Lecture No. 14

Functions of Two Random Variables

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- There are many situations in which we observe two random variables and use their values to compute a new random variable.

- For example, we can describe the amplitude of the signal transmitted by a radio station as a random variable, X.

- We can describe the attenuation of the signal as it travels to the antenna of a moving car as another random variable, Y .

- In this case the amplitude of the signal at the radio receiver in the car is the random variable $\boldsymbol{W = X/Y}$..

- Other practical examples appear in cellular telephone base stations with two antennas [MIMO systems contain multiple antennas].

- The amplitudes of the signals arriving at the two antennas are modeled as random variables X and Y. .

- The radio receiver connected to the two antennas can use the received signals in a variety of ways.

  - It can choose the signal with the larger amplitude and ignore the other one. In this case, the receiver produces the random variable $\boldsymbol{W = X \; if} \; |\boldsymbol{X}| > |\boldsymbol{Y}| \; and \; \boldsymbol{W = Y}$, otherwise. This is an example of **selection diversity combining**.

- The receiver can add the two signals and use $W = X + Y$. This process is referred to as **equal gain combining** because it treats both signals equally.

- A third alternative is to combine the two signals unequally in order to give less weight to the signal considered to be more distorted. In this case $\boldsymbol{W = aX + bY}$. If a and b are optimized, the receiver performs **maximal ratio combining**.

- All three combining processes appear in practical radio receivers.

Theorem

For discrete random variables X and Y, the derived random variable $W = g(X, Y)$ has PMF

$$P_W(w) = \sum_{(x,y):g(x,y)=w} P_{X,Y}(x, y)$$

Example

A firm sends out two kinds of promotional facsimiles. One kind contains only text and requires 40 seconds to transmit each page. The other kind contains grayscale pictures that take 60 seconds per page. Faxes can be 1, 2, or 3 pages long.

Let the random variable L represent the length of a fax in pages. $S_L = \{1, 2, 3\}$.

Let the random variable T represent the time to send each page. $S_T = \{40, 60\}$.

After observing many fax transmissions, the firm derives the following probability model:

$P_{L,T}(l,t)$	$t = 40$ sec	$t = 60$ sec
$l = 1$ page	0.15	0.1
$l = 2$ pages	0.3	0.2
$l = 3$ pages	0.15	0.1

Let $\mathbf{D} = \mathbf{g}(\mathbf{L}, \mathbf{T}) = \mathbf{LT}$ be the total duration in seconds of a fax transmission.

Find the range S_D, the PMF $P_D(\mathrm{d})$, and the expected value E[D].

Solution

By examining the six possible combinations of L and T we find that the possible values of D are $\boldsymbol{S_D = \{40, 60, 80, 120, 180\}}$.

For the five elements of $\boldsymbol{S_D}$, we find the following probabilities:

$$
\begin{aligned}
P_D(40) &= P_{L,T}(1,40) = 0.15, & P_D(120) &= P_{L,T}(3,40) + P_{L,T}(2,60) = 0.35, \\
P_D(60) &= P_{L,T}(1,60) = 0.1, & P_D(180) &= P_{L,T}(3,60) = 0.1, \\
P_D(80) &= P_{L,T}(2,40) = 0.3, & P_D(d) &= 0; \quad d \neq 40, 60, 80, 120, 180.
\end{aligned}
$$

Solution

The expected duration of a fax transmission is

$$
\begin{aligned}
E[D] &= \sum_{d \in S_D} d P_D(d) \\
&= (40)(0.15) + 60(0.1) + 80(0.3) + 120(0.35) + 180(0.1) = 96 \text{ sec.}
\end{aligned}
$$

Theorem

For continuous random variables X and Y , the CDF of W is

$$F_W(w) = F_{X,Y}(w, w) = \int_{-\infty}^{w} \int_{-\infty}^{w} f_{X,Y}(x, y)\, dx\, dy.$$

- There are many situations in which we are interested only in the expected value of a derived random variable $W = g(X, Y)$ not the entire probability model.

- In these situations, we can obtain the expected value directly from $P_{X,Y}(x, y)$ or $f_{X,Y}(x, y)$ without taking the trouble to compute $P_W(w)$ or $f_W(w)$.

$$\textit{Discrete:} \quad E[W] = \sum_{x \in S_X} \sum_{y \in S_Y} g(x, y) P_{X,Y}(x, y),$$

$$\textit{Continuous:} \; E[W] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) f_{X,Y}(x, y)\, dx\, dy$$

Theorem **For any two random variables X and Y**

$$E[X+Y] = E[X] + E[Y]$$

- An important consequence of this theorem is that we can find the expected sum of two random variables from the separate probability models: $P_X(x)$ and $P_Y(y)$ or $f_X(x)$ and $f_Y(y)$.

- We do not need a complete probability model embodied in $P_{X,Y}(x,y)$ or $f_{X,Y}(x,y)$.

- By contrast, the variance of $\mathbf{X + Y}$ depends on the entire joint PMF or joint CDF.

Theorem   Variance of the sum of two random variables

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2E[(X-\mu_X)(Y-\mu_Y)]$$

Variance is the second central moment

Proof: Since $E[X+Y] = u_X + u_Y$

$$Var[X+Y] = E[\left(X+Y-(u_X+u_Y)\right)^2]$$

$$Var[X+Y] = E[\left((X-u_X)+(Y-u_Y)\right)^2]$$

$$= E[(X-u_X)^2 + (Y-u_Y)^2 + 2(X-u_X)(Y-u_Y)]$$

$$= E[(X-u_X)^2] + E[(Y-u_Y)^2] + 2E[(X-u_X)(Y-u_Y)]$$

$$= Var[X] + Var[Y] + 2E[(X-u_X)(Y-u_Y)]$$

Adopt a growth mindset. When you open yourself up to challenges that sharpen your skills and you can master anything.