Lecture : 1

Instructor

Azeem Iqbal

azeem.iqbal@uet.edu.pk

Textbooks:

REQUIRED:

Introduction to Probability by Dimitri P. Bertsekas and John N. Tsitsiklis, 2nd Edition, Athena Scientific, 2008

OPTIONAL:

Probability, Random Variables and Stochastic Processes by Athanasios Papoulis and S. U. Pillai, 4th Edition, McGraw Hill, 2002

Course Objectives:

The objectives of this course are to teach the Students:

1) Compute simple and conditional probabilities in different situations

2) Efficiently use the concept of random variables

3) Apply suitable distribution in solving real life problems

Course Learning Outcomes :

Students shall be able to:

CLO1 : Apply elements of probability theory to various problems in engineering

CLO2 : Determine densities/distributions and expectations of discrete and continuous, single and multiple, random variables.

CLO3 : Analyze and understand random processes and apply second moment theory to random processes

Grading Policy

Quizzes	30%
Midterm	30%
Final	40%

Instructions:

- Probability is a tricky subject.
- Problem solving is the key part of this class. (**Practice – Practice - Practice**)
- We will study basic concepts of probability and a lot of formulas in this class.
- This is not the plug-in formulas class, where you are given a list of formulas and numbers, you plug in numbers and get answers.
- You have to choose formula for every situation.
- A problem may have multiple solutions.
  - Short solutions need in depth understanding.

Introduction

- We live in a world of uncertainty.

- We study probability models to study the nature of uncertainty in the world.

Applications

- **Communication:** Noise is random. You have to study the noise in order to improve your communication system.
- **Management:** You have to study customer demands which are random. You also need help of probability for Investment planning.
- **Finance:** Markets are uncertain, and whoever has the best methods to analyze financial data has an advantage.
- **Computer Science:** Probability also plays an important role in machine learning and artificial intelligent.
- **Transportation Systems:** Random disruptions due to weather or accidents are a major concern.

Applications

- I could go on and on, giving you many more examples.
- But the message is hopefully clear.
- Most phenomena of interest involve significant randomness.
- And the only reason we collect and manipulate data is because we want to fight this randomness as much as we can.
- And the first step in fighting an enemy like randomness is to study and understand your enemy.

Elementary Concepts

- Sample Space

- Probability Laws
  - Axioms
  - Properties that follows from axioms

- Examples
  - Discrete
  - Continuous

Set Theory

Definition 1: A set is a collection of objects, which are called the elements of the set.

Ex: **A** = {1, 2, 3, . . .,}

B = {Monday, Wednesday, Friday},

C = {real numbers (x, y): min (x, y) $\leq$ 2}.

(Finite, Countably Infinite, Uncountably Infinite)

Null set=empty set=$\varnothing$={}

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Set Theory

The **universal set (Ω):** The set which contains all the elements under investigation.

Some relations

- A is a subset of B ($A \subset B$) if every element of A is also an element of B.
- A and B are equal ($A = B$) if they have the same elements.

Set Operations

1. UNION
2. INTERSECTION
3. COMPLEMENT of a set
4. DIFFERENCE

- Two sets are called disjoint or **mutually exclusive** if $\mathbf{A \cap B = \varnothing}$.

- A collection of sets is said to be a **partition** of a set S if the sets in the collection are disjoint and their union is S.

Probability Models

A probabilistic model is a mathematical description of an uncertain situation. A probability model consists of an experiment, a sample space, and a probability law.

Experiment

Every probabilistic model involves an underlying process called the experiment.

Sample Space

The list (set) of all possible results (OUTCOMES) of an experiment is called the SAMPLE SPACE (Ω) of the experiment.

Probability Models

List must be:

- **Collectively exhaustive** (No matter what happened, you will get one of the outcomes in the sample space)
- **Mutually exclusive** (If outcome A is happened than outcome B could not be happened)

Ex: List the sample spaces corresponding to the following experiments:

➢ **Experiment 1**: Toss a coin and look at the outcome.

$\Omega = \{H, T\}$ – Finite

➢ **Experiment 2**: Toss a coin until you get "Heads".

$\Omega = \{TH, TTH, TTTH, TTTTH, \ldots\ldots\ldots\ldots\}$ – Countably Infinite

Probability Models

- **Experiment 3**: Throw a dart into a circular region of radius r, and check how far it fell from the center.

  $\Omega = \{x: 0 \le x \le r, x \in R\}$ – Uncountably Infinite

- **Experiment 4**: Pick a point (x, y) on the unit square.

  $\Omega = \{(x, y) \in R^2, -\frac{1}{2} \le x \le \frac{1}{2}, -\frac{1}{2} \le y \le \frac{1}{2}\}$ – Uncountably Infinite

- **Experiment 5**: A family has two children.

  Ω = {GG, BB,GB,BG}

Probability Models

Definition 2: An event is a subset of the sample space Ω.

Ω: certain event **$\varnothing$:** impossible event

TRIAL: Single performance of an experiment

$\Rightarrow$ An event A is said to have OCCURRED if the outcome of the trial is in A.

Probability Law

- The probability law assigns to every event A a nonnegative number **P(A)** called the probability of event A.

- Intuitively, this specifies the "likelihood" of any outcome, or of any set of possible outcomes.

Probability Axioms

Event: a subset of the sample space

1. **(Nonnegativity):** $P(A) \geq 0$ for every event A

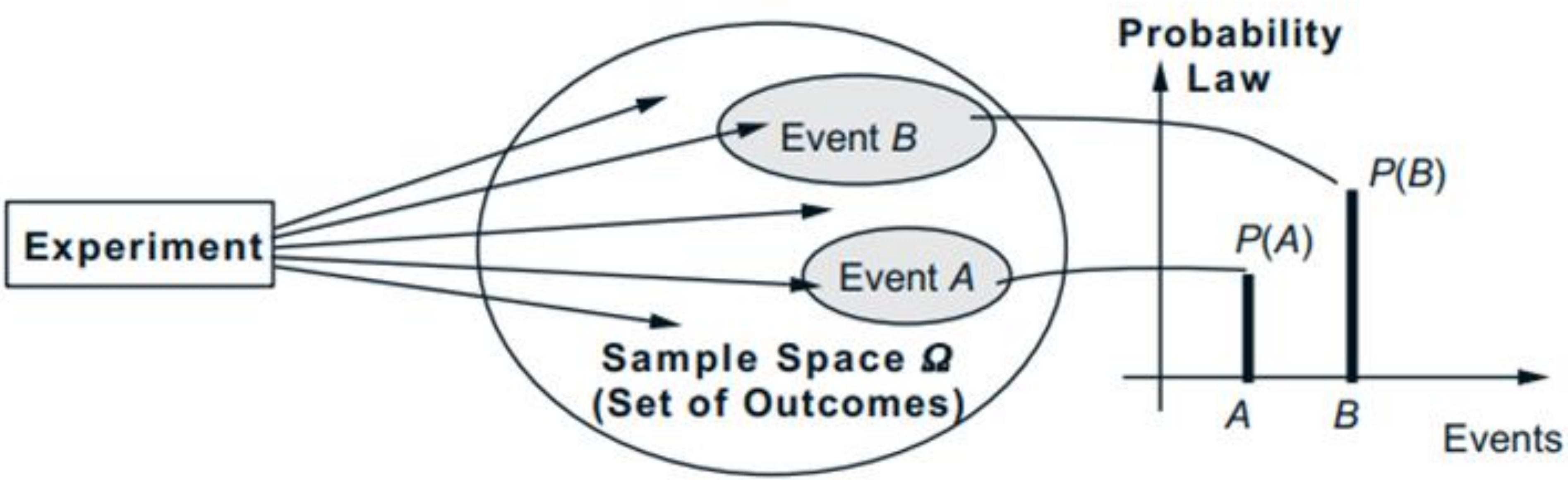
2. **(Additivity):** If A and B are two disjoint events, then

$$P(A \cup B) = P(A) + P(B).$$

More generally, if the sample space has an infinite number of elements and $A1, A2, \ldots$ is a sequence of disjoint events, then $P(A1 \cup A2 \cup \ldots) = P(A1) + P(A2) + \ldots.$

3. **(Normalization):** $P(\Omega) = 1$

Probability Axioms



Properties of Probability Laws

$(a)\ \ P(\emptyset) = 0,$

$$\Omega = \Omega \cup \phi\ (disjoint\ events)$$

$\Rightarrow\ P(\Omega) = P(\Omega \cup \phi)$

$\Rightarrow\ P(\Omega) = P(\Omega) + P(\emptyset)$ (Additivity axiom)

$\Rightarrow 1 = 1 + P(\emptyset) \Rightarrow P(\emptyset) = 0$ (Normalization axiom)

(b) $P(A^c) = 1 - P(A)$

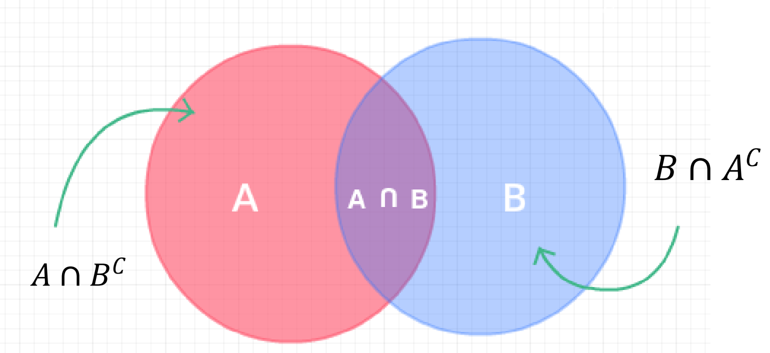
P(Ω) = 1 (Normalization axiom)

$\Rightarrow P(A \cup A^c) = 1$

$\Rightarrow P(A) + P(A^c) = 1\ (Additivity\ axiom)$

Properties of Probability Laws

(c) P (A ∪ B) = P(A) + P(B) – P (A ∩ B) [if sets are not disjoint]



$$A \cup B = (A \cap B^C) \cup (A \cap B) \cup (B \cap A^C)\ [These\ all\ disjoint\ sets]$$

$$P(A \cup B) = P(A \cap B^C) + P(A \cap B) + P(B \cap A^C)\quad [additivity\ axiom]$$

$$P(A \cup B) = P(A \cap B^C) + P(A \cap B) + P(B \cap A^C) + P(A \cap B) - P(A \cap B)$$

$$P(A) = P(A \cap B^C) + P(A \cap B)$$

$$P(B) = P(B \cap A^C) + P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

(d) $A \subset B \Rightarrow P(A) \le P(B)$

Practice Problem

Let A and B be the events on the same sample space, with P(A)=0.6, P(B)=0.7. Can these events be disjoint?

Solution: If two events are disjoint, then additivity axiom would imply

$\boldsymbol{P(A \cup B) = P(A) + P(B\ ) = 1.3 > 1} \Rightarrow$ Which contradicts the Normalization Theorem

Discrete Probability Models

The sample space is a countable (finite or infinite) set in discrete models.

Example: An experiment involving a single coin toss. We say that the coin is "fair", equal probabilities are assigned to the possible outcomes. That is, P(H) = P(T) = 1/2.

Discrete Probability Law

If the sample space consists of a finite number of possible outcomes, then the probability law is specified by the probabilities of the events that consist of a single element. In particular, the probability of any event $\{s_1, s_2, \ldots, s_n\}$ is the sum of the probabilities of its elements:

$$\mathbf{P}(\{s_1, s_2, \ldots, s_n\}) = \mathbf{P}(\{s_1\}) + \mathbf{P}(\{s_2\}) + \cdots + \mathbf{P}(\{s_n\}).$$

Discrete Probability Models

Discrete Uniform Probability Law

If the sample space consists of n possible outcomes which are equally likely (i.e., all single-element events have the same probability), then the probability of any event A is given by

$$\mathbf{P}(A) = \frac{\text{Number of elements of } A}{n}.$$

Discrete Probability Models

Sample space: discrete/finite example

- Two rolls of a tetrahedral die

Example 1.3.

Dice. Consider the experiment of rolling a pair of 4-sided dice

Solution:

P{the sum of the rolls is even}= 8/16 = 1/2,
P{the sum of the rolls is odd}= 8/16 = 1/2,
P{the first roll is equal to the second}= 4/16 = 1/4,
P{the first roll is larger than the second}= 6/16 = 3/8,
P{at least one roll is equal to 4}= 7/16.

Y = Second roll

4	(1,4)	(2,4)	(3,4)	(4,4)
3	(1,3)	(2,3)	(3,3)	(4,3)
2	(1,2)	(2,2)	(3,2)	(4,2)
1	(1,1)	(2,1)	(3,1)	(4,1)
	1	2	3	4

X = First roll

Discrete Probability Models

Another Example:

- Two rolls of tetrahedral die
- Let every possible outcome have probability 1/16
    - $P(X = 1) = \frac{1}{16} + \frac{1}{16} + \frac{1}{16} + \frac{1}{16} = \frac{1}{4}$
    - $Let\ Z = \min(X, Y)$
    - $P(Z = 4) = \frac{1}{16}$
    - $P(Z = 2) = 5 * \frac{1}{16}$

Y = Second roll				
4	(1,4)	(2,4)	(3,4)	(4,4)
3	(1,3)	(2,3)	(3,3)	(4,3)
2	(1,2)	(2,2)	(3,2)	(4,2)
1	(1,1)	(2,1)	(3,1)	(4,1)
	1	2	3	4

X = First roll

Discrete Probability Models

Sample space: discrete but infinite

- Sample space: $\{1, 2, ...\}$
- We are given $P(n) = \frac{1}{2^n}$ $, n = 1,2,3, \ldots ..$

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots \quad = \frac{1}{2}\left[\frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \cdots\right] = \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n}$$

$$= \frac{1}{2}\left[\frac{1}{1 - (\frac{1}{2})}\right] = 1 \ \ (geometric\ series)$$

Discrete Probability Models

- $P(outcome\ is\ even) = ?\,, P(\{2,4,6,8,\ldots..\})$

$$P(\{2,4,6,8,\ldots..\}) = \mathrm{P}(\{2\}\mathrm{U}\{4\}\mathrm{U}\{6\}\ldots\ldots\ldots.)$$

$$= \mathrm{P}\{2\} + \mathrm{P}\{4\} + \mathrm{P}\{6\}\ldots\ldots\ldots$$

$$= \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \cdots$$

$$= \frac{1}{4}\left[1 + \frac{1}{4} + \frac{1}{4^2} + \cdots\right]$$

$$= \frac{1}{4}\left[\frac{1}{1-(\frac{1}{4})}\right] = 1/3$$