A Guide to ISE Entrance Exams

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2013

Introduction

The purpose of these materials is to help students to prepare themselves to the ISE Entrance Exams in Mathematics.

The first part of the e-book considers some relevant theoretical topics required for successful passing the entrance exam. This part also contains important Descartes' theory on tangents to algebraic curves, including parabolas and hyperbolas.

The second part contains eight real entrance exams for 2013.

Finally, in the third part solutions to some typical problems from exams for 2013 are given. Problems for entrance exams were taken from the following books:

1. Лысенко Ф.Ф., Кулабухова С.Ю. (ред). Математика. Подготовка к ЕГЭ-2013, Ростов-на-Дону, Легион 2012, 413стр.

2. Рутюмова И.П., Рустюмова С.Т. Пособие для подготовки к ЕНТ, 4-ое изд., Алматы 2010, 714стр.

3. Рутюмова И.П., Рустюмова С.Т. Тренажер по Математике для подготовки к ЕНТ, 3-ье изд., Алматы 2011, 626стр.

Part I
Some Theory

GUIDANCE TO LINES

Fall 2013

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Study theory and memorize formulae. Study sample homework problems.

THEORETICAL BACKGROUND.

In applications one often studies dependence of one parameter (considered as dependent) from another parameter (considered as independent). In Mathematics the dependence of one parameter from another is expressed by functions and is denoted as $y = f(x)$. The main idea of Calculus is to replace complicated functions with simplest ones. The simplest functions are linear functions

$$y = m \cdot x + b \ , \tag{1}$$

where m and b are some constants and x is an independent variable. We begin our study of Calculus with the most important properties of linear functions.

From school we know that functions can be identified with their graphs on the coordinate plane. Recall that the coordinate plane is just a plane on which two perpendicular axis are fixed. The x-axis is used to mark the values of the independent variable x, where as the y-axis is used to mark the values of the dependent variable y.

Each point P of the plane is uniquely determined by its coordinates (x, y). On Fig.1 we marked the point P with the cartesian coordinates $x = 2$, $y = 1$. The graph of the linear function (1) is the line on the coordinate plane consisting of points with the cartesian coordinates satisfying (1).

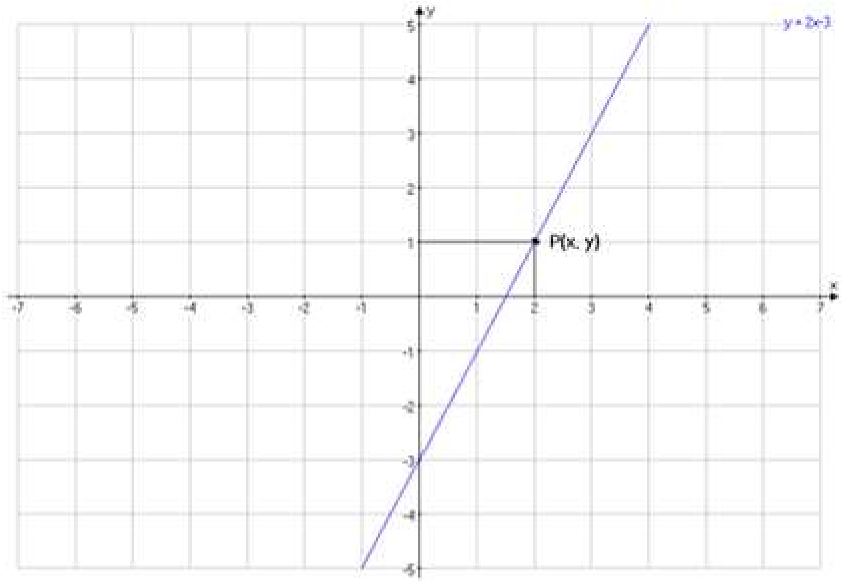

FIG.1

For example, the marked point P with cartesian coordinates $(2, 1)$ is on the graph of $y = 2x - 3$.

When a particle is moving along the coordinate plane from point (x_1, y_1) to the point (x_2, y_2), the increment in its coordinates are

$$\boxed{\Delta x = x_2 - x_1 \text{ and } \Delta y = y_2 - y_1\ .} \tag{2}$$

Thinking in terms of increments is important for understanding of Calculus. The first important application of the notion of increment is the notion of the slope of a non-vertical line.

Definition. let $P(x_1, y_1)$ and $P(x_2, y_2)$ be points on a non-vertical line L. Then the **slope** m of L is

$$\boxed{m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}\ .} \tag{3}$$

Using increments and slopes one can easily plot lines on the coordinate plane.

Definition. The equation

$$\boxed{y = m(x - x_1) + y_1} \tag{4}$$

is called the **point-slope equation** of the line passing through the point (x_1, y_1) with slope m.

Thus if you know the slope of a line and the coordinates of any point on it, then using (4), you can obtain the equation of a line. For instance, the line plotted on Fig. 1 passes through two points $(2, 1)$ and $(3, 3)$. It follows that the slope of the line is

$$m = \frac{3-1}{3-2} = 2\ .$$

Then by (4) we obtain that the equation of the line is $y = 2(x - 2) + 1 = 2x - 3$.

Definition. The equation

$$(5) \qquad \boxed{y = mx + b}$$

is called the **slope-intercept equation** of the line with slope m and y-intercept b.

This terminology roots in the fact that b is the intercept of the line with the y-axis (to see this just put $x = 0$ in (5)).

The two forms of equations for lines on plane may be put together by the **General Line Equation.**

Definition. The equation

$$\boxed{Ax + By = C\ ,}$$

where A and B cannot be both zero, is called a **general linear equation** in x and in y.

The notions discussed above are used for plotting lines on the coordinate plane. They are also used to write down the equation of a line plotted. This very old theory has important applications in Linear Regression Analysis.

In practice functions are often given by their values at some particular points. This is common, for instance, for recording the results of experiments. Therefor instead of a curve representing the graph of an unknown function we have only a finite number of points on the plane (which are supposed to be on the graph of the function studied).

The problem of Linear Regression Analysis is to plot a line on the coordinate plane which is close to the data marked the best possible way. In practice it is not necessary to plot the best possible line. Any line fitting the data plotted rather well does the job. The reason for this is that the data are usually not at all reflect the function studied very precisely due to inevitable mistakes of measurements.

The line plotted can be fixed by finding its equation. This allows one not only to keep big tables of data in a compact form, but also to make predictions and interpolate the data. The point-slope equation and point-intercept equation of lines are used to record just graphically plotted practical lines which fit the data given.

Let us illustrate this method with Example 9 from the book (see pp. 6-7).

In this Example we study the dependence of the minimal postage cost in the US starting from 1885. The data are arranged in the table on page 6 of the textbook. A brief look at this table shows that the cost varies very little for the period 1885-1968. Since noticeable changes took place only after 1968, we may represent the time variable as $1968 + x$, where $x = 0, 3, 6, 7, 9, 13, 17, 19, 23, 27, 30$. Then our data can be plotted on the coordinate plane (see Fig. 2).

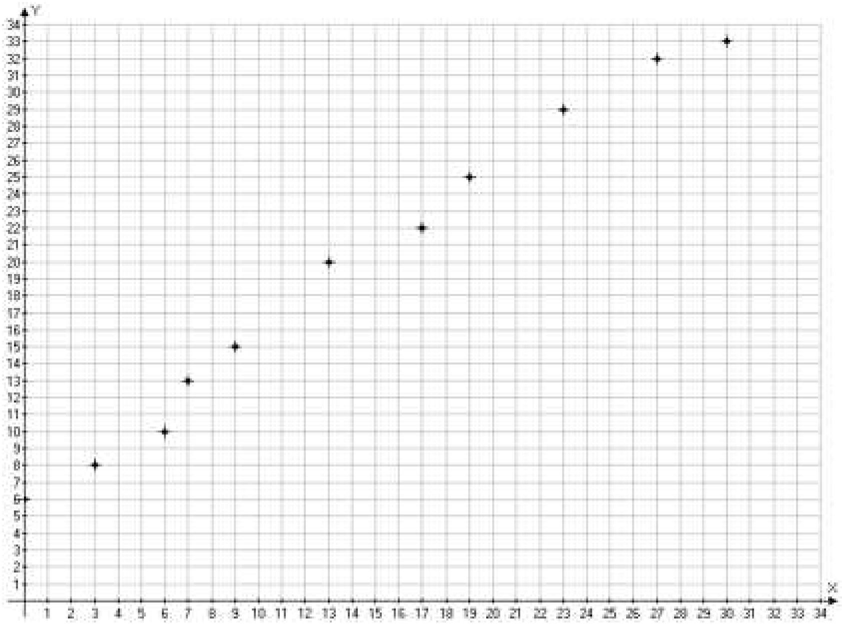

FIG.2

With the help of ruler we can arrange a line which fits the data plotted the best possible way. Next, using the slope-intercept equation, we can write down the equation of the line required. Calculus allows one to give precise formulae for the best line fitting the data. Special software can even plot this line and find its equation. On Fig. 3 we present the graph of this line.

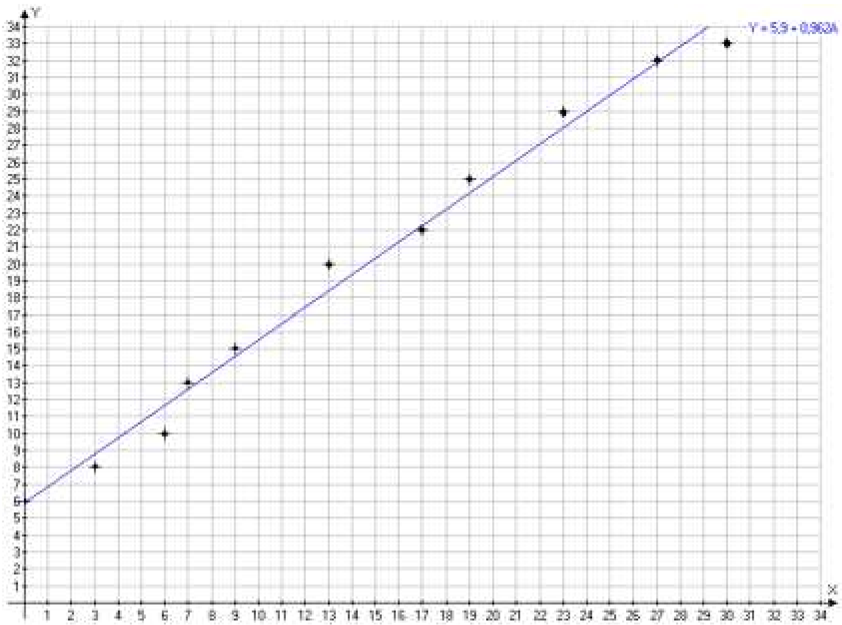

FIG.3

In your homework you can either plot the regression line with a ruler, or use special scientific calculators (or even computer) to determine the equation of the regression line.

EXERCISES 1

Problem 1 (b). *Find the coordinate increments from $A(-3, 2)$ to $B(-1, -2)$.*

SOLUTION. Since the coordinates increase from A to B, we have

$$\begin{array}{ll} x_2 = -1 & y_2 = -2 \\ x_1 = -3 & y_1 = 2 \end{array}$$

It follows that

$$\boxed{\Delta x = (-1 - (-3)) = 2} \text{ and } \boxed{\Delta y = (-2 - 2) = -4} \quad \square$$

Problem 3 (b). *Let L be the line determined by the points $A(-2, -1)$ and $B(1, -2)$.*

(1) *Plot A and B.*
(2) *Find the slope of L.*
(3) *Draw the graph of L.*

SOLUTION.

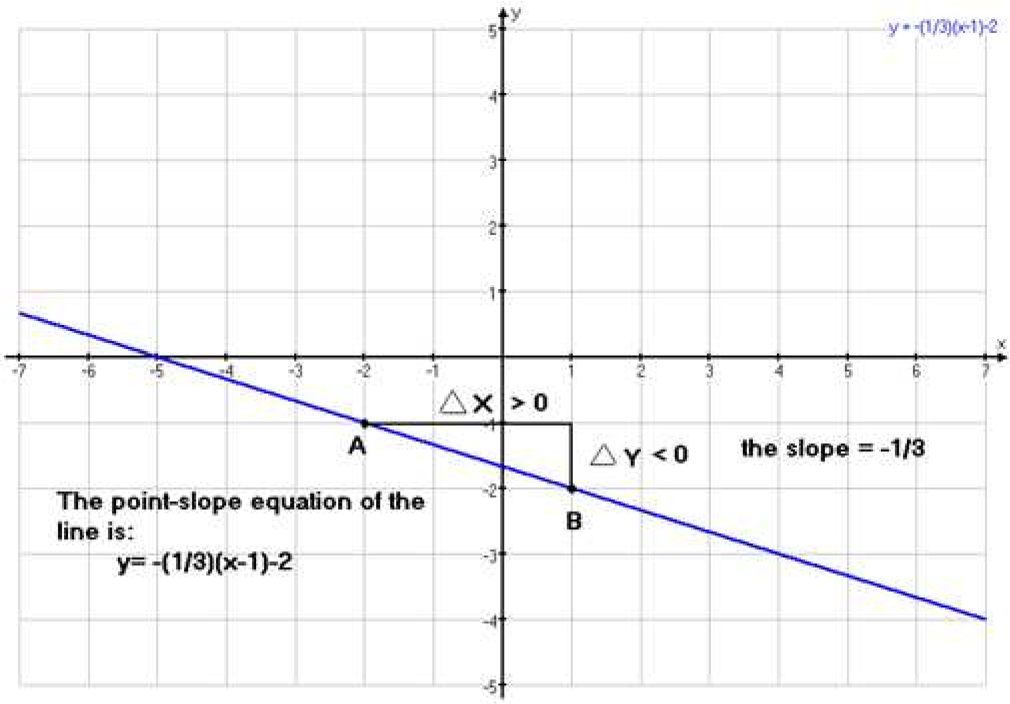

FIG.4

Problem 7(a). *Write the point-slope equation for the line through the point $P(1,1)$ with slope $m = 1$.*

SOLUTION. The point slope equation of the line is

$$y = m(x-1) + 1 = (x-1) + 1 \ . \quad \square$$

Problem 9(b). *Write a general linear equation for the line through $A = (1,1)$ and $B = (2,1)$.*

SOLUTION.

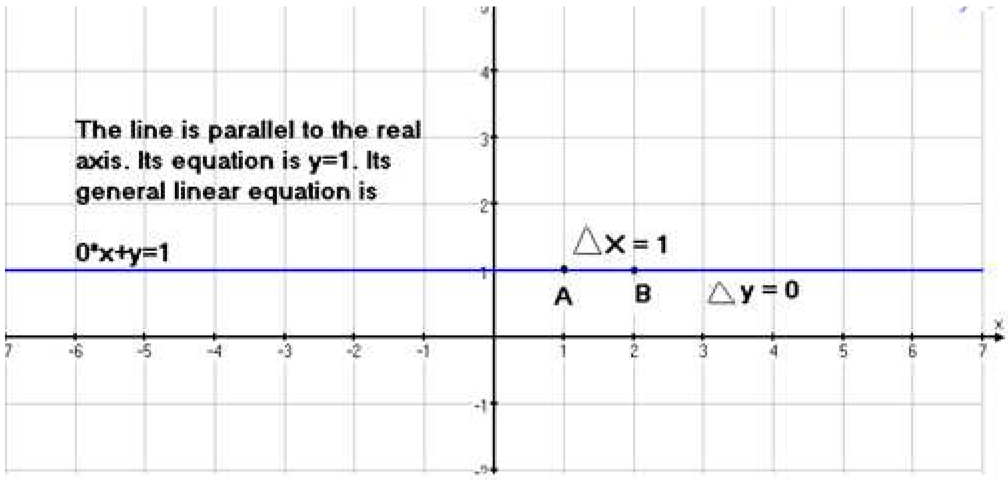

FIG.5

Problem 10(b). *Write a general linear equation for the line through $A = (-2,1)$ and $B = (2,-2)$.*

SOLUTION. First we find the increments $\Delta x = 2 - (-2) = 4$ and $\Delta y = -2 - 1 = -3$. Then we find the slope

$$m = \frac{\Delta y}{\Delta x} = \frac{-3}{4} = \frac{3}{4} \ .$$

Next, we may write the point-slope equation of the line through $(2,-2)$:

$$y = \frac{3}{4}(x-2) - 2 \ .$$

By elementary algebraic operations we transform this equation to the form of a general linear equation (see Fig. 6 below):

$$3 \cdot x + 4 \cdot y = -2 \ .$$

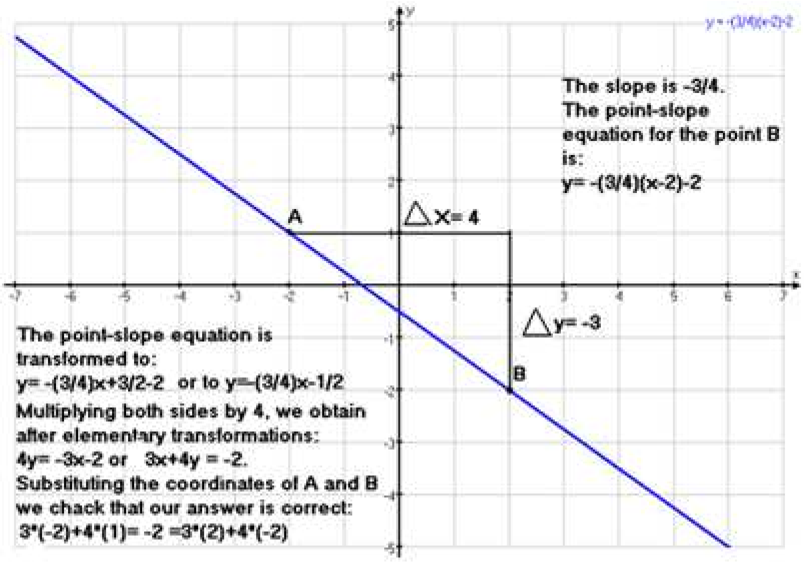

FIG.6

Problem 16(b). *Find the slope, y-intercept of the line $y = 2x + 4$ and plot the graph of this line.*

SOLUTION. The y-intercept is the value of y corresponding to $x = 0$. If we put $x = 0$ in our equation, then we obtain that $y = 2 \cdot 0 + 4 = 4$. Hence the y-intercept is 4. The slope of our line equals the coefficient at x, i.e., $m = 2$. For plotting the graph of our line it is useful to find its x-intercept. To find it we just put $y = 0$ in the equation of the line (that is exactly the case for the point of the intersection with the x-axis). Then we find that $x = -4/2 = -2$. The graph looks as it is pictured on Fig. 7.

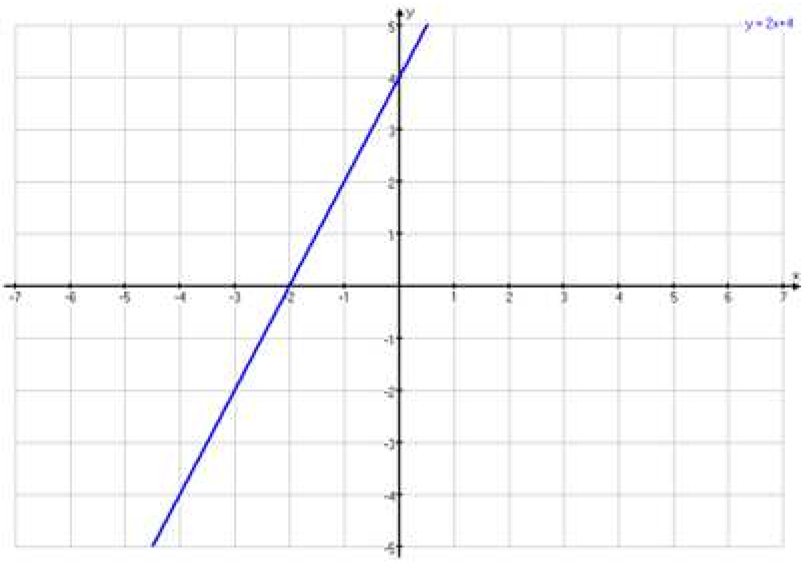

FIG.7

GUIDANCE TO
FUNCTIONS AND GRAPHS
Fall 2013

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Study theory and memorize formulae. Study sample homework problems.

THEORETICAL BACKGROUND.

In applications we usual deal with two variables, one of which is independent and another one is dependent. For instance, if we measure the distance s covered by a moving car, then we must take into consideration the time t elapsed from the moment when this car started its movement. In this model time t is considered as an independent variable and the variable s depends on t.

In 18th century one of the creators of Calculus Leonard Euler introduced in Mathematics a convenient notation:

$$s = f(t) \ .$$

This notation says that s is a *function* of t.

In practice functions are usually given by formulae:

$$s = t^2 - 1 \quad \text{or} \quad s = \frac{t^2 - 1}{t + 1} \quad \text{or else} \quad s = 10\sin(t + 1) \ .$$

The advantage of Euler's notation is that it allows one to concentrate on the study of general properties of functions, not related with particular properties of the formulae involved in their definitions.

Later Euler's approach was made even more systematic. Together with a function $y = f(x)$ one also considers its **domain** $\mathcal{D} = \mathcal{D}(f)$, and its **range** $\mathcal{R} = \mathcal{R}(f)$.

The domain $\mathcal{D}(f)$ is defined as a set of points x, to which f may be applied. It is usually determined in practice together with f. In the above example with car the domain is the closed interval $[0, T]$, where T is the time of observation.

As soon as $\mathcal{D}(f)$ is defined one associates with EVERY element x in $\mathcal{D}(f)$ a unique element denoted by $f(x)$. The set of all elements $f(x)$, when x runs over $\mathcal{D}(f)$, makes up the range $\mathcal{R}(f)$ of the function $y = f(x)$.

Returning back to our example $\mathcal{R}(s)$ is the set of all possible distances from the starting point passed by our car during the time of observation.

Since in applications functions are often determined by formulae, these functions play a very important role in Calculus.

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If $y = f(x)$ is determined by some formula, then one may talk on the so-called **natural domain** of f. The natural domain is the set of all x such that the formula defining f makes sense.

By some reasons, which we cannot touch at this very moment, the natural domains and ranges of all functions, defined by formulae, are finite unions of intervals on the real axis. Please, read attentively p. 12 of the text-book.

It is common to study functions with their graphs. The graph $G(f)$ of a function $y = f(x)$ is the line (not necessarily straight) on the coordinate plane, which is made of the points $(x, f(x))$, when x ranges over $\mathcal{D}(f)$.

Since often functions are represented by combinations of elementary functions such as polynomials, sines, cosines, etc, the skill of plotting the graphs of complex functions starting from the known graphs of elementary components is important for practical applications.

In this section of the textbook you find a number of tricks which are very helpful in solving problems on plotting graphs.

In practice graphs are also used to define functions. You may be asked if a given line on a plane is a graph of some function. In such a case apply the **vertical line test.**

The vertical line test says that **a subset Γ of the coordinate plane is the graph of some function if and only if every vertical line $x = a$ intersects Γ at most at one point.**

In fact, the proof of the test is evident, since points of any graph have the form $(a, f(a))$.

Indeed, assume that Γ is a graph of $y = f(x)$. Then any vertical line $x = a$ either intersects Γ or not. If it intersects the graph, then $(a, f(a))$ is on $x = a$. There may be no other point of Γ in this intersection, since f associates only one element $f(a)$ with every $a \in \mathcal{D}(f)$.

In case $x = a$ intersects Γ at most at one point, we can easily define the function with graph Γ. Namely, we define $\mathcal{D}(f)$ to be the set of all a such that $x = a$ intersects Γ. Then $f(a)$ is defined to be such that $(a, f(a)) \in \Gamma$.

Apply this arguments to Problems 5(a-b), 6(a-b) of EXERCISES 2, p. 20 of the textbook.

Problem 2(Exercises 2, p.20)**.** *Express the side length of a square as a function of the length d of the square's diagonal. Then express the area as a function of the diagonal length.*

SOLUTION. Let y be the side length of a square with the diagonal length d. Applying Pythagorean theorem, we have

$$y^2 + y^2 = d^2 ,$$

which implies that

$$y = \frac{d}{\sqrt{2}} .$$

Here by the geometrical sense of the problem $d > 0$. In Geometry we usually do not consider squares with zero diagonals. Therefore the domain of our function is $(0, +\infty)$.

From Geometry we know that the area A of the square is y^2. It follows that

$$A = \frac{d^2}{2} . \quad \square$$

Problem 7(a-b)(Exercises 2, p.20). *Find the domain and range of each function*

$$\textbf{(a)} \quad f(x) = 1 + x^2 \qquad\qquad \textbf{(b)} \quad f(x) = 1 - \sqrt{x}\ .$$

SOLUTION. (a) The function is defined by the algebraic formula (the quadratic polynomial), which makes sense for any vaue of x. Therefore $\mathcal{D}(f) = (-\infty, +\infty)$. To find $\mathcal{R}(f)$ we apply the **horizontal line** test. We imagine that we plotted the graph of this function on the coordinate plane. Now, if b is such that the horizontal line $y = b$ intersects the graph $\Gamma(f)$, then there is $x \in \mathcal{D}(f)$ such that $b = f(x)$ and therefore $b \in \mathcal{R}(f)$. If this line does not intersect the graph, then the equation $b = f(x)$ has no solutions. Thus we see that our problem is reduced to the question of the solvability of the equation

$$b = f(x)$$

in x. If the equation has a solution, then b is in the range. If it does not have, then b is not in the range. Let us write down the equation for our case. Then we have

$$b = x^2 + 1 \Longleftrightarrow b - 1 = x^2\ ,$$

which obviously has a solution if and only if $b - 1 \geqslant 0$. Therefore the range is $[1, +\infty)$.

(b) Since the square root is determined only for nonnegative numbers, we obtain that $\mathcal{D}(f) = [0, +\infty)$. Now, the equation

$$b = 1 - \sqrt{x} \quad \text{is equivalent to} \quad \sqrt{x} = 1 - b\ ,$$

which in turn is solvable in x if and only if $1 \geqslant b$. Hence $\mathcal{R}(f) = (-\infty, 1]$. □

Problem 12(b). *Graph the function $y = -1/x^2$. What symmetries, if any, do the graph have?*

SOLUTION. It is clear that the formula, which defines our function $y(x)$, makes sense for every $x \neq 0$. It follows that the natural domain of the function is $(-\infty, +\infty)$.

Next, we observe that, $y(-x) = y(x)$. Therefore the graph of $y(x)$ is symmetric with respect to the y-axis (indeed, if $(x, y(x))$ is on the graph, then $(-x, y(-x)) = (-x, y(x))$ is on the graph too. Hence it is sufficient to plat the graph over $(0, +\infty)$ and then reflect it symmetrically with respect to the y-axis.

If x increases along $(0, +\infty)$, then x^2 increases too. Therefore $1/x^2$ decreases and $y(-x) = -1/x^2$ increases (we changed the sign, and any change of sign in an inequality turns the sign of the inequality to the opposite sign).

When x is close to 0, then the values of $y(-x)$ are negative numbers with a very big modulus. Indeed,

$$y(0.1) = -100\ ,\ y(0.01) = -10000\ ,\ y(0.001) = -1000000, \cdots\ .$$

On the other hand, if x is a big positive number. than the value of $y(x)$ is small:

$$y(10) = -0.01\ ,\ y(100) = -0.00001\ ,\ y(1000) = -0.0000001\ , \cdots\ .$$

Taking into account this behavior of the function about 0 and $+\infty$ and the fact that $y(x)$ increases on $(0, +\infty)$, we plot the graph over $(0, +\infty)$. After that we reflect it symmetrically and obtain the following plot:

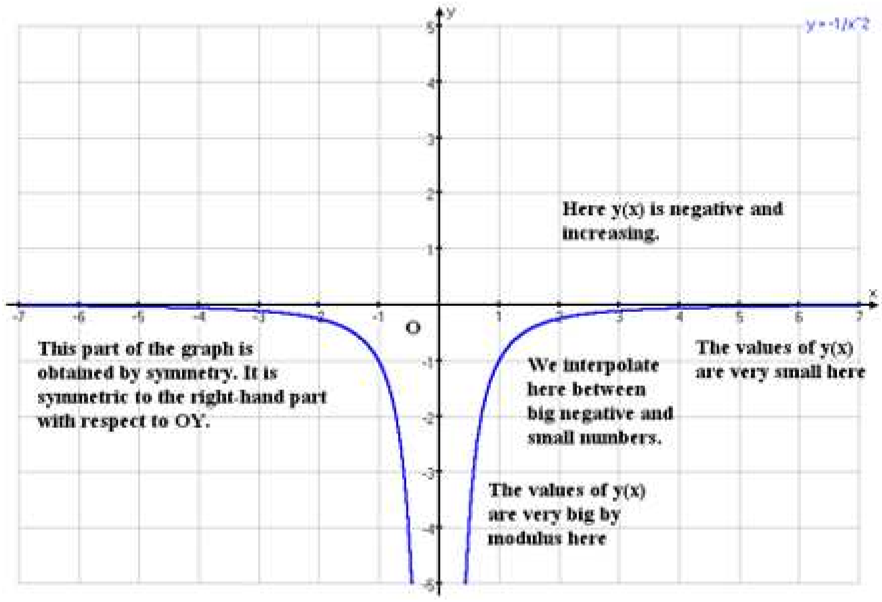

FIG.1

Problem 17. *Say whether the function defined below is even, odd, or neither.*

$$\textbf{(a)} \quad f(x) = x^3 + x \qquad\qquad \textbf{(b)} \quad g(x) = x^4 + 3x^2 - 1 \ .$$

SOLUTION. (a) We have $f(-x) = (-x)^3 + (-x) = -(x^3 + x) = -f(x)$, which implies that f is an odd function.

(b) We have $g(-x) = (-x)^4 + 3(-x)^2 - 1 = x^4 + 3x^2 - 1 = g(x)$, which implies that g is an even function. $\square$

Problem 21(b). *Draw the graph of the function defined by*

$$f(x) = 2|x + 4| - 3$$

and find its domain and range.

SOLUTION. Since the formula which defines the function makes sense for every value of x, the domain $\mathcal{D}(f)$ is $(-\infty, +\infty)$.

To plot the graph let us observe that it is important first find the point on the real axis at which $x + 4$ changes sign. It is clear that this point is the zero of the linear function $x + 4$, i.e., it is $x = -4$. Then by the definition of the modulus we have

$$|x + 4| = \begin{cases} (x + 4) \text{ if } x \geqslant -4 \\ -(x + 4) \text{ if } x < -4 \end{cases}$$

Therefore

$$f(x) = \begin{cases} 2x + 2 \cdot 4 - 3 = 2x + 5 \text{ if } x \geqslant -4 \\ -2x - 2 \cdot 4 - 3 = -2x - 11 \text{ if } x < -4 \end{cases}$$

It follows that the graph of our function is made from two pieces of lines, each of which we can easily plot. The result is given on Fig. 2 below.

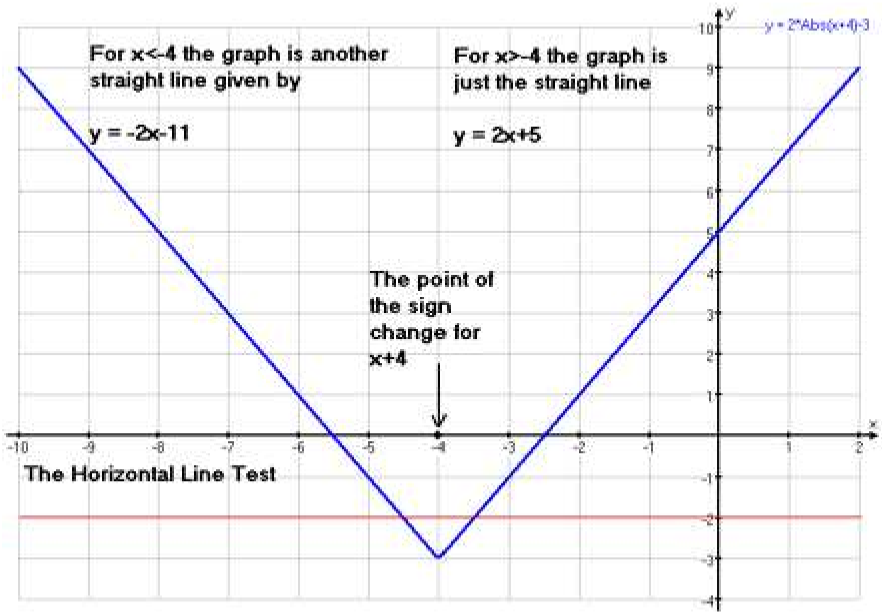

FIG.2

The horizontal line test shows that the horizontal line $y = b$ intersects the graph if and only if $b \geqslant -3$. On Fig. 2 you see the horizontal line $y = -2$ intersecting the graph. Hence the range of our function is $\boxed{\mathcal{R}(f) = [-3, +\infty)}$. $\square$

Problem 27(c). *Write a piecewise formula for the function defined by the graph*

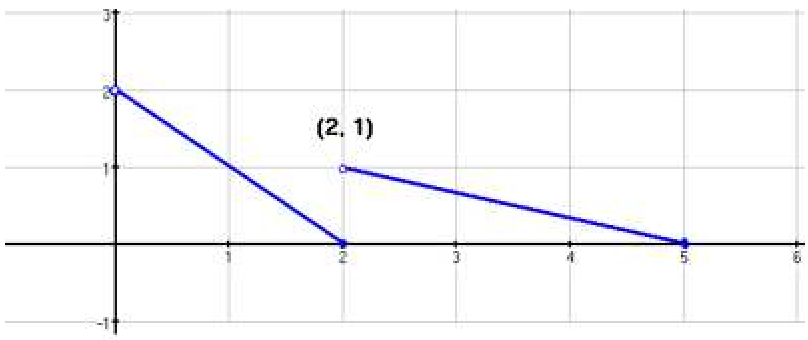

FIG.3

SOLUTION. We see that the graph consists of two pieces of lines. Both lines have negative slopes m_1 and m_2. The slopes of the lines are determined by

$$m_1 = \frac{0-2}{2-0} = -1 \qquad\qquad m_2 = \frac{0-1}{5-2} = -\frac{1}{3}\ .$$

Using the method of point-slope equations we find the equations of these two lines:

$$y = -(x-0)+2 = -x+2 \qquad\qquad y = -\frac{1}{3}(x-5)+0 = -\frac{1}{3}x+\frac{5}{3}\ .$$

Now we can write down the formula, which determines this graph. One should only take some care on the values of our function at $x = 0$ and at $x = 2$, which are stressed on the graph. The result is given by:

$$f(x) = \begin{cases} -x+2 \text{ if } 0 < x \leqslant 2 \\ -\dfrac{1}{3}x+\dfrac{5}{3} \text{ if } 2 < x \leqslant 5 \end{cases} \qquad \square$$

Problem 34. *Plot the graph of $y = -\sqrt{x}$ shifted to the right by 3 units and write down the formula for the function determined by the shifted graph.*

SOLUTION.

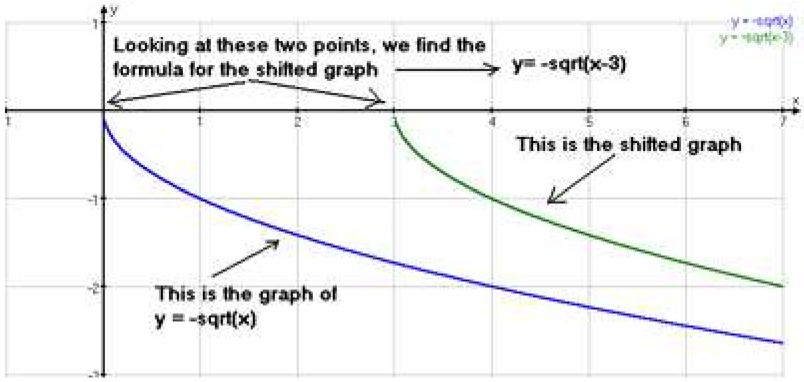

FIG.4

$\square$

Problem 40. *If $f(x) = \sqrt{x}$, $g(x) = x/4$, and $h(x) = 4x - 8$, find the formulas for the following:* **(a)** h(g(f(x))), **(b)** h(f(g(x))), **(c)** g(h(f(x))), **(d)** g(f(h(x))), **(e)** f(g(h(x))), **(f)** f(h(g(x))).

SOLUTION. We know from the book that the combination of the functions given above are called **superpositions** of functions. Any three general functions make up 6 possible superpositions. So, in this problem we compute all possible elementary superpositions from three given functions.

(a) We just make a substitution of one formulas in another:

$$h(g(f(x))) = 4g(f(x)) - 8 = 4f(x)/4 - 8 = f(x) - 8 = \sqrt{x} - 8\ .$$

(b) The same method of consecutive substitutions gives:

$$h(f(g(x))) = 4f(g(x)) - 8 = 4\sqrt{g(x)} - 8 = 4\sqrt{x/4} - 8 = 2\sqrt{x} - 8\ .$$

(c) We have

$$g(h(f(x))) = \frac{h(f(x))}{4} = \frac{4f(x) - 8}{4} = f(x) - 2 = \sqrt{x} - 2 \ .$$

(d) Similarly

$$g(f(h(x))) = \frac{f(h(x))}{4}4 = \frac{\sqrt{4x - 8}}{4} = \frac{\sqrt{x - 2}}{2} \ .$$

(e) Next,

$$f(g(h(x))) = \sqrt{g(h(x))} = \sqrt{\frac{h(x)}{4}} = \sqrt{\frac{4x - 8}{4}} = \sqrt{x - 2} \ .$$

(f) Finally,

$$f(h(g(x))) = \sqrt{h(g(x))} = \sqrt{4g(x) - 8} = \sqrt{4(\frac{x}{4}) - 8} = \sqrt{x - 8} \ .$$

□

The examples in Problem 40 show that the ORDER in which you apply the superpositions of given functions is IMPORTANT"!

GUIDANCE TO EXPONENTIAL FUNCTIONS Fall 2013

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Study theory and memorize formulae. Study sample homework problems.

THEORETICAL BACKGROUND.

1. Let us consider the sequence $\{2^n\}_{n\geqslant 1}$, where $n = 1, 2, 3, \cdots$:

$$2^1 = 2\ ,\ 2^2 = 4\ ,\ 2^3 = 8\ ,\ 2^4 = 16\ ,\ 2^5 = 32\ ,\ 2^6 = 64\ ,\ 2^7 = 128\ ,\ 2^8 = 256\ ,$$
$$2^9 = 512\ ,\ 2^{10} = 1024\ ,\ 2^{11} = 2048\ ,\ 2^{12} = 4096\ ,\ 2^{13} = 8192\ ,\ 2^{14} = 16384 \cdots$$

If $y_n = 2^n$, then $y_{n+1} = 2 \cdot y_n$. Therefore plotting the scatter plot (see section P1, p.5 for the definition) for the data (n, y_n), $n = 1, 2, ...$ faces serious difficulties. There is no problem in marking the x-coordinates of the scatter plot on the x-axis. Any standard sheet of paper with width 21 cm can be used for this. However, the calculations above show that for plotting the pair $(14, y_{14})$ one needs the height along the y-axis exceeding 16384 cm = 163.84 meters (!!!). With plotting of $(64, y_{64})$ the situation is even worse, since

$$2^{64} = 18\quad 446\quad 744\quad 073\quad 709\quad 551\quad 616\ .$$

To get the length of 64 centimeters one needs only four widths of the standard $A4$ paper. But what is about the height?

Such a phenomenon is described in science as an **Exponential Growth.** In practice models of exponential growth are used to describe phenomenons of catastrophic character. The catastrophic character of exponential models can be easily seen from the above example. Plotting the first 4 points does not create serious problems. You can do this on one sheet of paper. But after that with every step the difficulties increase tremendously. Squeezing the scale of the y-axis 1000 times (!!!), we get a plot which can be saved on this sheet of paper (see Fig. 1 below).

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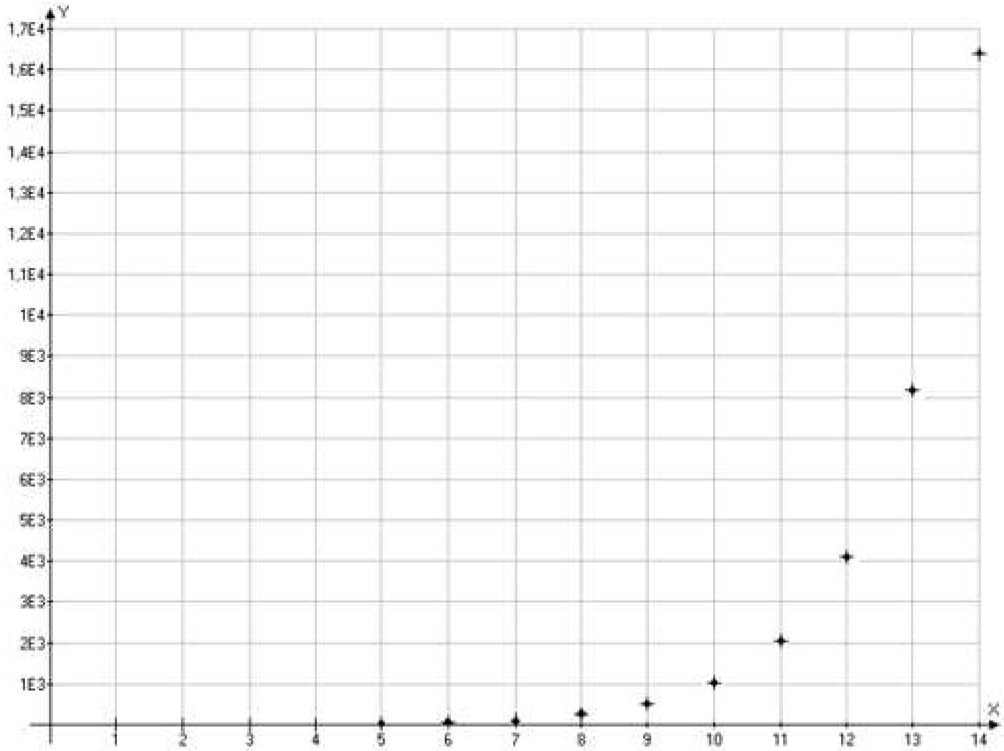

FIG.1

2. Section P1 of the textbook considers **Linear Regression Analysis** (see pp. 5-7). In Linear Regression Analysis the regression curve for data is a straight line. It is clear that Linear Regression Analysis cannot be applied to the data plotted on Fig. 1. Is it still possible to find some regression curve for these data? The answer to this question is "Yes". One can arrange such a curve. It is given by the graph of an exponential function

$$y = 2^x \ .$$

For positive integer values of x this function is defined by induction (see above). If $x = p/q$ is a positive rational number represented in lowest terms, then

$$2^{\left(\frac{p}{q}\right)} = \sqrt[q]{2^p} \ .$$

If x is an irrational number, such as $\sqrt{2}$, then we can approximate x by a sequence of rational numbers x_n. It is shown in the rigorous theory of real numbers that in this case 2^{x_n} approximate some real number, which is defined to be 2^x. For negative $x = -a$ we put $2^x = 1/2^a$. The graph of $y = 2^x$ is plotted on Fig. 2.

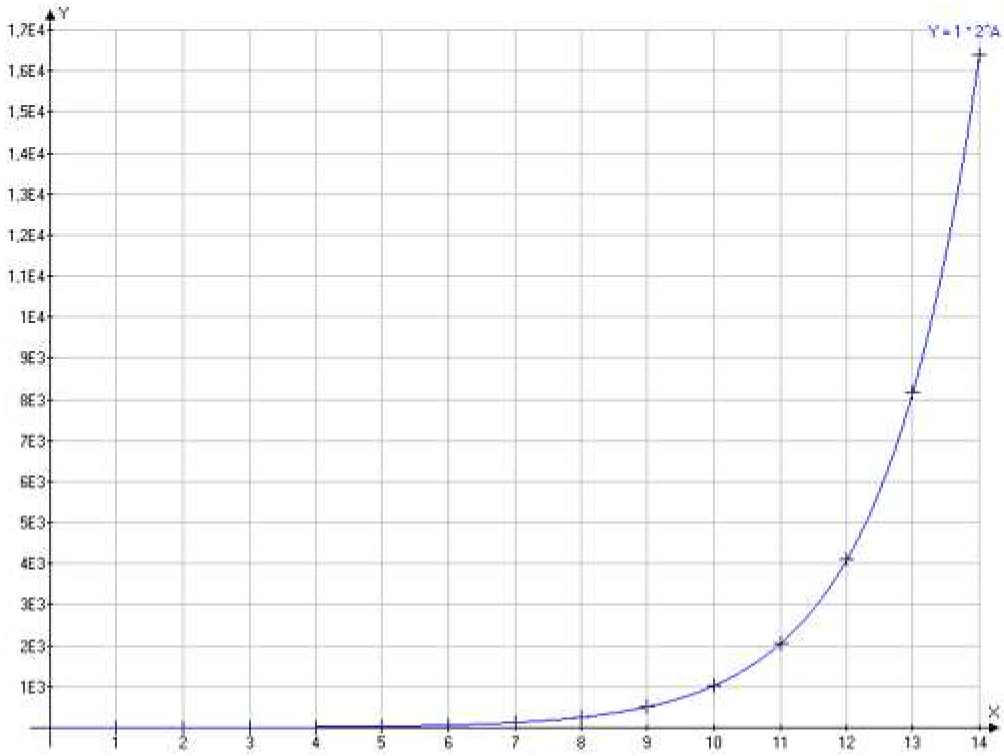

FIG.2

Due to the catastrophic character of the problem considered this graph looks misleading. One could conclude from it that $2^0 = 0$. **It is not the case!** In fact $\mathbf{2}^0 = 1$. Remember that to put scatter plot of our data on this sheet of paper we were forced to squeeze the scale of the y-axis 1000 times, which resulted in the new scale, in which one unit equals 0.001 cm = 0.01 millimeter. Therefore the values of 2^x for $x < 6$ on our graph look so close to zero.

Notice that the choice of the base 2 was not very important. Any positive number $a \neq 1$ can be taken as the **base** of the exponential function $y = a^x$. IT IS IMPORTANT TO REMEMBER THAT THE CASE $a = 1$ IS EXCLUDED FROM CONSIDERATIONS.

3. The characteristic property of an exponential function $y = a^x$ is the **addition formula** :

$$\boxed{a^{x+y} = a^x a^y\ .} \tag{1}$$

It is obvious for positive integer integers x and y. Applying roots of arbitrary integer order to the already proved identity, we obtain the formula for rational x and y. Finally, the process of approximation is applied.

A practically important partial case of the addition formula reduces to the definition of exponential functions for **negative** values of x. If we assume that $y = -x$ in the addition formula, then we obtain that

$$1 = a^0 = a^{x-x} = a^x a^{-x}\ ,$$

which implies that

$$a^{-x} = \frac{1}{a^x} \ .$$

Since the addition formula is the characteristic property of an exponential function, it is used in Data Analysis to determine whether a given scatter plot can be treated with the Exponential Regression.

Suppose that our data are represented by points (k, y_k), where $k = n, n + 1, \cdots, n + N$. If these data can be in principle well approximated by an exponential function $y(x) = ca^x$, where a, c some unknown constants at the moment, then $y_k \approx ca^k$. By the addition formula we have

$$\frac{y_{k+1}}{y_k} \approx \frac{ca^{k+1}}{ca^k} = \frac{ca^k \cdot a}{ca^k} = a \ .$$

This important observation shows that to check whether the data given to you fit the Exponential Regression, you must first calculate the quotients of the consecutive ordinates:

$$\frac{y_{n+1}}{y_n} \ , \ \frac{y_{n+2}}{y_{n+1}} \ , \ \frac{y_{n+3}}{y_{n+2}} \ , \ \cdots \frac{y_{n+N}}{y_{n+N-1}} \ .$$

If these numbers are more or less close to one number, then these data can be treated with the Exponential Regression. Moreover, this number a is the base of the exponent in the equation of the regression curve: $y = c \cdot a^x$. After a is found you can easily find c from the equation

$$y_n = c \cdot a^n \ .$$

Revise the data given in Table 9, p. 27 of the textbook. In this case $n = 1986$ and $N = 5$.

It is important to observe that this method can be used directly only if the x-coordinates of the points given have equal increments:

$$\Delta x_{n+1} = x_{n+1} - x_n = \Delta x_{n+2} = x_{n+2} - x_{n+1} = \cdots \ .$$

4. All rules for exponential functions can be derived from the addition formula. Try to do this for the rules boxed on page 26 of the book.

5. It may sound strange but exponential models, which as it was shown above describe catastrophic processes, are widely used in the financial world of banking systems. However, the main purpose of these applications of exponential functions is to avoid catastrophes, i.e. the bankruptcy of a bank.

To each investor of I_0 dollars a bank returns in one year

$$I_1 = I_0 + I_0 \cdot \frac{r}{100} = I_0 \cdot \left(1 + \frac{r}{100}\right) \ .$$

Here r is called the **interest**. The interest is announced in percents, which explains the appearance of the factor $1/100$ in the above formula. Suppose that an investor decides to continue with this bank one year more. Than in a new year his initial investment will be $I_1 > I_0$. Assuming that the interest is the same, we obtain that in two years after his investment I_0, the sum to be returned by the bank equals

$$I_2 = I_1 \cdot \left(1 + \frac{r}{100}\right) = I_0 \cdot \left(1 + \frac{r}{100}\right)^2 \ .$$

Now one can easily see that the amount to be returned to the customer in n years equals

$$I_n = I_0 \cdot \left(1 + \frac{r}{100}\right)^n . \tag{2}$$

Notice that if the interest $r = 100\%$ and $I_0 = \$1$, then we obtain by the above formula that $I_n = \$2^n$. The calculation made at the beginning of this section show that in 10 years this investor will get back \$1024. But this will happen only in the case this bank would be able to meet the catastrophic growth of its financial obligations. Therefore the main skill in banking is to determine the rate in such a way that it is still attractive for bank's customers but, on the other hand, is not very high to avoid bankruptcy.

6. There is one number in the infinite set of all possible bases of exponential functions which is of the great importance for Mathematics and its applications.

We already know that the annual growth of the initial investment I_0 is described by (2). Now an educated customer may argue the following way. The bank claims that it works day and night to increase the investment. This work of the bank continues uniformly during the whole year. Therefore the interest promised to customers grows linearly with time. Let us split one year into relatively small equal intervals of time of some positive length Δt. Then the interest earned for a customer for any period of time Δt equals $r \cdot \Delta t$. Applying formula (2), we obtain that the investment by the moment of $k \cdot \Delta t$, where k is a positive integer, reaches the value determined by the formula

$$I_0 \left(1 + \frac{r \cdot \Delta t}{100}\right)^k = I_0 \left(1 + \frac{1}{\frac{100}{r \cdot \Delta t}}\right)^{\frac{100}{r\cdot\Delta t}\cdot\frac{r}{100}\cdot k\cdot\Delta t} . \tag{3}$$

Next, when Δt is getting smaller and smaller, the value

$$X = \frac{100}{r \cdot \Delta t}$$

is getting bigger and bigger.

We are interested in the value of the investment by some moment t. Making Δt smaller and smaller, we see that $t, k, \Delta t$ in (3) are related by the equation $t = k \cdot \Delta t$.

Taking these observations into account, we obtain from (3) that the value $I(t)$ of the investment by the moment t is approximated by

$$\left(1 + \frac{1}{X}\right)^{\frac{1}{X}\cdot\frac{r}{100}\cdot t} .$$

Finally, calculations show (see Fig. 27 on p. 27 of the text-book) that

$$\boxed{\left(1 + \frac{1}{X}\right)^X \longrightarrow e = 2.71828182845904523536028747135266249775 7\cdots .}$$

Hence the following formula for $I(t)$ holds:

$$\boxed{I(t) = e^{\left(\frac{r}{100}\right)\cdot t} .} \tag{4}$$

In banking business the interest obtained by (2) is called the **interest compounded annually**. The interest obtained by (3) is called the **interest compounded continuously**. It can be shown that the interest compounded continuously is always greater than the interest compounded annually.

It is traditional to express the interest compounded annually in percents. At the same time the interest k compounded continuously is usually given in decimals:

$$k = \frac{r}{100} \ .$$

In science such processes like bacterial growth are described by mathematical models involving the number e:

$$y(t) = P \cdot e^{k \cdot t} \ ,$$

where $P = y(0)$ is the initial quantity of bacteria units and k is a decimal coefficient, which depends on the population considered.

The exponential model with the base e is also used in physics in the theory of decay of radioactive elements. On p. 27 of the textbook you may find the formula (notice an irritating mistake in it):

$$y(t) = A \cdot e^{-1.2 \times 10^{-4} \cdot t} \ ,$$

representing how Carbon-14 (a radioactive element) decays over time. Here A is the initial value of Carbon-14.

7.

Problem 3 (Exercises 3, pp. 29-31)**.** *Match the graph of $y = -3^{-x}$ with the graphs from Fig. 29, p. 29.*

SOLUTION. It is clear from the formula given for $y(x)$, that $y(x)$ is negative for every x. It follows that its graph must be below the coordinate axis. Hence our choice is restricted between (c) and (e).

Next,

$$y(x) = -\frac{1}{3^x} \ ,$$

which shows that for big positive values of x the function $y(x)$ must approach the zero value from below. Of two graphs only the graph plotted in (e) satisfies this property. hence the graph of $y(x)$ matches with the graph on (e). $\square$

Problem 9. *Graph the function $y = 3 \cdot e^{-x} - 2$ and find its domain, range, and intercept.*

SOLUTION. First we plot the graph of $y = e^x$:

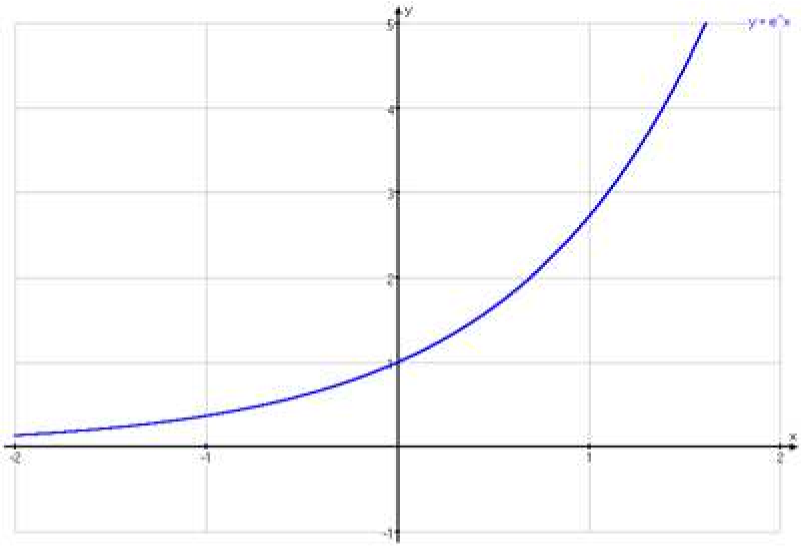

FIG.3

Applying elementary transforms of graphs, we obtain the required graph as indicated in Fig. 4.

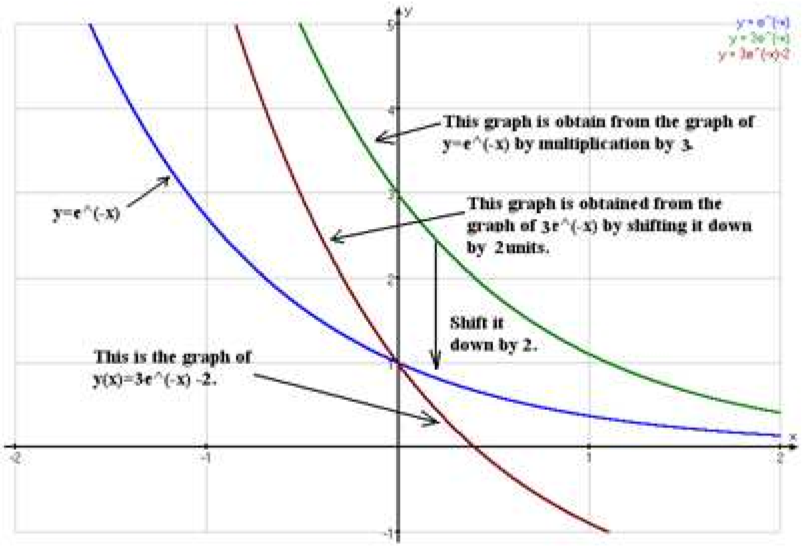

FIG. 4

Since the formula $3e^x - 2$ determining $y(x)$ makes sense for every real x, we conclude that $\mathcal{D}(y) = (-\infty, +\infty)$. To find the range we may apply both graphical and algebraic arguments.

One may conclude from the graph that the range of $3e^{(-x)}$ is $(0, +\infty)$. since the graph of $y(X)$ is obtained from the graph of $3e^{(-x)}$ by shifting it 2 units down, we see that $\mathcal{R}(y) = (-2, +\infty)$.

On the other hand by the algebraic method we obtain that b is in the range of $y(x)$ if and only if the equation

$$b = 3 \cdot e^{(-x)} - 2$$

is solvable in x. Elementary algebraic transformations show that this equation is equivalent to

$$\frac{b+2}{3} = e^{(-x)} \left(= \frac{1}{e^x} \right) \Longleftrightarrow e^x = \frac{3}{b+2} \ ,$$

which is solvable if and only if $b > -2$. $\square$

Problem 11. *Rewrite the exponential expression* 9^{2x} *to have base* 3

SOLUTION. We apply **Rule 3** for exponents (p. 26), saying that

$$(a^x)^y = (a^y)^x = a^{xy} \ .$$

Observing that $9 = 3 \cdot 3 = 3^2$, we obtain

$$9^{2x} = (3^2)^{2x} = 3^{2 \cdot 2x} = 3^{4x} \ . \quad \square$$

Problem 32. *If John invests* \$2300 *in a saving account with* 6% *interest rate* **compounded annually** how long will it take until John's account has a balance of \$4150?

SOLUTION. The model for accounts compounded annually is given by (2). Therefore the value I_n of John's account in n years will be

$$I_n = 2300 \cdot (1 + 0.06)^n = 2300 \cdot (1.06)^n \ .$$

We find the integer such that

$$I_{n-1} = 2300 \cdot (1.06)^{n-1} < 4150 < 2300 \cdot (1.06)^n \ . \tag{4}$$

Let us divide (4) by 4150. Then we obtain

$$\frac{46}{83} \cdot (1.06)^{n-1} < 1 < \frac{46}{83} \cdot (1.06)^n \ . \tag{5}$$

By the Binomial Theorem we have

$$(1 + x)^n > 1 + nx \ , \ x > 0 \ .$$

It follows that the right-hand inequality in (5) takes place for sure if

$$1 + n \cdot 0.06 > \frac{83}{46} = 1.8043478 \cdot \ ,$$

which happens for the first time for $n = 13$. Now using a calculator, we find that

$$(1.06)^{13} = 2.132 \ , \ (1.06)^10 = 1.7980 \ , \ (1.06)^{11} = 1.8982985 \cdots .(1.06)^12 = 2.012 \cdots \ .$$

This shows that the balance of John's account will reach \$4150 on the eleventh year of banking. $\square$

Problem 41. *The population of Mexico for* 1950 − −1990 *years is presented in the following table*

Year	*Population in millions*
1950	25.8
1960	34.9
1970	48.2
1980	66.8
1990	81.1

(*a*) *Let* $x = 0$ *represent* 1900*,* $x = 1$ *represents* 1901*, and so forth. Find an exponential regression equation for the data and superimpose its graph on a scatter plot of the data.*

(*b*) *Use the exponential regression equation to estimate the population of Mexico in* 1900*. How close is the estimate to the actual population in* 1900 *of* 13 607 272*?*

(*c*) *Use the exponential regression equation to estimate the annual rate of growth of the population of Mexico.*

SOLUTION. First we find the decimals for consequitive quotients of our data:

$$\frac{34.9}{25.8} = 1.35271$$
$$\frac{48.2}{34.9} = 1.38109$$
$$\frac{68.8}{48.2} = 1.42739$$
$$\frac{81.1}{66.8} = 1.21407$$

These calculations show that the quotients vary about some constant d. We can take the mean value for this constant:

$$d = \frac{1.35271 + 1.38109 + 1.42739 + 1.21407}{4} = \frac{5.37526}{4} = 1.34382 \ .$$

Therefore if $y = c \cdot a^x$ is the exponential regression equation, then

$$\frac{c \cdot a^{x+10}}{c \cdot a^x} = a^{10} = d = 1.34382 \ .$$

It follows that

$$a = (1.34382)^{1/10} = 1.02999 \approx 1.03 \ .$$

It is clear that 1950 corresponds to $x = 50$. Then

$$25.8 \approx c \cdot a^{50} = c \cdot \left(a^{10}\right)^5 = c \cdot d^5 \approx c \cdot (1.34382)^5 \ .$$

It follows that

$$c = \frac{25.8}{(1.34382)^5} = 5.79119 \approx 5.79 \ .$$

Hence the regression equation is given by

$$\boxed{y(x) = 5.79 \cdot (1.03)^x \ .}$$

(b) The exponential regression says that the population of Mexico in 1900 was $y(0) = 5.79$ million. The relative mistake of the prediction is

$$\frac{13.6 - 5.79}{13.6} = 0.574265 \approx 57\% \ .$$

(c) The annual rate of growth is

$$\frac{y(x+1) - y(x)}{y(x)} = a - 1 = 0.03 = 3\% \ . \quad \square$$

Let us observe that in our solution of Problem 41 we didn't use the best exponential regression. However, the results obtained match pretty well the results obtained with a computer program:

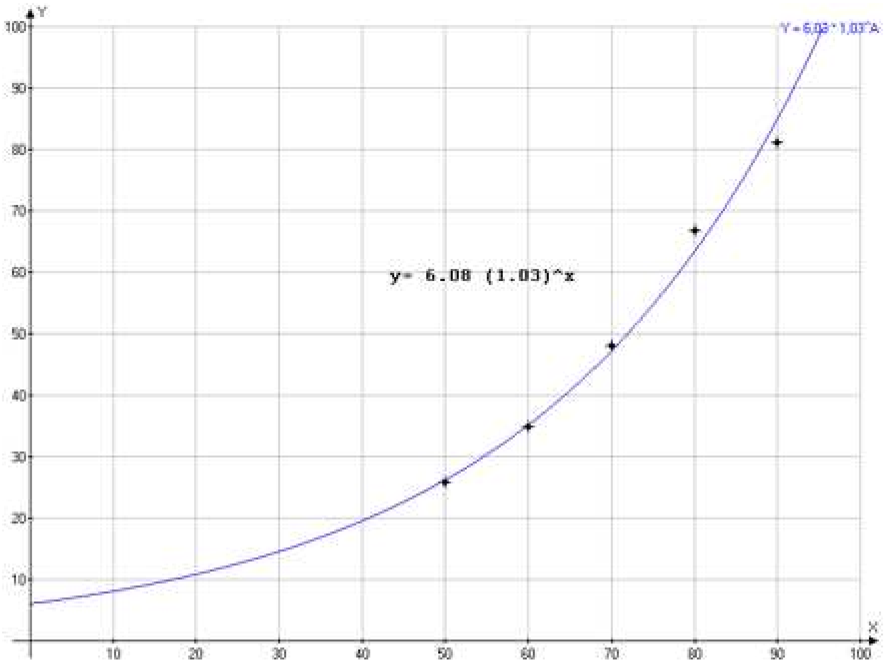

FIG. 5

GUIDANCE TO
INVERSE FUNCTIONS
AND LOGARITHMS
Fall 2013

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THEORETICAL BACKGROUND.

1. The first row of the table

2	4	8	16	32	64	128	256	512	1024	2048	4096	$\cdots$
1	2	3	4	5	6	7	8	9	10	11	12	$\cdots$

lists the consecutive powers of two, whereas the second row presents the corresponding powers. The catastrophic character of an exponential growth was discussed in $P3$. Since the numbers in the second row of the above table are much smaller than the corresponding numbers in the first row, the idea to consider the powers n instead of the powers a^n of a looks very attractive.

Suppose for simplicity that $a > 1$. Then the exponential function $y = a^x$ increases on $(-\infty, +\infty)$. The rigorous theory of real numbers says that every value in $(0, +\infty)$ is taken by an exponential function exactly one time. This means that the equation

$$y = a^x \tag{1}$$

has exactly one solution for every $y > 0$. In other words, for every $y > 0$ there is exactly one x such that (1) is satisfied.

If we consider the above statement attracting the concept of function, then we may conclude that in fact some function on the interval $(0, +\infty)$ has been defined. This function is denoted by

$$x = \log_a y$$

and is called the **logarithmic function with base a.**

It is clear that the logarithmic function satisfies the identity

$$\boxed{y = a^{\log_a y} \ , \ y > 0 \ .} \tag{2}$$

In operations with logarithms the identity (2) is of the great importance and **must be memorized.**

2.

Typeset by $\mathcal{AMS}$-TEX

Properties of Logarithms. *Let $a \neq 1$ be a positive number and let $x > 0$, $y > 0$. Then*

1. **Product Rule:** $\log_a(xy) = \log_a x + \log_a y$
2. **Quotient Rule:** $\log_a(\frac{x}{y}) = \log_a x - \log_a y$
3. **Power Rule:** $\log_a(x^y) = y \cdot \log_a x$

PROOF. 1. The Product Rule follows from the addition formula for exponential functions. Indeed, we have by (2) that

$$a^{\log_a xy} = xy = a^{\log_a x} a^{\log_a y} = a^{\log_a x + \log_a y} \ .$$

Since any exponential function takes every its value only once, we conclude that

$$\log_a(xy) = \log_a x + \log_a y \ .$$

as stated.

2. The Quotient Rule. Similar arguments show that

$$a^{\log_a(\frac{x}{y})} = \frac{x}{y} = \frac{a^{\log_a x}}{a^{\log_a y}} = a^{\log_a x - \log_a y} \ ,$$

which implies the Quotient Rule by the same observation as in 1.

3. The Power Rule. We have

$$a^{\log_a x^y} = x^y = ((a^{\log_a x})^y = a^{y \log_a x} \ ,$$

which implies the Power Rule. $\square$

Notice the correspondence between the properties of exponential functions and of logarithms.

$a^{x+y} = a^x \cdot a^y$	$\log_a xy = \log_a x + \log_a y$
$a^{x-y} = \dfrac{a^x}{a^y}$	$\log_a \left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
$(a^x)^y = a^{xy}$	$\log_a x^y = y \cdot \log_a x$

VERY OFTEN STUDENT CONFUSE THE FORMULAS FOR EXPONENTIAL FUNCTIONS WITH THE FORMULAS FOR LOGARITHMIC FUNCTIONS. DO NOT FOLLOW THIS POOR MISTAKES! MEMORIZE THE BOXED FORMULAS CORRECTLY!

3. Similar to exponential functions with base e the logarithmic function with base e is of big importance. It is denoted by

$$\boxed{\ln x = \log_e x \ .,}$$

and called the **natural logarithm** of x.

Using natural logarithms and Rule 4 for exponents (see p.26), one can easily write any exponential function as an exponential function with base e:

$$\boxed{a^x = (e^{\ln a})^x = e^{x \ln a} \ .}$$

EXAMPLE (see p. 37). We have

$$2^x = (e^{\ln 2})^x = e^{x \ln 2}$$
$$5^{-3x} = (e^{\ln 5})^{-3x} = e^{-3x \ln 5} \ .$$

STUDY EXAMPLE 7 on p. 38 of the textbook.
Observing that

$$\ln x = \ln a^{\log_a x} = (\log_a x)(\ln x) \ ,$$

we obtain an important formula

$$\boxed{\log_a x = \frac{\ln x}{\ln a} \ .}$$

STUDY EXAMPLE 9 on p. 39 of the textbook.
STUDY EXAMPLE 12 on p. 40 of the textbook.

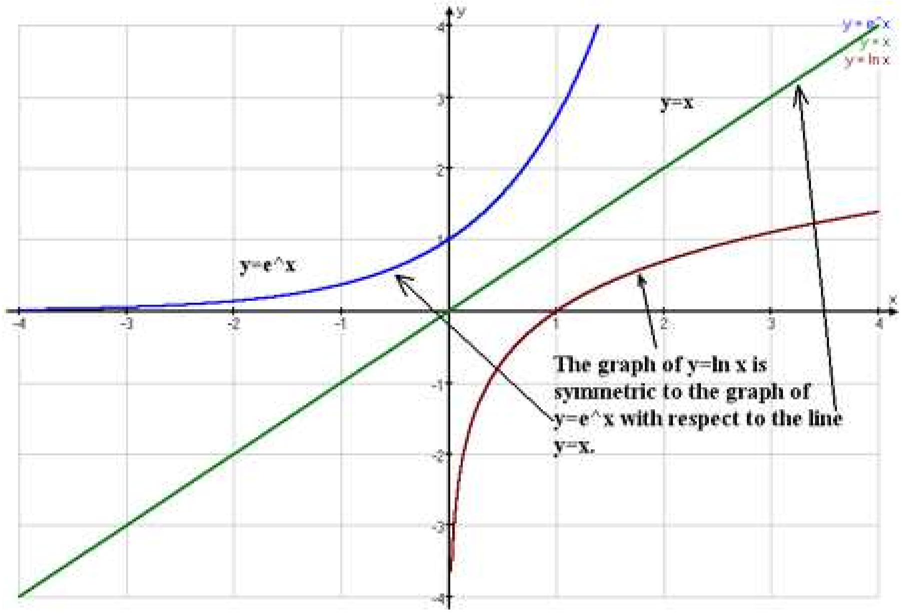


FIG.1

4. It is clear from Fig. 1 that the graph of $y = \ln x$ is symmetric to the graph of $y = e^x$ with respect to the line $y = x$. To prove this let us consider an arbitrary point (a, b) of the graph of $y = e^x$. Then a and b are related by

$$b = e^a \Longleftrightarrow a = \ln b \ .$$

It follows that (b, a) belongs to the graph of the logarithmic function if and only if (a, b) belongs to the graph of the exponential function e^x.

The picture on Fig. 2. shows that (b, a) and (a, b) are opposite vertices of a square with one diagonal on the line $y = x$. Since we know from Geometry that the diagonals of any square split into equal parts by the point of their intersection, the vertices (b, a) and (a, b) are symmetric with respect to the diagonal of the square lying on $y = x$.

Now, when (b, a) moves along the graph $y = e^x$, the point (a, b) moves along the graph of $y = \ln x$. Hence these graphs are symmetric with respect to the line $y = x$ as stated.

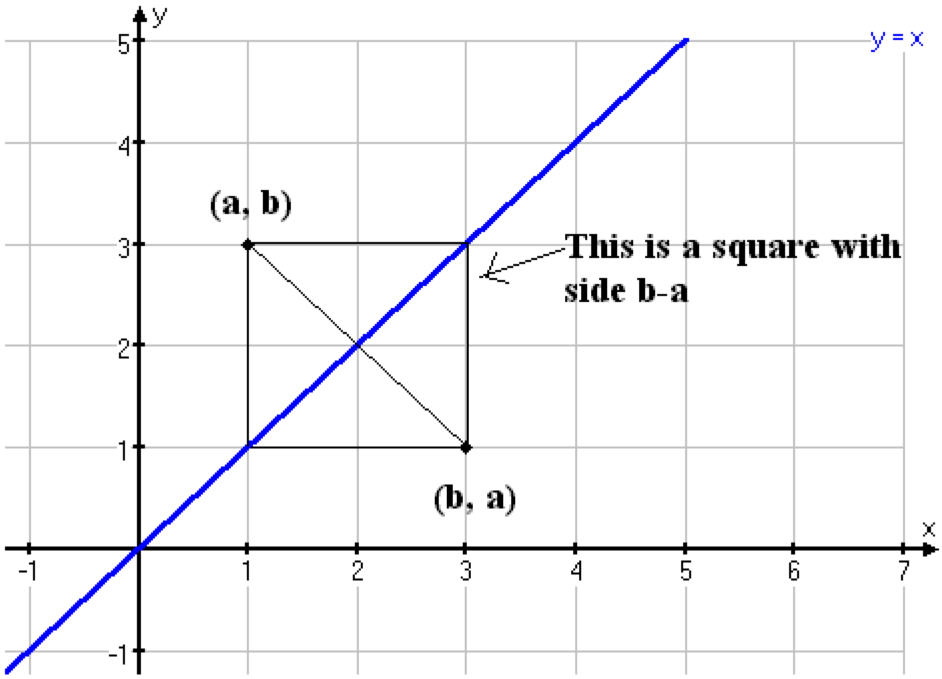


FIG.2

5. The theory of logarithms developed above can be easily generalized. Let us observe that the arguments applied to define logarithms used only one property of an exponential function. Namely, for every value y in the range $\mathcal{R}(f)$ of an exponential function the equation

$$y = f(x)$$

has a unique solution in x. Exponential functions are not the only ones sharing this property. For instance, $y = x^3$ has it too. Then we can define a function $g(x)$ by $g(x) = x^{1/3}$. It is clear that

$$f(g(x)) = g(x)^3 = (x^{1/3})^3 = x^{3 \cdot \frac{1}{3}} = x^1 = x \ .$$

The graphs on Fig. 3 are symmetric with respect to $y = x$, which which is not surprising, since the proof of this fact follows the same lines as the proof of the symmetry of exponential and logarithmic graphs.

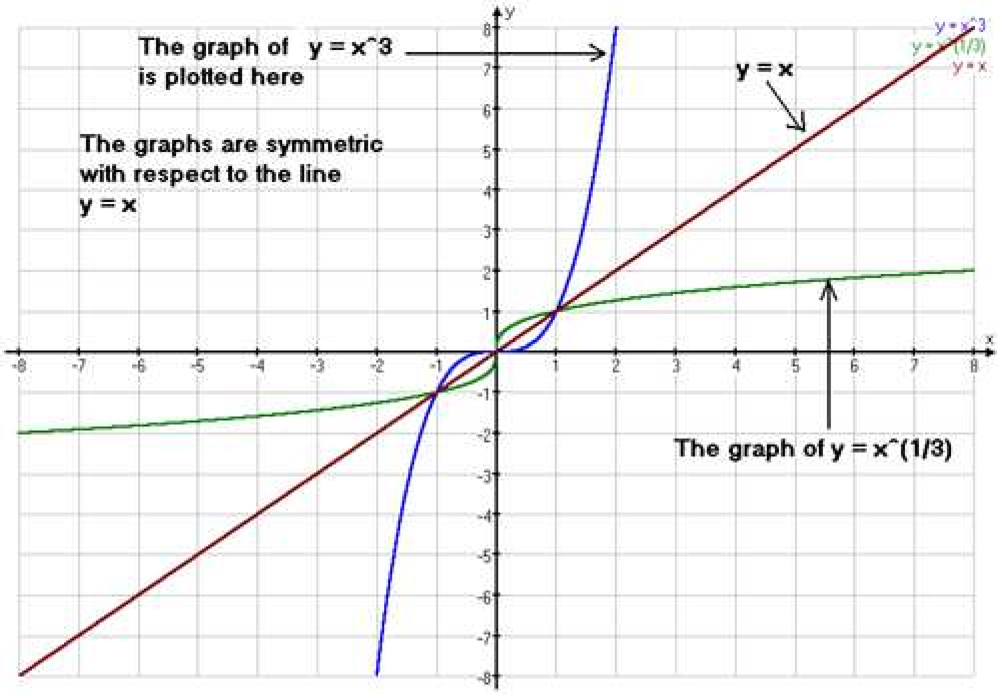


FIG.3

Definition One-to-One Function. A function $f(x)$ is **one-to-one** on its domain $\mathcal{D}(f)$ if $f(a) \neq f(b)$ whenever $a \neq b$.

Check that this definition is equivalent to the statement that for every $y \in \mathcal{R}(f)$ the equation $y = f(x)$ has a unique solution in x.

The last statement has a nice geometrical interpretation.

The Horizontal Line Text. *A function $y = f(x)$ is one-to-one if and only if every horizontal line $y = b$ intersects the graph of $y = f(x)$ at most at one point.*

Proof. If a line $y = b$ intersects the graph of f, then the points of intersection are of the form $(a, f(a))$, where a is a solution to the equation

$$b = f(a) \ .$$

This equation has a unique solution if and only if the line $y = b$ intersects the graph of f exactly at one point. $\square$

Let f be a one-to-one function on its domain $\mathcal{D}(f)$. If we assign with each element y in $\mathcal{R}(f)$ the unique solution x to the equation $y = f(x)$, then we obtain a function $x = g(y)$ with the domain $\mathcal{D}(g) = \mathcal{R}(f)$ and with the range $\mathcal{R}(g) = \mathcal{D}(f)$ such that

$$x = f(g(x)) \ , \ x \in \mathcal{D}(g) = \mathcal{R}(f) \ .$$

The function g is called the **inverse of** f and is denoted by $g = f^{-1}$.

In case $f(x) = e^x$ we obtain that $g(y) = \ln y$ is the inverse of the exponential function $y = e^x$.

A Test for Inverse. *Functions f and g are an inverse pair if and only if*

$$f(g(x)) = x \text{ and } g(f(x)) = x \ .$$

In this case $g = f^{-1}$ and $f = g^{-1}$.

Problem 28. *Find the inverse function f^{-1} and verify that $(f \circ f^{-1})(x) = (f^{-1} \circ f)(x) = x$ for*

$$f(x) = \frac{x+3}{x-2} \ .$$

SOLUTION. We have $\mathcal{D}(f) = (-\infty, 2) \cup (2, +\infty)$. Let y be a real number. We are going to study when the equation

$$y = \frac{x+3}{x-2}$$

has a unique solution in x. Since y is a finite real number, x cannot be 2. Therefore $x - 2 \neq 0$ and multiplying the both side of the above equation by $(x-2)$, we arrive at an equivalent equation

$$y(x-2) = x + 3 \ ,$$

which is a liner equation in x equivalent to

$$x(y-1) = 2y + 3 \ .$$

If $y = 1$, then the left-hand side of the equation is 0 for every value of x , whereas the right-hand side is 5. It follows that for $y = 1$ the equation has no solutions in x.

On the other hand, if $y \neq 1$, then there is a unique solution given by

$$x = \frac{2y+3}{y-1} \ .$$

We see that the range of $f(x)$ is $(-\infty, 1) \cup (1 + \infty)$. Since we proved that for every $y \in \mathcal{R}(f)$ the equation $y = f(x)$ has a unique solution, we can conclude that f is one-to-one and that

$$f^{-1}(x) = \frac{2x+3}{x-1} \ .$$

Now, we have

$$(f \circ f^{-1})(x) = f\left(\frac{2x+3}{x-1}\right) = \frac{\frac{2x+3}{x-1} + 3}{\frac{2x+3}{x-1} - 2} = \frac{2x+3+3(x-1)}{2x+3-2(x-1)} = \frac{5x}{5} = x \ .$$

Similarly,

$$(f^{-1} \circ f)(x) = \frac{2f(x)+3}{f(x)-1} = \frac{2\frac{x+3}{x-2} + 3}{\frac{x+3}{x-2} - 1} = \frac{2(x+3)+3(x-2)}{(x+3)-(x-2)} = \frac{5x}{5} = x \ . \quad \square$$

Problem 34. *Solve the equation* $e^{0.05t} = 3$ *algebraically.*

SOLUTION. Using the basic property of logarithms, we obtain

$$0.05t = \ln\left(e^{0.05t}\right) = \ln 3 \ ,$$

which implies that

$$t = \frac{\ln 3}{0.05} = 21.97224577 \cdots \ . \quad \square$$

Problem 35. *Solve the equation* $e^{x} + e^{-x} = 3$ *algebraically.*

SOLUTION. Let $z = e^x$. Then $e^{-x} = 1/e^x = 1/z$ and

$$z + \frac{1}{z} = 3 \Longleftrightarrow z^2 - 3z + 1 = 0 \ .$$

The roots of the above quadratic equation are given by

$$z_{1,2} = \frac{3 \pm \sqrt{3^2 - 4}}{2} = \frac{3 \pm \sqrt{5}}{2} \ .$$

It follows that

$$x_{1,2} = \ln(z_{1,2}) = \ln\left(\frac{3 \pm \sqrt{5}}{2}\right) \ .$$

Observing that $z_1 \cdot z_2 = 1$ by Vieta Theorem, we obtain that

$$\begin{aligned} x_1 &= \ln\left(\frac{3 + \sqrt{5}}{2}\right) = 0.9624236501 \cdots \\ x_2 &= -0.9624236501 \cdots \ . \end{aligned}$$

Problem 41. *The half-life of a certain radioactive substance is* 12 *hours. There are* 8 *grams present initially.*

- **(a)** *Express the amount of substance remaining as a function of time* t.
- **(b)** *When there will be* 1 *gram remaining?*

SOLUTION. According the theory of radioactive decay the amount $I(t)$ of any radioactive substance remaining in t hours is given by the formula

$$I(t) = I(0) \cdot e^{-k \cdot t} \ ,$$

where k is a positive constant which depends on a radioactive substance.

The half-life of a radioactive substance is the time required for it to diminish to the half of its initial weight.

We know that the half-life of this particular substance is 12 hours. We can use this condition to determine an unknown parameter k from the equation

$$I(12) = \frac{I(0)}{2} \ ,$$

which is equivalent to

$$e^{-12k} = \frac{1}{2} \ .$$

Applying the natural logarithm to the both sides of the above equation we obtain that

$$-12 \cdot k = \ln\left(\frac{1}{2}\right) \Longleftrightarrow k = \frac{\ln 2}{12} = 0.05776226505 \cdots \ .$$

To find the time required the radioactive substance to reach the weight of 1 gram, we observe that $I(0) = 8$ grams. Then t satisfies the equation:

$$8e^{-\frac{\ln 2}{12} \cdot t} = 1 \ ,$$

which implies that

$$t = \frac{\ln 8}{(\ln 2)/12} = 12\frac{3\ln 2}{\ln 2} = 36$$

hours. $\square$

GUIDANCE TO
TRIGONOMETRIC FUNCTIONS
AND THEIR INVERSES
Fall 2013

Sergey Khrushchev, Professor of ISE, KBTU

THEORETICAL BACKGROUND.

1. Angles. In Geometry an angle is defined as a figure formed by two rays starting at a common point called the **vertex** of an angle. Two angles on a plane are said to be equal if one can make them identical by a movement in the plane. Any angle is equal to an angle with the vertex at the origin of the coordinate plane and with one side directed along the positive x-axis.

In Trigonometry angles have directions. One ray is considered as the **initial** side (ray) of the angle. Then the other side is called the **terminal** side (ray). We say that an angle is in a standard position if its vertex is at the origin and the initial side is directed along the positive x-axis.

One indicates which side is which by drawing a curved arrow from the initial side to the terminal side.

If the curved arrow is pointing counterclockwise, then the angle is called **positive**. If the arrow points in the clockwise direction, the angle is **negative**.

Angles are classified as being a first, second, third, or forth-quadrant angle depending on what quadrant its terminal side is in. The quadrants are numbered by Romans I, II, III, IV.

2. Angle's Measurement. The size of an angle is measured by the amount the terminal ray moved if started where the initial ray is. Angles are measured either in **degrees** or in **radians.**

An angle in the standard position is called **full** if its terminal side makes the whole circle and coincides with the initial side. The full angle makes $360°$. Then the angle with the initial side OX and the terminal side $-OX$ makes the half of the full angle, which is $180°$. The standard angle with the terminal side OY equals $180°/2 = 90°$.

To measure angles in radians one draws a circle of radius 1 centered at the origin of the rectangular coordinates system (see Fig. 1). This circle is called a unit circle. The terminal side of any angle in the standard position intersects the unit circle at one point $A = (x, y)$. This point at the same time is the end-point of the circular arc started at $D = (1, 0)$. Therefore one can measure angle $\angle DOA$ with the length of the corresponding arc. This measure is called the **radian measure** of an angle.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

The full angle measured in radians makes 2π. Its half equals π. The direct angle is $\pi/2$. The following table establishes the correspondence between degree and radian measures:

Degrees	$-180°$	$-135°$	$-90°$	$-45°$	$0°$	$30°$	$45°$	$60°$	$90°$	$180°$
Radians	$-\pi$	$-3\pi/4$	$-\pi/2$	$-\pi/4$	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	π

Since both measures are linear functions of the corresponding measures, we obtain the following conversion formulae

$$\boxed{1 \text{ degree} = \frac{\pi}{180} \approx 0.02 \text{ radians} \ , \ 1 \text{ radian} = \frac{180}{\pi} \approx 57° \ .}$$

3. Trigonometric Functions. The perpendicular line from A to x-axis intersects x-axis at point B. The rectangular triangle $\triangle OAB$ is called the **reference triangle** for that angle. Traditionally trigonometric functions are determined first for the angles of rectangular triangles. After that using the method of reference triangle they are defined for arbitrary angles. Recall that the sine of an angle in a rectangular triangle is the quotient of the leg opposite to the angle to the hypothenuse. Cosine is defined as the quotient of the leg adjusting to the vertex of the angle to the hypothenuse.

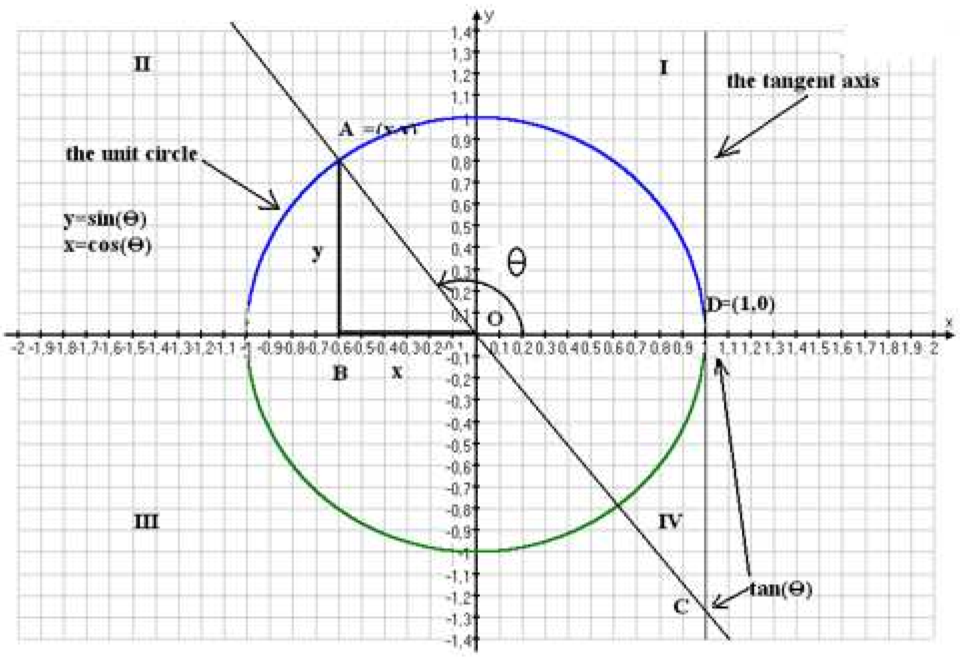


FIG.1

There is however a more direct way to do this. We just say that

$$\boxed{\sin\theta = y \ , \ \cos\theta = x \ ,}$$

where x and y are the coordinates of the point of the intersection of the terminal side of the angle with the unit circle (see Fig. 1).

The third important trigonometric function is defined by

$$\boxed{\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}\ .}$$

Since $\triangle AOB \backsim \triangle DOC$, we see that the length of the segment DC taken with an appropriate sign (in the case of Fig. 1 it is minus) equals $\tan\theta$. Therefore vertical line $x = 1$ is called the **tangent axis.**

There are three other trigonometric functions associated with sine, cosine and tangent.

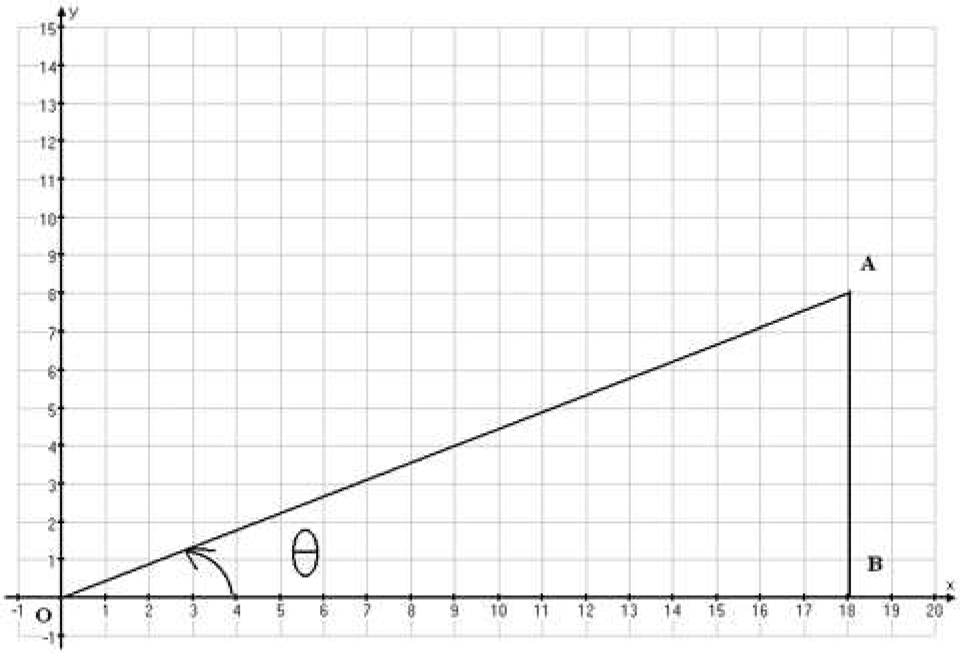


FIG.2

Suppose that we are measuring the distance from O to the top point A. The distance OB is known and is equal to 18 (see Fig. 2). Then it is clear from the definition of cosine that

$$|OA| = \frac{18}{\cos\theta}\ .$$

It is clear that if one knows the table of values for cosine, then the distance is determined by a simple division. However, multiplication is easier to do. Therefore a separate trigonometric function called **secant** is also considered:

$$\boxed{\sec\theta = \frac{1}{\cos\theta}\ .}$$

Similarly, **cosecant** and **cotangent** are defined as

$$\boxed{\csc\theta = \frac{1}{\sin\theta}\ ,\ \cot\theta = \frac{1}{\tan\theta}\ .}$$

4. General Properties of Trigonometric Functions. Since trigonometric functions of angles are defined by the coordinates of the point A at which the terminal side of the angle intersects the unit circle, the values of trigonometric functions remain unchanged after full rotations:

$$\sin(x+2\pi) = \sin x \tag{1.1}$$

$$\cos(x+2\pi) = \cos x \tag{1.2}$$

$$\tan(x+2\pi) = \tan x \tag{1.3}$$

It can be seen from the definition of sine and cosine that 2π is the smallest positive number satisfying (1.1) and (1.2). It is called the **period** of sine and cosine. It is not the case for tangent however. This can be seen easily if we apply the definition of tangent with the tangent line. Adding a half of the full angle to the terminal side of any angle moves this line into itself. Therefore

$$\boxed{\tan(x+\pi) = \tan x\ .}$$

It follows that the period of tangent is π.

Sine is an **odd** function:

$$\boxed{\sin(-\theta) = -\sin\theta\ .,}$$

and cosine is an **even** function:

$$\boxed{\cos(-\theta) = \cos\theta\ .}$$

It follows that tangent is an odd function.

5. The Addition Formulas. We know that exponential functions are characterized by the addition formula:

$$a^{(x+y)} = a^x \cdot a^y\ , a \neq 1\ ,\ a > 0\ .$$

The addition formulas for sine and cosine look as follows:

$$\boxed{\begin{aligned} \sin(x+y) &= \sin x\cos y + \cos x\sin y \\ \cos(x+y) &= \cos x\cos y - \sin x\sin y \end{aligned}} \tag{2}$$

Let us replace y in the second formula (2) with $-y$. Observing that $\cos 0 = 1$ and that sine is an odd function, we obtain that

$$\boxed{\cos^2 x + \sin^2 x = 1\ .} \tag{3}$$

Putting $x = y$ in (2), we have

$$\boxed{\begin{aligned} \sin 2x &= 2\sin x \cos x \\ \cos 2x &= \cos^2 x - \sin^2 x\ . \end{aligned}} \tag{4}$$

The Addition Formula for tangent follows from (2):

$$\boxed{\tan(x+y) = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y} = \frac{\tan x + \tan y}{1 - \tan x \tan y}} \tag{5}$$

(we divided both the numerator and denominator in (5) by $\cos x \cos y$).

Comparing the Addition formulas for sine and cosine with the Addition Formula for exponents, one may find out a surprising similarity. This observation led L. Euler to a remarkable formula:

$$\boxed{e^{i\cdot x} = \cos x + i \cdot \sin x\ ,}$$

where $i^2 = -1$. Euler's formula has many applications including engineering mathematics. You may find more details in **A.4** of the text-book, pp. 1151-1161.

6. Inverse Trigonometric Functions. The graph of sine is plotted on Fig. 3:

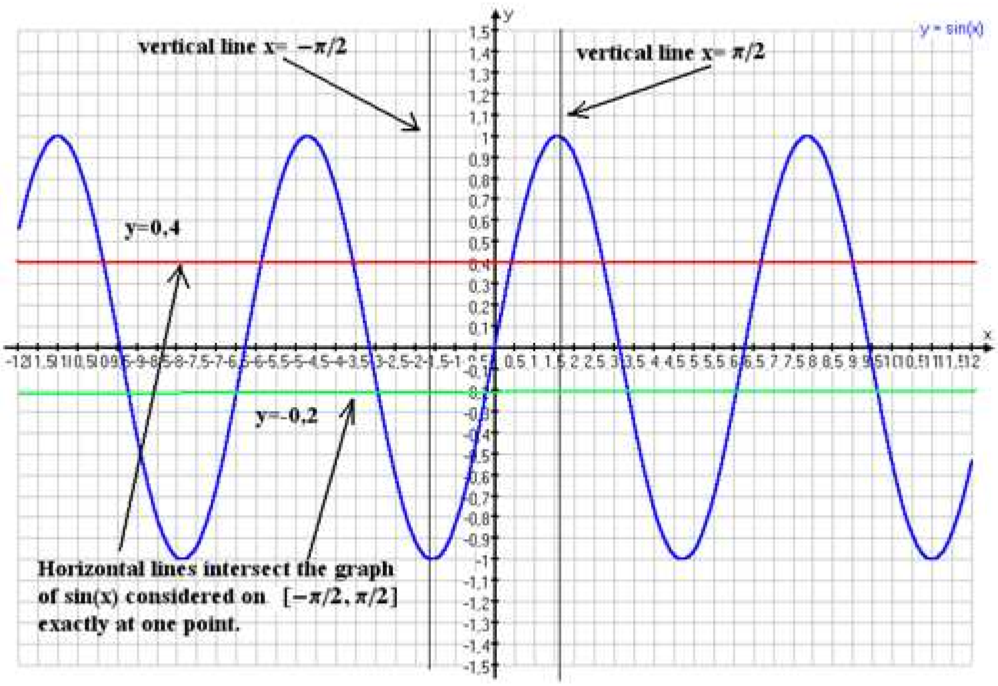


FIG.3

It is clear that if we consider $sin(x)$ on $[-\pi/2, \pi/2]$, then this function is one-to-one by the horizontal line test. Therefore there exists an inverse function, which is denoted by $\arcsin(x)$. Its graph is plotted on Fig. 4.

Similar arguments allows one to plot the graph of $\arccos(x)$ and of $\arctan(x)$. See the details and graphs on p. 52 of the text-book.

It is important to memorize the graphs of $\arcsin(x)$, $\arccos(x)$, and $\arctan(x)$.

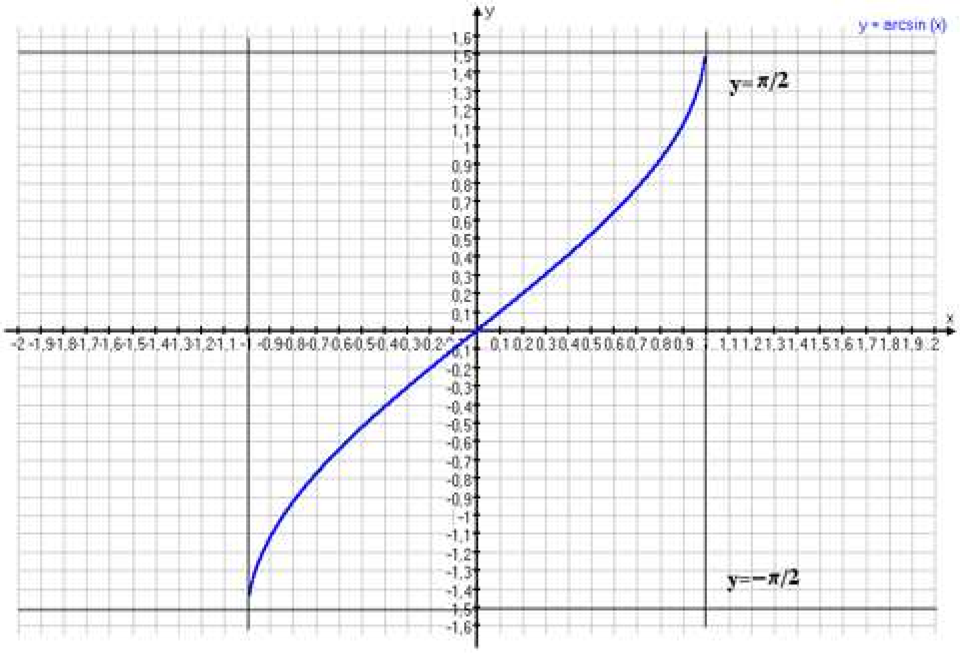


FIG.4

Problem 1 (Exercises 5, p. 55). *On a circle of radius* 10 *meters, how long is an arc that substends a central angle of*

(a) $4\pi/5$ *radians;*
(b) $110°$*?*

SOLUTION. (a) The length of the circle of radius r is $2\pi r$. This length corresponds to the full angle of 2π radians. The angle of

$$\frac{4\pi}{5} = (2\pi) \cdot \frac{2}{5}$$

radians makes $2/5$ of the full angle. It follows that the length of the arc considered equals

$$\boxed{2\pi \cdot 10 \cdot \frac{2}{5} = 8\pi \text{ m.}}$$

(b) An angle of $110°$ makes $110/360 = 11/36$ part of the full angle. Hence the length is

$$\boxed{2\pi \cdot 10 \cdot \frac{11}{36} = \frac{55}{9}\pi}$$

meters. □

Problem 3, (Exercises 5, p. 55). *Copy and complete the following table of function values (see Fig. 5). If the function is undefined at a given angle, enter "UND". Do not use a calculator or tables.*

SOLUTION. Let us complete first the column for $\theta = -\pi$. The terminal side of the angle $-\pi$ in the standard position is directed along the negative x-axis. It intersects the unit circle at $(-1, 0)$. It follows that

$$\cos(-\pi) = -1 \ , \quad \sin(-\pi) = 0 \ , \quad \tan(-\pi) = 0$$
$$\cot(-\pi) = \text{"UND"}, \ \sec(-\pi) = -1 \ , \ \csc(-\pi) = \text{"UND"}.$$

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$	0	-sqrt(3)/2	0	1	1/sqrt(2)
$\cos\theta$	−1	-1/2	1	0	-1/sqrt(2)
$\tan\theta$	0	sqrt(3)	0	UND	-1
$\cot\theta$	UND	1/sqrt(3)	UND	0	-1
$\sec\theta$	-1	-2	1	UND	-sqrt(2)
$\csc\theta$	0	-2/sqrt(3)	UND	1	sqrt(2)

FIG. 5

The column for $\theta = 0$ is easy to complete. The terminal side coincides with the initial side. Hence the coordinates of the intersection point are $(0, 0)$, which implies that

$$\cos(0) = 1 \ , \quad \sin(0) = 0 \ , \ \tan(0) = 0$$
$$\cot(0) = \text{"UND"}, \ \sec(0) = 1 \ , \ \csc(0) = \text{"UND"}.$$

To complete the column headed by $\theta = \pi/2$, we observe that the coordinates of the intersection the terminal side and the unit circle are $(0, 1)$. Therefore

$$\cos(\pi/2) = 0 \ , \ \sin(\pi/2) = 1 \ , \quad \tan(\pi/2) = \text{"UND"}$$
$$\cot(\pi/2) = 0 \ , \ \sec(\pi/2) = \text{"UND"} \ , \ \csc(\pi/2) = 1 \ .$$

Next, we have

$$\frac{3\pi}{4} = \pi - \frac{\pi}{4} = \frac{\pi}{2} + \frac{\pi}{4} \ .$$

This shows that the terminal side is in the II quadrant and is directed along the line $y = -x$. By Pythagorean Theorem the intersection point has coordinates $(-1/\sqrt{2}, 1/\sqrt{2})$.

Descartes' Theory of Tangents

Sergey Khrushchev, Math Professor of ISE

In 1637 Descartes published a Book on Geometry. This book influenced the development of Mathematics to a great extend. To begin with the idea to represent points as points on the coordinate plane originates in this book. Similarly, the idea to describe functions with their graphs also can be found in this book by Descartes. Finally, Descartes proposed a beautiful method of finding equations of tangents to algebraic curves.

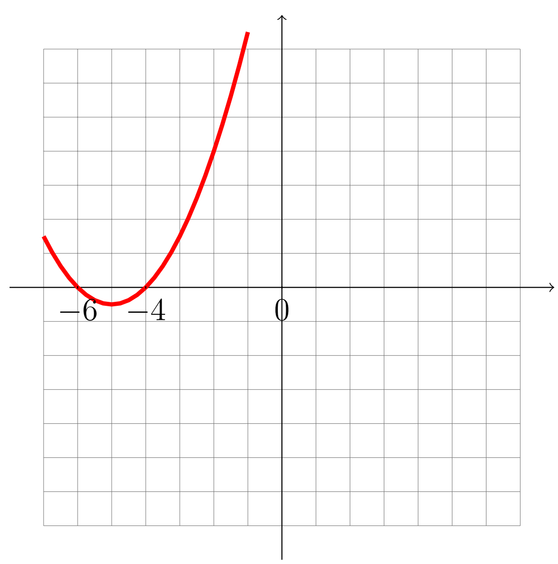
Following Descartes we illustrate his theory on examples.

Problem 1. Find equations of the tangent lines to the parabola

$$y = -x^2 - 10x - 24,$$

passing through the point with the coordinates $(0, -1)$.

SOLUTION: To plot the graph of $y = -x^2 - 10x - 24$



we observe that

$$x^2 + 10x + 24 = (x^2 + 2 \cdot 5 \cdot x + 5^2) - 1 = (x+5)^2 - 1 = (x+6)(x+4).$$

This parabola is directed upwards, and the x-coordinate of its vertex is given by:

$$-5 = \frac{(-6) + (-4)}{2},$$

i.e it is the middle point of two roots -6 and -4.

Exercise. *Show that the x-coordinate of the vertex of a parabola*

$$y = a(x - x_1)(x - x_2)$$

is given by the formula:

$$x = \frac{x_1 + x_2}{2}.$$

It is clear that the parabola $y = -(x+6)(x+4)$ is directed downwards, the x-coordinate of the vertex is the same, namely -5, and the y-intercepts is $y(0) = -24$.

The point-slope equation of a line through $(0, -1)$ is

$$y = ax - 1.$$

Descartes' Rule of tangents says that $y = ax - 1$ is a tangent to the parabola $y = -x^2 - 10x - 24$ if and only if the quadratic equation

$$-x^2 - 10x - 24 = ax - 1$$

has only one real root (it is the x-coordinate of the point of tangency on the parabola).

We know from Algebra that this happens if and only if the discriminant of the quadratic equation

$$x^2 + (10 + a)x + 23 = 0$$

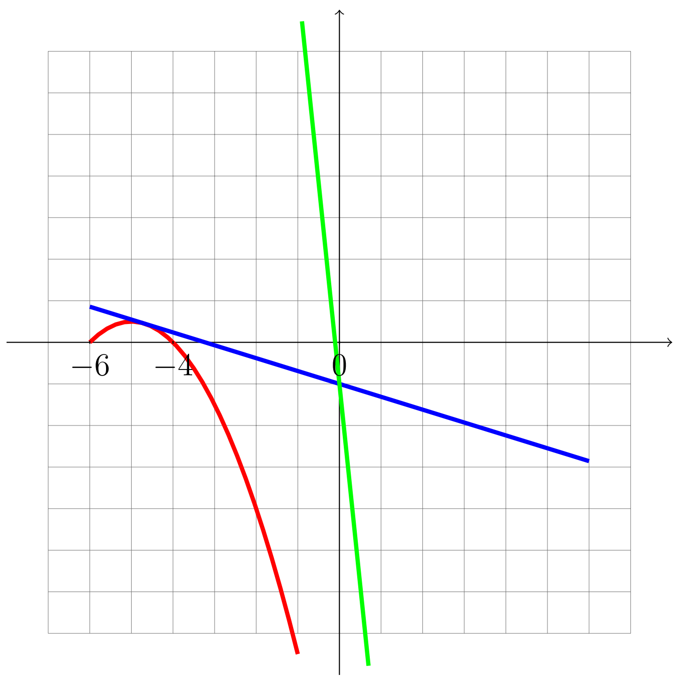
is zero. The discriminat is

$$\mathbf{D} = (10 + a)^2 - 4 \cdot 23 = 0 \Leftrightarrow (10 + a)^2 = 4 \cdot 23.$$

It follows that

$$a = \begin{cases} -10 + 2\sqrt{23} \\ -10 - 2\sqrt{23} \end{cases}.$$

Since $y(0) = -24 < -1$ we see that the point $(0, -1)$ is placed outside of the parabola and therefore there are two tangents to it. One tangent with negative slope $-10-2\sqrt{23}$ goes steep down (plotted green below) and another one with slope $-10 + 2\sqrt{23} < 0$ (plotted blue below) touches the parabola at a point with the x-coordinate in the interval $(-6; -4)$.



The equations of the tangents are:

$$y = (-10 - 2\sqrt{23})x - 1\,;$$
$$y = (-10 + 2\sqrt{23})x - 1\,.$$

To find the tangency points, we observe that for both values of a found above the discriminant $\mathbf{D}$ is zero. Therefore we can find the x-coordinates of the points of tangency by the formula for the solutions of quadratic equations:

$$x = \frac{-(10 + a) \pm \sqrt{\mathbf{D}}}{2} = -\frac{10 + a}{2}.$$

It follows that the line with slope $a = -10+2\sqrt{23}$ is a tangent to the parabola at the point with the coordinates:

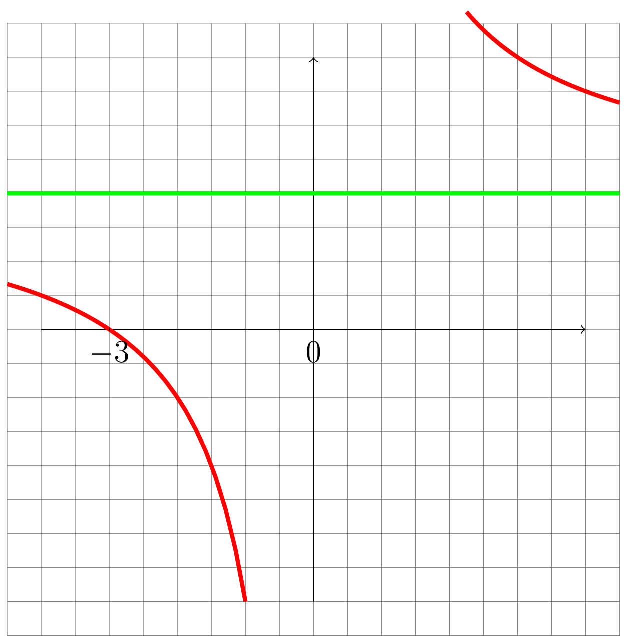
$$x = -\sqrt{23} = -4.795\ldots;$$
$$y = 10\sqrt{23} - 47 = 0.958\ldots.$$

Problem 2. Find the equation of the tangent to

$$y = \frac{6}{x} + 2$$

passing through the point $(0, -1)$.

SOLUTION: To plot the graph of $y = 2+6/x$ we observe that $y(x) > 2$ for $x > 0$ and $y(x) < 2$ for $x < 0$. If $|x|$ is big, then $y(x)$ is very close to 2. If $|x|$ is small, then $|y(x)|$ is very big. On the picture below the graph of $y = 2 + 6/x$ is plotted in red, and the graph of the horizontal asymptote $y = 2$ is plotted in green.



The point-slope equation of a line through $(0, -1)$ is given by

$$y = ax - 1.$$

Descartes' Rule of tangents says that $y = ax - 1$ is a tangent to the hyperbola $y = 2 + 6/x$ if and only if the equation

$$\frac{6}{x} + 2 = ax - 1$$

has a **multiple** root.

Let us rewrite this equation in the standard form:

$$ax^2 - 3x - 6 = 0.$$

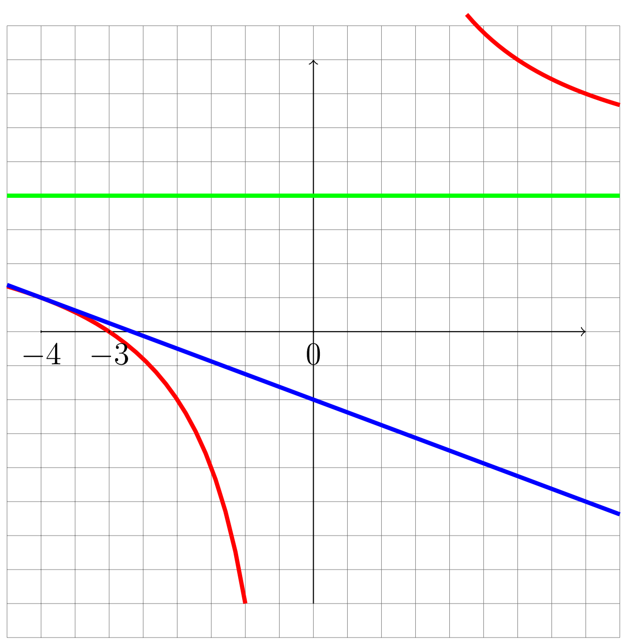
This quadratic equation has a multiple root if and only if its discriminant $\mathbf{D} = 3^2 + 4 \cdot 6 \cdot a$ is zero, implying that

$$a = -\frac{3 \cdot 3}{4 \cdot 6} = -\frac{3}{8} = -0.375.$$

Therefore the equation of the tangent is

$$y = -0.375x - 1.$$

It is plotted in blue in the picture below.



To find the x-coordinate of the tangency point we observe that for $a = -3/8$ the quadratic equation

$$ax^2 - 3x - 6 = 0$$

has a zero discriminant and therefore

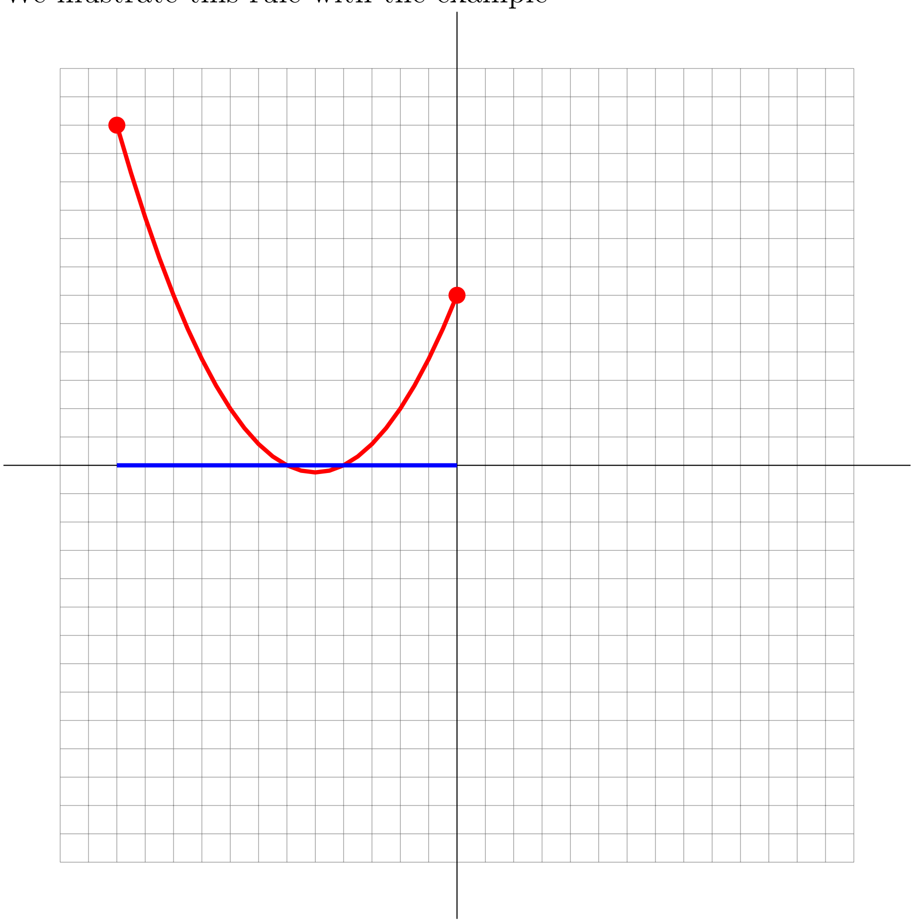
$$x = \frac{3}{2a} = -\frac{3 \cdot 8}{2 \cdot 3} = -4.$$

Extremal Properties of Parabolas

Sergey Khrushchev, Math Professor of ISE

Rule of Maximum. The maximal value of any parabola $y = ax^2 + bx + c$ directed upwards ($a > 0$) on a closed segment $[\alpha; \beta]$ of a real line equals the maximum of the values $y(\alpha)$ and $y(\beta)$.

We illustrate this rule with the example



Here $y = (x + 1.5)(x + 1)$, $\alpha = -3$, $\beta = 0$.

This Rule can be easily proved. There are three cases.

Case 1. The x-coordinate v of the vertex of the parabola $y = y(x)$ is in the interval (α, β). Then the function $y = y(x)$ decreases on $[\alpha, v]$ and increases on $[v, \beta]$. Therefore the maximal value of $y(x)$ on $[\alpha, \beta]$ is either $y(\alpha)$ or $y(\beta)$.

Case 2 $v \leq \alpha$. Then $y(x)$ increases on $[\alpha, \beta]$ and the maximal value is $y(\beta)$.

Case 3.$v \geq \beta$. Then $y(x)$ decreases on $[\alpha, \beta]$ and the maximal value is $y(\alpha)$.

Rule of Minimum. The minimal value of any parabola $y = ax^2 + bx + c$ directed downwards ($a < 0$) on a closed segment $[\alpha; \beta]$ of a real line equals the minimum of the values $y(\alpha)$ and $y(\beta)$.

Exercise. *Prove the Rule of Minimum.*

Part II
2013 Entrance Exams

KBTU, ISE, Entrance Exam

10 Questions - 120 minutes

February 23, 2013

Name :

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1. Solve the system

$$\begin{cases} \dfrac{2^{x+1}+31}{32-2^x} \geq 1 \\ \log_{(x-1)^2}(x+5) \leq 1 \end{cases}$$

2. Evaluate

$$\frac{\sqrt{45}}{\sqrt{65}-\sqrt{45}} + \frac{\sqrt{65}}{\sqrt{65}+\sqrt{45}}.$$

3. Find all values of a such that the equation

$$ax - 1 = |x^2 + 10x + 24|$$

has two roots.

4. The sum of five consecutive elements of an arithmetic progression equals the root of the equation $8^{2x+1} = (0,125)^{4-3x}$. The last term equals the sum of the infinite geometric progression

$$\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \dots.$$

Find the arithmetic progression.

5. Two bicyclists started simultaneously from A to B. The first one made the whole route with a constant velocity, whereas the second made half of the route with 15 km/h and the rest with the velocity which exceeded the velocity of the first by $4,5$ km/h. Find the velocity of the first bicyclist if it is known that both arrived at B at the same time.

6. A cell company invested half of the total revenue for the previous year to extend its network. It simultaneously reduced its monthly tariff 50%. The volume of sales increased three times as a result. In how many months does the additional profit obtained by this cell company compensate all its investments on the extension of business ?

7. The base of a right pyramid is a regular triangle. The cylinder whose base is the circle inscribed in the triangle has the same height as the pyramid. Find the volume the cylinder if the volume of the pyramid is $\sqrt{3}/\pi$.

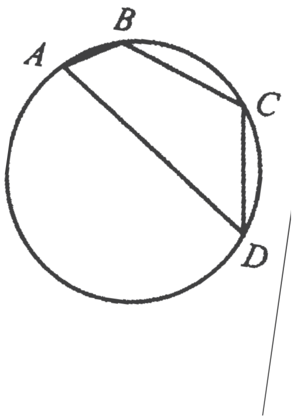
8. Solve the equation

$$4\cos^3 x - (\sin x + \cos x) = 0.$$

9. Solve the inequality

$$\sqrt{\log^2_{\frac{1}{3}} x + 2\log_3 \sqrt{x} - \log_x x - 1} > \log_{27} x^3.$$

10. A quadrilateral $ABCD$ is inscribed inside a circle (see the picture below). Two its angles are 29° and 43°. Evaluate the greatest angle.



KBTU, ISE, Entrance Exam in Astana

10 Questions - 120 minutes

March 10, 2013

Name:

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1. Find the number of integer solutions to the inequality

$$\log_{9x^2+24x+17} \frac{8x-13}{(x-1)^2(x^2-3x-4)} + \log_{9x^2+24x+17}(x^2-5x+4) \geq 0.$$

2. Evaluate
$$\frac{\cos 71^\circ \cdot \cos 10^\circ + \cos 80^\circ \cdot \cos 19^\circ}{2\cos 69^\circ \cdot \cos 8^\circ + 2\cos 82^\circ \cdot \cos 21^\circ}.$$

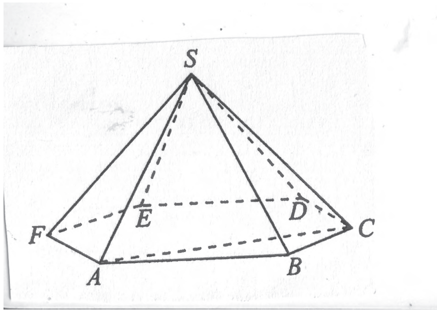
3. Find all values of a real parameter a such that the minimal value of $f(x) = -x^2 + 2x + a^2$ on the segment $[-1, 0]$ is smaller or equal 1.

4. To prepare to entrance exams in mathematics Paul solved problems from his textbook for 30 days. To make a good progress he increased daily the number of solved problems by one and the same number. At the end, it turned out that for the initial twenty days he solved as many problems as for the final ten. By how many percents does the number of problems solved for the last fifteen days exceed the number of problems solved for the first fifteen days?

5. A woodworker makes 72 chairs four hours faster than his partner. What is the productivity of the partner if the productivity of the woodworker is three chairs per hour more than the productivity of its partner?

6. A monopoly can sell q thousand units of its production on the market for $p = 8 - 0.05q$ dollars per unit every month. Its trading policy is to keep the price p on the product as high as possible but still to get revenue not less than \$300 000. Which price satisfies the strategy of the monopoly?

7. The volume of the triangular pyramid $SABC$, which is a part of a right hexagonal pyramid $SABCDEF$, equals 2. Find the volume of the pyramid $SABCDEF$.



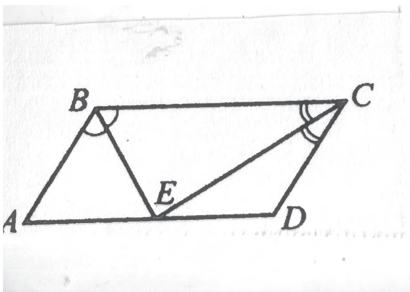
8. Solve the equation

$$24\tan^2 x - 9\sin^2 x = 2.$$

9. Solve the equation

$$\sqrt{x^2 - 7x + 1} + \sqrt{-x^2 + 11x - 10} = \sqrt{4x - 9}.$$

10. Two bisectors of consecutive interior angles in a parallelogram intersect at a point E, which is placed on the opposite side AD. The greatest side of the parallelogram has length 14. Find the length of the smallest side.



KBTU, ISE, Entrance Exam

10 Questions - 120 minutes

March 30, 2013

Name :

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1. Solve the system

$$\begin{cases} \dfrac{\log_x(x+2) - 4\log_{x+2} x}{x(x+2)} \geq 0 \\ 2^x + 2^{-x} \leq 3 \end{cases}$$

2. Evaluate

$$\frac{15\sin 68^\circ}{\sin 34^\circ \cdot \sin 56^\circ}.$$

3. Find all values of a such that the equation

$$\left|\frac{6}{x} + 2\right| = ax - 1$$

has more than two roots.

4. A lorry transports 392 tons of sand increasing the daily norm by one and the same amount. For the first day, the lory transported 2 tons of sand. The job was finished in 16 days. How much did the lorry transport for the 12th day ?

5. The productivity of the first pipe to a basin is 15 liters per minute less than of the second. What is the productivity of the second pipe if it can fill a basin of volume 300 liters 18 minutes faster than the first ?

6. A company plans the 10% growth of its production in 2013. Evaluate the increase (in percents) of companies's profit in 2013 compared with 2012 if its product price increased in 2013 by 15% and the cost of production, which in 2012 was 3/4 of the sale price, increased in 2013 by 20%.

7. All pairs of lateral sides of a triangular pyramid are perpendicular at its vertex. Their lengths are equal 6. Find the volume of the pyramid.

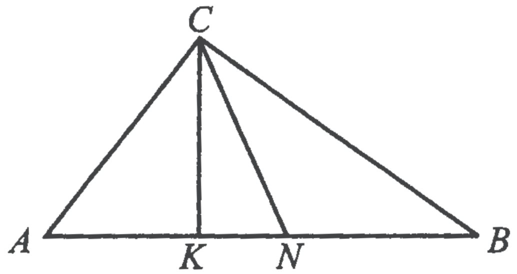
8. Solve the equation

$$\frac{1-\cos^2 2x}{\tan x} + 2\cos^2 x \cdot \tan x = 3\cos x \cdot \sin 2x.$$

9. Solve the equation

$$\sqrt{-x^2+15x-14} + \sqrt{x^2-3x-1} = \sqrt{12x-15}.$$

10. The acute angles in a right triangle are 34° and 56°. Evaluate the angle between the median and the height of the triangle to the hypothenuse.



KBTU, ISE, Entrance Exam in Taldykorgan

10 Questions - 120 minutes

April 13, 2013

Name:

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1. Solve the inequality

$$\log_{1-x}(1+x-2x^2)+\frac{1}{4}\log_{1+2x}(x^2-2x+1)^2 \geq -1.$$

2. Evaluate

$$\frac{6\sin 37^\circ \cdot \sin 53^\circ}{\sin 74^\circ}.$$

3. Find all values of a such that the minimal value of

$$f(x) = 2ax + 5 + |x^2 + 6x + 5|$$

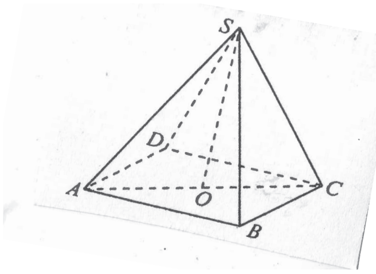
is smaller than 1.

4. A forestry farm sold in a nearby city Christmas trees. Every day, the number of sold trees increased by 200%. How many days did the farm supply trees if on the second day it sent to the city 12 trees, and on the last day 2916 trees?

5. An experienced worker produces 40 widgets two hours faster than a younger worker makes 30 widgets. How long will it take for them both to produce 120 widgets if the first worker for one hour makes five widgets more than the second one?

6. An antique shop sold a cigar-case and a statuette with 40% profit. It is known that the cigar-case was sold with 35%, whereas the statuette with 60% profit. In how many times did the shop pay more for the cigar-case than for the statuette?

7. The diagonal AC of the base of the right square pyramid $SABCD$ equals 24. The length of a lateral edge is 13. Evaluate the height SO.



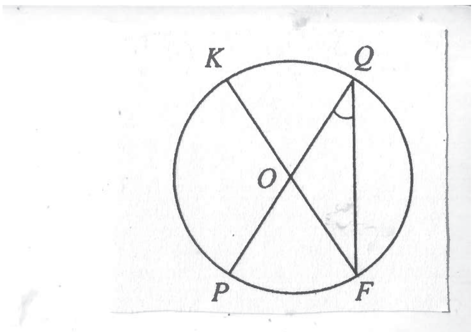
8. Solve the equation

$$8 - 4\sin^2 x = \sin 2x \cot x - 9\cos x.$$

9. Solve the equation

$$(2x-5)\sqrt{2x^2-9x+4} + 10 = 4x.$$

10. PQ and KF are diameters of a circle centered at O. The angle $\angle PQF$ equals $42°$. Evaluate the angle $\angle KOP$.



KBTU, ISE, Entrance Exam
10 Questions - 120 minutes

April 20, 2013

Name:

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1. Solve the inequality

$$\frac{1}{2}\log_{4+x}(x^2+2x+1)+\log_{-x-1}(-x^2-5x-4)\leq 3.$$

2. Evaluate

$$1-(\cos 45^\circ \cdot \cos 15^\circ - \sin 15^\circ \cdot \sin 45^\circ)^2.$$

3. Find all solutions to the equation

$$k\log_3 x^2 + 3k\log_x 3 + \log_x 9 = 2k+8$$

for those values of the parameter k for which the maximal solution is three times more than the minimal solution.

4. A positive integer x satisfies the equation

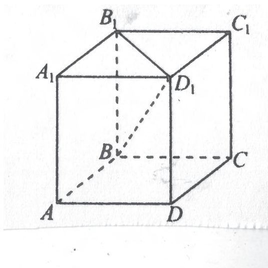
$$\frac{x-1}{x}+\frac{x-2}{x}+\cdots+\frac{1}{x}=3.$$

Find x.

5. The first worker makes 84 details three hours faster than the second. How many details does the second worker produce per hour if another worker makes nine details more for the same time?

6. Alexandra wants to take a loan in a bank under 20% of annual interest rate for two years. The installment (partial repayment) she can pay to the bank at the end of each year is bounded by 90000 KZT. Find the maximal sum which Alexandra can borrow at the bank.

7. In a rectangular parallelepiped $ABCDA_1B_1C_1D_1$, the lengths of segments are $BD_1 = 27$, $BB_1 = 2\sqrt{26}$. $AB = 24$. Find the length of AD.



8. Solve the equation

$$\sqrt{1 + \frac{1}{\cos x}} = \tan x.$$

9. Solve the equation

$$(x+1)\sqrt{3x^2 + 17x + 10} - 4 = 4x.$$

10. A vertex C of an equilateral triangle $\triangle ABC$ is connected to a point P of the opposite side AB by a line segment. The point P divides AB in the ratio 3:5. The area of the smaller disc among of two inscribed in $\triangle APC$ and $\triangle PBC$ is 36. Find the area of the second disc.

KBTU, ISE, Entrance Exam

10 Questions - 120 minutes

May 18, 2013

Name:

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1. Solve the inequality

$$\frac{1+\log_{x+2}(x-2)}{1+\log_{x+2}^2(x-2)} \cdot \big(\log_{x+2}(x-2)+\log_{x-2}(x+2)\big) \geq \log_{x^2-4}(x^2+4x+4).$$

2. Evaluate

$$\left(\sqrt{1+\sqrt{6}}-\sqrt{\sqrt{150}+\sqrt{25}}\right)\cdot\sqrt{\sqrt{6}-1}+1-\sqrt{5}.$$

3. Find all values of the parameter a such that the system

$$\begin{cases} x^2+y^2=a, \\ y+a=|x+1| \end{cases}$$

has exactly one solution.

4. The sum of the first 50 elements of an arithmetic progression equals its 99-th element. Find the element in this progression which equals zero.

5. A boat has passed 10 km upstream of a river, and then 45 km downstream in two hours. The speed of the currents in the river is five km per hour. Find boat's speed.

6. A printing-house spends 1250000 KZT per month for lease, utility payments and other permanent expenditures. Per each book, the company spends extra 250 KZT plus 1% of the total number of printed books in KZT for transportation and store-keeping. Book stores pay 750 KZT to the company per book. Evaluate the maximal monthly profit of the printing-house.

7. The volume of a right circular cone is 100π and its height is 12. Find the ratio of the lateral surface area to the area of the base of this cone.

8. Find $\sin 2x$ if x is a solution to the equation

$$\cos x + 1 = \frac{2}{\cos x}.$$

9. Solve the equation

$$3x + 2\sqrt{2x^2 + 3x - 5} = 12.$$

10. In a trapezoid $ABCD$ the bases AB and DC have lengths 8 and $5,5$. The diagonals AC and BD intersect at a point O and make a direct angle. Evaluate

$$AO \cdot OC + BO \cdot OD.$$

KBTU, ISE, Entrance Exam

10 Questions - 120 minutes

June 29, 2013

Name:

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1. Solve the equation:

$$3x - 4x^3 = (1 - 4x^2)\sqrt{1 - x^2}.$$

2. Evaluate

$$3^{\log_3 2} - \sqrt{0.09} + 3\log_9 \frac{1}{3\sqrt[3]{3}}.$$

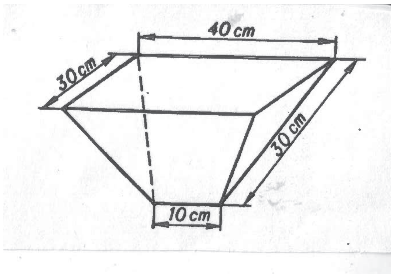
3. Find all values $x > 2$ such that the greatest of two numbers $a = 2\log_x 27 - \log_3(81x)$ and $b = \log_3^2 x^3 - 28\log_3 x$ is bigger than -3.

4. Fresh figs contain 70% of water and dried figs 3.4% of water. How many kilos of fresh figs are required to obtain 10 kilos of dried figs?

5. An apple farmer can produce 600 apples from each of his apple trees if no more than 20 trees are planted. If he plants more than 20 trees, the yield per tree will decrease. In fact, he figures out that for each extra tree he plants, his yield per tree will decrease by 15 apples. How many trees he should plant in order to obtain the maximal number of apples?

6. A manufacturer can produce tires at a cost of $20 apiece. It is estimated that if the tires are sold for p dollars apiece, consumers will buy $1\,560 - 12p$ of them each month. How many tires will be sold each month at the optimal price maximizing the profit of the manufacturer?

7. Determine the capacity of the plastic container whose shape and dimensions are shown on the figure below.



8. Solve the equation

$$4\sin^3 x - (\sin x + \cos x) = 0.$$

9. Solve the system of equations:

$$\begin{cases} \log_{0.5}\left(\dfrac{y}{x}\right) + \log_2(y+1) = \log_2 3 \\ 4^{(x+y)} \cdot 4^y - 2^{y^2} = \log_{0.2} 1. \end{cases}$$

10. The diagonals of a parallelogram $ABCD$ are perpendicular, its acute angle $\angle A$ is $60°$. Evaluate the distance of the center of the circumscribed circle about the triangle ΔABC to the center of the circle inscribed into $ABCD$ if $AB = 2\sqrt{3}$.

KBTU, ISE, Entrance Exam

10 Questions - 120 minutes

July 27, 2013

Name:

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1. For every real number a solve the equation:

$$\sqrt{x^2 - 7ax + 10a^2} - \sqrt{x^2 + ax - 6a^2} = x - 2a.$$

2. Evaluate

$$\sqrt{0.04} + \log_4(2\sqrt{2}) + 2^{\log_2 3}.$$

3. Solve the inequality

$$\frac{1}{1 + \log_3 x} + \log_x 27 \geq \frac{11}{6}.$$

4. If a ship and boat move downstream, the ship arrives from A to B one and a half times faster than the boat. In this case, the boat lags behind the ship on 8 km every hour. If they move upstream, then the ship makes the whole distance from B to A twice as fast as the boat. Find the speeds of the ship and boat in still water.

5. A hungry hunter came upon two shepherds, one of whom had 3 small loaves of bread, and the other 5, all of the same size. The loaves were divided equally among the three, and the hunter paid 8 cents for his share. How should the shepherds divide the money?

6. A store bought apples at wholesale price. If the price for one kilo would be $\$\,2.8$, the store would have profit equal the 1/6th of the total sum paid by the store for apples. If the price were $ 2.2 per kilogram, the store would have lost $ 300. How many kilograms of apples did the store buy?

7. Drawn from the vertex A of a cube are diagonals of the faces containing the point A. The end-points B, C and D of the diagonals are jointed to one another. Find the volume of the tetrahedron $ABCD$ if the edge of the cube is a.

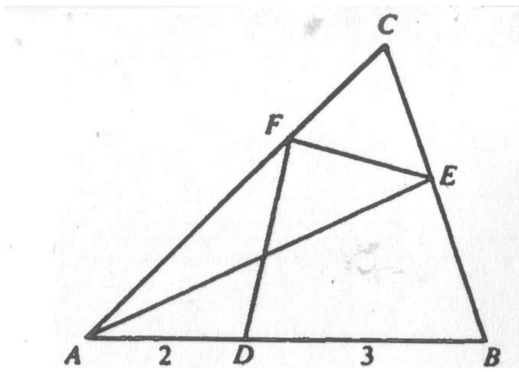
8. Solve the equation

$$1 - \sin^4 x - \frac{5}{3}\cos^4 x = 0.$$

9. Solve the system of inequalities:

$$\begin{cases} 5^x + 5^{x+1} \le 36 \\ \sqrt{9 - x^2}\,\log_x 2 > 0. \end{cases}$$

10. Triangle $\triangle ABC$ in the figure below has area 10. Points D, E and F, all distinct from A, B and C, are on the sides AB, BC and CA respectively, and $AD = 2$, $DB = 3$. If triangle $\triangle ABE$ and quadrilateral $DBEF$ have equal areas then what is this area?



Part III
Solutions

Problem 1

Sergey Khrushchev, Math Professor of ISE

Problem 1 (February 23, 2013) Solve the system

$$\begin{cases} \dfrac{2^{x+1}+31}{32-2^x} \geq 1 \\ \log_{(x-1)^2}(x+5) \leq 1 \end{cases}$$

Solution: First we determine the domain of the system, i.e. the set of all x such that all expressions of the system make sense.

The logarithm $\log_a(x+5)$ is defined for $\boxed{x > -5}$, its base $a = (x-1)^2 \neq 1$, i.e. $\boxed{x \neq 0}$ and $\boxed{x \neq 2}$. Also $a \neq 0$, equivalently $x \neq 1$.

Thus the domain of the system is:

$$D = (-5; 0) \cup (0; 1) \cup (1; 2) \cup (2, +\infty).$$

Since $2^{x+1} > 0$, we see that the numerator $2^{x+1} + 31 > 0$. Therefore

$$\frac{2^{x+1}+31}{32-2^x} \geq 1 \Rightarrow 2^x < 32 = 2^5 \Leftrightarrow x < 5.$$

We see that the solutions must be in the interval $(-5, 5)$.

The inequality

$$\frac{2^{x+1}+31}{32-2^x} \geq 1$$

is equivalent to

$$2^{x+1} + 31 \geq 32 - 2^x \Leftrightarrow 3 \cdot 2^x \geq 1 \Leftrightarrow x \geq -\log_2 3.$$

We see that the solutions to the first inequality are exhausted by the numbers in the segment

$$\boxed{[\log_2 3; 5)}.$$

To solve the second inequality we consider two cases.

Case 1 $0 < |x-1| < 1$. Then by the properties of the logarithm the second inequality is equivalent to

$$x+5 = [(x-1)^2]^{\log_{(x-1)^2}(x+5} \geq (x-1)^2 \Leftrightarrow$$
$$x+5 \geq x^2-2x+1 \Leftrightarrow x^2-3x-4 \leq 0 \Leftrightarrow (x+1)(x-4) \leq 0 \Leftrightarrow -1 \leq x \leq 4.$$

Together with the assumption $0 < |x-1| < 1$ this results in two intervals:

$$\boxed{(0;1) \cup (1;2)}.$$

Case 1 $|x-1| > 1$. Then

$$x+5 = [(x-1)^2]^{\log_{(x-1)^2}(x+5} \leq (x-1)^2 \Leftrightarrow 0 \leq (x+1)(x-4) \Leftrightarrow$$
$$x \in (-\infty; -1] \cup [4; +\infty).$$

We have

$$-\log_2 3 = \log_2 \frac{1}{3} < \log_2 \frac{1}{2} = -1.$$

Combining these results with the solutions to the first inequality we get

Answer: $\boxed{[-\log_2 3; -1) \cup (0;1) \cup (1;2) \cup [4,5)}$.

Problem 1 (March 10, 2013) Find the number of integer solutions to the inequality

$$\log_{9x^2+24x+17} \frac{8x-13}{(x-1)^2(x^2-3x-4)} + \log_{9x^2+24x+17}(x^2-5x+4) \geq 0.$$

Solutions: First we determine the values of x for which all terms involved in the inequality make sense. Let us consider the base of the logarithms:

$$9x^2+24x+17 = (3x)^2 + 2\cdot 4\cdot(3x) + 4^2 + 1 = (3x+4)^2 + 1 \geq 1.$$

since the base of logarithms cannot be equal 1, we obtain the first restriction

$$\boxed{x \neq -\frac{4}{3}}.$$

Now we study the expressions under the sign of logarithms:

$$x^2-3x-4 = (x+1)(x-4), \quad x^2-5x+4 = (x-1)(x-4).$$

Since the argument of any logarithm must be positive, we obtain the second restriction:

$$x^2 - 5x + 4 = (x-1)(x-4) > 0 \Leftrightarrow \boxed{x \in (-\infty; 1) \cup (4; +\infty)}.$$

Similarly, the consideration of the argument of the first logarithm results in the formula:

$$\frac{8x - 13}{(x-1)^2(x^2 - 3x - 4)} = \frac{8(x+1)(x-3/8)(x-4)}{(x-1)^2(x2 - 3x - 4)^2)}.$$

Applying the standard method of intervals, we obtain the following restriction:

$$\boxed{x \in (-1; 3/8) \cup (4; +\infty)}.$$

Notice that

$$(-1; 3/8) \subset (-\infty; 1).$$

Now we can apply the addition formula for logarithms:

$$\log_{9x^2+24x+17} \frac{(8x-13)(x-1)(x-4)}{(x-1)^2(x+1)(x-4)} \geq 0.$$

Since $9x^2 + 24x + 17 > 1$ on the domain of the inequality, the last inequality is equivalent to

$$\frac{8x - 13}{x^2 - 1} \geq 1.$$

There is only one integer $x = 0$ in the interval $(-1; 8/3)$ and $x = 0$ obviously satisfies the inequality.

If $x \in (4; +\infty)$, then $x^2 - 1 > 0$ and the inequality is equivalent to

$$8x - 13 \geq x^2 - 1 \Leftrightarrow x^2 - 8x + 12 \leq 0 \Leftrightarrow (x-2)(x-6) \leq 0$$

Therefore the interval $(4; +\infty)$ reduces to $(4; 6]$. The last interval contains two integer solutions $x = 5$ and $x = 6$.

Answer: $\boxed{\mathbf{3}}$.

Problem 1 (April 13, 2013) Solve the inequality

$$\log_{1-x}(1+x-2x^2)+\frac{1}{4}\log_{1+2x}(x^2-2x+1)^2 \geq -1.$$

Solution: The bases $1-x$ and $1+2x$ must be positive and not equal 1. hence

$$\boxed{x \in (-0.5;0)\cup(0;1)}.$$

We observe that

$$(1+x-2x^2)=(1+2x)(1-x)>0$$

on the domain specified above. We have

$$\log_{1-x}(1+x-2x^2)=\log_{1-x}(1+2x)+\log_{1-x}(1-x)=\log_{1-x}(1+2x)+1,$$

$$\frac{1}{4}\log_{1+2x}(x^2-2x+1)^2=\frac{1}{4}\log_{1+2x}(1-x)^4=\log_{1+2x}(1-x).$$

It follows that the inequality is equivalent to

$$\log_{1-x}(1+2x)+1+\log_{1+2x}(1-x)\geq -1.$$

Let $z=\log_{1-x}(1+2x)$. Then

$$\log_{1+2x}(1-x)=\frac{1}{\log_{1-x}(1+2x)}=\frac{1}{z}.$$

We obtain the inequality

$$z+2+\frac{1}{z}\geq 0 \Leftrightarrow \frac{(z+1)^2}{z}\geq 0.$$

We see that $z=-1$ is a solution. Then

$$\log_{1-x}(1+2x)=-1 \Leftrightarrow (1-x)(1+2x)=1 \Leftrightarrow x(1-2x)=0 \Rightarrow \mathbf{x}=\mathbf{0.5}.$$

It remains to solve

$$\log_{1-x}(1+2x)>0$$

on the domain $(-0.5;0)\cup(0;1)$.

Case 1 $x\in(-0.5;0)$. Then $1-x>1$ and $2x+1\in(0;1)$ implying that the logarithm is negative. There are no solutions in this case.

Case 2 $x\in(0;1)$. Then $1-x\in(0,1)$ and $1+2x>1$ implying that the logarithm is negative. Tere is no solutions in this case also.

Answer: $\boxed{\mathbf{x}=\mathbf{0.5}}$.

Problem 3.

Sergey Khrushchev, Math Professor of ISE

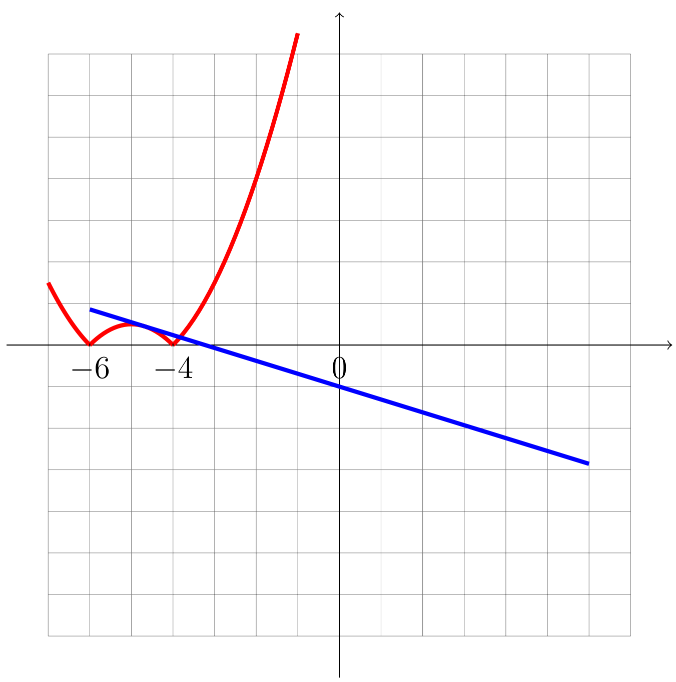
Problem 3. (February 23, 2013). Find all values of a such that the equation

$$ax - 1 = |x^2 + 10x + 24|$$

has two roots.

Solution: To plot the graph of $y = |x^2 + 10x + 24|$ we observe that by the properties of the modulus this function can be defined by two formulas:

$$y = \begin{cases} x^2 + 10x + 24 & \text{if } x \text{ is not in } [-6, -4] \\ -\left(x^2 + 10x + 24\right) & \text{if } x \text{ is in } [-6, -4] \end{cases}$$



The blue line on the picture is the tangent to

$$y = -(x^2 + 10x + 24),$$

passing through the point $(0, -1)$. Applying Descartes' Rule of tangents we find its equation:

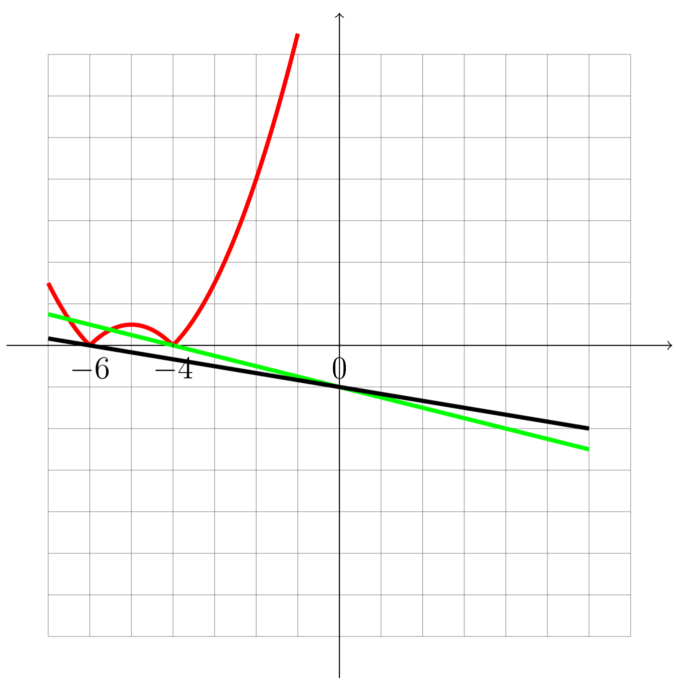
$$y = (-10 + 2\sqrt{23})x - 1.$$

This line intersects the graph of $y = |x^2 + 10x + 24|$ at three points implying that the equation

$$ax - 1 = |x^2 + 10x + 24|$$

has three solutions. If $a < -10 + 2\sqrt{23}$, the the line $y = ax - 1$ passes above the tangent on the interval $(-\infty, 0)$. Then there will be two solutions and we found the first interval:

$$\boxed{(-\infty; -10 + 2\sqrt{23})}.$$



The slope of the green line is:

$$\frac{\Delta y}{\Delta x} = \frac{(-1) - 0}{0 - (-4)} = \boxed{-\frac{1}{4}}.$$

The slope of the black line is:

$$\frac{\Delta y}{\Delta x} = \frac{(-1) - 0}{0 - (-6)} = \boxed{-\frac{1}{6}}.$$

It follows that any line through $(0, -1)$ with the slope from the interval

$$\boxed{\left(-\frac{1}{4}; -\frac{1}{6}\right)}$$

crosses the graph of $y = |x^2 + 10x + 24|$ at two points.

If $a = 0$, then we obtain the equation

$$-1 = |x^2 + 10x + 24|,$$

which has no solutions.

It remains to find the tangent $y = ax - 1$ to

$$y = x^2 + 10x + 24$$

with positive slope $a > 0$. We apply Descartes' Rule of Tangents:

$$ax - 1 = x^2 + 10x + 24 \Leftrightarrow x^2 + (10 - a)x + 25 = 0.$$

The discriminant of this quadratic equation is:

$$\mathbf{D} = (10 - a)^2 - 100.$$

It is zero if and only if

$$10 - a = \pm 10 \Leftrightarrow a = \begin{cases} 0; \\ 20. \end{cases}$$

The required solution is $a = 20$. Every line through $(0, -1)$ with positive slope greater than 20 intersects the parabola exactly at two points. Hence the third interval is

$$\boxed{(20; +\infty).}$$

Answer: $(-\infty; -10 + 2\sqrt{23}) \cup (-1/4; -1/6) \cup (20; +\infty)$.

Problem 3 (March 10, 213) Find all values of a real parameter a such that the minimal value of

$$f(x) = -x^2 + 2x + a^2$$

on the segment $[-1, 0]$ is smaller or equal 1.

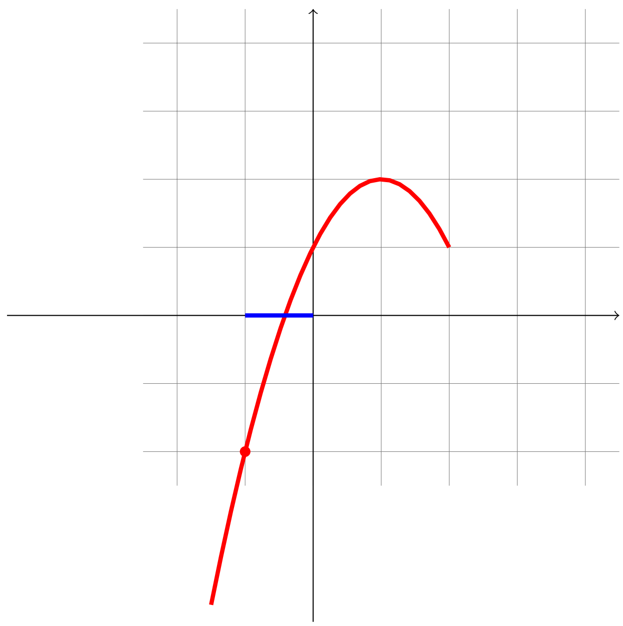
Solution: First we write:

$$f(x) = -x^2 + 2x - 1 + 1 + a^2 = -(x-1)^2 + 1 + a^2.$$

The parabola is directed downwards and its vertex has coordinates:

$$x = 1, \quad y = 1 + a^2.$$

It follows that the function $y = f(x)$ increases on the interval $[-1, 0]$.



Hence

$$\min_{-1 \le x \le 0} f(x) = f(-1) = -3 + a^2.$$

Then

$$-3 + a^2 \le 1 \Leftrightarrow a^2 \le 4.$$

Answer: $\boxed{-2 \le a \le 2}$.

Problem 3. (April 20, 2013) Find all solutions to the equation

$$k \log_3 x^2 + 3k \log_x 3 + \log_x 9 = 2k + 8$$

for those values of the parameter k for which the maximal solution is three times more than the minimal solution.

Solution: By the definition of logarithm $x > 0$. By the properties of logarithms

$$\log_3 x^2 = 2\log_3 x; \ \log_x 9 = 2\log_x 3; \ \log_x 3 = \frac{1}{\log_3 x}.$$

We can rewrite the equation as follows:

$$2k(\log_3 x) + \frac{3k}{\log_3 x} + \frac{2}{\log_3 x} = 2k + 8.$$

Let $z = \log_3 x$. Then

$$2kz + \frac{3k}{z} + \frac{2}{z} = 2k + 8,$$

$$2kz^2 - 2(k+4)z + 3k + 2 = 0.$$

This equation is quadratic and therefore has two roots

$$z_1 = \log_3 x_1, \quad z_2 = \log_3 x_2,$$

if any. Then

$$x_2 = 3x_1 \Leftrightarrow z_2 = \log_3(3x_1) = \log_3 3 + \log_3 x_1.$$

Equivalently

$$z_2 - z_1 = 1.$$

Case 1. $k > 0$.

$$z_2 = \frac{(k+4) + \sqrt{(k+4)^2 - 2k(3k+2)}}{2k}$$

$$z_1 = \frac{(k+4) - \sqrt{(k+4)^2 - 2k(3k+2)}}{2k}.$$

$$z_2 - z_1 = \frac{\sqrt{(k+4)^2 - 2k(3k+2)}}{k} = 1$$

Squaring the last equation, we obtain

$$(k+4)^2 - 2k(3k+2) = k^2 \Leftrightarrow 3k^2 - 2k - 8 = 0.$$

Then

$$k_{1,2} = \frac{1 \pm \sqrt{1+3\cdot 8}}{3} = \begin{cases} 2 \\ -\dfrac{4}{3} \end{cases}$$

Since $k > 0$ we obtain only one solution $\boxed{k=2}$. Then

$$(k+4)^2 - 2k(3k+2) = k^2\Big|_{k=2} = 2^2 = 4.$$

It follows that

$$z_2 = \frac{6+2}{4} = 2, \quad z_1 = \frac{6-2}{4} = 1.$$

Hence

$$\boxed{x_1 = 3^{z_1} = 3, \quad x_2 = 3^{z_2} = 9}.$$

Case 2. $k < 0$.

$$z_1 = \frac{(k+4) + \sqrt{(k+4)^2 - 2k(3k+2)}}{2k}$$
$$z_2 = \frac{(k+4) - \sqrt{(k+4)^2 - 2k(3k+2)}}{2k}.$$

$$z_2 - z_1 = \frac{\sqrt{(k+4)^2 - 2k(3k+2)}}{k} = 1$$

We already solved this equation. Since now $k < 0$, we find that

$$\boxed{k = -\frac{4}{3}}$$

Then

$$(k+4)^2 - 2k(3k+2) = k^2\Big|_{k=-4/3} = \frac{16}{9},$$

$$z_1 = \frac{(4-4/3)+4/3}{-8/3} = -\frac{3}{2}.$$

$$z_2 = \frac{(4-4/3)-4/3}{-8/3} = -\frac{1}{2}.$$

Hence

$$\boxed{x_1 = 3^{z_1} = \frac{\sqrt{3}}{9}, \quad x_2 = 3^{z_2} = \frac{\sqrt{3}}{3}}.$$

Case 3. $k = 0$. Then there is only one solution.

Problem 4

Sergey Khrushchev, Math Professor of ISE

Problem 4 (March 10, 2013) To prepare to entrance exams in mathematics Paul solved problems from his textbook for 30 days. To make a good progress he increased daily the number of solved problems by one and the same number. At the end, it turned out that for the initial twenty days he solved as many problems as for the final ten. By how many percents does the number of problems solved for the last fifteen days exceed the number of problems solved for the first fifteen days?

Solution:

$$\begin{gathered}
\text{1st day} a_1 \text{ problems} \\
\text{2nd day} a_1 + d \text{ problems} \\
\text{3rd day} a_1 + 2d \text{ problems} \\
\dots\dots\dots\dots \\
\text{20th day} a_1 + 19d \text{ problems} \\
\text{21st day} a_1 + 20d \text{ problems} \\
\dots\dots\dots\dots \\
\text{30th day} a_1 + 29d \text{ problems.}
\end{gathered}$$

For the first 20 days Paul solved

$$a_1 + \cdots + a_20 = \frac{a_1 + a_{20}}{2} \cdot 20 = 10(2a_1 + 19d)$$

problems. For the last 10 days he solved

$$a_{21} + \cdots + a_{30} = \frac{a_{21} + a_{30}}{2} \cdots 10 = 5(2a_1 + 49d)$$

problems. The equation of balance is:

$$2(2a_1 + 19d) = (2a_1 + 49d) \Rightarrow \boxed{a_1 = \frac{11}{2}d}.$$

Now

$$\frac{a_{16}+a_{17}+\cdots+a_{30}}{a_1+a_2+\cdots+a_{15}} = \frac{a_{16}+a_{30}}{a_1+a_{15}} = \frac{a_1+15d+a_1+29d}{a_1+a_1+14d} =$$

$$\frac{2a_1+44d}{2a_1+14d} = \frac{11d+44d}{11d+14d} = \frac{55}{25} = \frac{11}{5} = 2.2.$$

Answer: $\boxed{\mathbf{(2.2-1)\times 100\% = 120\%}}$.

Problem 4 (June 29, 2013) Fresh figs contain 70% of water and dried figs 3.4% of water. How many kilos of fresh figs are required to obtain 10 kilos of dried figs?

Solution: Ten kilograms of dried figs contain

$$10 \times 0.034\text{kg} = 340\text{gr}$$

of water. Hence the mass of absolutely dry figs is

$$10\,000 - 340 = 9660\text{gr}.$$

Now

$$\text{Fresh figs} = \underbrace{70\%}_{\text{of water}} + \underbrace{30\%}_{\text{absolutely dry substance}}.$$

Then 9.66 kg is 30% of fresh figs:

$$\frac{9.66}{0.3} = \frac{3.22}{0.1} = \mathbf{32.2kg}.$$

Answer: $\boxed{32.2\text{kg}}$.

Problem 5

Sergey Khrushchev, Math Professor of ISE

Problem 5 (June 29, 2013) An apple farmer can produce 600 apples from each of his apple trees if no more than 20 trees are planted. If he plants more than 20 trees, the yield per tree will decrease. In fact, he figures out that for each extra tree he plants, his yield per tree will decrease by 15 apples. How many trees he should plant in order to obtain the maximal number of apples?

Solution: Let us make first steps:

$$\begin{array}{cc}
0\,\text{trees added} & \text{Harvest} = 600 \times 20 \\
1\,\text{tree added} & \text{Harvest} = (600 - 15) \times (20 + 1) \\
2\,\text{trees added} & \text{Harvest} = (600 - 2 \cdot 15) \times (20 + 2) \\
3\,\text{trees added} & \text{Harvest} = (600 - 3 \cdot 15) \times (20 + 3) \\
\dots & \\
x\,\text{trees added} & \text{Harvest} = (600 - x \cdot 15) \times (20 + x)
\end{array}$$

It follows that we must find the maximal value of the function

$$H(x) = (600 - x \cdot 15) \times (20 + x) = -15(x + 20)(x - 40).$$

The x-coordinate of this parabola directed downwards equals

$$\frac{40 - 20}{2} = \mathbf{10}.$$

Answer: $\boxed{\mathbf{20 + 10 = 30}}$.

Problem 6.

Sergey Khrushchev, Math Professor of ISE

In this section solutions to economical problems are presented. As it is common for all text problem one should first eliminate all words and state the problem in terms of equations. To do this unknown variables must be properly denoted so that one look on the letter of the corresponding variable explained it meaning.

Problem 6. (February 23, 2013) A cell company extended its network and simultaneously reduced the monthly tariff on its services 50%. The cost of extension was the half of the total revenue of the company for the previous year. As result the monthly volume of sales increased three times. In how many months does the additional profit obtained by this cell company compensate its investment on the extension?

Solution: We can easily find the revenue R of the company for the previous year if we denote by q the quantity of customers for the previous year and by p (p stands for "price") the monthly payments of customers. Then

$$R = \boxed{12pq}.$$

After the extension of the network the tariff was reduced 50%. Therefore the new tariff is:

$$p \to p - 0.5p = \boxed{\mathbf{0.5p}}.$$

The number of customers increased three times:

$$q \to \boxed{\mathbf{3q}}.$$

Using this notations we can write the formula for the size of the investment I made:

$$I = \frac{1}{2}R = \frac{1}{2}12pq = \boxed{\mathbf{6pq}}.$$

Te formula for the monthly additional profit obtained by the company in view of its extension policy is given by

$$\underbrace{0.5p}_{\textbf{new tariff}} \times \underbrace{3q}_{\textbf{new customers}} - \underbrace{pq}_{\textbf{old monthly revenue}} = 1.5pq - pq = \boxed{\mathbf{0.5pq}}.$$

Let x be the number of months required to compensate the investment $I = 6pq$ by the additional profit. Then

$$6pq = x \cdot 0.5pq \Rightarrow x = \frac{6}{0.5} = \boxed{\mathbf{12}}.$$

Answer: 12.

Problem 6 (March 10, 2013). A monopoly can sell q thousand units of its production on the market for $p = 8 - 0.05q$ dollars per unit every month. Its trading policy is to keep the price p on the product as high as possible but still to get revenue not less than \$300 000. What price satisfies the stratage of the monopoly?

Solution: keeping the same notation as in the first problem we have:

$$R = pq.$$

Since we must determine the optimal price p, we express q in terms of p using the equation $p = 8 - 0.05q$:

$$q = \frac{8-p}{0.05} = 20(8-p).$$

It follows that

$$R = pq = 20p(8-p) \text{ in thousand dollars!}$$

Now the policy of the monopoly can be described by the inequality:

$$160p - 20p^2 \geq 300.$$

This inequality is equivqlent to

$$p^2 - 8p + 15 \leq 0 \Leftrightarrow (p-3)(p-5) \leq 0 \Leftrightarrow 3 \leq p \leq 5.$$

Answer: $\boxed{\mathbf{p = 5}}$.

Problem 6 (March 30, 2013). A company plans the 10% growth of its production in 2013. Evaluate the increase (in percents) of company's profit in 2013 compared with 2012 if its product price increased in 2013 by 15% and the cost of production, which in 2012 was 3/4 of the sale price, increased in 2013 by 20%.

Solution: Let
q be the level of production in 2012;
p the price per unit in 2012;
C the cost of production in 2012 of one unit.
Then in 2013:

$$q \longrightarrow q + 10\%q = \mathbf{1.1q};$$

$$p \longrightarrow p + 15\%p = \mathbf{1.15p};$$

$$C = \frac{3}{4}p \longrightarrow \frac{3}{4}p + 20\%\frac{3}{4}p = \mathbf{1.2\frac{3}{4}p}.$$

The profit Π_{2012} for 2012 is given by

$$\Pi_{2012} = (p - C)q = \left(p - \frac{3}{4}p\right)p = \frac{\mathbf{pq}}{\mathbf{4}}.$$

It follows that

$$\Pi_{2013} = \left(1.15p - 1.2\frac{3}{4}p\right)(1.1q) = (1.15 - 3 \cdot 0.3) \cdot (1.1)pq =$$
$$(1.15 - 0.9) \cdot (1.1)pq = (1.1) \cdot (0.25)pq = 1.1\frac{pq}{4} = 1.1 \cdot \Pi_{2012}.$$

Hence the profit in 2013 increased by

$$\left(\Pi_{2013} - \Pi_{2012}\right) \cdot 100\% = 10\%\Pi_{2012}.$$

Answer: 10%.

Problem 6 (April 13, 2013). An antique shop sold a cigar-case and a statuette with 40% profit. It is known that the cigar-case was sold with 35%, whereas the statuette with 60% profit. In how many times did the shop pay more for the cigar-case than for the statuette?

Solution: Let
p_c be the price paid by the antique shop for the cigar-case;
P_c the price for which the cigar-case was sold;
p_s be the price paid by the antique shop for the statuette;
P_c the price for which the statuette was sold.
Then the equation of the profit for two items is given by:

$$P_c + P_s = 1.4(p_c + p_s).$$

The equations of the profit for the cigar-case and the statuette:

$$P_c = 1.35p_c, \quad P_s = 1.6p_s.$$

Substituting these expressions into the main equation, we obtain:

$$1.35p_c + 1.6p_s = 1.4p_c + 1.4p_s.$$

It follows that

$$(1.6 - 1.4)p_s = (1.4 - 1.35)p_c \Rightarrow \mathbf{0.2p_s = 0.05p_c}.$$

Hence

$$\frac{p_c}{p_s} = \frac{0.2}{0.05} = \mathbf{4}.$$

Answer: $\boxed{\mathbf{4}}$.

Problem 6 (April 13, 2013). Alexandra wants to take a loan in a bank under 20% of annual interest rate for two years. The installment (partial repayment) she can pay to the bank at the end of each year is bounded by 90 000 KZT. Find the maximal sum, which Alexandra can borrow in the bank.

Solution: Let x be the maximal size of Alexandra's loan in hundred thousands KZT. Then Alexandra's debt to the bank at the end of the first year is

$$1.2x - 0.9.$$

By the end of the second year the debt must be zero:

$$(1.2x - 0.9) \cdot 1.2 - 0.9 = 0.$$

It follows that

$$1.44x = 0.9 \cdot 2.2 \Rightarrow x = \frac{1.98}{1.44} = \frac{198}{144} = \frac{11}{8} = 1.375.$$

Answer: $\boxed{\mathbf{100\,000 \times 1.375 = 137\,500\,KZT}}$.

Problem 6 (May 18, 2013). A printing-house spends 1 250 000 KZT per month for lease, utility payments and other permanent expenditures. Per each book, the company spends extra 250 KZT plus 1% of the total number printed books in KZT for transportation and store-keeping. Book stores pay 750 KZT to the company per book. Evaluate the maximal monthly profit of the printing-house.

Solution: Let q be the number of books printed. Then the production cost can be represented as follows:

$$C(q) = \underbrace{1\,250\,000}_{\textbf{fixed cost } = C(0)} + \Big(\underbrace{250}_{\textbf{extra production expenditures per book}} + \underbrace{0.01q}_{\textbf{transportation and storing}}\Big) \cdot q = 1\,250\,000 + 250q + 0.01q^2.$$

The revenue is given by

$$R(q) = 750q.$$

Then the profit can be expressed as follows:

$$\Pi(q) = R(q) - C(q) = 750q - (1\,250\,000 + 250q + 0.01q^2) = \\ -0.01q^2 + 500q - 1\,250\,000 = \mathbf{0.01q(50\,000 - q) - 1\,250\,000}.$$

The graph of $y = \Pi(q)$ is the parabola directed downwards. The q-coordinate of its vertex is:

$$\frac{0 + 50\,000}{2} = \mathbf{25\,000}.$$

Therefore the maximal value of the profit is:

$$\Pi(25\,000) = 0.01 \cdot 25\,000 \cdot (50\,000 - 25\,000) - 1\,250\,000 = \\ 250 \times 25\,000 - 1\,250\,000 = 6\,250\,000 - 1\,250\,000 = \mathbf{5\,000\,000}.$$

Answer: $\boxed{\mathbf{5\,000\,000}}$.

Problem 6 (June 29, 2013). A manufacturer can produce tires at a cost of \$20 apiece. It is estimated that if the tires are sold for p dollars apiece, consumers will buy $1560 - 12p$ of them each month. How many tires will be sold each month at the optimal price maximizing the profit of the manufacturer?

Solution: Let q be the number of tires sold. Then $q = 1560 - 12p$ and the production cost is:

$$C(q) = 20q = 20(1560 - 12p).$$

The revenue is:

$$R(q) = pq = p(1560 - 12p).$$

Then the profit can be written as follows

$$\Pi(q) = R(q) - C(q) = p(1560 - 12p) - 20(1560 - 12p) = -12(p - 20)(p - 130).$$

Since the graph of the profit is the parabola directed downwards the profit will be maximal at p defined by

$$p_{\max} = \frac{20 + 130}{2} = \mathbf{75}.$$

Then

$$q_{\max} = 1560 - 12p_{\max} = \mathbf{660}$$

Answer: $\boxed{\mathbf{660}}$.

PROBLEMS TO SOLVE

Problem A. An algebra student has won \$100 000 in a lottery and wishes to deposit it in saving accounts in two financial institutions. One account pays 8% simple interest, but deposits are insured only to \$50 000. The second account pays 6.4% simple interest, and deposits are insured up to \$100 000. Determine whether the money can be deposited so that it is fully insured and earns annual interest of \$7 500.

Problem B. A workman's basic hourly wage is \$10, but he receives one and a half time his hourly rate for any hours worked in excess of 40 per week. If his paycheck for the week is \$595, how many hours of overtime did he work?

Problem C. A man spends 1/3 of his money and loses 2/3 of the remainder. He then has \$12. How much money had he at first?

Problem D. A merchant visited three fairs. At the first he doubled his money and spent \$30, at the second he tripled his money and spent \$54, at the third he quadrupled his money and spent \$72, and then had \$48 left. With how many money did he start?