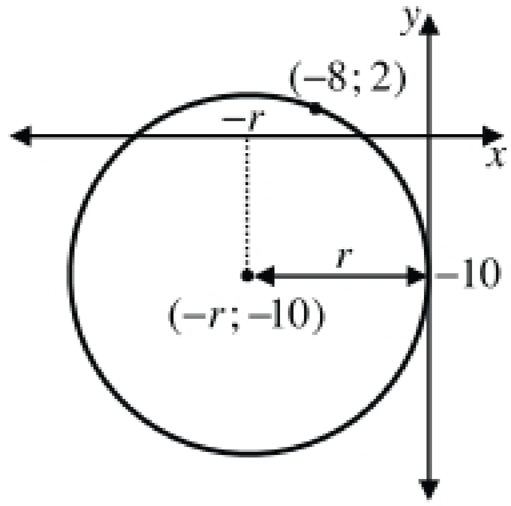
$\therefore 16+64-16r+r^2=r^2$

$\therefore -16r=-80$

$\therefore r=5$

$\therefore (x+4)^2+(y-5)^2=25$

(d)



$(x+r)^2+(y+10)^2=r^2$

Substitute $(-8;2)$ into the equation:

$(-8+r)^2+(2+10)^2=r^2$

$\therefore 64-16r+r^2+144=r^2$

$\therefore -16r=-208$

$\therefore r=13$

$\therefore (x+13)^2+(y+10)^2=169$

EXERCISE 1

(a) Write down the radius and the coordinates of the centre of the circles with the following equations:

(1) $x^2+y^2=4$

(2) $(x-2)^2+(y-1)^2=9$

(3) $(x+5)^2+(y+2)^2=36$

(4) $(x+3)^2+(y-1)^2=16$

(5) $x^2+(y-7)^2=81$

(6) $(x+2)^2+y^2=49$

(7) $(x-5)^2+(y+3)^2=20$

(8)* $(2x-2)^2+4(-y-3)^2=12$

(b) Determine the equation of the circle with

(1) centre $(0;0)$ and radius 8 units.

(2) centre $(2;-1)$ and radius 3 units.

(3) centre $(-3;5)$ and diameter 10 units.

(4) centre $(1;0)$ and radius $\sqrt{7}$ units.

(5) centre $(0;0)$ and passing through the point $(1;-2)$.

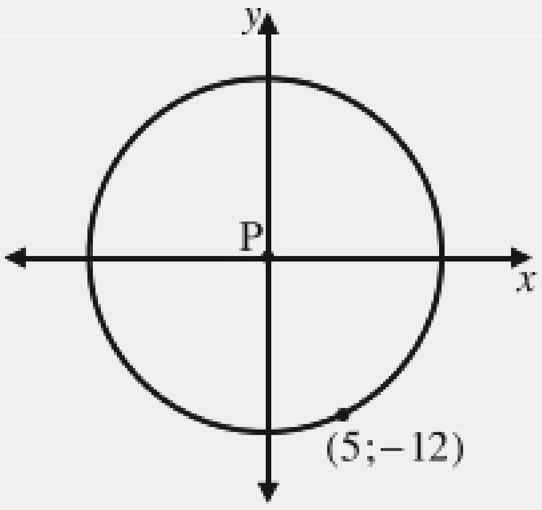
(6) centre $(-4;1)$ and passing through the point $(-1;-3)$.

(7) centre $(0;2)$ and passing through the point $(\sqrt{3};\sqrt{2}+2)$.

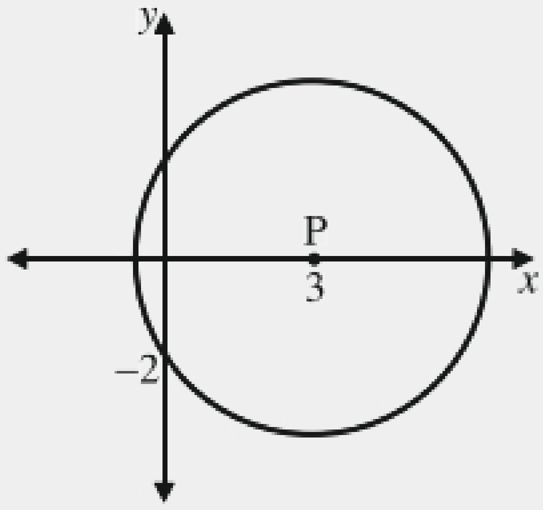
(8) $(2;-3)$ and $(8;-1)$ the endpoints of one of its diameters.

(c) Determine the equation of the circle in each one of the following sketches.
(P is the centre of the circle in each case.)

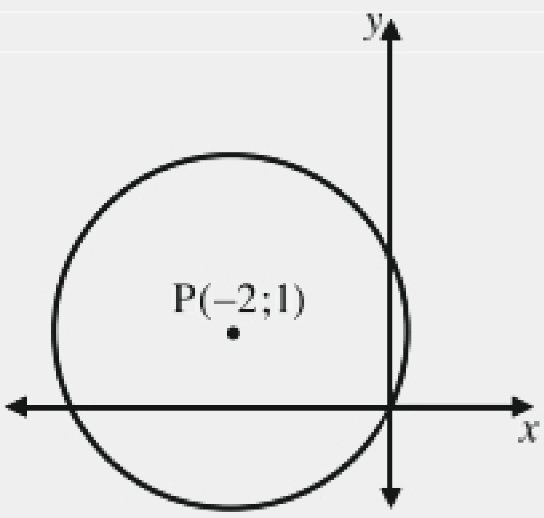
(1)



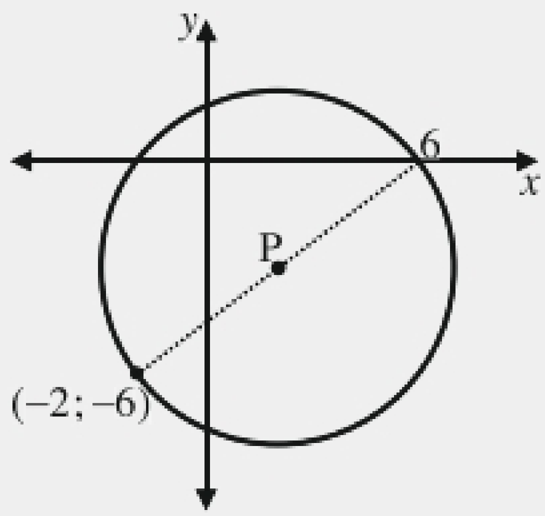
(2)



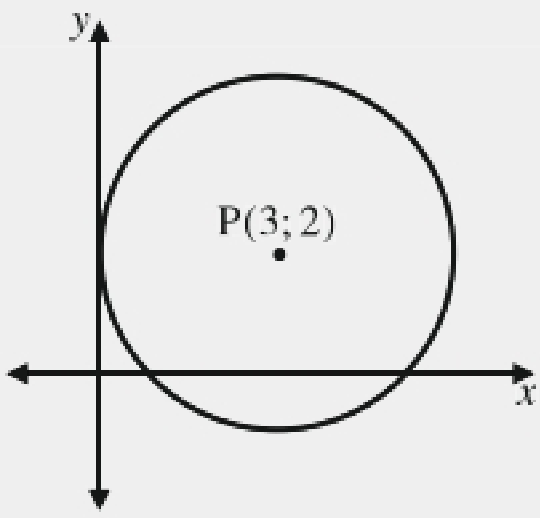
(3)



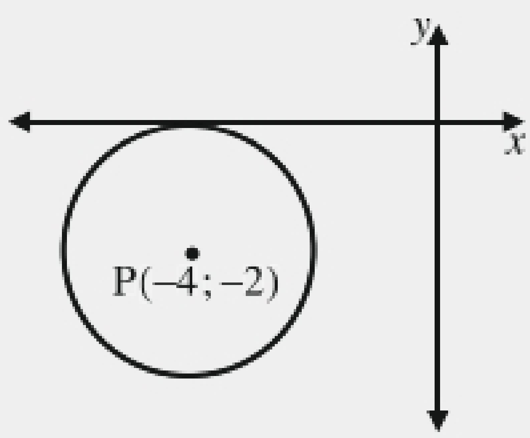
(4)



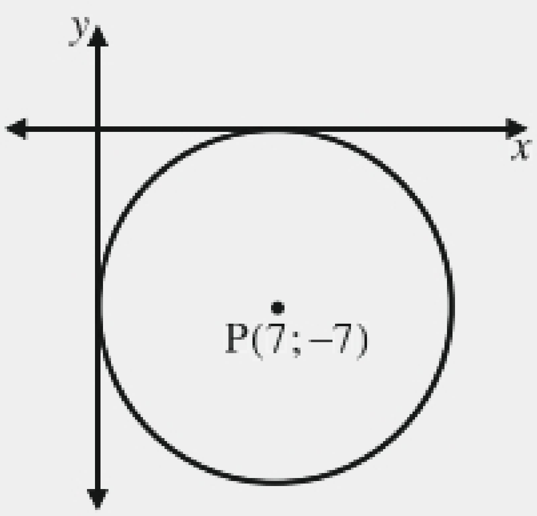
(5)



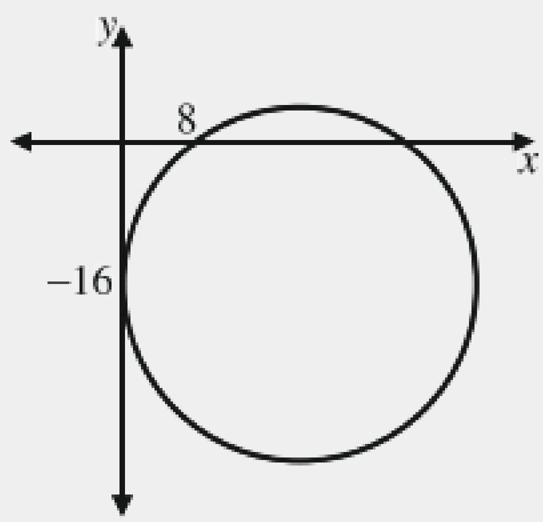
(6)



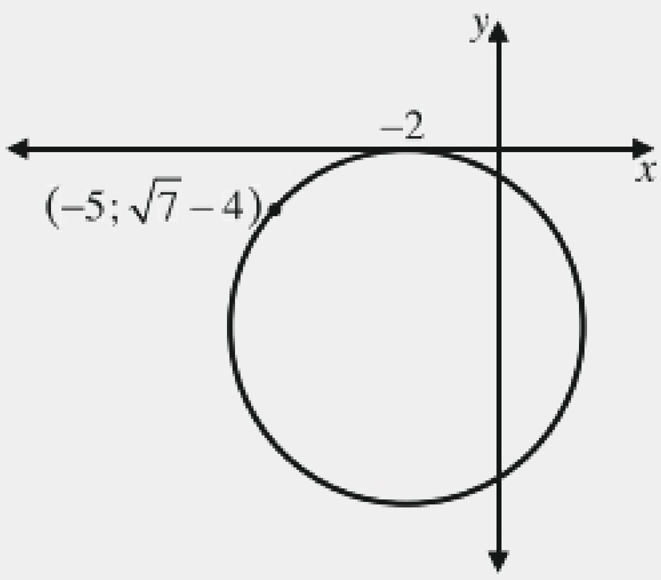
(7)



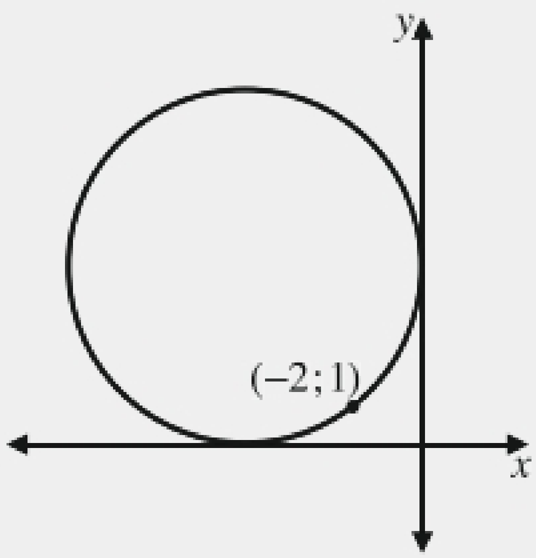
(8)*



(9)*



(10)*



(d)* Determine the equations of two possible circles of which the radius is 5, the x-intercept is 1 and the x-coordinate of the centre is 3.

(e)* Determine the equation of a circle passing through the points $(5;7)$ and $(-1;-1)$ and with its centre on the line $y = x+1$.

(f)* A circle passes through the point $(9;-1)$, has a radius of $\sqrt{113}$ and its centre lies on the line $y - 2x = 3$. The coordinates of the centre are integers. Determine the equation of the circle.

EQUATIONS NOT GIVEN IN THE STANDARD FORM

The equation of a circle is often given in the form $x^2 + y^2 + ax + by + c = 0$. In this form, the centre and the radius of the circle can not be seen immediately. We use the method of **"completing the square"** to rewrite these equations to the standard form:

EXAMPLE 7

Determine the radius and the coordinates of the centre of the circle with equation $x^2 + y^2 - 6x + 2y - 6 = 0$.

Solution

$x^2 - 6x + y^2 + 2y = 6$

Rearrange :
x's together and y's together and transpose constant term to RHS.

$\therefore x^2 - 6x + \boxed{(-3)^2} + y^2 + 2y + 1^2 = 6 + \boxed{(-3)^2} + 1^2$

halve & square   halve & square

- **Complete the square** for x and y.
- Add on RHS what you add on LHS.

$\therefore (x^2 - 6x + 9) + (y^2 + 2y + 1) = 6 + 9 + 1$

$\therefore (x-3)^2 + (y+1)^2 = 16$

$\therefore$ The centre is $(3;-1)$ and the radius is 4.

EXAMPLE 8*

The equation of a circle is $x^2+2kx+y^2-6y+4=0$.

(a) Express the radius of the circle in terms of k.

(b) If the radius is 3 units and $k<0$, determine the coordinates of the centre.

Solution

(a) $x^2+2kx+y^2-6y+4=0$

$\therefore x^2+2kx+y^2-6y=-4$

$\therefore x^2+2kx+k^2+y^2-6y+9=-4+k^2+9$

$\therefore (x+k)^2+(y-3)^2=k^2+5$

$\therefore r=\sqrt{k^2+5}$

(b) $\sqrt{k^2+5}=3$

$\therefore k^2+5=9$

$\therefore k^2=4$

$\therefore k=\pm 2$, but since $k<0$, $k=-2$.

$\therefore$ The equation of the circle is $(x-2)^2+(y-3)^2=9$.

$\therefore$ The centre is $(2;3)$.

EXERCISE 2

(a) Determine the radius and the coordinates of the centre of the circles with the following equations:

(1) $x^2+2x+y^2-8y=-8$ (2) $x^2+y^2-4x+6y-3=0$

(3) $x^2+y^2+6x=-8$ (4) $x^2+y^2=10y$

(5) $x^2+y^2-10x+14y+66=0$ (6) $x^2-3x+y^2+2y=\frac{3}{4}$

(7) $12x^2+12y^2+36x-8y-5=0$ (8) $(x-y)^2+(y+x)^2=50$

(b) The circle $x^2+y^2+2p^2+4px-py=0$, with $p>1$, passes through the point $(-1;1)$. Determine

(1) the value of p.

(2) the radius and coordinates of the centre of the circle.

(c)* The equation of a circle is $x^2+y^2-6x+4y=k$.

(1) Determine the coordinates of the centre of the circle.

(2) If the radius of the circle is 13 units, determine the value of k.

(d)* The circle defined by $x^2+2ax+y^2+4by+9=0$ has centre $(-3;2)$.

(1) Determine the values of a and b.

(2) Calculate the radius of the circle.

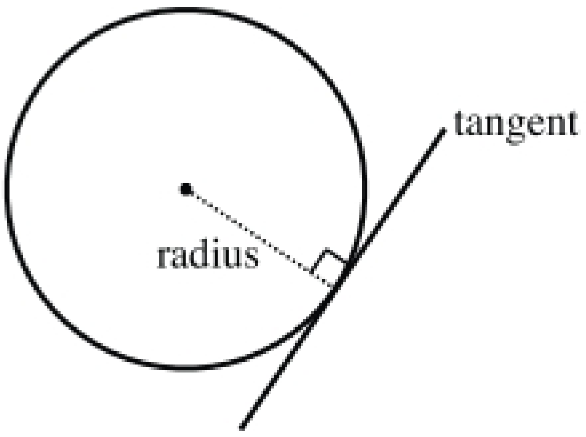
(e)* The equation of a circle, with a diameter of $4\sqrt{3}$ units, is $x^2+y^2-2mx+7m-4=0$.

(1) Determine the value of m.

(2) Write down the coordinates of the centre of the circle.

TANGENTS

A *tangent* to a circle is a line **touching** the circle (in **one point**). A tangent is always **perpendicular** to the radius drawn to the point of contact. This means that the product of the gradient of the tangent and the gradient of the radius is -1:



$$\text{tangent} \perp \text{radius} \qquad \therefore m_{\text{radius}} \times m_{\text{tangent}} = -1$$

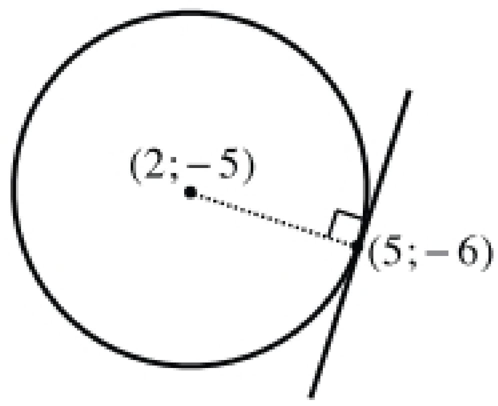
We can use this fact to determine the equation of the tangent to a given circle at any point on the circumference of the circle:

EXAMPLE 9

Determine the equation of the tangent to the circle $(x-2)^2+(y+5)^2=10$ at the point $(5;-6)$.

Solution

Centre: $(2;-5)$



$m_{\text{rad}} = \dfrac{-6-(-5)}{5-2} = -\dfrac{1}{3}$

$\therefore m_{\tan} = +3$ (tangent $\perp$ radius)

$y - y_1 = m(x - x_1)$

$\therefore y + 6 = 3(x-5)$ The tangent passes through the point $(5;-6)$

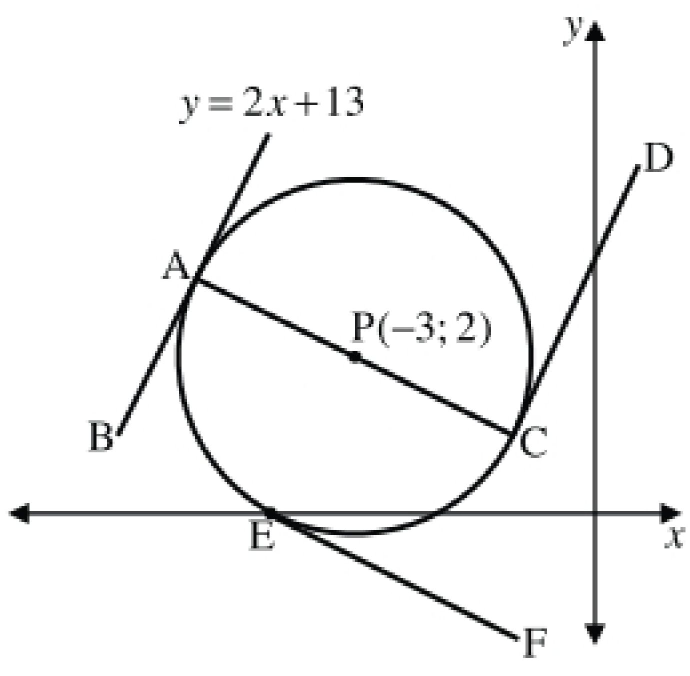
$\therefore y = 3x - 21$

In the following example the equation of a tangent is given:

EXAMPLE 10

In the sketch alongside, $P(-3;2)$ is the centre of the circle. E is an x-intercept of the circle. AB, CD and EF are tangents to the circle. The equation of AB is $y = 2x + 13$.

Determine
(a) the equation of AC.
(b) the coordinates of A.
(c) the equation of CD.
(d) the equation of the circle.
(e) the equation of EF.



Solution

(a) $m_{AB} = 2 \quad \rightarrow \quad m_{AC} = -\frac{1}{2}$ (tangent $\perp$ radius)

$y - y_1 = m(x - x_1)$

$\therefore y - 2 = -\frac{1}{2}(x + 3)$

$\therefore y = -\frac{1}{2}x + \frac{1}{2}$

(b) A is the point of intersection between $y = 2x + 13$ and $y = -\frac{1}{2}x + \frac{1}{2}$:

$2x + 13 = -\frac{1}{2}x + \frac{1}{2}$

$\therefore 4x + 26 = -x + 1$

$\therefore 5x = -25$

$\therefore x = -5$

$\therefore y = 2(-5) + 13 = 3$

$\therefore$ The coordinates of A are $(-5; 3)$.

(c) Find the coordinates of C:

P is the midpoint of AC.

$\therefore \frac{x_A + x_C}{2} = x_P$ and $\frac{y_A + y_C}{2} = y_P$

$\therefore \frac{-5 + x_C}{2} = -3$ and $\frac{3 + y_C}{2} = 2$

$\therefore x_C = -1$ and $y_C = 1$

$\therefore$ The coordinates of C are $(-1; 1)$.

$m_{CD} = 2$ (tangent $\perp$ radius)

$y - y_1 = m(x - x_1)$

$\therefore y - 1 = 2(x + 1)$

$\therefore y = 2x + 3$

(d) $(x + 3)^2 + (y - 2)^2 = r^2$

Substitute the point $A(-5; 3)$:

$(-5 + 3)^2 + (3 - 2)^2 = r^2$

$\therefore r^2 = 5$

$\therefore (x + 3)^2 + (y - 2)^2 = 5$

(e) For E, let $y = 0$ in $(x + 3)^2 + (y - 2)^2 = 5$:

$(x + 3)^2 + (0 - 2)^2 = 5$

$\therefore (x + 3)^2 = 1$

$\therefore x + 3 = \pm 1$

$\therefore x = -2$ or $x = -4 \quad \rightarrow \quad E(-4; 0)$

$$m_{PE} = \frac{0-2}{-4+3} = 2$$

$$\therefore m_{EF} = -\frac{1}{2} \qquad \text{(tangent} \perp \text{radius)}$$

$$y - 0 = -\frac{1}{2}(x+4)$$

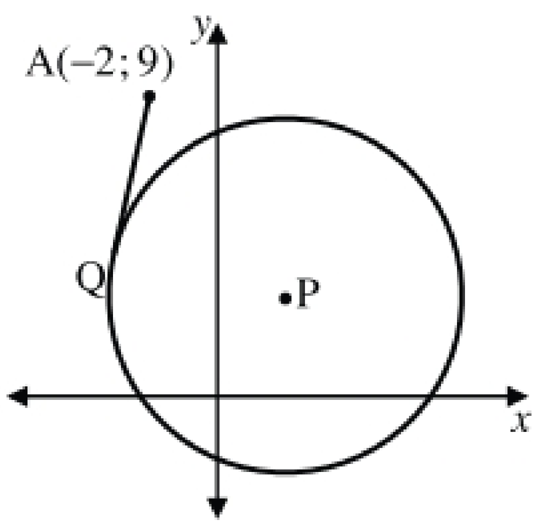
$$\therefore y = -\frac{1}{2}x - 2$$

THE LENGTH OF A TANGENT

To determine the length of a tangent to a given circle, from a given point, we make use of the distance formula and the theorem of Pythagoras:

EXAMPLE 11

In the sketch alongside, the equation of the circle is $x^2 + y^2 - 4x - 6y - 14 = 0$. P is the centre of the circle. AQ is a tangent to the circle at Q.



Determine
(a) the coordinates of P.
(b) the length of AQ.

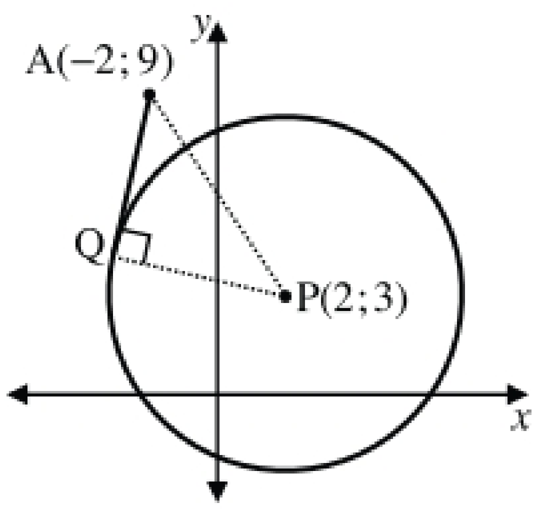
Solution

(a) $x^2 - 4x + y^2 - 6y = 14$

$\therefore x^2 - 4x + 4 + y^2 - 6y + 9 = 14 + 4 + 9$

$\therefore (x-2)^2 + (y-3)^2 = 27 \quad \rightarrow \quad P(2;3)$

(b) Draw AP and QP.



$AP^2 = (-2-2)^2 + (9-3)^2 = 52$

$QP^2 = r^2 = 27$

$AQ^2 = AP^2 - QP^2$ (tan AQ ⊥ rad QP ; Pythag)

$\therefore AQ^2 = 52 - 27$

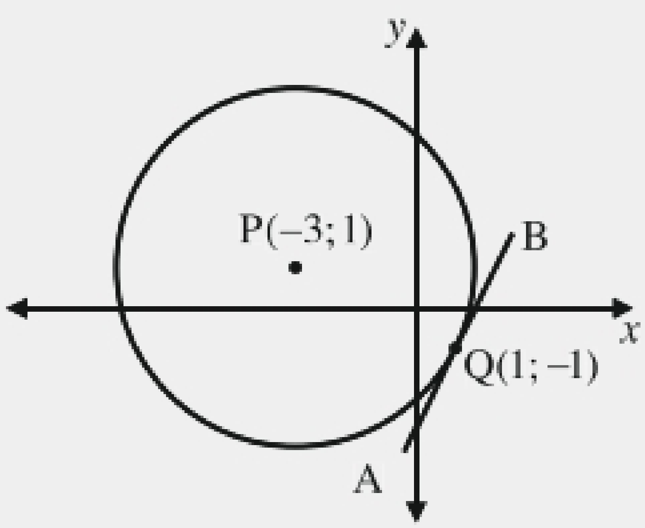
$\therefore AQ^2 = 25$

$\therefore AQ = 5$ units

EXERCISE 3

(a) In the sketch alongside, $P(-3;1)$ is the centre of the circle. AB is a tangent to the circle at $Q(1;-1)$.

Determine
(1) the equation of circle P.
(2) the equation of tangent AQB.



(b) Determine the equation of the tangent to the circle $(x+4)^2+(y+2)^2=50$ at the point

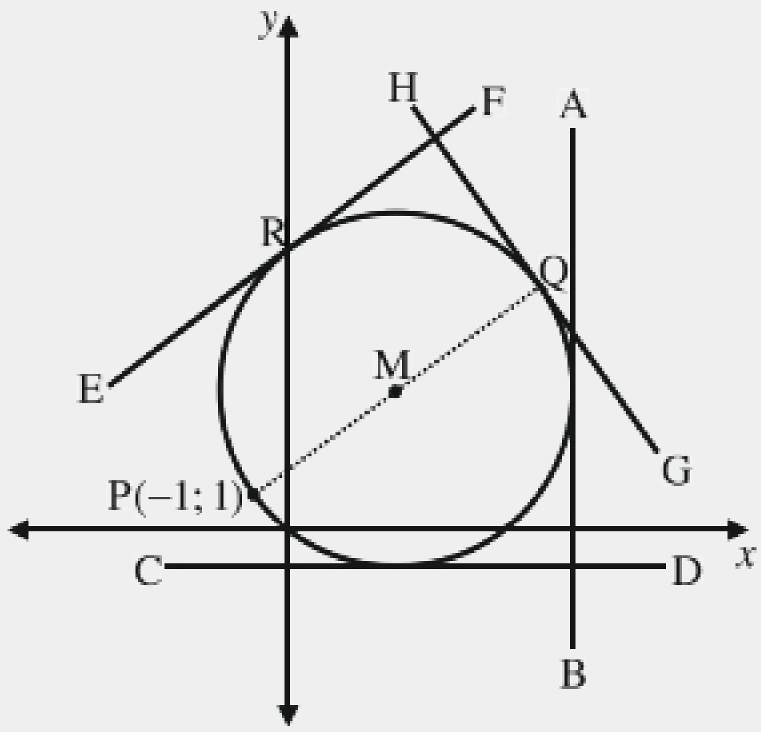
(1) $(1;3)$ (2) $(-9;3)$

(c) Determine the equation of the tangent to the circle $x^2+y^2+2x-6y-4=0$, at $(-6;2)$.

(d) Determine the equation of the tangent to the circle $(x-7)^2+y^2=13$, at the point

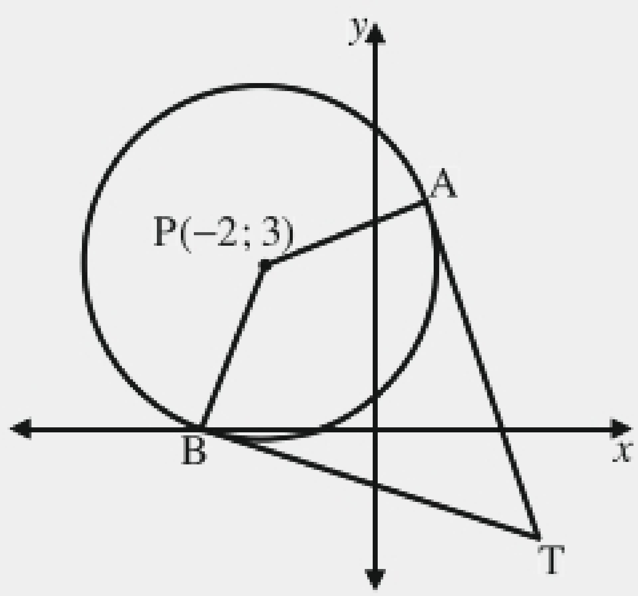
(1) $(p;-2)$ where $p>7$ (2) $(5;q)$ where $q>0$

(e) In the sketch alongside, the equation of the circle is $x^2-6x+y^2+ay=0$.
M is the centre of the circle passing through P, R and Q. R is a point on the y-axis and P has coordinates $(-1;1)$.
PQ is a diameter of the circle. AB, CD, ERF and HQG are all tangents to the circle. AB is parallel to the y-axis and CD is parallel to the x-axis.



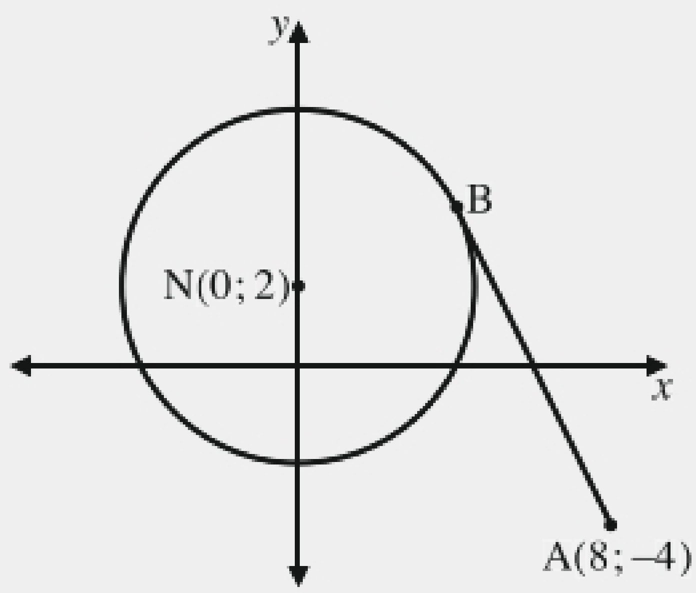
(1) Show that $a=-8$.
(2) Determine the coordinates of M.
(3) Determine the equation of
 (i) AB (ii) CD
(4) Determine the equation of
 (i) ERF in the form $ax+by+c=0$.
 (ii) HQG in the form $y=mx+c$.

(f) In the sketch alongside, the point $P(-2;3)$ is the centre of the circle. AT and BT are tangents to the circle at A and B respectively.
The equation of AT is $3x+y=7$.



(1) Determine the equation of AP.
(2) Determine the equation of the circle.
(3) If the equation of the circle is $(x+2)^2+(y-3)^2=10$, determine
 (i) the coordinates of B.
 (ii) the equation of BT.
 (iii) the coordinates of T.
 (iv) the length of BT.

(g) In the sketch alongside $N(0;2)$ is the centre of the circle. The radius of the circle is $\sqrt{20}$ units. AB is a tangent to the circle at B.
The coordinates of A are $(8;-4)$.

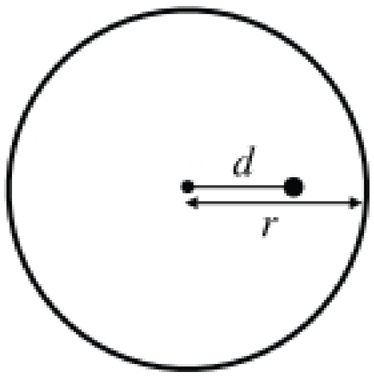


(1) Write down the equation of the circle.
(2) Determine the length of AB.
(3) If the coordinates of B are $(4;4)$, determine the equation of AB.

THE RELATIONSHIP BETWEEN A CIRCLE AND A POINT

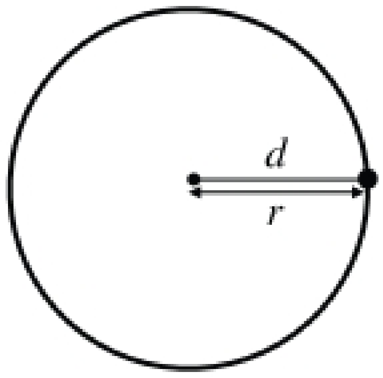
Let r be the radius of a circle and d the distance between the centre of the circle and a point.

If $d < r$, the point is **inside** the circle.



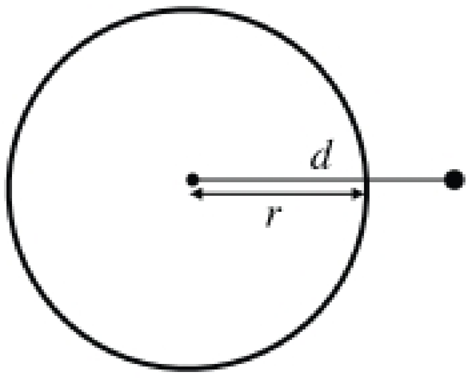
$\boxed{d < r}$

If $d = r$, the point is **on** the circumference of the circle.



$\boxed{d = r}$

If $d > r$, the point is **outside** the circle.



$\boxed{d > r}$

EXAMPLE 12

Determine whether each of the following points is inside, outside or on the circumference of the circle $(x+1)^2 + (y-2)^2 = 25$:

(a) $(-3;6)$

(b) $(3;-1)$

(c) $(4;0)$

Solution

(a) $r = 5$

Centre: $(-1;2)$

Point: $(-3;6)$

$d = \sqrt{(-3+1)^2 + (6-2)^2} = \sqrt{20} \approx 4{,}47$

$\therefore d < r$

$\therefore$ The point is **inside** the circle.

(b) Point: $(3;-1)$

$d = \sqrt{(3+1)^2 + (-1-2)^2} = 5$

$\therefore d = r$

$\therefore$ The point is **on the circumference** of the circle.

(c) Point: $(4;0)$

$d = \sqrt{(4+1)^2 + (0-2)^2} = \sqrt{29} \approx 5{,}39$

$\therefore d > r$

$\therefore$ The point is **outside** the circle.

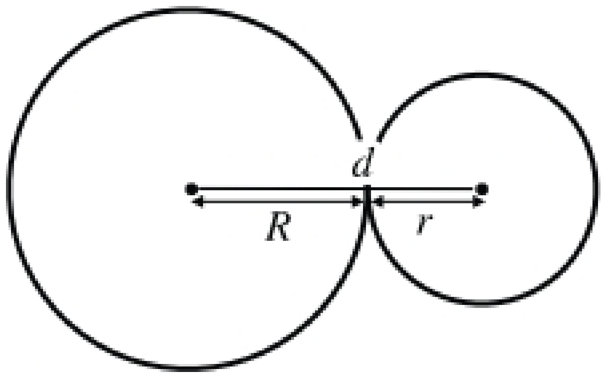
THE RELATIONSHIP BETWEEN TWO CIRCLES

We can determine the relationship between any two circles by comparing the distance between their centres to the sum and/or difference of their radii.

Let
R = the radius of the larger circle
r = the radius of the smaller circle
d = the distance between the centres of the two circles

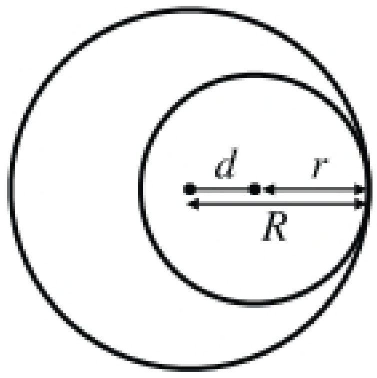
Circles that touch

If $d = R + r$, the circles **touch externally**.



$\boxed{d = R + r}$

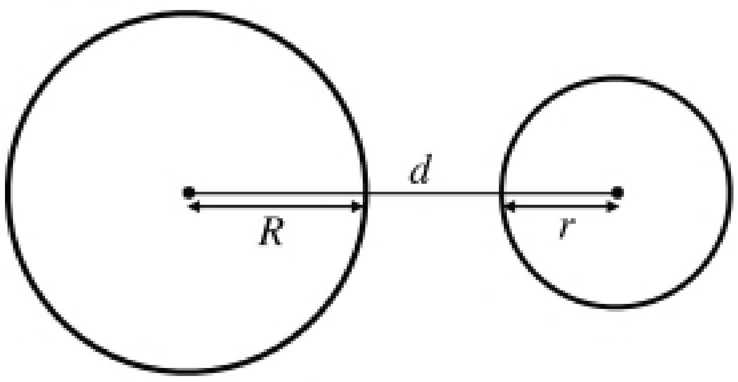
If $d = R - r$, the circles **touch internally**.



$\boxed{d = R - r}$

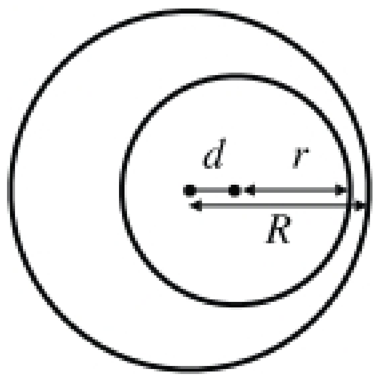
Circles that don't intersect

If $d > R + r$, the circles **don't intersect** and the smaller circle lies **outside** the larger circle.



$\boxed{d > R + r}$

If $d < R - r$, the circles **don't intersect** and the smaller circle lies **inside** the larger circle.

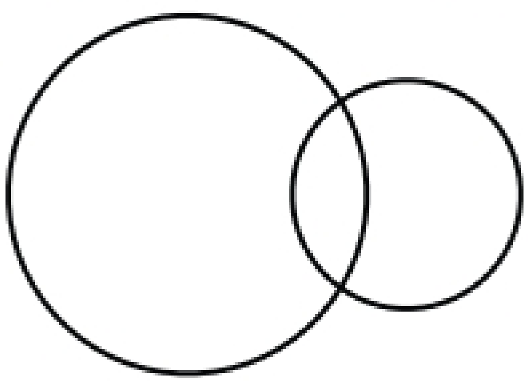


$\boxed{d < R - r}$

Circles that intersect in two points

Let R be the radius of the larger circle, r the radius of the smaller circle and d the distance between the centres of the circles:

If $R - r < d < R + r$, the two circles **intersect in two points**.



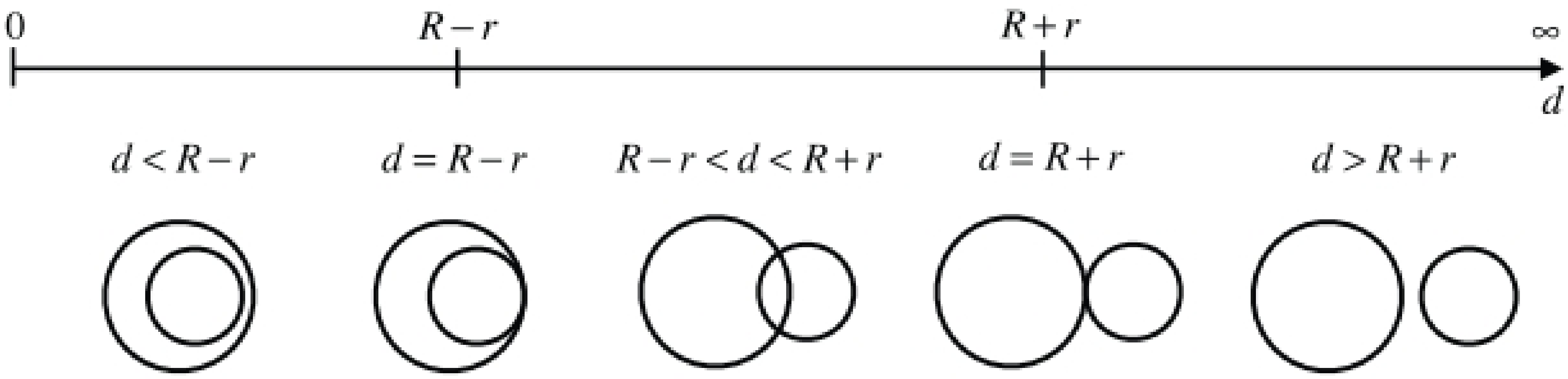
$\boxed{R - r < d < R + r}$

SUMMARY OF THE DIFFERENT RELATIONSHIPS BETWEEN CIRCLES

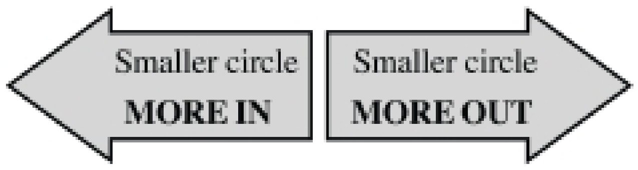
d = the distance between the centres

R = the radius of the larger circle $\qquad$ r = the radius of the smaller circle

Locate the value of d on the following number line to determine which relationship applies:



Don't intersect (Inside)	**Touch internally**	**Intersect in 2 points**	**Touch externally**	**Don't intersect (Outside)**



EXAMPLE 13

Describe the relationship between the two circles in each case:

(a) $(x+1)^2+(y+3)^2=9$ and $(x-2)^2+(y-1)^2=4$

(b) $(x+1)^2+(y-3)^2=9$ and $(x+3)^2+(y-2)^2=36$

(c) $x^2-10x+y^2-14y+49=0$ and $x^2+y^2=9$

Solution

(a) $R=3$ and $r=2$

$R+r=5$

$R-r=1$

Centres: $(-1;-3)$ and $(2;1)$

$d=\sqrt{(-1-2)^2+(-3-1)^2}=5$

$d=R+r$

$\therefore$ The circles **touch externally**.

(b) $R=6$ and $r=3$

$R+r=9$

$R-r=3$

Centres: $(-1;3)$ and $(-3;2)$

$d=\sqrt{(-1+3)^2+(3-2)^2}=\sqrt{5}\approx 2{,}24$

$d<R-r$

$\therefore$ The circles **don't intersect** and the smaller circle lies **inside** the larger circle.

(c) $x^2 - 10x + y^2 - 14y + 49 = 0$

$\therefore x^2 - 10x + 25 + y^2 - 14y + 49 = 25$

$\therefore (x-5)^2 + (y-7)^2 = 25$

and $x^2 + y^2 = 9$

$R = 5$ and $r = 3$

$R + r = 8$

$R - r = 2$

Centres: $(5;7)$ and $(0;0)$

$d = \sqrt{(5-0)^2 + (7-0)^2} = \sqrt{74} \approx 8{,}60$

$d > R + r$

$\therefore$ The circles **don't intersect** and the smaller circle lies **outside** the larger circle.

EXAMPLE 14

Show that the circles $(x-1)^2 + (y-2)^2 = 25$ and $(x+1)^2 + (y-1)^2 = 16$ intersect in two points without calculating the points of intersection.

Solution

$R = 5$ and $r = 4$

$R + r = 9$

$R - r = 1$

Centres: $(1;2)$ and $(-1;1)$

$d = \sqrt{(1+1)^2 + (2-1)^2} = \sqrt{5} \approx 2{,}24$

$R - r < d < R + r$

$\therefore$ The circles **intersect in two points**.

EXAMPLE 15

For which value of k will the circle $(x-1)^2 + (y+3)^2 = k$ touch the circle $(x+2)^2 + (y-1)^2 = 49$ internally if $k < 49$?

Solution

$R = 7$ and $r = \sqrt{k}$

Centres: $(1;-3)$ and $(-2;1)$

$d = \sqrt{(1+2)^2 + (-3-1)^2} = 5$

For the circles to touch internally $d = R - r$

$\therefore 5 = 7 - \sqrt{k}$

$\therefore \sqrt{k} = 2$

$\therefore k = 4$

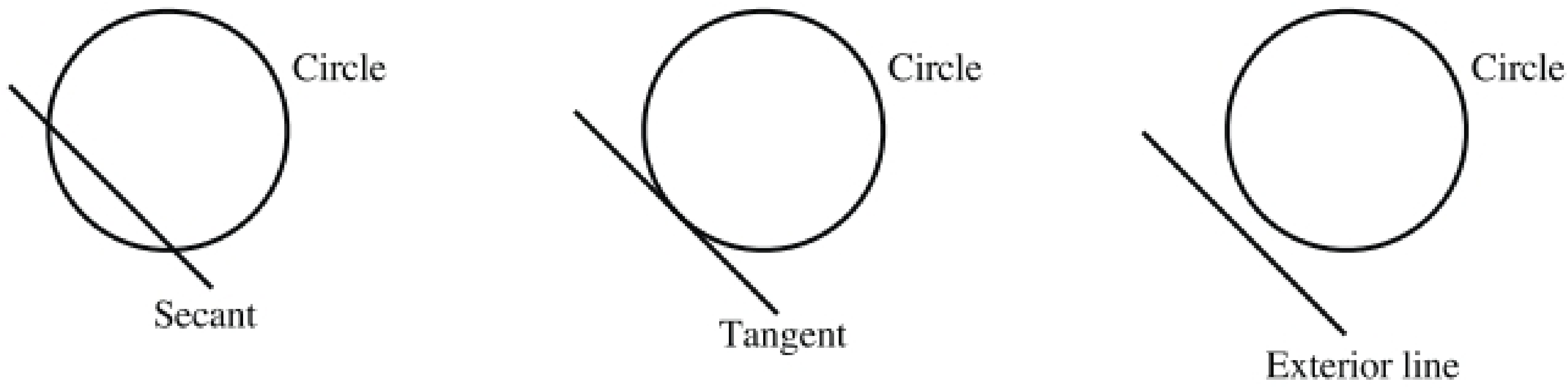
THE RELATIONSHIP BETWEEN A CIRCLE AND A LINE

A circle and a line can have **two**, **one** or **no** points of intersection:

If a circle and a line have **two points of intersection** the line is called a **secant** of the circle.

If a circle and a line have **one point of intersection** the line is a **tangent** of the circle.

If a circle and a line have **no points of intersection** the line is called an **exterior line** of the circle.



To find possible point(s) of intersection between a circle and a line, we **solve** the equation of the circle and the equation of the line **simultaneously**. This always leads to a **quadratic equation**. We know that a quadratic equation can have **two**, **one** or **no** real roots. The number of real roots indicates how many points of intersection the circle and the line have and this determines whether the line is a secant, a tangent or an exterior line of the circle:

EXAMPLE 16

Given a circle with equation $x^2 + y^2 = 25$ and a line with equation $y = x + 1$.

(a) Determine the point(s) of intersection of the circle and the line.

(b) Is the line a secant, a tangent or an exterior line of the circle?

Solution

(a) Solve simultaneously:

$x^2 + y^2 = 25$ and $y = x + 1$

$\therefore x^2 + (x+1)^2 = 25$

$\therefore x^2 + x^2 + 2x + 1 = 25$

$\therefore 2x^2 + 2x - 24 = 0$

$\therefore x^2 + x - 12 = 0$

$\therefore (x+4)(x-3) = 0$

$\therefore x = -4 \quad \text{or} \quad x = 3$

$\therefore y = -4 + 1 \qquad y = 3 + 1$

$\therefore y = -3 \qquad y = 4$

The points of intersection are $(-4; -3)$ and $(3; 4)$.

(b) The line $y = x + 1$ is **secant** of $x^2 + y^2 = 25$, since there are **two** points of intersection between the line and the circle.

EXAMPLE 17

Given a circle with equation $(x-1)^2+(y+4)^2=2$.

(a) Determine whether the line $y=2x-1$ is a secant, a tangent or an exterior line of the circle.

(b) Show that the line $y=x-7$ is a tangent to the circle.

Solution

(a) $(x-1)^2+(y+4)^2=2$ and $y=2x-1$

$\therefore (x-1)^2+(2x-1+4)^2=2$

$\therefore (x-1)^2+(2x+3)^2=2$

$\therefore x^2-2x+1+4x^2+12x+9=2$

$\therefore 5x^2+10x+8=0$

$$\therefore x=\frac{-10\pm\sqrt{10^2-4(5)(8)}}{2(5)}$$

$$=\frac{-10\pm\sqrt{-60}}{10}$$

The equation has **no** real solutions.

$\therefore$ The line is an **exterior line** of the circle.

(b) $(x-1)^2+(y+4)^2=2$ and $y=2x-1$

$\therefore (x-1)^2+(x-7+4)^2=2$

$\therefore (x-1)^2+(x-3)^2=2$

$\therefore x^2-2x+1+x^2-6x+9=2$

$\therefore 2x^2-8x+8=0$

$\therefore x^2-4x+4=0$

$\therefore (x-2)(x-2)=0$

$\therefore x=2$

The equation has only **one** real solution.

$\therefore$ The line is a **tangent** to the circle.

EXERCISE 4

(a) Determine whether each of the following points is inside, outside or on the circumference of the circle $(x-3)^2+(y+5)^2=25$:

(1) $(-2;-3)$ (2) $(7;-2)$

(3) $(5;-1)$ (4) $(3;0)$

(b) Determine whether the point $(-3;1)$ is inside, outside or on the circumference of each of the following circles:

(1) $(x+1)^2+(y-3)^2=9$ (2) $(x-1)^2+(y-4)^2=25$

(3) $x^2+y^2=8y$ (4) $x^2+y^2+4x-6y+5=0$

(c) Describe the relationship between the two circles in each case:

(1) $(x+3)^2+(y-1)^2=1$ and $(x-1)^2+(y-4)^2=16$

(2) $(x+2)^2+y^2=4$ and $(x-3)^2+(y+3)^2=9$

(3) $x^2+(y-2)^2=64$ and $(x-3)^2+(y+2)^2=9$

(4) $x^2+4x+y^2+6y=12$ and $x^2+2x+y^2+4y=-1$

(5) $x^2+y^2-6x+4y-12=0$ and $x^2+y^2+4x+2y-95=0$

(6) $x^2+y^2=20$ and $(x+3)^2+(y+6)^2=5$ (without the use of a calculator)

(7) $x^2-2x+y^2-4y=3$ and $x^2+y^2-2y=17$ (without the use of a calculator)

(d) Given a circle with equation $x^2+y^2=13$ and a line with equation $y=2x+1$.

(1) Determine the point(s) of intersection of the circle and the line.

(2) Is the line a secant, a tangent or an exterior line of the circle?

(e) Given a circle with equation $x^2+y^2-4x+6y=3$ and a line with equation $y=-7$.

(1) Determine the point(s) of intersection of the circle and the line.

(2) Is the line a secant, a tangent or an exterior line of the circle?

(f) Determine whether each of the following lines is a secant, tangent or exterior line of the circle $(x+2)^2+(y-3)^2=10$:

(1) $y=x-1$ (2) $y=5-x$

(3) $3x+y=7$ (4) $x=2$

(g) Show that the line $3x+2y=8$ is a tangent to the circle $x^2-2x+y^2+8y+4=0$.

(h) Given circle C, with equation $(x-14)^2+(y-10)^2=100$.

(1) Describe the relationship between circle C and

(i) the x-axis (ii) the y-axis (iii) the circle $x^2+y^2=36$

(2) Determine the value of a if the circle $x^2-4x+y^2-2y=a$ touches circle C externally.

(i) For which value of p will the circle $(x+5)^2+(y+2p)^2=225$ touch the circle $x^2+(y-p)^2=4$ internally?

(j) Given the line $y=k-x$ and the circle $x^2+y^2+4x-2y=13$.

For which value(s) of k will the line be a

(1) tangent to the circle? (2) secant of the circle?

(3) exterior line of the circle?

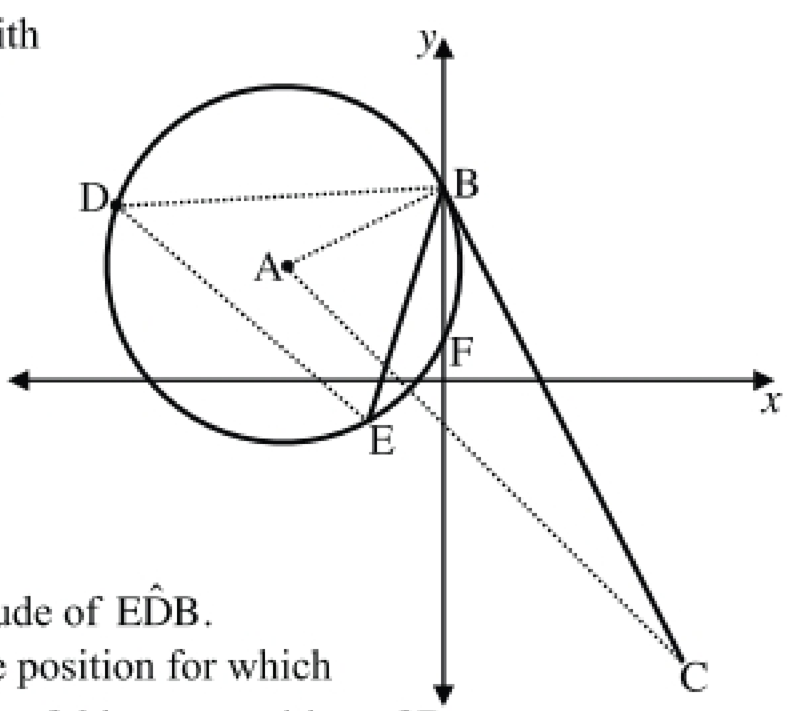
(k)* Determine the equations of the tangents to the circle $x^2+y^2=20$ with a gradient of -2.

INTEGRATION OF CONCEPTS (EXAMINATION QUESTIONS)

Examination questions usually require the use of concepts from both Grade 11 and Grade 12 Analytical Geometry, and often also concepts from Euclidean Geometry:

EXAMPLE 18

In the sketch alongside, A is the centre of the circle with equation $x^2+y^2+8x-6y+5=0$. The circle cuts the y-axis at B and F. BC is a tangent to the circle at B. The equation of BE is $y=3x+5$.



(a) Determine the coordinates of A.

(b) Determine the length of BF.

(c) Determine the equation of BC.

(d) Calculate the coordinates of E.

(e) Calculate the magnitude of $\hat{EBC}$.

(f) D is any point on major arc BE.

(1) Write down, with a reason, the magnitude of $\hat{EDB}$.

(2) D is moved, along major arc BE, to the position for which $\hat{DBE}=90°$. Determine the coordinates of this new position of D.

(g) If it is given that the coordinates of C are $(6;-7)$, determine

(1) the equation of the circle passing through A, B and C.

(2) the equation of the median of ΔABC, from vertex B.

(3) the area of ΔABC.

Solution

(a) $x^2+y^2+8x-6y=-5$

$\therefore x^2+8x+16+y^2-6y+9=-5+16+9$

$\therefore (x+4)^2+(y-3)^2=20 \quad \rightarrow \quad \text{A}(-4;3)$

(b) For y-intercepts, let $x=0$ in $x^2+y^2+8x-6y+5=0$:

$y^2-6y+5=0$

$\therefore (y-5)(y-1)$

$\therefore y=5$ or $y=1$

$y_{\text{B}}=5$ and $y_{\text{F}}=1$

$\therefore$ BF $=4$ units

(c) $m_{\text{AB}}=\dfrac{5-3}{0-(-4)}=\dfrac{1}{2}$

$\therefore m_{\text{EF}}=-2$ (tangent $\perp$ radius)

$\therefore y=-2x+5$ * y-intercept is 5

(d) Solve simultaneously:

$x^2+y^2+8x-6y+5=0$ and $y=3x+5$

$\therefore x^2+(3x+5)^2+8x-6(3x+5)+5=0$

$\therefore x^2+9x^2+30x+25+8x-18x-30+5=0$

$\therefore 10x^2+20x=0$

$\therefore x^2+2x=0$

$\therefore x(x+2)=0$

$\therefore x=0$ or $x=-2$

$\therefore y=3(0)+5$ $\quad y=3(-2)+5$

$\therefore y=5$ $\quad y=-1$

The points of intersection are $(0;5)$ and $(-2;-1)$.

$\therefore$ The coordintes of E are $(-2;-1)$.

(e) $\tan\theta_{\text{BE}}=m_{\text{BE}}$

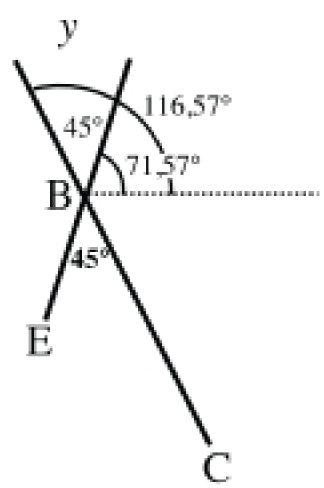
$\therefore \tan\theta_{\text{BE}}=3$

$\therefore \theta_{\text{BE}}=71{,}57°$

$\tan\theta_{\text{BC}}=m_{\text{BC}}$

$\therefore \tan\theta_{\text{BC}}=-2$

$\therefore \theta_{\text{BC}}=116{,}57°$



$\theta_{\text{BC}}-\theta_{\text{BE}}=116{,}57°-71{,}57°=45°$

$\therefore \text{E}\hat{\text{B}}\text{C}=116{,}57°-71{,}57°=45°$ (vert opp $\angle$s)

(f) (1) $\text{E}\hat{\text{D}}\text{B}=\text{E}\hat{\text{B}}\text{C}=45°$ (tan chord thm)

(2) If $\text{D}\hat{\text{B}}\text{E}=90°$ then DE is a diameter of the circle (chord subtends 90°)

$\therefore$ A will be the midpoint of DE.

$\therefore \dfrac{x_{\text{D}}+x_{\text{E}}}{2}=x_{\text{A}}$ and $\dfrac{y_{\text{D}}+y_{\text{E}}}{2}=y_{\text{A}}$

$\therefore \dfrac{x_{\text{D}}-2}{2}=-4$ and $\dfrac{y_{\text{D}}-1}{2}=3$

$\therefore x_{\text{D}}=-6$ and $y_{\text{D}}=7$ $\quad\therefore$ The coordinates of D are $(-6;7)$.

(g) (1) $A\hat{B}C = 90°$ (tangent $\perp$ radius)

$\therefore$ AC is the diameter of circle ABC (chord subtends $90°$)

$\therefore$ The centre of circle ABC is the midpoint of AC:

$$\left(\frac{x_A + x_C}{2}; \frac{y_A + y_C}{2}\right) = \left(\frac{-4+6}{2}; \frac{3-7}{2}\right) = (1; -2)$$

$\therefore$ Equation of the circle:

$(x-1)^2 + (y+2)^2 = r^2$

Substitute $A(-4; 3)$:

$(-4-1)^2 + (3+2)^2 = 50$

$\therefore$ The equation of circle ABC is $(x-1)^2 + (y+2)^2 = 50$.

(2) The median passes through $B(0;5)$ and the midpoint of AC, which is $(1; -2)$.

$$m = \frac{-2-5}{1-0} = -7$$

$\therefore y = -7x + 5$ * y-intercept is 5

(3) AB = radius of circle DBE

$\therefore AB = \sqrt{20} = 2\sqrt{5}$

$BC = \sqrt{(x_C - x_B)^2 + (y_C - y_B)^2}$

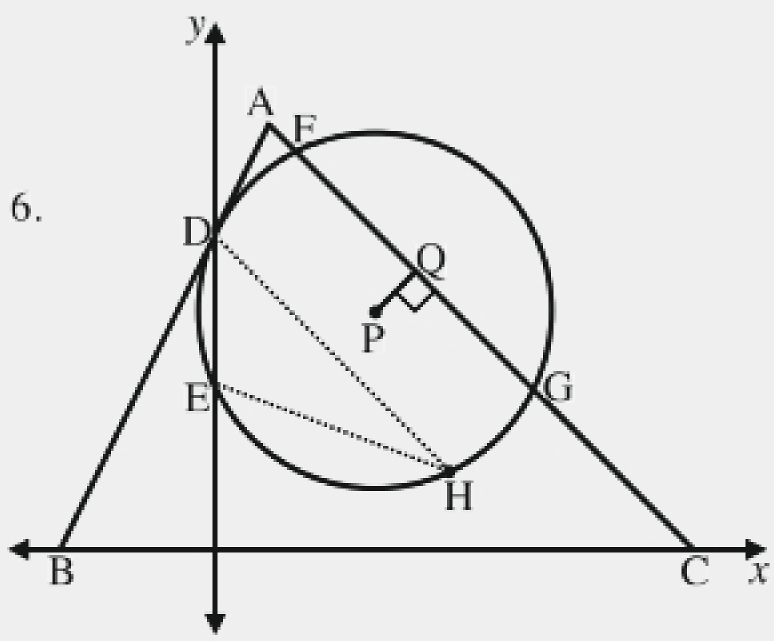
$\therefore BC = \sqrt{(0-6)^2 + (5+7)^2}$

$= \sqrt{180} = 6\sqrt{5}$

$$\text{Area of } \Delta ABC = \frac{1}{2} \cdot AB \cdot BC$$
$$= \frac{1}{2} \cdot 2\sqrt{5} \cdot 6\sqrt{5}$$
$$= 30 \text{ units}^2$$

EXERCISE 5

(a) In the sketch alongside, P is the centre of the circle with equation $(x-2)^2 + (y-3)^2 = 5$. The circle cuts the y-axis at D. AB is a tangent to the circle at D. The equation of AC is $x + y = 6$. AC cuts the circle at F and G. $PQ \perp FG$.



(1) Write down the coordinates of P.

(2) Calculate the coordinates of D and E.

(3) Determine the equation of AB.

(4) Calculate the magnitude of $B\hat{A}C$.

(5) Calculate the coordinates of F and G.

(6) Calculate the length of PQ in surd form.

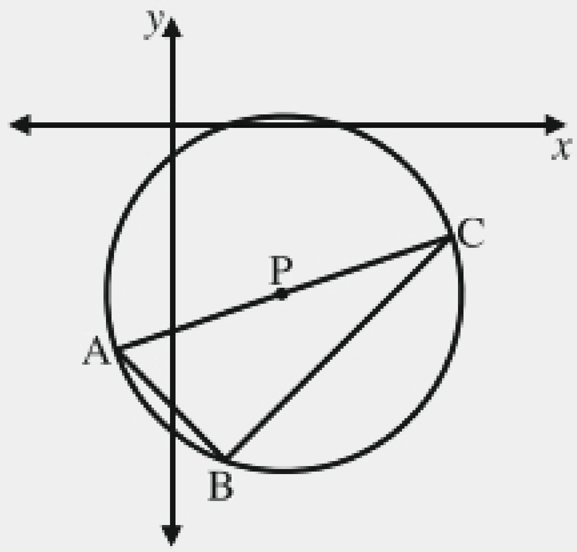
(7) If H is any point on major arc DE, calculate the magnitude of $D\hat{H}E$.

(8) If H has coordinates $(3; 1)$,

(i) show that $DH \parallel AC$.

(ii) calculate the area of ΔDEH.

(b) In the sketch alongside, A is the point $(-1; -4)$, B is the point $(1; k)$ and C is the point $(5; -2)$. P is the centre of the circle.



(1) Calculate the length of AC in surd form.

(2) Calculate the coordinates of P.

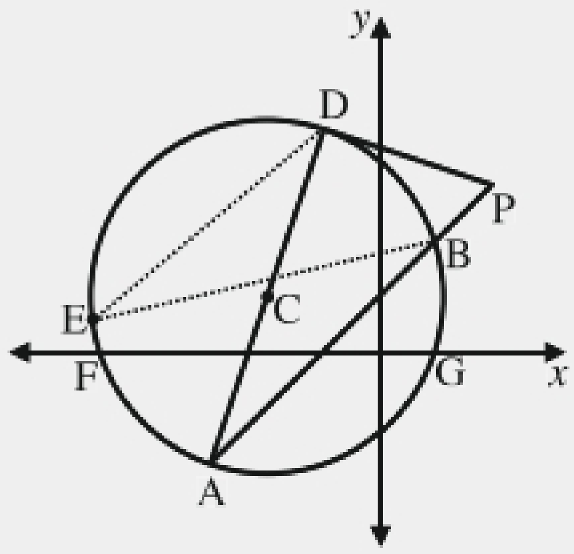
(3) Write down the equation of the circle.

(4) Show, with reasons, that $k = -6$.

(5) Determine the equation of
 (i) the median of ΔABC, from vertex B.
 (ii) the altitude of ΔABC, from vertex B.
 (iii) the perpendicular bisector of AC.

(6) Calculate the
 (i) perimeter of ΔABC. (Round to 2 decimals.)
 (ii) area of ΔABC.

(c) In the sketch alongside, the equation of the circle is $x^2 + 4x + y^2 - 2y = 5$. The equation of line AP is $y = x + 1$. C is the centre of the circle and AD is a diameter of the circle. DP is a tangent to the circle at D. The circle cuts the x-axis at F and G.



(1) Calculate the length of FG.

(2) Calculate the coordinates of A and B.

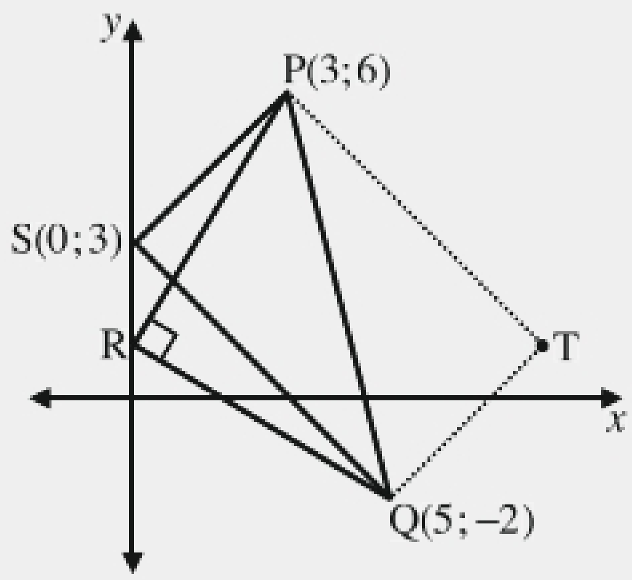
(3) Determine the coordinates of C.

(4) Determine the equation of DP in the form $ax + by + c = 0$.

(5) If E is any point on major arc DB, determine the magnitude of $D\hat{E}B$.

(6) Given that the coordinates of P are $(2; 3)$, determine the area of ΔADP.

(d) In the sketch alongside, the coordinates of P, Q and S are $(3;6)$, $(5;-2)$ and $(0; 3)$ respectively. R is a point on the y-axis, such that $PR \perp RQ$.



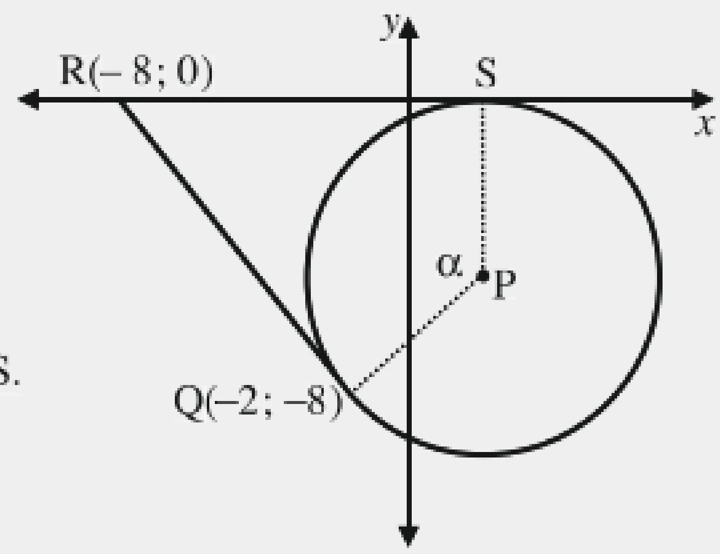
(1) T is a point such that PSQT is a parallelogram.
 (i) Determine the coordinates of T.
 (ii) Prove that PSQT is a rectangle.

(2) Determine the equation of the circle passing through P, Q and S.

(3) Does point R lie on circle PQS? Motivate.

(4) Calculate the length of SR.

(e) In the sketch alongside, P is the centre of the circle. The circle touches the x-axis at S. RQ is a tangent to the circle at Q. The coordinates of R and S are $(-8; 0)$ and $(-2; -8)$ respectively. $\hat{SPQ} = \alpha$.



(1) Calculate the length of RQ.
(2) Write down, with a reason, the length of RS.
(3) Determine the equation of the circle.
(4) Determine the size of α.
(5) Write down the equations of the two tangents to the circle that are parallel to the y-axis.

CONSOLIDATION AND EXTENSION EXERCISE

(a) Write down the radius and the coordinates of the centre of the circles with the following equations:

(1) $x^2 + y^2 = 25$ (2) $(x+3)^2 + (y-7)^2 = 9$

(3) $(x-5)^2 + y^2 = 16$ (4) $x^2 + (y+2)^2 = 32$

(b) Determine the equation of a circle with
(1) centre $(0; 0)$ and radius 7 units.
(2) centre $(5; -2)$ and radius 2 units.
(3) centre $(0; -1)$ and radius $\sqrt{5}$ units.
(4) centre $(0; 0)$ and passing through the point $(-1; 3)$.
(5) centre $(2; -3)$ and passing through the point $(-2; -5)$.
(6) centre $(5; 0)$ and passing through the point $(\sqrt{2} + 5; \sqrt{5})$.
(7) $(-3; 5)$ and $(-7; -1)$ the endpoints of one of its diameters.

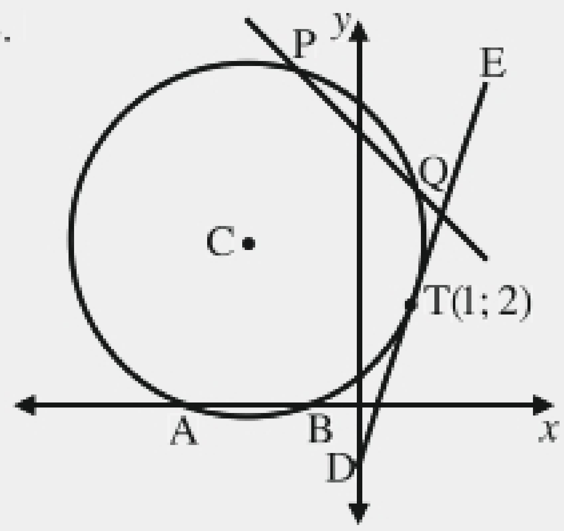
(c) Determine the radius and the coordinates of the centre of the circles with the following equations:

(1) $x^2 + y^2 - 12x + 2y + 33 = 0$ (2) $4x^2 + 4x + 4y^2 = 8y - 1$

(d) The equation of a circle is $(x+1)^2 + (y-2)^2 - 2(x-y) = t$.
(1) Determine the coordinates of the centre of the circle.
(2) If the radius of the circle is $\sqrt{5}$ units, determine the value of t.

(e) Determine the equation of the tangent to the circle $(x+3)^2 + (y-4)^2 = 17$ at point $(1; 5)$.

(f) The sketch alongside shows the circle with centre C and equation $x^2 + 4x + y^2 - 6y = -3$. The circle cuts the x-axis at A and B. A straight line, $x + y = 5$, intersects the circle in points P and Q. DE is a tangent to the circle at $T(1; 2)$.



(1) Determine the coordinates of
 (i) A and B (ii) P and Q
(2) Determine the equation of DE.
(3) Show that C lies on AP.
(4) Write down, with a reason, the size of $\hat{AQP}$.
(5) Is the point $(-5; 1)$ inside, outside or on the circumference of the circle?

(g) In the sketch alongside, $N(5;n)$ is the centre of the circle. $M(m;4)$ is the midpoint of chord AB, with endpoints $A(-1;a)$ and $B(5;1)$.

(1) Determine the values of a and m.

(2) Show that $n=7$.

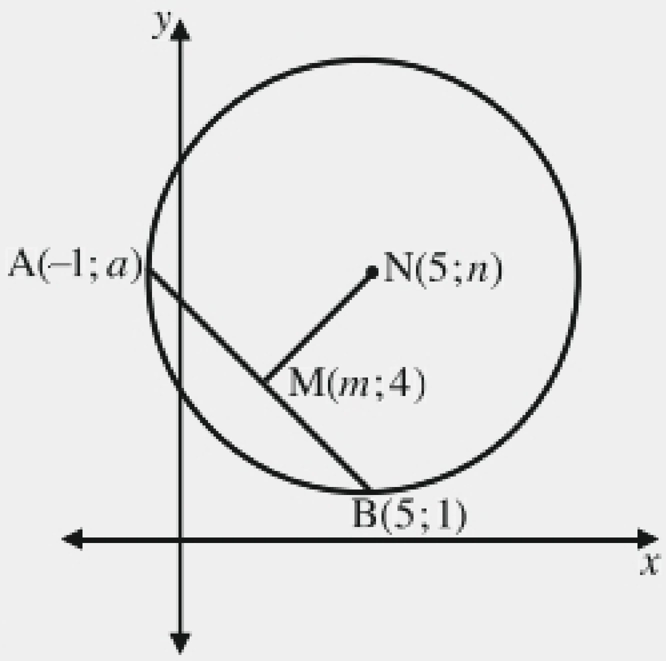
(3) Determine the equation of the circle.

(4) Write down the equation of the tangent to the circle through

(i) A (ii) B

(5) Determine the coordinates of the point of intersection of the tangents through A and B.

(6) Show that the point of intersection referred to in (5) lies on the line through N and M.



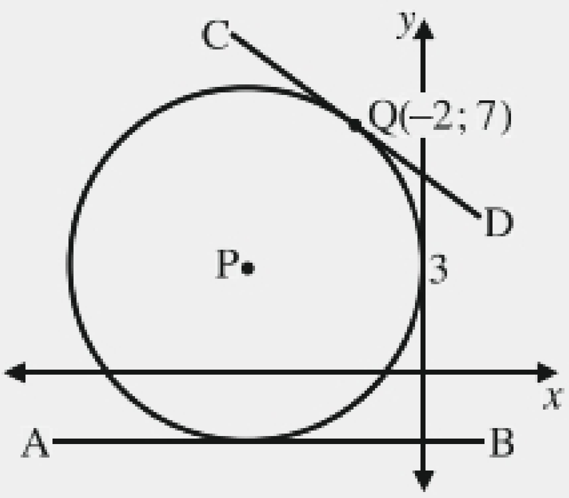
(h) In the sketch alongside, P is the centre of the circle that touches the y-axis at $(0;3)$.
AB is a tangent to the circle, parallel to the x-axis.
CD is a tangent to the circle at the point $Q(-2;7)$.

(1) Determine the equation of the circle.

(2) Determine the equation of

(i) AB

(ii) CD, in the form $ax+by+c=0$.

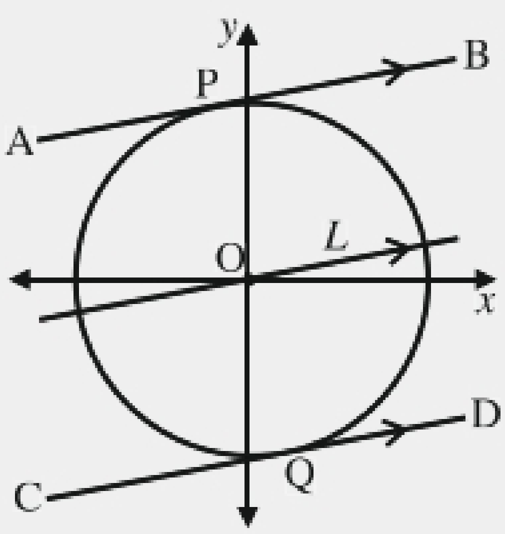


(i) The sketch alongside shows the circle defined by $x^2+y^2=26$ and two tangents to the circle, parallel to the line $L: x-5y=0$. These two tangents, AB and CD, touch the circle at P and Q respectively. O is the origin.

(1) Explain why P, O and Q are collinear.

(2) Determine the coordinates of P and Q.

(3) Determine the equations of tangents AB and CD, in the form $ax+by+c=0$.



(j) The sketch alongside shows ΔPQR, with **median** PS and **altitude** RT. The coordinates of P, Q and R are $(-2;1)$, $(-4;-5)$ and $(2;-3)$ respectively.

(1) Calculate the length of QR.

(2) Determine the equation of

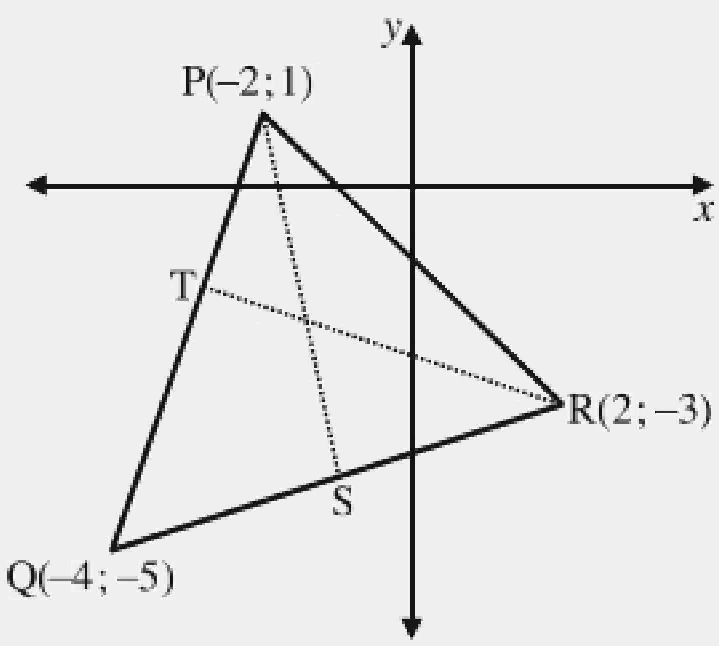
(i) PS

(ii) RT, in the form $ax+by+c=0$.

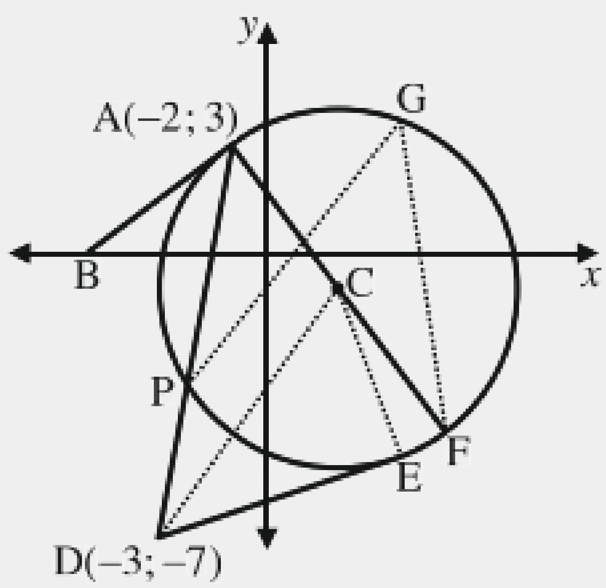
(3) Calculate the size of $P\hat{Q}R$.

(4) Determine the equation of the circle passing through the points Q, T and R.

(5) If PRAQ is a parallelogram, calculate the coordinates of A.

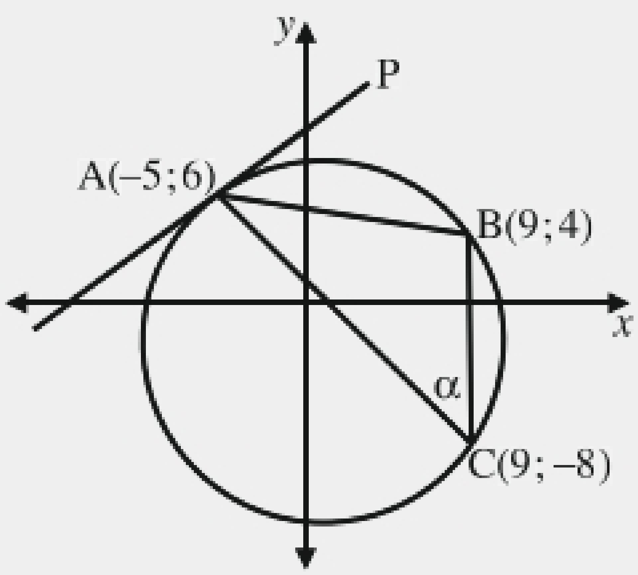


(k) In the sketch alongside, C is the centre of the circle with equation $x^2 + y^2 - 4x + 2y = 20$. AB is a tangent to the circle at $A(-2; 3)$ and DE is a tangent to the circle at E. The coordinates of D are $(-3; -7)$.
P and F are points on the circumference of the circle, such that APD and ACF are straight lines.



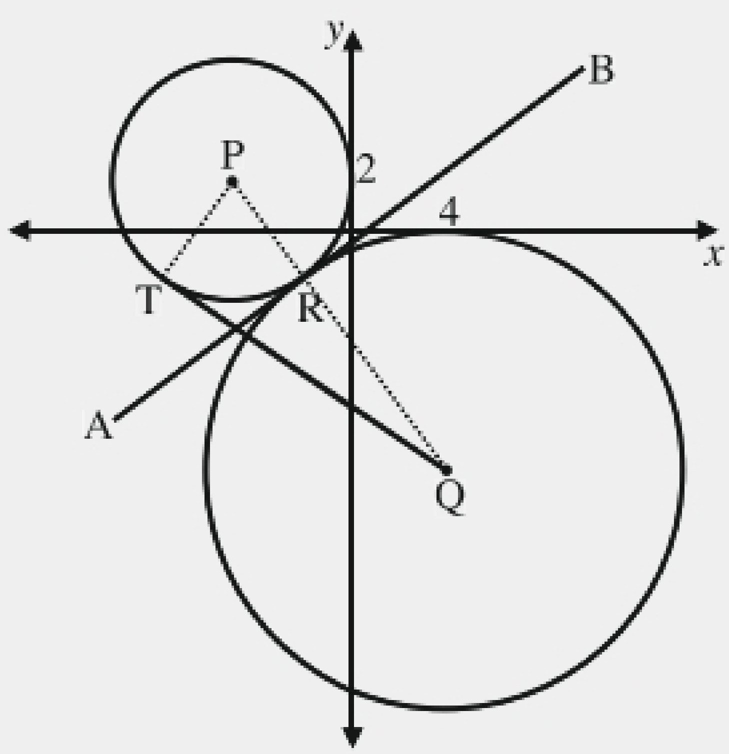
(1) Write the equation of the circle in the form $(x-a)^2 + (y-b)^2 = r^2$.
(2) Determine the equation of AB.
(3) Determine the coordinates of F.
(4) If G is any point on major arc PF, determine the magnitude of $P\hat{G}F$.
(5) Calculate the length of tangent DE.
(6) Calculate the area of ΔCDE.

(l)* The sketch alongside shows a circle passing through the three points $A(-5;6)$, $B(9;4)$ and $C(9;-8)$.
AP is a tangent to the circle at A. $A\hat{C}B = \alpha$.



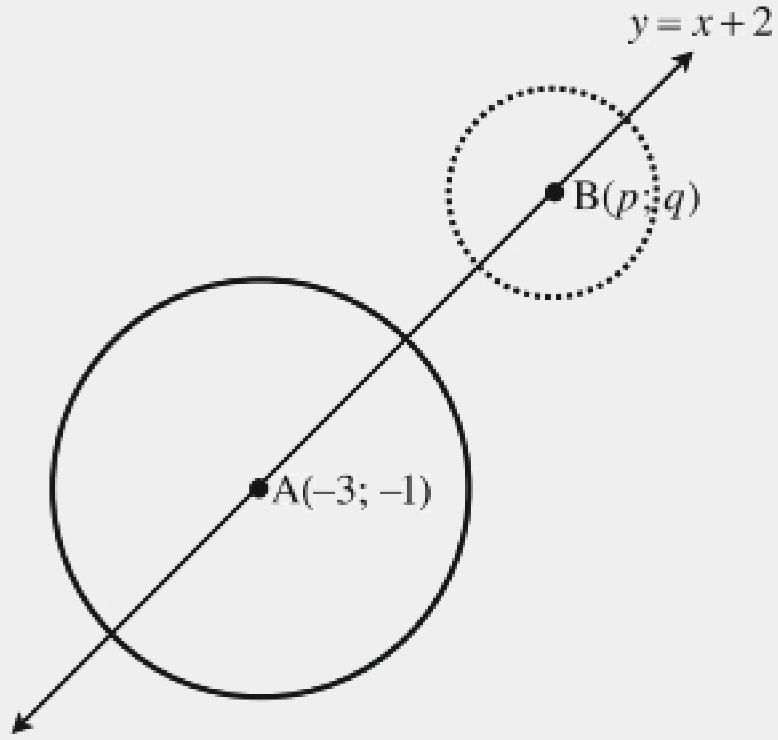
(1) Calculate α.
(2) Write down, with a reason, the size of $P\hat{A}B$.
(3) Determine the coordinates of the midpoint of AB.
(4) Let the coordinates of the centre be $(a;b)$.
  (i) Show that $b = 7a - 9$.
  (ii) Determine the values of a and b.
(5) Determine the equation of
  (i) the circle. (ii) AP

(m)* In the sketch alongside, the circle with centre P has equation $(x-a)^2 + (y-b)^2 = 25$ and touches the y-axis at $(2;0)$. The circle with centre Q touches the x-axis at $(4;0)$. The two circles touch each other at R. AB is a common tangent of the two circles. TQ is a tangent to the circle, with centre P, at T.
The length of PQ is 15 units. Determine



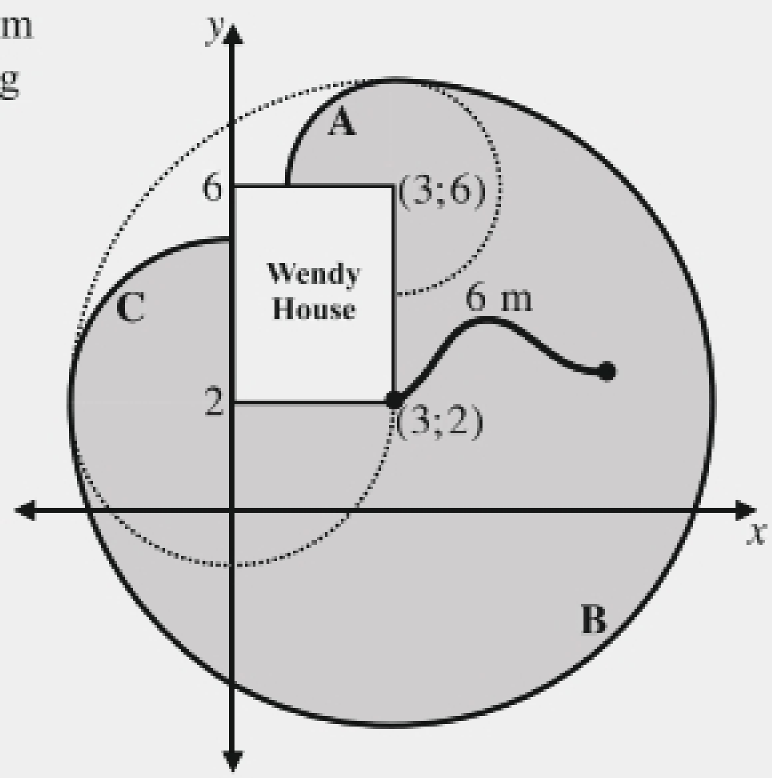
(1) the values of a and b.
(2) the equation of the circle with centre Q.
(3) the length of TQ in surd form.
(4) the area of ΔPTQ.
(5) the equation of PQ.
(6) the coordinates of R.
(7) the equation of AB.

(n)* The sketch alongside shows two circles:
Circle A has equation $(x+3)^2+(y+1)^2=32$ and remains fixed in this position.
Circle B has equation $(x-p)^2+(y-q)^2=8$ and can move to any position such that its centre remains on the line $y=x+2$.



(1) Describe the relationship between the two circles when
(i) $p=4$ (ii) $q=3$
(2) For which value(s) of p will the circles touch externally?
(3) For which value(s) of q will the circles touch internally?
(4) For which value(s) of k will the line $y=-x+k$ be
(i) a tangent to circle A?
(ii) a secant of circle A?
(iii) an exterior line of circle A?

(o)* Thabiso owns two big dogs. He has to leave them alone for the day, but cannot leave them roaming free, because they might fight. He temporarily ties one of the dogs, with a 6-m-long chain, to a corner of a wendy house at $(3;2)$. The four corners of the wendy house are at $(0;2)$, $(0;6)$, $(3;6)$ and $(3;2)$.
This dog is now restricted to the shaded region in the sketch alongside.
The curve by which this region is bounded consists of three sections:
A - a quarter of a circle,
B - three quarters of a circle and
C - a quarter of a circle.
(The units on the axes are metres.)



(1) Write down the equations of the three circles of which sections A, B and C form part.
(2) If the dog's bowl is placed at the point $(-2;-1)$, will the dog be able to access its bowl?
(3) The second dog is tied to a tree (with a chain) so that its movement is restricted to the region bounded by the circle $x^2+y^2-18x-20y+172=0$.
(i) At what point is the tree located?
(ii) Will the dogs be able to get together? Motivate by calculation.

(p)* There are 64 squares (8×8) on a chess board. Suppose a circle is drawn on the chess board, touching all four sides of the chess board (inscribed circle). How many of the 64 squares of the chess board are fully contained inside this circle? (Parts of squares don't count!)

Try to solve this problem by using Analytical Geometry.
Hint: Use the circle $x^2+y^2=16$.

CHAPTER EIGHT

Euclidean Geometry

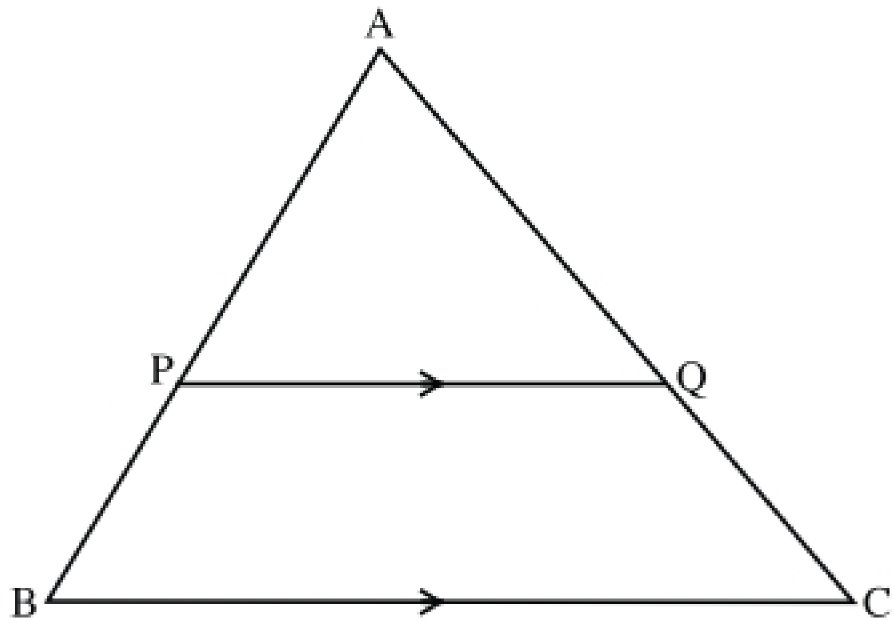
In this chapter, we will study the concepts of *ratio*, *proportionality* and *similarity*. We will focus on triangles. You will learn three new theorems. **At the end of this chapter you will find a summary of principles/theorems from earlier grades.**

THE TRIANGLE PROPORTIONALITY THEOREM

A *proportion* is a ratio that remains constant. When we say that two line segments are divided *proportionally* (or in proportion) it means that they are divided in the same ratio.

THEOREM 1

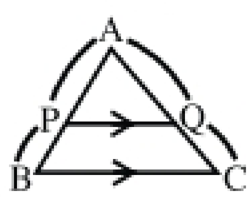
A line drawn parallel to one side of a triangle divides the other two sides proportionally.



Given: ΔABC with $PQ \parallel BC$

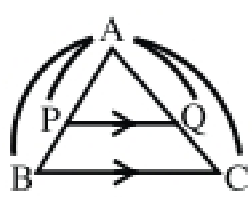
Conclusion: $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$ **Reason:** line $\parallel$ side of Δ

Note: This proportionality can be expressed in six different ways:



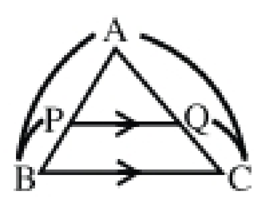
$$\frac{AP}{PB} = \frac{AQ}{QC}$$

$$\frac{PB}{AP} = \frac{QC}{AQ}$$



$$\frac{AP}{AB} = \frac{AQ}{AC}$$

$$\frac{AB}{AP} = \frac{AC}{AQ}$$



$$\frac{PB}{AB} = \frac{QC}{AC}$$

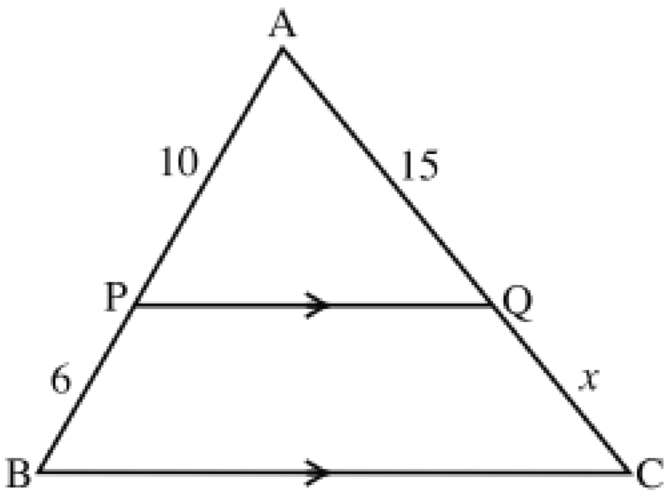
$$\frac{AB}{PB} = \frac{AC}{QC}$$

COROLLARY (Midpoint theorem):

A line drawn through the midpoint of one side of a triangle, parallel to another side, bisects the third side. $\left(\dfrac{AP}{PB} = \dfrac{AQ}{QC} = \dfrac{1}{1}\right)$

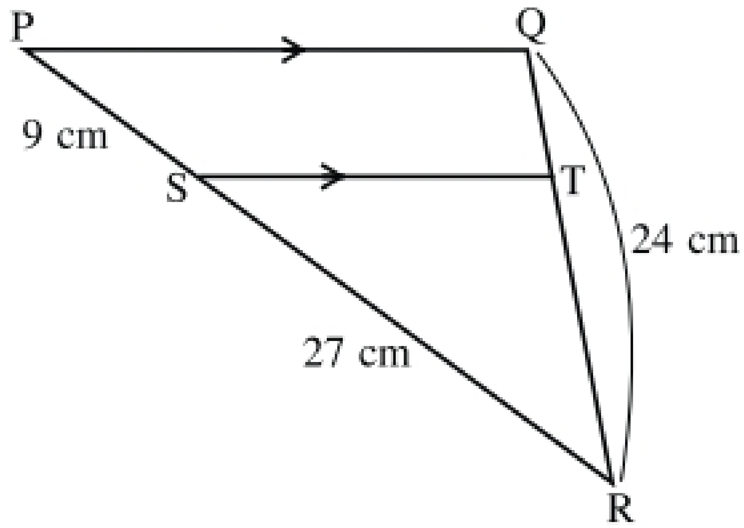
EXAMPLE 1

(a) In the following sketch, $PQ \parallel BC$.
$AP = 10$ units, $PB = 6$ units,
$AQ = 15$ units and $QC = x$.



Determine the value of x.

(b) In the following sketch, $ST \parallel PQ$.
$PS = 9$ cm, $SR = 27$ cm
and $QR = 24$ cm.



Determine the length of TR.

Solution

(a) $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$ (line $\parallel$ side of Δ)

$\therefore \dfrac{10}{6} = \dfrac{15}{x}$

$\therefore 10x = 90$

$\therefore x = 9$

(b) $\dfrac{TR}{QR} = \dfrac{SR}{PR}$ (line $\parallel$ side of Δ)

$\therefore \dfrac{TR}{24} = \dfrac{27}{36}$

$\therefore 36TR = 648$

$\therefore TR = 18$ cm

RATIOS AND FRACTIONS

Remember that a ratio can be expressed as a fraction and *vice versa*:

$$\boxed{AB : BC = m : n \quad \leftrightarrow \quad \frac{AB}{BC} = \frac{m}{n}}$$

For example, if $AB : BC = 2 : 3$, then $\dfrac{AB}{BC} = \dfrac{2}{3}$.

PROBLEMS INVOLVING RATIOS AND LENGTHS

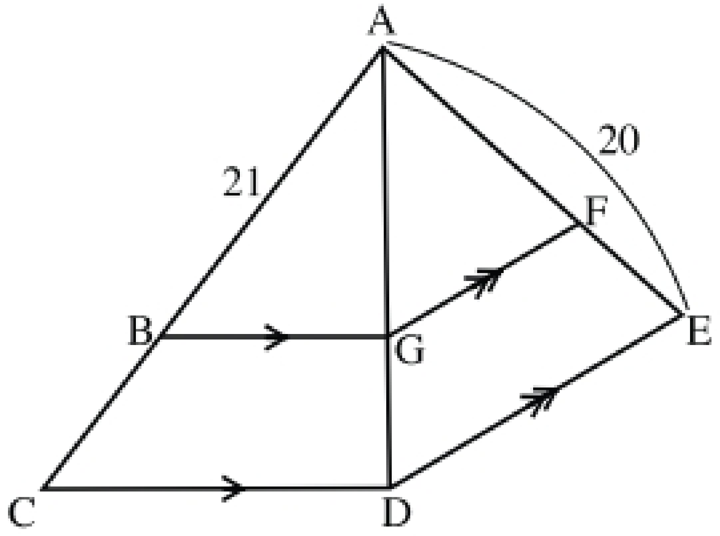
If two line segments are in a ratio $m : n$, we can represent their lengths by mx and nx respectively.

$$\boxed{\text{If } AB : BC = m : n, \text{ let } AB = mx \text{ and } BC = nx.}$$

For example, if $AB : BC = 2 : 3$, we let $AB = 2x$ and $BC = 3x$.

This technique is very useful in solving problems where a combination of ratios and lengths are given.

EXAMPLE 2



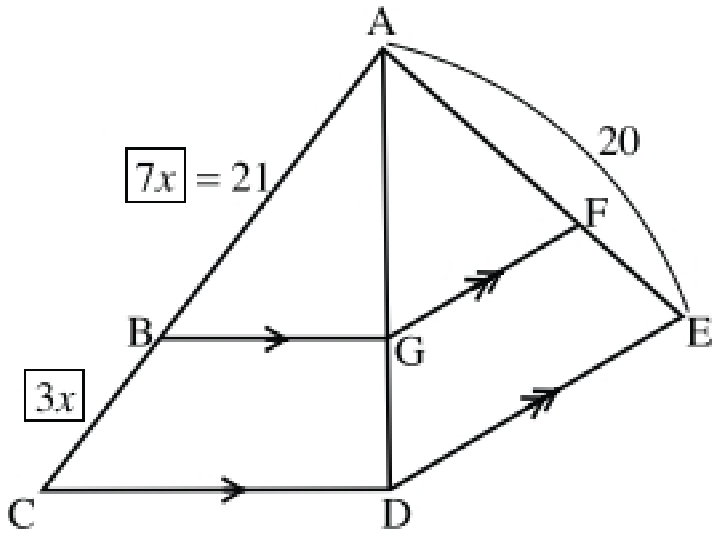
In the sketch alongside, $AB : BC = 7 : 3$.
$AB = 21$ units and $AE = 20$ units.

Determine

(a) the length of BC. (b) the value of $\frac{AG}{GD}$.

(c) the length of FE.

Solution



(a) $AB : BC = 7 : 3$
Let $AB = 7x$ and $BC = 3x$
But $AB = 21$
$\therefore 7x = 21$
$\therefore x = 3$
$BC = 3x = 3(3) = 9$ units

(b) $AB : BC = 7 : 3 \rightarrow \frac{AB}{BC} = \frac{7}{3}$

$\frac{AB}{BC} = \frac{AG}{GD}$ (line $\parallel$ side of Δ)

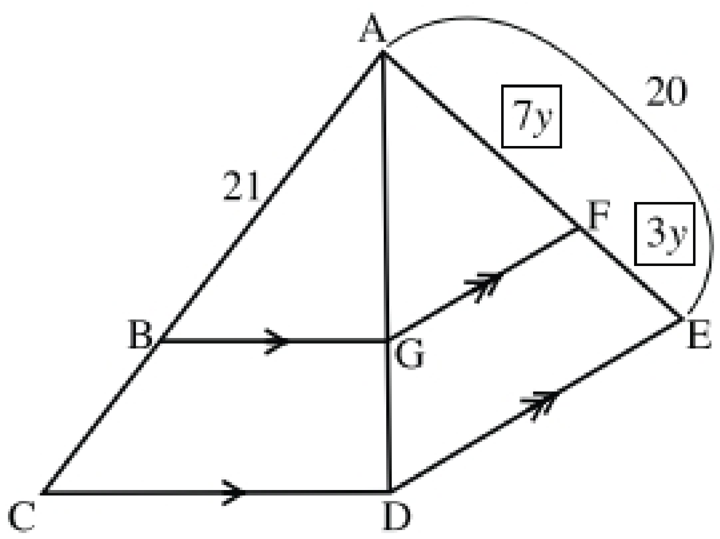
$\therefore \frac{AG}{GD} = \frac{7}{3}$

(c) $\frac{AG}{GD} = \frac{AF}{FE}$ (line $\parallel$ side of Δ)

$\therefore \frac{AF}{FE} = \frac{7}{3} \rightarrow AF : FE = 7 : 3$

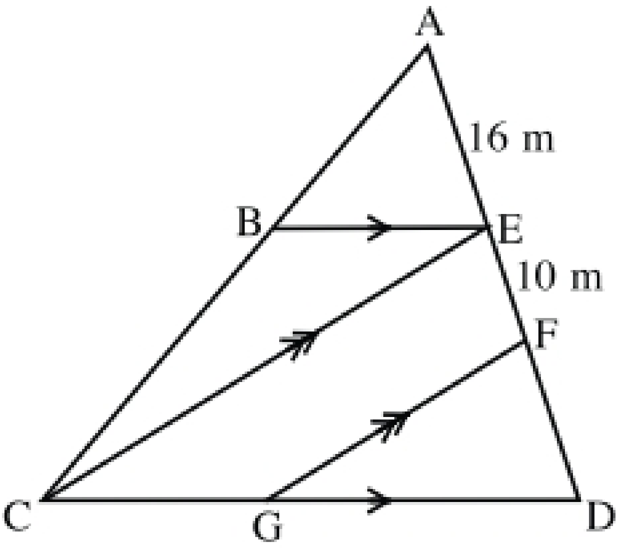
Let $AF = 7y$ and $FE = 3y$

$AE = 7y + 3y = 10y$
But $AE = 20$
$\therefore 10y = 20$
$\therefore y = 2$
$FE = 3(2) = 6$ units



EXAMPLE 3

In the sketch alongside, $BE \parallel CD$ and $CE \parallel GF$.
$AE = 16$ m and $EF = 10$ m.
$BC = \frac{3}{5}AC$.



Determine
(a) the length of ED.
(b) CG : GD

Solution

(a) $\frac{BC}{AC} = \frac{3}{5}$

$\frac{BC}{AC} = \frac{ED}{AD}$ (line $\parallel$ side of Δ)

$\therefore \frac{ED}{AD} = \frac{3}{5} \rightarrow ED : AD = 3:5$

Let $ED = 3x$ and $AD = 5x$

$\therefore AE = 5x - 3x = 2x$

But $AE = 16$

$\therefore 2x = 16$

$\therefore x = 8$

$\therefore ED = 3x = 3(8) = 24$ m

(b) $\frac{CG}{GD} = \frac{EF}{FD}$ (line $\parallel$ side of Δ)

But $EF = 10$ m and $FD = 24 - 10 = 14$ m

$\therefore \frac{EF}{FD} = \frac{10}{14} = \frac{5}{7}$

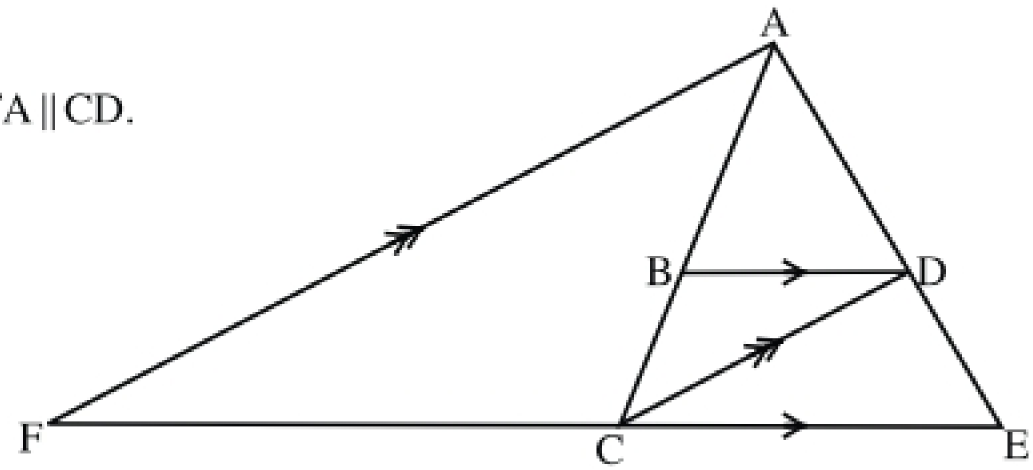
$\therefore \frac{CG}{GD} = \frac{5}{7} \rightarrow CG : GD = 5:7$

PROVING THAT RATIOS ARE EQUAL

EXAMPLE 4

In the sketch alongside, $BD \parallel CE$ and $FA \parallel CD$.

Prove that $\frac{AB}{BC} = \frac{FC}{CE}$.



Solution

In ΔABC: $\frac{AB}{BC} = \frac{AD}{DE}$ (line $\parallel$ side of Δ)

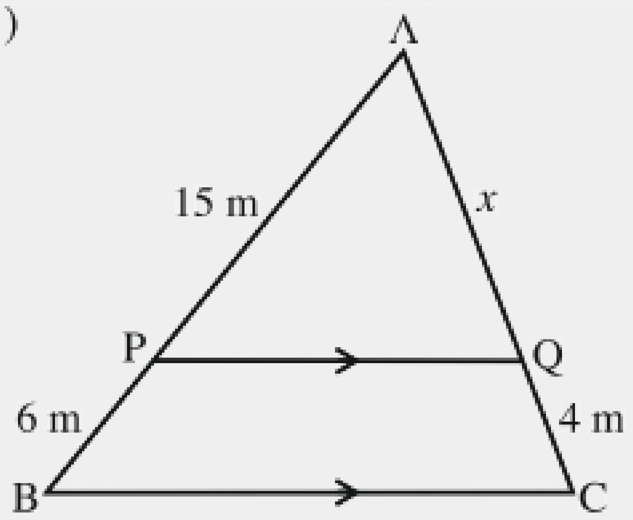
In ΔEAF: $\frac{AD}{DE} = \frac{FC}{CE}$ (line $\parallel$ side of Δ)

$\therefore \frac{AB}{BC} = \frac{FC}{CE}$

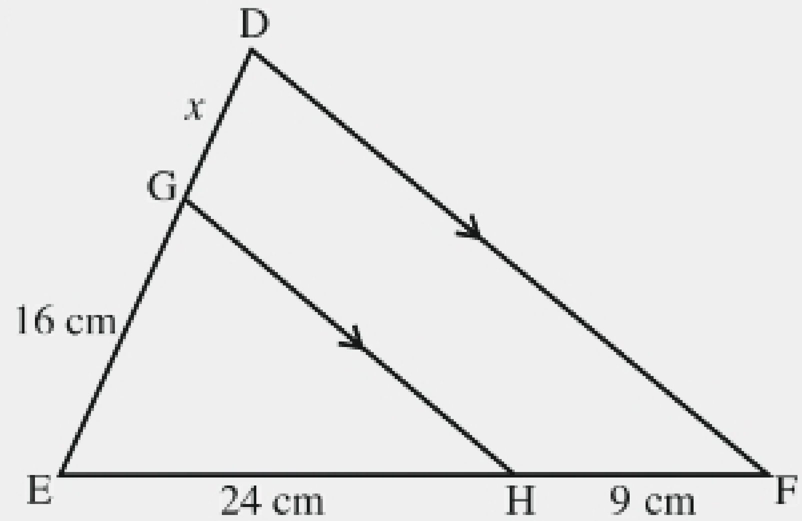
EXERCISE 1

(a) Determine the value of x in each of the following sketches:

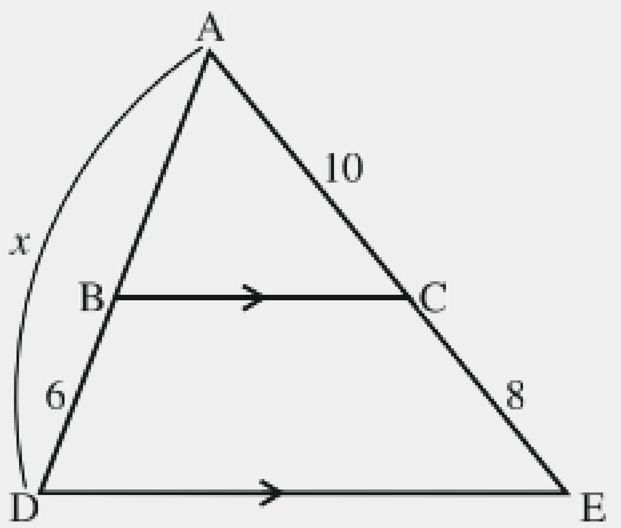
(1)



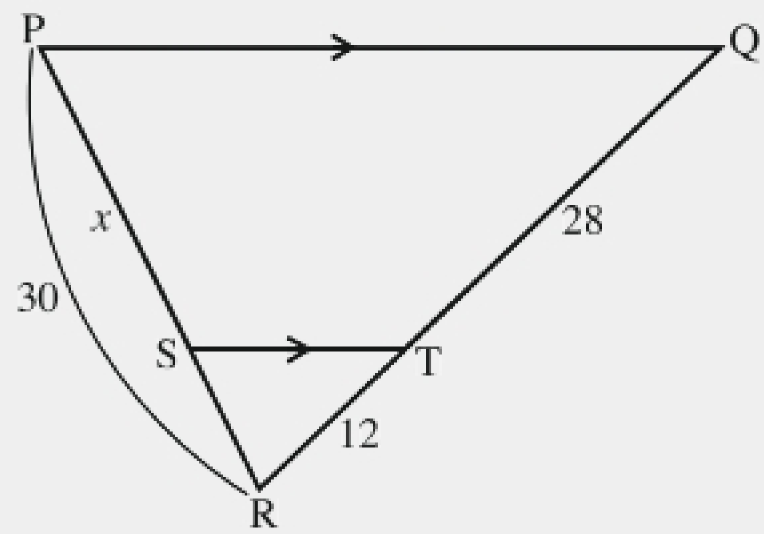
(2)



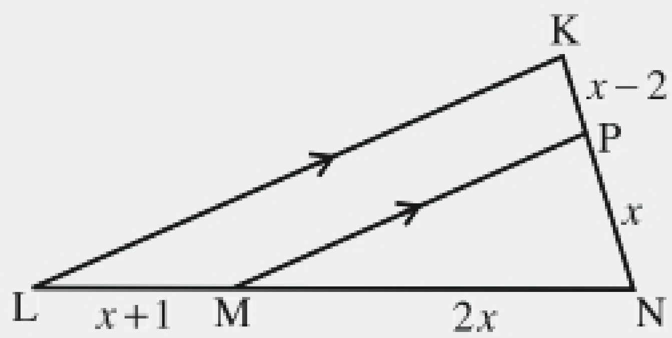
(3)



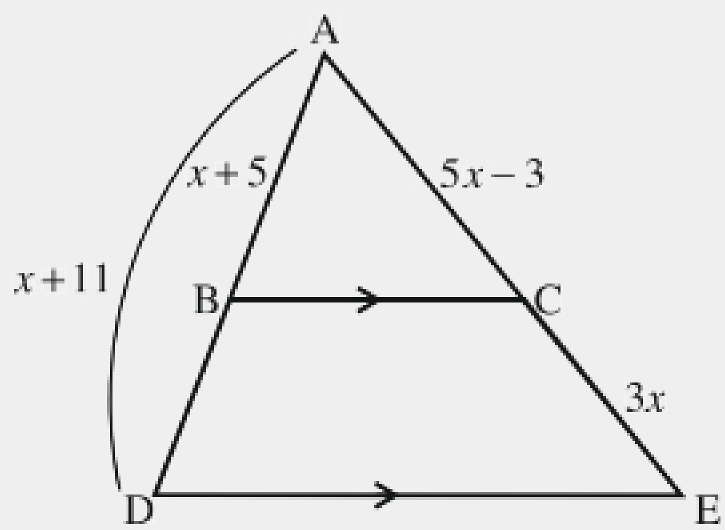
(4)



(5)



(6)



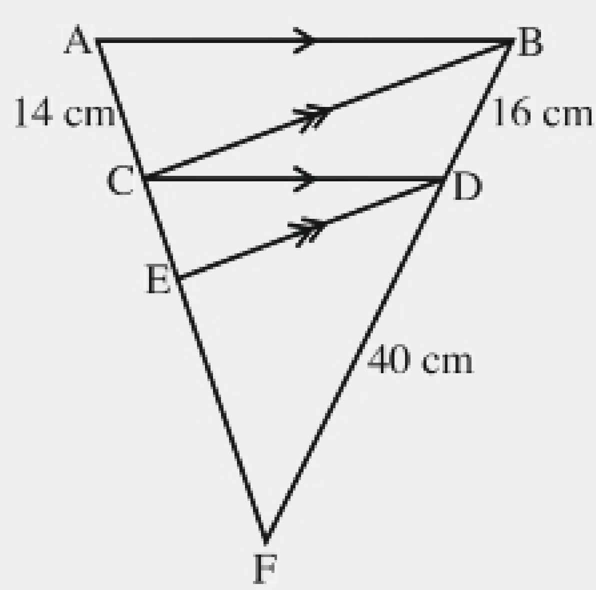
(b) In the sketch alongside, $AB \parallel CD$ and $CB \parallel ED$. $AC = 14$ cm, $BD = 16$ cm and $DF = 40$ cm.

Determine the length of

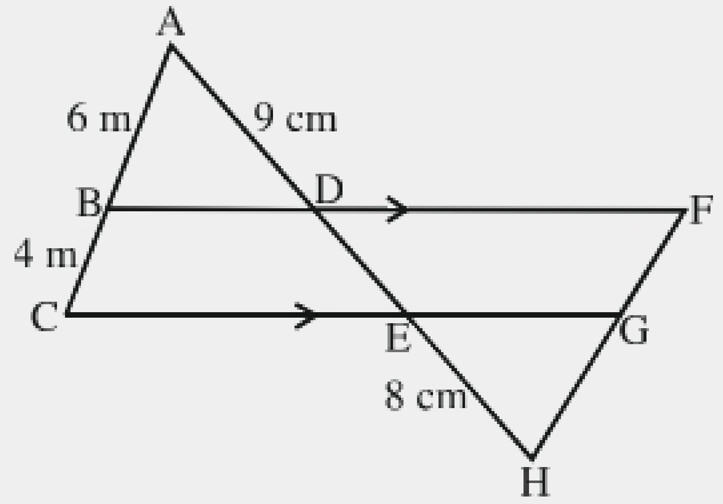
(1) CF

(2) CE

(3) EF



(c) In the sketch alongside, $BDF \parallel CEG$.
$AB = 6$ m, $BC = 4$ m, $AD = 9$ m and $EH = 8$ m.

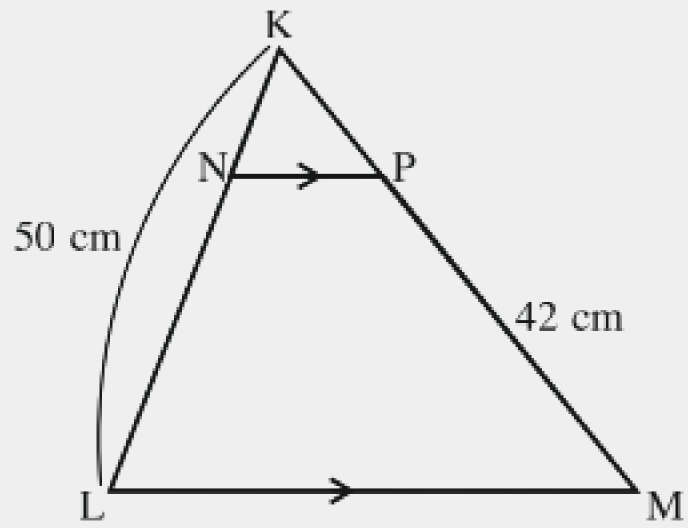


Determine the following ratios:

(1) AD : DE

(2) FG : FH

(d) In the sketch alongside, $KN : NL = 3 : 7$.
$NP \parallel LM$, $KL = 50$ cm and $PM = 42$ cm.

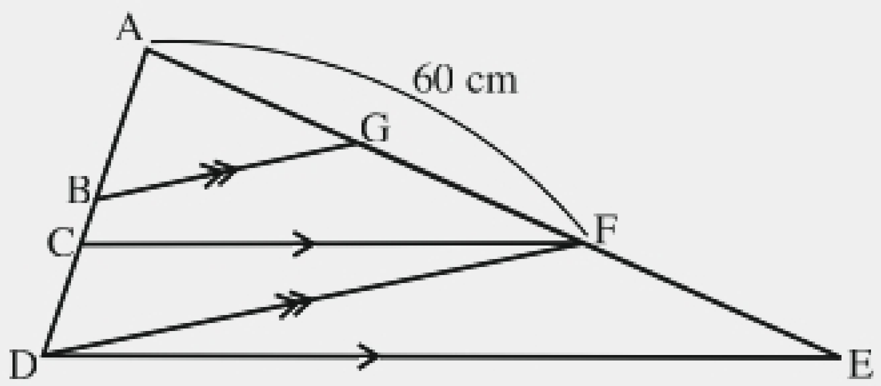


Determine the length of

(1) NL

(2) KP

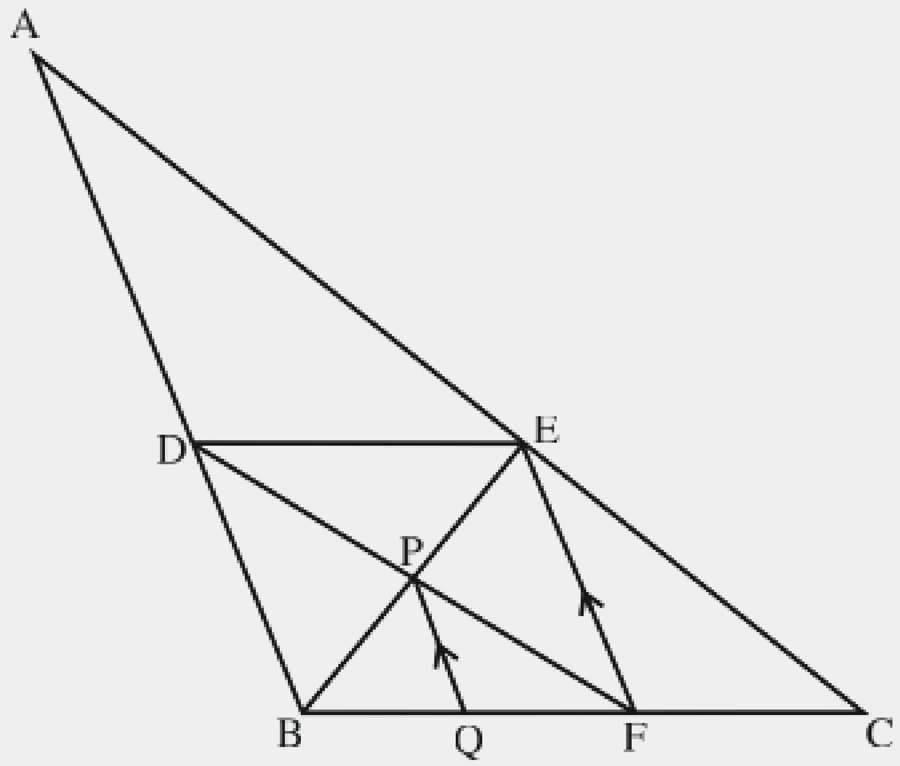
(e) In the sketch alongside, $CF \parallel DE$ and $BG \parallel DF$. $AF = 60$ cm.
$AB : BD = 7 : 8$ and $AF = \frac{3}{2}FE$.



(1) Determine the length of
  (i) FE
  (ii) AG

(2) If $AB = 21$ cm, determine the length of
  (i) AD
  (ii) BC

(f) In the sketch alongside, BDEF is a parallelogram. $16AD = 9AB$.
$PQ \parallel EF$.



(1) Determine BF : FC.

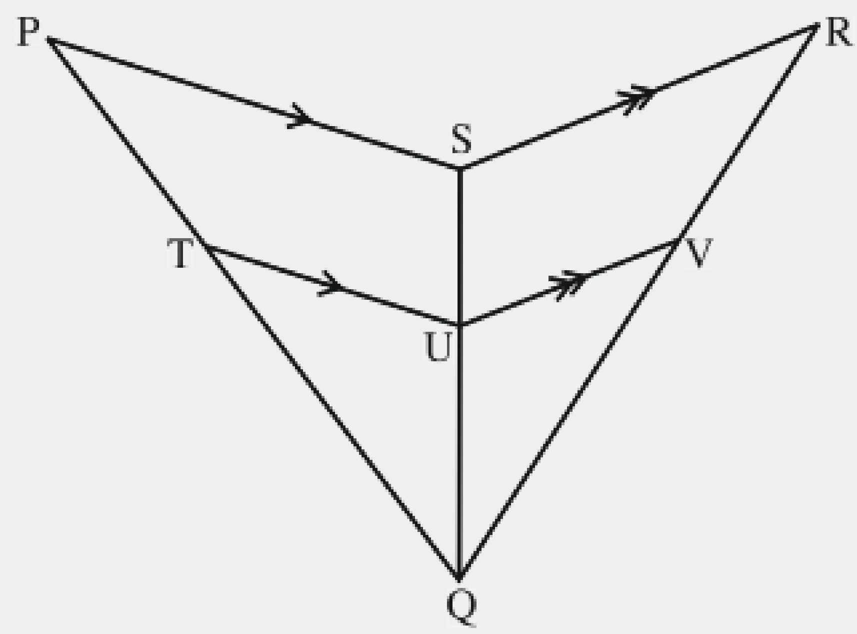
(2) Is $DF \parallel AC$? Motivate.

(3) Determine the following ratios:
  (i) BQ : QF
  (ii) BQ : BC

(4) If $DE = 45$ m and $EC = 56$ m, determine the length of
  (i) AE
  (ii) BC
  (iii) QC

(g) In the sketch alongside, $PS \parallel TU$ and $SR \parallel UV$.

Prove that $\dfrac{PT}{TQ} = \dfrac{RV}{VQ}$.

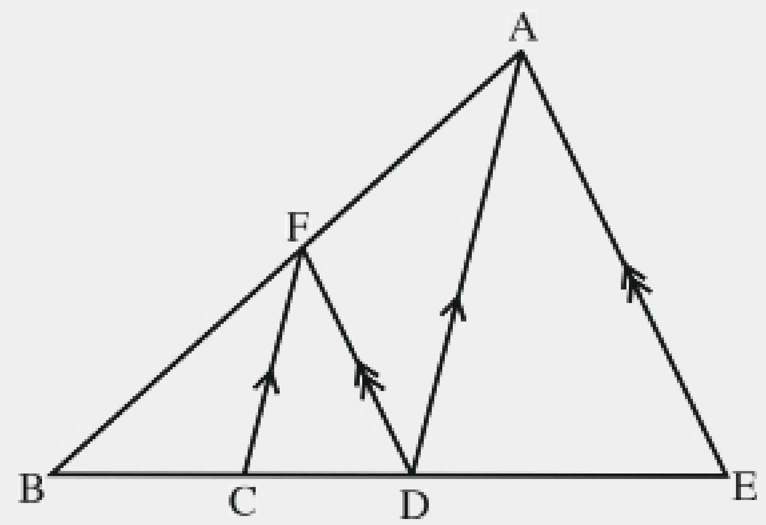


(h) In the sketch alongside, $FC \parallel AD$ and $FD \parallel AE$.

Prove that

(1) $\dfrac{BD}{DE} = \dfrac{BC}{CD}$

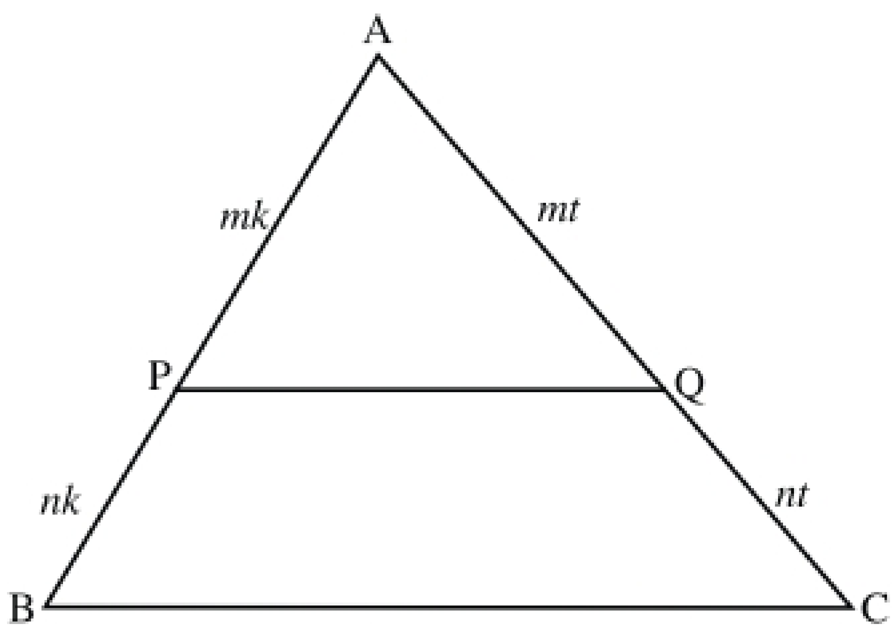
(2) $BD^2 = BE \cdot BC$



THE CONVERSE OF THE TRIANGLE PROPORTIONALITY THEOREM

CONVERSE OF THEOREM 1

If a line divides two sides of a triangle proportionally, then that line is parallel to the third side.



Given: ΔABC with $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$ $\left(\text{or } \dfrac{AP}{AB} = \dfrac{AQ}{AC} \text{ or } \dfrac{PB}{AB} = \dfrac{QC}{AC} \right)$

Conclusion: $PQ \parallel BC$ **Reason:** line divides 2 sides of Δ in prop

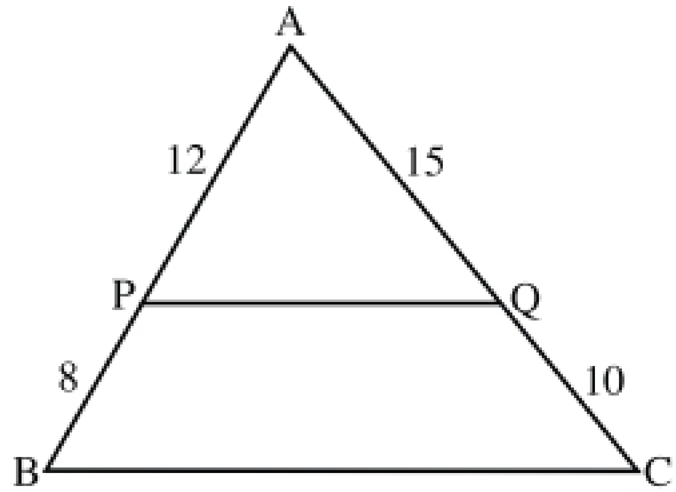
COROLLARY (Midpoint Theorem):

A line drawn through the midpoints of two sides of a triangle is parallel to the third side.

EXAMPLE 5

In the sketch alongside, $AP = 12$ units, $PB = 8$ units, $AQ = 15$ units and $QC = 10$ units.

Prove that $PQ \parallel BC$.



Solution

$$\frac{AP}{PB} = \frac{12}{8} = \frac{3}{2}$$

$$\frac{AQ}{QC} = \frac{15}{10} = \frac{3}{2}$$

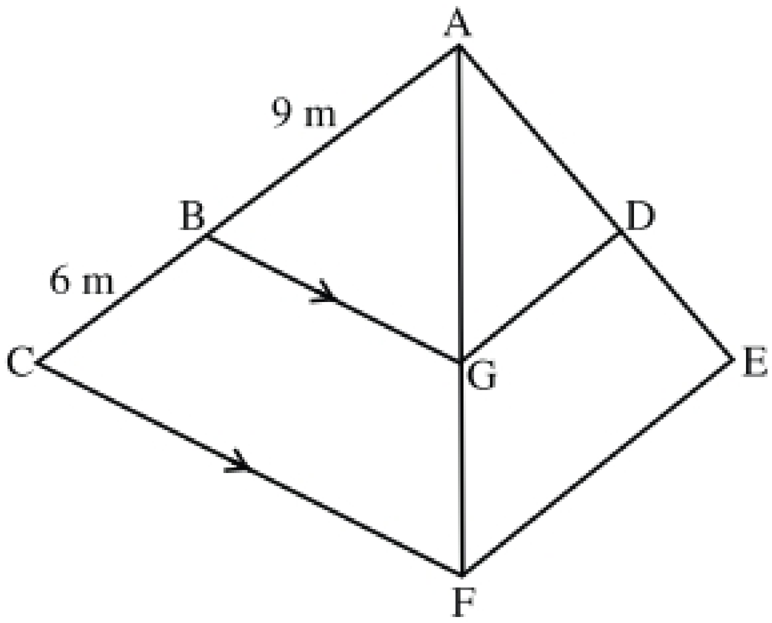
$$\therefore \frac{AP}{PB} = \frac{AQ}{QC}$$

$\therefore PQ \parallel BC$ (line divides 2 sides of Δ in prop)

EXAMPLE 6

In the sketch alongside, $AB = 9$ m, $BC = 6$ m and $AD = \frac{3}{5}AE$.

Prove that $GD \parallel FE$.



Solution

$\frac{AG}{AF} = \frac{AB}{AC} = \frac{9}{15}$ (line $\parallel$ side of Δ)

$$\therefore \frac{AG}{AF} = \frac{3}{5}$$

$AD = \frac{3}{5}AE$ (Given)

$$\therefore \frac{AD}{AE} = \frac{3}{5}$$

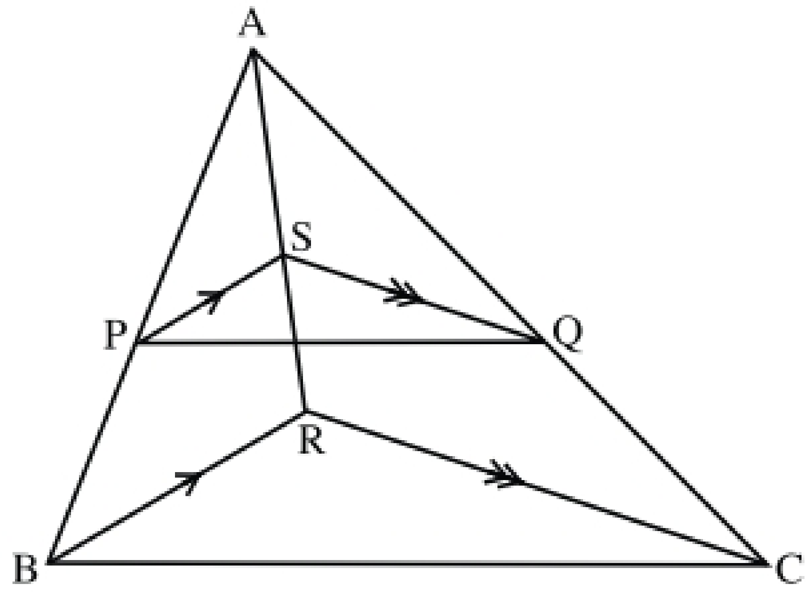
$$\therefore \frac{AD}{AE} = \frac{AG}{AF}$$

$\therefore GD \parallel FE$ (line divides 2 sides of Δ in prop)

EXAMPLE 7

In the sketch alongside, $PS \parallel BR$ and $SQ \parallel RC$.

Prove that $PQ \parallel BC$.



Solution

$\dfrac{AP}{PB} = \dfrac{AS}{SR}$ (line $\parallel$ side of Δ)

and $\dfrac{AS}{SR} = \dfrac{AQ}{QC}$ (line $\parallel$ side of Δ)

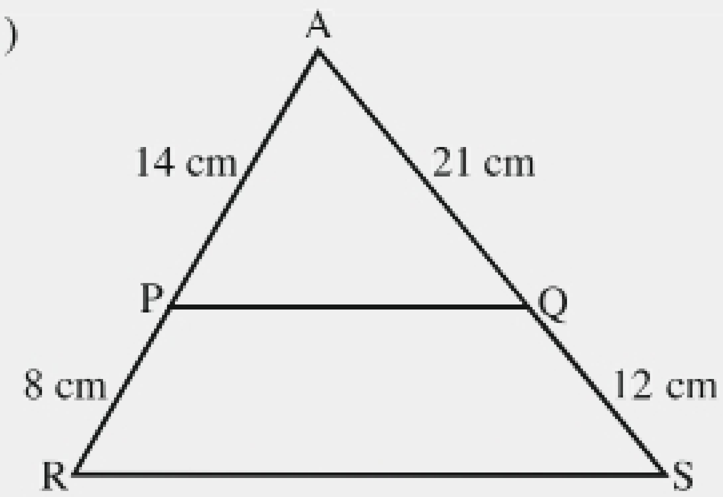
$\therefore \dfrac{AP}{PB} = \dfrac{AQ}{QC}$

$\therefore PQ \parallel BC$ (line divides 2 sides of Δ in prop)

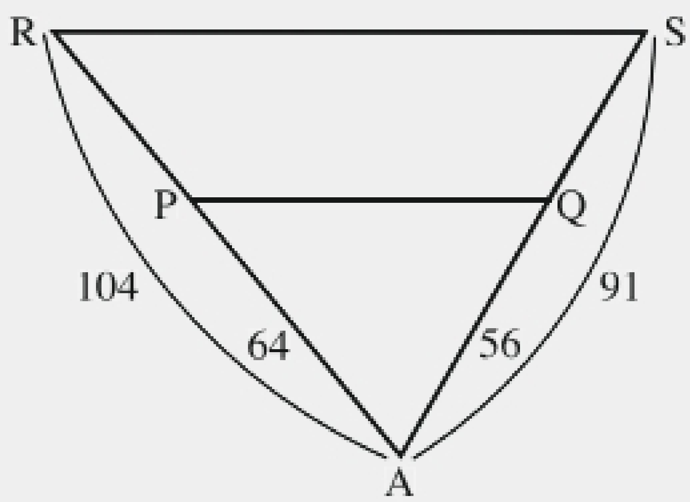
EXERCISE 2

(a) In each of the following sketches, prove that $PQ \parallel RS$:

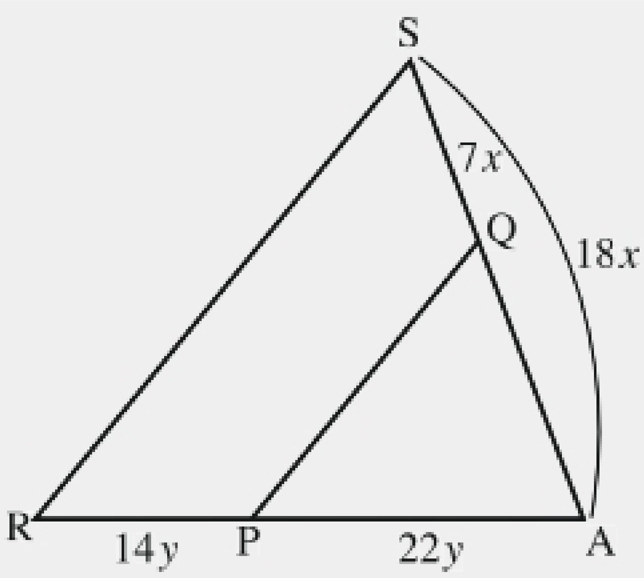
(1)



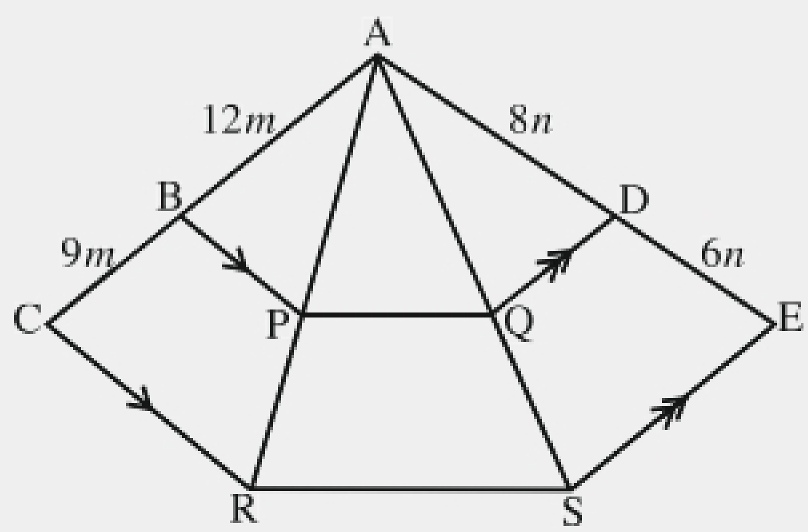
(2)



(3)

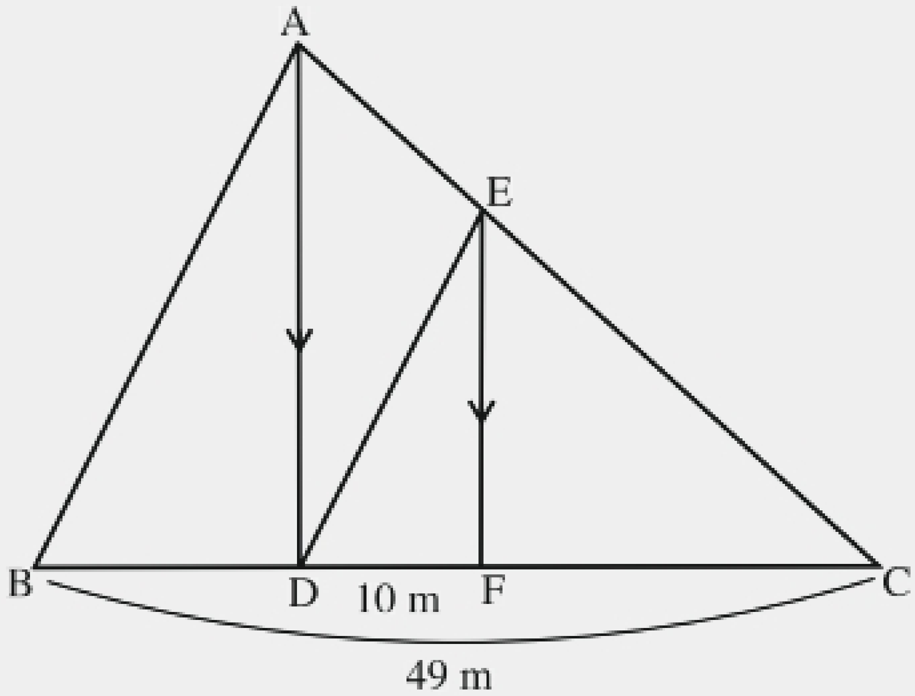


(4)



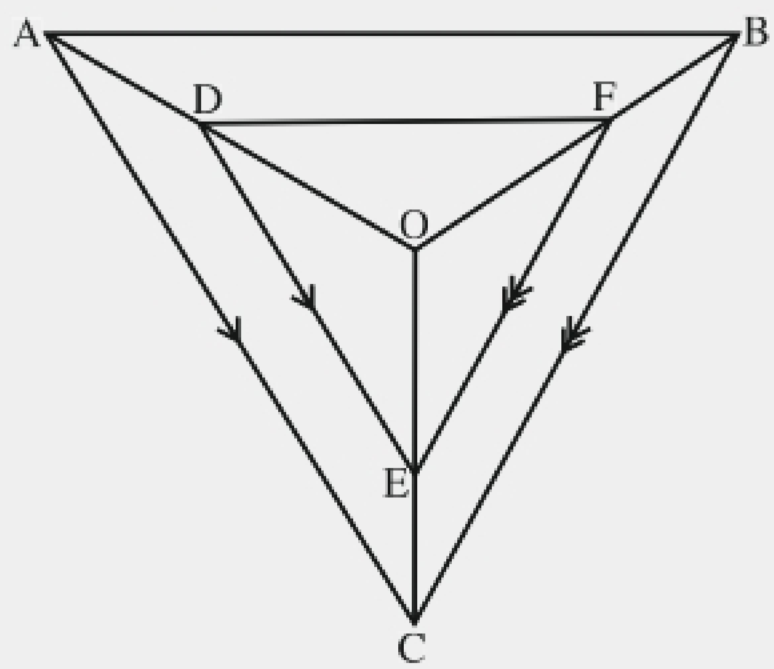
(b) In the sketch alongside, $AE : EC = 2:5$. $BC = 49$ m and $DF = 10$ m. $AD \parallel EF$.

Prove that $AB \parallel DE$.



(c) In the sketch alongside, $DE \parallel AC$ and $EF \parallel BC$.

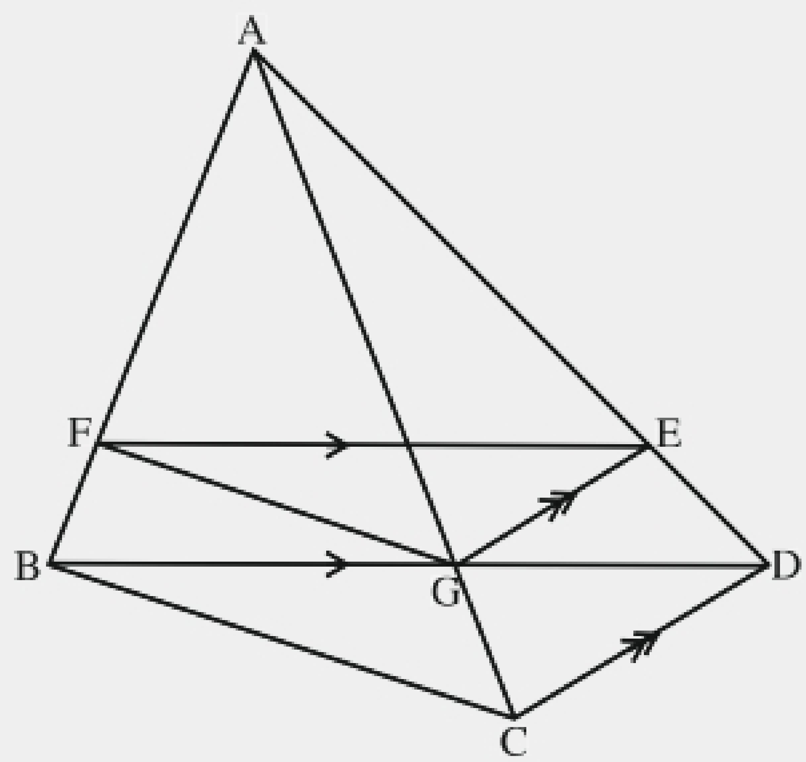
Prove that $DF \parallel AB$.



(d) In the sketch alongside, $FE \parallel BD$ and $GE \parallel CD$.

(1) Prove that $FG \parallel BC$.

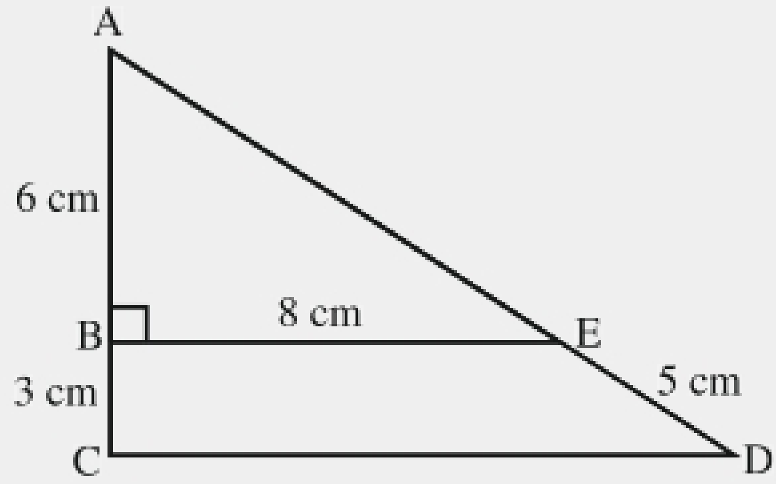
(2) If $AF = 12$ cm, $FB = 4$ cm, $AC = 20$ cm and $AE = 18$ cm, determine the length of
(i) ED
(ii) AG



(e) In the sketch alongside, $A\hat{B}E = 90°$. $AB = 6$ cm, $BC = 3$ cm, $BE = 8$ cm and $ED = 5$ cm.

(1) Prove that $A\hat{C}D = 90°$

(2) Determine the length of CD.

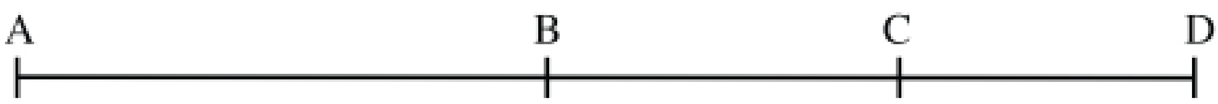


MULTIPLE RATIOS ON A LINE

Before attempting more advanced problems on the proportionality theorem, you will have to know how to deal with multiple ratios on the same line. The following example shows an effective technique:

EXAMPLE 8

In the following sketch, $AB : BC = 3:2$ and $BC : CD = 6:5$.



Determine AB : CD.

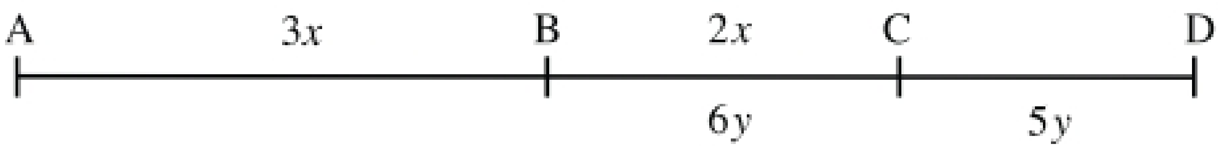
Solution

Step 1: Represent the lengths in terms of variables according to the given ratios:

$AB : BC = 3:2 \rightarrow$ Let $AB = 3x$ and $BC = 2x$.

$BC : CD = 6:5 \rightarrow$ Let $BC = 6y$ and $CD = 5y$.

Step 2: Write the lengths (in terms of variables) on the given line:



Step 3: Find a **line segment** on the sketch that is expressed **in terms of both variables**, **equate the two expressions** and make one of the variables the subject:

$2x = 6y$ (Both $= BC$)

$\therefore x = 3y$

Step 4: Express the required ratio as a fraction by using the lengths on the sketch:

$$\frac{AB}{CD} = \frac{3x}{5y}$$

Step 5: Substitute the result from step 3 into the expression in step 4 and simplify:

$$\frac{AB}{CD} = \frac{3x}{5y} = \frac{3(3y)}{5y} = \frac{9y}{5y} = \frac{9}{5} \rightarrow AB : CD = 9:5$$

Let us practise some more:

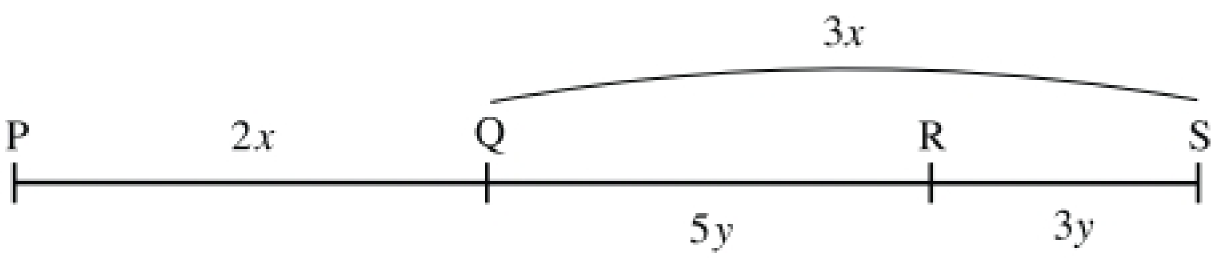
EXAMPLE 9

In the following sketch, $PQ : QS = 2:3$ and $QR : RS = 5:3$.

Calculate $\dfrac{QR}{PQ}$.

Solution

$PQ : QS = 2:3 \rightarrow$ Let $PQ = 2x$ and $QS = 3x$.
$QR : RS = 5:3 \rightarrow$ Let $QR = 5y$ and $RS = 3y$.

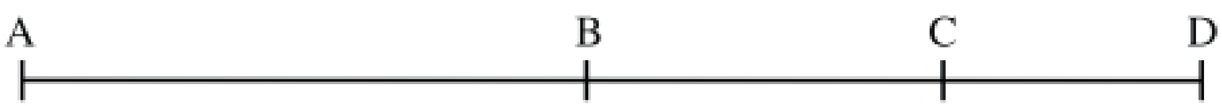


$5y + 3y = 3x$ (QS)
$\therefore 8y = 3x$
$\therefore y = \dfrac{3x}{8}$

$$\frac{QR}{PQ} = \frac{5y}{2x} = \frac{5\left(\dfrac{3x}{8}\right)}{2x} = \frac{15x}{8} \times \frac{1}{2x} = \frac{15}{16}$$

EXAMPLE 10

(a) In the following sketch, $AB : BC = 3:2$ and $AC : CD = 10:3$.



Determine $AB : BD$.

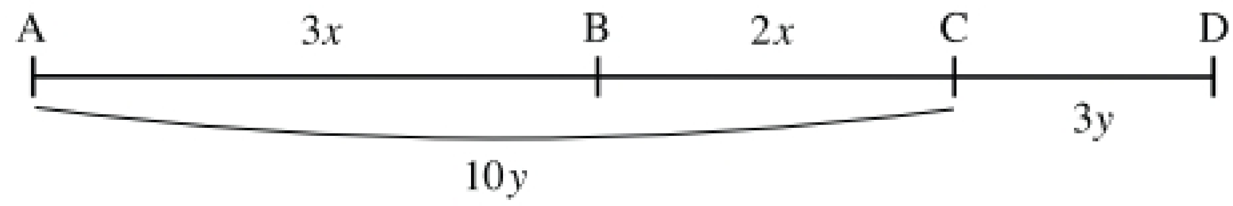
(b) In the following sketch, $KL : LN = 3:13$ and $KM : MN = 5:3$.

Calculate $\dfrac{KL}{LM}$.

Solution

(a) AB : BC $= 3:2 \rightarrow$ Let AB $= 3x$ and BC $= 2x$.

AC : CD $= 10:3 \rightarrow$ Let AC $= 10y$ and CD $= 3y$.



$3x + 2x = 10y$ (AC)

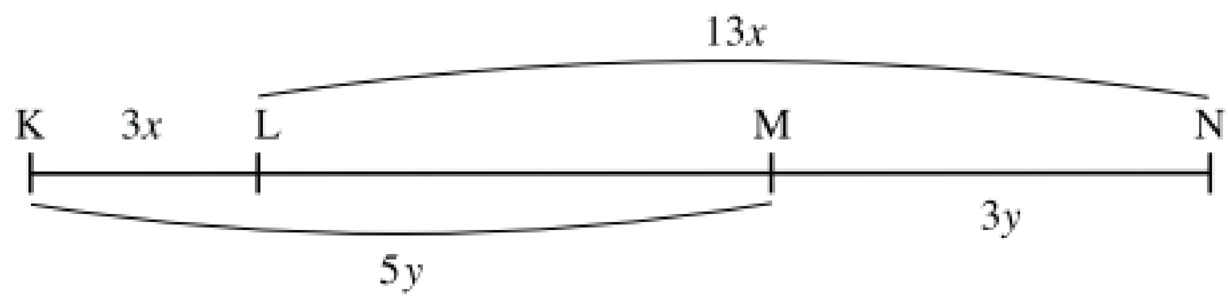
$\therefore 5x = 10y$

$\therefore x = 2y$

$$\frac{\text{AB}}{\text{BD}} = \frac{3x}{2x+3y} = \frac{3(2y)}{2(2y)+3y} = \frac{6y}{7y} = \frac{6}{7} \rightarrow \text{AB : BD} = 6:7$$

(b) KL : LN $= 3:13 \rightarrow$ Let KL $= 3x$ and LN $= 13x$.

KM : MN $= 5:3 \rightarrow$ Let KM $= 5y$ and MN $= 3y$.



$3x + 13x = 5y + 3y$ (KN)

$\therefore 16x = 8y$

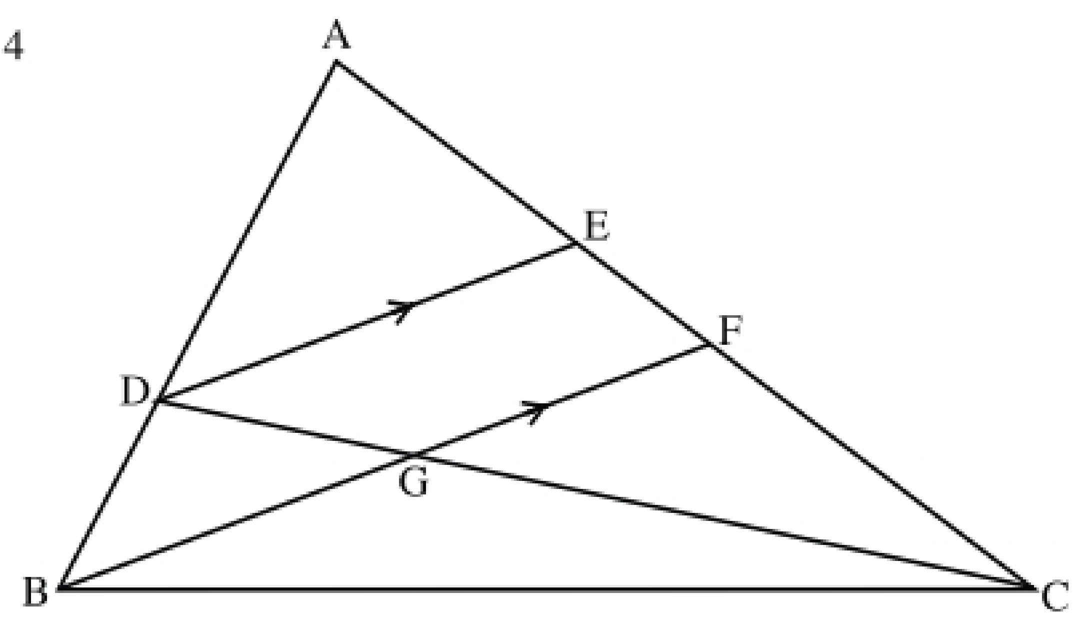
$\therefore y = 2x$

$$\frac{\text{KL}}{\text{LM}} = \frac{3x}{5y-3x} = \frac{3x}{5(2x)-3x} = \frac{3x}{7x} = \frac{3}{7}$$

EXAMPLE 11

In the sketch alongside, AD : DB $= 7:4$ and DG : GC $= 2:5$.

Determine AF : FC.



Solution

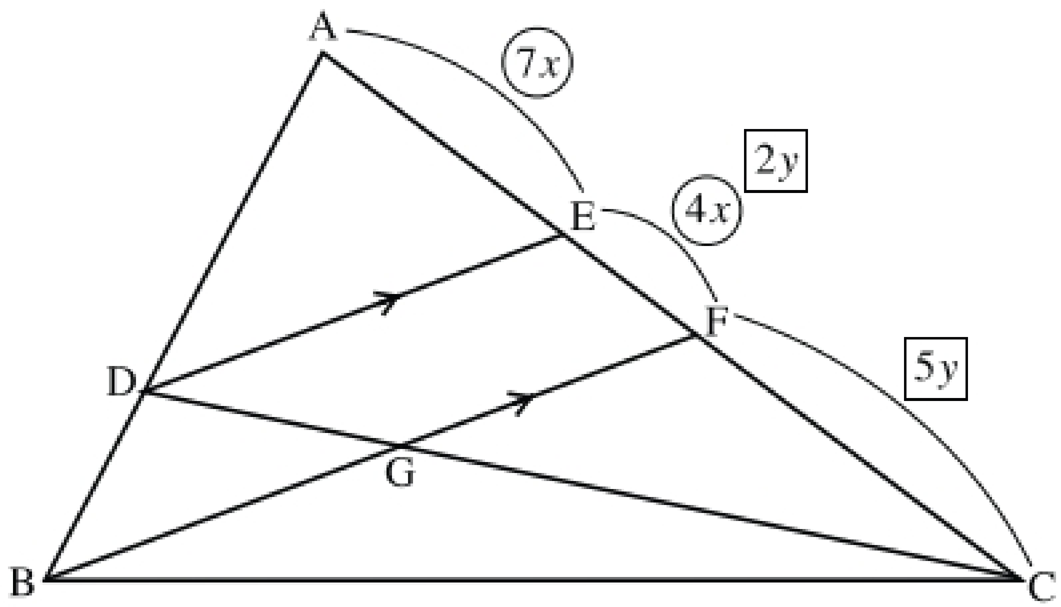
Transfer both ratios onto the same line segment (AC):

$\dfrac{AE}{EF}=\dfrac{AD}{DB}=\dfrac{7}{4}$ (line $\parallel$ side of Δ)

$\dfrac{EF}{FC}=\dfrac{DG}{GC}=\dfrac{2}{5}$ (line $\parallel$ side of Δ)

Let $AE=7x$ and $EF=4x$

Let $EF=2y$ and $FC=5y$



$4x=2y$ (EF)

$\therefore y=2x$

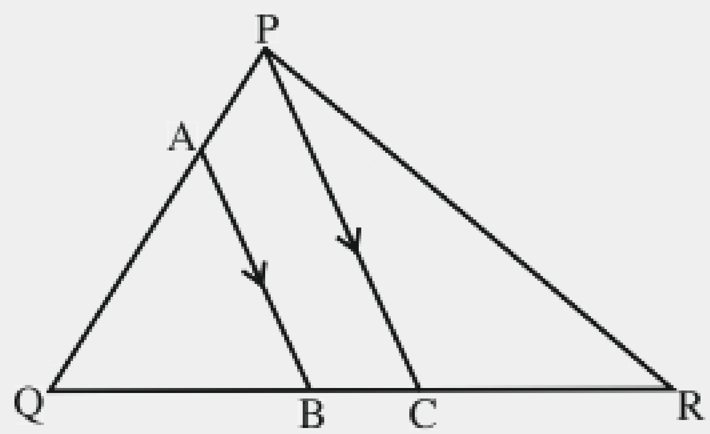
$\dfrac{AF}{FC}=\dfrac{7x+4x}{5y}=\dfrac{11x}{5y}$

$\therefore \dfrac{AF}{FC}=\dfrac{11x}{5(2x)}=\dfrac{11x}{10x}=\dfrac{11}{10} \rightarrow AF:FC=11:10$

EXERCISE 3

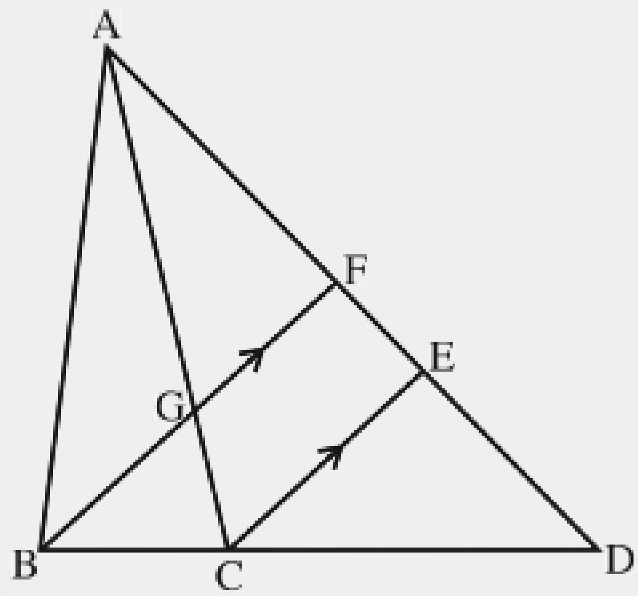
(a) In the sketch alongside, $AB \parallel PC$.
$PA:AQ=2:5$ and $BC:CR=4:9$.

Determine
(1) QB : CR
(2) CR : QC



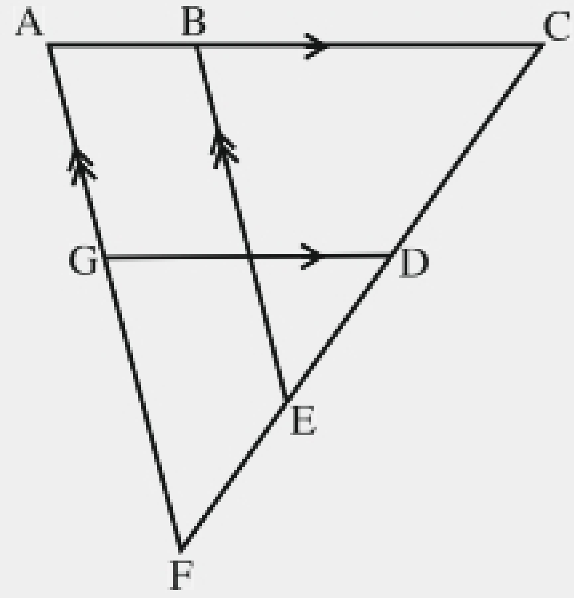
(b) In the sketch alongside, $BF \parallel CE$.
$AF:AD=5:11$ and $CD=2BC$.

Determine
(1) AF : ED
(2) AG : AC



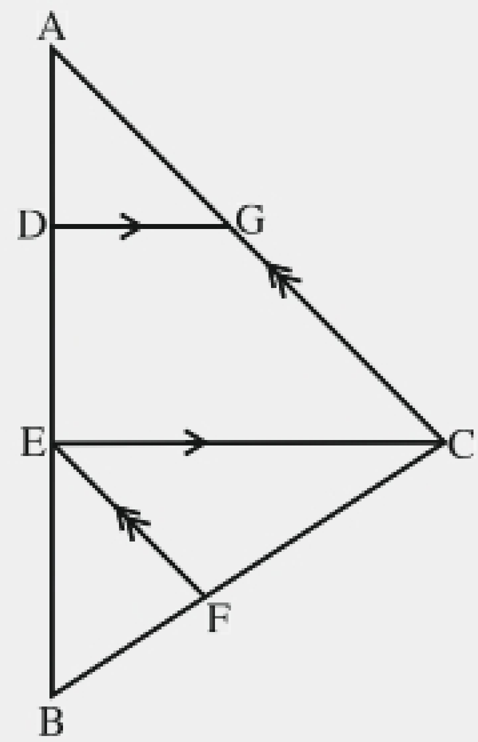
(c) In the sketch alongside, $AC \parallel GD$ and $AF \parallel BE$. $\frac{BC}{AB} = 3$ and $AG = \frac{4}{5}GF$.

Calculate $\frac{FE}{DC}$.



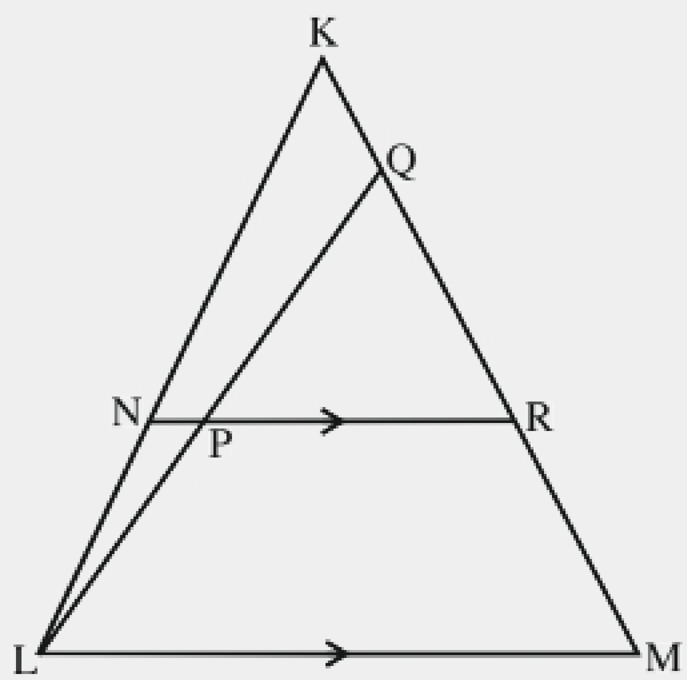
(d) In the sketch alongside, $DG \parallel EC$ and $EF \parallel AC$. $AG : GC = 4 : 5$ and $BF : FC = 2 : 3$.

Determine
(1) $AD : DB$
(2) $DE : AB$



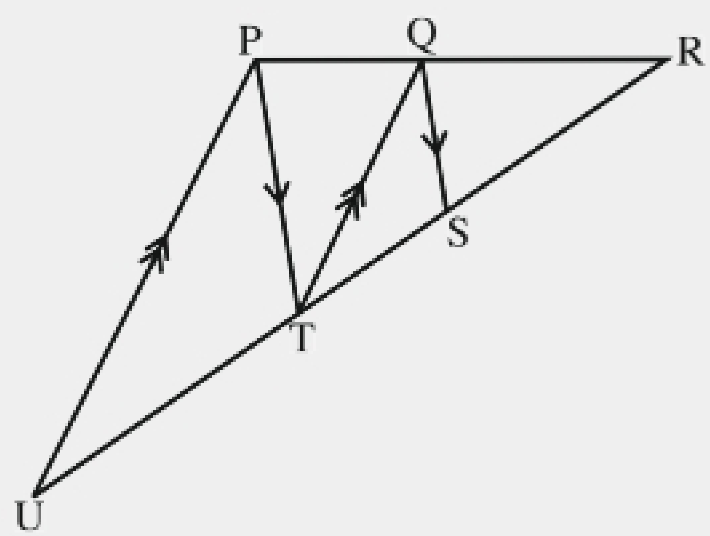
(e) In the sketch alongside, $NR \parallel LM$. $KN : NL = 5 : 3$ and $LP : PQ = 6 : 7$.

Determine
(1) $QM : KR$
(2) $KQ : RM$



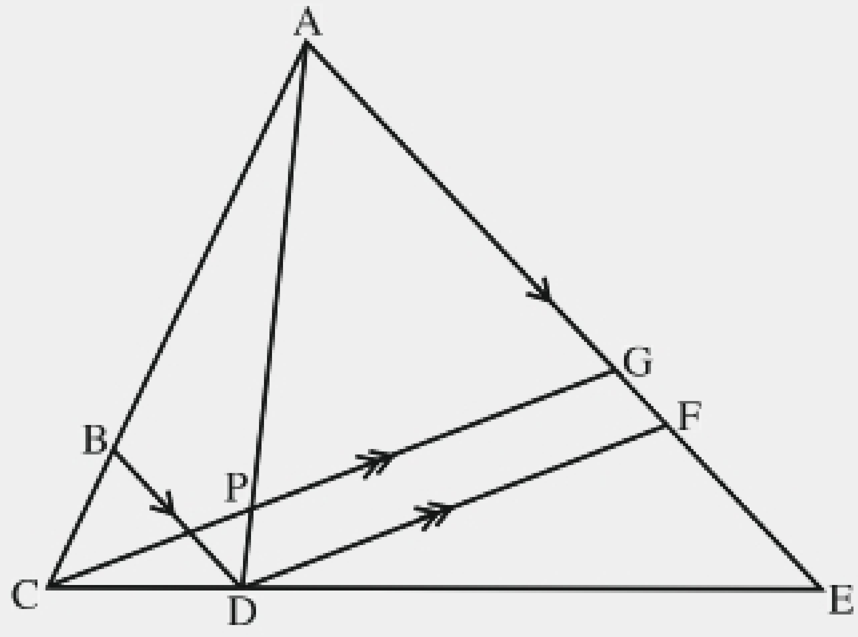
(f) In the sketch alongside, $PU \parallel QT$ and $PT \parallel QS$. $5PQ = 2PR$.

Calculate $\frac{US}{SR}$.



(g) In the sketch alongside, $AB = 42$ cm and $BC = 14$ cm. $BD \parallel AE$ and $CG \parallel DF$. $\frac{AG}{GE} = \frac{3}{2}$.

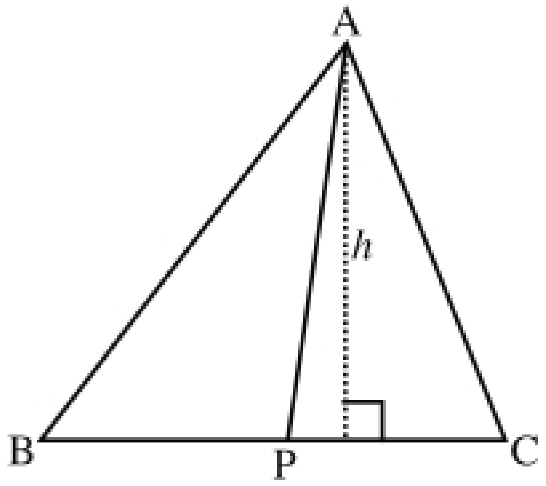
Determine AD : PD.



AREA RATIOS

In this section you will learn how to calculate the ratio of the areas of two triangles under specific conditions:

TYPE 1 **SAME PERPENDICULAR HEIGHT**



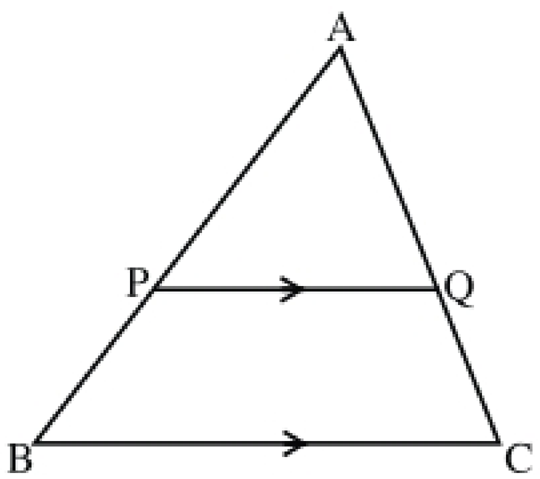
$$\frac{\text{Area of } \Delta ABP}{\text{Area of } \Delta ACP} = \frac{\frac{1}{2} \cdot BP \cdot h}{\frac{1}{2} \cdot PC \cdot h} = \frac{BP}{PC}$$

$$\frac{\text{Area of } \Delta ABP}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} \cdot BP \cdot h}{\frac{1}{2} \cdot BC \cdot h} = \frac{BP}{BC}$$

$$\frac{\text{Area of } \Delta ACP}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} \cdot PC \cdot h}{\frac{1}{2} \cdot BC \cdot h} = \frac{PC}{BC}$$

$$\boxed{\text{Ratio of areas} = \text{Ratio of bases}}$$

TYPE 2 **COMMON ANGLE; PARALLEL LINES**



$$\frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta ABC} = \frac{\frac{1}{2} \cdot AP \cdot AQ \cdot \sin A}{\frac{1}{2} \cdot AB \cdot AC \cdot \sin A} = \frac{AP}{AB} \cdot \frac{AQ}{AC}$$

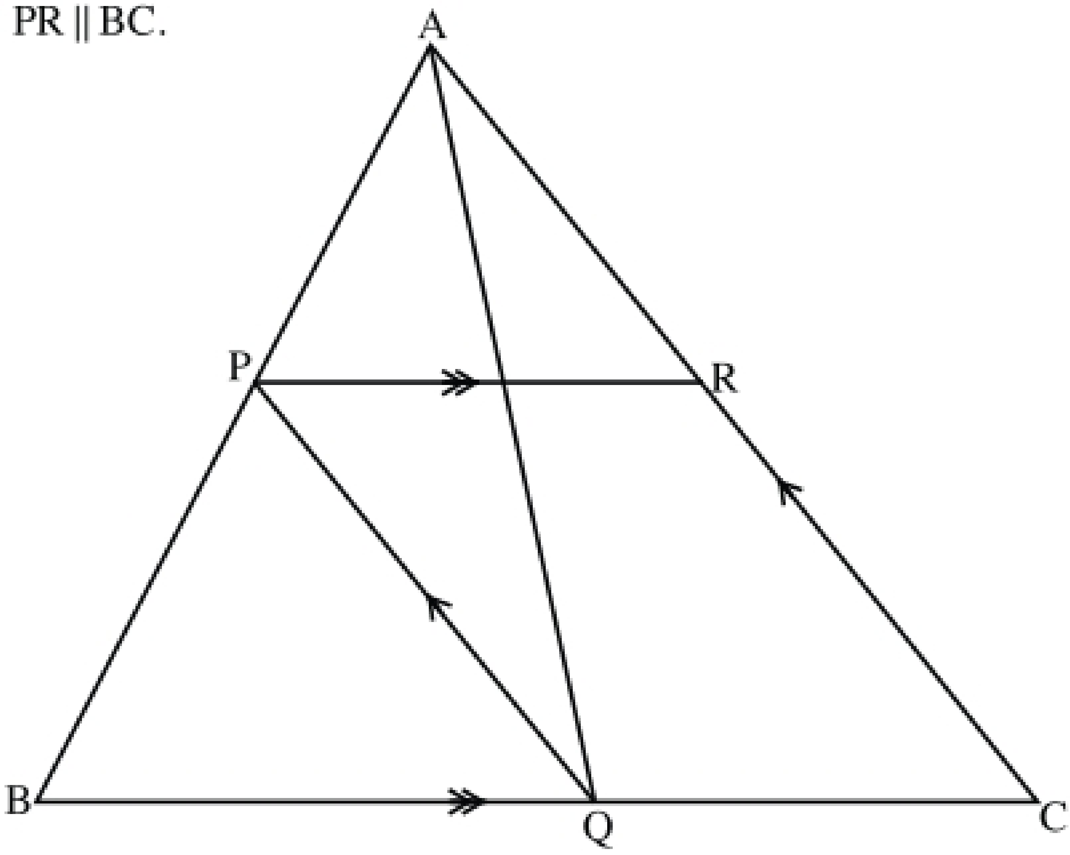
But $\frac{AP}{AB} = \frac{AQ}{AC}$ (line $\parallel$ side of Δ)

$$\therefore \frac{\text{Area of } \Delta APQ}{\text{Area of } \Delta ABC} = \left(\frac{AP}{AB}\right)^2 = \left(\frac{AQ}{AC}\right)^2$$

$$\boxed{\text{Ratio of areas} = (\text{Ratio of corresponding sides})^2}$$

EXAMPLE 12

In the sketch alongside $PQ \parallel AC$ and $PR \parallel BC$.
$BQ : QC = 4 : 3$.



Determine

(a) $\dfrac{\text{Area of } \Delta ABQ}{\text{Area of } \Delta ACQ}$

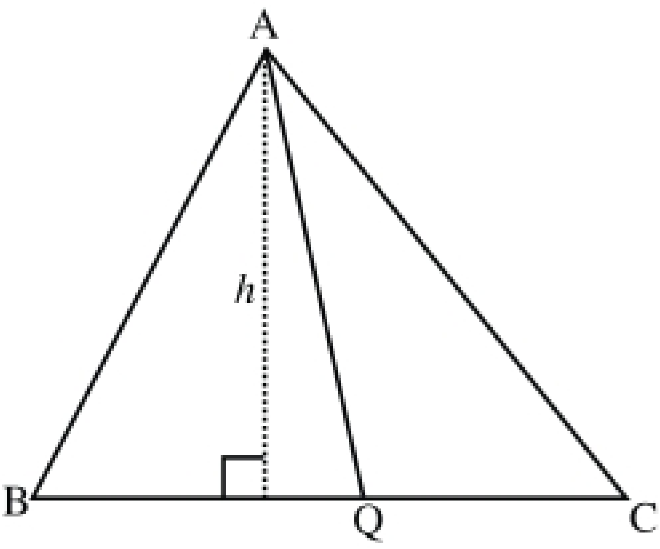
(b) $\dfrac{\text{Area of } \Delta BPQ}{\text{Area of } \Delta ABQ}$

(c) $\dfrac{\text{Area of } \Delta APR}{\text{Area of } \Delta ABC}$

(d) $\dfrac{\text{Area of } \Delta BPQ}{\text{Area of } \Delta ACQ}$

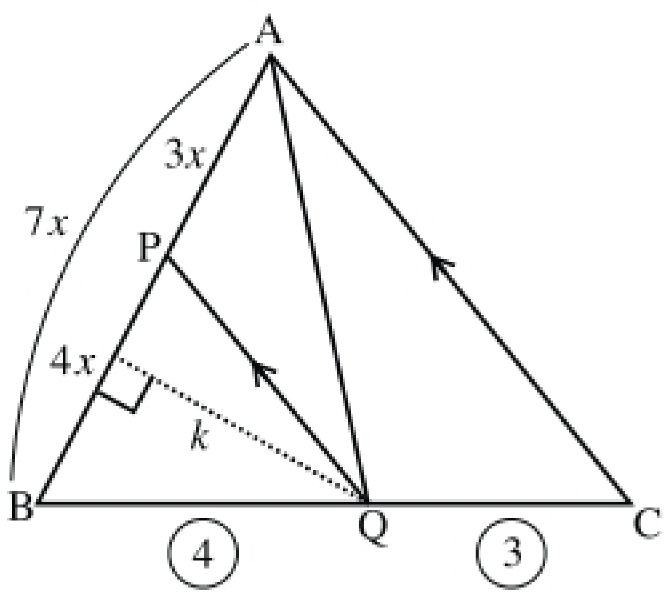
Solution

(a) With BQ as base of ΔABQ and QC as base of ΔACQ, these triangles have the same perpendicular height. Let the common height be h.



$$\frac{\text{Area of } \Delta ABQ}{\text{Area of } \Delta ACQ} = \frac{\frac{1}{2} \cdot BQ \cdot h}{\frac{1}{2} \cdot QC \cdot h} = \frac{BQ}{QC} = \frac{4}{3}$$

(b) With BP as base of ΔBPQ and AB as base of ΔABQ, these triangles have the same perpendicular height. Let the common height be k.



$$\frac{BQ}{QC} = \frac{BP}{PA} = \frac{4}{3} \qquad (\text{line} \parallel \text{side of } \Delta)$$

Let $BP = 4x$ and $PA = 3x$

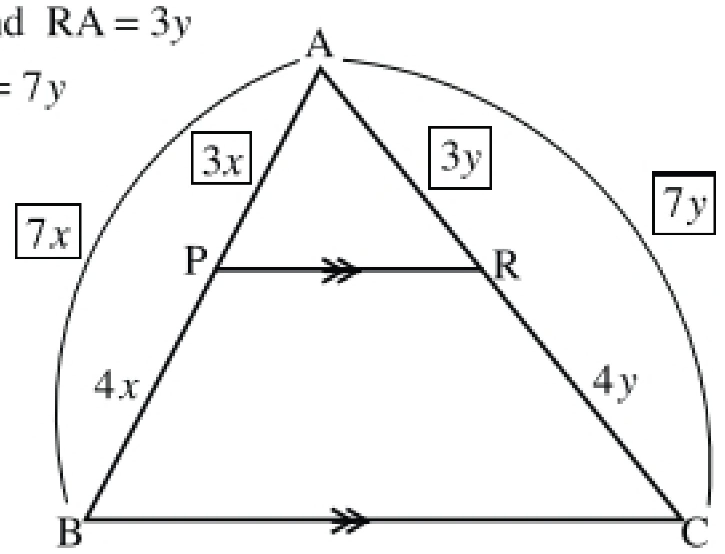
$\therefore BA = 4x + 3x = 7x$

$$\frac{\text{Area of } \Delta BPQ}{\text{Area of } \Delta ABQ} = \frac{\frac{1}{2} \cdot BP \cdot k}{\frac{1}{2} \cdot BA \cdot k} = \frac{BP}{BA} = \frac{4x}{7x} = \frac{4}{7}$$

(c) $\dfrac{BP}{PA}=\dfrac{CR}{RA}=\dfrac{4}{3}$ (line $\parallel$ side of Δ)

Let $BP=4x$ and $PA=3x$ Let $CR=4y$ and $RA=3y$

$\therefore BA=4x+3x=7x$ $\therefore CA=4y+3y=7y$



$$\frac{\text{Area of }\Delta APR}{\text{Area of }\Delta ABC}=\frac{\frac{1}{2}\cdot PA\cdot RA\cdot\sin A}{\frac{1}{2}\cdot BA\cdot CA\cdot\sin A}$$

$$=\frac{PA\cdot RA}{BA\cdot CA}$$

$$=\frac{3x\cdot 3y}{7x\cdot 7y}=\frac{9}{49}$$

(d) We have already found the ratio of each one of these areas to the area of ΔABQ:

$$\frac{\text{Area of }\Delta BPQ}{\text{Area of }\Delta ABQ}=\frac{4}{7} \qquad \frac{\text{Area of }\Delta ABQ}{\text{Area of }\Delta ACQ}=\frac{4}{3}$$

$$\therefore \text{Area of }\Delta BPQ=\frac{4}{7}\text{Area of }\Delta ABQ \qquad \therefore \text{Area of }\Delta ABQ=\frac{4}{3}\text{Area of }\Delta ACQ$$

$$\therefore \text{Area of }\Delta BPQ=\frac{4}{7}\left(\frac{4}{3}\text{Area of }\Delta ACQ\right)$$

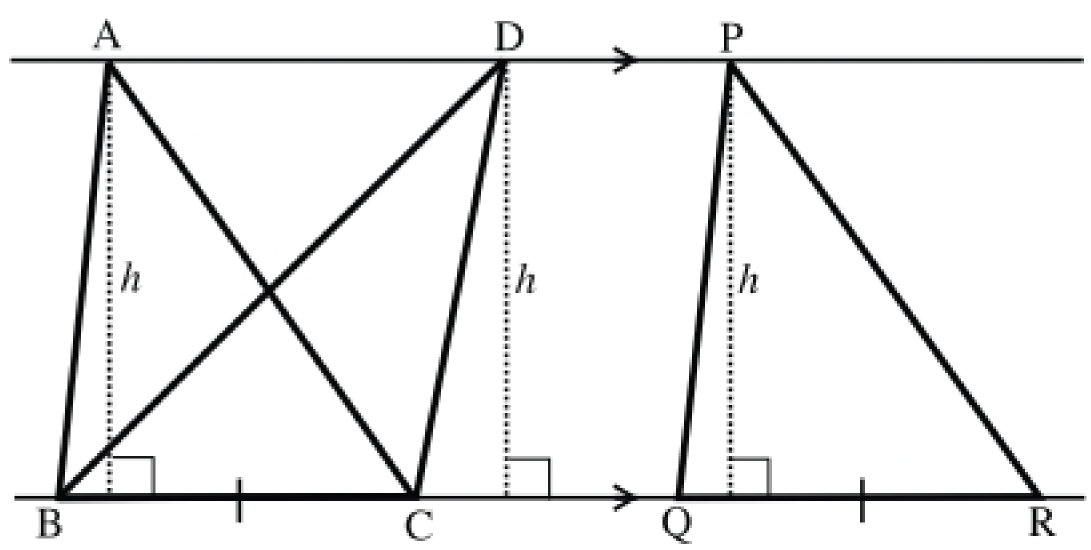
$$\therefore \text{Area of }\Delta BPQ=\frac{16}{21}\text{Area of }\Delta ACQ$$

$$\therefore \frac{\text{Area of }\Delta BPQ}{\text{Area of }\Delta ACQ}=\frac{16}{21}$$

Alternatively: $\dfrac{\text{Area of }\Delta BPQ}{\text{Area of }\Delta ACQ}=\dfrac{\text{Area of }\Delta BPQ}{\text{Area of }\Delta ABQ}\times\dfrac{\text{Area of }\Delta ABQ}{\text{Area of }\Delta ACQ}=\dfrac{4}{7}\times\dfrac{4}{3}=\dfrac{16}{21}$

TRIANGLES BETWEEN PARALLEL LINES

Triangles with the same base (or equal bases) and between the same parallel lines have equal areas:

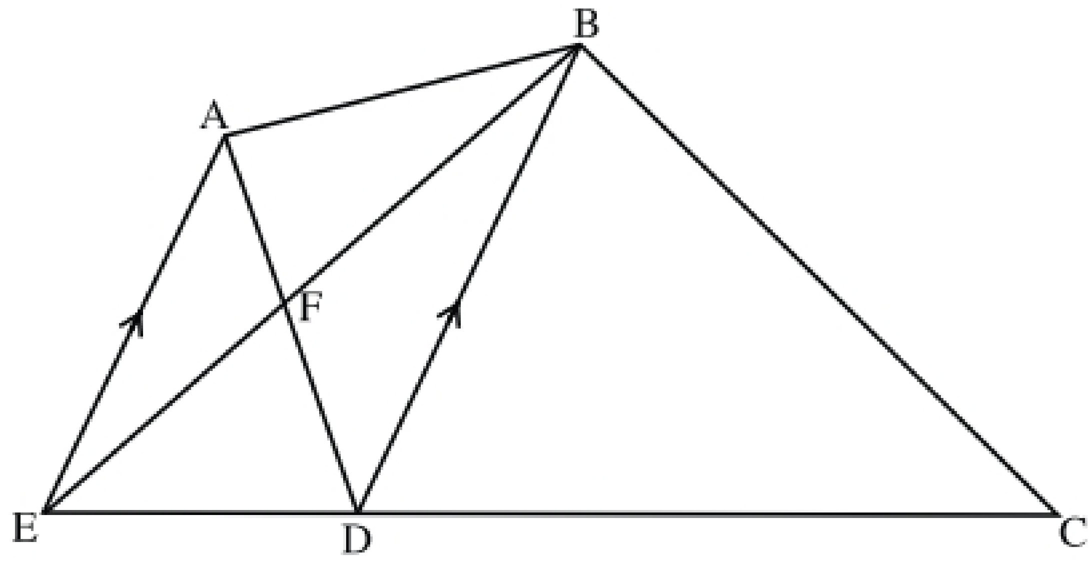


Area of ΔABC = Area of ΔDBC = Area of ΔPQR **Reason :** same base; same height

EXAMPLE 13

In the sketch alongside, $AE \parallel BD$.
$ED : DC = 5 : 12$.

Determine $\dfrac{\text{Area of } \Delta ABD}{\text{Area of } \Delta BDC}$.



Solution

Strategy: Notice that ΔEBD and ΔABD have the same base (BD) and lie between parallel lines and thus they will have equal areas. This means that we can calculate $\dfrac{\text{Area of } \Delta EBD}{\text{Area of } \Delta BDC}$ as this will have the same value as $\dfrac{\text{Area of } \Delta ABD}{\text{Area of } \Delta BDC}$.

With ED as base of ΔEBD and DC as base of ΔBDC, these triangles have the same perpendicular height. Let the common height be h.

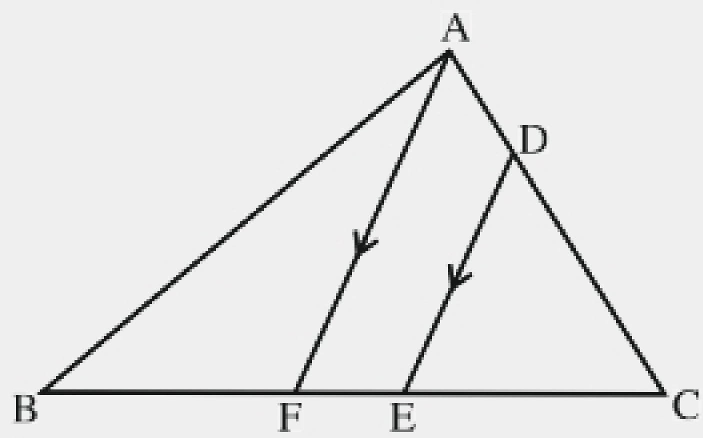
$$\frac{\text{Area of } \Delta EBD}{\text{Area of } \Delta BDC} = \frac{\frac{1}{2} \cdot ED \cdot h}{\frac{1}{2} \cdot DC \cdot h} = \frac{ED}{DC} = \frac{5}{12}$$

But Area of ΔEBD = Area of ΔABD (same base; same height)

$$\therefore \frac{\text{Area of } \Delta ABD}{\text{Area of } \Delta BDC} = \frac{5}{12}$$

EXERCISE 4

(a) In the sketch alongside, $AF \parallel DE$.
$AD : DC = 2 : 5$ and $BF : FC = 2 : 3$.



Determine

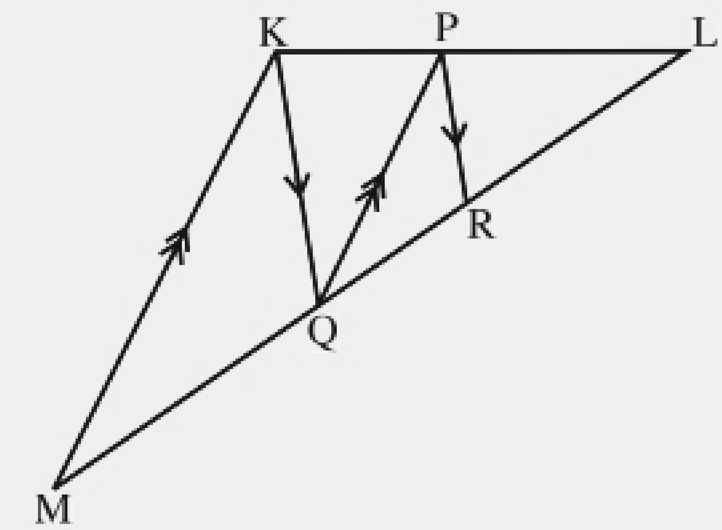
(1) $\dfrac{\text{Area of } \Delta ABF}{\text{Area of } \Delta ACF}$

(2) $\dfrac{\text{Area of } \Delta ACF}{\text{Area of } \Delta ABC}$

(3) $\dfrac{\text{Area of } \Delta ACF}{\text{Area of } \Delta DCE}$

(4)* $\dfrac{\text{Area of } \Delta ABF}{\text{Area of } \Delta DCE}$

(b) In the sketch alongside, $KM \parallel PQ$ and $KQ \parallel PR$. $KP : PL = 5 : 7$



Determine

(1) $\dfrac{\text{Area of } \Delta PRL}{\text{Area of } \Delta KQL}$

(2) $\dfrac{\text{Area of } \Delta KPQ}{\text{Area of } \Delta KLQ}$

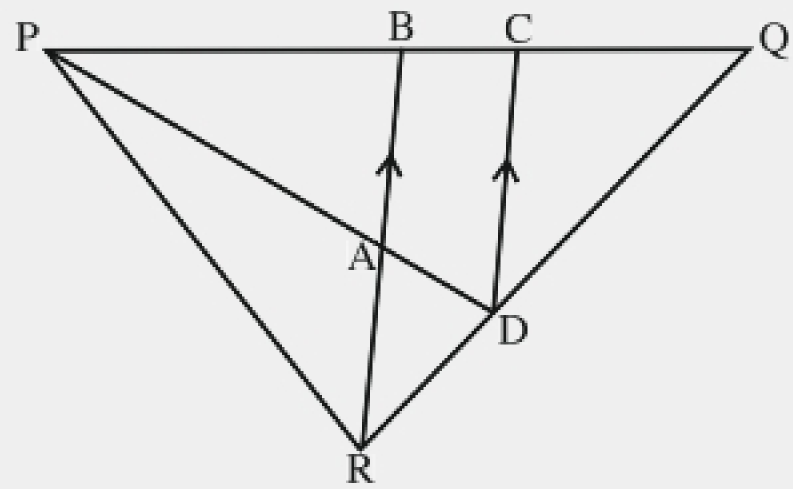
(3) $\dfrac{\text{Area of } \Delta KQL}{\text{Area of } \Delta KQM}$

(4)* $\dfrac{\text{Area of } \Delta PRL}{\text{Area of } \Delta KQM}$

(5)* $\dfrac{\text{Area of } \Delta KPQ}{\text{Area of } \Delta KML}$

(c) In the sketch alongside, $BR \parallel CD$.

$BC = \dfrac{5}{13} BQ$ and $PB = BQ$.



Determine

(1) $\dfrac{\text{Area of } \Delta CQD}{\text{Area of } \Delta BQR}$

(2) $\dfrac{\text{Area of } \Delta QPD}{\text{Area of } \Delta RPD}$

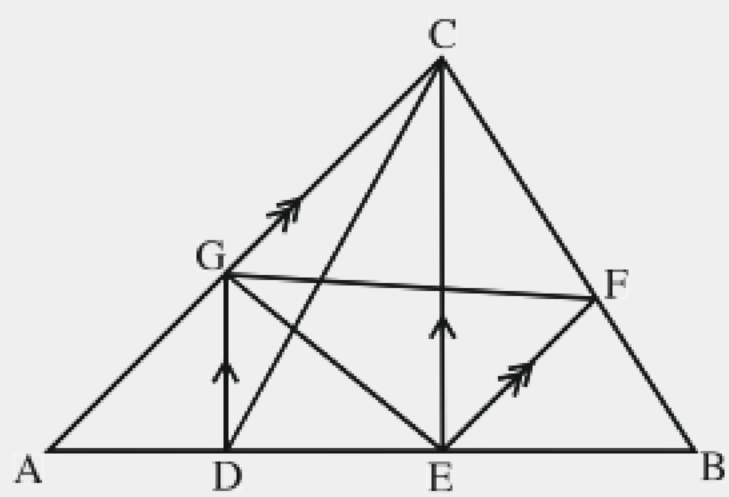
(3)* $\dfrac{\text{Area of } \Delta CQD}{\text{Area of } \Delta PQR}$

(4)* $\dfrac{\text{Area of } \Delta CQD}{\text{Area of } \Delta PQD}$

(5)* $\dfrac{\text{Area of } \Delta CQD}{\text{Area of } \Delta RPD}$

(d) In the sketch alongside, $DG \parallel EC$ and $EF \parallel AC$.

$AG = \dfrac{4}{5} GC$ and $BF : FC = 2 : 3$.



Determine

(1) $\dfrac{\text{Area of } \Delta ACE}{\text{Area of } \Delta AGD}$

(2) $\dfrac{\text{Area of } \Delta CEF}{\text{Area of } \Delta BEF}$

(3) $\dfrac{\text{Area of } \Delta GEF}{\text{Area of } \Delta BEF}$

(4)* $\dfrac{\text{Area of } \Delta CGE}{\text{Area of } \Delta CDA}$

SIMILARITY

POLYGONS

Polygons are said to be *similar* if they have the **same shape** (but not necessarily the same size). If two polygons are similar, the one is an enlargement of the other.

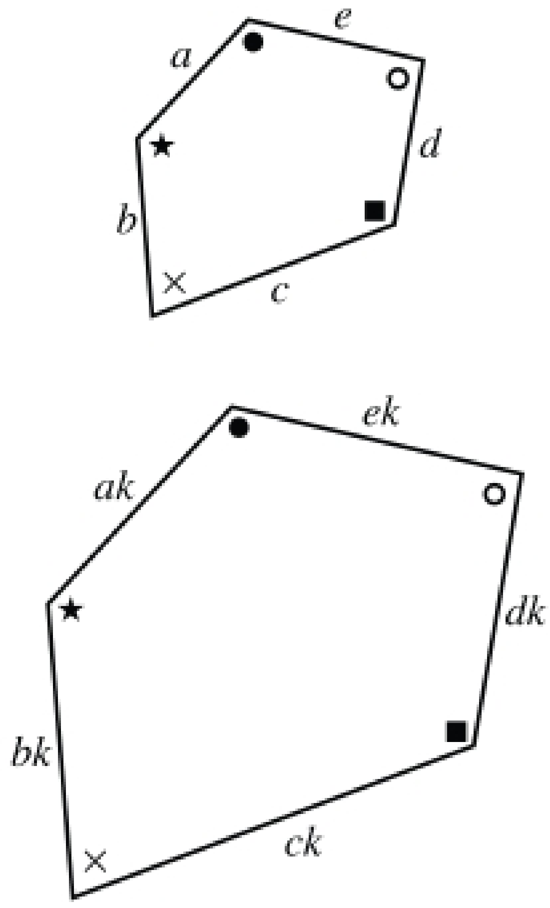
Formally, we say that two polygons are similar if and only if

- all pairs of corresponding angles are equal

AND

- all pairs of corresponding sides are in the same proportion.

Both of these conditions have to be met for two polygons to be similar.

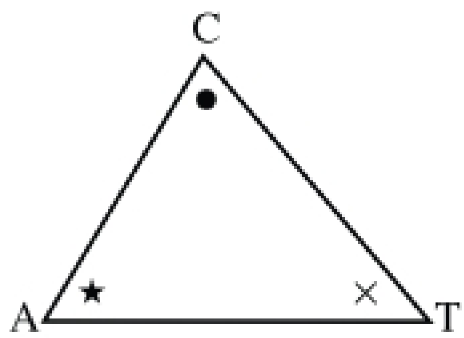


TRIANGLES

For **triangles**, any **one** of the two conditions (equal angles or proportional sides) is sufficient to guarantee similarity. There are theorems which state that if any one of the conditions apply, then the other condition will automatically apply as well. This is **only** true for **triangles**. For all other polygons, the two conditions for similarity have to be established separately.

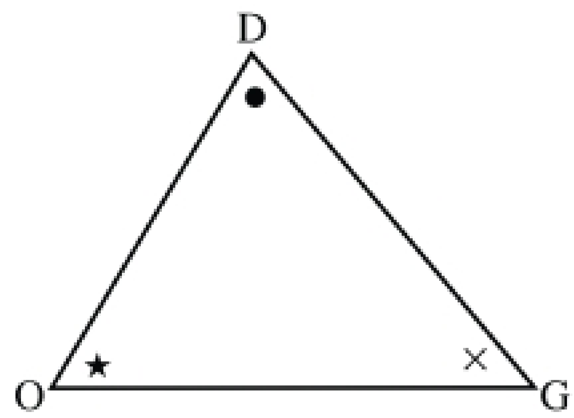
NOTATION

$\Delta CAT \,|||\, \Delta DOG$ means "triangle CAT is similar to triangle DOG". In this notation, the **order** in which the letters are written is important, as it indicates which **angles** are equal:



$$\Delta CAT \,|||\, \Delta DOG$$

$$\hat{C} = \hat{D}$$
$$\hat{A} = \hat{O}$$
$$\hat{T} = \hat{G}$$



The order also indicates which **ratios of sides** are equal:

$$\Delta CAT \,|||\, \Delta DOG$$

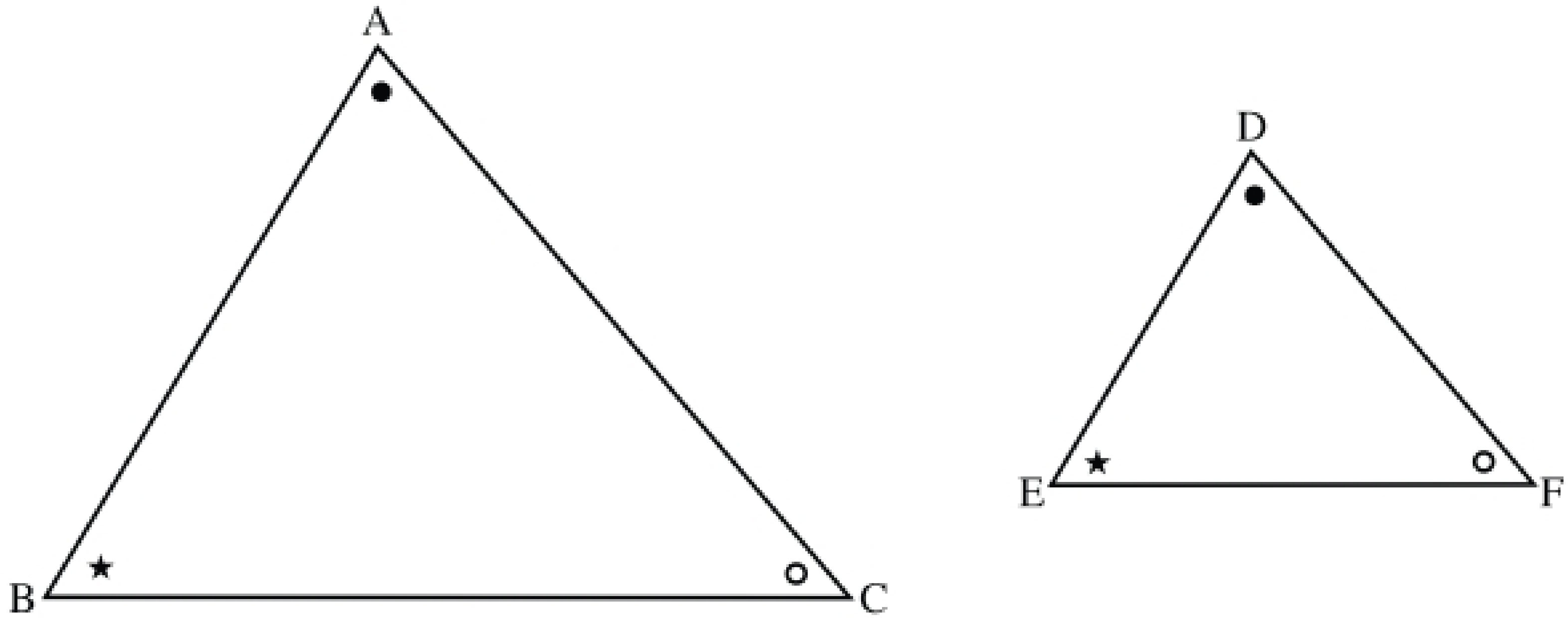
$$\frac{CA}{DO} = \frac{CT}{DG} = \frac{AT}{OG}$$

$$\frac{CA}{AT} = \frac{DO}{OG} \quad \text{and} \quad \frac{CA}{CT} = \frac{DO}{DG} \quad \text{and} \quad \frac{AT}{CT} = \frac{OG}{DG}$$

THE TRIANGLE SIMILARITY THEOREM

THEOREM 2

If two triangles are equiangular, then their corresponding sides are in the same proportion and hence the triangles are similar.



Given: ΔABC and ΔDEF with $\hat{A}=\hat{D}$, $\hat{B}=\hat{E}$ and $\hat{C}=\hat{F}$.

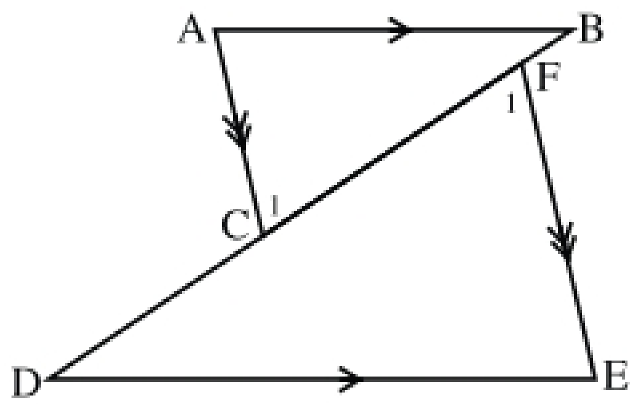
Conclusion: $\frac{AB}{DE}=\frac{AC}{DF}=\frac{BC}{EF}$ and hence $\Delta ABC \mathbin{|||} \Delta DEF$. **Reason:** $\angle\angle\angle$

USING ANGLES TO PROVE THAT TRIANGLES ARE SIMILAR

We can prove that two triangles are similar by showing that all the pairs of corresponding angles are equal. If we can show that any **two pairs** of corresponding angles are equal, then the **third pair** of angles will be equal as well (since the three angles of both triangles must add up to $180°$):

EXAMPLE 14

In the sketch alongside, $AB \parallel DE$ and $AC \parallel FE$.



Prove that
(a) $\Delta BCA \mathbin{|||} \Delta DFE$
(b) $AB \cdot EF = AC \cdot ED$

Solution

(a) In ΔBCA and ΔDFE:

$\hat{B} = \hat{D}$ (alt $\angle$s ; $AB \parallel DE$)

$\hat{C}_1 = \hat{F}$ (alt $\angle$s ; $AC \parallel FE$)

$\hat{A} = \hat{E}$ (3^{rd} $\angle$ of Δ)

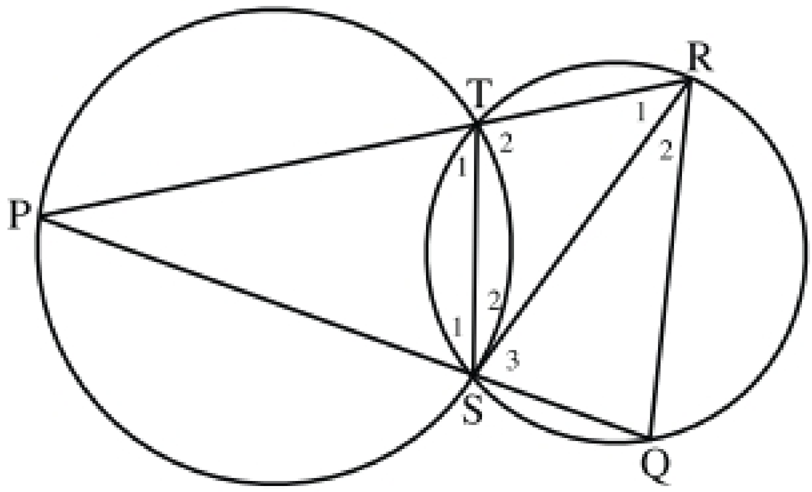
$\therefore \Delta BCA \mathbin{|||} \Delta DFE$ ($\angle\angle\angle$)

(b) $\frac{BC}{DF}=\boxed{\frac{CA}{FE}}=\boxed{\frac{BA}{DE}}$ ($|||$ Δs)

$\frac{AC}{EF}=\frac{AB}{ED}$ $\therefore AB \cdot EF = AC \cdot ED$

EXAMPLE 15

In the sketch alongside, SR is a tangent to circle PST at S.



Prove that

(a) $\Delta PRS\ |||\ \Delta SRT$

(b) $RS^2 = PR \cdot RT$

(c) $\Delta PQR\ |||\ \Delta PTS$

(d) $\dfrac{RT}{PT} = \dfrac{RS^2}{PQ \cdot PS}$

Solution

(a) In ΔPRS and ΔSRT:

$$\hat{P} = \hat{S}_2 \quad \text{(tan chord thm)}$$
$$\hat{R}_1 = \hat{R}_1 \quad \text{(common)}$$
$$\hat{S}_1 + \hat{S}_2 = \hat{T}_2 \quad (3^{rd} \angle \text{ of } \Delta)$$
$$\therefore \Delta PRS\ |||\ \Delta SRT \quad (\angle\angle\angle)$$

(b) $$\boxed{\frac{PR}{SR}} = \boxed{\frac{RS}{RT}} = \frac{PS}{ST} \quad (|||\ \Delta s)$$

$$\frac{PR}{RS} = \frac{RS}{RT}$$
$$\therefore RS^2 = PR \cdot RT$$

(c) In ΔPQR and ΔPTS:

$$\hat{P} = \hat{P} \quad \text{(common)}$$
$$\hat{Q} = \hat{T}_1 \quad \text{(ext } \angle \text{ of cyclic quad)}$$
$$\hat{R}_1 + \hat{R}_2 = \hat{S}_1 \quad (3^{rd} \angle \text{ of } \Delta)$$
$$\therefore \Delta PQR\ |||\ \Delta PTS \quad (\angle\angle\angle)$$

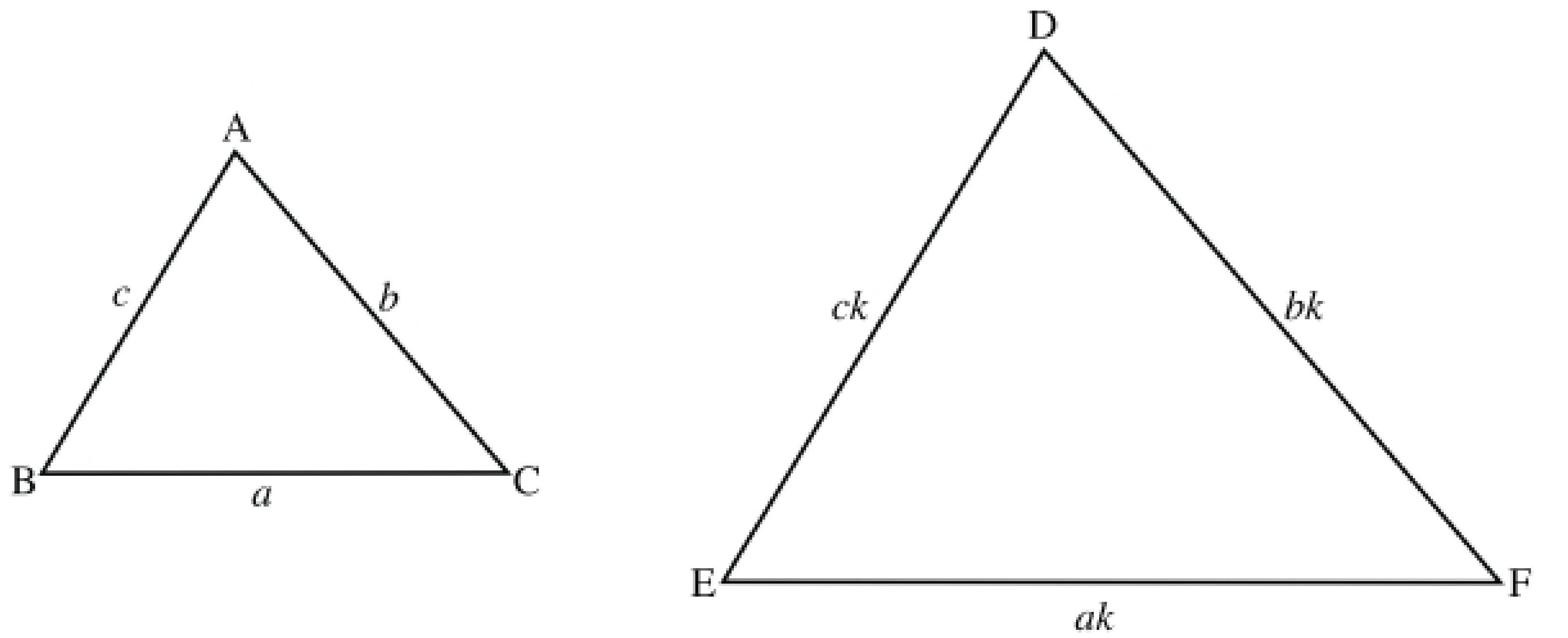
(d) $$\boxed{\frac{PQ}{PT}} = \frac{QR}{TS} = \boxed{\frac{PR}{PS}}$$

$$\frac{PQ}{PT} = \frac{PR}{PS}$$
$$\therefore PR = \frac{PQ \cdot PS}{PT} \quad \text{and} \quad RS^2 = PR \cdot RT \quad \text{(from (b))}$$
$$\therefore RS^2 = \frac{PQ \cdot PS}{PT} \cdot RT$$
$$\therefore \frac{RT}{PT} = \frac{RS^2}{PQ \cdot PS}$$

USING SIDES TO PROVE THAT TRIANGLES ARE SIMILAR

CONVERSE OF THEOREM 2

If the corresponding sides of two triangles are in the same proportion, then the triangles are equiangular and hence similar.



Given: ΔABC and ΔDEF with $\dfrac{AB}{DE} = \dfrac{AC}{DF} = \dfrac{BC}{EF}$.

Conclusion: $\hat{A} = \hat{D},\ \hat{B} = \hat{E},\ \hat{C} = \hat{F}$ and hence $\Delta ABC \,|||\, \Delta DEF$. **Reason:** sides of Δs in prop

We can prove that two triangles are similar by showing that **all three** pairs of corresponding sides are in the same proportion:

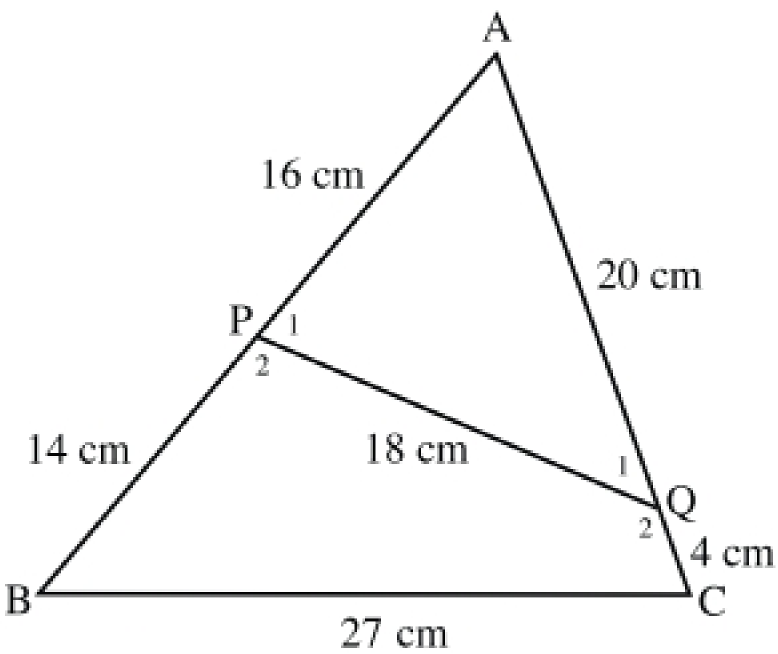
EXAMPLE 16

In the sketch alongside, $AP = 16$ cm, $PB = 14$ cm, $AQ = 20$ cm, $QC = 4$ cm and $BC = 27$ cm.

Prove that

(a) $\Delta APQ \,|||\, \Delta ACB$

(b) PBCQ is a cyclic quadrilateral.



Solution

(a) $\dfrac{AP}{AC} = \dfrac{16 \text{ cm}}{24 \text{ cm}} = \dfrac{2}{3}$

$\dfrac{PQ}{CB} = \dfrac{18 \text{ cm}}{27 \text{ cm}} = \dfrac{2}{3}$

$\dfrac{AQ}{AB} = \dfrac{20 \text{ cm}}{30 \text{ cm}} = \dfrac{2}{3}$

$\therefore \dfrac{AP}{AC} = \dfrac{PQ}{CB} = \dfrac{AQ}{AB}$

$\therefore \Delta APQ \,|||\, \Delta ACB$ (sides of Δs in prop)

(b)	$\hat{P}_1 = \hat{C}$	$(\|\|\|\ \Delta s)$
	$\therefore$ PBCQ is a cyclic quad.	(ext $\angle$ of quad = opp int $\angle$)

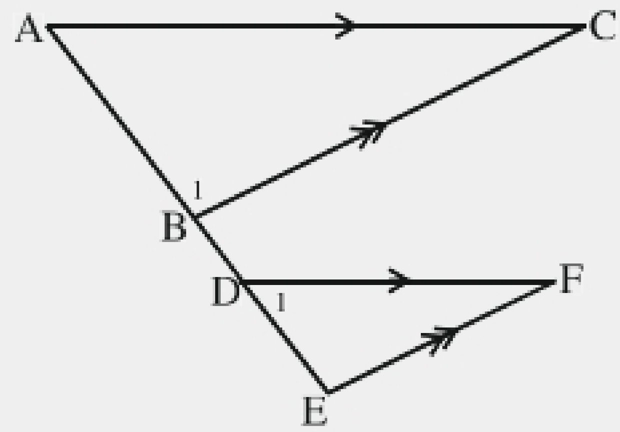
EXERCISE 5

(a) In the sketch alongside, $AC \parallel DF$ and $BC \parallel EF$.

Prove that

(1) $\Delta ABC\ |||\ \Delta DEF$

(2) $AB \cdot EF = BC \cdot DE$

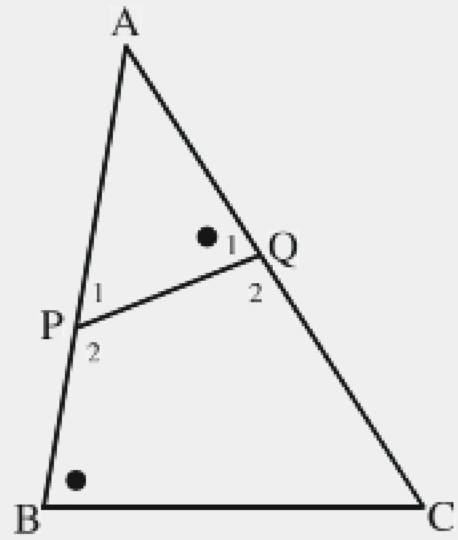


(b) In the sketch alongside, $\hat{B} = \hat{Q}_1$.

(1) Prove that $\Delta ABC\ |||\ \Delta AQP$.

(2) Prove that $AB \cdot AP = AQ \cdot AC$.

(3) If $AP = 8$ cm, $AQ = 6$ cm and $QC = 10$ cm, calculate the length of AB.

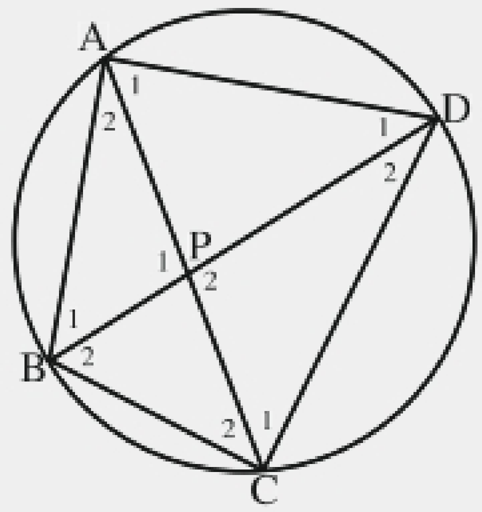


(c) In the sketch alongside, A, B, C and D are four points on the circumference of the circle.

Prove that

(1) $\Delta APB\ |||\ \Delta DPC$

(2) $DC \cdot BP = AB \cdot CP$

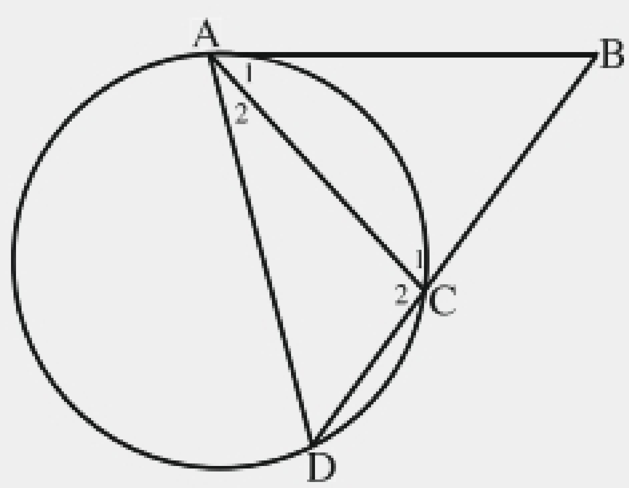


(d) In the sketch alongside, AB is a tangent to the circle at A.

(1) Prove that $\Delta ABC\ |||\ \Delta DBA$.

(2) Prove that $AB^2 = BD \cdot BC$.

(3) If $AB = 12$ m, $CD = 7$ m and $BC = x$, determine the value of x.

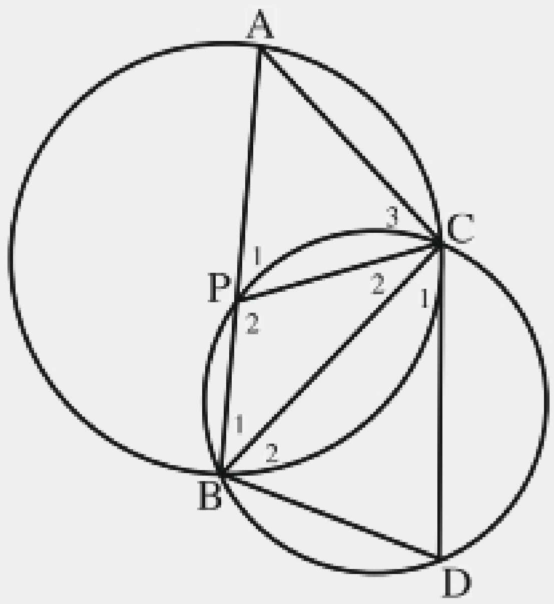


(e) In the sketch alongside, CD is a tangent to circle ACB at C.

Prove that

(1) $\Delta APC\ |||\ \Delta CDB$

(2) $AP \cdot BC = AC \cdot CD$

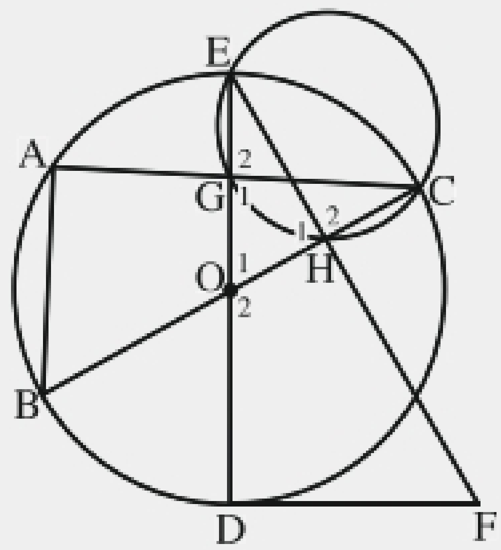


(f) In the sketch alongside, O is the centre of the larger circle. DF is a tangent to the larger circle at D.

Prove that

(1) $\Delta ABC\ |||\ \Delta DFE$

(2) $\Delta CGO\ |||\ \Delta EHO$



(g) In the sketch alongside, AB is a tangent to the circle at B and CE is a tangent to the circle at C. $AC \parallel BE$.

Prove that

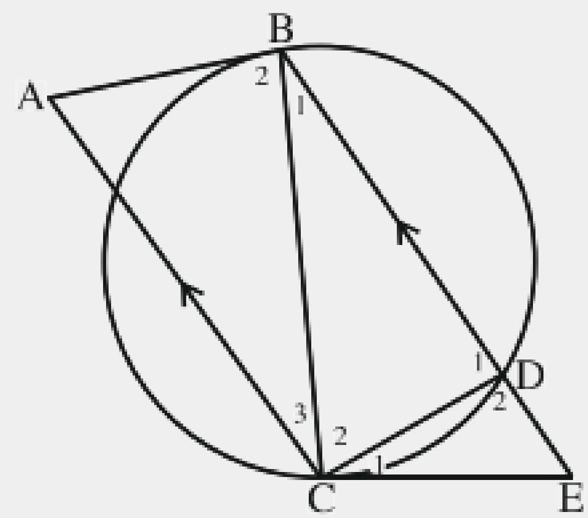
(1) $\Delta ABC\ |||\ \Delta CDB$

(2) $CD \cdot BC = AB \cdot BD$

(3) $\Delta CDE\ |||\ \Delta BCE$

(4) $BC \cdot CE = CD \cdot BE$

(5) $\dfrac{BC}{CE} = \dfrac{AC \cdot BD}{CD \cdot BE}$

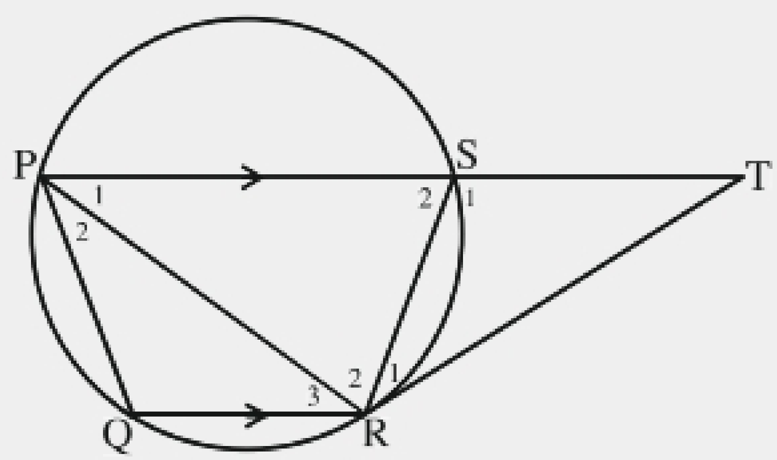


(h) In the sketch alongside, RT is a tangent to the circle at R. $PT \parallel QR$.

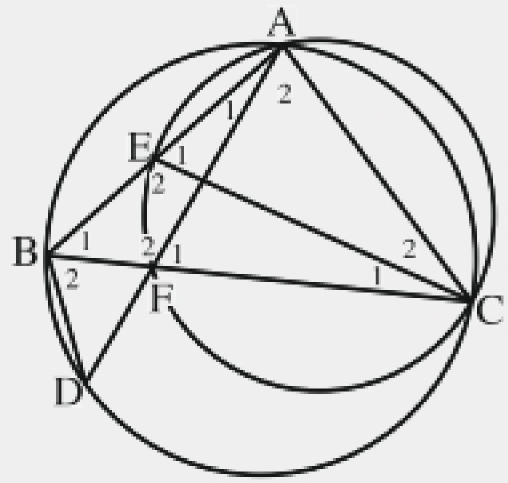
Prove that

(1) $\Delta PQR\ |||\ \Delta TSR$

(2) $PR \cdot SR = TR \cdot QR$



(i) In the sketch alongside, $AB = AC$.



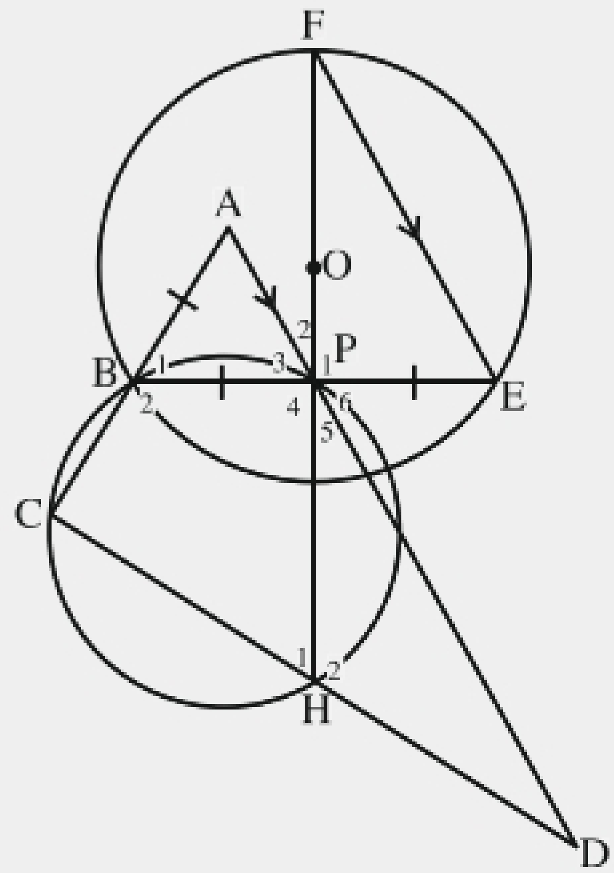
Prove that

(1) $\Delta ABD\ |||\ \Delta CEB$

(2) $\dfrac{AC}{CE} = \dfrac{AD}{BC}$

(3)* $BD \cdot CE - AE \cdot BE = BE^2$

(j) In the sketch alongside, O is the centre of the larger circle. $AB = BP = PE$ and $AD \parallel FE$.



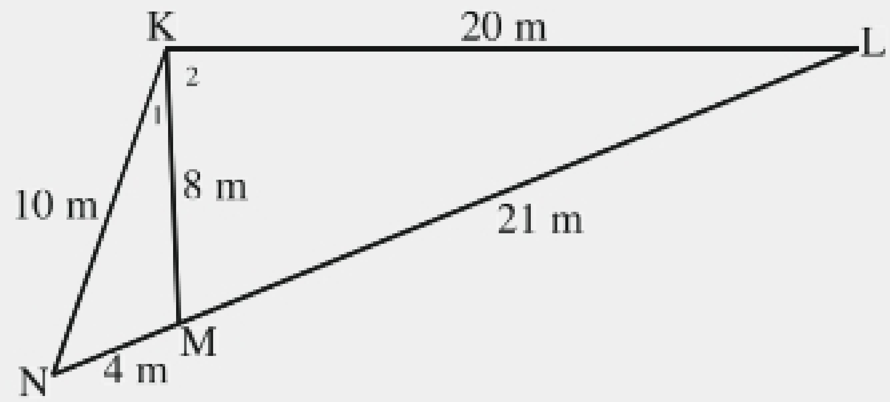
Prove that

(1) $\Delta ACD\ |||\ \Delta EPF$

(2) $AC \cdot EF = AD \cdot EP$

(3)* $AD - EF = \dfrac{BC \cdot EF}{AB}$

(k) In the sketch alongside, $KL = 20$ m, $KN = 10$ m, $MN = 4$ m, $KM = 8$ m and $LM = 21$ m.

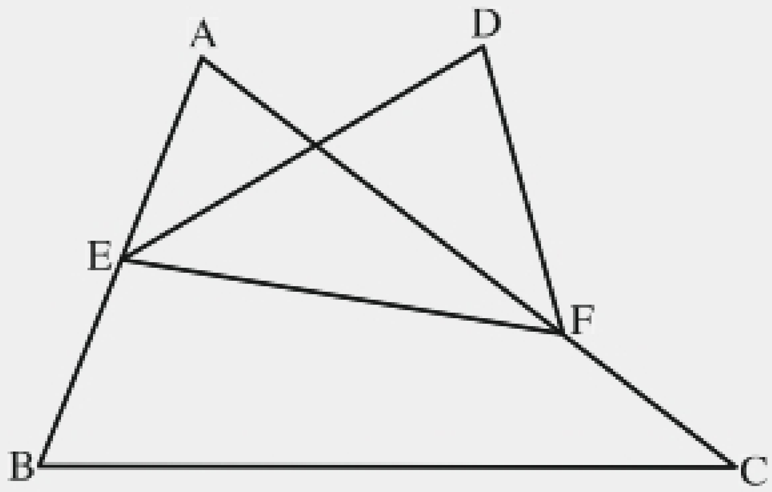


Prove that

(1) $\Delta KMN\ |||\ \Delta LKN$

(2) KN is a tangent to circle LKM at K.

(l) In the sketch alongside, $AB \cdot EF = BC \cdot DF$ and $AB \cdot DE = AC \cdot DF$.



Prove that

(1) $\Delta ABC\ |||\ \Delta DFE$

(2) ADFE is a cyclic quadrilateral.

IDENTIFYING TRIANGLES

We are sometimes required to prove the equality of ratios and/or products, where the question doesn't state which triangles to prove similar. In such cases we have to identify the triangles first.

Suppose, for example, you have to prove that $\frac{AB}{AC}=\frac{BC}{CD}$. There are two possible ways of identifying triangles in order to prove that these ratios are equal:

1. Top triangle, bottom triangle

$$\frac{\overset{\Delta ABC}{\overline{\quad}}}{}\qquad \overline{\underline{\frac{AB}{AD}=\frac{BC}{CD}}}\qquad \Delta ACD$$

$$\begin{array}{c}\Delta ABC\\ \hline \dfrac{AB}{AD}=\dfrac{BC}{CD}\\ \hline \Delta ACD\end{array}$$

Are the top sides (AB and BC) sides of one triangle and the bottom sides (AD and CD) sides of another triangle?

If each of these pairs are sides of a triangle in the sketch, then you can proceed to try proving these triangles similar.

2. Left triangle, right triangle

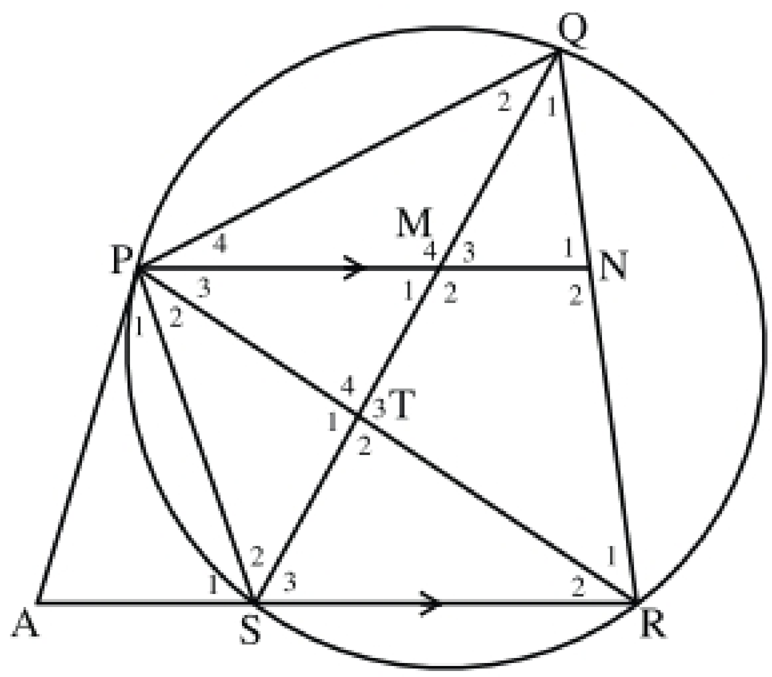
$$\Delta ABD\left|\frac{AB}{AD}=\frac{BC}{CD}\right|\Delta BCD$$

Are the left sides (AB and AD) sides of one triangle and the right sides (BC and CD) sides of another triangle?

If each of these pairs are sides of a triangle in the sketch, then you can proceed to try proving these triangles similar.

EXAMPLE 17

In the sketch alongside, AP is a tangent to the circle at P. $PN \parallel SR$.



Prove that

(a) $\frac{PS}{QR}=\frac{ST}{RT}$

(b) $\frac{PQ}{PT}=\frac{SR}{ST}$

(c) $PM\cdot ST=RS\cdot MT$

(d) $AP^2=AR\cdot AS$

Solution

(a) **PLANNING**

Top triangle, bottom triangle:

$$\begin{array}{c}\Delta PST\checkmark\\ \hline \dfrac{PS}{QR}=\dfrac{ST}{RT}\\ \hline \Delta QRT\checkmark\end{array}$$

ΔPST and ΔQRT are both triangles in the sketch, hence we will attempt to prove that ΔPST and ΔQRT are similar.

In ΔPST and ΔQRT:

$$\hat{P}_2 = \hat{Q}_1 \quad (\angle\text{s in same segment})$$
$$\hat{S}_2 = \hat{R}_1 \quad (\angle\text{s in same segment})$$
$$\hat{T}_1 = \hat{T}_3 \quad (3^{rd} \angle \text{ of } \Delta)$$
$$\therefore \Delta PST \; ||| \; \Delta QRT \quad (\angle\angle\angle)$$

Pay attention to the order of letters!!!

$$\therefore \frac{PS}{QR} = \frac{ST}{RT} \quad (|||\ \Delta s)$$

(b) **PLANNING**

Top triangle, bottom triangle:

$\Delta PQRS$??? ✗

$$\frac{PQ}{PT} = \frac{SR}{ST}$$

ΔPST ✓

The top sides don't give a triangle in the sketch.

Left triangle, right triangle:

ΔPQT ✓ $\left|\dfrac{PQ}{PT} = \dfrac{SR}{ST}\right|$ ΔRST ✓

ΔPQT and ΔRST are both triangles in the sketch, hence we will attempt to prove that ΔPQT and ΔRST are similar.

In ΔPQT and ΔRST:

$$\hat{P}_3 + \hat{P}_4 = \hat{S}_3 \quad (\angle\text{s in same segment})$$
$$\hat{Q}_2 = \hat{R}_2 \quad (\angle\text{s in same segment})$$
$$\hat{T}_4 = \hat{T}_2 \quad (3^{rd} \angle \text{ of } \Delta)$$
$$\therefore \Delta PQT \; ||| \; \Delta SRT \quad (\angle\angle\angle)$$

Pay attention to the order of letters!!!

$$\therefore \frac{PQ}{PT} = \frac{SR}{ST} \quad (|||\ \Delta s)$$

(c) **PLANNING**

Rewrite as an equality of <u>ratios</u>:

$$\underline{PM} \cdot \boxed{ST} = \boxed{RS} \cdot \underline{MT} \quad \rightarrow \quad \frac{PM}{RS} = \frac{MT}{ST}$$

Top triangle, bottom triangle:

ΔMPT ✓

$$\frac{PM}{RS} = \frac{MT}{ST}$$

ΔRST ✓

Prove that ΔMPT and ΔRST are similar.

In ΔMPT and ΔRST:

$$
\begin{array}{lcll}
\hat{P}_3 & = & \hat{R}_2 & \text{(alt } \angle\text{s ; PN} \parallel \text{SR)} \\
\hat{M}_1 & = & \hat{S}_3 & \text{(alt } \angle\text{s ; PN} \parallel \text{SR)} \\
\hat{T}_4 & = & \hat{T}_2 & (3^{\text{rd}}\ \angle \text{ of } \Delta) \\
\therefore \Delta PMT & ||| & \Delta RST & (\angle\angle\angle)
\end{array}
$$

$$\therefore \frac{PM}{RS} = \frac{MT}{ST} \qquad (|||\ \Delta s)$$

$$\therefore PM \cdot ST = RS \cdot MT$$

(d) **PLANNING**

Rewrite the square as a product:

$$AP^2 = AR \cdot AS$$
$$\therefore AP \cdot AP = AR \cdot AS$$

Rewrite as an equality of <u>ratios</u>:

$$\underline{AP} \cdot \boxed{AP} = \boxed{AR} \cdot \underline{AS} \quad \rightarrow \quad \frac{AP}{AR} = \frac{AS}{AP}$$

Top triangle, bottom triangle:

$$\begin{array}{c} \Delta APS ✓ \\ \hline \dfrac{AP}{AR} = \dfrac{AS}{AP} \\ \hline \Delta APR ✓ \end{array}$$

Prove that ΔAPS and ΔAPR are similar.

In ΔAPS and ΔAPR:

$$
\begin{array}{lcll}
\hat{P}_1 & = & \hat{R}_2 & \text{(tan chord thm)} \\
\hat{A} & = & \hat{A} & \text{(common)} \\
\hat{S}_1 & = & \hat{P}_1 + \hat{P}_2 & (3^{\text{rd}}\ \angle \text{ of } \Delta) \\
\therefore \Delta PAS & ||| & \Delta RAP & (\angle\angle\angle)
\end{array}
$$

$$\therefore \frac{PA}{RA} = \frac{AS}{AP} \qquad (|||\ \Delta s)$$

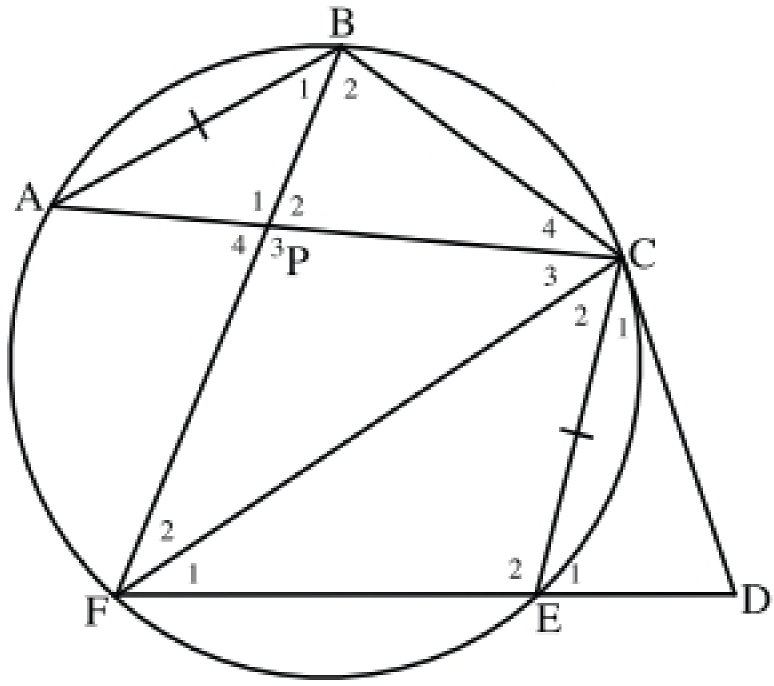
$$\therefore AP^2 = AR \cdot AS$$

The strategy that we have used in the previous example doesn't always work immediately. It is often necessary to replace one of the sides with a side of equal length:

EXAMPLE 18

In the sketch alongside, CD is a tangent to the circle at C. $AB = CE$.

Prove that $\frac{AB}{CB} = \frac{ED}{BP}$.



Solution

PLANNING

Top triangle, bottom triangle:

ΔABDE??? ✘

$\frac{AB}{CB} = \frac{ED}{BP}$

ΔBCP ✓

Left triangle, right triangle:

ΔABC ✓ $\left| \frac{AB}{CB} = \frac{ED}{BP} \right|$ ΔBDEP??? ✘

Neither gives two triangles on the sketch!

Replace a side:

Replace AB by CE, since $AB = CE$ (given): $\frac{CE}{CB} = \frac{ED}{BP}$

Try again:

ΔCDE ✓

Top triangle, bottom triangle: $\frac{CE}{CB} = \frac{ED}{BP}$ Prove that ΔCDE and ΔBCP are similar.

ΔBCP ✓

In ΔCDE and ΔBCP:

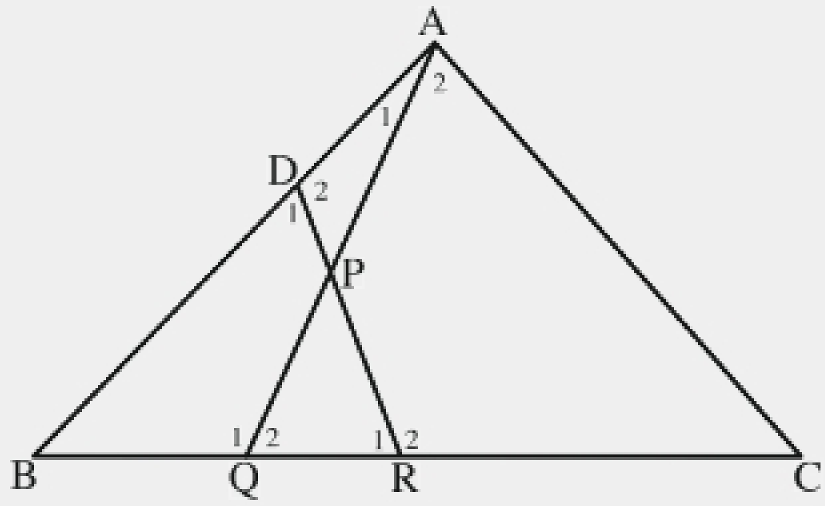
$\hat{C}_1$	$=$	$\hat{F}_1$	(tan chord thm)	
$\hat{F}_1$	$=$	$\hat{C}_4$	(equal chords $\rightarrow$ equal $\angle$s)	
$\therefore \hat{C}_1$	$=$	$\hat{C}_4$		$\angle$
$\hat{E}_1$	$=$	$\hat{B}_2$	(ext $\angle$ of cyclic quad)	$\angle$
$\hat{D}$	$=$	$\hat{P}_2$	(3^{rd} $\angle$ of Δ)	$\angle$
$\therefore \Delta CED$	$\vert\vert\vert$	ΔCBP	($\angle\angle\angle$)	

$\therefore \frac{CE}{CB} = \frac{ED}{BP}$ ($\vert\vert\vert$ Δs)

$\therefore \frac{AB}{CB} = \frac{ED}{BP}$ ($CE = AB$)

EXERCISE 6

(a) In the sketch alongside, $AB = AC$ and $PQ = PR$. Prove that $\dfrac{BD}{AC} = \dfrac{DR}{AQ}$.



(b) In the diagram alongside, KLMN is a parallelogram. $\hat{K}_1 = \hat{K}_3$.

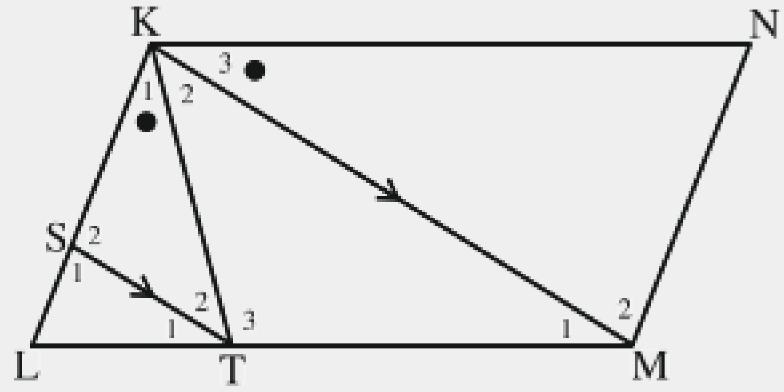
Prove that

(1) $\dfrac{KL}{KT} = \dfrac{KN}{KM}$

(2) $KN \cdot ST = KM \cdot LT$

(3) $KT^2 = MK \cdot TS$

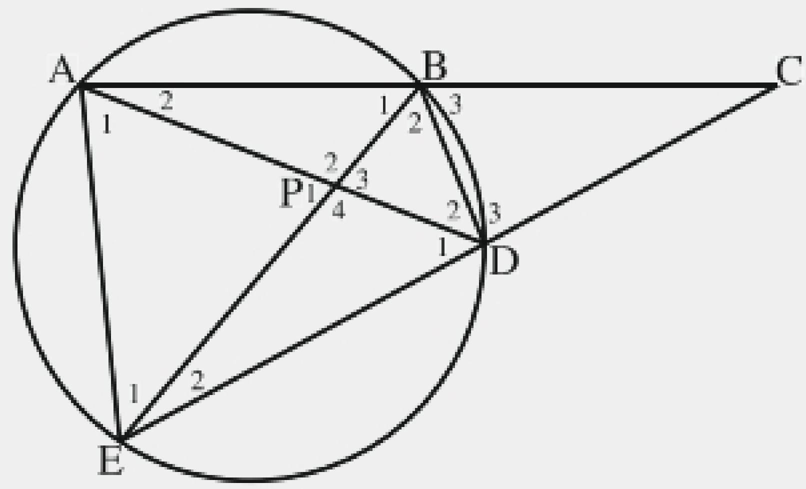
(4) $\dfrac{KN}{LT} = \left(\dfrac{KT}{ST}\right)^2$



(c) In the sketch alongside, prove that

(1) $\dfrac{AB}{ED} = \dfrac{AP}{EP}$

(2) $AE \cdot CD = AC \cdot BD$



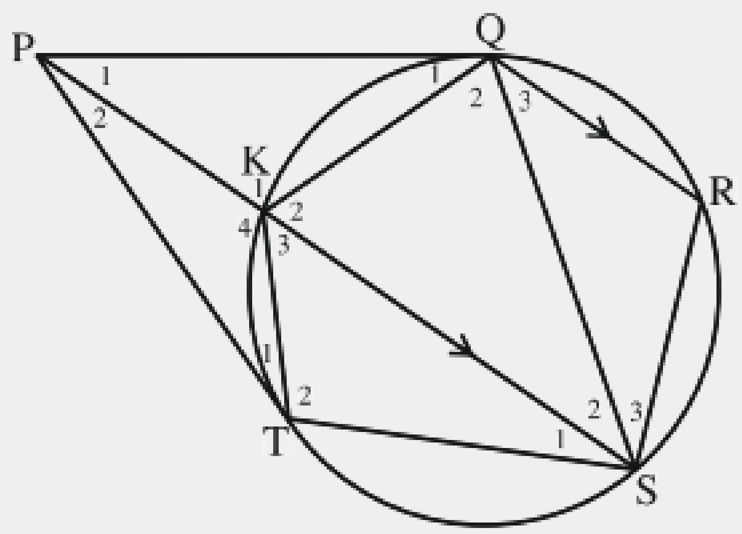
(d) In the sketch alongside, PQ and PT are both tangents to the circle. $PS \parallel QR$.

Prove that

(1) $\dfrac{PQ}{QS} = \dfrac{RS}{QR}$

(2) $PT^2 = PK \cdot PS$

(3) $KQ \cdot RS = KP \cdot QR$

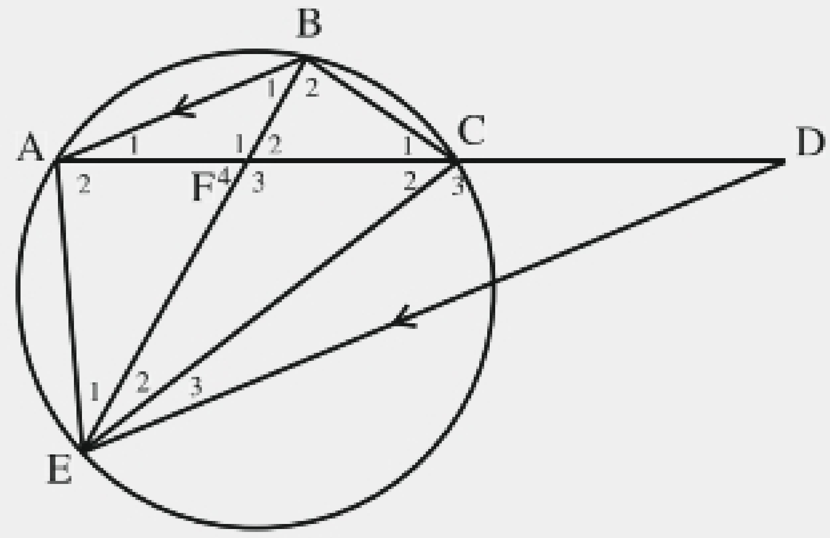


(e) In the sketch alongside, $AB \parallel ED$.

Prove that

(1) $EF^2 = DF \cdot CF$

(2) EF is a tangent to circle CED at E.



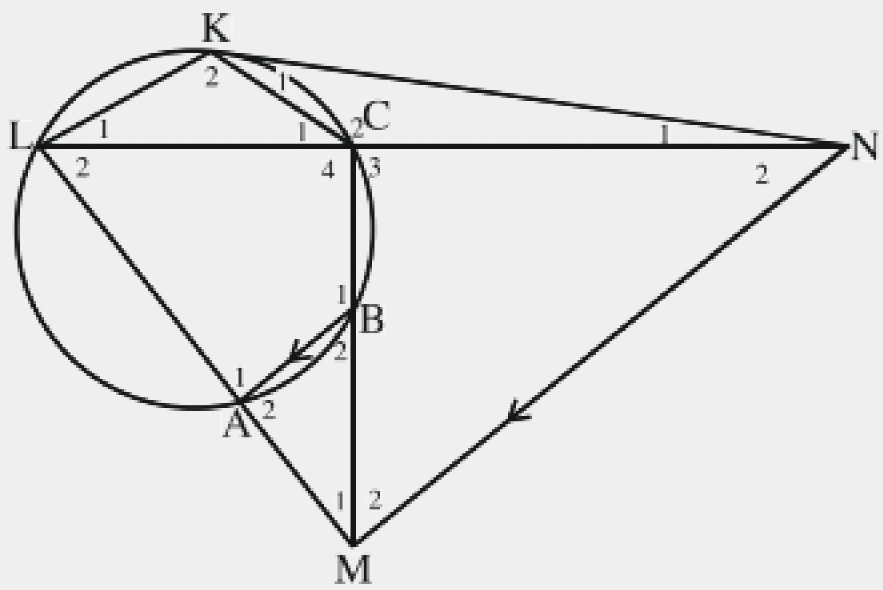
(f) In the sketch alongside, KN is a tangent to the circle at K. $AB \parallel MN$.

Prove that

(1) $\Delta NCM \,|||\, \Delta NML$

(2) $KN^2 = CN \cdot LN$

(3) $KN = MN$

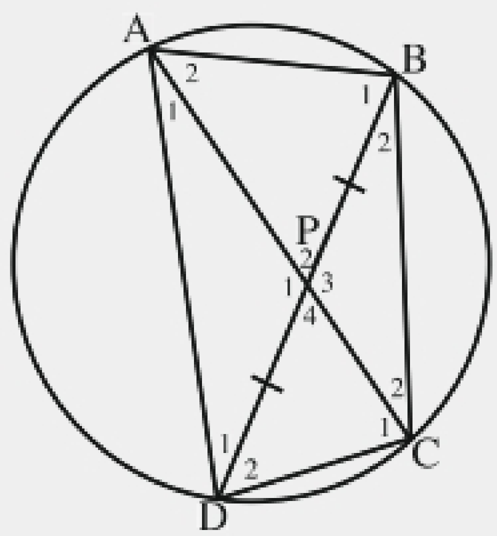


(g) In the sketch alongside, $DP = PB$.

Prove that

(1) $\dfrac{AD}{BC} = \dfrac{BP}{CP}$

(2) $AB \cdot CP = CD \cdot DP$

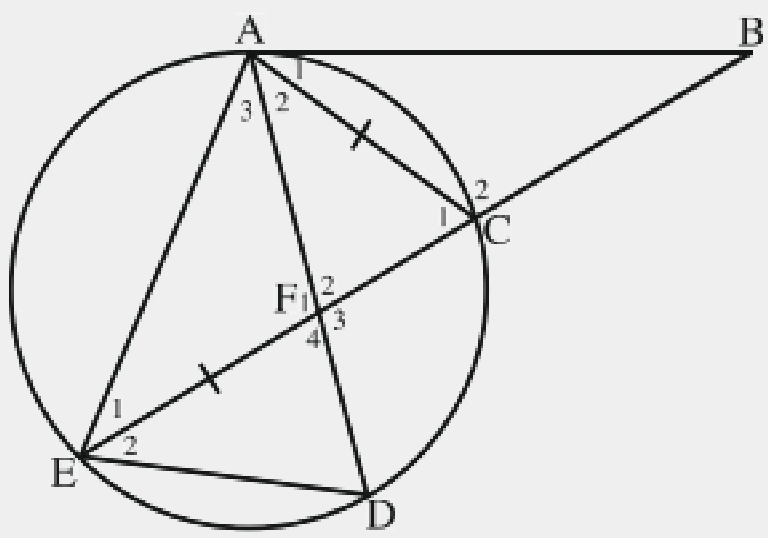


(h) In the sketch alongside, AB is a tangent to the circle at A. $AC = EF$.

Prove that

(1) $AB^2 = EB \cdot BC$

(2) $AC^2 = ED \cdot AF$

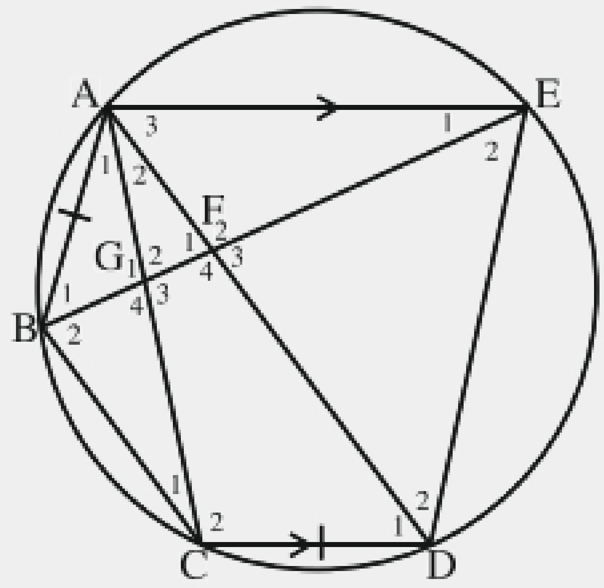


(i) In the sketch alongside, $AE \parallel CD$ and $AB = CD$.

Prove that

(1) $AE \cdot CD = AF \cdot AD$

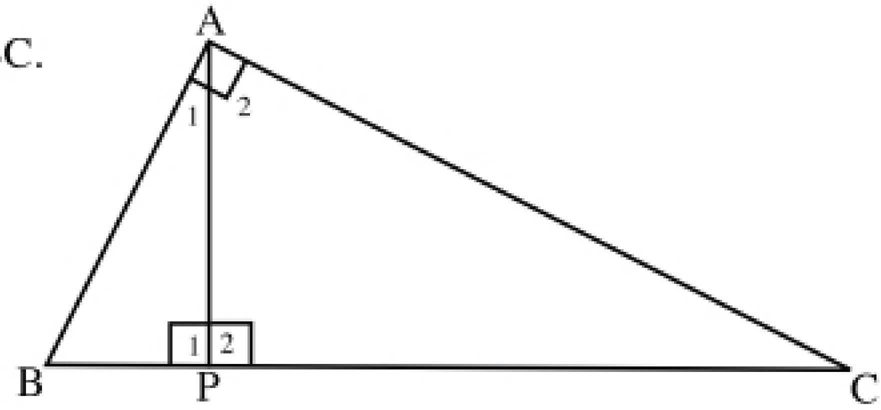
(2) $CD \cdot EF = AF \cdot DE$



RIGHT ANGLED TRIANGLES

EXAMPLE 20

In the sketch alongside, $B\hat{A}C = 90^\circ$ and $AP \perp BC$.



Prove that

(a) $\Delta ABC\ |||\ \Delta PBA$

(b) $\Delta ABC\ |||\ \Delta PAC$

(c) $\Delta PBA\ |||\ \Delta PAC$

Solution

(a) In ΔABC and ΔPBA:

$\hat{A}_1 + \hat{A}_2 = \hat{P}_1 = 90^\circ$ (given)

$\hat{B} = \hat{B}$ (common)

$\hat{C} = \hat{A}_1$ (3^{rd} $\angle$ of Δ)

$\therefore \Delta ABC\ |||\ \Delta PBA$ ($\angle\angle\angle$)

(b) In ΔABC and ΔPAC:

$\hat{A}_1 + \hat{A}_2 = \hat{P}_2 = 90^\circ$ (given)

$\hat{C} = \hat{C}$ (common)

$\hat{B} = \hat{A}_2$ (3^{rd} $\angle$ of Δ)

$\therefore \Delta ABC\ |||\ \Delta PAC$ ($\angle\angle\angle$)

(c) $\Delta PBA\ |||\ \Delta ABC$ (proven)

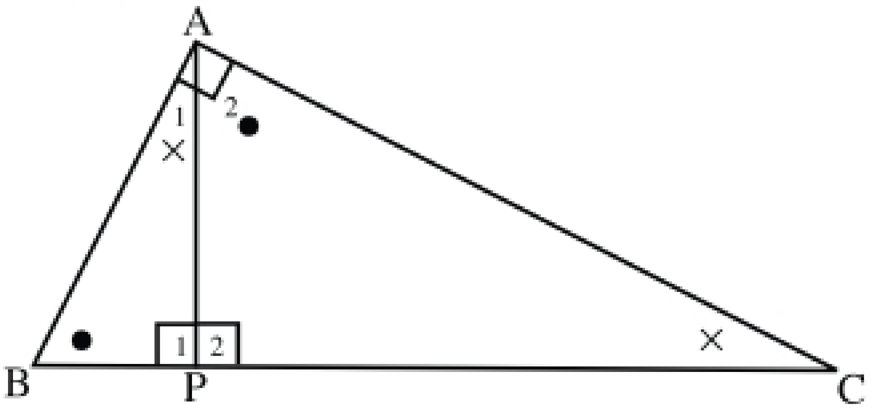
and $\Delta ABC\ |||\ \Delta PAC$ (proven)

$\therefore \Delta PBA\ |||\ \Delta PAC$ (both $|||$ ΔABC)

The results from *Example 20* (above) can be used in proofs and/or calculations involving right angled triangles and are summarised by the following **theorem**:

THEOREM 3

A line drawn from the vertex at the right angle of a right angled triangle, perpendicular to the hypotenuse, divides the triangle into two triangles which are similar to each other and to the original triangle.



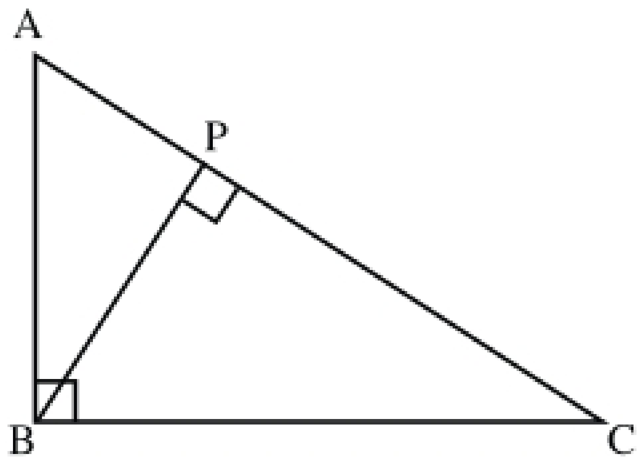
Each triangle has: $90°$, $\bullet$ and $\times$

$\boxed{\bullet}+\boxed{\times}=90°$

$\underset{\urcorner\,\bullet\,\times}{\Delta ABC}\ |||\ \underset{\urcorner\,\bullet\,\times}{\Delta PBA}\ |||\ \underset{\urcorner\,\bullet\,\times}{\Delta PAC}$ **Reason :** line from right $\angle$ vertex $\perp$ hypotenuse

EXAMPLE 21

In the sketch alongside, ΔABC is a right angled triangle with $A\hat{B}C = 90°$. Line BP is drawn such that $BP \perp AC$.



(a) Prove that

(1) $AB^2 = AP \cdot AC$

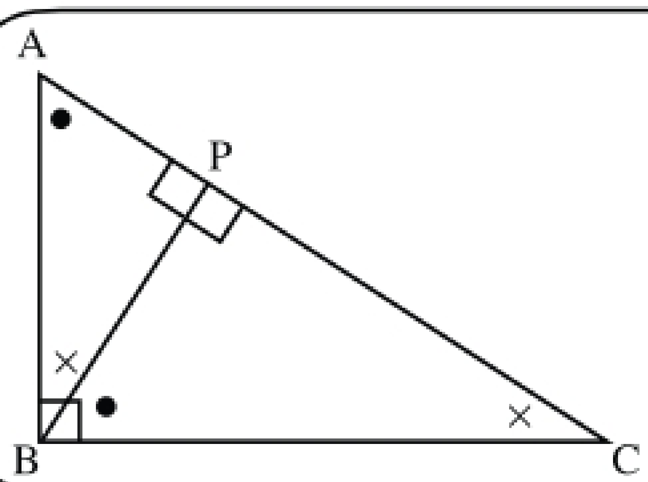
(2) $BC^2 = PC \cdot AC$

(b) Use the results from (a) to prove that

$AB^2 + BC^2 = AC^2$

Solution

(a)



$\underset{\bullet\,\urcorner\,\times}{\Delta ABC}\ |||\ \underset{\bullet\,\urcorner\,\times}{\Delta APB}\ |||\ \underset{\bullet\,\urcorner\,\times}{\Delta BPC}$

(1) $\Delta ABC\ |||\ \Delta APB$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$\therefore \dfrac{AB}{AP} = \dfrac{AC}{AB}$ ($|||\ \Delta$s)

$\therefore AB^2 = AP \cdot AC$

(2) $\Delta ABC\ |||\ \Delta BPC$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$\therefore \dfrac{BC}{PC} = \dfrac{AC}{BC}$ ($|||\ \Delta$s)

$\therefore BC^2 = PC \cdot AC$

In each case, use the two triangles that contain the **squared side**.

(b) $AB^2 + BC^2 = AP \cdot AC + PC \cdot AC$

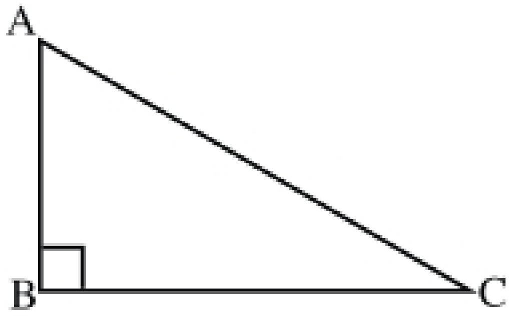
$= AC \cdot (AP + PC)$

$= AC \cdot AC = AC^2$

In *Example 21* above, we have actually proved the well-known **theorem of Pythagoras**:

THEOREM 4 (THE THEOREM OF PYTHAGORAS)

In a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides:

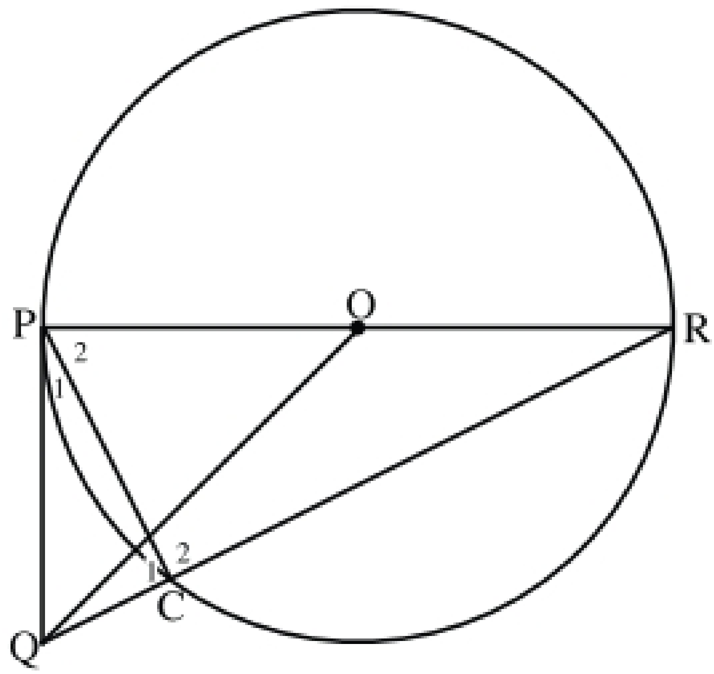


$AC^2 = AB^2 + BC^2$ **Reason:** Pythag

The theorem of Pythagoras is often used in proofs and/or calculations.

EXAMPLE 22

In the sketch alongside, O is the centre of the circle.
PQ is a tangent to the circle at P.



(a) Prove that

(1) $PQ^2 = CQ \cdot QR$

(2) $\dfrac{CP^2}{PQ^2} = \dfrac{CR}{QR}$

(3) $OP^2 = \dfrac{CR \cdot QR}{4}$

It is further given that $QC : CR = 1 : 4$ and $QC = x$.

(b) Express the following lengths in terms of x:

(1) QR (2) CP

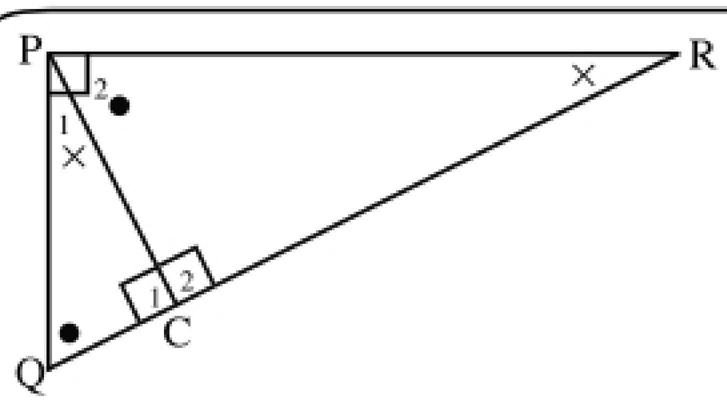
(c) Prove that

(1) $OP = PQ$ (2) $2OQ^2 = CR \cdot QR$

Solution

(a) $\hat{P}_1 + \hat{P}_2 = 90°$ (rad $\perp$ tan)

$\hat{C}_2 = 90°$ ($\angle$ in semi $\odot$)



$\Delta PQR \;|||\; \Delta CQP \;|||\; \Delta CPR$

(1) $\Delta PQR \;|||\; \Delta CQP$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$\therefore \dfrac{PQ}{CQ} = \dfrac{QR}{QP}$ ($|||$ Δs)

$\therefore PQ^2 = CQ \cdot QR$

(2) $\Delta CQP \parallel\!\mid \Delta CPR$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$\therefore \dfrac{CP}{CR} = \dfrac{CQ}{CP}$ ($\parallel\!\mid$ Δs)

$\therefore CP^2 = CR \cdot CQ$ and $PQ^2 = CQ \cdot QR$ (from (a)(1))

$\therefore \dfrac{CP^2}{PQ^2} = \dfrac{CR \cdot CQ}{CQ \cdot QR}$

$\therefore \dfrac{CP^2}{PQ^2} = \dfrac{CR}{QR}$

(3) $\Delta PQR \parallel\!\mid \Delta CPR$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$\therefore \dfrac{PR}{CR} = \dfrac{QR}{PR}$ ($\parallel\!\mid$ Δs)

$\therefore PR^2 = CR \cdot QR$

and $PR = 2OP$ (diameter $= 2 \times$ radius)

$\therefore (2OP)^2 = CR \cdot QR$

$\therefore 4OP^2 = CR \cdot QR$

$\therefore OP^2 = \dfrac{CR \cdot QR}{4}$

(b) (1) $QC : CR = 1:4$

$QC = x \quad \therefore CR = 4x$

$QR = QC + CR = x + 4x$

$\therefore QR = 5x$

(2) $CP^2 = CQ \cdot CR$ (proven)

$= x \cdot 4x$

$= 4x^2$

$\therefore CP = 2x$

(c) $OP^2 = \dfrac{CR \cdot QR}{4}$ (proven)

$= \dfrac{4x \cdot 5x}{4}$

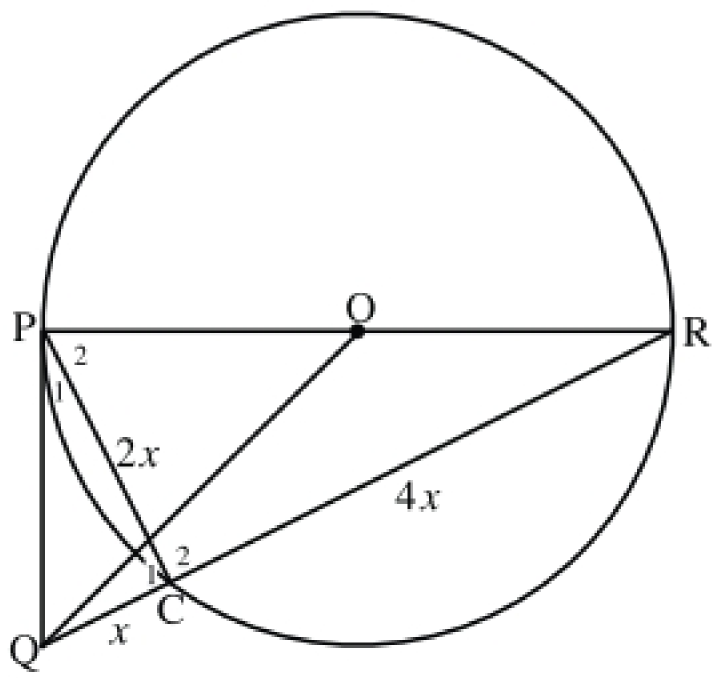
$= 5x^2$

$PQ^2 = CQ \cdot QR$ (proven)

$= x \cdot 5x$

$= 5x^2$

$\therefore OP^2 = PQ^2 \qquad \therefore OP = PQ$



(d) $OQ^2 = OP^2 + PQ^2$ (Pythagoras)

and $PQ^2 = OP^2$ (proven)

$\therefore OQ^2 = OP^2 + OP^2$

$\therefore OQ^2 = 2OP^2$

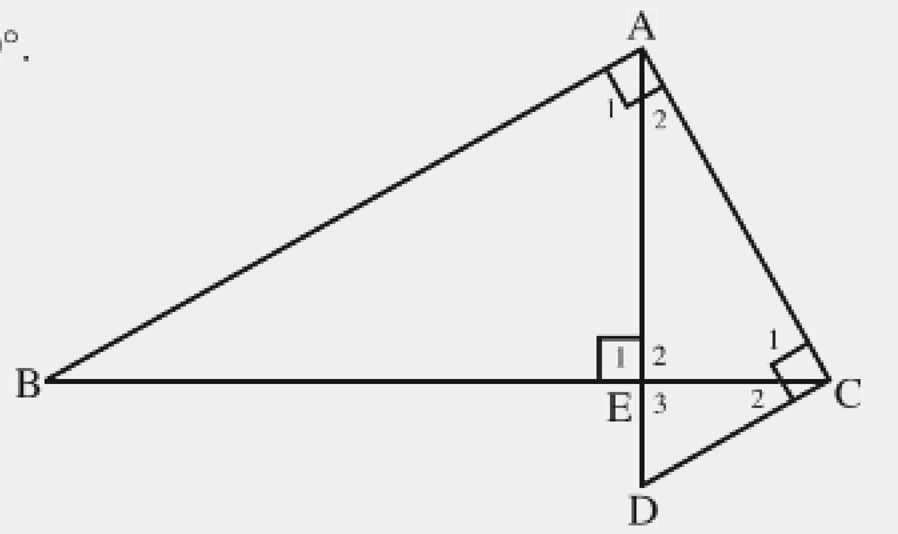
$\therefore 2OQ^2 = 4OP^2$

But $OP^2 = \dfrac{CR \cdot QR}{4}$

$\therefore 2OQ^2 = 4\left(\dfrac{CR \cdot QR}{4}\right) = CR \cdot QR$

EXERCISE 7

(a) In the sketch alongside, $B\hat{A}C = A\hat{C}D = 90°$. AD and BC intersect at E. $AD \perp BC$.



Prove that

(1) $AB \cdot AE = AC \cdot BE$

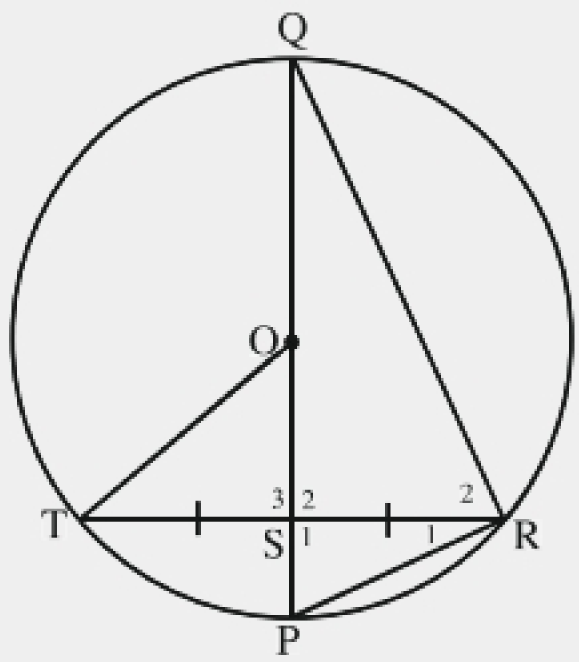
(2) $AC^2 = BC \cdot CE$

(3) $EC^2 = AE \cdot DE$

(4)* $BC \cdot CE = AD \cdot AE$

(5)* $BE \cdot CE + AE \cdot DE = BC \cdot CE$

(b) In the sketch alongside, O is the centre of the circle. $TS = SR$.



(1) Prove that

(i) $PR^2 = PS \cdot PQ$

(ii) $ST^2 = PS \cdot QS$

(iii) $\dfrac{PR^2}{QR^2} = \dfrac{PS}{QS}$

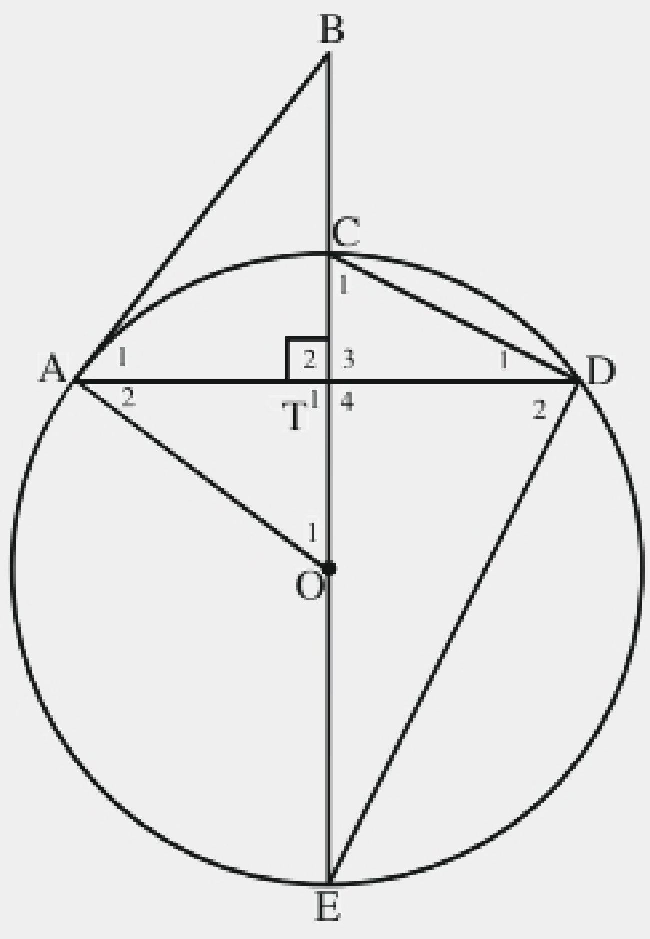
(2) It is further given that $PR = 2\sqrt{5}$ units, $QS = 8$ units and $PS = x$. Use the results in (1) to calculate

(i) the value of x.

(ii) the length of ST.

(iii) the length of QR

(c) In the sketch alongside, O is the centre of the circle. BA is a tangent to the circle at A. $BE \perp AD$.



Prove that

(1) $AB^2 = BT \cdot BO$

(2) $AT^2 = BT \cdot OT$

(3) $\dfrac{AB^2}{OE^2} = \dfrac{BT}{OT}$

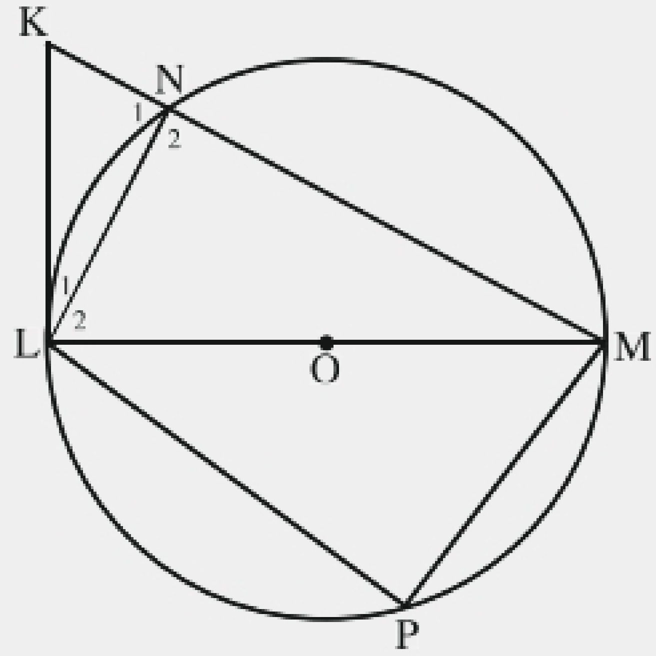
(4) $DT^2 = CT \cdot ET$

(5) $DE^2 = 2OE \cdot ET$

(6) $BT \cdot OT = CT \cdot ET$

(7)* $BC \cdot OT = CT \cdot OE$

(d) In the sketch alongside, O is the centre of the circle. KL is a tangent to the circle at L.



(1) Prove that $LM^2 = MN \cdot MK$

It is further given that $KN : NM = 1:3$ and $KN = x$.

(2) Express the following in terms of x:

(i) KM

(ii) KL

(iii) $LP^2 + MP^2$

(3) Prove that $LN = LO$.

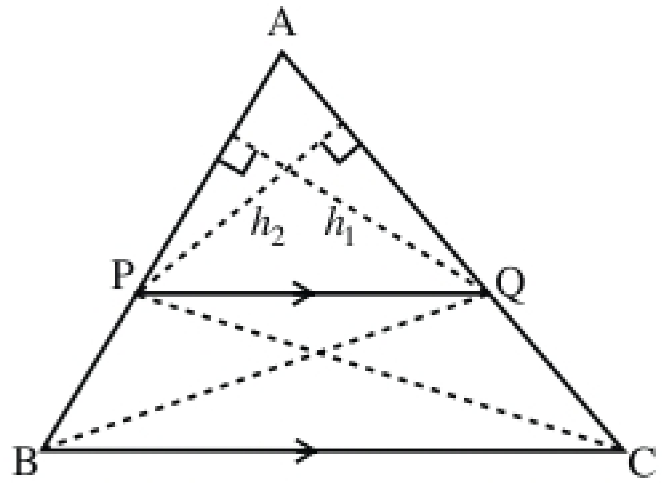
FORMAL PROOFS OF THE THEOREMS

THEOREM 1

A line drawn parallel to one side of a triangle divides the other two sides proportionally.

Given: ΔABC with $PQ \parallel BC$

Required to prove: $\dfrac{AP}{PB} = \dfrac{AQ}{QC}$



Proof

Construction: In ΔAPQ , draw height h_1 with base AP and height h_2 with base AQ. Draw BQ and CP.

$$\frac{\text{Area } \Delta APQ}{\text{Area } \Delta BPQ} = \frac{\frac{1}{2} \cdot AP \cdot h_1}{\frac{1}{2} \cdot PB \cdot h_1} = \frac{AP}{PB}$$

$$\frac{\text{Area } \Delta APQ}{\text{Area } \Delta CQP} = \frac{\frac{1}{2} \cdot AQ \cdot h_2}{\frac{1}{2} \cdot QC \cdot h_2} = \frac{AQ}{QC}$$

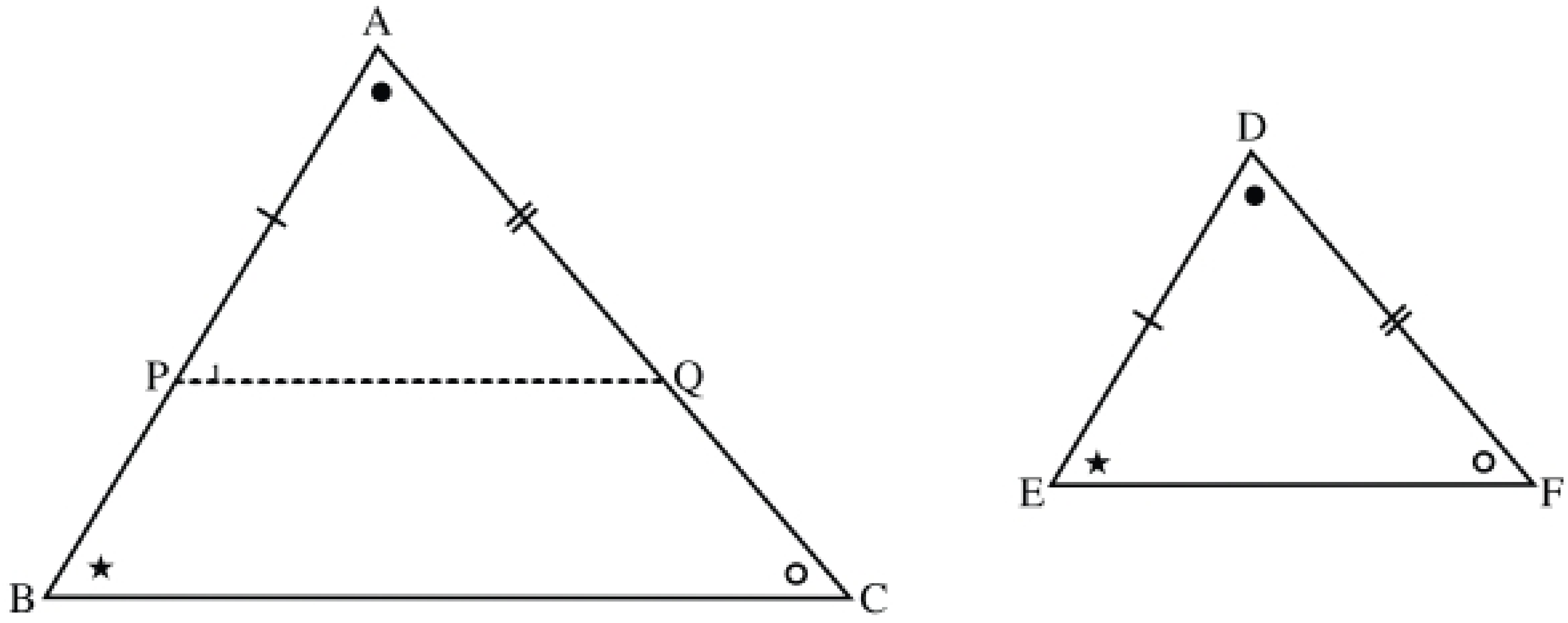
ΔBPQ and ΔCQP have the same base PQ and lie between the same parallel lines PQ and BC, thus Area ΔBPQ = Area ΔCQP (same base; same height)

$$\therefore \frac{\text{Area } \Delta APQ}{\text{Area } \Delta BPQ} = \frac{\text{Area } \Delta APQ}{\text{Area } \Delta CQP}$$

$$\therefore \frac{AP}{PB} = \frac{AQ}{QC}$$

THEOREM 2

If two triangles are equiangular, then their corresponding sides are in the same proportion and hence the triangles are similar.



Given: ΔABC and ΔDEF with $\hat{A} = \hat{D}$, $\hat{B} = \hat{E}$ and $\hat{C} = \hat{F}$.

Required to prove: $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$ and hence $\Delta ABC \mid\mid\mid \Delta DEF$.

Proof

Construction: Mark off P on AB and Q on AC, such that $AP = DE$ and $AQ = DF$. Draw PQ.

In ΔAPQ and ΔDEF:

$\hat{A} = \hat{D}$ (given)

$AP = DE$ (construction)

$AQ = DF$ (construction)

$\therefore \Delta APQ \equiv \Delta DEF$ (S $\angle$ S)

$\therefore \hat{P}_1 = \hat{E}$ ($\equiv$ Δs)

and $\hat{E} = \hat{B}$ (given)

$\therefore \hat{P}_1 = \hat{B}$

$\therefore PQ \parallel BC$ (corresp $\angle$s =)

$\therefore \frac{AB}{AP} = \frac{AC}{AQ}$ (line $\parallel$ side of Δ)

But $AP = DE$ and $AQ = DF$

$\therefore \frac{AB}{DE} = \frac{AC}{DF}$

Similarly, by marking of P on BA and Q on BC, such that $BP = ED$ and $BQ = EF$, it can be shown that $\frac{AB}{DE} = \frac{BC}{EF}$.

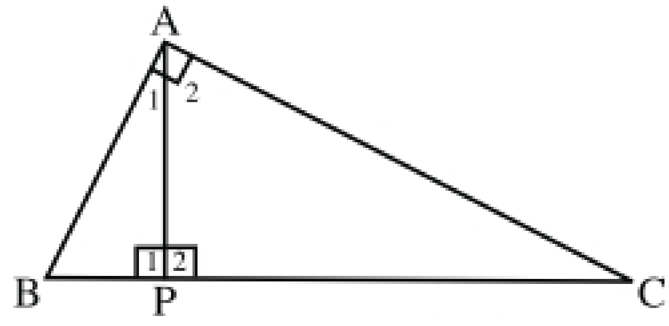
$\therefore \frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$

THEOREM 3

A line drawn from the vertex at the right angle of a right angled triangle, perpendicular to the hypotenuse, divides the triangle into two triangles which are similar to each other and to the original triangle.

Given: ΔABC with $B\hat{A}C = 90°$ and $AP \perp BC$.

Required to prove: $\Delta ABC \,|||\, \Delta PBA \,|||\, \Delta PAC$



Proof

In ΔABC and ΔPBA:

$$\begin{array}{rclll} \hat{A}_1 + \hat{A}_2 & = & \hat{P}_1 = 90° & \text{(given)} \\ \hat{B} & = & \hat{B} & \text{(common)} \\ \hat{C} & = & \hat{A}_1 & (3^{rd} \angle \text{ of } \Delta) \\ \therefore \Delta ABC & ||| & \Delta PBA & (\angle\angle\angle) \end{array}$$

In ΔABC and ΔPAC:

$$\begin{array}{rclll} \hat{A}_1 + \hat{A}_2 & = & \hat{P}_2 = 90° & \text{(given)} \\ \hat{C} & = & \hat{C} & \text{(common)} \\ \hat{B} & = & \hat{A}_2 & (3^{rd} \angle \text{ of } \Delta) \\ \therefore \Delta ABC & ||| & \Delta PAC & (\angle\angle\angle) \end{array}$$

$\therefore \Delta PBA \,|||\, \Delta PAC$ (both $|||\ \Delta ABC$)

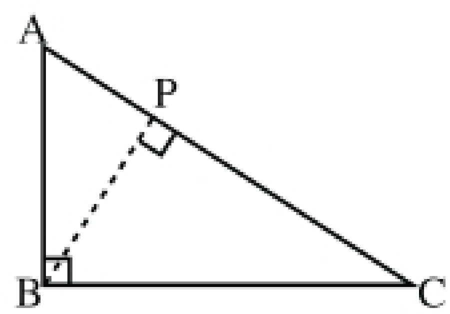
$\therefore \Delta ABC \,|||\, \Delta PBA \,|||\, \Delta PAC$

THEOREM 4 (THE THEOREM OF PYTHAGORAS)

In a right angled triangle the square of the hypotenuse is equal to the sum of the squares of the other two sides:

Given: ΔABC with $A\hat{B}C = 90°$

Required to prove: $AB^2 + BC^2 = AC^2$



Proof

Construction: Draw BP, with P on AC, such that $BP \perp AC$.

$\Delta ABC \,|||\, \Delta APB$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$$\therefore \frac{AB}{AP} = \frac{AC}{AB} \quad (|||\ \Delta s)$$

$\therefore AB^2 = AP \cdot AC$

$\Delta ABC \,|||\, \Delta BPC$ (line from right $\angle$ vertex $\perp$ hypotenuse)

$$\therefore \frac{BC}{PC} = \frac{AC}{BC} \quad (|||\ \Delta s)$$

$\therefore BC^2 = PC \cdot AC$

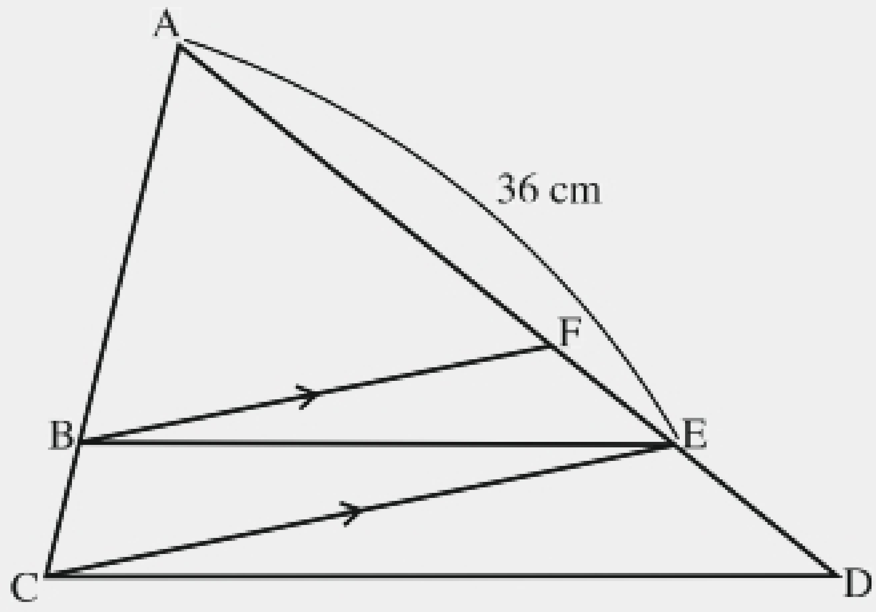
$$\begin{aligned} \therefore AB^2 + BC^2 &= AP \cdot AC + PC \cdot AC \\ &= AC \cdot (AP + PC) \\ &= AC \cdot AC = AC^2 \end{aligned}$$

CONSOLIDATION AND EXTENSION EXERCISE

(a) In the sketch alongside, $BF \parallel CE$.
$AB = 3BC$ and $FE : ED = 3:4$.
$AE = 36$ cm.

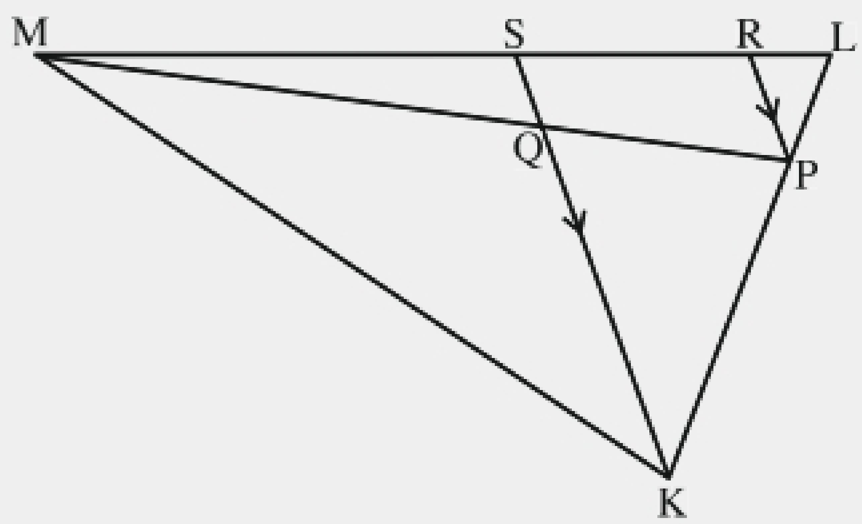
(1) Determine the length of
(i) AF
(ii) ED

(2) Prove that $BE \parallel CD$.



(b) In the sketch alongside, $SK \parallel RP$.
$ML : SL = 5:2$ and $KP : PL = 3:1$.

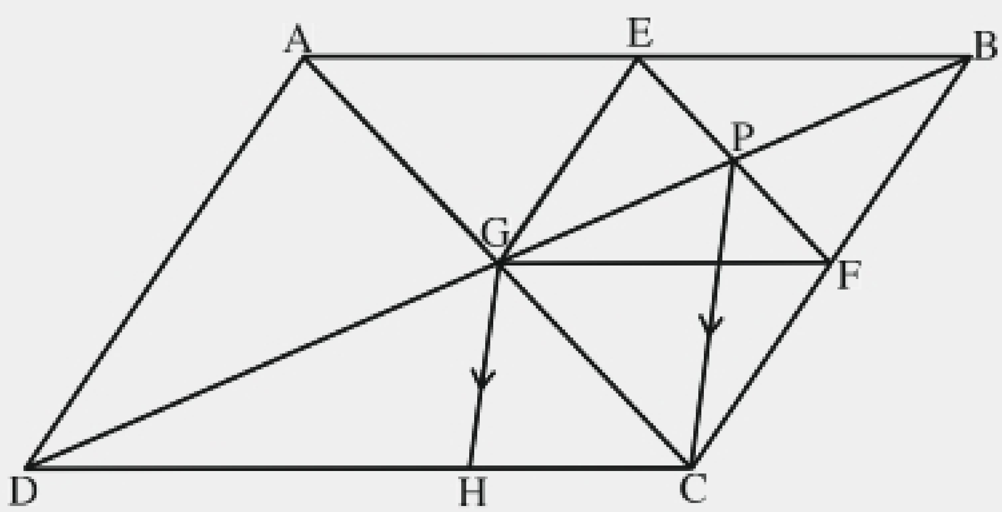
Determine
(1) MS : RL
(2) MQ : QP
(3) MR : RL
(4) $\dfrac{\text{Area of } \Delta MKP}{\text{Area of } \Delta MKL}$
(5) $\dfrac{\text{Area of } \Delta LRP}{\text{Area of } \Delta LSK}$
(6) $\dfrac{\text{Area of } \Delta RLP}{\text{Area of } \Delta MRP}$
(7) $\dfrac{\text{Area of } \Delta MRP}{\text{Area of } \Delta MSQ}$
(8) $\dfrac{\text{Area of } \Delta RLP}{\text{Area of } \Delta MSQ}$



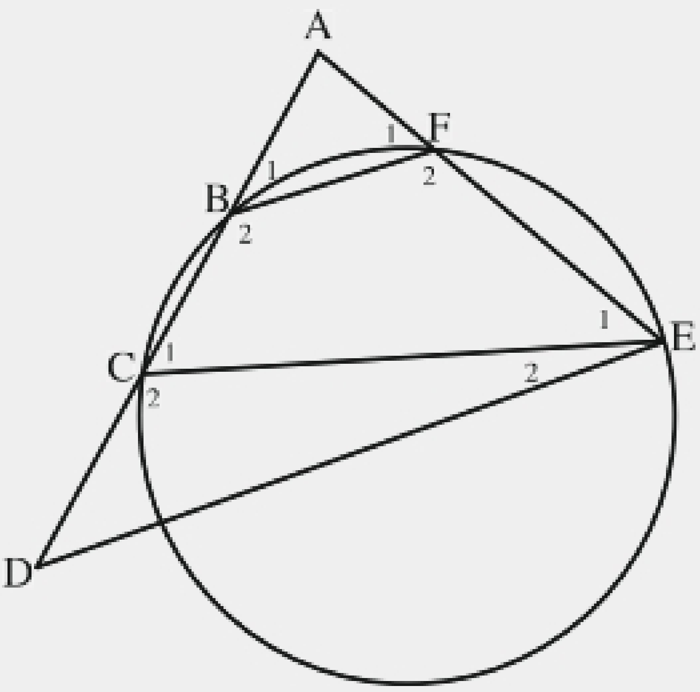
(c) In the sketch alongside, ABCD and EBFG are parallelograms. $GH \parallel PC$.

(1) Prove that $EF \parallel AC$.

(2) Determine the value of
(i) $\dfrac{DH}{DC}$
(ii) $\dfrac{\text{Area of } \Delta PDC}{\text{Area of } \Delta GDH}$
(ii) $\dfrac{\text{Area of } \Delta PBC}{\text{Area of } \Delta PDC}$



(d) Consider the sketch alongside.



(1) Prove that $\Delta ABF\,|||\,\Delta AEC$.

(2) It is given that $AB = 25$ cm, $AF = 20$ cm, $FE = 40$ cm and $CD = 27$ cm.

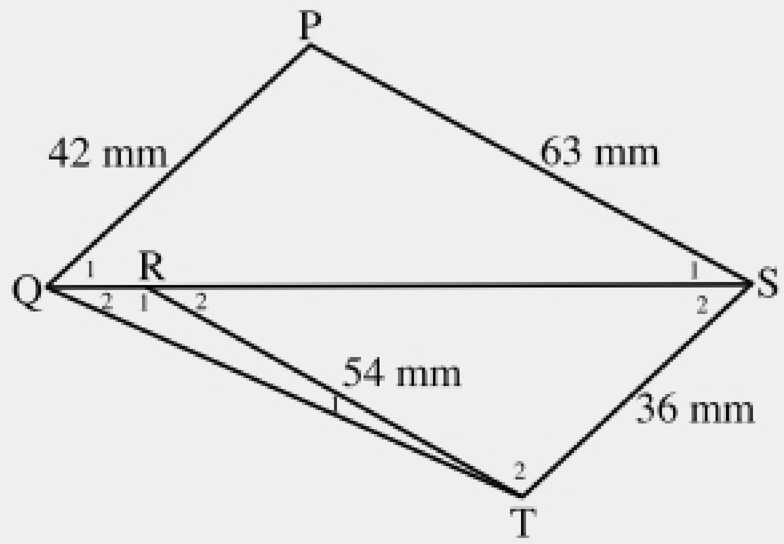
(i) Determine the length of BC.

(ii) Prove that $BF \parallel DE$.

(iii) Prove that $\Delta AEC\,|||\,\Delta ADE$.

(iv) Calculate the ratio DE : BF.

(e) In the sketch alongside, $PQ = 42$ mm, $PS = 63$ mm, $RT = 54$ mm and $TS = 36$ mm. $QR : RS = 1:6$.



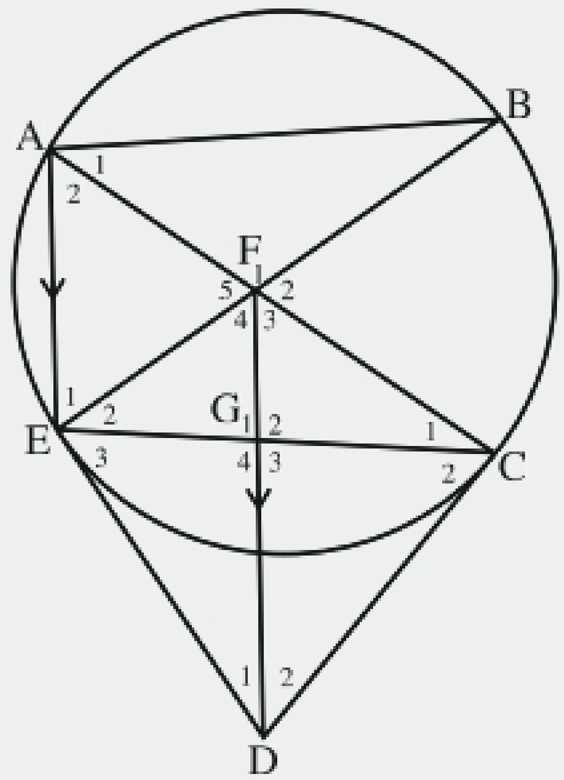
Prove that

(1) $\Delta PQS\,|||\,\Delta TSR$

(2) $PQ \parallel ST$

(3) $\hat{S}_1 = \hat{T}_1 + \hat{Q}_2$

(f) In the sketch alongside, DE and DC are tangents to the circle at E and C respectively. $AE \parallel FD$.



Prove that

(1) $\Delta AEC\,|||\,\Delta EGD$

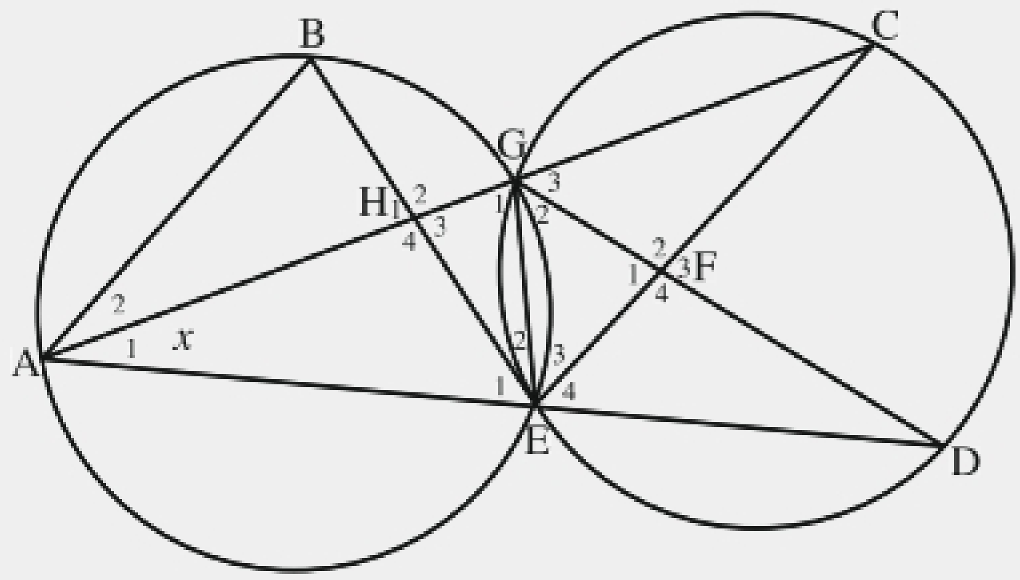
(2) $\dfrac{AF}{EF} = \dfrac{AB}{EC}$

(3) CDEF is a cyclic quadrilateral.

(4) DC is a tangent to circle FCG.

(5) $EF \cdot CF = DF \cdot GF$

(g) In the sketch alongside, BE is a tangent to circle CEG at E. $AE = EC$. $\hat{A}_1 = x$.



(1) Name, with reasons, four other angles equal to x.

(2) Prove that $\dfrac{AB}{AG} = \dfrac{AH}{EC}$.

(3) Prove that

(i) $AG = GD$

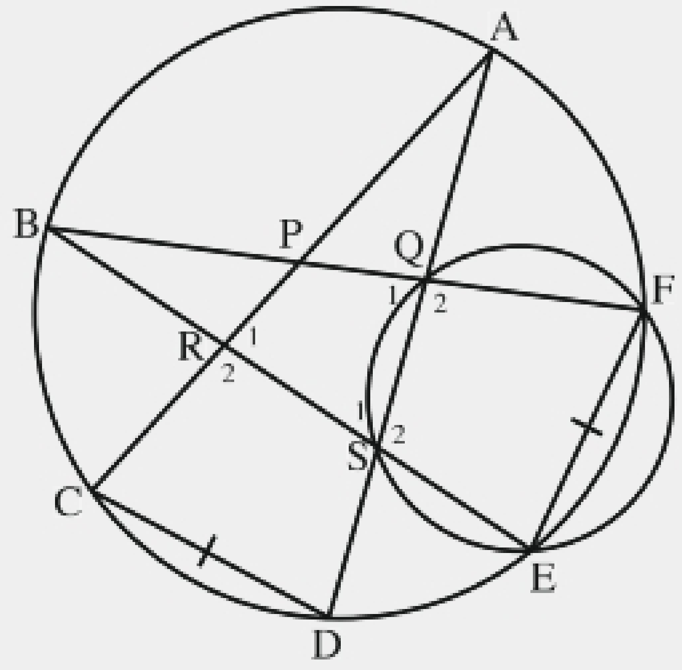
(ii) $EG^2 = GD \cdot GH$

(h) In the sketch alongside, $CD = EF$.

Prove that

(1) $\Delta ARS\ |||\ \Delta BEF$

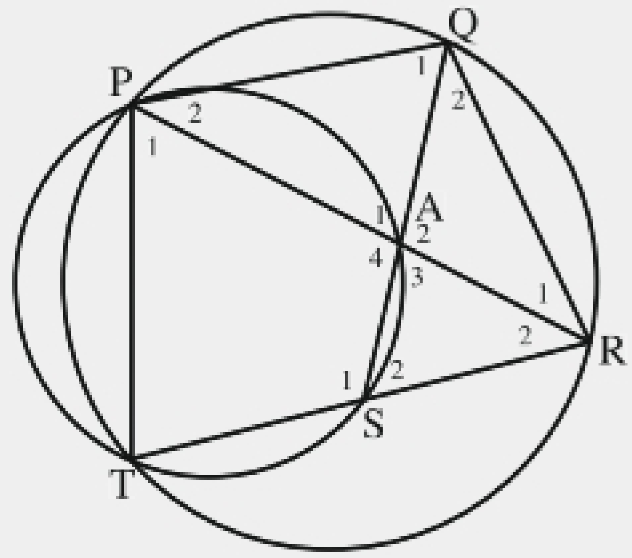
(2) $BF \cdot SQ = BS \cdot CD$



(i)* In the sketch alongside two intersecting circles are shown. PQ is **not** a tangent to circle PTSA. Prove that

(1) $QR^2 = AR \cdot PR$

(2) $QR^2 - AR^2 = PA \cdot AR$



(j) In the sketch alongside, O is the centre of the semi-circle. $EB \parallel DO$. AB is a tangent to the circle at B.

(1) Prove that

(i) $BC^2 = CE \cdot CA$.

(ii)* $CA \cdot DE = 2OE^2$

(2) If $AE : ED = 9 : 8$ and $OE = 40$ mm, Calculate the length of

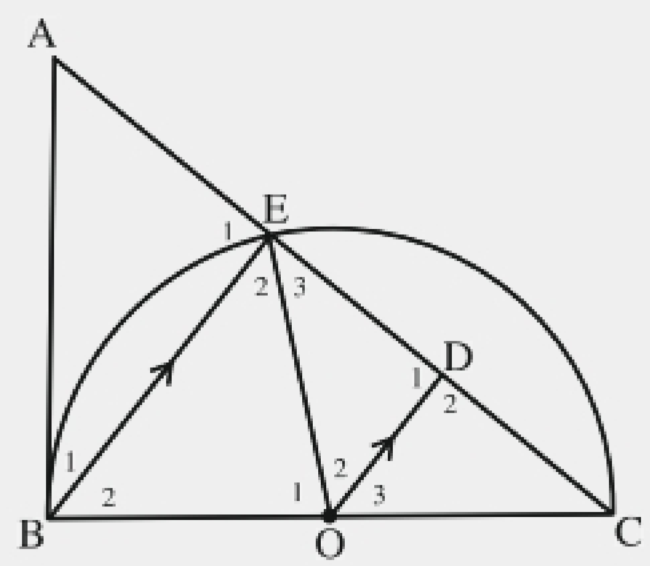
(i) BC

(ii) AC

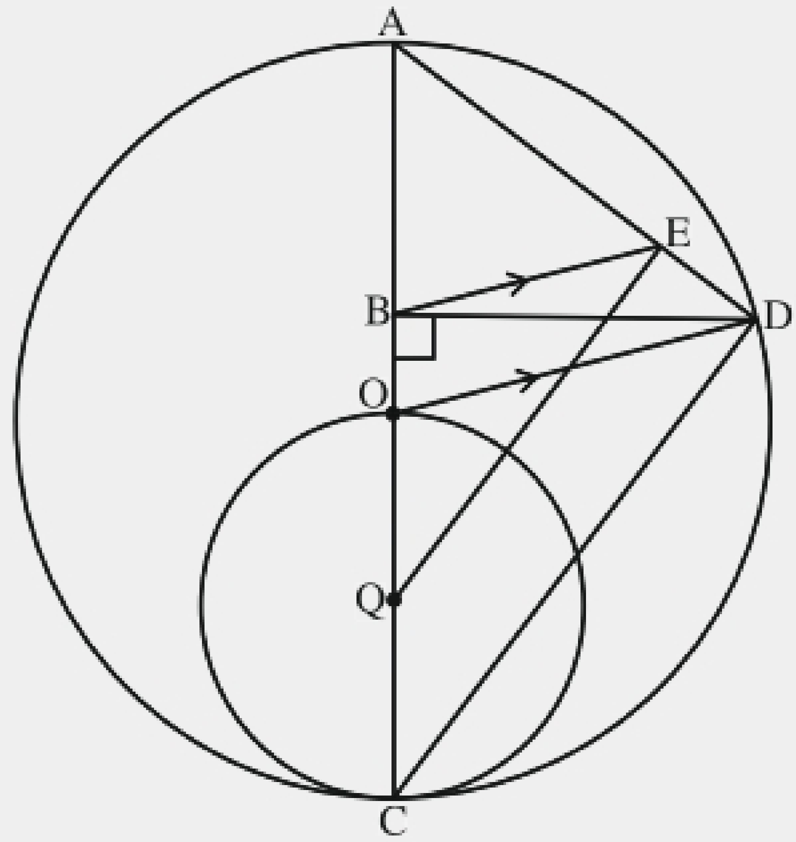
(iii) AB

(iv) BE

(v) OD



(k) In the sketch alongside, O is the centre of the larger circle and Q is the centre of the smaller circle. $BE \parallel OD$ and $DB \perp AC$. $AB : BO = 3 : 1$. $BO = x$.



(1) Express the following lengths in terms of x:
 (i) AO (ii) QC
 (iii) AQ

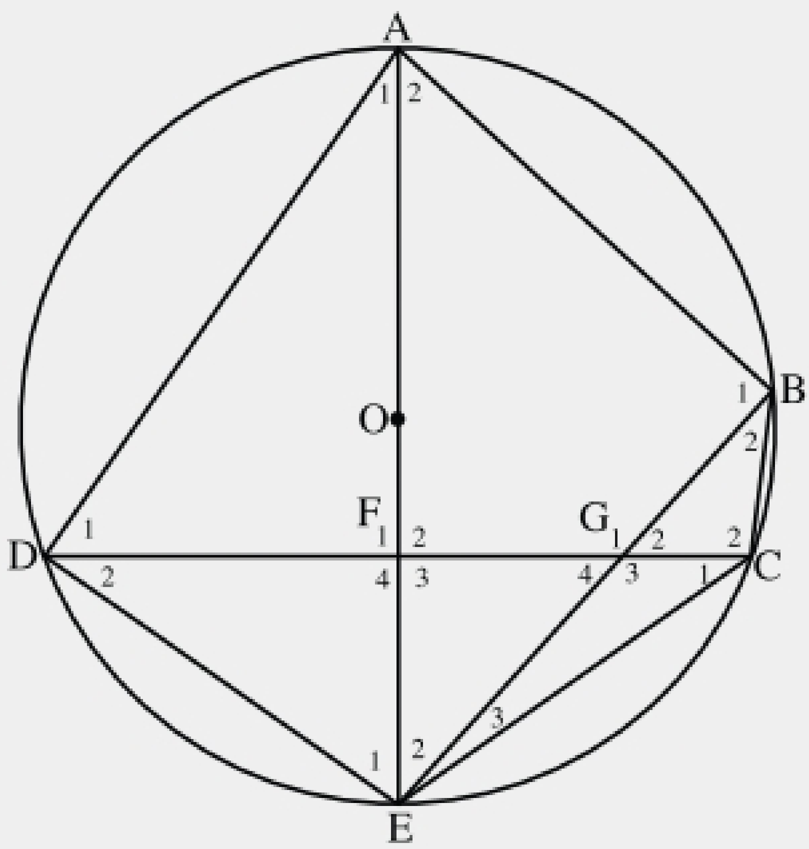
(2) Prove that $QE \parallel CD$.

(3) Prove that
 (i) $AD^2 = AB \cdot AC$
 (ii) $BD^2 = AB \cdot BC$

(4) Express the following lengths in terms of x:
 (i) AD
 (ii) BD

(5) Determine the ratio AB : ED.

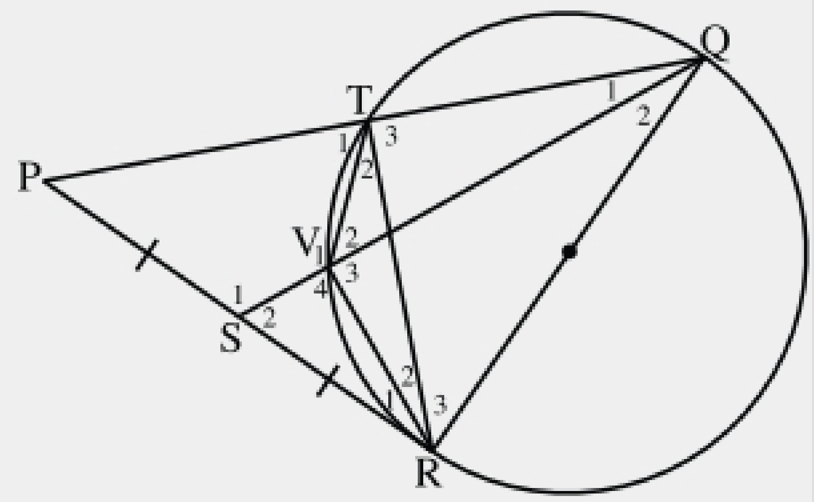
(l) In the sketch alongside, O is the centre of the circle. $DF = FC$.



(1) Prove that
 (i) $DE = EC$
 (ii) BAFG is a cyclic quadrilateral.
 (iii) $GE \cdot BC = GC \cdot EC$
 (iv) DE is a tangent to circle ADF.
 (v) $\Delta EFG \,|||\, \Delta EBA$
 (vi) $DE^2 = EB \cdot EG$

(2) If $EF = 30$ cm, $AB = 60$ cm and the diameter of the circle is 90 cm, determine the following in simplest surd form:
 (i) DF
 (ii) GE
 (iii) $\dfrac{\text{Area of } \Delta DFE}{\text{Area of } \Delta ABE}$

(m) In the sketch alongside, QR is a diameter of the circle. PR is a tangent to the circle at R. $PS = SR$.



Prove that

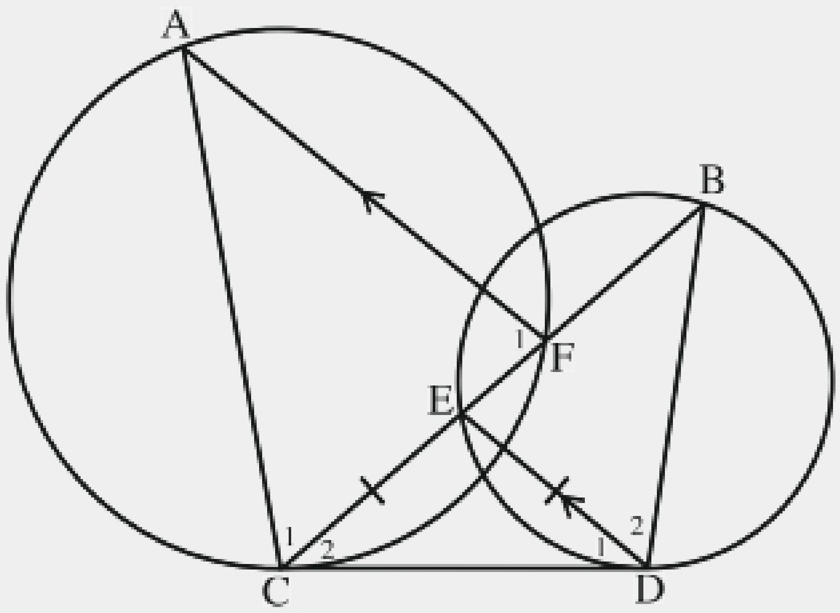
(1) $\Delta VSR \, ||| \, \Delta RSQ$

(2) PTVS is a cyclic quadrilateral.

(3)* $PQ^2 = QS^2 + 3VS \cdot QS$

(4) $\dfrac{QS}{QT} = \dfrac{RS}{TV}$

(n)* In the sketch alongside, CD is a tangent to both circles (at C and D). $EC = ED$ and $AF \parallel ED$.



(1) Prove that

(i) $\dfrac{CF}{CE} = \dfrac{AC}{BD}$

(ii) $CE = \dfrac{BD \cdot EF}{AC - BD}$

(2) If $AC : BD = 3 : 2$ and $EF : FB = 5 : 9$, determine

(i) $CE : EF$

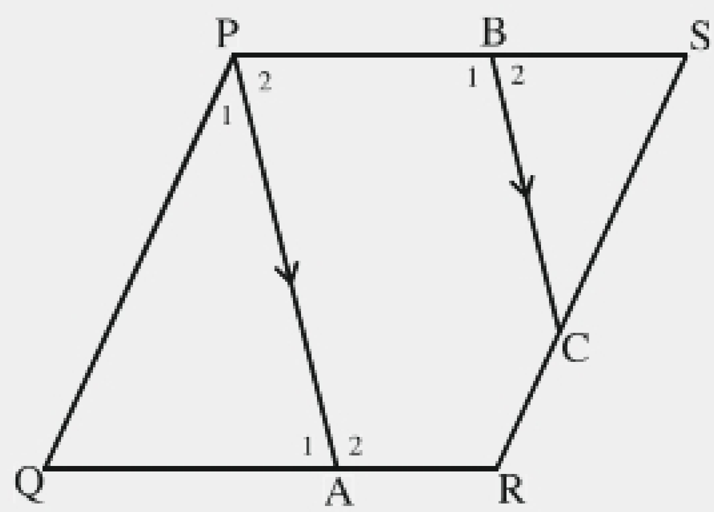
(ii) $CE : EB$

(iii) $\dfrac{\text{Area of } \Delta BED}{\text{Area of } \Delta AFC}$

(iv) $\dfrac{\text{Area of } \Delta CED}{\text{Area of } \Delta BED}$

(v) $\dfrac{\text{Area of } \Delta CED}{\text{Area of } \Delta AFC}$

(o)* In the sketch alongside, PQRS is a rhombus. $PA \parallel BC$.



(1) Prove that $\Delta PAQ \, ||| \, \Delta CBS$.

(2) Given that $PB : BS = 5 : 4$ and $RC : CS = 1 : 2$, determine

(i) $QA : BS$

(ii) $PB : AR$

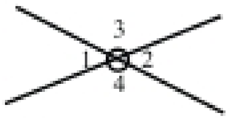
(iii) $\dfrac{\text{Area of } \Delta PAQ}{\text{Area of } \Delta CBS}$

(iv) $\dfrac{\text{Area of } \Delta PQA}{\text{Area of rhombus PQRS}}$

SUMMARY OF PRINCIPLES/THEOREMS FROM EARLIER GRADES

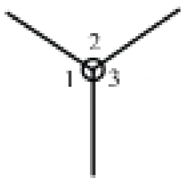
BASIC PRINCIPLES

Points



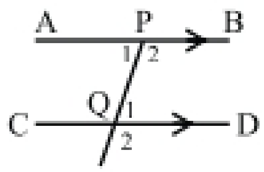
$\hat{O}_1 = \hat{O}_2$ and $\hat{O}_3 = \hat{O}_4$ (vert opp $\angle$s)

$\hat{O}_1 + \hat{O}_3 = 180°$ ($\angle$s on a str ln)



$\hat{O}_1 + \hat{O}_2 + \hat{O}_3 = 360°$ ($\angle$s round a pt)

Parallel lines

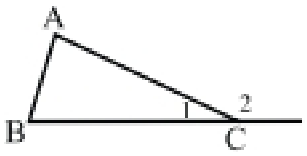


$\hat{P}_2 = \hat{Q}_2$ (corresp $\angle$s; $AB \parallel CD$)

$\hat{P}_1 = \hat{Q}_1$ (alt $\angle$s; $AB \parallel CD$)

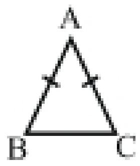
$\hat{P}_2 + \hat{Q}_1 = 180°$ (co-int $\angle$s; $AB \parallel CD$)

Triangles

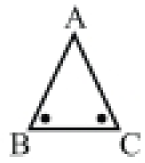


$\hat{A} + \hat{B} + \hat{C}_1 = 180°$ ($\angle$s of Δ)

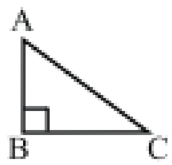
$\hat{C}_2 = \hat{A} + \hat{B}$ (ext $\angle$ of Δ)



$\hat{B} = \hat{C}$ ($\angle$s opp = sides)



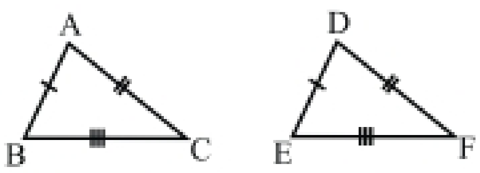
$AB = AC$ (sides opp = $\angle$s)



$AC^2 = AB^2 + BC^2$ (Pythag)

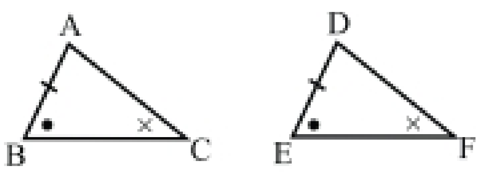
CONGRUENCY

SSS



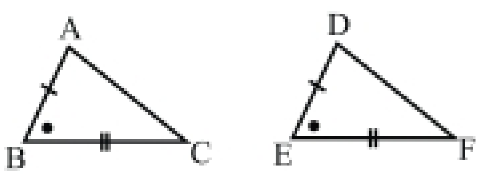
$\Delta ABC \equiv \Delta DEF$ (S S S)

AAS



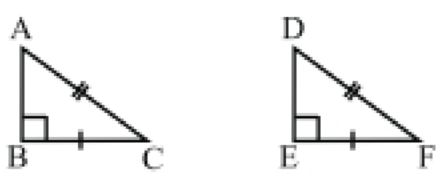
$\Delta ABC \equiv \Delta DEF$ ($\angle$ $\angle$ S)

SAS



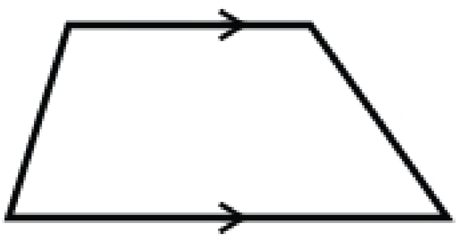
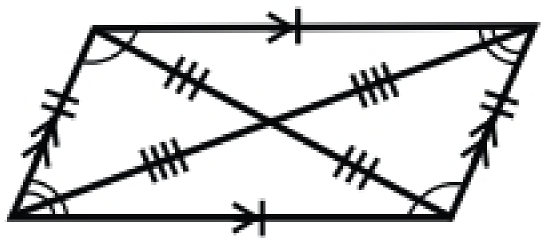
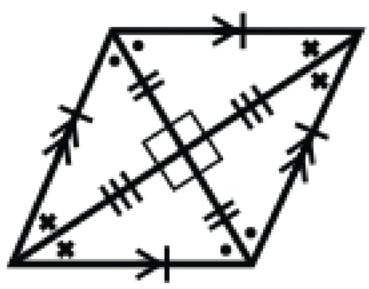
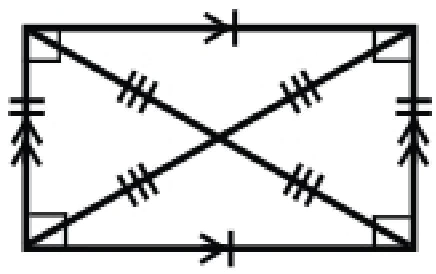
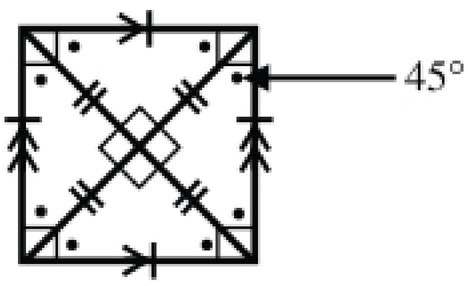
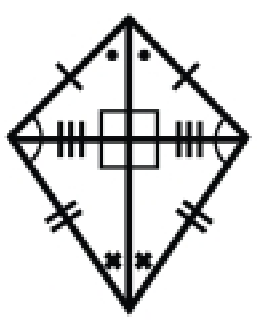
$\Delta ABC \equiv \Delta DEF$ (S $\angle$ S)

RHS

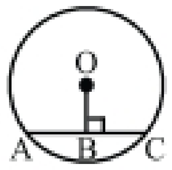
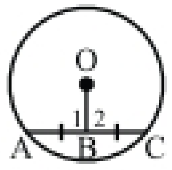
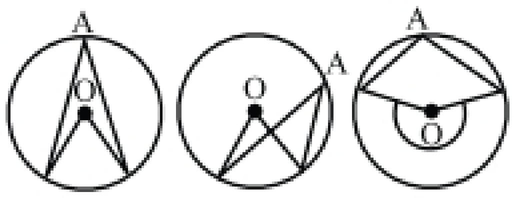
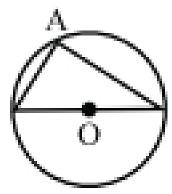
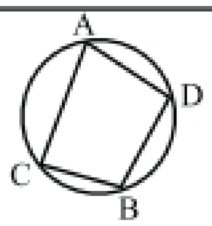
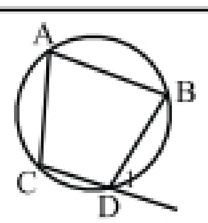
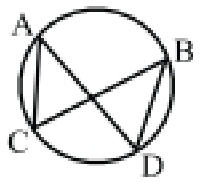
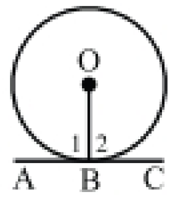
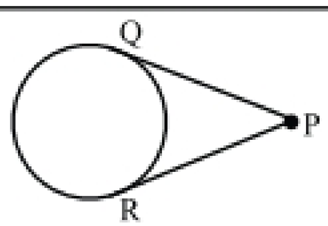
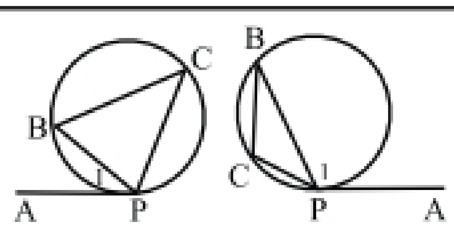


$\Delta ABC \equiv \Delta DEF$ (R H S)

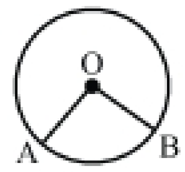
PROPERTIES OF QUADRILATERALS

Quadrilateral	Properties
Trapezium 	• One pair of opposite sides parallel
Parallelogram 	• Both pairs of opposite sides parallel • Both pairs of opposite sides equal • Both pairs of opposite angles equal • Diagonals bisect each other
Rhombus 	• All four sides equal • Diagonals are perpendicular to each other • Diagonals bisect the angles (In addition to this, the rhombus also inherits all the properties of the **parallelogram**)
Rectangle 	• All four angles are 90º • Diagonals are equal in length (In addition to this, the rectangle also inherits all the properties of the **parallelogram**)
Square 	• All four sides are equal • All four angles are 90º (The square inherits ALL the properties of BOTH the **rhombus** and **rectangle**, and therefore also of the **parallelogram**)
Kite 	• Two pairs of adjacent sides equal • Diagonals are perpendicular to each other • The main diagonal bisects the other diagonal • The main diagonal bisects angles • The angles opposite the main diagonal are equal

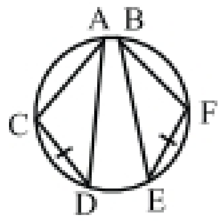
CIRCLE THEOREMS

GROUP 1: CENTRE **Key words:** centre, radius, diameter	
1 	$AB = BC$ (line from centre $\perp$ chord)
	$\hat{B}_1 = \hat{B}_2 = 90^\circ$ (line from centre to mpt chord)
2 	$\hat{O} = 2\hat{A}$ ($\angle$ at centre $= 2 \times \angle$ at circ)
3 	$\hat{A} = 90^\circ$ ($\angle$ in semi $\odot$)
GROUP 2: CYCLIC QUADRILATERALS **Key words:** cyclic quadrilateral, concyclic (cyclic quads are often not mentioned in the wording of a question - LOOK FOR four points on the circumference of a circle:  or )	
1 	$\hat{A} + \hat{B} = 180^\circ$ and $\hat{C} + \hat{D} = 180^\circ$ (opp $\angle$s of cycl quad)
2 	$\hat{D}_1 = \hat{A}$ (ext $\angle$ of cycl quad)
3 	$\hat{A} = \hat{B}$ and $\hat{C} = \hat{D}$ ($\angle$s in same segment)
GROUP 3: TANGENTS **Key words:** tangent, touch	
1 	$\hat{B}_1 = \hat{B}_2 = 90^\circ$ (rad $\perp$ tan)
2 	$PQ = PR$ (tangents from same pt)
3 	$\hat{P}_1 = \hat{C}$ (tan chord thm)

Also remember:

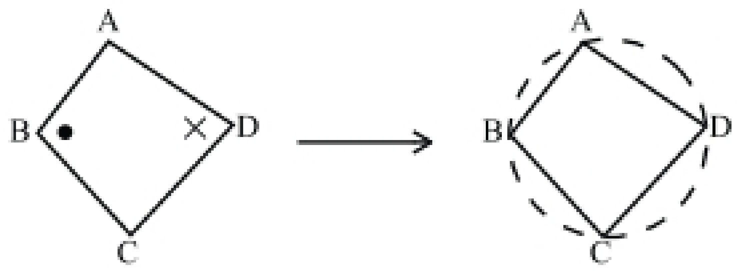


$OA = OB$
(radii)



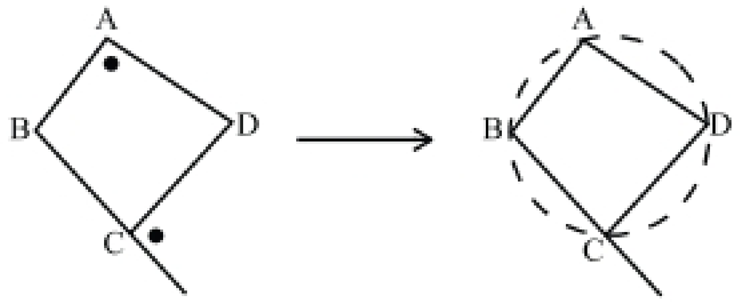
$\hat{A} = \hat{B}$
(equal chords $\rightarrow$ equal $\angle$s)

TO PROVE THAT A QUADRILATERAL IS CYCLIC:

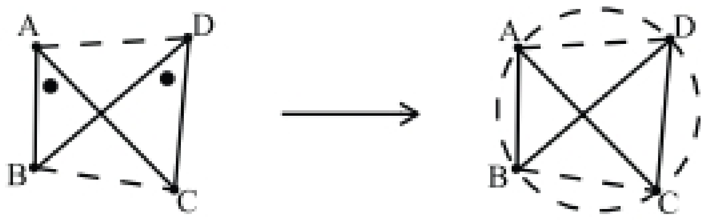


$\bullet + \times = 180°$

ABCD is a cyclic quad
(opp $\angle$s of quad suppl)

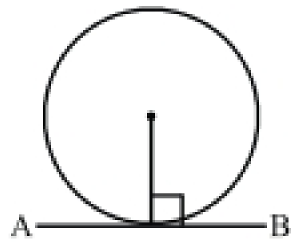


ABCD is a cyclic quad
(ext $\angle$ of quad = opp int $\angle$)

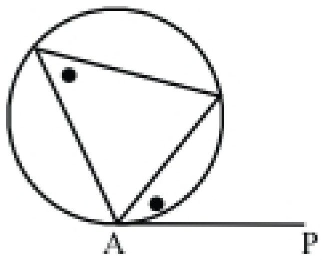


ABCD is a cyclic quad
(line subtends $= \angle$s)

TO PROVE THAT A LINE IS A TANGENT:



AB is a tangent
(line $\perp$ radius)



AP is a tangent
(converse tan chord thm)

CHAPTER NINE

Financial Mathematics

FORMULAE FROM PREVIOUS GRADES

Simple Interest	$A = P(1 + in)$	A is the final (accumulated) amount. P is the original (principal) amount. i is the interest rate per period. n is the number of periods.
Compound Interest or Inflation	$A = P(1 + i)^n$	A is the final (accumulated) amount. P is the original (principal) amount. i is the interest rate per compounding period. n is the number of periods.
Straight Line Depreciation	$A = P(1 - in)$	A is the final (depreciated) value. P is the original value. i is the depreciation rate per period. n is the number of periods.
Reducing Balance Depreciation	$A = P(1 - i)^n$	A is the final (depreciated) value. P is the original value. i is the depreciation rate per period. n is the number of periods.
Effective and Nominal Interest Conversion	$1 + i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^m$	i_{eff} is the effective annual interest rate. i_{nom} is the nominal interest rate (rate p.a. compounded more than once a year). m is the number of times interest is compounded in one year.

CALCULATING THE VALUE OF n

With our knowledge of logs, we are now able to calculate the value of n in the formulae $A = P(1 + i)^n$ and $A = P(1 - i)^n$.

EXAMPLE 1

Amanda deposits R10 000 into a savings account. Interest is calculated at 12% p.a. compounded monthly. How long will it take for her savings to grow to R15 000?

Solution

$$A = P(1 + i)^n$$

$$\therefore 15\ 000 = 10\ 000\left(1 + \frac{0{,}12}{12}\right)^n$$

$$\therefore \frac{3}{2} = (1{,}01)^n$$

$$\therefore n = \log_{1,01}\left(\frac{3}{2}\right)$$

$$\therefore n = 40,74890716$$

n is the number of compounding periods (months in this case). To convert to years:

$$\frac{40,74890716}{12} = 3,395742263 \text{ years}$$

It will take approximately 3 years and 5 months ($0,395742263 \times 12 \approx 5$).

EXAMPLE 2

A company purchases a truck for R800 000. The value of the truck depreciates at 20% p.a. on the reducing balance. How long will it take for the truck's value to depreciate to R500 000?

Solution

$$A = P(1-i)^n$$

$$\therefore 500\ 000 = 800\ 000(1-0,2)^n$$

$$\therefore \frac{500\ 000}{800\ 000} = (0,8)^n$$

$$\therefore \frac{5}{8} = (0,8)^n$$

$$\therefore n = \log_{0,8}\left(\frac{5}{8}\right)$$

$$\therefore n = 2,10628372$$

$\therefore$ Approximately 2 years and 1 month ($0,10628372 \times 12 \approx 1$).

EXERCISE 1

(a) Sandy deposits R25 000 into a savings account. The interest rate is 18% p.a. compounded monthly. How long will it take for her savings to grow to R40 000?

(b) A school purchases a photocopier for R150 000. The value of the photocopier depreciates at 15% p.a. on the reducing balance. How long will it take for the photocopier's value to depreciate to R90 000?

(c) Refilwe deposits R100 000 into a savings account which offers an interest rate of 12% p.a. compounded quarterly. How long will she have to wait to have R180 000 in the account?

(d) A manufacturing machine is purchased for R1 000 000. Its value depreciates at 35% p.a. on the reducing balance. The machine will be sold once its value reaches R600 000. How long will the machine be used before it is sold?

(e) How long will it take for the value of an investment to treble, if interest is calculated at 22% p.a. compounded semi-annually?

(f) How long will it take for a bowling machine to halve in value, if its value depreciates at 25% p.a. on the reducing balance?

In Grade 11 you learnt to deal with **individual** payments and withdrawals separately. However, if **the same** payment or withdrawal is made **at regular intervals**, special formulae can be used to obtain the result in a single calculation.

ANNUITIES

In Financial Mathematics, an *annuity* is a series of equal payments made at regular time intervals. Examples include regular savings, retirement annuities, and home/vehicle loan repayments.

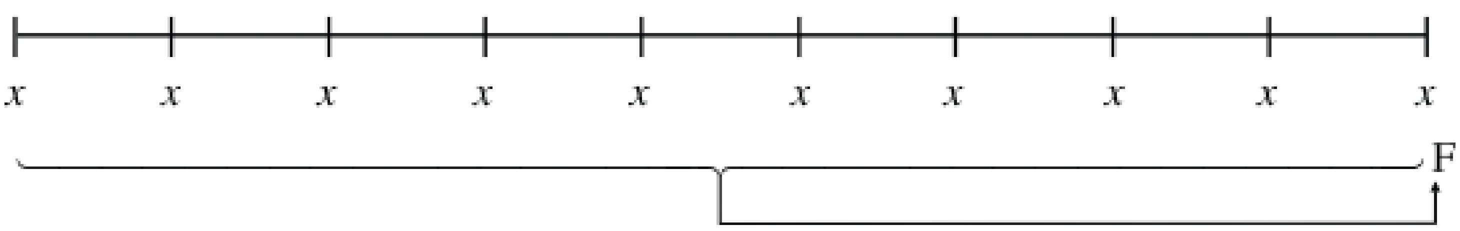
The annuity formulae you will learn this year are used under the following conditions:

- all payments are equal,
- the payments are made at regular intervals,
- the interest rate remains fixed and
- the compounding period for interest is the same as the payment intervals.

THE FUTURE VALUE FORMULA

The *future value* of an annuity is defined as the collective value of all the payments, including interest, **immediately after the last payment**.

On a timeline:



We can use the following formula to calculate the future value of an annuity:

$$\boxed{\mathrm{F} = \frac{x\left[(1+i)^n - 1\right]}{i}}$$

F is the future value.
x is the payment.
i is the interest rate per interval.
n is the number of payments.

The future value formula is used whenever the payments (small amounts) are made first and the total (large amount) is accumulated/collected later. This would include regular savings for some future purpose like education, a holiday trip, retirement, replacing equipment etc.

EXAMPLE 3

Mokgeseng wants to save up money for an overseas vacation after he graduates from university. He opens a savings account and immediately deposits R500. After this he continues to deposit R500 at the end of each month for 4 years. The interest rate remains fixed at 8% p.a. compounded monthly.

(a) How much will he have in the savings account at the end of the 4 years?

(b) At the end of the 4 years, Mokgeseng stops making payments. He leaves the money in the account for another 2 years, while he completes a postgraduate degree. How much money will he have in the account at the end of these two years?

Solution

(a)

T_0	T_1	T_2	…	T_{48}
R500	R500	R500		R500

F = ? (at T_{48})

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$= \frac{500\left[\left(1+\frac{0,08}{12}\right)^{49} - 1\right]}{\frac{0,08}{12}}$$

$$= R28\ 862,79$$

(b) $A = P(1+i)^n$

$$= 28\ 862,79\left(1+\frac{0,08}{12}\right)^{24}$$

$$= R33\ 852,82$$

EXAMPLE 4

Meghan wants to save up R1 million by her 30th birthday. On her 18th birthday, she opens a savings account to save for this goal. She makes equal monthly payments into the account, starting one month after the account was opened. The last payment is made on her 30th birthday. The account pays interest at 15% p.a. compounded monthly. How much money will she have to save each month to reach her goal by her 30th birthday?

Solution

T_0	T_1	T_2	T_3	…	T_{144}
	x	x	x		x

F = R1 000 000 (at T_{144})

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore 1\ 000\ 000 = \frac{x\left[\left(1+\frac{0,15}{12}\right)^{144} - 1\right]}{\frac{0,15}{12}}$$

$$\therefore x = \frac{1\ 000\ 000 \times \frac{0,15}{12}}{\left[\left(1+\frac{0,15}{12}\right)^{144} - 1\right]}$$

$$\therefore x = R2\ 508,77$$

EXAMPLE 5

Linda wants to save up R30 000 to buy a new gaming laptop. She can afford to save R3 000 every six months. Interest is calculated at 10,2% p.a. compounded semi-annually. How many payments will she have to make to have at least R30 000 in savings?

Solution

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore 30\ 000 = \frac{3\ 000\left[\left(1+\frac{0{,}102}{2}\right)^n - 1\right]}{\frac{0{,}102}{2}}$$

$$\therefore \frac{30\ 000 \times \frac{0{,}102}{2}}{3\ 000} = \left(1+\frac{0{,}102}{2}\right)^n - 1$$

$$\therefore \frac{51}{100} = \left(1+\frac{0{,}102}{2}\right)^n - 1$$

$$\therefore \frac{151}{100} = \left(\frac{1\,051}{1\,000}\right)^n$$

$$\therefore n = \log_{\frac{1\,051}{1\,000}}\left(\frac{151}{100}\right)$$

$$\therefore n = 8{,}28$$

She will have to make 9 payments to have at least R30 000 in savings.

EXERCISE 2

(a) Lebo decides to save R200 per month for the next 3 years to create an emergency fund for unforeseen expenses. She makes the first deposit immediately and the last deposit at the end of the 3 years. Interest is calculated at 8,7% p.a. compounded monthly. How much will she have saved up by the end of the 3 year period?

(b) Gerrit, a medical doctor, wants to save up R2 million to buy his partner's share of their practice, when he retires in 10 years' time. He immediately opens a savings account and makes the first deposit into the account one month after opening the account. He continues making equal monthly deposits. The last deposit is made at the end of the 10 years. Interest is calculated at 6,5% p.a. compounded monthly. How much does Gerrit have to save each month?

(c) Griffen wants to move out of his parents' house when he is done with school. He decides to save R2 500 per quarter for the next two years, starting immediately and ending at the end of the two years. How much will he have saved up by the end of the 2 years, if interest is calculated at 9% p.a. compounded quarterly?

(d) Katlego opens a savings account to save for an engagement ring for his girlfriend. One month after opening the account, he deposits R400 and continues to deposit R400 at the end of each month thereafter. Interest is calculated at 11,5% p.a. compounded monthly. How much will he have in the account 3 years after it was opened?

(e) On her 25th birthday, Jeanine decides to start saving for retirement. One month after her 25th birthday, she takes out a retirement annuity and immediately pays R1 000 into the account. She continues making monthly payments of R1 000 until she retires on her 65th birthday. Interest is calculated at 7,3% p.a. compounded monthly. How much money will be in her annuity when she retires?

(f) Amahle wants to save up R100 000 to pay a deposit on a house in 4 years' time. She makes equal quarterly payments into a savings account, starting immediately. She makes the last payment at the end of the 4 years. How much must she save each quarter, if interest is calculated at 12% p.a. compounded quarterly?

(g) Richard pays R1 200 per month into a savings account, starting immediately and ending at the end of 8 years. Thereafter he leaves the accumulated amount in the account for another 2 years. If interest is calculated at 10,4% p.a. compounded monthly, how much will he have in the account after 10 years?

(h) Lerato pays R2 000 into a savings account at the end of each month, for 10 years. Thereafter she leaves the accumulated amount in the account for another 4 years.

The interest rate is:

- 9,8% p.a. compounded monthly for the first 12 years
- 11% p.a. effective for the last 2 years

What is the balance on Lerato's account after 14 years?

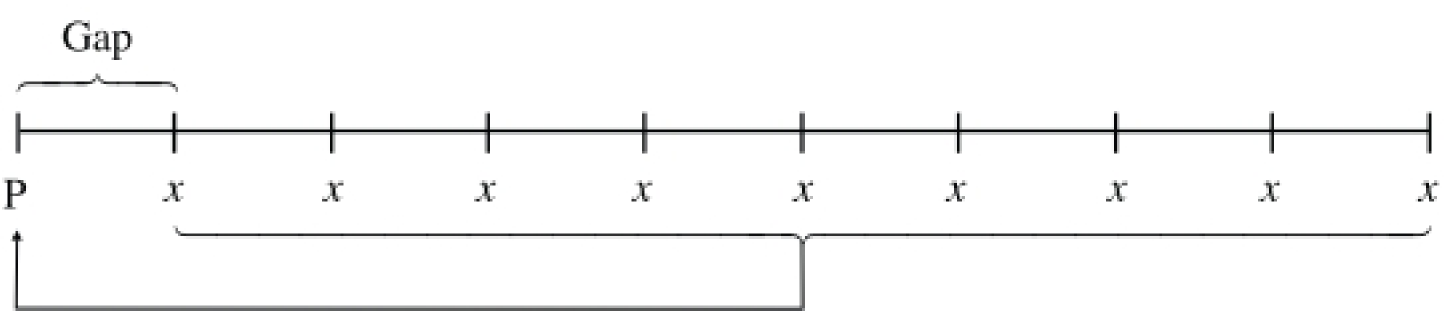
(i) Jaco wants to save up R25 000 to redo his swimming pool. He can afford to save R1 000 per month and the interest rate on his savings account is 6,9% p.a. compounded monthly. How long will Jaco have to save before he will have at least R25 000 in his savings account?

(j) On the 1st of January 2019, Mia's parents start saving R500 per month towards her tertiary education. They continue saving R500 at the end of month until the 31st of December 2036. Interest is calculated at 14,2% p.a. compounded monthly. At that time, they estimate that university fees will be R500 000. Will they have saved enough money to pay for her studies?

<u>THE PRESENT VALUE FORMULA</u>

The present *value* of an annuity is defined as the collective value of all the payments, with interest removed/reversed, **exactly one interval before the first payment**.

On a timeline:



We can use the following formula to calculate the present value of an annuity:

$$\boxed{P = \frac{x\left[1-(1+i)^{-n}\right]}{i}}$$

P is the present value.
x is the payment.
i is the interest rate per interval.
n is the number of payments.

The present value formula is used whenever the total (large amount) is paid/received first and payments (small amounts) are made later. This includes all loans. It is also used when a sum of money is set aside in order to make regular withdrawals in the future.

EXAMPLE 6

Mpho can afford to pay back at most R8 000 per month on a car loan. The interest on a car loan is 15% p.a. compounded monthly. He will make his first payment one month from now and will repay the loan over a period of 5 years. What is the price of the most expensive car Mpho can afford?

Solution

$$P = \frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$= \frac{8\,000\left[1-\left(1+\frac{0,15}{12}\right)^{-60}\right]}{\frac{0,15}{12}}$$

$$= R336\ 276,73$$

EXAMPLE 7

Jenny wishes to repay a loan of R150 000, by means of 16 equal quarterly payments, starting three months from now. The interest rate on the loan is 21,5% p.a. compounded quarterly.

(a) Calculate what Jenny's quarterly payment will be.
(b) Calculate the total interest that Jenny will pay on the loan.

Solution

(a) Notice that 3 months = 1 quarter, which means payments start one period after the loan was made as is normally the case.

$$P = \frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$\therefore 150\ 000 = \frac{x\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]}{\frac{0,215}{4}}$$

$$\therefore \frac{150\ 000 \times \frac{0,215}{4}}{\left[1-\left(1+\frac{0,215}{4}\right)^{-16}\right]} = x$$

$$\therefore x = \text{R}14\ 212,35$$

(b) Total Repayments $= 14\ 212,35 \times 16 = \text{R}227\ 397,60$

Interest = Total Repayments – Loan Value

$= \text{R}227\ 397,60 - \text{R}150\ 000$

$= \text{R}77\ 397,60$

<u>EXAMPLE 8</u>

Karel has to pay off a loan of R75 000. He can afford to pay R1 500 per month. The interest rate is 16,2% p.a. compounded monthly. How many payments will he have to make?

<u>Solution</u>

$$P = \frac{x\left[(1-(1+i)^{-n}\right]}{i}$$

$$\therefore 75\ 000 = \frac{1\ 500\left[1-\left(1+\frac{0,162}{12}\right)^{-n}\right]}{\frac{0,162}{12}}$$

$$\therefore \frac{75\ 000 \times \frac{0,162}{12}}{1\ 500} = 1-\left(1+\frac{0,162}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0,162}{12}\right)^{-n} = 1-\frac{75\ 000 \times \frac{0,162}{12}}{1\ 500}$$

$$\therefore \left(\frac{2\ 027}{2\ 000}\right)^{-n} = \frac{13}{40}$$

$$\therefore -n = \log_{\frac{2\ 027}{2\ 000}} \frac{13}{40}$$

$$\therefore n = -\log_{\frac{2\ 027}{2\ 000}} \frac{13}{40} = 83,81$$

He will make 84 payments, consisting of 83 payments of R1500 and one lesser payment*.

*In the next section you will learn how to calculate the value of this lesser payment.

EXERCISE 3

(a) Madeline's monthly repayment on her student loan is R3 500 per month. She repays this loan by means of 48 equal monthly payments. Interest on her loan is calculated at 12,5% p.a. compounded monthly. What is the value of Madeline's student loan?

(b) Donald wants to buy a house worth R1,2 million. His bank offers him a home loan at an interest rate of 11,5% p.a. compounded monthly. He will pay off the loan with monthly payments over a period of 20 years. He will make his first payment a month after taking out the loan. Calculate
(1) Donald's monthly payment. (2) the total interest on the loan.

(c) Otto repays a loan over a period of 5 years, by means of equal semi-annual payments of R15 000, starting 6 months after the loan was granted. What is the value of the loan, if the interest rate is 14,8% p.a. compounded semi-annually?

(d) Matthew takes out a loan of R175 000, at 18% p.a. compounded quarterly. He repays the loan over a period of 4 years, starting 3 months after the loan was granted. Calculate
(1) Matthew's quarterly payment. (2) the total interest on the loan.

(e) Julius wants to deposit an amount of money into a savings account, paying 6% interest p.a. compounded monthly, to provide for his mother for the next five years. She will withdraw R10 000 from the account on a monthly basis, starting one month after Julius makes the deposit. What amount must Julius deposit into the account?

(f) Keabetswe purchases a car and pays a deposit of R36 000, which is 8% of the price of the car. The remainder of the purchase price is financed through a loan, with an interest rate of 15% p.a. compounded monthly, over a period of 5 years. The first repayment is made one month after the car was purchased. Calculate
(1) the original purchase price of the car. (2) the loan amount.
(3) the monthly repayment. (4) the total amount of interest paid.

(g) How long will it take Mohammed to repay a loan of R1 million if he pays R16 000 per month? The interest rate on the loan is 11,2% p.a. compounded monthly.

(h) Caitlyn wants to repay a loan of R250 000 by means of equal semi-annual payments of R20 000 each. The interest rate on the loan is 14% p.a. compounded semi-annually. How many payments will Caitlyn have to make to repay the loan?

THE OUTSTANDING BALANCE ON A LOAN

The *outstanding balance* on a loan, at a given moment, is the amount that has to be paid to settle the loan. The outstanding balance is given by the following formula:

Outstanding Balance	=	Loan with interest to date (Using Compound Interest)	−	Repayments with interest to date (Using the Future Value formula)

For a normal loan, where the first payment is made one interval after the loan was received, and no payments are missed, the outstanding balance immediately after the *m*-th payment is given by:

$$\mathrm{OB} = \mathrm{P}(1+i)^m - \frac{x\left[(1+i)^m - 1\right]}{i}$$

EXAMPLE 9

Sibusiso took out a loan of R250 000. He makes monthly payments of R5 000 on the loan and interest is calculated at 15% p.a. compounded monthly. What is the outstanding balance on the loan after 3 years (directly after the payment made at the end of the 3rd year)?

Solution

$$\text{OB} = \text{P}(1+i)^m - \frac{x\left[(1+i)^m - 1\right]}{i}$$

$$\therefore \text{OB} = 250\ 000\left(1+\frac{0{,}15}{12}\right)^{36} - \frac{5\ 000\left[\left(1+\frac{0{,}15}{12}\right)^{36} - 1\right]}{\frac{0{,}15}{12}}$$

$$\therefore \text{OB} = \text{R}165\ 408{,}43$$

THE LAST PAYMENT

When the payment amount is fixed beforehand, it is usually not possible to settle the loan by an integer number of equal payments. In such cases, there will be a last payment, smaller than the others, to settle the loan. This payment is made one period after the last full payment:

$$\boxed{\text{Last payment} = \text{Outstanding balance after the last full payment} \times (1+i)}$$

EXAMPLE 10

George has to repay a loan of R375 000. The interest rate is 14% p.a. compounded monthly. He pays back R7 500 per month.

(a) How many payments will George have to make? (b) What will his last payment be?

Solution

(a) $$375\ 000 = \frac{7\ 500\left[1-\left(1+\frac{0{,}14}{12}\right)^{-n}\right]}{\frac{0{,}14}{12}}$$

$$\therefore \frac{375\ 000 \times \frac{0{,}14}{12}}{7\ 500} = 1-\left(1+\frac{0{,}14}{12}\right)^{-n}$$

$$\therefore \left(1+\frac{0{,}14}{12}\right)^{-n} = 1-\frac{375\ 000 \times \frac{0{,}14}{12}}{7\ 500}$$

$$\therefore \left(\frac{607}{600}\right)^{-n} = \frac{5}{12}$$

$$\therefore -n = \log_{\frac{607}{600}}\left(\frac{5}{12}\right)$$

$$\therefore n = -\log_{\frac{607}{600}}\left(\frac{5}{12}\right) = 75{,}47706563$$

He will make 76 payments (75 payments of R7 500 and 1 lesser payment).

(b) Calculate the outstanding balance after the last full payment (the 75th payment):

$$\text{OB} = \text{P}(1+i)^m - \frac{x\left[(1+i)^m - 1\right]}{i}$$

$$\therefore \text{OB} = 375\ 000\left(1+\frac{0{,}14}{12}\right)^{75} - \frac{7\ 500\left[\left(1+\frac{0{,}14}{12}\right)^{75} - 1\right]}{\frac{0{,}14}{12}}$$

$$\therefore \text{OB} = \text{R}3\ 547{,}457499$$

$$\text{Last Payment} = 3547{,}457499 \times \left(1+\frac{0{,}14}{12}\right)$$

$$= \text{R}3\ 588{,}84$$

EXERCISE 4

(a) Thabo pays off a loan of R80 000 over a period of 5 years. He makes half-yearly payments, starting 6 months after the loan was granted and ending at the end of the 5 year period. The interest rate is 18% p.a. compounded semi-annually.
(1) What is his half-yearly payment?
(2) How much does he owe immediately after the 10th payment?

(b) Sanele pays off his R2 000 000 home loan, with monthly payments, over a period of 20 years. The interest on the home loan is 10,8% p.a. compounded monthly.
(1) What is his monthly payment?
(2) What is the outstanding balance on his loan after 10 years?
(3)* What percentage of his payments during the first 10 years went towards interest?

(c) Alicia wants to pay off a loan of R450 000. She can afford to pay back R11 000 per month. Interest is calculated at 12,8% p.a. compounded monthly.
(1) How many payments will Alicia have to make?
(2) What is the outstanding balance on the loan immediately after the last full payment?
(3) Determine the value of the last payment, made one month after the last full payment.
(4) How much interest will Alicia pay on this loan?

(d) Petro pays back R7 500 per quarter on a loan of R75 000. The interest rate on the loan is 8,2% p.a. compounded quarterly.
(1) How many payments will Petro have to make?
(2) Calculate the value of Petro's last payment, made one quarter after her last payment of R7 500.

SINKING FUNDS

When equipment is bought, provision has to be made to replace the equipment in the future. This is done by means of a sinking fund. Regular payments are made to save up for the replacement of equipment.

To set up a sinking fund, the following has to be calculated:

- What the old equipment will be sold for in the future. (Use **Depreciation**.)
- What new equipment will cost in the future. (Use **Inflation**.)

The difference between these two values (inflated value − depreciated value) is the **future value** that has to be saved up by the time the equipment has to be replaced.

EXAMPLE 11

A company purchases an excavator for R900 000. The value of the machine depreciates at a rate of 10% p.a. on the reducing balance. The company wants to buy a new excavator in 10 years' time. Inflation is estimated at 7% p.a. The old excavator will be sold at scrap value after 10 years. To purchase a new machine, the money obtained from selling the old excavator will be used. A sinking fund is set up to finance the balance. The interest rate for the fund is 9% p.a. compounded monthly. The first payment into the sinking fund is made immediately and the last payment at the end of the 10 years. How much must the company pay into the fund per month?

Solution

Depreciation:
$$\begin{aligned} A &= P(1-i)^n \\ &= 900\,000(1-0{,}1)^{10} \\ &= R313\,810{,}60 \end{aligned}$$

Inflation:
$$\begin{aligned} A &= P(1+i)^n \\ &= 900\,000(1+0{,}07)^{10} \\ &= R1\,770\,436{,}22 \end{aligned}$$

Difference: $1\,770\,436{,}22 - 313\,810{,}60 = R1\,456\,625{,}62$

Monthly Payment:
$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$\therefore 1\,456\,625{,}62 = \frac{x\left[\left(1+\frac{0{,}09}{12}\right)^{121} - 1\right]}{\frac{0{,}09}{12}}$$

$$\therefore x = \frac{1\,456\,625{,}62 \times \frac{0{,}09}{12}}{\left[\left(1+\frac{0{,}09}{12}\right)^{121} - 1\right]}$$

$$\therefore x = R7\,433{,}07$$

EXERCISE 5

(a) Cleaning equipment is bought for R120 000. The value of the equipment depreciates at 15% p.a. on the reducing balance. The inflation rate is 9% p.a. In 5 years' time, the old cleaning equipment will be sold at scrap value. The proceeds of the sale, together with money saved up in a sinking fund, will be used to purchase replacement cleaning equipment.

(1) Calculate the scrap value of the old equipment after 5 years.

(2) What will the new equipment cost in 5 years' time?

(3) What amount should the company budget for in 5 years' time?

(4) Calculate the monthly payment to be paid into a sinking fund, paying 12,5% p.a. compounded monthly, in order to have enough money to replace the equipment in 5 years' time? (The first payment is made immediately, and the last payment is made at the end of the 5 years.)

(b) A delivery vehicle is purchased for R250 000. After 4 years, it is sold for R100 000 and a replacement vehicle is purchased. The inflation rate is 8,6% p.a. A sinking fund is set up on the day the original delivery vehicle is purchased, in order to save for the replacement of the vehicle in 4 years' time. The company makes quarterly payments into the fund, starting 3 months after the original vehicle is purchased and ending at the end of the 4 years. The interest rate is 11% p.a. compounded quarterly.

(1) What is the depreciation rate? (Depreciation is calculated on the reducing balance.)

(2) What will the replacement delivery vehicle cost in 4 years' time?

(3) What quarterly payment should be made into the sinking fund to have enough funds to replace the delivery vehicle in 4 years' time?

(c) *Jan's Jam* has to replace their canning machine in 10 years' time. Their current canning machine is valued at R270 000 and depreciates at 17% p.a. on the reducing balance. The price of a replacement canning machine increases by 12% p.a. The old machine will be sold at scrap value and the proceeds used toward purchasing the new machine. The company decides to set up a sinking fund to cover the replacement cost of the machine. Payments are made into the sinking fund on a monthly basis. The first payment is made one month after the purchase of the original canning machine and the last payment at the end of the 10 years. Calculate the monthly payment into the sinking fund if the interest rate is 13,75% p.a. compounded monthly.

(d) A farmer just bought a new tractor for R550 000. The farmer will use the tractor for 8 years and then sell it at scrap value. The depreciation on the tractor is estimated at 10% p.a. using the straight line method. The inflation rate is 7,6% p.a. In order to afford a new tractor in 8 years' time, the proceeds from the sale of the old tractor will be used and a sinking fund is set up to fund the balance: half-yearly payments are made into an account paying 9,8% p.a. compounded semi-annually. The first payment is made immediately, and the last payment is made at the end of the 8 years. Calculate the value of the half-yearly payment.

DEVIATIONS FROM THE NORMAL PATTERN

NO PAYMENT AT THE END

In a future value problem, if no payment is made at the end of the last period, we simply calculate the normal future value (immediately after the last payment) and then apply one period of interest to the future value.

EXAMPLE 12

Hannes invests R1 000 per month for 4 years. He pays at the beginning of each month. No payment is made at the end of the last month of the four year period. The interest rate is 9,5% p.a. compounded monthly. How much money will he have at the end of the 4 years?

Solution

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$= \frac{1000\left[\left(1+\frac{0{,}095}{12}\right)^{48} - 1\right]}{\frac{0{,}095}{12}}$$

$$= R58\,117{,}67$$

Amount at T_{48} :

$$A = P(1+i)^n$$

$$= 58\,117{,}67\left(1+\frac{0{,}095}{12}\right)^{1}$$

$$= R58\,577{,}77$$

DEFERRED ANNUITIES

When the first payment of a loan is made more than one period after the loan was received, we refer to the payment annuity as a *deferred annuity*. Note that the present value of the annuity will still be located one period before the first payment. We will apply compound interest to the loan to move it to the same point on the timeline as the present value of the annuity.

EXAMPLE 13

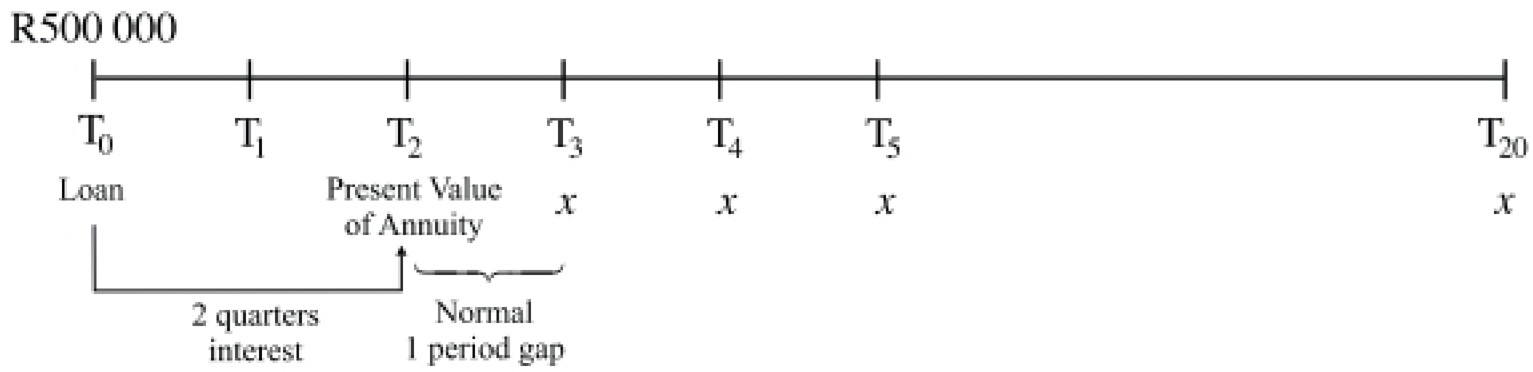
A loan of R500 000 is repaid by means of equal quarterly payments, starting 9 months after the loan is granted and ending 5 years after the loan is granted. The interest rate is 8% p.a. compounded quarterly.

(a) What are the quarterly payments?

(b) What is the outstanding balance on the loan directly after the 6th payment has been made?

Solution

(a) Repayments start 3 quarters after the loan was made. (9 months = 3 quarters)



Apply 2 quarters of interest to the loan:

$$A = P(1+i)^n$$

$$= 500\,000\left(1+\frac{0{,}08}{4}\right)^{2}$$

$$= R520\,200 \text{ (This is the present value of the annuity.)}$$

Calculate the payment:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

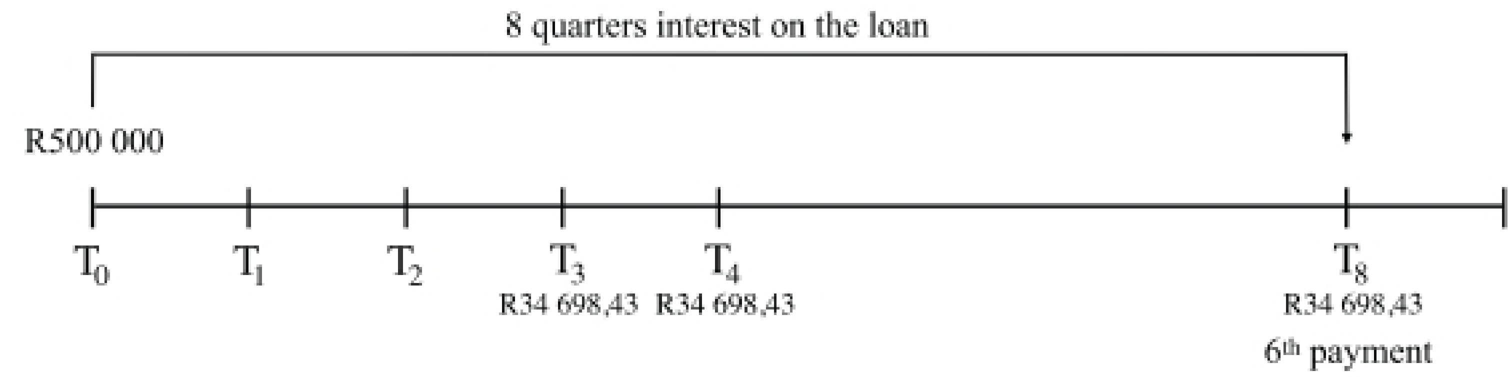
$n = 20 - 2 = 18$ (2 payments were skipped in the beginning)

$$\therefore 520\ 200 = \frac{x\left[1-\left(1+\frac{0,08}{4}\right)^{-18}\right]}{\frac{0,08}{4}}$$

$$\therefore \frac{520\ 200 \times \frac{0,08}{4}}{\left[1-\left(1+\frac{0,08}{4}\right)^{-18}\right]} = x$$

$$\therefore x = \text{R}34\ 698{,}43$$

(b)



Outstanding Balance = Loan with interest up to T_8 – Repayments with interest up to T_8

$$\therefore \text{OB} = 500\ 000\left(1+\frac{0,08}{4}\right)^{8} - \frac{34\ 698{,}43\left[\left(1+\frac{0,08}{4}\right)^{6}-1\right]}{\frac{0,08}{4}}$$

$$\therefore \text{OB} = \text{R}366\ 947{,}80$$

MISSED PAYMENTS

It can happen that, due to financial difficulty, a person is unable to make payments on a loan for some time. As soon as the person's situation improves, payments will resume. The new payment will be higher, since fewer payments will now be made in total. Also, the loan has accumulated interest in the time when no payments were made.

To calculate the new payment, we calculate the outstanding balance immediately after the last payment made, before payments ceased. We then apply compound interest to this outstanding balance, till one period before payments resume. The result is the present value of the new annuity consisting of all the remaining payments.

EXAMPLE 14

James takes out a loan of R100 000. He is to repay the loan by means of equal monthly payments, starting one month after the loan was granted. The loan is to be repaid over a period of 4 years, at an interest rate of 10% p.a. compounded monthly.

(a) What is his monthly payment?

(b) James lost his job and is unable to make the 20th, 21st and 22nd payments. He wishes to still repay the loan by the end of the original 4 years. What will his new monthly payment be?

Solution

(a)

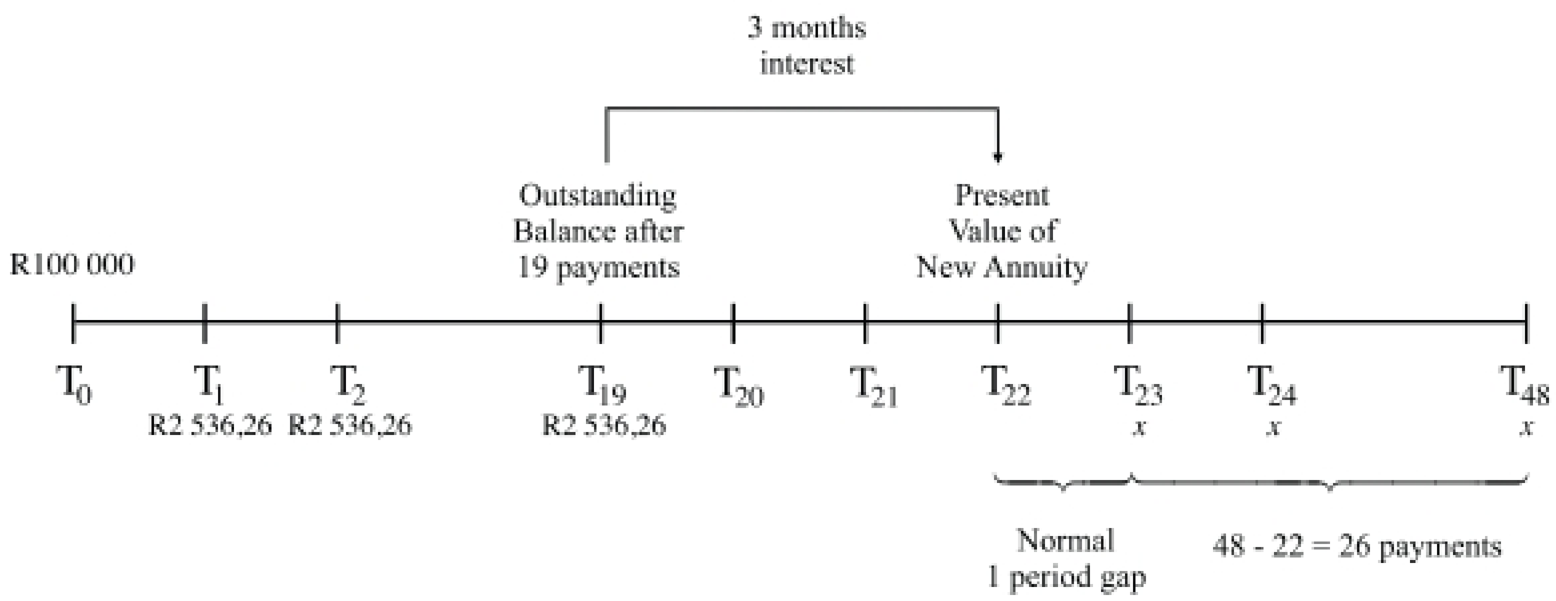
$$P=\frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 100\ 000=\frac{x\left[1-\left(1+\frac{0,1}{12}\right)^{-48}\right]}{\frac{0,1}{12}}$$

$$\therefore \frac{100\ 000\times\frac{0,1}{12}}{1-\left(1+\frac{0,1}{12}\right)^{-48}}=x$$

$$\therefore x=\text{R}2\ 536,26$$

(b)



Outstanding Balance after 19 payments:

$$\text{OB}=100\ 000\left(1+\frac{0,1}{12}\right)^{19}-\frac{2\ 536,26\left[\left(1+\frac{0,1}{12}\right)^{19}-1\right]}{\frac{0,1}{12}}$$

$$=\text{R}65\ 099,21$$

Add 3 months interest:

$$A = P(1+i)^n$$
$$= 65\,099{,}21\left(1+\frac{0{,}1}{12}\right)^3$$
$$= \text{R}66\,740{,}29$$

Calculate the payment on the new annuity:

$$P = \frac{x\left[1-(1+i)^{-n}\right]}{i}$$

$$\therefore 66\,740{,}29 = \frac{x\left[1-\left(1+\frac{0{,}1}{12}\right)^{-26}\right]}{\frac{0{,}1}{12}}$$

$$\therefore \frac{66\,740{,}29 \times \frac{0{,}1}{12}}{\left[1-\left(1+\frac{0{,}1}{12}\right)^{-26}\right]} = x$$

$$\therefore x = \text{R}2\,865{,}69$$

CHANGES IN PAYMENT AND INTEREST RATE (ENRICHMENT)

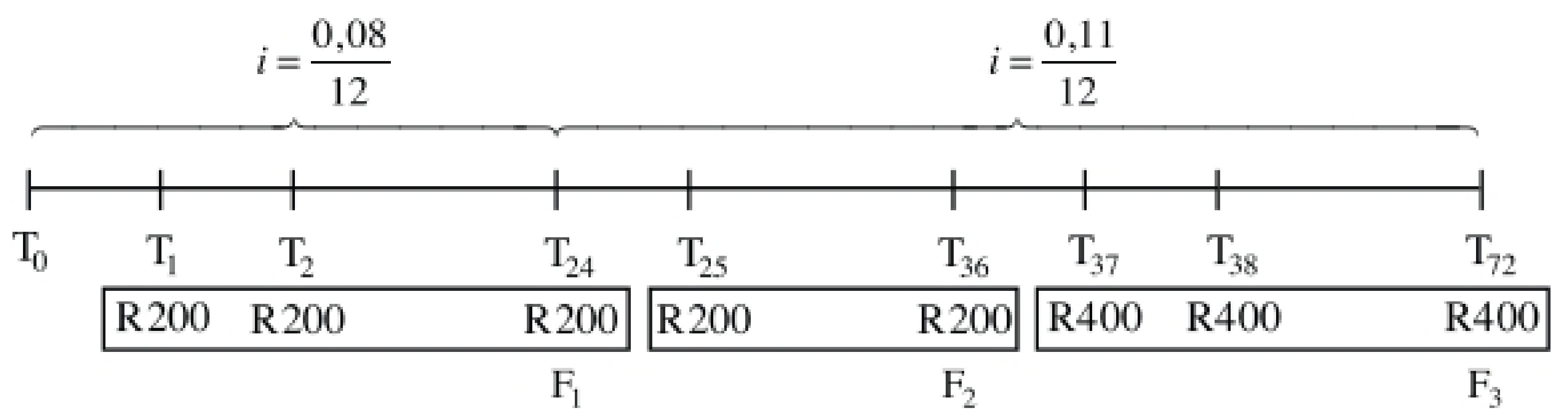
EXAMPLE 15*

Tristan makes the following deposits over 6 years:
- R200 per month, for the first 3 years, at the end of each month.
- R400 per month, for the next 3 years, at the end of each month.

The interest rates are:
- 8% p.a. compounded monthly for the first 2 years.
- 11% p.a. compounded monthly for the next 4 years.

How much money will Tristan have at the end of the 6-year period?

Solution



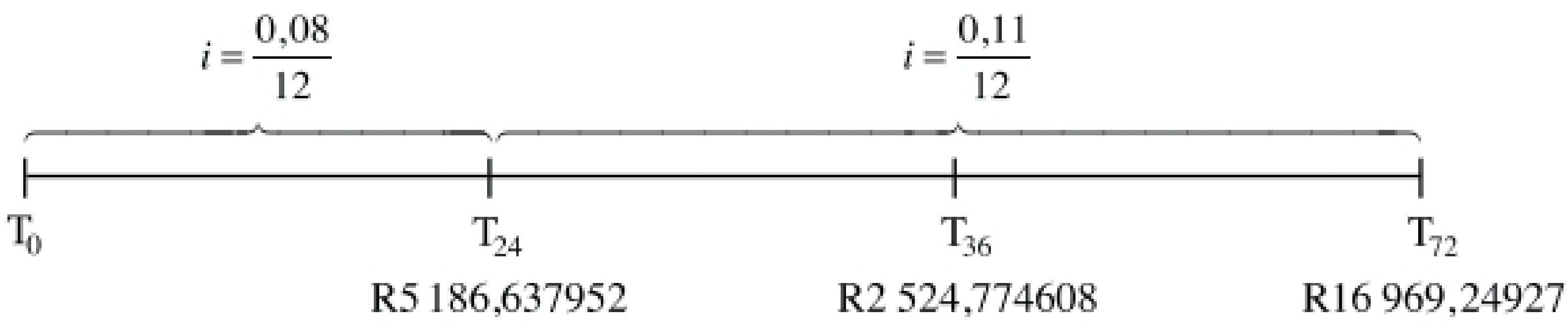
Whenever there is a change in the payments or the interest rate, a new annuity starts. There are therefore 3 annuities in this question, each with its own future value:

$$F_1 = \frac{x[(1+i)^n - 1]}{i}$$
$$= \frac{200\left[\left(1+\frac{0,08}{12}\right)^{24} - 1\right]}{\frac{0,08}{12}}$$
$$= \text{R}5\,186,637952 \quad (\text{at } T_{24})$$

$$F_2 = \frac{x\left[(1+i)^n - 1\right]}{i}$$
$$= \frac{200\left[\left(1+\frac{0,11}{12}\right)^{12} - 1\right]}{\frac{0,11}{12}}$$
$$= \text{R}2\,524,774608 \quad (\text{at } T_{36})$$

$$F_3 = \frac{x\left[(1+i)^n - 1\right]}{i}$$
$$= \frac{400\left[\left(1+\frac{0,11}{12}\right)^{36} - 1\right]}{\frac{0,11}{12}}$$
$$= \text{R}16\,969,24927 \quad (\text{at } T_{72})$$

Now work with these future values as single amounts:



Total at T_{72}

$$= 5\,186,637952\left(1+\frac{0,11}{12}\right)^{48} + 2\,524,774608\left(1+\frac{0,11}{12}\right)^{36} + 16\,969,24927$$
$$= \text{R}28\,513,06$$

<u>WHEN THE COMPOUNDING PERIOD DIFFERS FROM THE PAYMENT INTERVAL (ENRICHMENT)</u>

<u>EXAMPLE 16*</u>

Carl deposits R2 000 **per month** into a bank account paying 11,5% compounded **semi-annually**. He makes his first payment immediately and his last payment 4 years from now. How much will he have saved after 4 years?

<u>Solution</u>

We have to change the interest rate from a rate compounded semi-annually to a rate compounded monthly:

1. $$1 + i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^m$$
$$\therefore 1 + i_{eff} = \left(1 + \frac{0,115}{2}\right)^2$$
$$\therefore i_{eff} = \left(1 + \frac{0,115}{2}\right)^2 - 1$$
$$\therefore i_{eff} = 0,11830625$$

2. $$1 + i_{eff} = \left(1 + \frac{i_{nom}}{m}\right)^m$$
$$\therefore 1 + 0,11830625 = \left(1 + \frac{i_{nom}}{12}\right)^{12}$$
$$\therefore \sqrt[12]{1,11830625} = 1 + \frac{i_{nom}}{12}$$
$$\therefore \sqrt[12]{1,11830625} - 1 = \frac{i_{nom}}{12}$$
$$\therefore \left(\sqrt[12]{1,11830625} - 1\right) \times 12 = i_{nom}$$
$$\therefore i_{nom} = 0,1123378296$$

3. We now calculate the future value using the rate compounded monthly:

$$F = \frac{x\left[(1+i)^n - 1\right]}{i}$$

$$= \frac{2000\left[\left(1+\frac{0,1123378296}{12}\right)^{49} - 1\right]}{\frac{0,1123378296}{12}}$$

$$= R123\,626,55$$

EXERCISE 6

(a) Karabo invests R750 per month into a savings account for 6 years. His first payment is made immediately, and his last payment is made one month before the end of the 6 years. The interest rate is calculated at 11,2% p.a. compounded monthly. How much money will Karabo have at the end of the 6 years?

(b) Joyce invests R2 550 per quarter for 4 years. Her first payment is made immediately, and her last payment is made 3 months before the end of the 4 years. The interest rate is 10,5% p.a. compounded monthly. How much money will Joyce have saved after 4 years?

(c) Magda takes out a home loan of R1 000 000. She will repay the home loan by means of equal monthly payments starting 6 months after the loan was granted and ending 20 years after she took out the loan. The interest rate is calculated at 9,8% p.a. compounded monthly.
   (1) Calculate her monthly payment.
   (2) Calculate the outstanding balance on the loan immediately after the 100^{th} payment.

(d) On 1 January 2020, Celiwe takes out a loan of R250 000 to pay for her wedding. She will repay the loan by means of equal monthly payments, starting on 31 July 2020 and ending on 31 December 2023. The interest rate on the loan is 14% p.a. compounded monthly.
   (1) Calculate her monthly payment.
   (2) Calculate the outstanding balance on the loan on 30 September 2022, immediately after the payment has been made.

(e) Francois takes out a loan of R50 000, over a period of 2 years, to make repairs to his home. The loan is to be repaid by means of equal monthly payments, starting one month after the loan was granted. The interest rate is 14,5% p.a. compounded monthly.
   (1) Calculate his monthly payment.
   (2) Francois is unable to make the 11^{th}, 12^{th}, 13^{th} and 14^{th} payments. He still wants to repay the loan by the end of the original 2 years. Calculate his new monthly payment.

(f) Lesego is granted a loan of R150 000. Payments are to be made half-yearly, starting after 6 months and ending 5 years after the loan was granted. Interest is calculated at 12% p.a. compounded semi-annually.
   (1) Calculate his half-yearly payment.
   (2) Lesego misses the 3^{rd} and 4^{th} payments. Calculate his new monthly payment, if he still wants to repay the loan in the original time.

(g)* Maria makes the following deposits over 10 years:

- R1500 per month for 6 years starting immediately.
- R200 per month, at the end of each month, for the next 4 years.

The interest rates are:

- 14% p.a. compounded monthly for the first 5 years.
- 9% p.a. compounded monthly for the next 5 years.

How much money will Maria have at the end of the 10 year period?

(h)* Robert invests money for 8 years. During the first 3 years, he deposits R5 000 at the end of each quarter. Thereafter, he invests R8 000 per quarter. For the first 2 years interest is calculated at 12% p.a. compounded quarterly. The interest rate for the remainder of the term is 10,5% p.a. compounded quarterly. How much money will Robert have at the end of the 8 year period?

(i)* Johann invests R10 000 **per quarter**. The first deposit is made 3 months from now and the last deposit at the end of 7 years. The interest rate is 12,5% p.a. compounded **monthly**. How much money will Johann have saved at the end of the 7 years?

CONSOLIDATION AND EXTENSION EXERCISE

(a) How long will it take for an asset to depreciate to a quarter of its original value if it depreciates at 11,5% p.a. on the reducing balance?

(b) On 1 October 2020, Thando opens a savings account and makes a deposit of R2 000. He then makes monthly deposits of R2 000 at the end of each month. The last deposit is made on 31 March 2025. Interest is calculated at 9,8% p.a. compounded monthly. Determine the amount in the account immediately after the last payment.

(c) Interest is calculated at 8% p.a. compounded quarterly. Calculate
(1) the effective annual rate. (2) the rate p.a. compounded semi-annually.

(d) Jane pays off a loan of R150 000 by means of 20 equal quarterly payments, starting one year after the loan was taken out. The interest rate is 15,2% p.a. compounded quarterly.
(1) Calculate Jane's quarterly payment.
(2) Determine the outstanding balance after 11 payments.
(3) Jane is unable to make the 12^{th}, 13^{th} and 14^{th} payments. How much will her new monthly payment be, if she still wants to pay the loan off in the same amount of time?

(e) Armand pays off a loan of R550 000 by means of equal monthly payments of R11 000 each. The interest rate on the loan is 9,5% p.a. compounded monthly. The first payment is made 3 months after the loan was granted.
(1) How many payments will Armand have to make?
(2) What is the value of the last payment he will have to make to settle the loan one month after the last full payment?

(f) Lindiwe opens an investment account on her 21^{st} birthday. One month after her 21^{st} birthday, she deposits R1 000 into the account and continues making monthly deposits of R1 000 until her 30^{th} birthday. She makes no further deposits until her 35^{th} birthday. On her 35^{th} birthday she deposits a lump sum of R25 000 into the account. She makes no further deposits after this. How much money will she have in the account on her 40^{th} birthday, if the interest rate is 8% p.a. compounded monthly?

(g) *Pampered Paws Doggie Parlour* needs to replace grooming equipment, purchased for R250 000, in 4 years' time. A sinking fund, together with the proceeds from the sale of the current equipment (at scrap value), will be used to finance the replacement. The equipment is depreciated at 20% p.a. on the reducing balance and inflation is expected to be 8,5% p.a. The first monthly payment into the sinking fund is made immediately and the last payment at the end of the 4 years. The interest rate is 10% p.a. compounded monthly. How much should Pampered Paws pay into the sinking fund each month?

(h)* On 1 January 2010 Barbara decided to start saving R500 per month. She made the first deposit on 31 January 2010. On 1 January 2012 she changed the deposit to R1 000 per month. She made her last deposit on 31 December 2015.
Interest was calculated as follows:

- 1 January 2010 – 31 December 2013: 0,5% p.a. compounded monthly
- 1 January 2014 – 31 December 2015: 9,5% p.a. compounded monthly

How much money was in Barbara's savings account on 31 December 2015?

(i)** Yesheni deposits R1 000 per month, at the end of each month, into a bank account paying 5,4% p.a. compounded monthly over a period of 5 years. She makes the first deposit at the end of January of the first year. At the end of each year (31 December), an amount of R5 000 is paid to her father, from this account, as a new year's gift. How much money will Yesheni have in this account after ten years? (Take into account that she still makes the final R1 000 payment on the last day of the year and also that the last R5 000 is paid to her father on this day. Calculate the amount in the account after both these transactions have been completed.)

CHAPTER TEN

Statistics

In previous grades we usually worked with only one variable in each scenario. (There was only one measurement for each observation.)

We will now consider scenarios where there are two variables involved. (There will be two measurements for each observation.) We will also investigate the relationship between the two variables.

SCATTERPLOTS

Graphically, the relationship between two variables can be shown by means of a *scatterplot*. One variable is placed on the x-axis and one on the y-axis. Corresponding values of the variables are plotted as coordinates on the Cartesian plane:

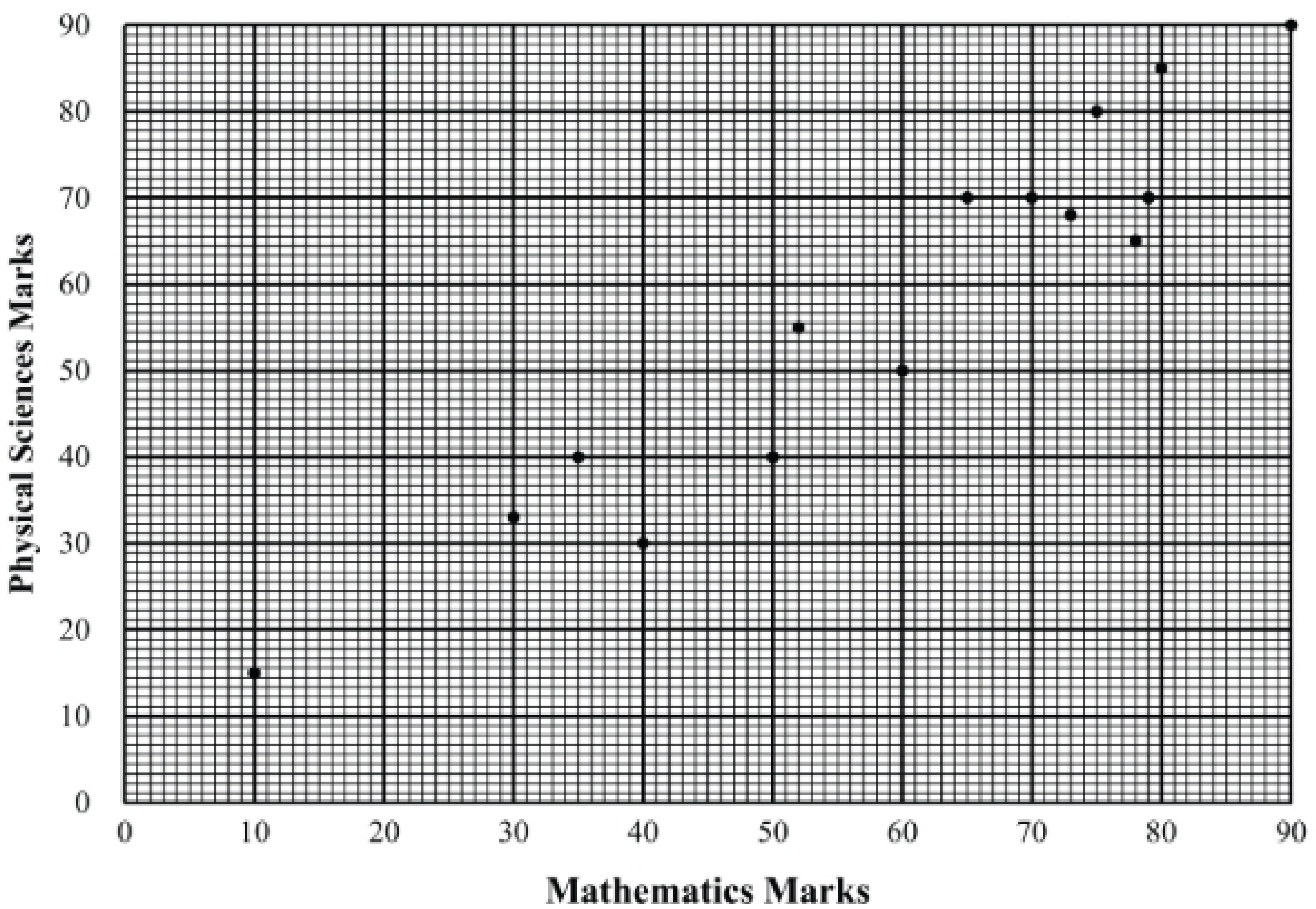
EXAMPLE 1

The Grade 12 Mathematics marks (x) and Grade 12 Physical Sciences marks (y) of the same fifteen learners are given in the table below:

x	10	30	35	40	50	52	60	65	70	73	75	78	79	80	90
y	15	33	40	30	40	55	50	70	70	68	80	65	70	85	90

Represent the marks of the learners by means of a scatterplot.

Solution



THE LINE OF BEST FIT (LEAST SQUARES REGRESSION LINE)

A **linear** relationship may exist between two variables. The straight line that best represents this relationship is called the *line of best fit* or the *least squares regression line*. Statistically, the line of best fit is such that the sum of the squares of the vertical distances between the points and the line is minimised. We will use a calculator to find the equation of the least squares regression line.

The equation of the least squares regression line is written in the form $\hat{y} = a + bx$, where a is the y-intercept of the line and b the gradient. The "hat" (^) on the y is to show that the equation represents a trend and doesn't give actual y-values but **predicted** y-values (estimates) based on the trend. More about this later. The following fact is very important:

> The least squares regression line always passes through the point $(\overline{x};\overline{y})$.

To sketch the least squares regression line, we plot the point $(\overline{x};\overline{y})$ and at least one other point - usually the y-intercept $(0; a)$. We then connect the points with a straight line.

EXAMPLE 2

The Grade 12 Mathematics marks (x) and Grade 12 Physical Sciences marks (y) of the same fifteen learners are given in the table below:

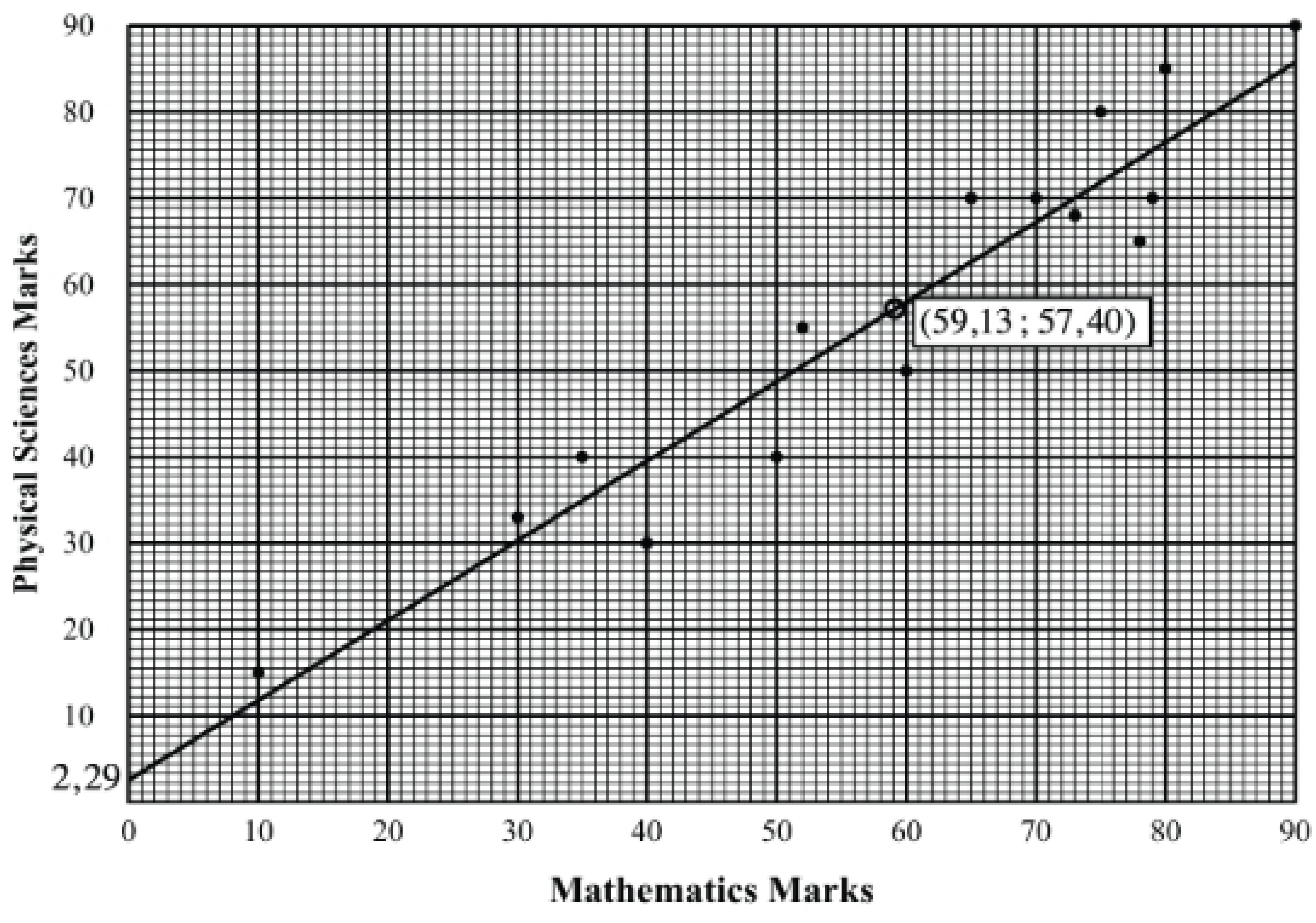
x	10	30	35	40	50	52	60	65	70	73	75	78	79	80	90
y	15	33	40	30	40	55	50	70	70	68	80	65	70	85	90

(a) Calculate the mean Mathematics and Physical Sciences marks of these learners.
(b) Determine the equation of the least squares regression line.
(c) Draw the least squares regression line on the scatterplot of the data (from Example 1).

Solution

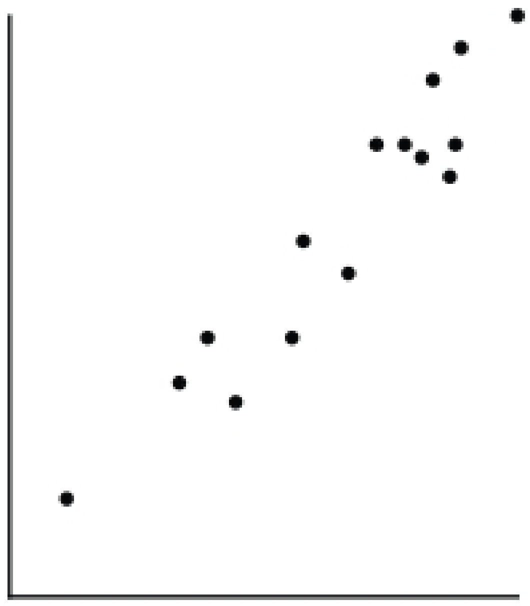
(a) $\overline{x} = 59{,}13$; $\overline{y} = 57{,}40$ (using a calculator) (b) $\hat{y} = 2{,}29 + 0{,}93x$ (using a calculator)

(c)

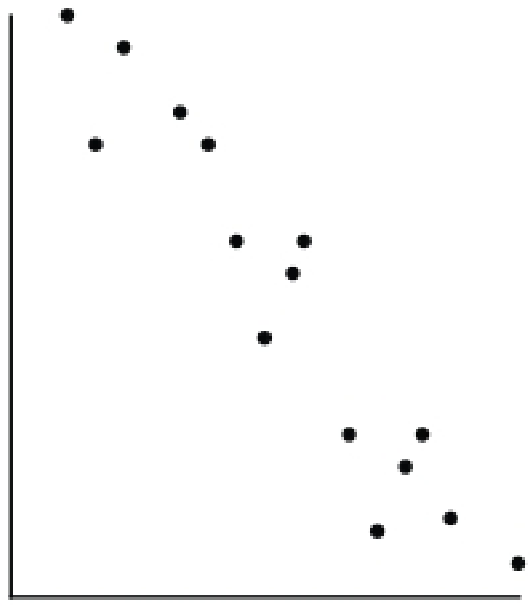


CORRELATION

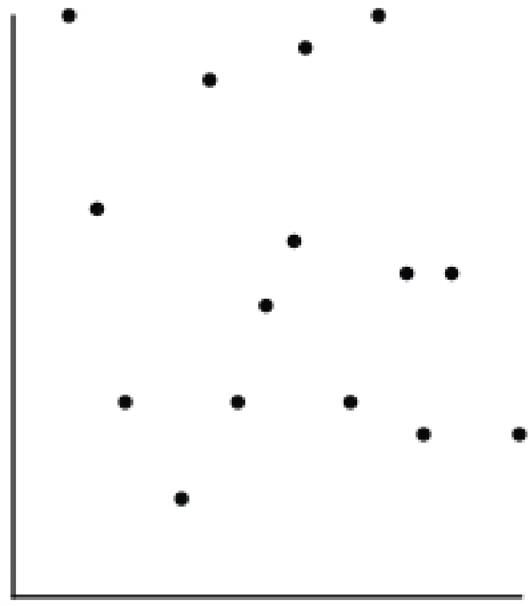
The *linear correlation* between two variables refers to the strength of the linear relationship between these variables. Variables can have a positive correlation, negative correlation or even no linear correlation:



Positive correlation
(as x increases y increases)



Negative correlation
(as x increases y decreases)



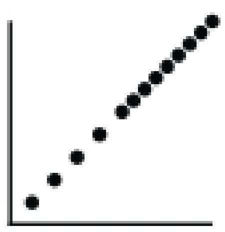
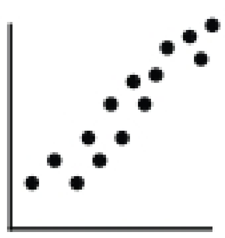
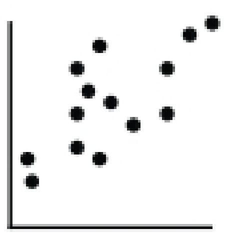
No correlation

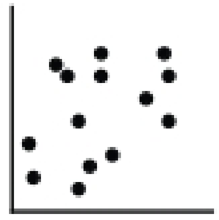
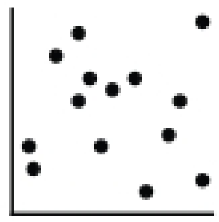
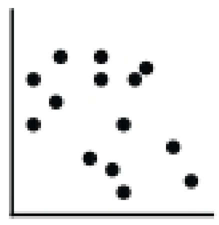
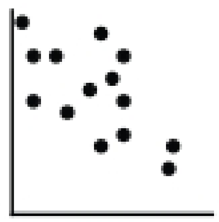
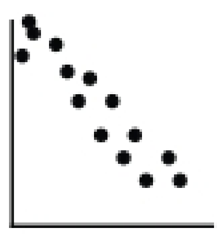
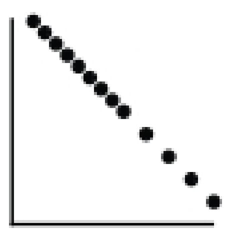
CORRELATION COEFFICIENT

The *correlation coefficient* (r) is a numerical value, between -1 and 1, that measures the strength of the linear relationship between two variables.

The correlation coefficient indicates how well the least squares regression line represents the data - roughly speaking, the more the points deviate from the least squares regression line, the lower the correlation coefficient. We will use a calculator to find the correlation coefficient.

INTERPRETING THE CORRELATION COEFFICIENT

Scatterplot	Value of r	Interpretation
	1	Perfect positive linear correlation
	Between 0,8 and 1	Strong positive linear correlation
	Between 0,5 and 0,8	Moderate positive linear correlation

Scatterplot	Value of *r*	Interpretation
	Between $0,2$ and $0,5$	Weak positive linear correlation
	Between $-0,2$ and $0,2$	Negligible linear correlation (No correlation when $r=0$)
	Between $-0,5$ and $-0,2$	Weak negative linear correlation
	Between $-0,8$ and $-0,5$	Moderate negative linear correlation
	Between -1 and $-0,8$	Strong negative linear correlation
	-1	Perfect negative linear correlation

EXAMPLE 3

The Grade 12 Mathematics marks (x) and Grade 12 Physical Sciences marks (y) of the same fifteen learners are given in the table below:

x	10	30	35	40	50	52	60	65	70	73	75	78	79	80	90
y	15	33	40	30	40	55	50	70	70	68	80	65	70	85	90

(a) Calculate the correlation coefficient.

(b) Comment on the relationship between the Mathematics and Physical Sciences marks.

Solution

(a) $r=0,95$ (using a calculator)

(b) There is a **strong positive linear** correlation between the Mathematics and Physical Sciences marks.

EXAMPLE 4

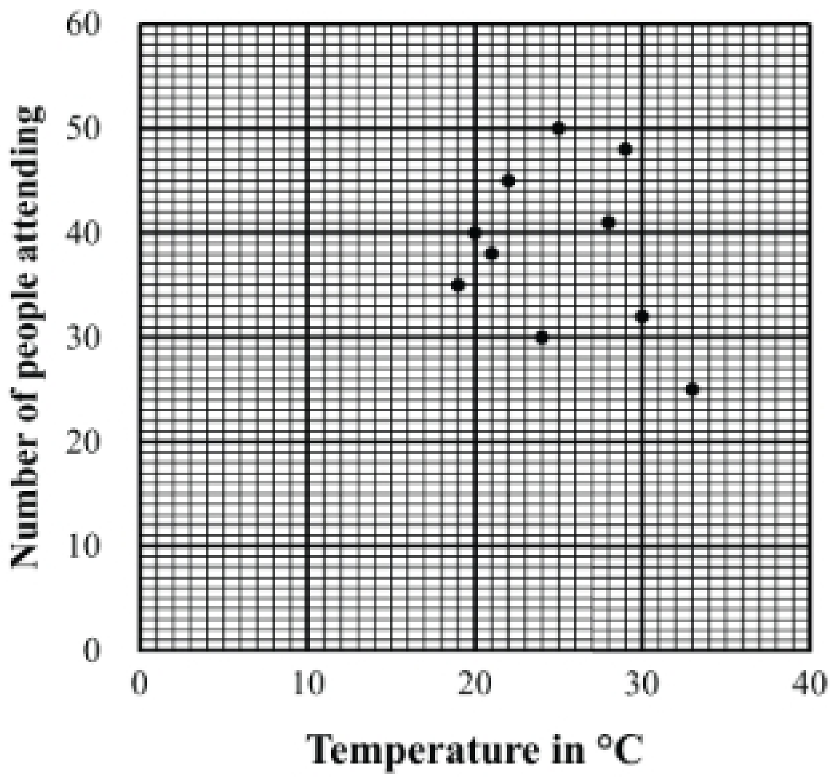
The drama department at *Umtiza High School* invited the community to attend a stage production on ten week nights in October, free of charge. Unfortunately, the air conditioning system in the hall was not working at the time. To investigate the possible effect that the temperature might have on the attendance, they measured the temperature at 18:00 and the number of people attending:

Temperature in °C	20	25	33	21	22	28	30	29	24	19
Number of people attending	40	50	25	38	45	41	32	48	30	35

(a) Represent the data by means of a scatterplot.
(b) Calculate the correlation coefficient.
(c) Comment on the relationship between the temperature at 18:00 and the number of people attending the production.

Solution

(a)



(b) $r = -0,26$

(c) There is a **weak negative linear** correlation between temperature at 18:00 and the number of people attending the production.

EXERCISE 1

(a) The ages and kilometres on the clock of ten second hand cars are given in the table below:

Age in years (x)	1	2	3,5	4	5	5	6	6,5	7	8
Kilometers in thousands (y)	15	33	50	65	73	95	84	100	95	110

(1) Represent this data by means of a scatterplot.
(2) Determine the equation of the least squares regression line.
(3) Sketch the least squares regression line on the scatterplot.
(4) Calculate the correlation coefficient.
(5) Comment on the relationship between the age and kilometres on the clock of the second hand cars.

(b) A coffee shop recorded the temperature on specific days and the number of cups of hot chocolate that were sold on these days:

Temperature in °C	14	15	18	19	20	21	23	23	24	27	28	30
Hot Chocolate Sold	60	85	70	80	75	60	50	40	55	30	80	40

(1) Determine the equation of the least squares regression line.
(2) Draw a scatterplot to represent this data and sketch the least squares regression line on the scatterplot.
(3) Calculate the correlation coefficient and comment on the relationship between the temperature and the sales of hot chocolate.

(c) The following table shows the ages of teenagers and the number of pets they own:

Age	12	13	14	14	15	16	16	17	17	18
Number of pets	1	3	2	4	3	5	2	3	5	4

(1) Determine the correlation coefficient and comment on the relationship between the age of a teenager and the number of pets owned.
(2) Determine the equation of the least squares regression line.
(3) Draw a scatterplot of the data and sketch the least squares regression line on the scatterplot.

(d) The table below shows the amount of time 15 learners spent preparing for a Mathematics test and the percentage each of them achieved for the test:

Time in hours (x)	1	2	2	2,5	3	4	6	7	8	9,5	10	11	12	13	15
Result (y)	15	30	10	40	20	50	50	65	60	80	75	85	65	90	100

(1) Determine the equation of the least squares regression line.
(2) Calculate the correlation coefficient and comment on the relationship between hours spent studying and results.

PREDICTIONS USING THE LEAST SQUARES REGRESSION LINE

We can use the least squares regression line to predict the most likely value of y, for a specific value of x. The stronger the correlation is, the more reliable our prediction.

EXAMPLE 5

The table below shows the number of cows a farmer owns and how many litres of milk are produced on the farm each day:

Number of Cows (x)	1	2	2	4	4	5	6	7	8	9	10	14	15	17	20
Litres of Milk (y)	7	20	15	30	50	50	45	100	50	80	150	140	300	280	320

(a) Determine the equation of the least squares regression line.
(b) Calculate the correlation coefficient.
(c) Predict the amount of milk the farmer would expect 16 cows to produce.
(d) How reliable is your prediction in (c)? Motivate.

Solution

(a) $\hat{y} = -33{,}95 + 17{,}31x$ (b) $r = 0{,}94$

(c) $\hat{y} = -33{,}95 + 17{,}31(16)$

$\therefore \hat{y} = 243{,}01$

$\therefore$ The farmer can expect 16 cows to produce 243,01 litres of milk per day.

(d) Very reliable. There is a **strong correlation** between the number of cows and the amount of milk produced.

When the correlation is strong, we can also use a specific value of y to determine a reasonably good estimate for x:

EXAMPLE 6

Let x be the number of cows a farmer owns and y the litres of milk produced on his farm. The least squares regression line is $\hat{y} = 17{,}31x - 33{,}95$ and the correlation coefficient is $0{,}94$. Estimate the number of cows needed to produce 200 litres of milk.

Solution

$200 = 17{,}31x - 33{,}95$

$\therefore 17{,}31x = 233{,}95$

$\therefore x = 13{,}52$

$\therefore$ The farmer would need 14 cows to produce 200 litres of milk.

INTERPOLATION AND EXTRAPOLATION

Interpolation is the practice of using the regression line to estimate a y-value for an x-value that lies **between the smallest x-value and the largest x-value** in the given data set. *Extrapolation* is the practice of using the least squares regression line to estimate a y-value for an x-value that is **greater than the greatest x-value**, or **smaller than the smallest x-value**, in the given data set. A linear trend, observed in a data set, does not always continue beyond the given range of values. For this reason, extrapolation only gives a reliable result if **the correlation is strong** and the x-value for which we estimate a y-value is **close to the x-values of the given data set**.

EXAMPLE 7

The table below shows the number of lengths girls of different ages can swim without taking a break:

Age (x)	13	13	14	14	14	15	16	16	17	18	18
Number of Lengths (y)	3	2	5	6	5	7	9	8	7	8	10

(a) Determine the equation of the least squares regression line.
(b) Determine the number of lengths a 19 year old girl should be able to swim.
(c) According to the least squares regression line, how many lengths should an 80 year old be able to swim?
(d) Is your answer in (c) reliable? Explain.

Solution

(a) $\hat{y} = -11{,}47 + 1{,}17x$

(b) $\hat{y} = -11{,}47 + 1{,}17(19)$
$\therefore \hat{y} = 10{,}76$
Approximately 11 lengths.

(c) $\hat{y} = -11{,}47 + 1{,}17(80)$
$\therefore \hat{y} = 82{,}13$
Approximately 82 lengths.

(d) The answer is not reliable. 80 is too far away from the other x-values. The least squares regression line cannot be used to determine the number of lengths an 80 year old will be able to swim.

OUTLIERS

An *outlier* is a data point that is very far from the other points. Outliers can have a significant effect on the correlation coefficient and on the equation of the least squares regression line. Statisticians often omit outliers from their calculations in order to make more accurate predictions. At school level you should **always include** outliers in your calculations, unless a question **specifically states** that you have to omit them.

EXAMPLE 8

The table below shows the rainfall on specific days in December and the number of umbrellas a shop sold on those days:

Rainfall in mm (x)	45	55	50	75	90	100	100	95	90	50	45
Umbrellas sold (y)	15	25	15	35	40	50	5	50	40	25	20

(a) Determine
 (1) the equation of the least squares regression line.
 (2) the correlation coefficient.
(b) Predict the number of umbrellas sold if 80 mm of rain was recorded.
(c) Draw a scatterplot of the data.
(d) Identify the outlier in the data set.
(e) If the outlier is removed from the data set, determine
 (1) the equation of the least squares regression line.
 (2) the correlation coefficient
(f) Use the least squares regression line in (e)(2) to predict the number of umbrellas sold if 80 mm of rain was recorded.

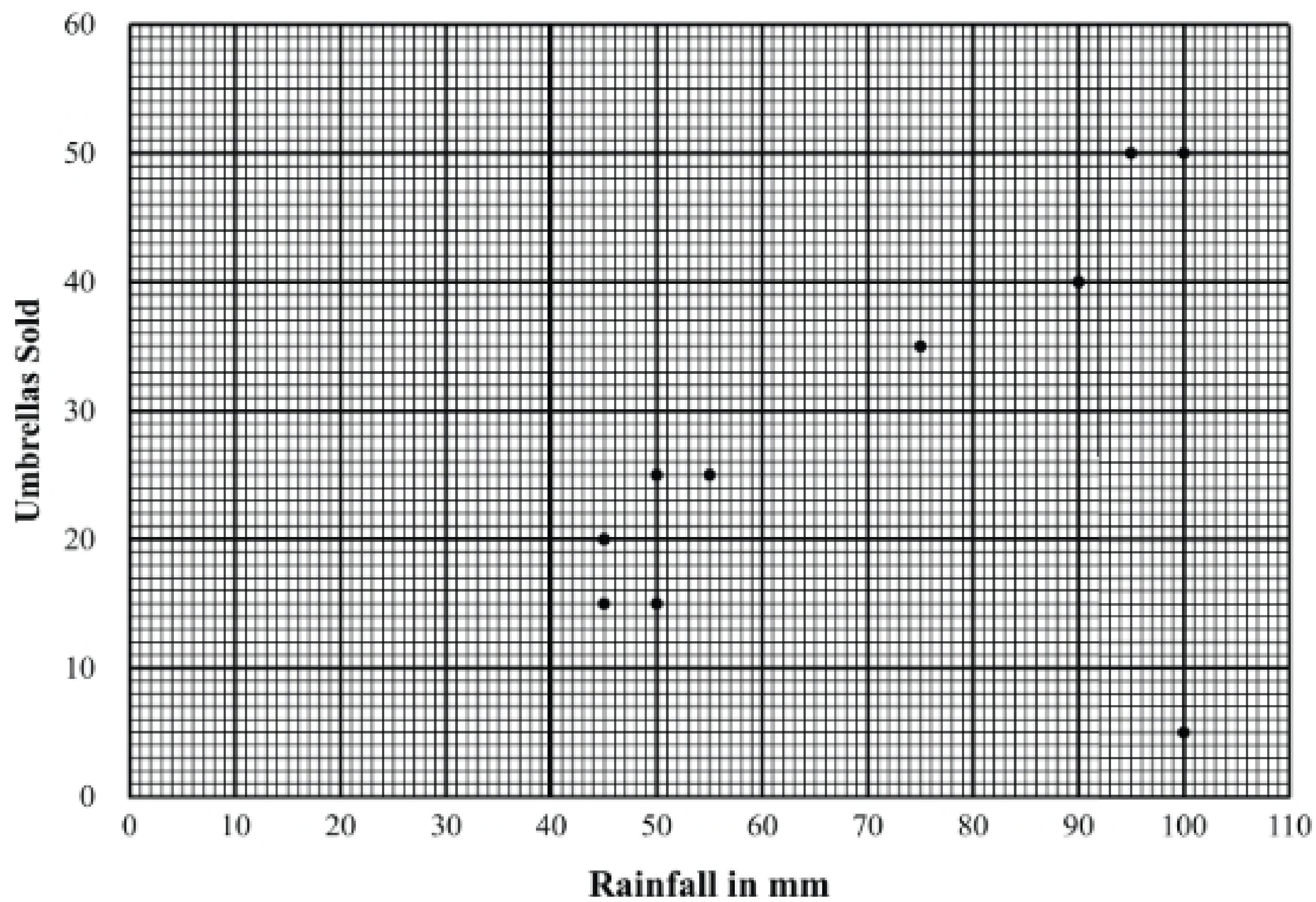
Solution

(a) (1) $\hat{y} = 4 + 0{,}35x$

 (2) $r = 0{,}54$

(b) $\hat{y} = 4 + 0{,}35(80)$
$\therefore \hat{y} = 32$
$\therefore$ Approximately 32 umbrellas will be sold.

(c)



(d) (100 ; 5)

(e) (1) $\hat{y} = -8,08 + 0,57x$

(2) $r = 0,97$

(f) $\hat{y} = -8,08 + 0,57(80)$

$\therefore \hat{y} = 37,52$

$\therefore$ Approximately 38 umbrellas will be sold.

EXERCISE 2

(a) The table below gives the ages and heights of boys between the ages of one and seventeen:

Age	1	3	4	7	9	11	14	15	17
Height in cm	70	100	100	120	145	140	160	150	170

(1) Determine the equation of the least squares regression line.
(2) Calculate the correlation coefficient and comment on the relationship between ages and heights.
(3) How tall would you expect a 13 year old to be?
(4) How reliable is your answer in (3)? Motivate.
(5) How tall would you expect an 18 year old to be?
(6) If a someone is 155 cm tall, how old would you expect him to be? Give your answer to the nearest whole number.
(7) Using the least square regression line, determine the expected height of a 40 year old man.
(8) Is your answer in (7) reliable? Motivate.

(b) The table below gives the years of experience of employees in the IT department of a company and their monthly salaries.

Years of experience	2	5	7	5	7	1	23	13	12	16	18	11	17	22	20
Salary in R' 000	15	20	50	35	45	10	5	90	80	110	125	80	140	150	140

(1) Represent this data by means of a scatterplot.
(2) Identify the outlier in the data set.
(3) Determine
  (i) the equation of the least squares regression line.
  (ii) the correlation coefficient.
(4) Estimate the monthly salary that a person who has 15 years of experience would earn.
(5) If the outlier is removed from the data set, determine
  (i) the equation of the least squares regression.
  (ii) the correlation coefficient.
(6) Use the least squares regression line in (5)(i) to predict how much a person would earn if they have 15 years experience.

(c) The following table shows the ages of ten men and the mass that they can benchpress:

Age (years)	30	32	33	38	42	49	52	53	59	60
Mass (kg)	75	50	40	50	62	68	60	45	40	38

(1) Draw a scatterplot to represent this data.
(2) Determine the equation of the least squares regression line.
(3) Sketch the least squares regression line on the scatterplot.
(4) Calculate the correlation coefficient and interpret the result.
(5) What would you estimate a 50 year old man would be able to bench press?
(6) How reliable is your answer in (5)? Motivate.
(7) Can the least squares regression line be used to predict the mass that an 8 year old can bench press? Motivate your answer Mathematically and practically.

(d) The table below gives the values of x and y. The equation of the least square regression line is $\hat{y} = 1,95 + 0,83x$:

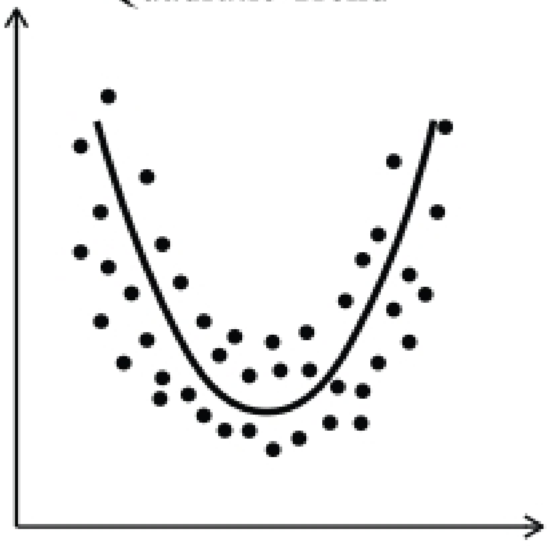
x	1	1	2	3	4	6	7	8	8	10
y	3	2	4	6	5	z	8	9	8	11

(1) Predict the value of y if x is 5.
(2) Predict the value of y if x is 12.
(3) Estimate the value of x if y is 10.
(4) Explain why the least squares regression line cannot be used to determine the missing value z.
(5) Determine the mean value of x.
(6) Determine the mean value of y.
(7) Determine the missing value z.

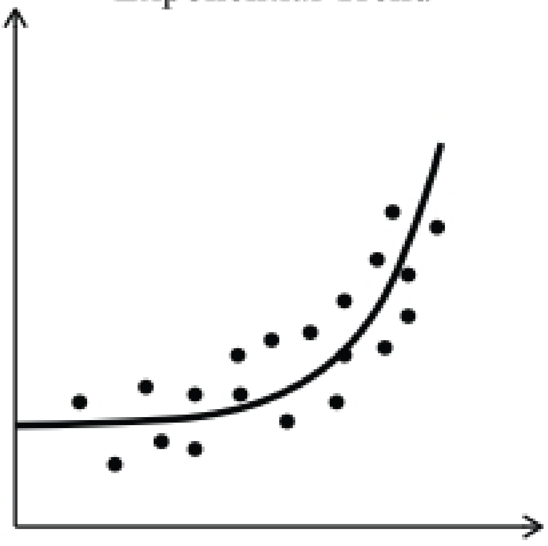
NON-LINEAR TRENDS

Thus far, we have worked with data with a linear trend, which is why we could draw a line of best fit. Many data sets do not follow a linear trend, for example:

Quadratic Trend



Exponential Trend



EXERCISE 3

(a) The table below shows the number of weeks since a new App had launched and the number of times it had been dowloaded that week:

Weeks since launch	1	3	4	5	6	8	10	15	17	22	25	27
Number of Downloads (in thousands)	4	50	75	100	110	30	180	210	190	145	110	55

(1) Represent this data on a scatterplot.
(2) What type of trend (linear, quadratic or exponential) best fits this data?
(3) Identify any outliers in this data set.

(b) Black rhinos have been reintroduced into Chad fifty years after they became extinct in the country. The table below shows the years since the reintroduction of the Rhinos and the population.

Years since reintroduction	0	5	10	15	20	25	30	35	40
Number of Rhinos	5	10	17	35	80	195	460	1100	2900

(1) Draw a scatterplot to represent this data set.
(2) What type of relationship exists between the years since reintroduction and the number of Rhinos in Chad?
(3) How many Rhinos were initially reintroduced into Chad?

(c) The table below shows the caloric requirement of males of different ages:

Age	13	15	16	18	20	25	35	45	55
Calories	2 200	2 600	2 800	3 200	3 000	2 800	2 600	2 400	2 000

(1) Represent this data on a scatterplot.
(2) What type of trend (linear, quadratic or exponential) best fits this data?
(3) Do you expect a 60 year old man to require more or less than 2000 calories a day?

CONSOLIDATION AND EXTENSION EXERCISE

(a) The table below shows the number of social media posts by people on a specific day and the number of "likes" they received:

Number of Posts	3	4	5	7	9	10	11	12	13	14	15	16	17	18	19	20
Number of Likes	5	20	20	50	90	100	120	140	155	200	225	270	290	325	375	380

(1) Draw a scatterplot to represent this data.
(2) Determine the equation of the least squares regression line and sketch the least squares regression line on the scatterplot.
(3) Calculate the correlation coefficient and comment on the relationship between the number of social media posts and the number of likes.
(4) Estimate the number of likes a person with 6 posts can expect to receive.
(5) How reliable is your estimate in (4)? Motivate.

(b) The heights (x) in centimetres and weights (y) in kilograms of some Springbok rugby players are given below:

x	186	205	186	184	189	200	210	183	182	187	184	183	178	185	180	176	170	175	189
y	90	122	93	93	117	119	117	96	100	106	125	117	117	118	89	106	80	85	101

(1) Consider the **heights** of the Springboks.
  (i) Calculate the mean and median of the heights.
  (ii) Comment on the distribution of the heights.
(2) Consider the **weights** of the Springboks.
  (i) Draw a box-and-whisker diagram of the weights.
  (ii) Comment on the distribution of the weights.
(3) Consider the **relationship** between the heights and the weights of the Springboks.
  (i)* Without calculating the correlation coefficient, would you expect this data to have a strong correlation or not? Motivate your answer.
  (ii) Calculate the correlation coefficient.
  (iii) Determine the equation of the least squares regression line.
  (iv) Predict the weight of a Springbok that is 190 cm tall.
  (v) How reliable is your prediction in (iv)?

(c) The temperature, in $°C$, at noon on ten days and the number of units of electricity used to heat a house on each of those 10 days are shown in the table below:

Noon Temperature in °C	7	8	9	2	4	z	0	9	2	3
Units of Electricity	32	20	27	37	32	28	41	23	33	36

The equation of the least squares regression line is $\hat{y} = -1{,}7766x + 39{,}78$.

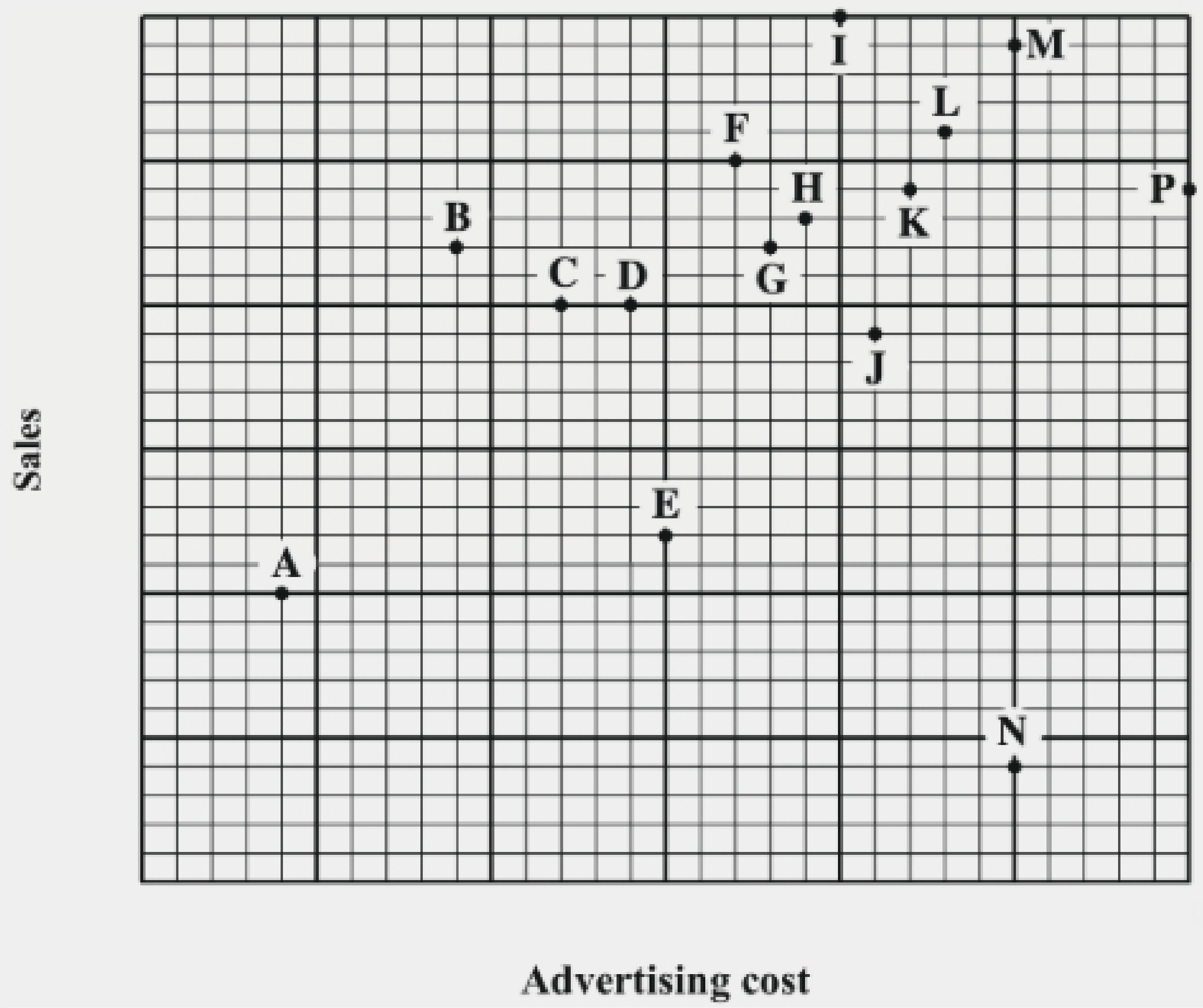
(1) Calculate the value of z, rounded to the nearest whole number.
(2) How many units of electricity would you expect to be used if the noon temperature is $6\ °C$?
(3) If $z = 6$, calculate the correlation coefficient between the temperature at noon and the number of units of electricity used and comment on the relationship.
(4) Predict how many units of electricity a household would use for heating if the noon temperature is $10\ °C$.
(5) Estimate the noon temperature on a day that 35 units of electricity was used.

(d) The following table shows the ages of females in years and their heights in cm:

Age in years	7	8	8	9	10	11	11	14	15	16
Height in cm	120	130	140	135	150	140	155	170	165	165

(1) Determine the equation of the least squares regression line.
(2) Determine the correlation coefficient and interpret the result.
(3) What height would you predict for a 13 year old girl?
(4) How reliable is your prediction in (3)?
(5) Using the least square regression line, determine the expected height of a 30 year old woman.
(6) Is your answer in (5) reliable? Motivate.

(e) The scatterplot below shows the amount spent on **advertising** by a company in **millions** of Rands and the resulting **sales** in **millions** of Rands:



The equation of the least squares regression line is $\hat{y} = 14{,}82 + 0{,}32x$.

(1) Which one of the following correlation coefficients fits the data best?
A. $r = 0{,}28$ B. $r = 0{,}82$ C. $r = -0{,}5$

(2) Does the data support the idea that spending more money on advertising would result in more sales? Motivate.

(3) What amount of sales would you predict if the company spends R20 million on advertising?

(4) Is your prediction in (3) reliable? Motivate.

(5) Which data point is an outlier?

(6) Would the correlation coefficient increase, decrease or remain the same if the outlier is removed from the data set?

CHAPTER ELEVEN

The Counting Principle

To calculate the probability of an event requires knowing the total number of possible outcomes of an experiment, as well as the number of outcomes under certain conditions. These numbers are not always obvious. In this chapter, we will study techniques used to calculate the number of outcomes in different scenarios. At the foundation of these techniques is the so-called *Counting Principle*.

THE COUNTING PRINCIPLE

If task 1 can be performed in n_1 ways and task 2 can be performed in n_2 ways, then the number of ways in which both tasks can be performed is

$$\boxed{n_1 \times n_2}$$

This can be extended to more than two tasks.

EXAMPLE 1

A couple decided on the following menu for their wedding reception:

MENU

Starter	Main Course	Dessert
Garlic Snails	Grilled Kingklip	Ice cream with chocolate sauce
Chicken Livers	Beef Fillet	Crème Brûlée
Calamari	Lamb Shanks	Malva Pudding
	Chicken Cordon Bleu	Peppermint Crisp Tart
	Vegetarian Pasta	

(a) How many different meal combinations can be chosen?
(b) A particular person wishes to have *Grilled Kingklip* as his main course. How many different meal combinations can he choose?

Solution

(a) There are 3 tasks:
Task 1 is to choose a starter: 3 ways.
Task 2 is to choose a main course: 5 ways.
Task 3 is to choose a dessert: 4 ways.

We draw a line to represent each task: ___ ___ ___

On each line we write the number of ways to perform the task: $\underline{\ 3\ }\ \underline{\ 5\ }\ \underline{\ 4\ }$

Applying the counting principle, we multiply these numbers together: $3 \times 5 \times 4 = 60$
$\therefore$ There are 60 possible meal combinations.

(b) Since the person has decided on his main course, there is only one option for this task:

$$\underline{\ 3\ }\quad\underline{\ 1\ }\quad\underline{\ 4\ }$$

Multiply the numbers: $3\times1\times4=12$
$\therefore$ There are 12 possible meal combinations.

The above example illustrates how the fundamental counting principle is applied. This principle can be applied in a variety of contexts. The problems we will encounter often involve arranging digits of numbers, letters of words or digits/letters comprising codes or passwords. In some instances the digits/letters may repeat and in other instances the digits/letters may not repeat.

WHEN REPETITION IS ALLOWED

EXAMPLE 2

How many 4-digit codes (using digits 0-9) are possible if each digit may be used any number of times?

Solution

In this example, the choice of each digit is regarded as a task. Since there are 4 digits, we have 4 tasks. For each task, we have 10 options (0, 1, 2, 3, 4, 5, 6, 7, 8 or 9):

$$\underline{\ 10\ }\quad\underline{\ 10\ }\quad\underline{\ 10\ }\quad\underline{\ 10\ }$$

$10\times10\times10\times10=10000$ possibilities.

Note: In cases where repetition is allowed, we can use:

$$\boxed{\text{number of possibilities} = n^r}$$

$n =$ the number of options to choose from

$r =$ the number of times we choose from these options

WHEN REPETITION IS NOT ALLOWED

When choosing from a group of different objects multiple times, and repetition is not allowed, the number of objects we choose from is reduced by 1 each time a selection is made.

EXAMPLE 3

How many 4-digit codes (using digits 0-9) are possible if no digit may be used more than once?

Solution

We will start with 10 options for the first digit and then reduce it by 1 for each digit after that:

$$\underline{\ 10\ }\quad\underline{\ 9\ }\quad\underline{\ 8\ }\quad\underline{\ 7\ }$$

$10\times9\times8\times7=5040$ possibilities.

EXAMPLE 4

How many ways can the letters of the word ACTION be arranged? Letters may not repeat.

Solution

There are 6 letters in the word ACTION. We will start with 6 options and then reduce it by 1 for each digit after that:

$$\underline{\;6\;}\quad\underline{\;5\;}\quad\underline{\;4\;}\quad\underline{\;3\;}\quad\underline{\;2\;}\quad\underline{\;1\;}$$

$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ possibilities.

FACTORIAL NOTATION

Note that, in example 4, we have used each letter exactly once. No letters were omitted. This means that we simply arranged the letters. This is called a pure **arrangement**. The calculation involves multiplying numbers that are reduced by 1 successively, till we get to the number 1. Such products are called **factorials** and indicated by the mathematical symbol "!":

$$6 \times 5 \times 4 \times 3 \times 2 \times 1 = 6!$$

Some more examples:

$4! = 4 \times 3 \times 2 \times 1 = 24$

$3! = 3 \times 2 \times 1 = 6$

$2! = 2 \times 1 = 2$

$1! = 1$

Factorials can be calculated by using the "! button" on a calculator.

EXERCISE 1

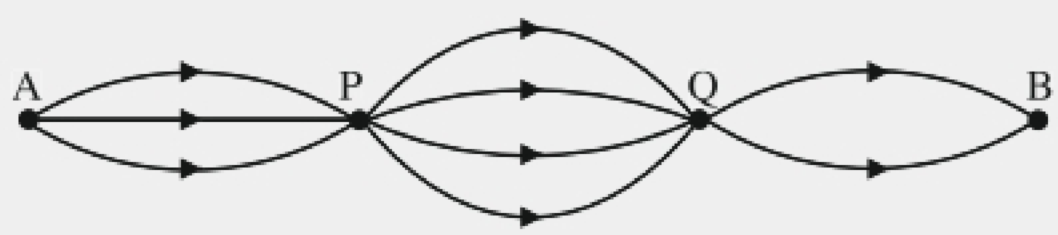
(a) There are three vacancies at a specific company. The company had shortlisted candidates for each position:

Post	Candidate
Admin Clerk	Ben, Zandile, Ann
Accountant	Donald, Thabo, Tebogo, Lindiwe
Personal Assistant for the CEO	Dorothy, Sibongile

(1) How many ways can these positions be filled?

(2) How many ways can the positions be filled if it is certain that Donald will get the job as accountant.

(b) In the diagram below, there are three routes from A to P, four routes from P to Q, and two routes from Q to B. How many different routes are there from A to B (via P and Q) in the direction of the arrows?



(c) Using the digits 1, 2, 3, 4, 5, 6 and 7,
  (1) how many two-digit numbers can be formed if digits may be repeated?
  (2) how many four-digit numbers can be formed if digits may **not** be repeated?

(d) Using the letters A, B, C, D and E,
  (1) how many three-letter 'words' can be formed if letters may be repeated?
  (2) how many four-letter 'words' can be formed if letters may **not** be repeated?
  (The 'words' don't have to mean anything.)

(e) All clients of a specific bank has a Personal Identity Number (PIN), which consists of five digits chosen from the digits 0 to 9. How many personal identity numbers are possible if
  (1) digits may be repeated?
  (2) digits may not be repeated?

(f) Consider the word COUNTED.
  (1) How many ways can the letters of the word be arranged if each letter must be used exactly once?
  (2) How many five-letter words can be formed from the letters of the word COUNTED if no letter may be used more than once?

(g) Seven learners are to be seated on seven chairs in the front row of a classroom. How many different ways can these learners be seated?

ARRANGEMENTS WITH RESTRICTIONS

In counting problems, we often encounter arrangements where there are certain restrictions which have to be adhered to. These restrictions may include restrictions on certain positions or the requirement for certain objects to be grouped together.

RESTRICTIONS ON POSITION

The following example illustrates how restrictions on position are dealt with:

EXAMPLE 5

How many ways can the letters of the word EQUATIONS be arranged if
(a) the first letter must be an E?
(b) the first letter must be an E and the last letter must be an S?
(c) the second letter must be a Q and the seventh letter must be an O?
(d) the first letter must be a vowel?
(e) the first and last letters must be vowels?

Solution

(a) The first letter is fixed, which means that there is only one option for the first letter and we have only 8 letters to arrange:

$$\underline{\mathbf{1}}\ \ \underline{8}\ \ \underline{7}\ \ \underline{6}\ \ \underline{5}\ \ \underline{4}\ \ \underline{3}\ \ \underline{2}\ \ \underline{1}$$

$1\times 8! = 40320$ possibilities.

(b) The first and last letters are fixed. We have only 7 letters to arrange:

$$\underline{\mathbf{1}}\ \ \underline{7}\ \ \underline{6}\ \ \underline{5}\ \ \underline{4}\ \ \underline{3}\ \ \underline{2}\ \ \underline{1}\ \ \underline{\mathbf{1}}$$

$1\times 1\times 7! = 5040$ possibilities.

(c) The second and seventh letters are fixed. We have only 7 letters to arrange:

$$\underline{\ 7\ }\ \ \underline{\ \mathbf{1}\ }\ \ \underline{\ 6\ }\ \ \underline{\ 5\ }\ \ \underline{\ 4\ }\ \ \underline{\ 3\ }\ \ \underline{\ \mathbf{1}\ }\ \ \underline{\ 2\ }\ \ \underline{\ 1\ }$$

$1\times1\times7!=5040$ possibilities.

(d) There are five vowels in the word EQUATIONS. This means there are 5 options for the first position. Whichever vowel we choose, we will have 8 letters left to arrange:

$$\underline{\ \mathbf{5}\ }\ \ \underline{\ 8\ }\ \ \underline{\ 7\ }\ \ \underline{\ 6\ }\ \ \underline{\ 5\ }\ \ \underline{\ 4\ }\ \ \underline{\ 3\ }\ \ \underline{\ 2\ }\ \ \underline{\ 1\ }$$

$5\times8!=201600$ possibilities.

(e) Since there are five vowels, we have 5 options for the first position. After choosing one of these five for the first position, we will have 4 options for the last position. We will then have 7 letters left to arrange:

$$\underline{\ \mathbf{5}\ }\ \ \underline{\ 7\ }\ \ \underline{\ 6\ }\ \ \underline{\ 5\ }\ \ \underline{\ 4\ }\ \ \underline{\ 3\ }\ \ \underline{\ 2\ }\ \ \underline{\ 1\ }\ \ \underline{\ \mathbf{4}\ }$$

$5\times4\times7!=100800$ possibilities.

<u>GROUPINGS</u>

The following example illustrates how the number of arrangements are calculated if certain objects have to be grouped together:

<u>EXAMPLE 6</u>

How many ways can the letters of the word PRODUCT be arranged if the P, O and C must be grouped together in any order?

<u>Solution</u>

We first calculate the number of possible arrangements of the letters P, O and C:

$$\underline{\ 3\ }\ \ \underline{\ 2\ }\ \ \underline{\ 1\ }\qquad 3!\text{ possibilities}$$

The letters P, O and C now form a group, which is counted as a single object in the final arrangement of all the letters:

$$\underset{1}{\boxed{\text{POC}}}\ \ \underset{2}{\boxed{\text{R}}}\ \ \underset{3}{\boxed{\text{D}}}\ \ \underset{4}{\boxed{\text{U}}}\ \ \underset{5}{\boxed{\text{T}}}$$

There are now 5 objects to arrange:

$$\underline{\ 5\ }\ \ \underline{\ 4\ }\ \ \underline{\ 3\ }\ \ \underline{\ 2\ }\ \ \underline{\ 1\ }\qquad 5!\text{ possibilities}$$

Notice that, in this process, we have two main tasks:

Task 1:	Arranging the letters P, O and C:	3!
Task 2:	Arranging 5 objects of which the group (P, O and C) is 1:	5!

According to the counting principle, we multiply the number of ways each of these tasks can be performed to get the total number of possibilities for the procedure:

$3!\times5!=720$ possibilities.

EXAMPLE 7

A group of eight people consists of four couples: four men and four women. (Each couple consists of a man and a woman.) These eight people are to be seated on a bench. How many possible seating arrangements are there if

(a) all the men have to sit together?

(b) all the men have to sit together **and** all the women have to sit together?

(c) each couple has to sit together?

(d) they have to alternate between men and women?

Solution

(a) Calculate the number of arrangements for the 4 men: $4!$

Now treat the group of four men as a single object:

$$\begin{array}{ccccc} \boxed{M_1\ M_2\ M_3\ M_4} & \boxed{W_1} & \boxed{W_2} & \boxed{W_3} & \boxed{W_4} \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

There are now 5 objects to arrange: $5!$

$4! \times 5! = 2880$ possibilities.

(b) Calculate the number of arrangements of the 4 men: $4!$

Calculate the number of arrangements of the 4 women: $4!$

Now treat the two groups as single objects:

$$\begin{array}{cc} \boxed{M_1\ M_2\ M_3\ M_4} & \boxed{W_1\ W_2\ W_3\ W_4} \\ 1 & 2 \end{array}$$

There are now 2 objects to arrange: $2!$

$4! \times 4! \times 2! = 1152$ possibilities.

(c) There are 2! arrangements for each couple.
Multiply the arrangements for the 4 couples together: $2! \times 2! \times 2! \times 2! = (2!)^4$

Now treat the couples as single objects:

$$\begin{array}{cccc} \boxed{M_1\ W_1} & \boxed{M_2\ W_2} & \boxed{M_3\ W_3} & \boxed{M_4\ W_4} \\ 1 & 2 & 3 & 4 \end{array}$$

There are now 4 objects to arrange: $4!$

$(2!)^4 \times 4! = 384$ possibilities.

(d) Starting with a man:

$$\begin{array}{cccccccc} M & W & M & W & M & W & M & W \\ \underline{4} & \underline{4} & \underline{3} & \underline{3} & \underline{2} & \underline{2} & \underline{1} & \underline{1} \end{array}$$

Starting with a woman:

$$\begin{array}{cccccccc} W & M & W & M & W & M & W & M \\ \underline{4} & \underline{4} & \underline{3} & \underline{3} & \underline{2} & \underline{2} & \underline{1} & \underline{1} \end{array}$$

There are 2 options and in each option there are $4! \times 4!$ possibilities.
This gives a total of $4! \times 4! \times 2 = 1152$ possibilities.

EXERCISE 2

(a) Consider the word CHEMISTRY.
   (1) How many ways can the letters of the word be arranged?
   (2) How many arrangements can be made starting with a C and ending with a Y?
   (3) How many arrangements can be made if the letters M, I and S must be grouped together in any order?

(b) Consider the word ANSWER.
   (1) How many six-letter arrangements can be made using all the letters of this word?
   (2) How many arrangements can be made if the letters A, N and S must be the first three letters (in any order)?

(c) Three boys and three girls are to be seated in the front row at assembly.
   (1) How many different ways can these six learners be seated?
   (2) How many different ways can they be seated if two particular learners must be seated together?
   (3) How many different ways can they be seated if all the boys must be seated together and all the girls must be seated together?
   (4) How many different ways can they be seated if they have to alternate between boys and girls?

(d) Six glasses and five bottles are to be arranged on a shelf. How many arrangements can be made if all the glasses must be grouped together and all the bottles must be grouped together?

(e) There are seven different Mathematics books and five different Science books on a bookshelf.
   (1) How many different ways can the books be arranged?
   (2) How many different ways can the books be arranged if a Mathematics book must be in the first position and a Science book must be in the last position?
   (3) How many different ways can the books be arranged if all the Mathematics books must be placed together and all the Science books must be placed together?

(f) A band is planning a concert tour with one performance in each of the cities: Johannesburg, Pretoria, Durban, Bloemfontein, Cape Town and Port Elizabeth. How many different ways can they arrange their itinerary if
   (1) there are no restrictions.
   (2) the first performance must be in Johannesburg and the last performance must be in Cape Town.
   (3) the performances in the non-coastal cities (Johannesburg, Pretoria and Bloemfontein) must be grouped together.

(g) Three Americans, five Germans, four British and six South Africans attend a conference in South Africa. These delegates are to be seated together in a row. How many ways can they be seated if
   (1) the South Africans are to occupy the first six seats?
   (2) people must be seated together by nationality (Americans together, Germans together *et cetera*).

PROBABILITY

The counting principle can be used in the calculation of probabilities. In a counting problem, we are often required to calculate the probability that the outcome will satisfy a certain condition/ restriction. The probability is then given by:

$$\boxed{\text{Probability} = \frac{\text{Number of possibilities meeting the condition}}{\text{Total number of possibilities}}}$$

EXAMPLE 8

Five men and four women are seated on a bench randomly. What is the probability that the four women will be seated together?

Solution

$\text{Total number of possibilities (no restrictions)} = 9!$

$\text{Number of possibilities if women are seated together} = 4! \times 6!$

$$\text{Probability} = \frac{\text{Number of possibilities if women are seated together}}{\text{Total number of possibilities}} = \frac{4! \times 6!}{9!} = \frac{1}{21}$$

EXAMPLE 9

Every client of a South African bank has a Personal Identity Number (PIN) which consists of five digits, chosen from the digits 0 to 9. Digits may not repeat. A PIN is generated randomly. What is the probability that it will start with 5 and end with 6?

If there are no restrictions:

$$\underline{10}\quad \underline{9}\quad \underline{8}\quad \underline{7}\quad \underline{6}$$

$\text{Total number of possibilities (no restrictions)} = 10 \times 9 \times 8 \times 7 \times 6$

If the code starts with a 5 and ends with a 6:

$$\underline{1}\quad \underline{8}\quad \underline{7}\quad \underline{6}\quad \underline{1}$$

$\text{Number of possibilities starting with 5 and ending with } 6 = 1 \times 1 \times 8 \times 7 \times 6$

$$\text{Probability} = \frac{8 \times 7 \times 6}{10 \times 9 \times 8 \times 7 \times 6} = \frac{1}{90}$$

EXERCISE 3

(a) Consider the word CAREFUL.
 (1) How many ways can the letters of this word be arranged?
 (2) If the letters of the word are arranged randomly, calculate the probability that the resulting word will start with R and end with U.

(b) Consider the word DANGEROUS.
 (1) How many ways can the letters of the word be arranged?
 (2) If the letters of the word are arranged randomly, what is the probability that the resulting word will start with the letters DAN (in this order)?

(c) If the letters of the word FACTOR are arranged randomly, calculate the probability that the letters F, A and C will be the first three letters, in any order.

(d) If five different Geography books and six different History books are arranged randomly on a bookshelf, calculate the probability that
 (1) the first and last positions will be occupied by Geography books.
 (2) all the Geography books will be grouped together.
 (3) all the Geography books will be grouped together and all the History books will be grouped together.
 (4) all the Geography books will be first, followed by all the History books.

(e) Three boys and four girls are seated randomly on a bench. Calculate
 (1) the total number of possible arrangements.
 (2) the probability that all the boys will be seated together.
 (3) the probability that they will be seated according to the pattern: girl, boy, girl, boy *et cetera*.

(f) Five Zimbabweans, four Kenyans and three South Africans are seated randomly in a row. What is the probability that they will be seated together by nationality (all the Zimbabweans together, all the Kenyans together and all the South Africans together)?

(g) The digits 1, 2, 3, 4, 5, 6, 7, 8 and 9 are used to generate a four-digit code randomly. Any digit may be used any number of times. Calculate the probability that
 (1) the code will consist only of even digits.
 (2) the digits of the code will all be different.
 (3) the code will consist of different even digits.

MORE ADVANCED COUNTING PROCEDURES

IDENTICAL OBJECTS

When n objects are arranged, where there are m_1 identical objects of type 1, m_2 identical objects of type 2 *et cetera*, then the number of different arrangements is given by

$$\boxed{\frac{n!}{m_1!\times m_2!\times ...}}$$

EXAMPLE 10

Consider the word SUBSTITUTE. (Repeating letters are regarded as identical.)

(a) How many different arrangements of the letters of this word are possible?

(b) If the letters of the word are arranged randomly, calculate the probability that the word will start and end with the letter S.

Solution

(a) There are 10 letters including 2 S's, 2 U's and 3 T's:

$$\frac{10!}{2!\times 2!\times 3!}=151\ 200 \text{ arrangements.}$$

(b) If the word starts and ends with an S, all that is left is to calculate the number of possible arrangements of the remaining letters UBTITUTE: $\dfrac{8!}{2!\times 3!}$

Now divide this by the total number of possibilities to obtain the probability:

$$\frac{\left(\dfrac{8!}{2!\times 3!}\right)}{\left(\dfrac{10!}{2!\times 2!\times 3!}\right)}=\frac{1}{45}$$

Alternative Solution:
When calculating probability, we may ignore the fact that some letters are identical and treat all the letters as different. We must, however ensure that we remain consistent in our reasoning. We regard all letters as different in the restricted as well as the total number of possibilities:

Total number of possibilities $= 10!$

Starting and ending with an S:

$\underline{2}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}\ \underline{1}$ $\quad 2\times 1\times 8!$ possibilities

$$\text{Probability}=\frac{2\times 8!}{10!}=\frac{1}{45}$$

COUNTING THE REST - THE "NOT" RULE

It is sometimes very difficult to count what is being described, but much easier to count the opposite of what is being described. In such cases we will use the rule:

$$\boxed{n(\text{not A})=n(\text{S})-n(\text{A})} \quad \text{or} \quad \boxed{n(\text{A})=n(\text{S})-n(\text{not A})}$$

Number of possibilities meeting a condition
= Total number of possibilities – Number of possiblities NOT meeting the condition

EXAMPLE 11

Six learners are to be seated on six chairs in an auditorium.
(a) How many ways can these six learners be seated?
(b) How many ways can they be seated if two particular learners refuse to sit together?

Solution

(a) $6! = 720$ arrangements.
(b) Number of ways the two learners can be seated together $= 2!\times 5!$
Number of arrangements in which the two learners don't sit together $= 6!-2!\times 5! = 480$

EXAMPLE 12

Using the digits 1 to 9, how many five-digit numbers can be constructed containing **at least one 6**? Digits may repeat.

Solution

There are many possibilities (one 6, two 6's, three 6's *et cetera*). To calculate the number of possibilities of all these would be very cumbersome. Instead, let us consider the opposite:
How many possibilities are there containing no 6's?

$$\underline{8}\ \ \underline{8}\ \ \underline{8}\ \ \underline{8}\ \ \underline{8} \qquad 8^5 \text{ possibilities}$$

How many possibilities are there overall (no restrictions)?

$$\underline{9}\ \ \underline{9}\ \ \underline{9}\ \ \underline{9}\ \ \underline{9} \qquad 9^5 \text{ possibilities}$$

Number of five-digit numbers with at least one 6

= Total number – Number containing no 6

$= 9^5 - 8^5$

$= 26\,281$

EXERCISE 4

(a) Consider the word PROBABILITY. (Repeating letters are regarded as identical.)
   (1) How many different arrangements of the letters of this word are possible?
   (2) If the letters of this word are arranged randomly, calculate the probability that the resulting word will start and end with the letter B.

(b) Five learners (three boys and two girls) are to be seated in a row for a photograph.
   (1) How many ways can these learners be arranged?
   (2) How many ways can they be arranged if the two girls refuse to sit together?
   (3) If the learners are arranged randomly, what is the probability that the two girls will **not** sit together?

(c) Consider the word EINSTEIN. (Repeated letters are regarded as identical.)
   (1) How many different arrangements of the letters of this word are possible?
   (2) If the letters of this word are arranged randomly, calculate the probability that the identical letters will be grouped together.

(d) Four-letter "words" are formed from the 26-letter alphabet. Letters may repeat. The "words" don't have to mean anything. How many such "words" can be formed containing at least one P.

(e) Consider the word BELLVILLE. (Repeating letters are regarded as identical.)
   (1) How many different arrangements of the letters of this word are possible?
   (2) If the letters of this word are arranged randomly, calculate the probability that
      (i) the resulting word will start and end with the letter L.
      (ii) all the L's will be grouped together.

(f) The digits 1 to 8 are used to create a random six-digit code. Calculate the probability that the code will contain at least one 5 if digits may repeat.

(g) Two friends, Thato and Ben, take part in the 400 m event at an athletics meeting. There are eight lanes and eight athletes. Calculate the number of arrangements in the starting line-up if

(1) Thato and Ben are placed next to each other.

(2) Thato and Ben are not placed next to each other.

(3) Thato in placed in Lane 1 and Ben is **not** placed next to Thato.

CONSOLIDATION AND EXTENSION EXERCISE

(a) A new series of vehicle registration numbers consist of

- two alphabet letters (excluding the vowels and the letter Q), followed by
- three numeric digits (0-9), followed by
- two alphabet letters.

How many such numbers are possible if

(1) repetition of letters and/or digits are allowed?

(2) repetition of letters and/or digits are not allowed?

(b) The digits 3, 4, 6, 7, 8 and 9 are used to construct a four-digit number. How many possibilities are there if

(1) repetition of digits is allowed?

(2) repetition of digits is not allowed?

(3) repetition of digits is allowed and the number must be divisible by 2?

(4) repetition of digits is not allowed and the number must be divisible by 2?

(c) The letters of the word MATRIX are arranged randomly.

(1) How many possible arrangements are there?

(2) What is the probability that the last letter will be a vowel?

(3) What is the probability that the two vowels will appear together?

(d) There are six seats in a row on an aeroplane. Six passengers have to be seated in this row:
Three women: Dorothy, Ann and Michelle
Three men: Eddy, Leslie and Simba

(1) How many possible ways can these passengers be arranged in the row?

(2) How many possibilities are there if all the men must be seated together and all the women must be seated together?

(3) How many possibilities are there if Dorothy must sit in the first seat and Ann must sit in the last seat?

(4) If the six passengers are arranged randomly in the row, what is the probability that the first seat as well as the last seat will be occupied by a woman?

(5) If the six passengers are arranged randomly in the row, what is the probability that the men and women will alternate?

(e) Consider the word JOHANNESBURG. (Repeating letters are regarded as identical.)

(1) How many different arrangements of the letters of this word are possible?

(2) If the letters of this word are arranged randomly, what is the probability that the resulting "word" will start with J and end with G?

(f) A random five-digit password is generated, using the digits 1 to 5.

(1) If digits may not be repeated, what is the probability that 1 and 2 will **not** appear together (next to each other) in the password?

(2) If digits may be repeated, what is the probability that the password will contain at least one odd digit?

(g) Consider the word MISSISSIPPI. (Repeated letters are regarded as identical.)

(1) How many different arrangements of the letters of this word are possible?

(2)** If the letters of this word are arranged randomly, what is the probability that the resulting "word" will start and end with the same letter?

(h)* A four-digit number is randomly formed, using only the digits 0, 1, 2, 4, 5, 6, 8 and 9. No digit is used more than once. Keep in mind that a number **may not** have 0 as the first digit.

(1) How many possible numbers can be formed?

(2) What is the probability that the number will be divisible by 10?

(3)** What is the probability that the number will be divisible by 5?

(i) Three people are to be selected from a group of eight to serve on a committee.

(1) If these three people must serve in three different positions on the committee, how many possibilities are there?

(2) Suppose three people were already selected to serve on the committee, but their positions are not yet allocated. How many possible ways are there to allocate these three people to the three positions?

(3)** Use your answers to (1) and (2) to calculate the number of possible combinations of three people that can be selected, from the group of eight, to serve on the committee, if the positions in which they serve are irrelevant.

Answers to Exercises

CHAPTER ONE

EXERCISE 1

(a) (1) $T_n = n^2 + 2n + 5$ (2) $T_n = -3n^2 + 2n - 1$ (3) $T_n = \frac{1}{2}n^2 + n - \frac{3}{2}$
(4) $T_n = -n^2 - \frac{1}{2}n + \frac{1}{2}$

(b) (1) 4801 (2) 60

(c) (1) −324 (2) 45

(d) (1) 16 (2) 2

(e) $x = 5; y = 35$

(f) (1) −2 (2) $T_{50} = 20$

(g) (1) $T_n = 2n^2 - 2n + 1$ (2) 11 400

(h) $a = -3; b = 13; c = -5$

(i) $T_n = 2n^2 - 6n + 8$

EXERCISE 2

(a) (1) $T_n = 6n - 4$ (2) $T_n = -3n + 13$ (3) $T_n = 2n - 5$
(4) $T_n = -2n$ (5) $T_n = 2n - \frac{3}{2}$ (6) $T_n = -\frac{2}{3}n + \frac{11}{3}$
(7) $T_n = \frac{3}{4}n - \frac{7}{4}$ (8) $T_n = -2n + \frac{11}{5}$

(b) (1) $T_n = 7n - 4$ (2) 241 (3) 28

(c) (1) $T_n = -\frac{3}{2}n + 2$ (2) −148 (3) 16

(d) (1) −76 (2) 46

(e) 82

(f) 74

(g) (1) 13 (2) $T_n = 8n - 3$ (3) 917

(h) (1) −3 (2) $T_n = -4n + 2$

(i) (1) 2 (2) $T_n = -4n + 7$

(j) $x = 2; y = 5$

EXERCISE 3

(a) (1) $T_n = 5(2)^{n-1}$ (2) $T_n = 4(-3)^{n-1}$ (3) $T_n = -3(2)^{n-1}$
(4) $T_n = -2(-5)^{n-1}$ (5) $T_n = 18\left(\frac{1}{3}\right)^{n-1}$ (6) $T_n = 20\left(-\frac{1}{2}\right)^{n-1}$
(7) $T_n = \frac{1}{2}(3)^{n-1}$ (8) $T_n = \frac{2}{3}\left(-\frac{1}{2}\right)^{n-1}$

(b) (1) $T_n = 3(2)^{n-1}$ (2) 96 (3) 8
(4) 10

(c) (1) $T_n = \frac{2}{9}(-3)^{n-1}$ (2) −4374 (3) 8
(4) 15

(d) (1) $\frac{1}{36}$ (2) 9 (3) 10

(e) (1) $\frac{5}{2}$ (2) 11 (3) 13

(f) $x = 8; x = -1$

(g) (1) $x = 3$ (2) $T_n = 4\left(\frac{1}{2}\right)^{n-1}$

(h) $x = 2; y = 4$

EXERCISE 4

(a)	(1) 1010	(2) -1085	(3) 345	(4) -1830
	(5) $10x+45$	(6) $800q$		
(b)	(1) 18	(2) 1107		
(c)	(1) 20	(2) -560		
(d)	(1) 2800	(2) -29650	(3) -13725	(4) 31275
	(5) 2730	(6) -270	(7) $1350p$	(8) $-550x$
(e)	(1) 20	(2) 17	(3) 100	(4) 50
(f)	(1) 102	(2) 1350	(3) 24	(4) 31
(g)	(1) $-5\ 760$	(2) 25	(3) 17	
(h)	(1) 51	(2) 28	(3) 32	
(i)	(1) -876	(2) 11		
(j)	(1) $T_n = 2x \cdot n + 1$	(2) 75	(3) 3	

EXERCISE 5

(a)	(1) 1 048 575	(2) 43 692	(3) $\frac{6375}{16}$	(4) $\frac{-14762}{729}$
	(5) $\frac{x\left(x^{20}-1\right)}{x-1}$	(6) $\frac{xy^{3}-xy^{11}}{1-y}$		
(b)	(1) 12282	(2) 10924	(3) -2735	(4) 90,91
	(5) 6690,08	(6) $\frac{-6141}{256}$	(7) $\frac{x-xy^{15}}{y^{12}(1-y)}$	(8) $\frac{1+x^{18}}{x^{11}+x^{13}}$
(c)	(1) 6	(2) 7	(3) 8	(4) 9
(d)	(1) 9840	(2) 10	(3) 9	
(e)	(1) $\frac{3069}{4}$	(2) 12	(3) 11	
(f)	(1) 100	(2) 8	(3) 5	

EXERCISE 6

(a)	(1) 32	(2) $\frac{45}{2}$	(3) $\frac{192}{5}$	(4) $\frac{-432}{7}$
	(5) $\frac{27}{4}$	(6) $\frac{125}{112}$		
(b)	(1) $\frac{4}{9}$	(2) $\frac{7}{9}$	(3) $\frac{25}{99}$	(4) $\frac{14}{33}$
	(5) $-2\frac{2}{3}$	(6) $1\frac{7}{30}$		
(c)	(1) $-\frac{1}{2} < x < \frac{1}{2}$	(2) $-1 < x < 1$	(3) $-2 < x < 0$	(4) $0 < x < 2$
	(5) $2 < x < 3$	(6) $3 < x < 4$	(7) $\frac{1}{3} < x < 1$	(8) $0 < x < 1$
(d)	(1) $3 < x < 5$	(2) $\frac{x+4}{5-x}$	(3) $\frac{7}{2}$	
(e)	Series diverges			

EXERCISE 7

(a)	(1) 3875	(2) 80	(3) 12276	(4) $\frac{3640}{27}$
	(5) -4305	(6) $\frac{255}{256}$	(7) $\frac{2}{3}$	(8) 35 495,95
	(9) 175	(10) 2046	(11) $\frac{243}{2}$	(12) -45
(b)	(1) 55	(2) 6	(3) 62	(4) 69
	(5) 9	(6) 5	(7) 30	(8) 30
	(9) 6			
(c)	(1) $\sum_{k=1}^{21}(7k-10)$	(2) $\sum_{k=1}^{15}2$	(3) $\sum_{k=1}^{\infty}6\left(\frac{1}{3}\right)^{k-1}$	(4) $\sum_{k=1}^{56}(-4k+14)$
	(5) $\sum_{k=1}^{6}(-2)(-7)^{k-1}$	(6) $\sum_{k=1}^{25}(k^2-3k+5)$	(7) $\sum_{k=1}^{41}(3k+x-3)$	(8) $\sum_{k=1}^{66}4x^{k-1}$
(d)	$2 < x < 4$			
(e)	$p = \frac{2}{3}$			

EXERCISE 8

(a)	6
(b)	2
(c)	$a = 10; d = -3$
(d)	$a = 3; r = -2$
(e)	$T_n = 8n - 7$
(f)	$-4; -12; -36$ or $4; -12; 36$

(g) $43+29+15+...$

(h) $3;6;12$ or $-9;18;-36$

(i) -520

(j) $\frac{425}{32}$

(k) $a=48; r=-\frac{1}{4}$

(l) $T_n=18\left(\frac{1}{3}\right)^{n-1}$ or $T_n=36\left(-\frac{1}{3}\right)^{n-1}$

(m) $S_{10}=\frac{1023}{16}$

(n) $S_{10}=116050$ or $S_{10}=-23210$

(o) $2;5;8$ or $8;5;2$

EXERCISE 9

(a)	(1) 510,996	(2) 740	(3) -30	(4) $\frac{58967}{3}$
	(5) 2760	(6) 2635	(7) $\frac{20451}{32}$	(8) -1404
(b)	(1) 12400	(2) 193	(3) 35	
(c)	(1) 50	(2) 40,5	(3) 129	(4) $T_n=8n-\frac{15}{2}$
(d)	(1) 8	(2) 4	(3) 2^{6-n}	(4) $\frac{1-p^2}{p}$
(e)	(1) $\sum_{k=1}^{30}(6k-1)(6k+1)$	(2) $\sum_{k=1}^{8}256\left(\frac{1}{2}\right)^{k-1}\cdot 3^k$	(3) $\sum_{k=1}^{9}\frac{2(3)^{k-1}}{-3k+123}$	(4) $\sum_{k=1}^{21}\frac{5k-3}{(1{,}01)^{k-1}}$
	(5) $\sum_{k=1}^{20}\frac{x+k}{x^{k+7}}$			
(f)	(1) 125250	(2) 62750	(3) 62500	
(g)	12498595			

EXERCISE 10

(a)	(1) 260 km	(2) 10	(3) 3850 km	(4) No
(b)	R10 737 418,23			
(c)	45			
(d)	(1) R121 550,63	(2) R1 337 500	(3) Mary	(4) Chris
(e)	9,5m			
(f)	30m			
(g)	(1) 36π cm	(2) $\frac{324}{5}\pi$ cm^2		
(h)	$\frac{4}{3}$ m^2			

CONSOLIDATION AND EXTENSION EXERCISE

(a)	(1) $T_n=4n^2-3n-3$	(2) $T_n=\frac{5}{3}(3)^{n-1}$	(3) $T_n=-n^2-\frac{1}{2}n+8$	(4) $T_n=\frac{64}{5}\left(-\frac{1}{2}\right)^{n-1}$
	(5) $T_n=n(x+5)-5$	(6) $T_n=x^{-n}$		
(b)	(1) $\frac{315}{2}$	(2) -29524	(3) 200	(4) -120
	(5) 120	(6) $\frac{3^9x^{10}-3x^2}{3x-1}$		
(c)	(1) $\frac{195312}{25}$	(2) 477	(3) 3	
(d)	(1) $\sum_{k=1}^{21}\left(\frac{1}{2}k-\frac{5}{2}\right)$	(2) $\sum_{k=1}^{11}9(2)^{k-1}$	(3) $\sum_{k=1}^{20}(-2k^2+9k+3)$	(4) $\sum_{k=1}^{\infty}5\left(\frac{1}{3}\right)^{k-1}$
	(5) $\sum_{k=1}^{20}(2k+3)\cdot 2^k$	(6) $\sum_{k=1}^{8}\frac{-5k+2}{5(3)^{k-1}}$		
(e)	(1) 37	(2) 5	(3) 6	(4) 99
(f)	(1) $T_n=\frac{3}{4}(4)^{n-1}$	(2) 3072	(3) 9	(4) $\frac{65535}{4}$
	(5) 5	(6) 8		
(g)	(1) $T_n=-5n+3$	(2) -137	(3) 18	(4) $S_{10}=-245$
	(5) 25	(6) 20		
(h)	(1) $T_n=n^2-42n+200$	(2) T_{21}	(3) $T_n=2n-41$	(4) 7
(i)	(1) 29	(2) $464x+464$	(3) 70	
(j)	(1) $2<x<4$	(2) $\frac{25}{6}$	(3) $x=\frac{7}{2}$	

(k)	(1) $x=-6$	(2) $x=4$	(3) $x=0;x=3$	
(l)	$-4;-1;2$			
(m)	$a=243$ and $r=\frac{1}{3}$; $a=486$ and $r=-\frac{1}{3}$			
(n)	$5;12;19$ or $19;12;5$			
(o)	$T_n=4n^2+6n-8$			
(p)	(1) $-\frac{795}{2}$	(2) 80	(3) $-\frac{111}{4}$	(4) $T_n=-n+\frac{9}{4}$
(q)	3745			
(r)	(1) 22	(2) 24		
(s)	(1) 6,29 kg	(2) 124,83 kg	(3) 150 kg	
(t)	Yes			
(u)	8			
(v)	10			
(w)	$20;60;200$			
(x)	$\frac{3t-1}{2}$			
(y)	(1) 162	(2) 5050	(3) 1958	

CHAPTER TWO

EXERCISE 1

(a)	(1) $f^{-1}(x)=x-1$	(2) $y=\pm\sqrt{x}$	(3) $h^{-1}(x)=\frac{x}{3}$
	(4) $y=\pm\sqrt{-x}$	(5) $h^{-1}(x)=\frac{x+5}{3}$	(6) $y=\pm\sqrt{\frac{x}{4}}$
	(7) $h^{-1}(x)=3x+2$	(8) $y=\pm\sqrt{-3x}$	(9) $g^{-1}(x)=\frac{-2x+8}{3}$
(b)	(1) $f^{-1}(x)=\log_3 x$	(2) $g^{-1}(x)=\log_{\frac{1}{2}} x$	(3) $h^{-1}(x)=-\log_6 x$
	(4) $g^{-1}(x)=\log_5(-x)$	(5) $f^{-1}(x)=-\log_7(-x)$	
(c)	(1) $f^{-1}(x)=7^x$	(2) $g^{-1}(x)=\left(\frac{1}{3}\right)^x$	(3) $h^{-1}(x)=5^{-x}$
	(4) $g^{-1}(x)=-4^x$	(5) $f^{-1}(x)=-\left(\frac{1}{2}\right)^{-x}$	

EXERCISE 2

(a) (1) $f(x)=3x-6$:
Straight line
Increasing
x-intercept: 2
y-intercept: -6

$f^{-1}(x)=\frac{x+6}{3}$:
Straight line
Increasing
x-intercept: -6
y-intercept: 2

Line of Symmetry: $y=x$

(2) $y=2x^2$:
Parabola
Positive Orientation
Points: $(-1;2)$; $(0;0)$; $(1;2)$

$y=\pm\sqrt{\frac{x}{2}}$:
Inverse parabola
x-values ≥ 0
Points: $(2;-1)$; $(0;0)$; $(2;1)$

Line of Symmetry: $y=x$

(3) $f(x)=3^x$:
Exponential curve
base >1
Asymptote: $y=0$
Points: $(-1;\frac{1}{3})$; $(0;1)$; $(1;3)$

$f^{-1}(x)=\log_3 x$:
Log curve
Increasing
Asymptote: $x=0$
Points: $(\frac{1}{3};-1)$; $(1;0)$; $(3;1)$

Line of Symmetry: $y=x$

(4) $y=-x^2$:
Parabola
Negative Orientation
Points: $(-1;-1)$; $(0;0)$; $(1;-1)$

$y=\pm\sqrt{-x}$:
Inverse parabola
x-values ≤ 0
Points: $(-1;-1)$; $(0;0)$; $(-1;1)$

Line of Symmetry: $y=x$

(5) $h(x)=\left(\frac{1}{3}\right)^x$:

Exponential curve
base <1
Asymptote: $y=0$
Points: $(-1;3)$; $(0;1)$; $(1;\frac{1}{3})$

$h^{-1}(x)=\log_{\frac{1}{3}} x$:

Log curve
Decreasing
Asymptote: $x=0$
Points: $(3;-1)$; $(1;0)$; $(\frac{1}{3};1)$

Line of Symmetry: $y=x$

(6) $g(x)=\log_2 x$:

Log curve
Increasing
Asymptote: $x=0$
Points: $(\frac{1}{2};-1)$; $(1;0)$; $(2;1)$

$g^{-1}(x)=2^x$:

Exponential curve
base >1
Asymptote: $y=0$
Points: $(-1;\frac{1}{2})$; $(0;1)$; $(1;2)$

Line of Symmetry: $y=x$

(7) $y=-\frac{1}{2}x^2$:

Parabola
Negative orientation
Points: $(-1;\frac{1}{2})$; $(0;0)$; $(1;\frac{1}{2})$

$y=\pm\sqrt{-2x}$:

Inverse parabola
x-values ≤ 0
Points: $(\frac{1}{2};-1)$; $(0;0)$; $(\frac{1}{2};1)$

Line of Symmetry: $y=x$

(8) $f(x)=\left(\frac{1}{4}\right)^x$:

Exponential curve
base <1
Asymptote: $y=0$
Points: $(-1;4)$; $(0;1)$; $(1;\frac{1}{4})$

$f^{-1}(x)=\log_{\frac{1}{4}} x$

Log curve
Decreasing
Asymptote: $x=0$
Points: $(4;-1)$; $(1;0)$; $(\frac{1}{4};1)$

Line of Symmetry: $y=x$

(9) $h(x)=\log_{\frac{1}{2}} x$:

Log curve
Decreasing
Asymptote: $x=0$
Points: $(2;-1)$; $(1;0)$; $(2;-1)$

$h^{-1}(x)=\left(\frac{1}{2}\right)^x$:

Exponential curve
base <1
Asymptote: $y=0$
Points: $(-1;2)$; $(0;1)$; $(1;\frac{1}{2})$

Line of Symmetry: $y=x$

(10) $f(x)=-3^x$:

Exponential curve
base >1
Negative orientation
Asymptote: $y=0$
Points: $(-1;-\frac{1}{3})$; $(0;-1)$; $(1;-3)$

$f^{-1}(x)=\log_3(-x)$:

Log curve
Decreasing
Negative x-values
Asymptote: $x=0$
Points: $(-\frac{1}{3};-1)$; $(-1;0)$; $(-3;-1)$

Line of Symmetry: $y=x$

(11) $h(x)=-\log_3 x$:

Log curve
Decreasing
Positive x-values
Asymptote: $x=0$
Points: $(\frac{1}{3};1)$; $(1;0)$; $(3;-1)$

$h^{-1}(x)=3^{-x}=\left(\frac{1}{3}\right)^x$:

Exponential curve

(12) $g(x)=\log_2(-x)$:

Log curve
Decreasing
Negative x-values
Asymptote: $x=0$
Points: $(-2;1)$; $(-1;0)$; $(-\frac{1}{2};-1)$

$g^{-1}(x)=-2^x$:

Exponential curve

base <1
Asymptote: $y=0$
Points: $(-1;3)\,;(0;1)\,;(1;\frac{1}{3})$

Line of Symmetry: $y=x$

base >1
Negative orientation
Asymptote: $y=0$
Points: $(-1;-\frac{1}{2})\,;(0;-1)\,;(1;-2)$

Line of Symmetry: $y=x$

(b) (1) $f^{-1}(x)=\frac{x-4}{4}$

(2) $f(x)=4x+4$:
Straight line
Increasing
x-intercept: -1
y-intercept: 4

$f^{-1}(x)=\frac{x-4}{4}$:
Straight line
Increasing
x-intercept: 4
y-intercept: -1

Line of Symmetry: $y=x$

(3) $\left(-\frac{4}{3}\,;-\frac{4}{3}\right)$

(c) (1) $y=\pm\sqrt{\frac{x}{3}}$

(2) $y=3x^2$:
Parabola
Positive orientation
Points: $(-1;3)\,;(0;0)\,;(1;3)$

$y=\pm\sqrt{\frac{x}{3}}$:
Inverse parabola
x-values ≥ 0
Points: $(3;-1)\,;(0;0)\,;(3;1)$

Line of Symmetry: $y=x$

(3) $(0\,;0)$ and $\left(\frac{1}{3}\,;\frac{1}{3}\right)$

EXERCISE 3

(a) (1) $g(x)=-4x+2$ (2) $h(x)=-4x-2$

(b) (1) $y=\frac{2}{3}x^2$ (2) $y=\pm\sqrt{-\frac{3x}{2}}$

(c) (1)(i) $y=-7^x$ (ii) $y=\left(\frac{1}{7}\right)^x$

(iii) $y=\log_7 x$ (2)(i) $y=\log_{\frac{1}{6}}(-x)$

(ii) $y=-\log_{\frac{1}{6}} x$ (iii) $y=\left(\frac{1}{6}\right)^x$

(d) (1) $b=4$ (2) $g(x)=\log_4 x$

(3) $h(x)=\left(\frac{1}{4}\right)^x$ (4)(i) A' $(2\,;\frac{1}{2})$

(ii) A'' $(-\frac{1}{2}\,;2)$

(5) $f(x)=4^x$:
Exponential curve
base >1
Asymptote: $y=0$
Points: $(-1;\frac{1}{4})\,;(0;1)\,;(1;4)$

$g(x)=\log_4 x$:
Log curve
Increasing
Asymptote: $x=0$
Points: $(\frac{1}{4};-1)\,;(1;0)\,;(4;1)$

$h(x)=\left(\frac{1}{4}\right)^x$:

Exponential curve
base <1
Asymptote: $y=0$
Points: $(-1;4)$; $(0;1)$; $(1;\frac{1}{4})$

(e) (1) $a=3$ (2) $f^{-1}(x)=3^x$

(3) $h(x)=\left(\frac{1}{3}\right)^x$ (4) $P'(9;-2)$

(5) $g(x)=\log_{\frac{1}{3}} x$

(f) (1) $a=\frac{1}{2}$ (2) $g(x)=-\left(\frac{1}{2}\right)^x$

(3)(i) $y>0$ (ii) $y<0$

(4) $h(x)=\log_{\frac{1}{2}} x$ (5) $x>0$

(6) $A(0;1)$; $B(0;-1)$; $C(1;0)$ (7) $P'\left(\frac{1}{8};3\right)$

(8)(i) $x>3$ (ii) $x>0$

(iii) $0<x\leq\frac{1}{8}$

(g) (1) $b=\frac{2}{3}$ (2) $g(x)=\log_{\frac{2}{3}}(-x)$

(3) $P(-1;0)$; $Q(0;1)$; $R(1;0)$ (4) $y>0$

(5)(i) $x>0$ (ii) $x<0$

(6)(i) $x>1$ (ii) $0<x\leq\frac{9}{4}$

(iii) $-1<x<0$ (7) $k=2$

(8) $\frac{3}{2}$

EXERCISE 4

(a) (1) $y\leq 0$ (2) $f^{-1}(x)=\sqrt{-x}$

(3) $g(x)=x^2$; $x\geq 0$

(4) $f(x)=-x^2$; $x\geq 0$:

Parabola
Negative orientation
Domain: $x\geq 0$
Points: $(0;0)$; $(1;-1)$; $(2;-4)$

$f^{-1}(x)=\sqrt{-x}$:

Inverse parabola
x-values ≤ 0
y-values ≥ 0
Points: $(0;0)$; $(-1;1)$; $(-4;2)$

$g(x)=x^2$; $x\geq 0$:

Parabola
Positive orientation
Domain: $x\geq 0$

Points: $(0;0)$; $(1;1)$; $(2;4)$

(b) (1) $g^{-1}(x)=-\sqrt{3x}$; $x\neq 0$

(2) $g(x)=\frac{1}{3}x^2$; $x<0$:

Parabola
Positive orientation
Domain: $x<0$
Points: $(-1;\frac{1}{3})$; $(-2;\frac{4}{3})$; $(-3;3)$
Exclude $(0;0)$

$g^{-1}(x)=-\sqrt{3x}$; $x\neq 0$:

Inverse parabola
x-values >0
y-values <0

Points: $(\frac{1}{3};-1)$; $(\frac{4}{3};-2)$; $(3;-3)$
Exclude $(0;0)$

(3) $h(x)=\frac{1}{3}x^2$; $x>0$

(4) $h(x)=\frac{1}{3}x^2$; $x>0$
Parabola
Positive orientation
Domain: $x>0$
Points: $(1;\frac{1}{3})$; $(2;\frac{4}{3})$; $(3;3)$
Exclude $(0;0)$

$h^{-1}(x)=\sqrt{3x}$; $x\neq 0$:
Inverse parabola
x-values >0
y-values >0
Points: $(\frac{1}{3};1)$; $(\frac{4}{3};2)$; $(3;3)$
Exclude $(0;0)$

(c) (1) Domain: $x\geq 0$
Range: $y\leq 0$

(2) $f^{-1}(x)=x^2$; $x\leq 0$

(3) $f(x)=-\sqrt{x}$:
Inverse parabola
x-values ≥ 0
y-values ≤ 0
Points: $(0;0)$; $(1;-1)$; $(4;-2)$

$f^{-1}(x)=x^2$; $x\leq 0$
Parabola
Positive orientation
Domain: $x\leq 0$
Points: $(0;0)$; $(-1;1)$; $(-2;4)$

(4)(i) $g(x)=\sqrt{x}$
(ii) $h(x)=-\sqrt{-x}$

(d) (1) $x\geq 0$ (2) $a=2$ (3) P $(\frac{1}{2};\frac{1}{2})$
(4) $x>\frac{1}{2}$

(e) (1) $g(x)=-\sqrt{2x}$ (2) $h(x)=-\frac{x^2}{2}$; $x\leq 0$ (3) Intersection on $y=x$
(4) P $(-2;-2)$ (5) $x\leq -2$

EXERCISE 5

(a) (1) Not a function (2) One-to-one function (3) Many-to-one function
(4) Not a function (5) Many-to-one function (6) Not a function
(7) One-to-one function (8) One-to-one function

(b) (1) One-to-one (2) Many-to-one (3) One-to-one
(4) One-to-one (5) One-to-one (6) Many-to-one
(7) One-to-one (8) One-to-one (9) One-to-one
(10) Many-to-one (11) One-to-one (12) Many-to-one

(c) (1) $f(x)=-2x^2$:
Parabola
Negative orientation
Turning point: $(0;0)$

$f^{-1}(x)=\pm\sqrt{-\frac{x}{2}}$:
Inverse parabola
x-values ≤ 0

(2) No

(3) $x\geq 0$ or $x\leq 0$

(4) For $x \geq 0$:

$f(x) = -2x^2$ Domain: $x \geq 0$ Range: $y \leq 0$

$f^{-1}(x) = \sqrt{-\frac{x}{2}}$ Domain: $x \leq 0$ Range: $y \geq 0$

For $x \leq 0$:

$f(x) = -2x^2$ Domain: $x \leq 0$ Range: $y \leq 0$

$f^{-1}(x) = -\sqrt{-\frac{x}{2}}$ Domain: $x \leq 0$ Range: $y \leq 0$

EXERCISE 6

(a) (1) $x = 3,70$ (2) $x = -2,26$ (3) $x = -0,72$
(4) $x = 2,55$ (5) $x = 2,58$ (6) $x = 2$ or $x = 0,63$

(b) (1) $x = 32$ (2) $x = 60$ (3) $x = \frac{1}{3}$
(4) $x = 2$ (5) $x = \frac{1}{3}$ (6) $x = 8$

(c) (1) $x = 2,24$ (2) $x = 0,88$

(d) (1) $x \leq 3$ (2) $x \leq 2$ and $x = -1$

(e) (1) $k = \frac{1}{4}$ (2) $a = 3\,000$ (3) 3 000
(4) 2 187 000 (5) 29,53 hours

(f) (1) 5,4 (2) $3,981 \times 10^{-3}$

CONSOLIDATION AND EXTENSION EXERCISE

(a) (1) $f^{-1}(x) = \frac{-x+4}{2}$ (2) $g(x) = 2x - 4$

(3) $f(x) = -2x + 4$:
Straight line
Decreasing
x-intercept: 2
y-intercept: 4

$f^{-1}(x) = \frac{-x+4}{2}$:
Straight line
Decreasing
x-intercept: 4
y-intercept: 2

$g(x) = 2x - 4$:
Straight line
Increasing
x-intercept: 2
y-intercept: -4

(4) $\left(\frac{4}{3}; \frac{4}{3}\right)$

(b) (1) $h^{-1}(x) = \log_{\frac{1}{4}} x$ (2) $a = 4$

(3) $h(x) = \left(\frac{1}{4}\right)^x$:
Exponential curve
base < 1
Asymptote: $y = 0$
Points: $(-1;4)$; $(0;1)$; $(1;\frac{1}{4})$

$h^{-1}(x) = \log_{\frac{1}{4}} x$:
Log curve
Decreasing
Asymptote: $x = 0$
Points: $(4;-1)$; $(1;0)$; $(\frac{1}{4};1)$

$f(x) = 4^x$:
Exponential curve
base > 1
Asymptote: $y = 0$
Points: $(-1;\frac{1}{4})$; $(0;1)$; $(1;4)$

(c) (1) $h(x)=2x^2$; $x \geq 0$

(2) $g^{-1}(x)=\sqrt{-\frac{x}{2}}$

(3) $g(x)=2x^2$; $x \geq 0$:

Parabola
Negative orientation
Domain: $x \geq 0$
Points: $(0;0)$; $(1;-2)$; $(2;-8)$

$g^{-1}(x)=\sqrt{-\frac{x}{2}}$:

Inverse parabola
x-values ≤ 0
y-values ≥ 0
Points: $(0;0)$; $(1;2)$; $(2;8)$

(4)(i) $f(x)=-2x^2$; $x \leq 0$

(ii) $f^{-1}(x)=-\sqrt{-\frac{x}{2}}$

(iii) $f(x)=-2x^2$; $x \leq 0$:

Parabola
Negative orientation
Domain: $x \leq 0$
Points: $(0;0)$; $(-1;-2)$; $(-2;-8)$

$f^{-1}(x)=-\sqrt{-\frac{x}{2}}$

Inverse parabola
x-values ≤ 0
y-values ≤ 0
Points: $(0;0)$; $(-2;-1)$; $(-8;-2)$

(d) (1) Domain: $x \geq 0$
Range: $y \leq 0$

(2) $f^{-1}(x)=\frac{x^2}{3}$; $x \leq 0$

(3) $f(x)=-\sqrt{3x}$:

Inverse parabola
x-values ≥ 0
y-values ≤ 0
Points: $(0;0)$; $(\frac{1}{3};-1)$; $(\frac{4}{3};-2)$

$f^{-1}(x)=\frac{x^2}{3}$; $x \leq 0$:

Parabola
Positive orientation
Domain: $x \leq 0$
Points: $(0;0)$; $(-1;\frac{1}{3})$; $(-2;\frac{4}{3})$

(4)(i) $g(x)=-\sqrt{-3x}$

(ii) $h(x)=\sqrt{3x}$

(e) (1)(i) h

(ii) g

(2) $y \geq 0$

(3) Yes

(4) $x \geq 0$ or $x \leq 0$

(f) (1) Domain: $x \leq 0$
Range: $y \leq 0$

(2) $g(x)=-\frac{x^2}{4}$; $x \leq 0$

(3) P $(-4;-4)$

(4) $-4 \leq x \leq 0$

(5) $h(x)=2\sqrt{-x}$

(g) (1) $a=\frac{1}{2}$

(2)(i) $f^{-1}(x)=-\sqrt{2x}$

(ii) $g(x)=-\sqrt{-2x}$

(iii) $h(x)=-\sqrt{2x+4}$

(3)(i) P' $(8;-4)$

(ii) P'' $(-8;-4)$

(iii) P''' $(6;-4)$

(4)(i) L $(0;-2)$

(ii) Q $(-1;-\sqrt{2})$

(h) (1) $b=\frac{1}{a}$

(2)(i) $y>0$

(ii) $x>0$

(3)(i) A $(0;1)$

(ii) B $(1;0)$

(4) $a=\frac{1}{3}$ and $b=3$

(5) $h(x)=\log_{\frac{1}{3}} x$

(6) Q $(\sqrt{27};-\frac{3}{2})$

(i)	(1) $b=\frac{1}{2}$	(2) $f^{-1}(x)=\left(\frac{1}{2}\right)^x$
	(3)(i) A (1;0)	(ii) B (0;1)
	(4)(i) $0<x\le 1$	(ii) $x\ge 0$
	(iii) $x>8$	(iv) $x<-3$
(j)	(1) $0<x\le 1$	(2) $a=8$
	(3) $\frac{1}{4}$	(4) $f^{-1}(x)=8^x$; $x\le 0$
	(5) $0<y\le 1$	
(k)	(1) $-4\le x\le 4$	(2) $-3\le y\le 3$
	(3) Yes	(4)(i) $(-4;-3)$; $(-2;3)$; $(0;0)$; $(2;3)$; $(4;-3)$
	(ii) $(-4;3)$; $(-2;-3)$; $(0;0)$; $(2;-3)$; $(4;3)$	(iii) $(3;-4)$; $(-3;-2)$; $(0;0)$; $(-3;2)$; $(3;4)$
	(5) $-4\le x\le -2$ and $2\le x\le 4$	
(l)	(1) $d=9$	(2) $m=\frac{1}{3}$ and $c=-3$
(m)	(1) $b=-a$	(2) $m=-1$
(n)	(1) $p=1-b$ and $q=1-b$	(2) No
	(3) $t=-\frac{b}{2}$	
(o)	(1) $-90°\le x\le 90°$	(2) $0°\le x\le 180°$
	(3) $-90°<x<90°$	

CHAPTER THREE

EXERCISE 1

(a)	$x(x-4)(x+3)$
(b)	$(x+1)(x+2)(x-2)$
(c)	$(2x+1)(4x^2-2x+1)$
(d)	$(x-10)(x^2+10x+100)$
(e)	$(x-3)(x^2+3)$
(f)	$(x-4)(x^2+x+7)$
(g)	$(4x+5)(16x^2-20x+25)$
(h)	$(x+5)(2x^2+1)$
(i)	$2x(2x-3)(4x^2+6x+9)$
(j)	$(x+3)(2x-1)(2x+1)$
(k)	$(2x-1)(3x^2-2)$
(l)	$16(x+1)(x^2-x+1)$
(m)	$(x+2)(x-1)(x+4)$
(n)	$x(x-2)(x-3)(x+3)$

EXERCISE 2

(a)	1		
(b)	8		
(c)	(1) $f(-4)=0$	(2) $(x+4)(x+2)(x-1)$	
(d)	$(3x-2)(x+5)(x-1)$		
(e)	(1) $(x-1)^2(x+3)$	(2) $(x+1)(x-3)(x-2)$	(3) $(x+2)^2(x-3)$
	(4) $(x+1)(x-4)(x+3)$	(5) $(x-2)(x-3)(x-4)$	(6) $-(x-2)(x+3)^2$
	(7) $-(x+4)(x-2)^2$	(8) $(x-2)(x-4)(x+5)$	(9) $(2x-1)(x-5)(x+2)$
	(10) $(3x-1)(x^2+x+1)$	(11) $(2x-1)(3x+2)(2x-3)$	(12) $(x+1)(x-7)(x-6)$

EXERCISE 3

(a)	0;8
(b)	−1;0;1
(c)	1;2;3
(d)	−1;4
(e)	1
(f)	−2;1;3

(g) -2
(h) $-1;2$
(i) $-2;2;3$
(j) $-3;2;5$
(k) $-\frac{1}{2};\frac{1}{2};2$
(l) $-\frac{3}{2};0;1$
(m) $-3;-2;-1$
(n) $-\frac{1}{2}$
(o) $-2{,}58;-0{,}58;1$
(p) $-5;\frac{1}{3};4$
(q) $-1;-0{,}26;0{,}55$
(r) $-2;0;\frac{1}{8};1$

EXERCISE 4

(a) $k=-4$
(b) $a=-2$
(c) $p=2q+9$
(d) $t=-4$
(e) $m=5;n=2$
(f) $r=2;s=-1$
(g) $a=1;b=-6$
(h) (1) 5 (2) $x-1$

CONSOLIDATION AND EXTENSION EXERCISE

(a) -2
(b) (1) $f(-1)=0$ (2) $(x+1)(x^2+2x-4)$ (3) $-1;-3{,}24;1{,}24$
(c) (1) $f(2)=0 \quad \therefore x-2$ is a factor (2) $2(x-2)(x+3)(x-1)$ (3) $-3;1;2$
(d) (1) $f\left(\frac{1}{2}\right)=0$ (2) $(2x-1)(x-4)(x+1)$
(e) (1) $x^2(x-3)$ (2) $2x(x-1)(x+5)$ (3) $-x(2x+5)(2x-5)$
(4) $(x-4)(x^2+4x+16)$ (5) $(x+3)(x-5)(x-3)$ (6) $(x-3)(x^2-4x+2)$
(7) $(x+1)(x-2)(x-3)$ (8) $-(x-1)(x+2)^2$ (9) $(x-1)(x+4)(x-3)$
(10) $2(x+2)^3$ (11) $(2x+1)(x+2)(x-3)$ (12) $(2x+1)(2x-3)(3x-1)$
(f) (1) $-9;0;9$ (2) $1;2$ (3) $-1;2;6$
(4) -3 (5) $-3;1;3$ (6) $-3;1;2$
(7) 3 (8) $0;1;4$ (9) 4
(10) $-2;3;-4$ (11) $-5;-1;6$ (12) $+2;3;5$
(13) $0;1$ (14) $-3;\frac{1}{2};4$ (15) $-2;-\frac{1}{2};3$
(16) $-\frac{2}{3};1;2$ (17) $\frac{1}{2}$ (18) $2;0{,}26;-0{,}55$
(19) $3;0{,}64;-1{,}24$ (20) $-3;-\frac{7}{8};0$
(g) $(x+1)(x^2-5x+8);-1$
(h) 10
(i) -2
(j) (1) $a=\frac{1}{2};b=-15$ (2) 17
(k) (1) 25 (2) $-5;1;\frac{5}{2}$
(l) $m=-19;n=-30$
(m) $b=6;c=-34;d=-60$
(n) $x=-\frac{2}{3}$
(o) (1) 19 (2) -11

CHAPTER FOUR

EXERCISE 1

(a) (1) 5 (2) -5

(b) (1) $-6-3h$ (2)(i) -9 (ii) $-6{,}3$ (iii) $-6{,}03$

(3) Gradient of the tangent to f at $x=1$ is -6

(c) (1) 1 (2) The gradient of the tangent to f at $x=2$ is 1.

EXERCISE 2

(a) (1) $2x$ (2) $6x$ (3) $-2x$ (4) $2x-4$

(5) $-2x+3$ (6) $2x+2$ (7) $6x^2$ (8) $-3x^2$

(9) $\frac{-3}{x^2}$ (10) $\frac{4}{x^2}$

(b) (1) -7 (2) $-2x-5$ (3) -9

(c) (1) $-\frac{8}{x^2}$ (2) -8

(3) Gradient of the tangent at $x=-1$ is -8

(4) $x=\pm 2$ (5) Derivative negative regardless of x-value

(d) (1) 3 (2) 0

(e) (1) 0 (2) m (3) $2ax$ (4) $2ax+b$

EXERCISE 3

(a) (1) $3x^2$ (2) $10x^4$ (3) -6 (4) 0

(5) $-6x^{-3}$ (6) $-3x^{\frac{1}{2}}$ (7) $\frac{x}{2}$ (8) $\frac{3x^2}{2}$

(9) $\frac{3x}{2}$ (10) $-3x^{-4}$ (11) $6x^{-2}$ (12) $-5x^{-3}$

(13) $2x$ (14) $-\frac{9}{2}x^{-4}$ (15) $\frac{2}{3}x^{-\frac{1}{3}}$ (16) $6x^{\frac{1}{2}}$

(17) $\frac{-3}{2}x^{-\frac{3}{2}}$ (18) $-4x^{-\frac{1}{3}}$

(b) (1) $-2x+3$ (2) $12x^2-4x+5$ (3) $-2x^{-3}+3x^{-2}$

(4) $\frac{1}{2}x^{-\frac{1}{2}}+\frac{5}{2}x^{-\frac{3}{2}}+2x^{-\frac{1}{3}}$ (5) $-6u^{-3}+\frac{1}{2}u^{-\frac{1}{2}}$ (6) $\frac{5}{3}a^{\frac{2}{3}}+2a^{-\frac{1}{2}}+2a^{-2}$

(7) $-2k^{-\frac{3}{2}}-3$ (8) $\frac{3}{2}t-\frac{2}{3}t^{-2}+6t^{-\frac{5}{3}}$ (9) $\frac{3}{2}\sqrt{2}x^{\frac{1}{2}}+\sqrt{2}x^{\sqrt{2}-1}$

(10) $2\pi x+\pi^2$

EXERCISE 4

(a) (1) $3x^2+1-6x$ (2) $-6x^2+8x-2$ (3) $2x+2x^{-3}$

(4) $1-x^{-2}$

(b) (1) $2x-x^{-2}$ (2) $\frac{2}{3}+\frac{1}{3}x^{-2}$ (3) $x^{-\frac{1}{2}}$

(4) $-\frac{3}{2}x^{-\frac{3}{2}}-\frac{2}{3}x^{-\frac{5}{3}}$ (5) $\frac{1}{2}-\frac{3}{4}x^{-\frac{1}{2}}$ (6) $\frac{20}{3}x^{\frac{2}{3}}+8x^{-\frac{1}{3}}-3x^{-\frac{4}{3}}$

(7) 1 (8) $-2x-2$ (9) $-\frac{1}{2}$

(10) $2x-2$

(c) (1) $3ax^2+2bx+c$ (2) $\frac{p}{2}x^{-\frac{1}{2}}+10qx-rx^{-2}$ (3) $3a^2x^2+3b-\frac{1}{c}$

(4) $3mk^2-n^3$ (5) $4\pi r+2\pi h$ (6) $u+at$

(d) (1) $2x+1$ (2) $-3x^2-6x$

(e) (1) $\frac{1}{2}x^{-\frac{1}{2}}+1$ (2) $\frac{1}{3}x^{-\frac{2}{3}}$ (3) $\frac{3}{2}t^{\frac{1}{2}}+3t^2$

EXERCISE 5

(a) (1) 3 (2) 1

(b) (1) $y=-x+9$ (2) $y=3x+12$

(c) $(2;2)$

(d) $y=4x-1$

(e) $y=-9x+11;y=-9x+7$

(f) (1) $y=8x+13$ (2) $y=8x+12$

(g) $a=-2$

(h) $a=-2;b=5$

(i) $p=2;q=3$

EXERCISE 6

(a) (1)(i) Decreasing (ii) Increasing (iii) Decreasing and Increasing

(2) $f''(x)=2$ f is always concave up

(b) (1)(i) Concave down (ii) Concave down (iii) Concave up

(2) $f''(1)=0$ f changes from concave down to concave up

(c) (1) $(-1;-7);(2;20)$ (2) $(-1;-7)$ Local minimum; $(2;20)$ Local maximum (3) $\left(\frac{1}{2};\frac{13}{2}\right)$

(4)(i) $-1\leq x\leq 2$ (ii) $x\leq -1$ or $x\geq 2$

(d) $(-1;4)$ stationary point of inflection

(e) (1) Positive (2) Negative (3) Zero

(4) Negative (5) Negative (6) Zero

(7) Zero (8) Zero (9) Zero

(10) Positive (11) Positive (12) Positive

EXERCISE 7

(a) Cubic curve
Positive orientation
$x-$intercepts: $0;3$
$y-$intercept: 0
Turning points: $(0;0);(2;-4)$
Point of inflection: $(1;-2)$

(b) Cubic curve
Negative orientation
$x-$intercepts: $0;3$
$y-$intercept: 0
Turning points: $(3;0);(1;-4)$
Point of inflection: $(2;-2)$

(c) Cubic curve
Positive orientation
$x-$intercepts: $-3;0;5$
$y-$intercept: 0
Turning points: $(-\frac{5}{3};\frac{400}{27});(3;-36)$
Point of inflection: $\left(\frac{2}{3};-\frac{286}{27}\right)$

(d) Cubic curve
Negative orientation
$x-$intercepts: $-1;2$
$y-$intercept: -4
Turning points: $(0;-4);(2;0)$
Point of inflection: $(1;-2)$

(e) Cubic curve
Positive orientation
$x-$intercepts: $-1;2$
$y-$intercepts: -2
Turning point: $(-1;0);(1;-4)$
Point of inflection: $(0;-2)$

(f) Cubic curve
Negative orientation
$x-$intercepts: $-3;1$
$y-$intercept: 9
Turning points: $(-\frac{1}{3};\frac{256}{27});(-3;0)$
Point of inflection: $(-\frac{5}{3};\frac{128}{27})$

(g) Cubic curve
Positive orientation
$x-$intercepts: $-1;2;7$
$y-$intercept: 14
Turning points: $(\frac{1}{3};\frac{400}{27});(5;-36)$
Point of inflection: $(\frac{8}{3};-\frac{286}{27})$

(h) Cubic curve
Positive orientation
$x-$intercepts: $-3;-\frac{1}{2};1$
$y-$intercept: -3
Turning points: $(-2;9);(\frac{1}{3};-\frac{100}{27})$
Point of inflection: $(-\frac{5}{6};\frac{143}{54})$

EXERCISE 8

(a) (1) Cubic curve
Positive orientation
$x-$intercept: 0
$y-$intercept: 0
Stationary point: $(0;0)$
Point of inflection: $(0;0)$

(2) Cubic curve
Negative orientation
$x-$intercept: 2
$y-$intercept: 8
Stationary point: $(0;8)$
Point of inflection: $(0;8)$

(3) Cubic curve
Positive orientation
$x-$intercept: -1
$y-$intercept: 1
Stationary point: $(-1;0)$
Point of inflection: $(-1;0)$

(4) Cubic curve
Positive orientation
$x-$intercept: 3
$y-$intercept: -9
Stationary point: $(1;-8)$
Point of inflection: $(1;-8)$

(5) Cubic curve
Negative orientation
$x-$intercept: -3
$y-$intercept: -27
Stationary point: $(-2;-3)$
Point of inflection: $(-2;-3)$

(b) (1) $(2;-1)$ (2) $(2;-1)$
(3) Cubic curve
Negative orientation
$x-$intercept: 1
$y-$intercept: 7
Stationary point: $(2;-1)$
Point of inflection: $(2;-1)$

(c) (1) $3x^2+6x+6=0$
No Solution
(2) $(-1;4)$
(3) Cubic curve
Positive orientation
$x-$intercept: -2
$y-$intercept: 8
Point of inflection: $(-1;4)$

EXERCISE 9

(a) (1) $y=2x^3-6x^2-2x+6$ (2) $y=-x^3+4x^2+3x-18$
(3) $y=-\frac{1}{2}x^3-4x^2-8x$ (4) $y=x^3-5x^2$

(b) (1) $y=2x^3-2$ (2) $y=-3(x-1)^3$
(3) $y=(x-2)^3+3$ (4) $y=\frac{1}{2}(x+2)^3-4$

(c) $a=2;b=12$

(d) $a=1;b=8;c=-12$

(e) $p=-1;q=-5;r=-3$

EXERCISE 10

(a) (1) C$(-6;0)$;D$(-1;0)$ and E$(2;0)$ (2) $(0;-12)$
(3) A$(-4;36)$ and B$(\frac{2}{3};-\frac{400}{27})$ (4) $x\in\left[-4;\frac{2}{3}\right]$ or $-4\leq x\leq\frac{2}{3}$
(5) $x=-\frac{5}{3}$ (6) $(3;36)$
(7) $y=17x+113$ (8)(i) $x\leq-6$ or $-1\leq x\leq 2$
(ii) $x<-4$ or $x>\frac{2}{3}$ (iii) $-6\leq x\leq-4$ or $-1\leq x\leq\frac{2}{3}$ or $x\geq 2$

(b) (2) $(-\frac{1}{3};-\frac{500}{27})$ (3) $x=\frac{4}{3}$
(4)(i) $-\frac{1}{3}\leq x\leq 3$ (ii) $x\geq\frac{4}{3}$
(5) $y=3x-18$ (6) $(4;-6)$
(7) $x>4$ (8) $0<k<\frac{500}{27}$
(9) $b<-2$

(c) (2) 5,12 (3) 3
(4) $y=24x-81$ (5) $(\frac{1}{2};-\frac{13}{2})$
(6)(i) $-20<t<7$ (ii) $t=7$ or $t=-20$
(iii) $t<-20$ or $t>7$

(d) (2) $x=\frac{-2\pm\sqrt{-8}}{2}$; no real root (3) $y=6x-1$
(4) $y=9x+4$ and $y=9x$

EXERCISE 11

(a) (1) Straight line graph
positive gradient
$x-$intercept: 4

(2) Straight line graph
negative gradient
$x-$intercept: -2

(3) Parabola
positive orientation
stationary point: $x=3$
$x-$intercepts: $1;5$

(4) Parabola
negative orientation
stationary point: $x=-4$
$x-$intercepts: $-6;-2$

(b) (1) 9
(2) $x=1;3$
(3) $x=2$
(4) -3
(5)(i) $x\leq 1$ or $x\geq 3$
(ii) $1\leq x\leq 3$
(6)(i) concave down
(ii) concave up
(7) Cubic graph
positive orientation
stationary point: $x=1$ and $x=3$
point of inflection: $x=2$
$y-$intercept of 0

(c) (1) 8
(2) $x=-2;4$
(3) $x=1$
(4) 8
(5) $x\leq -2$ or $x\geq 4$
(6) Cubic graph
negative orientation
stationary point: $x=-2;4$
point of inflection: $x=1$
negative $y-$intercept
(7)(i) $y=-x^2+2x+8$
(ii) $y=-\frac{1}{3}x^3+x^2+8x+2$

(d) (1) -4
(2) $x=1$
(3) B$(2;0)$
(4) $f(x)=2x^2-4x$
(5) horisontal line
intersecting $y-$axis: 4

(e) (1) Parabola
Negative orientation
Stationary point: $(-1;0)$
$y-$intercept: -3

(2) $y=-3(x+1)^2$
$=-3x^2-6x-3$

(3) Straight line graph
$x-$intercept: -1
$y-$intercept: -6

EXERCISE 12

(a) (1) 40 m^3
(2)(i) $16\text{ m}^3/\text{hour}$
(ii) $-80\text{ m}^3/\text{hour}$
(3)(i) 2 hours
(ii) 7 hours
(4) 5 hours
(5) 240 m^3

(b) (1) 49 °C
(2) -79 °C
(3) 9 °C/min
(4) -36 °C/min
(5) 4 minutes
(6) 81 °C
(7) $0\leq t\leq 4$

(c) (1) 35 m
(2) 30 m/s
(3) 10 m/s
(4) -10 m/s^2.
(5) After 3 seconds
(6) 80 m
(7) -40 m/s $\therefore$ speed = 40 m/s downwards
(8)(i) slowing down
(ii) speeding up

(d) (1) 12,5 m/s
(2) 22,5 m/s
(3) 3 m/s^2.
(4) speeding up
(5) After 20 seconds
(6) 400 m
(7) speeding up
(8) -90 m/s $\therefore$ 90 m/s southward
(9) 30 m/s

EXERCISE 13

(a)	2	
(b)	(1) $x = 3$	(2) -23
(c)	(2) $l = 20$ cm and $b = 20$ cm	(3) $400\ \text{m}^2$
(d)	(2) $1800\ \text{m}^2$	
(e)	(1) $100x - \frac{\pi x^2}{4}$	(2) $\frac{10000}{\pi}\ \text{mm}^2$
(f)	(2) $x = 6$ and $y = 12$	
(g)	(2) $x = 5$	(3) $150\ \text{cm}^2$
(h)	(2) $V = 50x - \frac{2}{3}x^3$	(3) $166{,}67\ \text{m}^3$
(i)	(2) $x = 5$	(3) $2250\ \text{cm}^3$
(j)	(2) $96\pi\ \text{cm}^2$	
(k)	(1) $h = \frac{54}{r} - \frac{r}{2}$	(3) $216\pi\ \text{cm}^2$
(l)	(1) $V = 10\pi r^2 - \frac{1}{3}\pi r^3$	(2) 10 cm
(m)	20 units	
(n)	(1) $AB = \frac{9}{x} - 4 + x$	(2) 2 units
	(3) $B(3;1)$	
(o)	(1) $-6a^3 + 72a$	(2) $a = 2$
	(3) $96\ \text{units}^2$	
(p)	$r = 4{,}30$ cm and $h = 8{,}61$ cm	
(q)	R350	

CONSOLIDATION AND EXTENSION EXERCISE

(a)	(1) -9	(2) 5	
(b)	(1) $-6x^2$	(2) $-\frac{5}{x^2}$	
(c)	(1) 2	(2) 5	(3) $f(x) = x^2 - 3x + 5$
(d)	(1) $12x^2 - 6x + 5$	(2) $\frac{1}{3}x^{-\frac{2}{3}} + 5x^{-2} + 2x - 1$	(3) $-2x^{-4} - \frac{10}{7}x - \frac{3}{2}x^{-\frac{1}{2}}$
	(4) $-9x^2 + 24x - 12$	(5) $\frac{2}{3}x^{-\frac{1}{3}} + x^{-2}$	(6) $3x^{-2} + 2x^{-3}$
	(7) $\frac{3}{2}x^{\frac{1}{2}} + x^{-\frac{1}{2}}$	(8) $-\frac{1}{6}a^{-\frac{4}{3}} + \frac{3}{2}a^{-2}$	(9) -2
	(10) $\frac{1}{2}p^{-\frac{1}{2}} + p^{-\frac{3}{2}}$	(11) $2x^{\sqrt{2}-1} + \frac{\sqrt[3]{2}}{3}x^{-\frac{2}{3}}$	(12) $2p^3x - q$
	(13) $n^2t^{n-1} - n^3$		
(e)	(1) $\frac{1}{3}x^{-\frac{2}{3}}$	(2) $2x + 1$	
(f)	$y = 3x - 4$		
(g)	$y = 9x - 16$ and $y = 9x + 16$		
(h)	$p = 0$ and $q = -3$		
(i)	$p = 3$ and $q = 12$		
(j)	(1) $a = -13$	(2) -5	

(k) (1) Cubic curve
Positive orientation
x – intercepts: $x = -1;4$
y – intercept: -4
Turning points: $(-1;0);\left(\frac{7}{3};\frac{-500}{27}\right)$
Point of inflection: $\left(\frac{2}{3};\frac{-250}{27}\right)$

(2) Cubic curve
Negative orientation
x – intercepts: $x = -2;1;6$
y – intercept: -12
Turning points: $\left(\frac{-2}{3};\frac{-400}{27}\right);(4;36)$
Point of inflection: $\left(\frac{5}{3};\frac{286}{27}\right)$

(l) (1) $(-1;2)$
(2) Cubic curve
Positive orientation
x – intercepts: $x = -2$
y – intercepts: 4
Stationary point of inflection: $(-1;2)$
(3) $f(x) = 2(x+1)^3 + 2$

(m) Cubic curve
Negative orientation
$x-$intercepts: $x=0;3$
$y-$intercepts: 0
Turning points: $x=1;3$
Point of inflection: $x=2$

(n) (2) $A\left(\frac{1}{3};\frac{500}{27}\right)$ (3) $\left(-\frac{4}{3};\frac{250}{27}\right)$ (4) $x=-4;-1;1$
(5) $y=7x+18$ (6)(i) $-3\leq x\leq\frac{1}{3}$ (ii) $x\leq-4$ or $-1\leq x\leq1$
(iii) $-3\leq x<0$ or $x>\frac{1}{3}$ (iv) $-3\leq x\leq\frac{1}{3}$ or $x\geq2$

(o) (2) A(2;5) (4) D(7;55) (5) $2\leq x\leq4$
(6) 32 units (7)(i) $k=1$ and $k=5$ (ii) $1<k<5$
(iii) $k<-5$ or $k>-1$

(p) (1)(i) -10 (ii) -10 (2) $x=1$; $x=5$
(3) $x=3$ (4) $x\leq1$ or $x\geq5$ (5)(i) concave up
(ii) concave down

(q) Cubic curve
$x-$intercepts: $x=-3$
$y-$intercepts: $y=9$
Stationary point of inflection: $(-2;1)$

(r) (1) 140 °C (2) 21 °C (3) -24 °C / hour
(4) At $t=3$ hours (5) 5 °C (6) At $t=4{,}5$ hours

(s) (1) 15 m/s (2) 5 m/s (3)(i) Slowing down
(ii) Speeding up (4) 1,5 s (5) 4 seconds
(6) 25 m/s

(t) (1) $x=4$ (2) 6

(u) (2) 1234 ml

(v) (2)(i) $x=14$ cm (ii) 56 cm and 44 cm

(w) $b=-9;c=27;d=-22$

(x) (2)(i) One (ii) None (3) $x=\frac{-b}{3a}$

(y) $k<3$

(z) (1) $x=0$; $x=\frac{4}{3}$
(2) Stationary point of inflection $(0;0)$
local minimum $(1;-1)$
(3) $(0;0)\left(\frac{2}{3};\frac{-16}{27}\right)$

CHAPTER FIVE

EXERCISE 1

(a) (1) $\cos\alpha\cos\beta-\sin\alpha\sin\beta$ (2) $\sin x\cos25°-\cos x\sin25°$ (3) $\frac{\sqrt{2}}{2}(\sin x+\cos x)$
(4) $\frac{\sqrt{3}}{2}\cos x+\frac{1}{2}\sin5x$

(b) (1) $\sin(x-y)$ (2) $\cos(\alpha-12°)$ (3) $\cos5\theta$
(4) $\sin10°$ (5) $\sin20°$ (6) $\cos x$
(7) $\sin x$

(c) (1) $\sqrt{2}\sin\theta$ (2) $-\sqrt{3}\sin\alpha$ (3) 0
(4) $\cos A-\sqrt{3}\sin A$ (5) $\frac{1}{2}$ (6) $\cos x$

(d) (1) $\frac{\sqrt{6}+\sqrt{2}}{4}$ (2) $\frac{\sqrt{6}+\sqrt{2}}{4}$ (3) $\frac{\sqrt{3}}{2}$
(4) $\frac{\sqrt{3}}{2}$ (5) $\frac{1}{2}$ (6) $-\frac{1}{2}$
(7) $-\frac{\sqrt{2}}{2}$ (8) $-\frac{1}{2}$

(e) $2\cos(x-30°)$

(f)	(1) $\cos 70°$	(2) $\cos(45° - \alpha)$	
(g)	(1) $2\sin\theta\cos x$	(2) $\cos 13°$	
(h)	1		

EXERCISE 2

(a)	(1) $\cos 2\alpha$	(2) $\sin 2x$	(3) $\cos 2\theta$
	(4) $\cos 2P$	(5) $\sin\frac{\theta}{2}$	(6) $\cos 10\alpha$
	(7) $\cos 22°$	(8) $\sin 10°$	(9) $\cos 70°$
(b)	(1) $\frac{\sqrt{2}}{2}$	(2) $\frac{\sqrt{2}}{2}$	(3) $-\frac{\sqrt{3}}{2}$
	(4) 1	(5) -1	(6) $\frac{\sqrt{2}}{2}$
	(7) $\frac{\sqrt{3}}{2}$	(8) $-\frac{\sqrt{3}}{2}$	(9) $-\frac{\sqrt{3}}{2}$
	(10) $\frac{1}{4}$	(11) $\frac{\sqrt{2}+2}{2}$	(12) $\frac{\sqrt{3}}{4}$
	(13) $\frac{3}{2}$	(14) $-\frac{\sqrt{3}}{2}$	
(c)	(1) $\tan x$	(2) $\cos\theta - \sin\theta$	(3) 2
	(4) -1	(5) 1	
(d)	2		

EXERCISE 3

(a)	(1) $\frac{-24}{25}$	(2) 1	(3) $\frac{-7}{25}$
	(4) $\frac{-24}{25}$		
(b)	(1) $\frac{-63}{65}$	(2) $\frac{56}{65}$	(3) $\frac{7}{25}$
	(4) $\frac{-24}{7}$		
(c)	(1) $\frac{\sqrt{5}}{3}$	(2) $\frac{\sqrt{5}}{6}$	(3) $\frac{1}{3}$
	(4) $\frac{1}{\sqrt{6}}$		
(d)	(1) $\frac{5}{6}$	(2) $\cos P = \sqrt{\frac{11}{12}}$	(3) $\sin P = \sqrt{\frac{1}{12}}$
(e)	(1) $\frac{\sqrt{7}-1}{4}$	(2) $\frac{\sqrt{7}}{4}$	

EXERCISE 4

(a)	(1) $\sin(\alpha+\beta)$	(2) $\cos(A+B)$	(3) $\cos(x-y)$	(4) $\sin(P-Q)$
	(5) $\cos 2\theta$	(6) $\sin 2x$	(7) 1	(8) $-\sin(\alpha+\beta)$
	(9) $-\cos 2x$	(10) $-\sin 2A$	(11) $-\cos 2\theta$	
(b)	(1) $\cos 2\theta$	(2) $\sin 2\alpha$	(3) 1	(4) -1
	(5) $\tan x$	(6) $\frac{1}{2}$		
(c)	(1) $\cos A\cos B - \sin A\sin B$			
	(2)(i) $\sin A\cos B + \cos A\sin B$		(ii) $\sin A\cos B - \cos A\sin B$	

EXERCISE 5

(a)	(1) 0	(2) $\frac{-\sqrt{3}}{2}$	(3) $\frac{\sqrt{3}}{2}$	(4) -1
	(5) $\frac{-\sqrt{2}}{2}$	(6) $\frac{\sqrt{2}}{2}$	(7) $-\frac{1}{4}$	(8) $-\frac{1}{8}$
(b)	(1) 2	(2) -1	(3) 1	(4) $-\frac{1}{2}$
(c)	(1) $\frac{1}{2}$	(2) -1	(3) $\frac{\sqrt{3}}{2}$	(4) 1
	(5) 1	(6) 2		
(d)	2			
(e)	1			

EXERCISE 6

(a)	(1) k	(2) k	(3) $\sqrt{1-k^2}$	(4) $2k\sqrt{1-k^2}$
	(5) $1-2k^2$	(6) $\frac{\sqrt{3}}{2}\cdot\sqrt{1-k^2}+\frac{1}{2}\cdot k$	(7) $\frac{-k}{\sqrt{1-k^2}}$	(8) $\frac{\sqrt{1-k^2}}{k}$
	(9) $\frac{2k\sqrt{1-k^2}}{1-2k^2}$	(10) k	(11) $\sqrt{1-k^2}$	(12) $1-k$

(b)	(1) $-p$	(2) p	(3) $-\sqrt{1-p^2}$	(4) $\frac{-\sqrt{1-p^2}}{p}$
	(5) $2p^2-1$	(6) $2p\sqrt{1-p^2}$	(7) $\frac{2p\sqrt{1-p^2}}{2p^2-1}$	(8) $\frac{\sqrt{3}}{2}p+\frac{1}{2}\sqrt{1-p^2}$
	(9) $\frac{\sqrt{3}}{2}p+\frac{1}{2}\sqrt{1-p^2}$	(10) $p+1$	(11) $\frac{1-p}{2}$	(12) $\sqrt{\frac{1-p}{2}}$
(c)	(1) $-t$	(2) $\frac{-t}{\sqrt{t^2+1}}$	(3) $\frac{1}{\sqrt{t^2+1}}$	(4) $\frac{2}{t^2+1}-1$
	(5) $\frac{-2t}{t^2+1}$	(6) $\frac{1+\sqrt{3}t}{2\sqrt{t^2+1}}$		
(d)	(1) $\frac{m}{\sqrt{1-m^2}}$	(2) $1-2m^2$	(3) $\frac{1}{2}m$	(4) $\sqrt{1-m^2}$
	(5) $-\sqrt{1-m^2}$	(6) m		
(e)	(1) $-k$	(2) $2k^2-1$	(3) $2k\sqrt{1-k^2}$	(4) $\frac{1}{2}\sqrt{1-k^2}$
	(5) $\sqrt{\frac{k+1}{2}}$	(6) $\sqrt{\frac{1-k}{2}}$		
(f)	(1) $1-t$	(2) $\sqrt{1-t}$	(3) $2t-1$	(4) $2\sqrt{t-t^2}$
	(5) $\frac{-2\sqrt{t-t^2}}{2t-1}$	(6) $\frac{\sqrt{3t}-\sqrt{1-t}}{2}$		
(g)	(1) $2p$	(2) $\sqrt{1-4p^2}$	(3) $1-8p^2$	(4) $\frac{\sqrt{1-4p^2}}{2p}$
	(5) $\frac{-\sqrt{3}\sqrt{1-4p^2}}{2}-p$	(6) $-\sqrt{1-4p^2}$		
(h)	(*1) $\frac{1}{2k}$	(2) $\frac{2k}{\sqrt{1-k^2}}$		
(i)	(1) m^2-1	(2) $\sqrt{2m^2-m^4}$	(3) $2\left(m^2-1\right)\sqrt{2m^2-m^4}$	(4) $\sqrt{2m^2-m^4}$
	(5) $\sqrt{2m^2-m^4}$	(6) $\sqrt{2-m^2}$		

EXERCISE 9

(a)
(1) $111{,}80°+k\cdot 360°;k\in\mathbb{Z}$; $291{,}8°+k\cdot 360°;k\in\mathbb{Z}$
(2) $26{,}87°+k\cdot 360°;k\in\mathbb{Z}$; $313{,}13°+k\cdot 360°;k\in\mathbb{Z}$
(3) $3{,}85°+k\cdot 120°;k\in\mathbb{Z}$; $56{,}15°+k\cdot 120°;k\in\mathbb{Z}$
(4) $3{,}73°;76{,}27°;183{,}73°;256{,}27°$
(5) $26{,}36°;45{,}64°$
(6) $15°+k\cdot 180°;k\in\mathbb{Z}$; $75°+k\cdot 180°;k\in\mathbb{Z}$; $105°+k\cdot 180°;k\in\mathbb{Z}$; $165°+k\cdot 180°;k\in\mathbb{Z}$

(b)
(1) $33{,}69°+k\cdot 360°;k\in\mathbb{Z}$; $213{,}69°+k\cdot 360°;k\in\mathbb{Z}$
(2) $-318{,}59°;-270°;-90°;-41{,}41°;41{,}41°;90°;270°;318{,}59°$
(3) $0°+k\cdot 180°;k\in\mathbb{Z}$; $210°+k\cdot 360°;k\in\mathbb{Z}$; $330°+k\cdot 360°;k\in\mathbb{Z}$
(4) $68{,}20°+k\cdot 360°;k\in\mathbb{Z}$; $248{,}20°+k\cdot 360°;k\in\mathbb{Z}$
(5) $60°+k\cdot 360°;k\in\mathbb{Z}$; $300°+k\cdot 360°;k\in\mathbb{Z}$
(6) $153{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $333{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $108{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $288{,}43°+k\cdot 360°;k\in\mathbb{Z}$
(7) $38{,}17°+k\cdot 360°;k\in\mathbb{Z}$; $141{,}83°+k\cdot 360°;k\in\mathbb{Z}$

(c)
(1) $90°+k\cdot 360°;k\in\mathbb{Z}$; $270°+k\cdot 360°;k\in\mathbb{Z}$
(2) $-95°;85°$
(3) $15°+k\cdot 120°;k\in\mathbb{Z}$; $165°+k\cdot 360°;k\in\mathbb{Z}$
(4) $147{,}5°+k\cdot 180°;k\in\mathbb{Z}$

(d) $153{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $333{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $19{,}47°+k\cdot 360°;k\in\mathbb{Z}$; $160{,}53°+k\cdot 360°;k\in\mathbb{Z}$

EXERCISE 10

(a) $0°+k\cdot 360°;k\in\mathbb{Z}$; $180°+k\cdot 360°;k\in\mathbb{Z}$; $70{,}53°+k\cdot 360°;k\in\mathbb{Z}$; $289{,}47°+k\cdot 360°;k\in\mathbb{Z}$
(b) $60°;300°$
(c) $20{,}91°+k\cdot 180°;k\in\mathbb{Z}$; $69{,}09°+k\cdot 180°;k\in\mathbb{Z}$
(d) $21{,}81°;118{,}19°$
(e) $33{,}69°+k\cdot 360°;k\in\mathbb{Z}$; $213{,}69°+k\cdot 360°;k\in\mathbb{Z}$; $153{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $333{,}43°+k\cdot 360°;k\in\mathbb{Z}$
(f) $75{,}96°+k\cdot 360°;k\in\mathbb{Z}$; $255{,}96°+k\cdot 360°;k\in\mathbb{Z}$; $116{,}57°+k\cdot 360°;k\in\mathbb{Z}$; $296{,}57°+k\cdot 360°;k\in\mathbb{Z}$

(g) $90°+k\cdot360°;k\in\mathbb{Z}$; $270°+k\cdot360°;k\in\mathbb{Z}$; $26{,}57°+k\cdot360°;k\in\mathbb{Z}$; $206{,}57°+k\cdot360°;k\in\mathbb{Z}$

(h) $-160{,}53°;-19{,}47°;30°;150°$

(i) $108{,}43°+k\cdot360°;k\in\mathbb{Z}$; $288{,}43°+k\cdot360°;k\in\mathbb{Z}$; $135°+k\cdot360°;k\in\mathbb{Z}$; $315°+k\cdot360°;k\in\mathbb{Z}$

(j) $5°+k\cdot120°;k\in\mathbb{Z}$; $-195°-k\cdot360°;k\in\mathbb{Z}$

(k) $48{,}19°+k\cdot360°;k\in\mathbb{Z}$; $311{,}81°+k\cdot360°;k\in\mathbb{Z}$

(l) $27{,}5°+k\cdot180°;k\in\mathbb{Z}$

(m) $0°+k\cdot360°;k\in\mathbb{Z}$; $180°+k\cdot360°;k\in\mathbb{Z}$; $45°+k\cdot360°;k\in\mathbb{Z}$; $135°+k\cdot360°$;
$225°+k\cdot360°;k\in\mathbb{Z}$; $315°+k\cdot360°;k\in\mathbb{Z}$

(n) $75°+k\cdot360°;k\in\mathbb{Z}$; $195°+k\cdot360°;k\in\mathbb{Z}$

(o) $0°+k\cdot360°;k\in\mathbb{Z}$; $180°+k\cdot360°;k\in\mathbb{Z}$; $63{,}43°+k\cdot360°;k\in\mathbb{Z}$; $243{,}43°+k\cdot360°;k\in\mathbb{Z}$

(p) $60°+k\cdot360°;k\in\mathbb{Z}$; $300°+k\cdot360°;k\in\mathbb{Z}$; $221{,}81°+k\cdot360°;k\in\mathbb{Z}$; $318{,}19°+k\cdot360°;k\in\mathbb{Z}$

(q) $90°+k\cdot360°;k\in\mathbb{Z}$; $270°+k\cdot360°;k\in\mathbb{Z}$; $45°+k\cdot180°;k\in\mathbb{Z}$

(r) $15°+k\cdot90°;k\in\mathbb{Z}$

EXERCISE 11

(a)
(1) $90°+k\cdot180°;k\in\mathbb{Z}$; $k\cdot90°;k\in\mathbb{Z}$
(2) $30°+k\cdot120°;k\in\mathbb{Z}$; $270°+k\cdot360°;k\in\mathbb{Z}$
(3) $90°+k\cdot180°;k\in\mathbb{Z}$; $0°+k\cdot180°;k\in\mathbb{Z}$
(4) $0°+k\cdot180°;k\in\mathbb{Z}$; $0°+k\cdot360°;k\in\mathbb{Z}$
(5) $45°+k\cdot360°;k\in\mathbb{Z}$; $225°+k\cdot360°;k\in\mathbb{Z}$; $45°+k\cdot90°;k\in\mathbb{Z}$; $90°+k\cdot180°;k\in\mathbb{Z}$
(6) $-90°-k\cdot180°;k\in\mathbb{Z}$; $30°+k\cdot30°;k\in\mathbb{Z}$; $k\cdot36°;k\in\mathbb{Z}$; $18°+k\cdot36°;k\in\mathbb{Z}$

(b)
(1) $0°;90°;180°;270°;360°$
(2) $45°;135°;225°;315°$

(c)
(1) $-90°;0°;90°;180°$
(2) $-90;90°$

EXERCISE 12

(a)
(1)(i) $180°$ (ii) 3 (2)(i) $A(90°;2)$ (ii) $B(90°;-1)$
(3) $P\left(30;\frac{1}{2}\right)Q\left(150;\frac{1}{2}\right)$ (4)(i) $x\in[45°;135°]$ (ii) $x\in[0°;30°)$ or $(150°;180°]$
(5)(i) $90°$ (ii) $30°;150°$ (6)(i) 1 unit up (ii) reflection in the x-axis

(b)
(1)(i) $360°$ (ii) 1 (2)(i) $(-45°;0)$ (ii) $(45°;0)$
(iii) $(30°;0)$ (3) $P(-20°;-0{,}78)Q(60°;0{,}5)R(100;0{,}94)$
(4)(i) $[-90°;-20°)$ or $(60°;100°)$ (ii) $[-45°;30°]$ or $[45°;135°]$
(5)(i) $-60°$ (ii) $-20°;60°;100°$ (6) $75°$ to the right (7) 1 unit up

(c)
(1) $30°$ (2) $\frac{1}{2}$ (3) $-120°$ (4)(i) $(-120°;-75°]$
(ii) $(-120°;-90°]$ or $[-30°;0°]$ (5) 1 unit up (6)(i) $-75°;-15°$ (ii) $-75°$

(d)
(1) $\frac{1}{2};-\frac{1}{2}$ (2) $3;-1$ (3) $1;-1$ (4) $2;0$
(5) $2;-2$ (6) $2;-2$ (7) $4;-4$ (8) $4;1$
(9) $5;3$ (10) $16;9$ (11) $2;10$ (12) $9;0$
(13) $6;2$

(e)
(1) Positive cos graph, amplitude of 1, period of $180°$, shifted 1 up.
(2) Positive cos graph, amplitude of 1, period of $180°$, shifted 1 down.
(3) Positive sin graph, amplitude of 2, period of $180°$.
(4) Positive sin graph, amplitude of $\frac{1}{4}$, period of $90°$.
(5) Positive sin graph, amplitude of 1, period of $90°$.
(6) Positive cos graph, amplitude of 1, period of $120°$.
(7) Positive tan graph, passing through $(22{,}5°;0{,}5)$, period of $90°$.
(8) Positive sin graph, amplitude of 1, period of $120°$, shifted $60°$ right.
(9) Negative sin graph, amplitude of 1, period of $360°$, shifted $60°$ right.
(10) Positive sin graph, amplitude of 2, period of $360°$, shifted $60°$ right.
(11) Positive sin graph, amplitude of 2, period of $360°$, shifted $45°$ left.
(12) Positive tan graph, passing through $(0;1)$, period of $180°$.

CONSOLIDATION AND EXTENSION EXERCISE

(a) (1) $-\frac{7}{25}$ (2) $\frac{24}{25}$ (3) $-\frac{7}{25}$

(b) (1) $\sqrt{26}$ (2) $\frac{5}{13}$ (3) $\sqrt{a^2+25}$ (4) -12
(5) $-\frac{5}{12}$

(c) $-\frac{1}{3}$

(d) (1) $\frac{1}{2}$ (2) 1

(e) (1) $\frac{\sqrt{6}-\sqrt{2}}{4}$ (2) $\frac{\sqrt{2}}{4}$ (3) $\frac{1}{2}$ (4) $\frac{\sqrt{3}}{2}$
(5) $\frac{\sqrt{3}}{2}$ (6) 2

(f) (1) $0<t<1$ (2)(i) $\frac{-\sqrt{1-t^2}}{t}$ (ii) $-2t\sqrt{1-t^2}$ (iii) $\frac{1}{2}t+\frac{\sqrt{3(1-t^2)}}{2}$

(g) (1) p (2) $-p$ (3) $\sqrt{1-p^2}$ (4) $2p^2-1$
(5) $2p\sqrt{1-p^2}$ (6) $\frac{1}{2}p-\frac{\sqrt{3}}{2}\sqrt{1-p^2}$ (7) $\frac{1}{2\sqrt{1-p^2}}$ (8) $\sqrt{\frac{p+1}{2}}$
(9) $\sqrt{\frac{1-p}{2}}$

(h) (1) $2t$ (2) $8t^2-1$ (3) $4t\sqrt{1-4t^2}$

(i) $3x-4x^3$

(k) (1) $135°+k\cdot 180°;k\in\mathbb{Z}$ (2) $k\cdot 60°;k\in\mathbb{Z}$; $18°+k\cdot 18°;k\in\mathbb{Z}$; $30°+k\cdot 60°;k\in\mathbb{Z}$

(l) (1) $-113{,}33°;-53{,}33°;6{,}67°;66{,}67°;126{,}67°$ (2) $-77{,}85°;-57{,}15°;12{,}15°;32{,}85;102{,}15°;122{,}85°$

(m) (1) $120°+k\cdot 360°;k\in\mathbb{Z}$; $240°+k\cdot 360°;k\in\mathbb{Z}$
(2) $108{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $288{,}43°+k\cdot 360°;k\in\mathbb{Z}$; $26{,}57°+k\cdot 360°;k\in\mathbb{Z}$;
$206{,}57°+k\cdot 360°;k\in\mathbb{Z}$
(3) $90°+k\cdot 360°;k\in\mathbb{Z}$; $210°+k\cdot 360°;k\in\mathbb{Z}$ (4) $24{,}4°+k\cdot 72°;k\in\mathbb{Z}$; $-122°-k\cdot 360°;k\in\mathbb{Z}$

(n) (2) $228{,}59°+k\cdot 360°;k\in\mathbb{Z}$; $311{,}41°+k\cdot 360°;k\in\mathbb{Z}$; $90°+k\cdot 360°$

(o) (2) $\frac{\sqrt{3}+2}{4}$

(p) (2)(i) $\frac{1}{2}$ (ii) 0

(q) $77°+k\cdot 360°;k\in\mathbb{Z}$; $103°+k\cdot 360°;k\in\mathbb{Z}$

(r) $\frac{1}{t}$

(s) (1) $\frac{1+\sqrt{5}}{4}$ (2) $\frac{1-\sqrt{5}}{4}$

(t) (1) $\sin 16°$ (2) $\sin 2^{n+1}$ (3)(i) $\sqrt{1-k^2}$ (ii) $\frac{1}{2}\sqrt{1-k^2}-\frac{\sqrt{3}}{2}k$
(iii) $-\sqrt{1-k^2}$

(u) (1) $180°$ (2) $a=2$ (3)(i) $(-45°;1)$ (ii) $(45°;-1)$
(iii) $(135°;1)$ (iv) $(30°;1)$ (4) $P(-20°;0{,}64);Q(100°;0{,}34)$
(5)(i) $[-20°;100°]$ (ii) $(-90°;-60°);(0°;90°);(120°;180°)$ (6)(i) $-45°;135°$
(ii) $x=30°$ (iii) $x=-20°;100°$ (7) Reflect in x-axis and move 1 up
(8) $15°$ to the right

(v) 3,606

CHAPTER SIX

EXERCISE 1

(a) (1) 6,22 cm (2) 9,30 cm (3) $41{,}95°$
(4) $79{,}95°$ (5) $49{,}68\ cm^2$

(b) (1) 6,08 m (2) $62{,}11°$ (3) $55{,}15°$
(4) $86{,}18°$

(c) (1) 9 m (2) 11,57 m (3) $26{,}91°$

(d) (1) $47{,}22°$ (2) 8,17 m (3) $53{,}24°$
(4) $32{,}80\ m^2$ (5) $49{,}40°$

EXERCISE 2

(a) (2) $\frac{5}{3}$ m
(b) (2) 24,51 cm
(d) (2) 40,15°
(e) (2) 18,87 m
(f) (2) 17,47 m

CONSOLIDATION AND EXTENSION EXERCISE

(a) (1) 4,36 m (2) 101,51° (3) 7,39 m^2 (4) 23,39°
(b) (1) 10,90 m (2) 9,36 m (3) 70,61° (4) 48,51°
(5) 118,43 m^2
(c) (1) 3,97 m (2) 78,75°
(d) (1) 26,57° (2) 11,31° (3) 88,29°
(e) (2) 55,70°
(h) (1)(i) $\sqrt{7}x$ (ii) $\frac{7\sqrt{3}}{4}x^2$ (2) 19,11°
(i) (1) 6,36 m (2) 23,56 m^2 (3) 54,11 m^2
(j) (1) 124,72 m^3 (2) 62,61° (3) 179,95 m^2
(k) (1) 36,21° (2) 2814,34 cm^2 (3) 9000 cm^3

CHAPTER SEVEN

EXERCISE 1

(a) (1) $2;(0;0)$ (2) $3;(2;1)$ (3) $6;(-5;-2)$
(4) $4;(-3;1)$ (5) $9;(0;7)$ (6) $7;(-2;0)$
(7) $2\sqrt{5};(5;-3)$ (8) $\sqrt{3};(1;-3)$
(b) (1) $x^2+y^2=64$ (2) $(x-2)^2+(y+1)^2=9$ (3) $(x+3)^2+(y-5)^2=25$
(4) $(x-1)^2+y^2=7$ (5) $x^2+y^2=5$ (6) $(x+4)^2+(y-1)^2=25$
(7) $x^2+(y-2)^2=5$ (8) $(x-5)^2+(y+2)^2=10$
(c) (1) $x^2+y^2=169$ (2) $(x-3)^2+y^2=13$ (3) $(x+2)^2+(y-1)^2=5$
(4) $(x-2)^2+(y+3)^2=25$ (5) $(x-3)^2+(y-2)^2=9$ (6) $(x+4)^2+(y+2)^2=4$
(7) $(x-7)^2+(y+7)^2=49$ (8) $(x-20)^2+(y+16)^2=400$ (9) $(x+2)^2+(y+4)^2=16$
(10) $(x+1)^2+(y-1)^2=1$
(d) $(x-3)^2+\left(y\pm\sqrt{21}\right)^2=25$
(e) $(x-2)^2+(y+3)^2=25$
(f) $(x-2)^2+(y-7)^2=113$

EXERCISE 2

(a) (1) $3;(-1;4)$ (2) $4;(2;-3)$ (3) $1;(-3;0)$
(4) $5;(0;5)$ (5) $2\sqrt{2};(5;-7)$ (6) $2;\left(\frac{3}{2};-1\right)$
(7) $\frac{5}{3};\left(-\frac{3}{2};\frac{1}{3}\right)$ (8) $5;(0;0)$
(b) (1) 2 (2) $3;(-4;1)$
(c) (1) $(3;-2)$ (2) 156
(d) (1) $a=3;b=-1$ (2) 2
(e) (1) 8 or -1 (2) $(8;0)$ or $(-1;0)$

EXERCISE 3

(a) (1) $(x+3)^2+(y-1)^2=20$ (2) $y=2x-3$
(b) (1) $y=-x+4$ (2) $y=x+12$
(c) $y=-5x-28$

(d) (1) $y=\frac{3}{2}x-17$ (2) $y=\frac{2}{3}x-\frac{1}{3}$

(e) (2) $(3;4)$ (3)(i) $x=8$ (ii) $y=-1$

(4)(i) $3x-4y+32=0$ (ii) $y=\frac{-4}{3}x+\frac{49}{3}$

(f) (1) $y=\frac{1}{3}x+\frac{11}{3}$ (2) $(x+2)^2+(y-3)^2=10$ (3)(i) $(-3;0)$

(ii) $y=-\frac{1}{3}x-1$ (iii) $(3;-2)$ (iv) $2\sqrt{10}$ units

(g) (1) $x^2+(y-2)^2=20$ (2) $4\sqrt{5}$ units (3) $y=-2x+12$

EXERCISE 4

(a) (1) Outside (2) On circumference (3) Inside
(4) On circumference

(b) (1) Inside (2) On circumference (3) Outside
(4) Inside

(c) (1) Touch externally (2) Small circle lies outside larger circle
(3) Touch internally (4) Don't intersect: small circle lies inside larger circle
(5) Intersect in two points (6) Touch externally (7) Touch internally

(d) (1) $(-2;-3)$ and $\left(\frac{6}{5};\frac{17}{5}\right)$ (2) Secant

(e) (1) $(2;-7)$ (2) Tangent

(f) (1) Exterior line (2) Secant (3) Tangent
(4) Exterior line

(h) (1)(i) The x-axis is tangent to circle (ii) The y-axis is an exterior line to circle
(iii) Circles lie outside one another (2) 20

(i) $p=\pm4$

(j) (1) $k=-7$ or $k=5$ (2) $k<-7$ or $k>5$ (3) $-7<k<5$

(k) $y=-2x+10$; $y=-2x-10$

EXERCISE 5

(a) (1) $(2;3)$ (2) $D(0;4)$ and $E(0;2)$ (3) $y=2x+4$

(4) $71{,}57°$ (5) $F(1;5)$ and $G(4;2)$ (6) $\frac{\sqrt{2}}{2}$

(7) $26{,}57°$ (8)(i) $m_{DH}=m_{AC}=-1$ (ii) 3 units^2

(b) (1) $2\sqrt{10}$ units (2) $(2;-3)$ (3) $(x-2)^2+(y+3)^2=10$

(5)(i) $y=3x-9$ (ii) $y=-3x-3$ (iii) $y=-3x+3$

(6)(i) 14,81 units (ii) 8 units^2

(c) (1) 6 units (2) $A(-3;-2)$ and $B(1;2)$ (3) $C(-2;1)$

(4) $x+3y-11=0$ (5) $26{,}57°$ (6) 10 units^2

(d) (1)(i) $T(8;1)$ (ii) $m_{PS}\times m_{SQ}=-1 \quad \therefore \hat{PSQ}=90°$ (2) $(x-4)^2+(y-2)^2=17$

(3) Yes. $\hat{PSQ}=90°=\hat{PRQ}$ (line subtends $=\angle$s) (4) 2 units

(e) (1) 10 units (2) 10 units (tangents from same point)

(3) $(x-2)^2+(y+5)^2=25$ (4) $126{,}87°$ (5) $y=-3$ and $y=7$

CONSOLIDATION AND EXTENSION EXERCISE

(a) (1) $r=5;(0;0)$ (2) $r=3;(-3;7)$ (3) $r=4;(5;0)$

(4) $r=\sqrt{32};(0;-2)$

(b) (1) $x^2+y^2=49$ (2) $(x-5)^2+(y+2)^2=4$ (3) $x^2+(y+1)^2=5$

(4) $x^2+y^2=10$ (5) $(x-2)^2+(y+3)^2=20$ (6) $(x-5)^2+y^2=7$

(7) $(x+5)^2+(y-2)^2=13$

(c) (1) $r=2;(6;-1)$ (2) $r=1;\left(-\frac{1}{2};1\right)$

(d) (1) $(0;1)$ (2) $t=9$

(e) $y=-4x+9$

(f) (1)(i) $A(-3;0)$ and $B(-1;0)$ (ii) $P(-1;6)$ and $Q(1;4)$ (2) $y=3x-1$

(4) $\hat{AQP}=90°$ ($\angle$ in semi$\odot$) (5) Outside

(g) (1) $a=7$ and $m=2$ (3) $(x-5)^2+(y-7)^2=36$ (4)(i) $x=-1$

	(ii) $y=1$	(5) $(-1;1)$	
(h)	(1) $(x+5)^2+(y-3)^2=25$	(2)(i) $y=-2$	(ii) $3x+4y-22=0$
(i)	(1) Gradients of both OP and OQ $=-5$	(2) P$(-1;5)$ and Q$(1;-5)$	
	(3) $x-5y+26=0$ and $x-5y-26=0$		
(j)	(1) $2\sqrt{10}$ units	(2)(i) $y=-5x-9$	(ii) $x+3y+7=0$
	(3) $53{,}14°$	(4) $(x+1)^2+(y+4)^2=10$	(5) A$(0;-9)$
(k)	(1) $(x-2)^2+(y+1)^2=25$	(2) $y=x+5$	(3) F$(6;-5)$
	(4) $50{,}71°$	(5) 6 units	(6) 15 units^2
(l)	(1) $45°$	(2) $45°$	(3) $(2;5)$
	(4)(ii) $a=1$ and $b=-2$	(5)(i) $(x-1)^2+(y+2)^2=100$	(ii) $y=\frac{3}{4}x+\frac{39}{4}$
(m)	(1) $a=-5$ and $b=2$	(2) $(x-4)^2+(y+10)^2=100$	(3) $10\sqrt{2}$ units
	(4) $25\sqrt{2}$ units^2	(5) $y=\frac{-4}{3}x-\frac{14}{3}$	(6) R$(-2;-2)$
	(7) $y=\frac{3}{4}x-\frac{1}{2}$		
(n)	(1)(i) Circles don't intersect	(ii) Circles intersect in two points	(2) $p=3$ or $p=-9$
	(3) $q=1$ or $q=-3$	(4)(i) $k=-12$ or $k=4$	(ii) $-12<k<4$
	(iii) $k<-12$ or $k>4$		
(o)	(1) $(x-3)^2+(y-6)^2=4$	(2) Yes	(3)(i) $(9;10)$
	$(x-3)^2+(y-2)^2=36$		(ii) No
	$x^2+(y-2)^2=9$		
(p)	32 squares		

CHAPTER EIGHT

EXERCISE 1

(a)	(1) 10 m	(2) 6 cm	(3) 13,5 units	(4) 21 units
	(5) 5 units	(6) 2 or 3 units		
(b)	(1) 35 cm	(2) 10 cm	(3) 25 cm	
(c)	(1) $3:2$	(2) $3:7$		
(d)	(1) 35 cm	(2) 18 cm		
(e)	(1)(i) 40 cm	(ii) 28 cm	(2)(i) 45 cm	(ii) 6 cm
(f)	(1) $9:7$	(2) No	(3)(i) $1:1$	(ii) $9:32$
	(4)(i) 72 m	(ii) 80 m	(iii) 57,5 m	

EXERCISE 2

(d)	(2)(i) 6 cm	(ii) 15 cm
(e)	(2) 12 cm	

EXERCISE 3

(a)	(1) $10:9$	(2) $9:14$
(b)	(1) $5:4$	(2) $5:7$
(c)	$9:16$	
(d)	(1) $4:11$	(2) $1:3$
(e)	(1) $13:10$	(2) $1:2$
(f)	$\frac{16}{9}$	
(g)	$7:1$	

EXERCISE 4

(a)	(1) $\frac{2}{3}$	(2) $\frac{3}{5}$	(3) $\frac{49}{25}$	(4) $\frac{98}{75}$
(b)	(1) $\frac{49}{144}$	(2) $\frac{5}{12}$	(3) $\frac{7}{5}$	(4) $\frac{343}{720}$
	(5) $\frac{35}{144}$			
(c)	(1) $\frac{64}{169}$	(2) $\frac{8}{5}$	(3) $\frac{32}{169}$	(4) $\frac{4}{13}$
	(5) $\frac{32}{65}$			
(d)	(1) $\frac{81}{16}$	(2) $\frac{3}{2}$	(3) $\frac{3}{2}$	(4) $\frac{5}{4}$

EXERCISE 5

(b) (3) 12 cm
(d) (3) 9 m

EXERCISE 7

(b) (2)(i) $x = 2$ (ii) 4 units (iii) $4\sqrt{5}$ units

CONSOLIDATION AND EXTENSION EXERCISE

(a) (1)(i) 27 cm (ii) 12 cm

(b) (1) $10:1$ (2) $10:3$ (3) $13:1$ (4) $\frac{3}{4}$
(5) $\frac{1}{16}$ (6) $\frac{1}{13}$ (7) $\frac{169}{100}$ (8) $\frac{13}{100}$

(c) (2)(i) $\frac{2}{1}$ (ii) $\frac{9}{4}$ (iii) $\frac{1}{3}$

(d) (2)(i) 48 cm (iv) $3:1$

(g) (1) $\hat{C} = \hat{D} = \hat{E}_2 = \hat{A}_2 = x$

(j) (2)(i) 80 mm (ii) 100 mm (iii) 60 mm (iv) 48 mm
(v) 24 mm

(k) (1)(i) $4x$ (ii) $2x$ (iii) $6x$ (5)(i) $2\sqrt{6}x$
(ii) $\sqrt{15}x$ (6) $6:\sqrt{6}$

(l) (2)(i) $30\sqrt{2}$ (ii) $18\sqrt{5}$ *(iii) $\frac{\sqrt{10}}{10}$

(n) (2)(i) $2:1$ (ii) $5:7$ (iii) $\frac{4}{9}$ (iv) $\frac{5}{7}$
(v) $\frac{20}{63}$

(o) (2)(i) $3:2$ (ii) $5:3$ (iii) $\frac{9}{4}$ (iv) $\frac{1}{3}$

CHAPTER NINE

EXERCISE 1

(a) 2 years, 8 months
(b) 3 years, 2 months
(c) 5 years
(d) 1 year, 2 months
(e) 5 years, 3 months
(f) 2 years, 5 months

EXERCISE 2

(a) R8 452,67
(b) R11 876,26
(c) R24 634,98
(d) R17 099,37

(e) R2 856 657,21
(f) R4 595,25
(g) R223 043,29
(h) R606 609,54
(i) 2 years
(j) No, they will only have saved R428 945,43

EXERCISE 3

(a) R131 677,96
(b) (1) R12 797,16 (2) R1 871 318,40
(c) R103 433
(d) (1) R15 577,69 (2) R74 243,04
(e) R517 255,61
(f) (1) R450 000 (2) R414 000 (3) R9 849,03 (4) R176 941, 80
(g) 7 years, 10 months
(h) 31 payments consisting of 30 full payments and 1 lesser payment

EXERCISE 4

(a) (1) R12 465,61 (2) Loan is paid in full

(b) (1) R20 372,24 (2) R 1 491 157,68 (3) 79,19%

(c) (1) 55 payments consisting of 54 full payments and 1 lesser payment (2) R408,61 (3) R412,97 (4) R144 412,97

(d) (1) 12 payments consisting of 11 full payments and 1 lesser payment (2) R2 305,19

EXERCISE 5

(a) (1) R53 244,64 (2) R184 634,87 (3) R131 390,23 (4) R1 552,43

(b) (1) 20,47% (2) R347 743,73 (3) R12 535,11

(c) R3 121,69

(d) R34 284,50

EXERCISE 6

(a) R77 218,57

(b) R51 317,73

(c) (1) R9981,70 (2) R814 591,46

(d) (1) R8 108,43 (2) R110 987,29

(e) (1) R2 412,47 (2) R3 461,57

(f) (1) R20 380,19 (2) R28 918,01

(g) R245 500,60

(h) R324 680,86

(i) R439 567,94

CONSOLIDATION AND EXTENSION EXERCISE

(a) 11 years, 4 months

(b) R138 159,16

(c) (1) 8,243216% (2) 8,08%

(d) (1) R12 126,31 (2) R90 990,58 (3) R12 421,81

(e) (1) 66 payments consisting of 65 full payments and 1 lesser payment
(2) R2 542,75

(f) R386 683,07

(g) R4 053,43

(h) R70 173,20

(i) R53 387,19

CHAPTER TEN

EXERCISE 1

(a) (2) $\hat{y} = 6{,}8 + 13{,}58x$ (3) y-intercept: $(0;6{,}8)$
$(\overline{x};\overline{y})$ $:(4{,}8;72)$ (4) 0,97
(5) Strong positive linear correlation. The older the car, the more kilometres.

(b) (1) $\hat{y} = 103{,}03 - 1{,}95x$ (2) y-intercept: $(0;103{,}03)$
$(\overline{x};\overline{y})$ $:(21{,}83;60{,}42)$ (3) $r = -0{,}54$
(4) Moderate negative linear correlation. The higher the temperature, the less hot chocolate is sold

(c) (1) $r = 0{,}59$ (2) $\hat{y} = -2{,}95 + 0{,}4x$ (3) x-intercept: $(7{,}38;0)$
Moderate strong linear correlation. The older the teenagers, the more pets they own.
$(\overline{x};\overline{y})$ $:(15{,}2;3{,}2)$

(d) (1) $\hat{y} = 14{,}54 + 5{,}82x$ (2) $r = 0{,}94$
Strong positive linear correlation. The more the learners spend on preparation, the higher their marks.

EXERCISE 2

(a) (1) $\hat{y} = 78{,}10 + 5{,}58x$ (2) $r = 0{,}96$
Strong positive linear correlation (3) 150,64 cm
(4) Very reliable. Strong correlation (5) 178,54 cm (6) 14 years
(7) 301,3 cm (8) No. Extropolated too far

(b) (2) (23;5) (3)(i) $\hat{y} = 15{,}56 + 4{,}81x$ (ii) 0,68
(4) R87 710 (5)(i) $\hat{y} = -2{,}41 + 7{,}20x$ (ii) $r = 0{,}99$
(6) R105 590

(c)	(2) $\hat{y} = 72{,}49 - 0{,}44x$	(3) y-intercept: $(0;72{,}49)$ (4) $r = -0{,}39$ $(\overline{x};\overline{y})$ $:(44{,}8;52{,}8)$	
	(5) 50,49 kg	(6) Not reliable. Weak correlation	(7) No. Extrapolated too far
(d)	(1) 6,1	(2) 11,91	(3) 9,70
	(4) It predicts a likely y-value when x is 6 and not the actual value.		(5) 5
	(6) 6,1	(7) 5	

EXERCISE 3

(a)	(2) Quadratic trend	(3) $(8;30)$
(b)	(2) Exponential relationship	(3) 5
(c)	(2) Quadratic trend	(3) Less than 2 000 calories

CONSOLIDATION AND EXTENSION EXERCISE

(a)	(2) $y = -103{,}73 + 22{,}93x$	(3) $r = 0{,}98$	(4) 34
	(5) Very reliable. Strong correlation		
(b)	(1)(i) Median 184 Mean 185,89	(ii) Positively skewed	(2)(i) Min: 80 Q_1 : 93 Q_2 : 106 Q_3 : 117 Max: 125
	(ii) Negatively skewed	(3)(i) No, height is positively skewed and weight is negatively skewed.	
	(ii) $r = 0{,}57$	(iii) $\hat{y} = -44{,}57 + 0{,}80x$	(iv) 107,43 kg
	(v) Not very reliable. Moderate correlation		
(c)	(2) 29,12 units	(3) $r = -0{,}89$ Correlation is strong negative linear. The higher the temperature, the less units used.	
	(4) 22,01 units	(5) 2,69 °C	
(d)	(1) $y = 94{,}03 + 4{,}86x$	(2) $r = 0{,}91$ Strong positive linear correlation	(3) 157,21 cm
	(4) Very reliable. Strong correlation	(5) 239,83 cm	(6) No. Extrapolated too far
(e)	(1) A	(2) No. Not a strong correlation between advertising spending and sales	
	(3) 21,22 million	(4) No. Weak correlation	(5) N
	(6) Increase		

CHAPTER ELEVEN

EXERCISE 1

(a)	(1) 24	(2) 6
(b)	24	
(c)	(1) 49	(2) 840
(d)	(1) 125	(2) 120
(e)	(1) 100000	(2) 30240
(f)	(1) 5040	(2) 2520
(g)	5040	

EXERCISE 2

(a)	(1) 362880	(2) 5040	(3) 30240
(b)	(1) 720	(2) 36	
(c)	(1) 720	(2) 240	(3) 72
	(4) 72		
(d)	172800		
(e)	(1) 479001600	(2) 127008000	(3) 1209600
(f)	(1) 720	(2) 24	(3) 144
(g)	(1) $3{,}4488 \times 10^{11}$	(2) 298598400	

EXERCISE 3

(a)	(1) 5040	(2) $\frac{1}{42}$
(b)	(1) 362880	(2) $\frac{1}{84}$

(c)	$\frac{1}{20}$		
(d)	(1) $\frac{2}{11}$	(2) $\frac{1}{66}$	(3) $\frac{1}{231}$
	(4) $\frac{1}{462}$		
(e)	(1) 5040	(2) $\frac{1}{7}$	(3) $\frac{1}{35}$
(f)	$\frac{1}{4620}$		
(g)	(1) $\frac{256}{6561}$	(2) $\frac{112}{243}$	(3) $\frac{8}{2187}$
(h)	$x = 2; y = 4$		

EXERCISE 4

(a)	(1) 9979200	(2) $\frac{1}{55}$	
(b)	(1) 120	(2) 72	(3) $\frac{3}{5}$
(c)	(1) 5040	(2) $\frac{1}{840}$	
(d)	66351		
(e)	(1) 7560	(2)(i) $\frac{1}{6}$	(iii) $\frac{1}{21}$
(f)	144495		
(g)	(1) 10080	(2) 30240	(3) 4320

CONSOLIDATION AND EXTENSION EXERCISE

(a)	(1) 160000000	(2) 83721600	
(b)	(1) 1296	(2) 360	(3) 648
	(4) 180		
(c)	(1) 720	(2) $\frac{1}{3}$	(3) $\frac{1}{3}$
(d)	(1) 720	(2) 72	(3) 24
	(4) $\frac{1}{5}$	(5) $\frac{1}{10}$	
(e)	(1) 239500800	(2) $\frac{1}{132}$	
(f)	(1) $\frac{3}{5}$	(2) $\frac{3093}{3125}$	
(g)	(1) 34650	(2) $\frac{13}{55}$	
(h)	(1) 1470	(2) $\frac{1}{7}$	(3) $\frac{13}{49}$
(i)	(1) 336	(2) 6	(3) 56