The Circle Packing Theorem and its Applications

Matthew Tze Ren Lee

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Contents

1 Abstract

In this thesis we prove the circle packing theorem (also known as the the Koebe-Andreev-Thurston theorem). We present preliminary material on Euclidean geometry and graph theory. We then prove the circle packing theorem. In particular, we prove the existence of a circle packing in the plane for any finite planar graph. Two corollaries to the circle packing theorem are presented: Fary's theorem and the Lipton-Tarjan separator theorem. Fary's theorem follows immediately, and we provide a proof of the separator theorem.

2 Acknowledgements

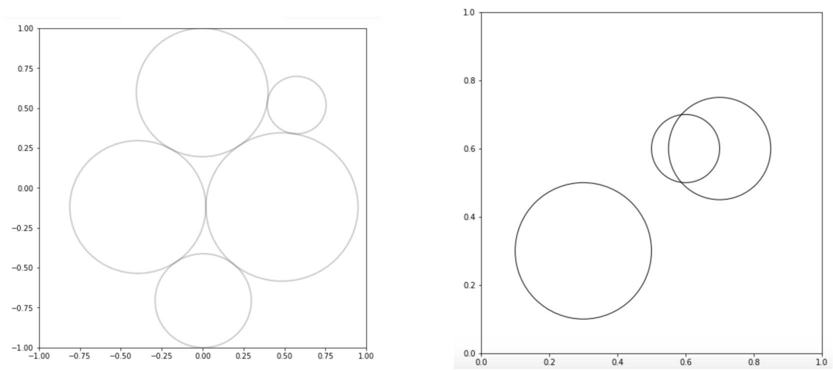
This report would not have been possible without the patience and guidance of Professor Dmitri Panov from King's College London. Without Professor Panov, many of the mathematical ideas and concepts presented here would have remained opaque to me. It has been a pleasure writing this under his guidance. I must also thank my parents for their support, and without whom the opportunity to write this thesis would not be possible.

3 Introduction

We begin with a central definition:

Definition 1. *A **Circle Packing** in the plane is an arrangement of circles such that no two overlap, and some circles are mutually tangent [11]*

The definition of a circle packing in the plane is exactly what we expect; it is a set of circles that 'touch', and without any circles 'separated' from the rest.



(a) A circle packing with five circles

(b) A set of three circles that do not form a packing

Figure 1

In Figure 1 (a), we have a valid circle packing. In Figure 1 (b), the set of three circles fails to be a circle packing since two of the circles overlap.

Note the phrasing '... a set of circles...' in Definition 1. Rigorously, we define a circle in the plane to be a subset of the plane, $\mathbb{R}^2$. For example, a circle of radius 5 centred at the origin would formally be defined as the subset $\{(x, y) : x^2 + y^2 = 5\}$. A circle of radius 5 centred at $(1, 0)$ is the subset $\{(x, y) : (x-1)^2 + y^2 = 5\}$. Clearly these are two different sets, and hence two different circles according to our definition.

By Definition 1, these two circles form two distinct (one element) circle packings. Consider also Figure 1 (a): if every circle was translated to the right by a fixed distance, we would have, by Definition 1, a new circle packing.

Note however we will generally not be concerned about the difference between circle packings that have been translated or rotated in the plane. The fundamental characteristic of a circle packing lies in the tangency relations between the circles, rather than the specific position[1] of the circles[9, page 15]. In other words, circles and circle packings can be identified up to rotational and translational isometries.[2]

We will consider only circle packings in the plane. Note, however, that circle packings can take place in other geometries. Various generalisations of the circle packing theorem in these non-Euclidean geometries exist. For that reason we will briefly introduce these more general settings to establish some background. Note that in this introductory section, there will be terms appearing that are defined in subsequent sections.

3.1 Circle Packing in the Plane, the Riemann Sphere, and in Hyperbolic Geometry

As mentioned, a circle is often defined as a subset of some surface (i.e. a real 2-manifold) that satisfies the fact that every point of the circle is the same 'distance' away from the centre of the circle. Of course, the metrics we use in the Riemann sphere and hyperbolic models are different from that of the Euclidean plane. Nonetheless, such well defined metrics exist, so the notion of circles in these geometries is valid. In particular, definition 1 may be extended to these new geometries by dropping the condition requiring the circles to be in the plane, and ensuring clarity of what a 'circle' is in these geometries.

In particular, it is valid to talk about about circle packings on the sphere or on, say, a Poincare disk (a model of hyperbolic geometry). To this end, it is also valid to ask whether statements about circle packings in one setting hold in another.

3.2 Versions of the Circle Packing Theorem

We now introduce the version of the circle packing theorem we will prove in this paper:

Theorem 2 (The circle packing theorem)**.** *Given any simple connected planar graph, G, with vertex set $V = \{v_1, ...v_n\}$ and edge set E, there exists a circle packing of n circles, $\{C_i\}_{i=1}^n$ in the plane with the property that C_i and C_j touch[3] if and only if $v_iv_j \in E$.*

In this paper, we will prove the circle packing theorem, as prescribed in Theorem 2. However, various versions of the theorem exist in the literature, particularly with generalisations in different geometries.

For example, one variation is as follows: 'Let K be a combinatorial sphere. Then there exists an essentially unique univalent circle packing, P_K for K on the Riemann sphere $\mathbb{P}$' [9]. Historically, E.M. Andreev formulated the statement not on the Riemann sphere or

[1]Here 'position' refers to the coordinates of the circle.

[2]This may fail to be true when considering circle packings within a prescribed boundary, however we do not consider such cases in this paper.

[3]'Touching' here refers to the natural interpretation; two circles touch if and only if they intersect at a single point

the plane, but with regards to convex polyhedra in hyperbolic geometric space [1] .

A far more general form of the statement also exists. The discrete uniformization theorem [9] asserts that for any topological triangulation of some 2-manifold, S, there is such a Riemann surface homeomorphic to S that permits a circle packing in either a Euclidean, hyperbolic, or spherical metric. The theorem also asserts some form of uniqueness of this packing.

While the theory of circle packings in these general settings is both rich and important, we will not consider them further. In particular we leave the more general geometric objects undefined. We note, however, that there is an abundance of high quality and easily accessible literature on these topics.

4 Preliminaries

In this section we cover some preliminary results and definitions that will be useful for proving the circle packing theorem. In particular we cover results about Euclidean geometry, graph theory, and circle packings. We assume familiarity with basic results in topology, analysis, and algebra.

4.1 Euclidean Geometry

We will now introduce some definitions and results about Euclidean geometry. In particular we give rigorous definitions of the geometric objects we work with.

Definition 3. *We take the* ***Euclidean plane*** *to be the set* $\mathbb{R}^2$ *equipped with the dot product. This induces the relevant Euclidean metric and standard topology.*

Whenever we speak of continuity of functions whose domain or codomain is the plane, it will be with respect to the standard topology. It is also helpful to know the plane is a Hausdorff space (in fact all metric spaces).

Definition 4. *A* ***line*** *in the plane is a subset of* $\mathbb{R}^2$ *of the form* $\{\vec{u} + t\vec{v} : t \in \mathbb{R}\}$*, where* $\vec{u}$*,* $\vec{v}$ *are vectors in* $\mathbb{C}$*. A* ***line segment*** *is a subset of the form* $\{\vec{u} + t\vec{v} : t \in [0, 1]\}$*. The vectors* u *and* w *give the endpoints of the line segment.*

The following theorem will allow us to rigorously justify both geometric and graph theoretic notions. For example, we will see in section 4.2 it allows us to deduce that the interior faces of planar graphs are bounded.

Definition 5. *A* ***path*** *is the image of a continuous map* $f : [0, 1] \to \mathbb{R}^2$*.*

A ***Jordan curve*** *is the image of a continuous map* $\phi : [0, 1] \to \mathbb{R}^2$ *such that* $\phi(0) = \phi(1)$*, and* ϕ *is injective over* $[0, 1)$*. A* ***Jordan arc*** *is the image of a continuous, injective map,* $f : [0, 1] \to \mathbb{R}^2$*.*

A Jordan arc is homeomorphic to a line segment, and a Jordan curve is homeomorphic to the circle.

Clearly then a line segment is a Jordan arc. Further, consider any three distinct line segments in the plane such that the first and second segments intersect only at a common endpoint, and the third segment intersects the other two only at the third's endpoints. This is a Jordan curve.

Theorem 6 (The Jordan Curve Theorem)**.** *Let C be a Jordan curve. Then $\mathbb{R}^2 \setminus C$ is the union of two connected components. One component is bounded, while the other is unbounded. The bounded component is called the interior, while the unbounded component is called the exterior. The curve C is the boundary of both components.*

Now we make use of Theorem 6 to define a triangle.

Definition 7. *Consider three distinct line segments in the plane such that the first and second segments intersect only at a common endpoint, and the third segment intersects the other two only at the third's (two) endpoints. A **triangle** is the union of these three lines together with the interior, where the interior is defined as in Theorem 6.*

The following lemma is simple, but crucial for later constructions.

Lemma 8. *Given any three line segments L_1, L_2, L_3 in the plane, there exist three line segments $\bar{L_1}, \bar{L_2}, \bar{L_3}$, with $|L_i| = |\bar{L_i}|$, that form a triangle if and only if $|L_1| + |L_2| > |L_3|$, $|L_1| + |L_3| > |L_2|$, and $|L_2| + |L_3| > |L_1|$.*

Proof. ($\rightarrow$):
This is trivial from the metric properties of $\mathbb{C}$.

($\leftarrow$):
Let L_1, L_2, L_3 be line segments satisfying $|L_1| + |L_2| > |L_3|$, $|L_1| + |L_3| > |L_2|$, and $|L_2| + |L_3| > |L_1|$. Without loss of generality, suppose that the line segment L_1 is the longest. Suppose $\bar{L_1}$ is the line segment with endpoints $(0, 0)$ and $(0, |L_1|)$. Consider the following two circles: a circle of radius $|L_2|$ centred at the origin, and a circle of radius $|L_3|$ centred at $(0, |L_1|)$. Since L_1 is the longest line segment, and since $|L_2| + |L_3| > |L_1|$, we must have that the two circles intersect. But now we have our triangle: take the line segment L_1, and the two line segments $(0, |L_1|)$ to the point of intersection, and $(0, 0)$ to the point of intersection . These are precisely the line segments $\bar{L_1}$ and $\bar{L_2}$ respectively, and we have our triangle formed from the union of $\bar{L_1}$,$\bar{L_2}$, and $\bar{L_3}$. $\square$

The following image captures the idea of the proof of the reverse implication of Lemma 5.

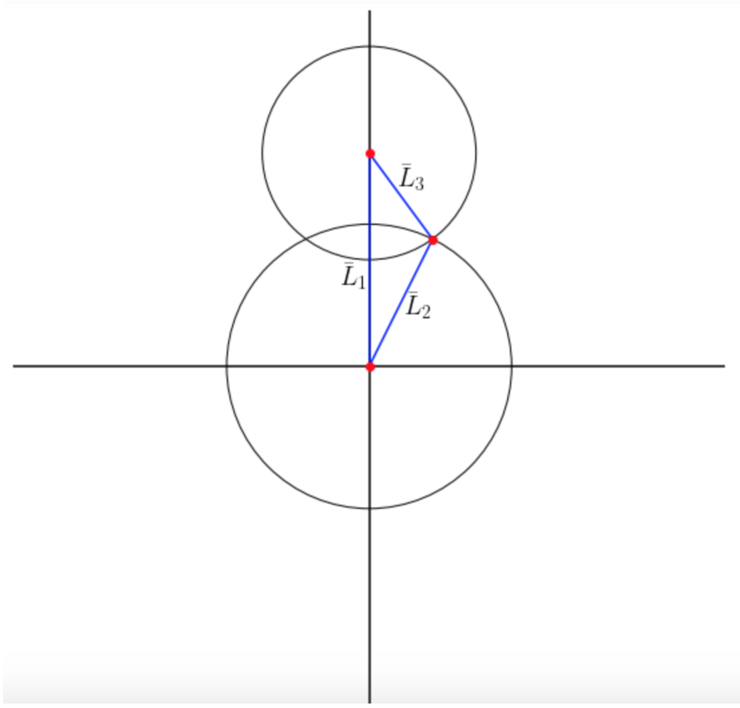


Figure 2: Constructing a triangle from line segments satisfying the specified inequalities

Note by Definition 7, triangles are a particular subset of the plane (i.e. a triangle has specific coordinates in the plane). However, we are often interested in the position of triangles (and other geometric objects) *relative* to each other, but without regard to where exactly they sit in the plane.

Consider the following: Let $\triangle_1, \triangle_2$ be two triangles (per Definition 7). We say $\triangle_1 \sim \triangle_2$ if and only if there exists an isometry[4] $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f(\triangle_1) = \triangle_2$. Then $\sim$ is obviously an equivalence relation. An equivalence class of $\sim$ is a collection of triangles in which every triangle is congruent. That is, any two triangles in the same equivalence class have the same edge lengths. This means we can refer to an equivalence class uniquely by specifying three values (assuming the values are valid, as prescribed in Lemma 8).

Consider a physical analogy: Suppose we cut a piece of paper into the shape of a triangle. This means the edges of the triangle have a fixed length. Now place the piece of paper down anywhere on a large table. Imagine the table has some axes, representing the plane. By Definition 7, a different initial choice of where we place the triangle gives us a different triangle, formally speaking. The same thing occurs if we slide and rotate the piece of paper across the table.

Sometimes we will care about the 'shape' of the paper triangle (i.e. the lengths of its edges), but we will not care where we place the piece of paper on the table. In this analogy, the paper triangle itself represents an entire equivalence class. The placement of the paper triangle at different positions on the table, or moving it across the table, represents different elements of the equivalence class.

The above discussion will be relevant when we prove the circle packing theorem (Theorem

[4]An isometry is a bijective, distance preserving map

2). At times it will be convenient to describe triangles via edge lengths only, and without providing exact coordinates of the triangle.

Theorem 9 below will be important for proving the circle packing theorem. It is quite a strong topological result, that is likely at least as hard to prove as the Brouwer fixed point theorem [10]. We will take advantage of the result without proving it, though many proofs exist in the literature (such as by Tao [10]).

Theorem 9 (Invariance of Domain). *If U is an open subset of $\mathbb{R}^n$, and $f : U \to \mathbb{R}^n$ is an injective, continuous map, then $f(U)$ is open in $\mathbb{R}^n$, and f is a homeomorphism between U and $f(U)$.*

We also present a result that will be used to prove a corollary to the Circle Packing Theorem. For a proof see Pach [8, Lemma 8.4]

Lemma 10 (The Centrepoint Theorem). *For any n-element subset P of $\mathbb{R}^d$, there exists a point $q \in \mathbb{R}^d$ with the property that any closed half-space[5] that does not contain q covers at most $\frac{dn}{d+1}$ elements of P. (Such a point is called a centrepoint of P) [8, Lemma 8.4].*

4.2 Graph Theory

4.2.1 Basic Notions

Definition 11. *A **graph** is a tuple (V, E). V is an arbitrary set, and E is a collection of two element subsets of V. V is called the vertex set, and E is called the edge set.[6]. We will take all graphs to have both finite vertex and edge sets.*

We use the following notation: suppose $v_1, v_2 \in V$, and $\{v_1, v_2\} = \{v_2, v_1\} \in E$. Then we write $v_1v_2 \in E$, or equivalently, $v_2v_1 \in E$. Note we say 'v_1 is adjacent to v_2' whenever $v_1v_2 \in E$.

Definition 12. *Suppose G is a graph.*

*(i) A **walk** between vertices v_0 and v_k is a sequence $(v_0, v_1, ..., v_k)$ of vertices of G and a sequence $(e_1, ..., e_k)$ of edges of G such that $e_i = v_{i-1}v_i$ whenever $1 \leq i \leq k$.*

*(ii) A **trail** is a walk with no repeated edges in the sequence $(e_1, ..., e_k)$ [14]*

*(iii) A **path** is a trail in which every vertex in the sequence of vertices is distinct. A **closed path** is a trail in which the first and last vertices are the same, but every other vertex is distinct.*

[5]A plane partitions $\mathbb{R}^3$ into two parts. Either of these parts together with the plane is a closed half-space.

[6]Note that according to this definition, the edge set E cannot contain loops, nor can it contain more than one edge between two vertices. Some refer to such graphs as *simple graphs*.

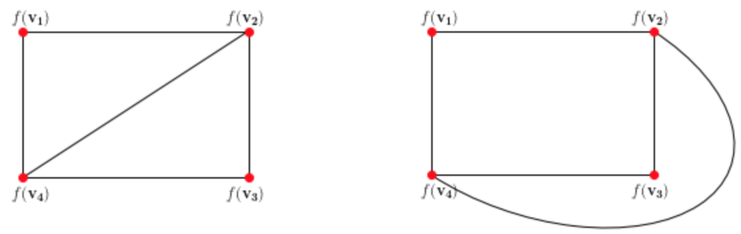
Definition 13. *We say a graph $G = (V, E)$ is* ***connected****, if there is a path between any two vertices of G.*

Definition 14. *Let $G = (V, E)$ be a graph. We say G is* ***planar*** *if the following hold: (i) there exists an injective function $f : V \to \mathbb{R}^2$, (ii) for every edge $v_iv_j \in E$, there exists a Jordan arc from $f(v_i)$ to $f(v_j)$, and, no two such Jordan arcs intersect except possibly at the image of a vertex under f.*

If $G = (V, E)$ is a connected planar graph, there exists some injective function $f : V \to \mathbb{R}^2$, and some appropriate Jordan arcs between points in the image of f (according to the conditions set out in Definition 14). Note the injective function and Jordan arcs between points is not necessarily unique. Given any possible f and collection of Jordan arcs, we denote by $\tilde{G}$ the union of $f(V)$ and the collection of arcs. In particular, $\tilde{G}$ is a subset of the plane. We refer to $\tilde{G}$ as a 'planar embedding' of G, or sometimes just an 'embedding' of G.

We reiterate an important point: there are many possible planar embeddings of a given planar graph G. This is because there are many possible injections and Jordan arcs between points in the plane. This serves to illustrate the distinction between an abstract graph (which is just a set of vertices and edges), and the many possible embeddings of that graph in the plane.

Indeed, consider the graph $G = \big(\{v_1, v_2, v_3, v_4\}, \{v_1v_2, v_1v_4, v_2v_4, v_2v_3, v_3v_4\}\big)$. Figure 3 below shows two possible planar embeddings of G. Note in particular the different Jordan arcs between points $f(v_2)$ and $f(v_4)$



(a) (b)

Figure 3: Two different planar embeddings of the graph G

4.2.2 Faces of a Planar Graph

We dedicate this subsection to the development of the notion of a face of a planar graph. While the notion is 'obvious', formally defining the term is somewhat involved.

Definition 15. *Let $a, b \in \mathbb{R}^2$ be two distinct points. If there exists a Jordan arc given by $f : [0, 1] \to \mathbb{R}^2$, such that $f(0) = a$ and $f(1) = b$, we say a is* ***arc connected*** *to b. We also define every point to be arc connected to itself.*

An analogous notion of path connectedness can be defined (recall the definition of a path in Definition 5). A non-trivial result from topology tells us that in any Hausdorff space, path connectedness implies arcwise connectedness [4]. Arcwise connectedness implies path connectedness in general. Therefore 'arcwise connected' and 'path connected' are equivalent notions in the plane.

Proposition 16. *The relation $\sim$, given by $a \sim b$ iff a is arc connected to b, is an equivalence relation (on any subset of the plane).*

Proof. Let $a, b, c \in \mathbb{R}^2$. It is clear from the definition that $a \sim a$. Suppose $a \sim b$. Then there is some Jordan arc given by f, such that f is injective, continuous, and $f(0) = a$ and $f(1) = b$. Define $\tilde{f} : [0,1] \to \mathbb{R}^2$ given by $\tilde{f}(x) = f(1-x)$. Then $\tilde{f}$ is obviously continuous and injective, and we have $\tilde{f}(0) = b$, $\tilde{f}(1) = a$. Thus b is arc connected to a, so $b \sim a$.

Finally suppose $a \sim b$ and $b \sim c$. By the discussion prior to the proposition, it is sufficient to show that a and c are merely path connected. Suppose the path from a to b is given by f_1, and the path from b to c is given by f_2. Define $\tilde{f} : [0,1] \to \mathbb{R}^2$ by

$$\tilde{f}(x) = \begin{cases} f_1(2x), & \text{if } x \in [0, 0.5] \\ f_2(2x-1), & \text{if } x \in [0.5, 1] \end{cases}$$

By an application of the Pasting lemma, $\tilde{f}$ is continuous. So $\tilde{f}$ defines a path between a and c, and hence $a \sim c$. This completes the proof.

□

Definition 17. *Let X be a subset of $\mathbb{R}^2$. Then the elements of $X/_\sim$ are called the* ***arc connected regions*** *of X, where $\sim$ is the equivalence relation described in Proposition 16.*

Definition 18. *Let G be a connected planar graph. Suppose $\tilde{G}$ is some embedding of G into the plane. The arc connected regions of the set $\mathbb{R}^2 \setminus \tilde{G}$ are the* ***faces*** *of the embedding $\tilde{G}$.*

A remark on Definition 18: Note that 'faces' are defined with respect to some embedding of G. In this sense, the notion of a face is not intrinsic to a graph, but rather to a particular embedding. It is however a remarkable fact that the number of faces is invariant regardless of the choice of embedding (see Theorem 22). This allows us to refer to the 'faces of G' without always having to specify an embedding.

As a face is a subset of the plane, its topological boundary is well defined. This motivates the following definition.

Definition 19. *Suppose G is a planar graph, and $\tilde{G}$ is a planar embedding of G induced by f. We say a vertex, v, is* ***adjacent*** *to a face, F, if the topological boundary of F contains the point $f(v)$. If v is adjacent to F, we also say F is* ***incident*** *on v.*

We say an edge vw ***borders*** *a face F if the topological closure of F contains a Jordan arc between $f(v)$ and $f(w)$ in the embedding $\tilde{G}$.*

Definition 19 formalises the intuitive meaning of what it means for parts of the graph to be 'next to' a face. For example, in Figure 3 (b), v_1 is adjacent to the exterior[7] face, while v_3 is not. Similarly, the edge v_2v_3 borders the two internal faces, while the edge v_1v_4 borders the exterior face and one interior face.

4.2.3 More Useful Results from Graph Theory

Proposition 20. *Let G be a connected planar graph, and $\tilde{G}$ a planar embedding of G. Then we have the following:*

(i) There are a finite number of faces of $\tilde{G}$ (recall we only consider finite graphs in this paper).

(ii) There is one exactly one unbounded face in $\tilde{G}$, which we call the exterior face.

(iii) Every edge borders exactly one or two faces.

We now introduce the type of graph we will work with in the main proof.

Definition 21. *A **maximal planar graph** is a planar graph with the property that adding an edge breaks the planarity of the graph.*

We may interpret Definition 21 in terms of the graph embedding. A graph is maximal planar if regardless of the embedding into the plane, it is never possible to add a Jordan arc between two vertices without intersecting an already existing arc.

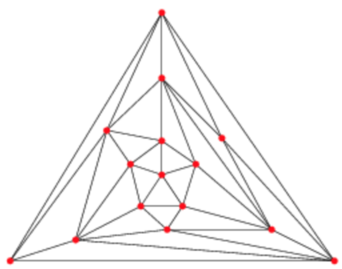


Figure 4: A representation of the maximal planar 'Poussin' graph. Taken from Wolfram MathWorld [12]

The following theorem is essential for the main proof. It establishes the relationship between the number of edges, vertices, and faces in a connected planar graph.

Theorem 22 (Euler's Formula)**.** *If G is a finite connected simple planar graph, with vertices V, edges E, and faces F, then $|V| + |F| - |E| = 2$.*

[7]We have technically not yet introduced what we mean by the exterior face. However we give a characterisation in Proposition 20.

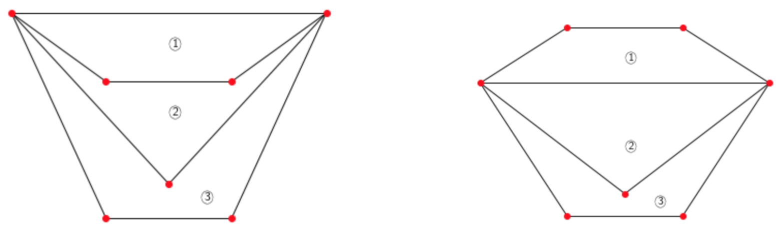
Before we give the next result, we need to introduce some relevant definitions.

Definition 23. *Let G be a connected planar graph. A face of a planar embedding of G is said to be* ***triangular*** *if the face is bounded by three edges and three vertices. This definition also applies to the external face of the graph. We sometimes refer to internal triangular faces simply as 'triangles'.*

Note that unlike the number of faces of a graph, the property of a face being triangular is *not* invariant to the embedding. For example, consider the graph

$$G = \{(v_1, v_2, v_3, v_4, v_5, v_6, v_7), (v_1v_2, v_1v_3, v_3v_4, v_2v_4, v_1v_5, v_2v_5, v_1v_6, v_6v_7, v_2v_7)\}$$

and Figure 5 below:



(a) (b)

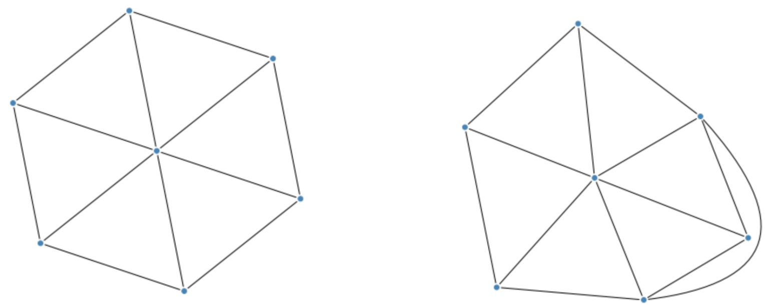
Figure 5: Two different embeddings of G in the plane, with the internal faces labelled.

Notice that in Figure 5 (a), none of the faces are triangular (including the exterior face). However in Figure 5 (b), the internal face ② is triangular.

Definition 24. *In a maximal planar graph, an* ***external vertex*** *is any vertex connected adjacent to the exterior face. An* ***internal vertex*** *is any vertex that is not an external vertex.*

Lemma 25. *Let G be a maximal planar graph, and let $\tilde{G}$ be any planar embedding of G. Every face of $\tilde{G}$ is triangular.*

Proof. For the sake of contradiction, suppose some face is not triangular. Then the face is bounded by an n-gon where $n \geq 4$. In particular, for every vertex that bounds the face, there exists another bounding vertex that is not adjacent to it. Regardless of whether the face is internal or external, choose any vertex and connect it to a non-adjacent vertex by an edge that lies in the face. Then the resulting graph is clearly still planar. However this is a contradiction as G is maximal planar. Hence every face must be triangular. □



(a) (b)

Figure 6: (a) is a subgraph of(b), but (b) is still planar.

We see above (in figure 3) an example of an edge we can add while still retaining planarity in the new graph. In this example, the outer face is the one not bounded by three edges and three vertices, so the new edged will be 'embedded' in the outer face to create the new graph.

Lemma 26. *Let $G = (V, E)$ be a finite maximal planar graph. Let v be an internal vertex of G. Define A_v to be the set of vertices adjacent to v (these adjacent vertices may be internal or external). Then the following hold:*

(i) deg$(v) \geq 3$

(ii) Every element of A_v is adjacent to exactly two other (distinct) elements of A_v

Where deg(v) denotes the number of adjacent vertices to v (read as 'degree of v).

Proof. (i): Suppose $\deg(v) = 0$. This is an immediate contradiction as G is connected. Suppose $\deg(v) = 1$. Then v cannot be adjacent to a triangular face, a contradiction. Finally suppose $\deg(v) = 2$. Since every face is triangular, we know the two vertices adjacent to v must themselves be adjacent. But this makes v an external vertex, a contradiction. Therefore $\deg(v) \geq 3$

(ii): Every face is triangular, so v is adjacent to $|A_v|$ triangular faces. Furthermore, every triangular face must share edges with exactly two other triangular faces that v is adjacent to. If this were not the case, there would be at least one non-triangular face, a contradiction. But now a vertex is adjacent to v if and only if it is one of the corners in such a triangular face. That is to say, the adjacent vertices are completely characterised by the triangular faces that v is adjacent to. But because there can be no gaps in the triangles, we can see that every adjacent vertex is connected to exactly two distinct adjacent vertices (because each corner of a triangle is connected, via an edge, to exactly two other corners). $\square$

Lemma 29 below will allow us to derive an inequality required in the proof of the Koebe-Andreev-Thurston theorem. Before stating and proving the result however we must first introduce the operation of edge contraction.

Definition 27. *Let $G = (V, E)$ be a graph containing an edge uv, where u and v are vertices in V. An* ***edge contraction*** *(along the edge uv) is an operation on G that results in a new graph $G' = (V', E')$. We further denote $G' = G \setminus uv$ to highlight the dependence on the choice of edge.*

We denote the by A_u the set of all vertices $x \in V$ such that x is adjacent to u. That is, $x \in A_u$ if and only if $xu \in E$. We analogously define A_v for the set of all vertices adjacent to v. We also let w be a new symbol (i.e. $w \notin V$).

We define V' as:

$$V' = V \setminus \{u, v\} \cup \{w\}$$

We define E' in the following way:

$$E_u = \{xu : x \in A_u\}\ ,\ E_v = \{xu : x \in A_v\}$$

$$\mathcal{E} = E_u \cup E_v$$

$$E' = \big(E \setminus \mathcal{E}\big) \cup \{xw : xu \in E \textit{ or } xv \in E \textit{ and } x \neq u, v\}$$

Let us illustrate Definition 27 with an example. Consider the graph $G = \big(\{v_1, v_2, v_3, v_4, v_5, v_6\}, \{v_1v_2, v_1v_3, v_2v_3, v_3v_4, v_1v_4, v_2v_4, v_4v_5, v_4v_6, v_5v_6\}\big)$. An example of an embedding of G is shown in Figure 7 (a) below. We will compute the contraction along the edge v_3v_4. That is, we will find the graph given by $G \setminus v_3v_4 = (V', E')$.

First we have $V' = V \setminus \{v_3, v_4\} \cup \{w\}$. In particular,

$$V' = \{v_1, v_2, w, v_5, v_6\}$$

We then evaluate E_{v_3} and E_{v_4}:

$$E_{v_3} = \{xv_3 : x \in A_{v_3}\} = \{v_1v_3, v_2v_3, v_3v_4\}$$

$$E_{v_4} = \{xv_4 : x \in A_{v_4}\} = \{v_1v_4, v_2v_4, v_3v_4, v_4v_5, v_4v_6\}$$

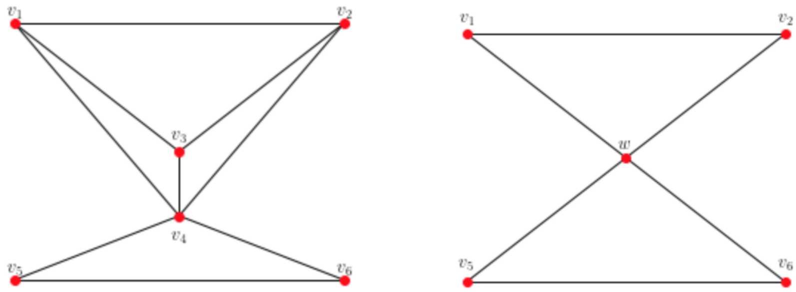
We also have

$$\{xw : xv_3 \in E \text{ or } xv_4 \in E \text{ and } x \neq v_3, v_4\} = \{v_1w, v_2w, v_5w, v_6w\}$$

Therefore, since $\mathcal{E} = E_{v_3} \cup E_{v_4}$, we evaluate E':

$$\begin{aligned} E' &= \big(E \setminus \mathcal{E}\big) \cup \{xw : xv_3 \in E \text{ or } xv_4 \in E \text{ and } x \neq v_3, v_4\} \\ &= \{v_1v_2, v_5v_6\} \cup \{v_1w, v_2w, v_5w, v_6w\} \\ &= \{v_1v_2, v_5v_6, v_1w, v_2w, v_5w, v_6w\} \end{aligned}$$

Therefore $G \setminus v_3v_4 = \big(\{v_1, v_2, w, v_5, v_6\}, \{v_1v_2, v_5v_6, v_1w, v_2w, v_5w, v_6w\}\big)$ in this particular example. See Figure 7 (b) above for an example of an embedding of the contracted graph.



(a) (b)

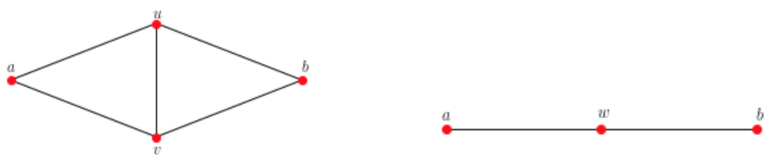
Figure 7: (a) An embedding of G (b) An embedding of $G \setminus v_3v_4$

In the literature, some authors define edge contraction in such a way that after contraction, there can be *multiple edges* between two vertices. However notice that in our definition of a graph (Definition 11), there is actually no way to introduce multiple edges. Indeed, suppose $u, v \in V$ and $uv \in E$. Formally, the edge uv is an element of E of the form $\{u, v\}$. Suppose we try to introduce another edge between u and v, say for example by adding $\{v, u\}$ to E. Notice that nothing in the graph changes. This is because $\{v, u\} = \{u, v\}$ and we know that $\{\{u, v\}, ...\} = \{\{u, v\}, \{v, u\}, ...\}$.

In the case of edge contraction, this works to our advantage because we would like to work with graphs with only one edge between vertices. The combination of Definition 11 and Definition 27 means that whenever we perform an edge contraction, the resulting graph only ever contains at most one edge between any two vertices.

Proposition 28. *Suppose G is a maximal planar graph. If e is an internal edge of G, then $G \setminus e$ is necessarily maximal planar.*

Proof. Suppose the edge $e = uv$, where u, v are two vertices of G. Since G is maximal planar, uv borders two triangular faces in any embedding. First we consider what happens locally (i.e. 'near' the contracted edge). See Figure 8 below.



(a) (b)

Figure 8: Before and after contracting along the edge uv

Note the figure is meant to represent only a portion of the graph; not every edge and vertex is shown.

Now the only thing to note is that the triangular face that was bounded by ua is still a triangular face after contraction, but is now bounded by wa. We apply an analogous argument to the edges ub, av, vb. Furthermore, any other triangular faces for which u or v were adjacent to, are now simply adjacent to w with no other change. Therefore in any case, all faces remain triangular after contraction, and this proves the proposition. ☐

The proof of Proposition 28 also makes it clear that when we perform an edge contraction on an internal edge in a maximal planar graph, the number of vertices decreases by 1, the number of faces by decreases by 2, and the number of edges decreases by 3. This fact can also be seen via Euler's formula (Theorem 22)

The following lemma can be found in the book by Sariel (see [5, Lemma 15.2.1])

Lemma 29. *Let $G = (V, E)$ be a maximal planar graph with n vertices, with an embedding $\tilde{G}$ in the plane. Let $X \subset V$ be a set of vertices such that $|X| \leq n - 3$. Denote the set of all faces that are incident on at least one vertex in X by $\mathcal{F}(X)$. Then $|\mathcal{F}(X)| > 2|X|$.*

Proof. If there exists an edge uv such that $u \in X$ and $v \in X$, then contract this edge. This contraction results in a new maximal planar graph, which has two less faces and one fewer vertex than the original. We update X by adding the new vertex introduced by the contraction, while removing the vertices u and v. This decreases $|X|$ by one. We correspondingly update $\mathcal{F}(X)$ with the new X, and hence $|\mathcal{F}(X)|$ decreases by two.

We repeat this process until no vertices in X are adjacent. We denote this final X by X'. Since $|X|$ is at most $n - 3$, we can perform the contraction at most $n - 4$ times (because once X' contains a single vertex, no more contractions can occur). Therefore there are at the fewest four remaining vertices in the graph.

Since there are at least four vertices, the three external vertices are obviously adjacent to at least three triangles. For any internal vertex, Lemma 26 tells us the degree is at least three. Therefore every internal vertex is adjacent to at least three triangles also. Note also that no triangle is adjacent to more than one vertex of X', as otherwise we may still contract an edge.

Therefore we have that

$$\mathcal{F}(X) \geq 2\big(|X| - |X'|\big) + 3|X'| = 2|X| + |X'|$$

Since X' contains at least one element, the claim $|\mathcal{F}(X)| > 2|X|$ holds. ☐

5 Proof of The Circle Packing Theorem

In this section we prove the circle packing theorem (Theorem 2). The proof presented here is based mainly on the proof provided by Pach ([8, Chapter 8, Koebe Representation Theorem]). Significant effort has been made to expound upon detail where appropriate. The work of Stephenson ([9]) and Sariel ([5, Chapter 15]) contributed significantly to resolving such details.

Before presenting the proof, we highlight the overall logic and approach we will take.

We will show it is sufficient to prove the result under the assumption that G is maximal planar. A key step is that we will assign positive real numbers to the vertices of G. We will interpret these values as 'radii' of circles, but formally we deal with a vector $r \in \mathbb{R}^n$, where n is the number of vertices in G.

By Lemma 25 we know every face is triangular, and bounded by three vertices. Then by Lemma 8, we show that for any three vertices v_1, v_2, v_3 that bound the same triangular face, it is possible to define a triangle with lengths $r_1 + r_2, r_1 + r_3, r_2 + r_3$, where the r_i are the 'radii' described in the previous paragraph. We will refer to such triangles as *cardboard triangles*[8]. We then define the *total angle* of a vertex, which is the sum of all the angles incident at that vertex, in all cardboard triangles that have that vertex as a corner.

We then establish that if the total angle of every internal vertex is 2π, a valid circle packing for G exists. We will then consider a function which we think of as mapping 'radii' to 'total angles'[9]. In particular, we will show this function is bijective. Hence a particular value of 'radii' exists such that the associated total angles of all internal vertices are 2π. This proves that there exists a vector r of radii that yield a valid circle packing. Note that in this proof we will not construct or explicitly compute a vector r that produces a circle packing; we simply prove such a collection of radii exist.

Proof of Thoerem 2 (The Koebe-Andreev-Thurston Theorem). We first note that it is sufficient to prove the result for maximal planar graphs. Indeed, suppose G is not maximal planar. We may extend G to a maximal planar graph by inserting vertices into non-triangular faces and adding edges from the new vertices to the existing vertices that bound the non-triangular face. If the resulting maximal planar graph has a circle packing, then we may simply remove the circles C_i whenever v_i is a newly added vertex. This obviously leaves us with a circle packing for the original G.

Let $G = (V, E)$ be a maximal planar graph with n vertices, and let $\tilde{G}$ be an arbitrary embedding of G. We note that G is obviously a connected planar graph. Let $r = (r_1, ..., r_n) \in \mathbb{R}^n$ such that $\sum_i^n r_i = 1$, and $r_i > 0$ for all i. Suppose we choose three vertices, v_1, v_2, v_3, such that $v_1v_2, v_1v_3, v_2v_3 \in E$. These three vertices together with these edges form a graph in their own right. That is, a graph $H = (\{v_1, v_2, v_3\}, \{v_1v_2, v_1v_3, v_2v_3\})$ (we refer to H as a *subgraph* of G). We assign to each vertex v_i in H the coordinate r_i in the vector r.

Note that it is possible to form a triangle with edge lengths given by $r_1+r_2, r_1+r_3, r_2+r_3$. Indeed, suppose we have three line segments L_1, L_2, L_3 of lengths $r_1 + r_2, r_1 + r_3, r_2 + r_3$ respectively. Since each r_i is positive, it is clear that these line segments form a triangle by Lemma 8.

In the proof of Lemma 8, we constructed a triangle. We will refer to *any* element of the equivalence class induced by this triangle (see discussion in Section 4.1) as a *cardboard triangle*[10] of vertices v_1, v_2, v_3 for the given vector r. We denote this by $\triangle v_1v_2v_3$.

The cardboard triangle $\triangle v_1v_2v_3$ (minus its interior) is an embedding of H into the plane, where $f(v_i)$ are the corners of the triangle, and the Jordan arcs are the line segments of

[8] The terminology 'cardboard triangle' was introduced in Pach [8]

[9] This statement is imprecise, but is formally clarified in the proof

[10] This terminology was introduced in Pach [8]

length $r_1+r_2, r_1+r_3, r_2+r_3$. In particular, for any choice of r_1, r_2 and r_3, there is a circle packing of the subgraph H, given by the three circles $\{C_1, C_2, C_3\}$ where C_i is centred at $f(v_i)$ and has radius r_i. Note that from this point on, we will stop writing $f(v_i)$ and instead simply write v_i, with the understanding that we now refer to embedded vertices of the graph.

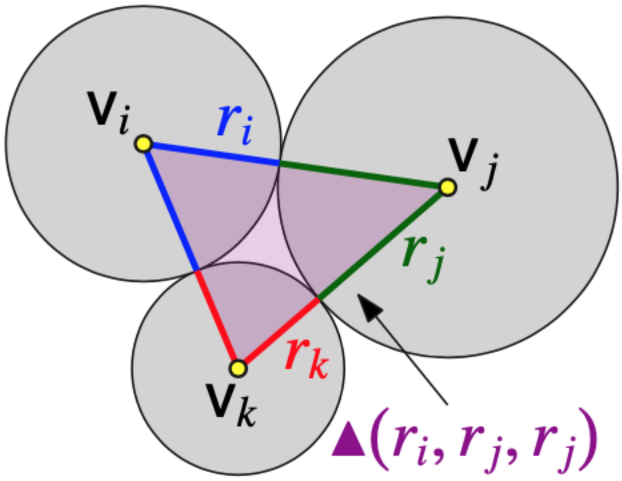


Figure 9: A 'Cardboard Triangle' and the associated circle packing. Taken from [5, Chapter 15]

We now define the *total angle* of a vertex. Let $v \in G$. We define the total angle of v to be the sum of all the angles at v, in all the cardboard triangles that have v as a corner (Note that these cardboard triangles correspond to the faces that v is adjacent to). We denote the total angle of v by $\sigma_r(v)$. The total angle depends on the choice of $r \in \mathbb{R}^n$.

Lemma 25 tells us that there is a one-to-one correspondence between the faces of $\tilde{G}$ and the cardboard triangles we can create. Additionally Lemma 26 tells us that for any internal vertex v, there are at least three cardboard triangles that can be constructed that have v as a corner.

Consider two faces that have a common edge bordering them. Then we may create the two cardboard triangles corresponding to these faces, and by construction these two cardboard triangles can be 'glued'[11] together along an edge. For example, suppose $r \in \mathbb{R}^n$ is fixed. Suppose also there is a (triangular) face bounded by v_1, v_2, v_3, and another bounded by v_2, v_3, v_4. Notice that the cardboard triangles that correspond to both faces will have

[11] A way to characterise the 'gluing' the two triangles in Figure 10 can be given by the following: Choose a cardboard triangle $\triangle v_1v_2v_3$ (an element of the relevant equivalence class). This triangle has specified coordinates in the plane (i.e. we have chosen where to 'place' the cardboard triangle in the plane). Choose a cardboard triangle $\triangle v_2v_3v_4$, which has some distinct coordinates in the plane. Then it is clear that there exists an isometry, f, such that $f(\triangle v_2v_3v_4)$ is another cardboard triangle of v_2, v_3, v_4, and the coordinates of exactly one edge of $f(\triangle v_2v_3v_4)$ and $\triangle v_1v_2v_3$ coincide. This formalisation is what we mean whenever we refer to gluing cardboard triangles along an edge.

an edge of the same length: $r_2 + r_3$.

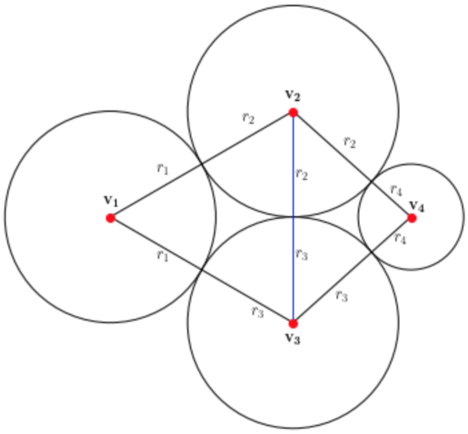


Figure 10: 'Gluing' two cardboard triangles along edges of equal length.

Note that there are in fact two possible orientations of a triangle in the plane once we have specified the coordinates of one of its edges. These orientations are given by a reflection along the fixed edge. In the case of two triangles, we can always choose an orientation that avoids any overlap of the interiors of the triangles.

It is immediately clear that the process of gluing can be extended beyond two triangles. In particular consider the following: Fix a particular internal face f_1 of $\tilde{G}$. Now place a cardboard triangle $\triangle f_1$ anywhere in the plane. Let f_k be any other face of $\tilde{G}$. Since G is connected, there will always exist a finite sequence of faces $(f_1, ..., f_k)$ with the property that each term[12] of the sequence shares an edge with the preceding term. In particular, since there is a one-to-one correspondence between the faces of $\tilde{G}$ and the possible cardboard triangles of G, there will be a sequence of cardboard triangles $(\triangle f_1, ..., \triangle f_k)$ such that each triangle can be glued to the previous one in the sequence. We say the sequence *tiles* the plane.

[12]Excluding f_1

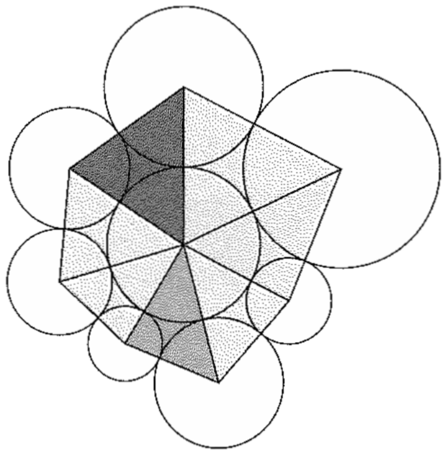


Figure 11: A circle packing and an associated set of cardboard triangles. Note the circle packing is for the graph whose vertex set is given by the centre of the circles, and the edges given by the tangency relations of the circles. Taken from [9, Chapter 5.2].

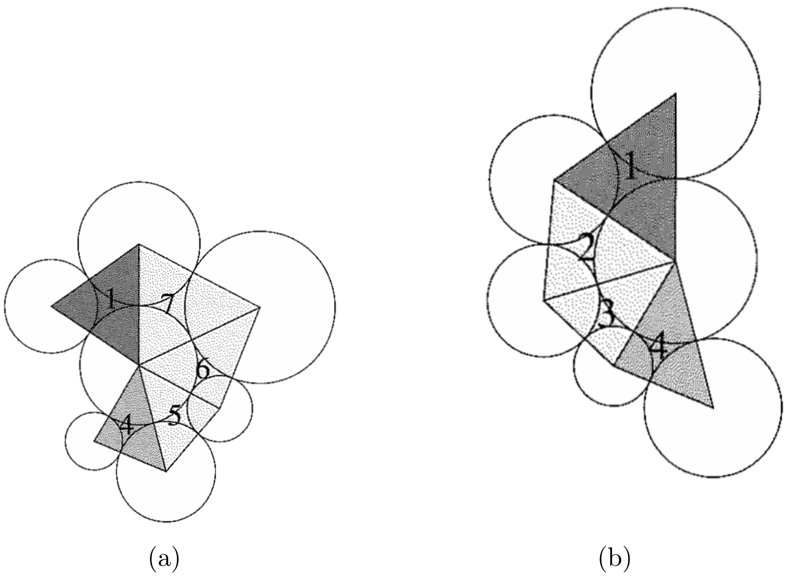


Figure 12: Two different sequences of cardboard triangles for the same planar graph. Taken from [9, Chapter 5.2].

Figures 11 and 12 above show two different examples of sequences of cardboard triangles. In Figure 12 (a) for example, there is a sequence of cardboard triangles $(1, 7, 6, 5, 4)$, while in (b) there is a sequence $(1, 2, 3, 4)$. In each sequence, the positions of all cardboard triangles are determined by original position of the triangle 1. However, it is not necessarily true that *every* sequence will lead to the same position of every cardboard triangle. Consider Figure 13 below:

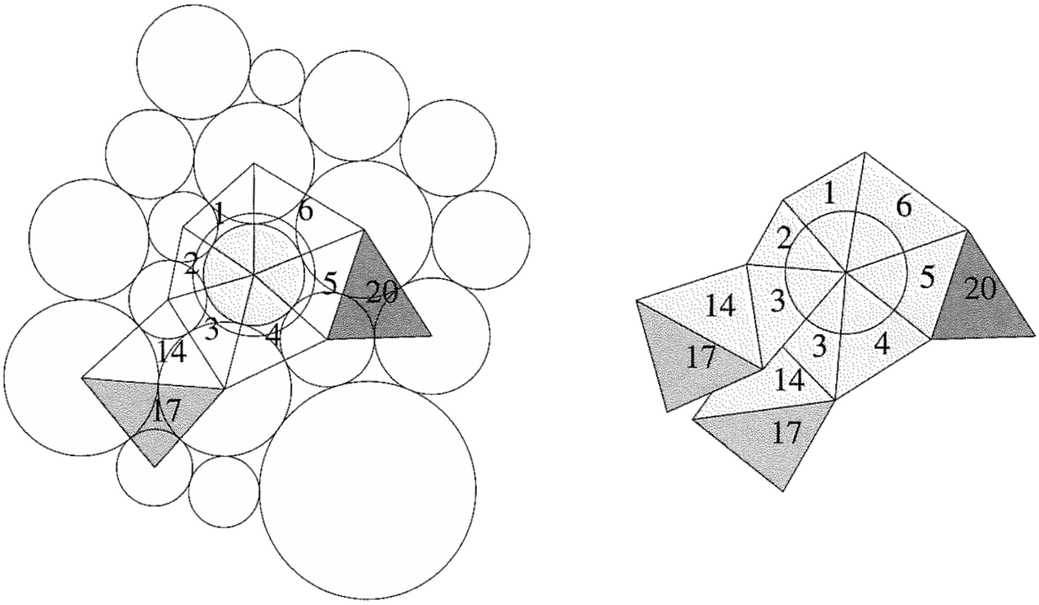


Figure 13: An example of when two different sequences of cardboard triangles lead to different positions of the triangles in the plane. Taken from [9, Chapter 5.2].

In Figure 13, the sequences $(1, 2, 3, 14, 17)$ and $(1, 6, 5, 4, 3, 14, 17)$ give different positions of cardboard triangles $3, 14$ and 17. Notice also the two sequences give rise to an overlap of cardboard triangles. This means the position of a face f is not necessarily unique across arbitrary sequences.

It is the case that whenever $r \in \mathbb{R}^n$ is such that $\sigma_r(v_i) = 2\pi$ for every internal vertex v_i, the following are true:

(i) Fix the position of a cardboard triangle $\triangle f_1$ (corresponding to a face f_1 of $\tilde{G}$). Then every cardboard triangle has a unique position in the plane relative to $\triangle f_1$. That is, every possible sequence of faces will tile the plane in a consistent way (see [9, Chapter 5]).

(ii) A circle packing of G exists (see [9, Chapter 5.2]).

A proof of *(i)* requires results from homotopy theory. For a proof see Chapter 5 of Pach [9, Sections 3.1.3 and Proof of Theorem 5.4]. Here we will take *(i)* for granted, i.e. if $r \in \mathbb{R}^n$ is such that $\sigma_r(v_i) = 2\pi$ for every internal vertex v_i, then *(i)* holds. However proving *(i)* implies *(ii)* is straightforward:

Claim: Let $r \in \mathbb{R}^n$ such that $\sigma_r(v_i) = 2\pi$, then a circle packing of G exists.

Proof. Fix a cardboard triangle associated to a face f_1. Then every cardboard triangle of G has a fixed position in the plane relative to $\triangle f_1$, by *(i)*. Note that it is not possible for any of the interiors of the cardboard triangles to intersect. If they do, this means there must be at least two different sequences of faces that give rise to a different location of one of the overlapping triangles, contradicting *(i)*. Hence there is a finite triangular tiling with no overlapping. Define a set of n circles $\{C_i\}_{i=1}^n$ such that each circle C_i is centred at v_i (which is a corner of some cardboard triangle). Let the radius of each circle C_i be

r_i. Then by construction of the cardboard triangles, a circle C_i touches another circle C_j if and only if the vertices $v_i v_j$ is an edge in G. But this exactly gives a circle packing of G. ■

Recall Theorem 22 states that for any finite connected graph, $|V|-|E|+|F|=2$. Combining this with the fact that in a maximal planar graph $3|F|=2|E|$, we obtain $|F|=2n-4$.

Notice that if we sum the total angle across all vertices, we are really summing up all the angles in all the cardboard triangles of G. But we know that the possible cardboard triangles are in one-to-one correspondence with the faces of G. Therefore, since the angles of a triangle sum up to π, and since the number of cardboard triangles is given by $|F|=2n-4$, we have that

$$\sum_{i=1}^{n} \sigma_r(v_i) = (2n-4)\pi. \tag{1}$$

This holds for any choice of $r \in \mathbb{R}^n$. We define the following sets

$$S = \Big\{ r = (r_1, ..., r_n) : \forall i (r_i > 0) \text{ and } \sum_{i=1}^{n} r_i = 1 \Big\} \subset \mathbb{R}^n,$$

$$H = \Big\{ x = (x_1, ..., x_n) : \sum_{i=1}^{n} x_i = (2n-4)\pi \Big\} \subset \mathbb{R}^n.$$

We refer to the set S as an $n-1$ dimensional simplex[13] (see for example [8, Chapter 8]). Consider the map $f : S \to H$ defined by

$$f(r) = (\sigma_r(v_1), ..., \sigma_r(v_n)).$$

The map f is clearly well defined by (1).

Claim: f is continuous.

Proof. Suppose $\triangle v_1 v_2 v_3$ is a cardboard triangle. Then the angle at v_1 in $\triangle v_1 v_2 v_3$ is given by

$$\angle v_2 v_1 v_3 = arccos\left(\frac{(r_i + r_k)^2 + (r_i + r_j)^2 - (r_j + r_k)^2}{2(r_i + r_k)(r_i + r_j)} \right) \tag{2}$$

Consider for example $\sigma_r(v_1)$. This is a map from $\mathbb{R}^n$ to $\mathbb{R}$. In particular, it takes the vector r and returns the sum of all angles at v_1 in all cardboard triangles that have v_1 as a corner. Without loss of generality suppose v_1 is adjacent to k vertices (and hence to k faces) which we label as $v_2, ..., v_{k+1}$. Then the only coordinates of r that affect $\sigma_r(v_1)$ are $r_1, ..., r_{k+1}$. Since (2) is clearly continuous, the sum of the angles at v over the k faces is clearly continuous over the $r_1, ..., r_{k+1}$. Therefore $\sigma_r(v_1)$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}$. We can apply analogous reasoning to all other $\sigma_r(v_i)$ to show they are continuous. Hence every component function of f is continuous, so f is itself continuous. ■

Now assume without loss of generality that v_1, v_2, v_3 are external vertices. As we have now established, it is sufficient to show that $x^* = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, \pi, ..., \pi)$ belongs to the image

[13] A simplex is a generalization of a triangle to Euclidean space of arbitrary dimensions.

of f[14].

Claim: f is injective.

Proof. Suppose $r, r' \in S$ with $r \neq r'$. Define I to be the set of indices such that $r_i < r'_i$. If I was the empty set, then $r_i \geq r'_i$ for every i. But this is impossible: if all coordinates of r are equal to those of r', we have a contradiction by the assumption that $r \neq r'$. So then at least some coordinates of r are larger than those in r', with all others being at least equal. But this is also a contradiction, as then $\sum r_i > 1$, which is impossible because $r \in S$. Therefore we have that $I \neq \emptyset$. Similarly $I \neq \{1, ..., n\}$ as then $\sum r_i < 1$ which is again impossible because $r \in S$.

We will now prove a bound for $\sum_{i \in I} \sigma_r(v_i)$ relative to r'. Let $\triangle v_i v_j v_k$ be a cardboard triangle. There are four possibilities: *(i)* None of the indices i, j, k are in I. *(ii)* Exactly one of i, j, k is in I. *(iii)* Exactly two of i, j, k are in I. *(iv)* All of i, j, k are in I.

We can calculate from Equation (2) that the partial derivatives of the the angle satisfy everywhere $\partial r_i < 0$, $\partial r_j > 0$, $\partial r_k > 0$ (knowing that all of r_i, r_j, r_k are positive). Let us now consider the cases.

(i): In this case, none of v_i, v_j, v_k contribute to the sum we are interested in.

(ii): Suppose without loss of generality that only $i \in I$. Consider that as r_i increases, while r_j and r_k either remain the same or decrease, then the angle at v_i must decrease. To see this, consider two values r, r' where $r_i < r'_i, r_j \geq r'_j, r_k \geq r'_k$. Let $\theta(r_i, r_j r_k)$ denote the angle at v_i in the cardboard triangle $\triangle v_i v_j v_k$, where r_i, r_j, r_k are associated to v_i, v_j, v_k respectively. Then by the partial derivatives, we have that $\theta(r_i, r_j, r_k) > \theta(r'_i, r_j, r_k) > \theta(r'_i, r'_j, r_k) > \theta(r'_i, r'_j, r'_k)$. I.e., if we increase r_i while decreasing or keeping the same r_j and r_k, the angle at v_i decreases.

(iii): Suppose without loss of generality that $i, j \in I, k \notin I$. Then if we increase r_i and r_j while r_k either decreases or remains the same, then the sum of the two angles at v_i and v_j will increase. Using an analogous argument to the previous case, we can use the partial derivatives to see that if we now consider the angle at v_k (previously we were concerned with the angle at v_i), then this angle must increase whenever we increase r_i and r_j while r_k either decreases or remains the same. Of course the sum of the other two angles must then increase.

(iv): In this case, all of $i, j, k \in I$. Notice that if we increase all of r_i, r_j, r_k, the sum of the three angles at v_i, v_j, v_k remains unchanged. Indeed, it must always sum to π as we still have a triangle.

Since $I \neq \emptyset$, it is impossible for case *(i)* to hold in every cardboard triangle. Also, since $I \neq \{1, ..., n\}$, it is impossible for case *(iv)* to hold in every cardboard triangle. Thus we have at least some instances of cases *(ii)* and *(iii)* holding. But this must mean that the

[14] In fact we showed that only the total angles of the internal vertices matter. The total angle of each external vertex being $\frac{2\pi}{3}$ has been chosen solely for convenience.

following holds

$$\sum_{i\in I}\sigma_r(v_i) > \sum_{i\in I}\sigma'_r(v_i) \tag{3}$$

But this means $f(r) \neq f(r')$, so f is injective. ■

We will proceed by showing some topological properties of the simplex S. First we will show it is open, and then we will give a characterisation of its boundary.

Note that S is contained in a particular[15] hyperplane[16] of $\mathbb{R}^n$. Since this hyperplane is a subset of $\mathbb{R}^n$, we may endow it with the subspace topology induced from $\mathbb{R}^n$. Observe that in this case, the hyperplane is homeomorphic to $\mathbb{R}^{n-1}$. In particular, we will prove S is open with respect to the hyperplane endowed with the subspace topology. We will also characterise the boundary of S with respect to the hyperplane. In other words, we will determine the topological properties of S with respect to $\mathbb{R}^{n-1}$ rather than $\mathbb{R}^n$. The above detail is subtle but important; Indeed in $\mathbb{R}^n$, S is neither even open nor closed (see for example [2]).

Claim: Let $P = \{(x_1, ..., x_n) : \sum x_i = 1\}$ be the unique hyperplane of $\mathbb{R}^n$ that contains S. Endow P with the subspace topology. That is, $\tau_P = \{O \cap P : O \text{ is open in } \mathbb{R}^n\}$. Then S is open with respect to P.

Proof. We will show that there exists an open set (of P) about every point of S, that is completely contained in S. In particular we will show that for every $r \in S$, there exists a positive ϵ_1 such that $P \cap B_{\mathbb{R}^n}(r, \epsilon_1) \subset S$. $B_{\mathbb{R}^n}(r, \epsilon)$ denotes the open ball in $\mathbb{R}^n$ about a point r of radius ϵ.

Let $r \in S$. Define $\Gamma = \{(x_1, ..., x_n) : \forall i(x_i > 0)\}$. Then $r \in \Gamma$, as $S \subset \Gamma$. Note that Γ is clearly an open set (of $\mathbb{R}^n$) as it is of the form $(0, \infty) \times ... \times (0, \infty)$, and $(0, \infty)$ is open in $\mathbb{R}$. Since Γ is an open set, there exists some ϵ_2 such that $B_{\mathbb{R}^n}(r, \epsilon_2) \subset \Gamma$.

Choose $\epsilon_1 = \epsilon_2$. We will now prove $P \cap B_{\mathbb{R}^n}(r, \epsilon_1) \subset S$. Let $r' \in P \cap B_{\mathbb{R}^n}(r, \epsilon_1)$. Since $r' \in P$, we have by definition of P that $\sum r'_i = 1$. But since $r' \in B_{\mathbb{R}^n}(r, \epsilon_1)$, we have that $r' \in \Gamma$. But by definition of Γ, this means $\forall i(x_i > 0)$. But now we have shown r' satisfies both $\sum r'_i = 1$ and $\forall i(x_i > 0)$, which exactly means $r' \in S$. Hence $P \cap B_{\mathbb{R}^n}(r, \epsilon_1) \subset S$, and this completes the proof.

■

Claim: Let P be as in the previous claim. The boundary of S (denoted ∂S) is characterised by $\partial S = \{(x_1, ..., x_n) : \sum x_i = 1 \text{ and } \exists i \in \{1, ..., n\} \text{ such that } x_i = 0\}$.

Proof. We will first show that any element $x = (x_1, ..., x_n)$ that satisfies $\sum x_i = 1$ and $\exists i \in \{1, ..., n\}$ such that $x_i = 0$, is a limit point of S. Indeed, let $(x_1, ..., x_n)$ be such an element. Let $1 \leq j < n$ and without loss of generality, suppose that $x_1, ..., x_j$ are all 0. That is we suppose the first j coordinates are 0. In particular we may define a sequence of points (x_k) where we define $x_k = (\frac{1}{k}, ..., \frac{1}{k}, \frac{k-j}{k(n-j)}, ..., \frac{k-j}{k(n-j)})$. Here the first j terms take the value $\frac{1}{k}$, and all other terms take the value $\frac{k-j}{k(n-j)}$. It is clear that the sum of

[15] I.e. the hyperplane $\{(x_1, ..., x_n) : \sum x_i = 1\}$

[16] A hyperplane of n-dimensional Euclidean space, $\mathbb{R}^n$, is an affine subspace of $\mathbb{R}^n$ that has dimension $n - 1$.

coordinates in x_k is 1, and for sufficiently large k it is clear that x_k will belong to S. Furthermore, it is immediate to see that the $\lim_{k\to\infty} x_k = (0, ..., 0, \frac{1}{n-j}, ..., \frac{1}{n-j})$. This argument will hold in general, and so we have proved that every element $(x_1, ..., x_n)$ that satisfies $\sum x_i = 1$ and $\exists i \in \{1, ..., n\}$ such that $x_i = 0$ is a limit point of S.

It now remains to show that these are the only limit points of S (other than S itself of course). In particular, suppose $x = (x_1, ..., x_n)$ is not an element of S, and does not satisfy the stated conditions. But then it is clear that for such an $x \in P$, there will always exist an open set of P containing x that does not intersect S, which by definition means x is not a limit point of S. For example one way to see this is as follows: Because x does not satisfy $\sum x_i = 1$ and $\exists i \in \{1, ..., n\}$ such that $x_i = 0$, it must be the case that $\inf \{x_i - s_i : s \in S\}$ is non-zero. But if we let ϵ be half this value, the set $P \cap B_{\mathbb{R}^n}(x, \epsilon)$ is an open set containing x, but will not intersect S. Hence x is not a limit point of S.

This completes the proof.

■

Let $s = (s_1, ..., s_n) \in \partial S$. Let I_s denote the set of all indices such that $s_i = 0$. Note from our characterisation of ∂S, it is impossible for I_s to be empty. It is also clearly impossible for I_s to be $\{1, ..., n\}$ as then the coordinates of s do not sum to 1.

Observe that if r tends to s, then in each cardboard triangle that has at least one vertex belonging to $\{v_i : i \in I_s\}$, the sum of angles at these vertices must tend to π. This is immediately obvious in the case that a cardboard triangle has just one vertex belonging to $\{v_i : i \in I_s\}$. However we can see also see this is true in the case where two or three vertices belong to $\{v_i : i \in I_s\}$. For the case of two vertices, notice that as the radii at these vertices shrink to zero, the third vertex (that is not in $\{v_i : i \in I_s\}$), must tend to 0. Hence the sum of the two vertices must tend to π. In the case of three vertices, the value simply converges to π, because for any values of r_i, the sum of the angles at the three vertices must always add up to π.

Hence we have the following:

$$\lim_{r\to s} \sum_{i\in I_s} \sigma_r(v_i) = |F(I_s)|\pi. \tag{4}$$

where $|F(I_s)|$ denotes the number of faces of $\tilde{G}$ with at least one vertex in $\{v_i : i \in I_s\}$.

Fix some $r \in S$, and choose any proper, non-empty subset $I \subset \{1, ..., n\}$. Notice there exists a point $s \in \partial S$ such that $s_i = 0$ for all $i \in I$, and $s_i > r_i$ whenever $i \notin I$. Indeed we can construct such a boundary point given r: Without loss of generality suppose $I = \{1, ..., j\}$. By our characterisation of the boundary, we know the point $\left(0, ..., 0, r_{j+1} + \frac{r_1+...+r_j}{n-j}, ..., r_n + \frac{r_1+...+r_j}{n-j}\right)$ belongs to it. But this point also satisfies our conditions.

We now consider what happens as we move from r to s along a line. In particular, we define the points r_t by $r_{t_i} = (1-t)r_i + ts_i$ for all $t \in (0, 1)$. By Equation 3, we know that for all $t \in (0, 1)$

$$\sum_{i\in I}\sigma_{r_t}(v_i) > \sum_{i\in I}\sigma_r(v_i)$$

We also know that by Equation 4

$$\lim_{t\to 1}\sum_{i\in I}\sigma_{r_t}(v_i) = \lim_{r\to s}\sum_{i\in I_s}\sigma_r(v_i) = |F(I_s)|\pi.$$

Which therefore gives us the following bound:

$$\sum_{i\in I}\sigma_r(v_i) < |F(I_s)|\pi.$$

In particular, have shown the image of f lies in the set

$$P^* = \left\{(x_1,...,x_n) : \sum_{i=1}^{n} x_i = (2n-4)\pi \text{ and for all proper subsets } I, \sum_{i\in I} x_i < |F(I_s)|\pi\right\}.$$

In fact, P^* *is* the image of f. To prove this we need to show a preliminary result (see [8, Exercise 8.2]):

Claim: Let f be a continuous injective map from the interior of the d-dimensional ball B^d ($d \geq 1$) to itself. If all accumulation points of $f(p)$ as p tends to the boundary of B^d lie on ∂B^d, then $f(B^d) = B^d$.

Proof. Note that since B^d is a connected set, it is sufficient to show that $f(B^d)$ is both open and closed in B^d (we take B^d to be endowed with the subspace topology). Indeed, the only clopen sets of a connected space are the whole set and the empty set, and B^d is obviously connected.

Since f is continuous and injective, and since B^d is an open subset of $\mathbb{R}^n$, by the Invariance of Domain (Theorem 9) $f(B^d)$ must be open.

For the sake of contradiction, suppose $f(B^d)$ is not closed. This means $f(B^d) \neq B^d$, so in particular there exists a point $\bar{y} \in B^d$ with $\bar{y} \notin f(B^d)$. Then there must also exist a point $y \in \partial f(B^d)$.

To see this, choose the point $f(\mathrm{O})$ (the image of the origin under f in $f(B^d)$). Now construct the line from this point to the point $\bar{y}$. That is, consider the points $r_t = (1-t)f(\mathrm{O}) + t\bar{y}$ as t varies over $[0,1]$. Consider the set $\inf\{t \in [0,1] : r_t \notin f(B^d)\}$. We know this set is non-empty because we know $r_1 = \bar{y}$ and $\bar{y} \notin f(B^d)$. It cannot be 0 either. Indeed, for the sake of contradiction suppose it was 0. By definition, $f(\mathrm{O}) \in f(B^d)$. We know $f(B^d)$ is open, and hence there exists some open ball (centred at $f(\mathrm{O})$), say of radius ϵ, that contains the point $f(\mathrm{O})$, and such that the ball is completely contained in $f(B^d)$. But now take a ball of positive radius that is strictly less than ϵ. This ball must intersect some r_t where $t > 0$. But by definition of the infimum, $r_t \notin f(B^d)$. But this is a contradiction, as r_t is contained in the open ball of radius ϵ which is itself contained in $f(B^d)$.

Therefore the infimum of our set is some value in $(0,1]$. Say the infimum is t^*. Consider the sequence (r_{t_n}) where $t_n = t^* - \frac{1}{n}$. For sufficiently large n, we have that t_n is necessarily positive, and clearly the sequence $r_{t_n} \to r_{t*}$. This means r_{t^*} is a limit point of $f(B^d)$. In an open set, a limit point is in the closure if and only if it is not in the interior of the set. So we show r_{t^*} cannot be in the interior. Indeed, suppose r_{t^*} is an interior point. Then there is an open ball containing it that is contained in $f(B^d)$. But then there is some t larger than t^* such that r_t intersects the ball. But this is again a contradiction, as t^* is the infimum. We take the point r_{t*} to be y.

Now $y \in B^d$ and $y \in \partial f(B^d)$. By definition of the closure, there must be a sequence of points contained in $f(B^d)$ converging onto y. Suppose $(y_k) \subset f(B^d)$ is some such sequence. Note that f is injective, and hence bijective onto its image. Therefore we may construct a new sequence of points $(p_k) = (f^{-1}(y_k)) \subset B^d$. Obviously $(p_k) \subset \bar{B}^d$ since a set is always contained in its closure[17]. We now note the following: in a compact set, every sequence has a convergent subsequence. In particular, $\bar{B}^d$ is obviously compact, and so (p_k) must have some convergent subsequence in $\bar{B}^d$.

But to be a convergent subsequence in $\bar{B}^d$ means one of two cases. Suppose p is some limit of a convergent subsequence (i.e. an accumulation point), then either $p \in \partial B^d$ or $p \in B$. Suppose $p \in B$. But now since f is continuous we have

$$\lim_{n\to\infty} f(p_{k_n}) = f(\lim_{n\to\infty} p_{k_n}) = f(p)$$

$$\lim_{n\to\infty} f(p_{k_n}) = \lim_{n\to\infty} f(f^{-1}(y_{k_n})) = \lim_{n\to\infty} y_{k_n} = y$$

where p_{k_n} is some convergent subsequence. But note this is a contradiction, as now $f(p) = y$, meaning $y \in f(B^d)$ which is impossible by construction of y.

Therefore it must mean the subsequence converges to the boundary of the ball. But note this is also a contradiction! If (p_k) converges to the boundary, there is an accumulation point at the boundary. But by the hypothesis of the theorem, $(f(p_k)) = (y_k)$ must also converge to the boundary by the hypothesis of the theorem. But we know $y \in B^d$, i.e. $y \notin \partial B^d$, giving us the contradiction. This means that $f(B^d)$ is in fact closed.

This proves that $f(B^d)$ is clopen, and so must in fact be all of B^d. I.e. we have shown $f(B^d) = B^d$. In particular we have shown f is surjective.

■

Claim: $f : S \to P^*$ is surjective.

Proof. Recall we proved that S is open (in a space homeomorphic to $\mathbb{R}^n$). We also note that P^* is open. Indeed we have

$$P^* = \left\{(x_1, ..., x_n) : \sum x_i = (2n-4)\pi\right\} \bigcap \left\{(x_1, ..., x_n) : \sum_{i\in I} < |F(I)|\pi\right\}$$

whenever I is a proper subset of $\{1, ..., n\}$. Note that $\{(x_1, ..., x_n) : \sum x_i = (2n-4)\pi\}$ is a hyperplane of $\mathbb{R}^n$. We can also see that $\{(x_1, ..., x_n) : \sum_{i\in I} < |F(I)|\pi\}$ is open: The number of possible proper subsets is finite, so in particular we may decompose the set into

[17] In topological settings, the closure of a set is denoted by the symbol representing that set with a bar on top.

a number of intersections, with each component reflecting corresponding to a particular proper subset I. Each of these is clearly open, and so the entire finite intersection is open. Thus P^* is open in the hyperplane $\{(x_1, ..., x_n) : \sum x_i = (2n-4)\pi\}$ endowed with the subspace topology. Note also that both S and P^* are convex subsets [18] of $\mathbb{R}^{n-1}$.

This shows S and P^* are both open sets in some hyperplanes of $\mathbb{R}^n$. Note that any hyperplane endowed with the subspace topology, is in fact homeomorphic to $\mathbb{R}^{n-1}$. So in particular, S and P^* can both be thought of as being open in $\mathbb{R}^{n-1}$[19].
A result from topology tells us that open convex sets in Euclidean space are homeomorphic to the open ball. Therefore S and P^* are homeomorphic to open balls in $\mathbb{R}^{n-1}$. We also proved f is continuous, and by Equation 4 it is quite clear that the accumulation points of $f(r)$ tend to the boundary of P^* whenever r tends to the boundary of S.

Therefore apply our earlier claim about injective continuous maps. Using this we have that $f(S) = P^*$, or in other words, that $f : S \to P^*$ is a surjective map.

■

Let us briefly recapitulate. We are to show that the element $x^* = (\frac{2\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}, ..., 2\pi, ..., 2\pi)$ belongs to the image of f. We have just shown that f is surjective onto P^*. Therefore the following claim completes the proof.

Claim: $x^* \in P^*$

Proof. It is clear that $\sum_{i=1}^{n} x^* = (2n-4)\pi$. Suppose I is some non-empty proper subset of $\{1, ..., n\}$. Suppose $|I|$ is either $n-1$ or $n-2$. Then every face of $\tilde{G}$ must have at least one vertex from $\{v_i : i \in I\}$ adjacent to it. So we have $|F(I)| = |F| = 2n-4$ in these cases.

Therefore we end up with the following bound for the cases that $|I|$ is $n-1$ or $n-2$

$$\sum_{i \in I} x^* \leq 2\pi(n-3) + \frac{4\pi}{3} < (2n-4)\pi = |F(I)|\pi$$

Therefore in these cases, x^* exactly satisfies the conditions to belong to P^*. But we can easily show this whenever $|I| \leq n-3$! We simply apply Lemma 29 and we immediately get

$$\sum_{i \in I} x^* \leq 2\pi|I| < |F(I)|\pi$$

But this shows that in all cases, $x^* \in P^*$.

This completes both the claim, and the proof of the Circle Packing Theorem. ■

□

[18] This is quite straightforward to see; simply take convex combinations of elements and show these belong to the sets.

[19] Formally speaking we would identify the sets S and P^* in $\mathbb{R}^{n-1}$ with their images under a homeomorphism. Note this homeomorphism would be from the hyperplane to $\mathbb{R}^{n-1}$

6 Corollaries to The Circle Packing Theorem

6.1 Fary's Theorem

Fary's Theorem states that for any finite planar graph, there is a straight line embedding of the graph in the plane (see for example [8, Theorem 8.2]). In particular this follows immediately from the circle packing theorem. Indeed given any G, we know there is some circle packing. Now connect the centre of every circle to the centre of circles adjacent to it with a straight line. This is clearly a straight line embedding of G! In fact, this embedding is also seen to be composed of exactly the cardboard triangles described in the proof of the circle packing theorem.

6.2 The Lipton-Tarjan Separator Theorem

A beautiful corollary to the Circle Packing theorem is presented below. Note the proof of the theorem is primarily based on the proof of found in Pach [8, Theorem 8.3]. However the proof we present here differs slightly in the treatment of certain bounds. See Appendix B for details on the changes.

Theorem 30 (Lipton-Tarjan Separator Theorem)**.** *Let $G = (V, E)$ be a planar graph with n vertices. Then V can be partitioned into three parts, A, B and C, such that $|A|, |B| \leq \frac{3}{4}n$, $|C| < 2\sqrt{n}$ and no vertex in A is adjacent to any vertex in B.*

Proof. We note first that in the case $n \leq 4$, the theorem trivially holds. So suppose $n > 4$.

Let S denote the unit sphere in $\mathbb{R}^3$ centred at $(0, 0, 1)$, such that its North, N, and South, S, poles are given by $(0, 0, 2)$ and $(0, 0, 0)$ respectively.

We will make use of the stereographic projection. This is defined by $\pi : \mathbb{R}^2 \to S$, that maps any point p of the plane to the intersection of the line segment Np with S. For example see Figure 14 below. We note that π maps circles in the plane to spherical caps[20] on the sphere and vice-versa (for a proof of this see for example [3, Pages 248-251]).

[20]A spherical cap is a portion of a sphere that lies above or below a plane[13].

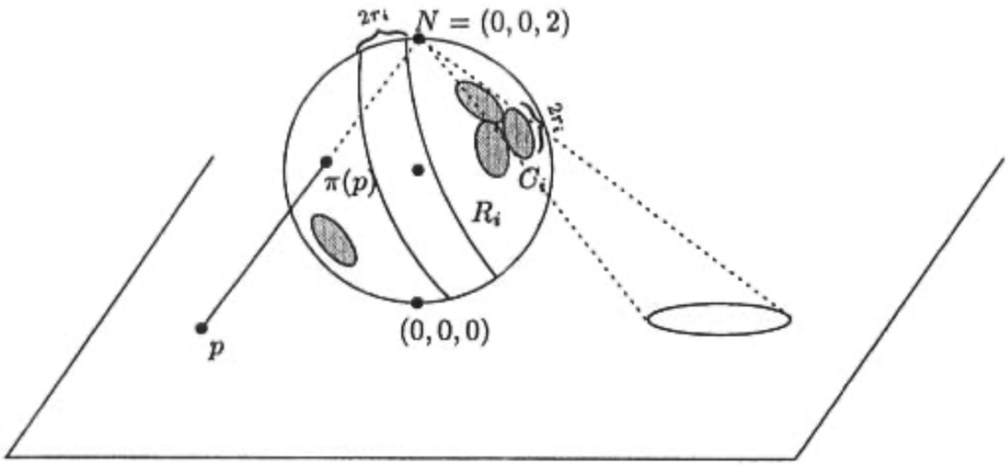


Figure 14: Illustration for the proof of Lipton-Tarjan. Taken from [8, Page 100]

.

By the circle packing theorem, we know there is a set of circles in the plane that correspond to G. We now project this onto the sphere, and obtain some spherical caps $C_1, ..., C_n \subset S$. From the definition of the stereographic projection, it is also clear that none of these spherical caps overlap, since none of the circles from the circle packing overlap in the plane. We also note the spherical caps C_i, C_j touch if and only if v_i is adjacent to v_j.

Choose any point p_i from each of the spherical caps C_i. Define $P = (p_1, ..., p_n) \subset S$. By Lemma 10, there exists a centrepoint, $q \in \mathbb{R}^3$, of P. Notice the following: q cannot be a point outside the closed unit ball centred at $(0, 0, 1)$, denoted S_B (this is the closed ball that has S as its closure). Indeed, suppose it is. Then take any closed half-space, H_S, that contains the closed ball, but does not contain q (it is immediately obvious such a plane exists because $q \notin S$). But by Lemma 10, we know that since H_S does not contain q, it can contain at most $\frac{3}{4}n$ points of P. But all of S is contained in H_S, so in fact H_S contains n points of P. Since $n > 4$, we have a contradiction. Therefore the centrepoint of P must lie within the closed ball.

Consider now any map $f : S_B \to S_B$ that is a rotation about the centre of the sphere (see Appendix A for details of such rotations). The map f is defined on the closed ball S_B. Take its restriction to S, $f|_S$. Then $f|_S$ is clearly an isometry because it is a composition of isometries. In particular, $f|_S : S \to S$ sends spherical caps to spherical caps, and preserves the tangency relations of the spherical caps.

Notice that since the rotation is an isometry, the following property must hold: Suppose f is a rotation of S_B about the centre of the sphere. If γ is a centrepoint of $\{x_1, ..., x_k\} \subset S$ for some $k \in \mathbb{N}$, then $f(\gamma)$ is a centrepoint of $\{f(x_1), ..., f(x_k)\}$. Indeed, take any plane, P through γ. Then since f is an isometry, $f(P)$ will be a plane containing $f(\gamma)$. But if there were j elements from $\{x_1, ..., x_k\}$ contained in either of the half-spaces induced by P, there must also be j elements of $\{f(x_1), ..., f(x_k)\}$ in the corresponding half-space induced by $f(P)$. If this was not the case, the isometry of f would clearly be violated. Therefore we see by definition that $f(\gamma)$ is necessarily a centrepoint of $\{f(x_1), ..., f(x_k)\}$.

Since $q \in S_B$, there exists some rotation, f, that sends q onto the line segment between $(0,0,1)$ and $(0,0,0)$. That is, there exists some rotation that sends the centrepoint to a point on the line segment between the centre and South pole of the sphere. By what we have shown, this new point will be the centrepoint of $\tilde{P} = \{f(p_1), ..., f(p_n)\}$, which are still points in some spherical caps that have the original tangency relations from G.

We may therefore assume without loss of generality that the centrepoint, q, lies on the line segment between the centre and South pole of the sphere. If q does not coincide with the centre of the sphere, it can be shown that there exists an appropriate map that sends spherical caps to spherical caps, while preserving tangency relations, and such that q is re-positioned to the centre of the sphere.

The idea is to use the inverse stereographic projection to map the sphere into the plane, then dilate the plane by an appropriate factor before mapping back onto the sphere by stereographic projection. The particular dilation factor on the plane is given by $\sqrt{\frac{1+d}{1-d}}$, where $d = |q - (0,0,1)|$. That is, we define a map

$$\sigma(p) = \begin{cases} \pi\left(\sqrt{\frac{1+d}{1-d}}\pi^{-1}(p)\right) & \text{if } p \neq N \\ N, & \text{if } p = N \end{cases}$$

The key point is that σ sends planes passing through the original centrepoint to planes passing through the centre of the sphere. In particular it is not hard to see that σ preserves spherical caps[21]. This means if two half-spaces are induced by a plane through the original centrepoint, then the number of points of P contained in the corresponding half-spaces is invariant under σ. Hence the centre of the sphere is a centrepoint for the set of points P under σ. More details can be found in Miller Et al. ([7, Lemma 2.3.3.1])

This allows us to assume without loss of generality that the centrepoint of P is located at the centre of the sphere. In particular, this means any half-space induced by a plane passing through $(0,0,1)$ contains at most $\frac{3}{4}n$ elements of P (by Lemma 10).

[21] Indeed we already know π maps circles to spherical caps and vice-versa. All we need to notice then is that dilations map circles to circles in the plane.

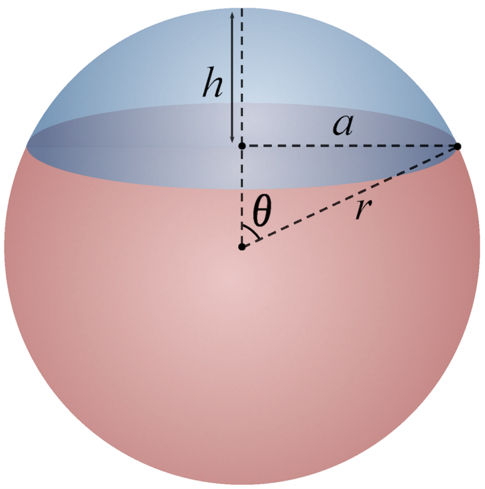


Figure 15: An example of a spherical cap. Taken from Wikimedia commons [6].

Let us briefly recapitulate: We have a set $\{C_1, ..., C_n\}$ of spherical caps from which we have chosen a set of points $P = \{p_1, ..., p_n\}$ with each $p_i \in C_i$, and the centrepoint of P can be assumed to be at the centre of the sphere.

We will refer to the length denoted by 'a' in Figure 15 as the *base* of the spherical cap, and 'h' the *height*. Note a formula for the area of spherical cap, C, is given by $A(C) = \pi(h^2 + a^2)$, where a, h are as in Figure 15. We may rewrite this as $A(C) = \pi r^2$ where the length r is the hypotenuse of the triangle induced by a and h. Clearly then we have that $\pi a^2 < \pi r^2$, i.e. the area of the spherical cap is an upper bound for the area of a circle with radius equal to that of the base of the spherical cap. Therefore letting a_i be the base of the spherical cap C_i, we obtain the bound

$$\sum_{i=1}^{n} \pi a_i^2 < \sum_{i=1}^{n} A(C_i) < 4\pi$$

The last inequality comes from the fact that the spherical caps form a packing, so in particular there is no overlap of caps. Therefore the sum of those areas must be bounded by the area of the unit sphere which is 4π.

We now make use of Jensen's inequality[22] (using the function $f(x) = x^2$ which is convex), and obtain

$$4 > \sum_{i=1}^{n} a_i^2 \geq n\left(\frac{\sum_{i=1}^{n} a_i}{n}\right)^2$$

and therefore

$$\sum_{i=1}^{n} a_i < 2\sqrt{n} \tag{5}$$

Consider now any plane passing through $(0, 0, 1)$. Let $u(H)$ denote the unit normal vector sitting at $(0, 0, 1)$. Consider the locus of the endpoint of $u(H)$ as we vary H over all planes

[22]This is a famous result that states: Given a convex function f, points $x_1, ..., x_n$ in the domain of f and positive weights $a_1, ..., a_n$, then $f\left(\frac{\sum a_i x_i}{\sum a_i}\right) \leq \frac{a_i f(x_i)}{\sum a_i}$

through the centre of the sphere that intersect a fixed spherical cap C_i. This locus will be a ringlike region, R_i, of the sphere, symmetric about a circle whose centre is the origin of the sphere. See Figure 14 for an example. Note that the spherical radius of the ringlike region R_i is equal to the spherical radius of the cap.

Consider now the sphere 'encased' in a cylinder. We will consider the cylinder whose height is $2a_i$, where a_i is the base of the spherical cap C_i, and that has a radius of 1. Consider Figure 16 below which shows a cross sectional view of the set-up.

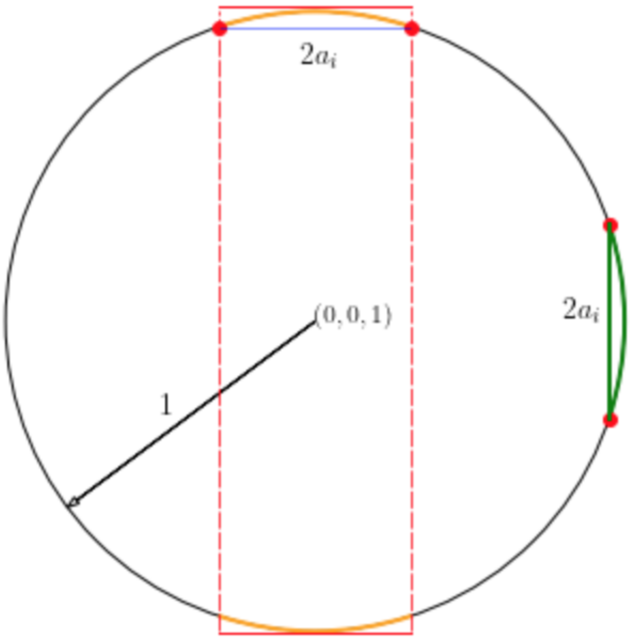


Figure 16: A cross-sectional view of a cylinder encasing a ringlike region.

In Figure 16, the region bounded by the green line and arc represents a (cross-sectional view of) the spherical cap C_i. The red solid and dashed lines represent a cylinder with height $2a_i$, and radius 1. The orange arcs on the sphere represent the spherical radii of the ringlike region R_i induced by C_i.

We now prove that the the area of R_i is exactly equal to the area of the cylinder as represented in Figure 16. Note the two dashed red lines can be thought of as defining two spherical caps of equal area. To calculate the area of $A(R_i)$, we simply need to subtract the area of these two caps from the area of the sphere. Another formula to calculate the area of spherical caps is given by $2\pi Rh$, where h is as in Figure 15, and R is the radius of the sphere, in our case 1. In our case, obviously $h = 1 - a_i$. Thus the area of R_i is given by

$$A(R_i) = 4\pi - 2(2\pi(1 - a_i)) = 4\pi a_i$$

But note, the area of our cylinder is exactly $(2\pi)(2a_i) = 4\pi a_i$. Therefore we have

$$A(R_i) = 4\pi a_i$$

With this, consider the following bound

$$\frac{\sum_{i=1}^{n} A(R_i)}{A(S)} = \frac{\sum_{i=1} 4\pi a_i}{4\pi} = \sum_{i=1}^{n} a_i < 2\sqrt{n} \tag{6}$$

Where the strict inequality is via Equation 5.

We now show that this must mean there exists at least one point in S which is covered by at most $2\sqrt{n}$ regions. To do this we construct a partition of the sphere. We define an equivalence relation $\sim$ on S by $x \sim y$ iff x and y are covered by exactly k ringlike regions, where $0 \leq k \leq n$. It is immediate to see this defines an equivalence relation, and hence a partition of the sphere. This partition can be thought of as 'labelled' in some sense: each element of the partition, that is an element of $S/_\sim$, has some unique non-negative integer associated to it. Notice also that since the number of regions is finite, $S/_\sim$ must also be finite. Suppose then $S/_\sim = \{E_1, ..., E_j\}$ for some positive integer j. By construction, we have

$$\sum_{l=1}^{j} A(E_l) = A(S)$$

$$\sum_{l=1}^{j} A(E_l)k_l = \sum_{i=1}^{n} A(R_i)$$

Where k_l is the number of ringlike regions that cover E_l.

Suppose for the sake of contradiction that every point lies in more than or equal to $2\sqrt{n}$ regions. Then we see that

$$2\sqrt{n}\sum_{l=1}^{j} A(E_l) \leq \sum_{l=1}^{j} A(E_l)k_l = \sum_{i=1}^{n} A(R_i)$$

$$\Rightarrow \qquad 2\sqrt{n} \leq \frac{\sum_{i=1}^{n} A(R_i)}{\sum_{l=1}^{j} A(E_l)}$$

$$\Rightarrow \qquad 2\sqrt{n} \leq \frac{\sum_{i=1}^{n} A(R_i)}{A(S)}$$

which is a contradiction by Equation 6. This proves that there exists at least one point,γ , in S covered by at most $2\sqrt{n}$ regions. But this implies that the plane, H_0, that goes through the centre of the sphere, and is normal to the vector going from $(0, 0, 1)$ to γ, intersects strictly fewer than $2\sqrt{n}$ spherical caps.

Finally, define A, B, C to be the vertices of G corresponding to the spherical caps C_i that lie entirely on one side of H_0, entirely on the other side, or meet H_0, respectively. From what we have shown, $|A|, |B| \leq \frac{3}{4}n$, and $|C| < 2\sqrt{n}$. Since A and B correspond to the spherical caps lying on opposite sides of H_0, no vertices in A and B can be adjacent. This is exactly the statement of the theorem and the proof is complete. $\square$

A Appendix A

Rotations in Three Dimensional Euclidean Space:

A rotation about a point (x, y, z) in three dimensional Euclidean space is given by a map $f = T^{-1} \circ R \circ T$, where

$$R = \begin{bmatrix} \cos(\gamma) & -\sin(\gamma) & 0 \\ \sin(\gamma) & \cos(\gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos(\beta) & 0 & \sin(\beta) \\ 0 & 1 & 0 \\ -\sin(\beta) & 0 & \cos(\beta) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & -\sin(\alpha) \\ 0 & \sin(\alpha) & \cos(\alpha) \end{bmatrix}$$

and T is a translation. In the case of our proof, the translation T is defined by sending $(0, 0, 1)$ to $(0, 0, 0)$.

B Appendix B

Changes made to the proof of The Lipton-Tarjan Theorem:

The proof presented in Pach ([8, Theorem 8.3]) asserts the bound $\sum_{i=1}^{n} \pi r_i^2 < 4\pi$, where here r_i refers to the *spherical* radius of the caps. However, it is not enitrely clear how this bound is obtained.

The following bounds are obvious (or at least easy to prove based on observations in the main text):

$$\sum_{i=1}^{n} A(C_i) < \sum_{i=1} \pi r_i^2$$

$$\sum_{i=1}^{n} A(C_i) < 4\pi$$

However the proof in Pach asserts

$$\sum_{i=1} \pi r_i^2 < 4\pi$$

In the source, it is claimed that this follows from the fact that the set of spherical caps forms a packing. However this fact manifests itself in the bound $\sum_{i=1}^{n} A(C_i) < 4\pi$. It allows us to do this because the the spherical caps live on the sphere. However, it is *not* clear how this fact can be used to establish the bound $\sum_{i=1} \pi r_i^2 < 4\pi$. Indeed, the (two dimensional) circle of radius r_i is not even homeomorphic to the sphere, so it is not obvious (at least geometrically) how to compare the two areas to obtain the bound.

Another bound given in Pach asserts that $A(R_i) < 4\pi r_i$. In fact in the main text we prove this is incorrect; there is an exact equality.

Regardless of the veracity of the bounds, the argument can be reformulated in terms of the bases of the spherical caps, and this is what we do in this text. This reformulation gives slightly different bounds compared to Pach, but the final result still holds.

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