Lecture No. 13
Joint Cumulative Density Function

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Definition

Joint Cumulative Density Function (CDF)

The joint CDF of the continuous random variables X and Y is a function $F_{X,Y}(x,y)$ with the property

$$F_{X,Y}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u,v)\, dv\, du$$

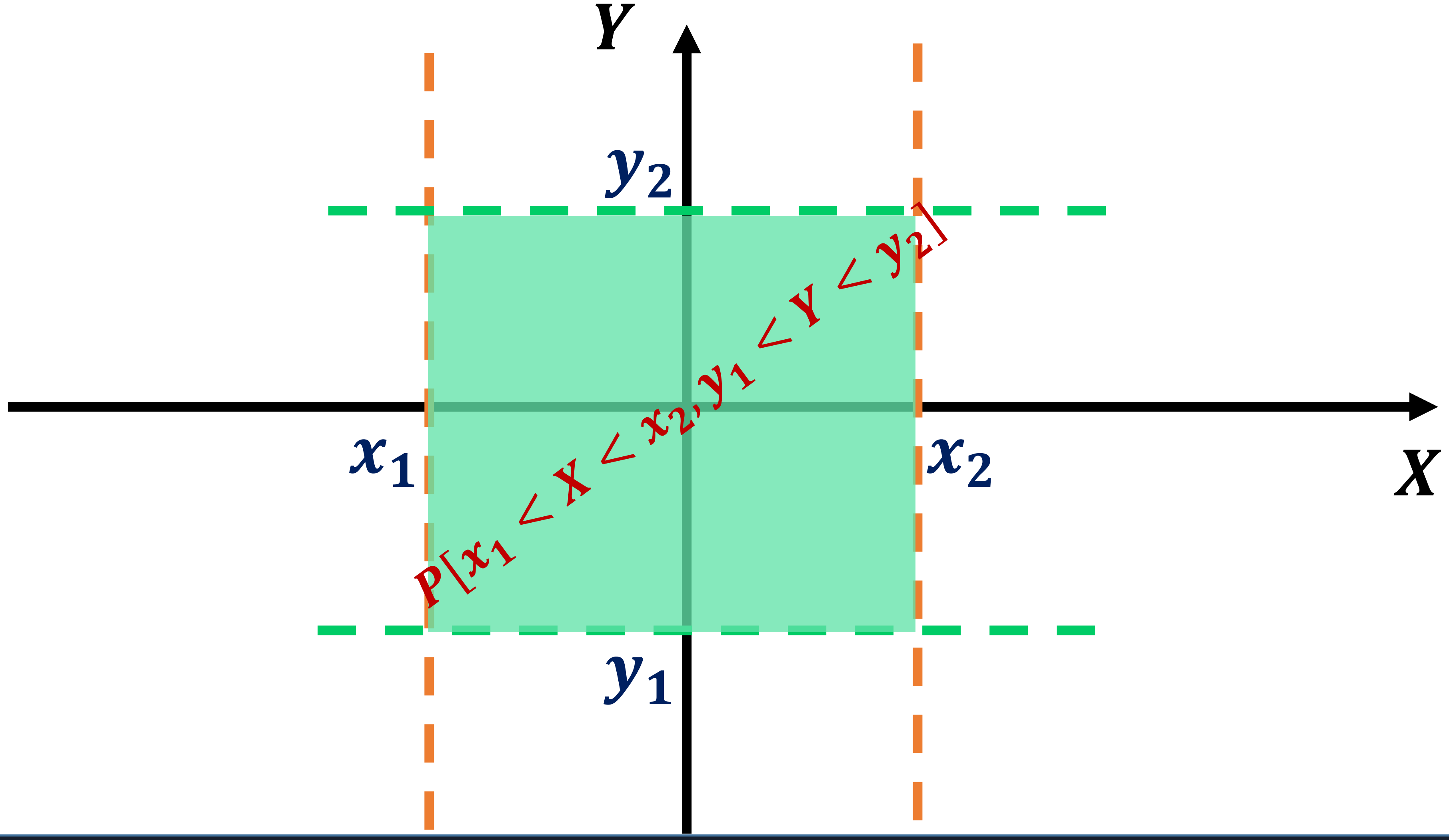
Theorem **Joint PDF and Joint CDF relation**

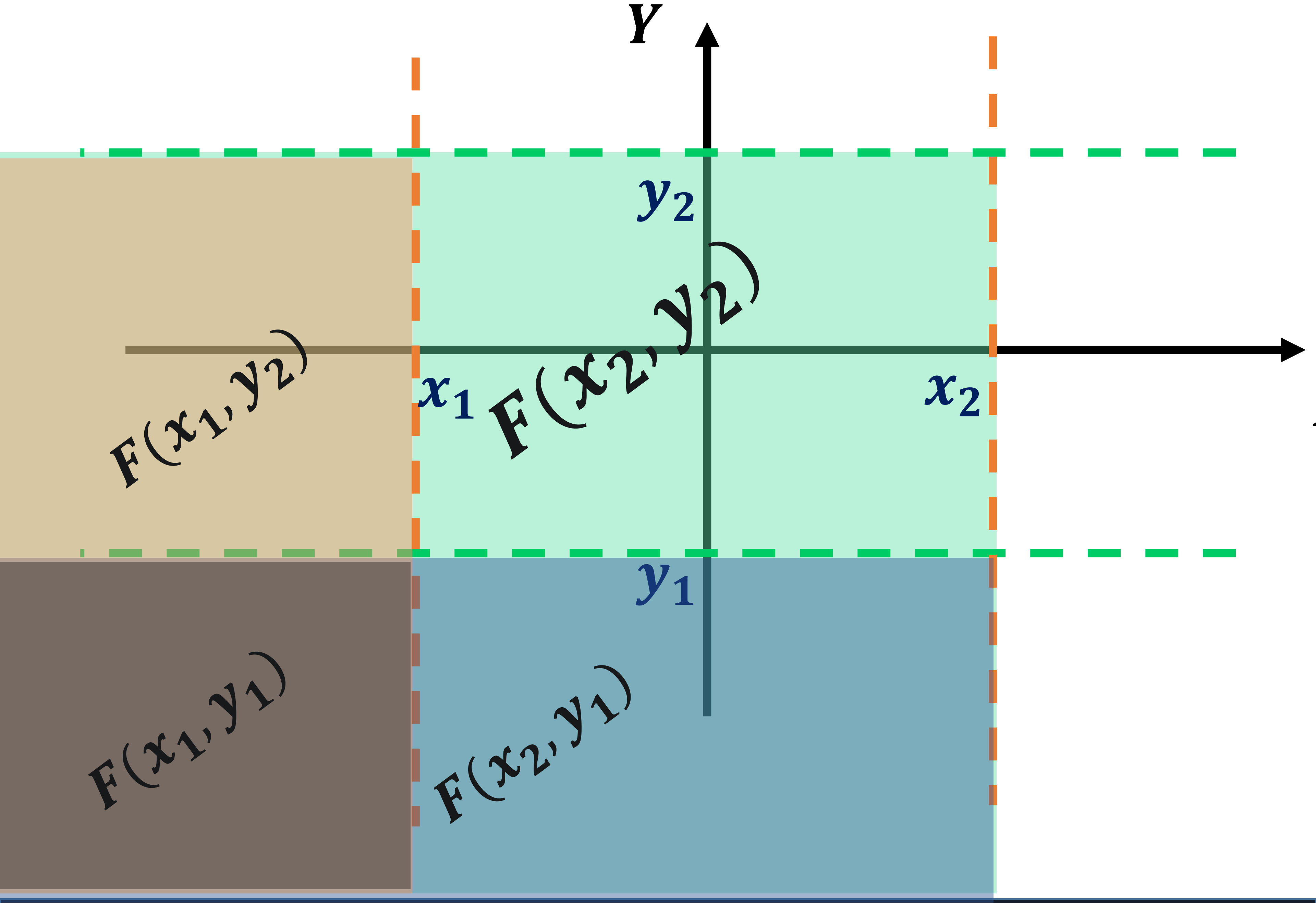
$$f_{X,Y}(x, y) = \frac{\partial^2 F_{X,Y}(x, y)}{\partial x \, \partial y}$$

Theorem **Probability of a finite rectangle in the X, Y plane in terms of the joint CDF**

$$P[x_1 < X \leq x_2, y_1 < Y \leq y_2] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$

$$P\left[x_1 < X \le x_2, y_1 < Y \le y_2\right] = F_{X,Y}(x_2, y_2) - F_{X,Y}(x_2, y_1) - F_{X,Y}(x_1, y_2) + F_{X,Y}(x_1, y_1)$$




Y
y_2
$F(x_2, y_2)$
$F(x_1, y_2)$
x_1
x_2
y_1
$F(x_1, y_1)$
$F(x_2, y_1)$

Theorem

A joint PDF $f_{X,Y}(x,y)$ has the following properties corresponding to first and second axioms of probability

$$(a)\quad f_{X,Y}(x,y) \geq 0 \text{ for all } (x,y),$$

$$(b)\quad \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

Theorem

The probability that the continuous random variables (X,Y) are in A is

$$P[A] = \iint_{A} f_{X,Y}(x,y)\,dx\,dy$$

Example

Random variables X and Y have join PDF

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 5, 0 \le y \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Find the

a) constant c

b) $P[A] = P[2 \le X < 3, 1 \le Y < 3]$

Solution

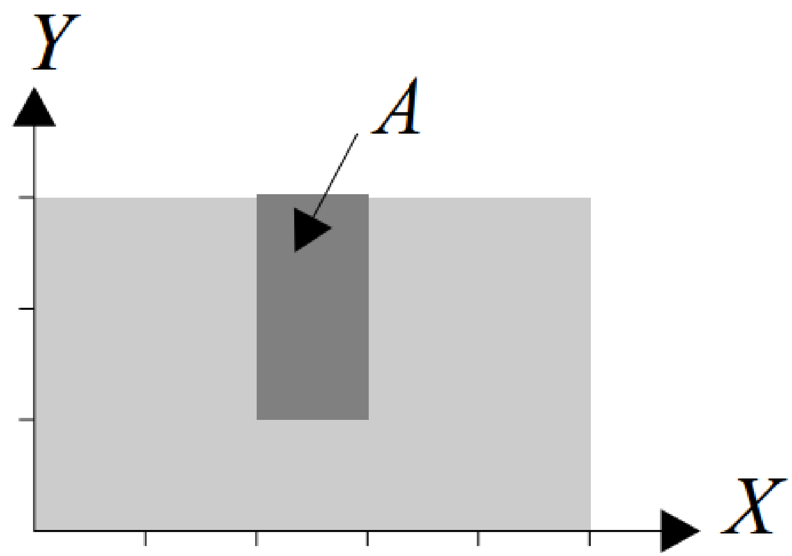
We studied that integral of the joint PDF over this rectangle is 1.

$$1 = \int_0^5 \int_0^3 c\, dy\, dx = 15c.$$

Therefore, **c = 1/15.**

Solution

The small dark rectangle in the diagram is the event
$\boldsymbol{A = \{2 \leq X < 3, 1 \leq Y < 3\}}$



$P[A]$ is the integral of the PDF over this rectangle, which is

$$P[A] = \int_2^3 \int_1^3 \frac{1}{15} \, dv \, du = 2/15.$$