Problem - 1

The cumulative distribution function of random variable X is

$$F_X(x) = \begin{cases} 0 & x < -1, \\ (x+1)/2 & -1 \le x < 1, \\ 1 & x \ge 1. \end{cases}$$

(a) What is $P[X > 1/2]$? Ans: 1/4

(b) What is $P[-1/2 < X \le 3/4]$? Ans: 5/8

(c) What is $P[|X| \le 1/2]$? Ans: 1/2

(d) What is the value of a such that $P[X \le a] = 0.8$? Ans: a=0.6

Solution

The CDF of X is

$$F_X(x) = \begin{cases} 0 & x < -1 \\ (x+1)/2 & -1 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1}$$

Each question can be answered by expressing the requested probability in terms of $F_X(x)$.

(a)

$$P[X > 1/2] = 1 - P[X \leq 1/2] = 1 - F_X(1/2) = 1 - 3/4 = 1/4 \tag{2}$$

(b) This is a little trickier than it should be. Being careful, we can write

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] + P[X = -1/2] - P[X = 3/4] \tag{3}$$

Since the CDF of X is a continuous function, the probability that X takes on any specific value is zero. This implies $P[X = 3/4] = 0$ and $P[X = -1/2] = 0$. (If this is not clear at this point, it will become clear in Section 3.6.) Thus,

$$P[-1/2 \leq X < 3/4] = P[-1/2 < X \leq 3/4] = F_X(3/4) - F_X(-1/2) = 5/8 \tag{4}$$

(c)

$$P[|X| \leq 1/2] = P[-1/2 \leq X \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] \tag{5}$$

Note that $P[X \leq 1/2] = F_X(1/2) = 3/4$. Since the probability that $P[X = -1/2] = 0$, $P[X < -1/2] = P[X \leq 1/2]$. Hence $P[X < -1/2] = F_X(-1/2) = 1/4$. This implies

$$P[|X| \leq 1/2] = P[X \leq 1/2] - P[X < -1/2] = 3/4 - 1/4 = 1/2 \tag{6}$$

(d) Since $F_X(1) = 1$, we must have $a \leq 1$. For $a \leq 1$, we need to satisfy

$$P[X \leq a] = F_X(a) = \frac{a+1}{2} = 0.8 \tag{7}$$

Thus $a = 0.6$.

Problem - 2

The random variable X has probability density function

$$f_X(x) = \begin{cases} cx & 0 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

Use the PDF to find

(a) the constant c, Ans: 1/2

(b) $P[0 \leq X \leq 1]$, Ans: 1/4

(c) $P[-1/2 \leq X \leq 1/2]$, Ans: 1/16

(d) the CDF $F_X(x)$.

Solution

$$f_X(x) = \begin{cases} cx & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) From the above PDF we can determine the value of c by integrating the PDF and setting it equal to 1.

$$\int_0^2 cx\,dx = 2c = 1 \tag{2}$$

Therefore $c = 1/2$.

(b) $P[0 \le X \le 1] = \int_0^1 \frac{x}{2}\,dx = 1/4$

(c) $P[-1/2 \le X \le 1/2] = \int_0^{1/2} \frac{x}{2}\,dx = 1/16$

(d) The CDF of X is found by integrating the PDF from 0 to x.

$$F_X(x) = \int_0^x f_X(x')\,dx' = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases} \tag{3}$$

Problem - 3

Find the PDF $f_W(w)$ of the random variable W

$$F_W(w) = \begin{cases} 0 & w < -5, \\ (w+5)/8 & -5 \le w < -3, \\ 1/4 & -3 \le w < 3, \\ 1/4 + 3(w-3)/8 & 3 \le w < 5, \\ 1 & w \ge 5. \end{cases}$$

Solution

We find the PDF by taking the derivative of $F_U(u)$ on each piece that $F_U(u)$ is defined. The CDF and corresponding PDF of U are

$$F_U(u) = \begin{cases} 0 & u < -5 \\ (u+5)/8 & -5 \leq u < -3 \\ 1/4 & -3 \leq u < 3 \\ 1/4 + 3(u-3)/8 & 3 \leq u < 5 \\ 1 & u \geq 5. \end{cases} \qquad f_U(u) = \begin{cases} 0 & u < -5 \\ 1/8 & -5 \leq u < -3 \\ 0 & -3 \leq u < 3 \\ 3/8 & 3 \leq u < 5 \\ 0 & u \geq 5. \end{cases} \tag{1}$$

Problem - 4

The peak temperature T , as measured in degrees Fahrenheit, on a July day in New Jersey is the Gaussian $(85, 10)$ random variable.

What is $P[T > 100]$, $P[T < 60]$, and $P[70 \leq T \leq 100]$?

Solution

$$P[T > 100] = 1 - P[T \leq 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100 - 85}{10}\right)$$

$$= 1 - \Phi(1.5) = 1 - 0.933 = 0.066$$

$$P[T < 60] = \Phi\left(\frac{60 - 85}{10}\right) = \Phi(-2.5)$$

$$= 1 - \Phi(2.5) = 1 - .993 = 0.007$$

$$P[70 \leq T \leq 100] = F_T(100) - F_T(70)$$

$$= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866$$

Problem - 5

The lifetime, in years, of some electronic component is a continuous random variable with the pdf

$$f(x) = \begin{cases} \dfrac{k}{x^3} & \text{for} \quad x \geq 1 \\ 0 & \text{for} \quad x < 1. \end{cases}$$

Find k, F (x), and compute the probability for the lifetime to exceed 5 years.

Solution

Solution. Find k from the condition $\int f(x)dx = 1$:

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{1}^{+\infty} \frac{k}{x^3}dx = -\frac{k}{2x^2}\Big|_{x=1}^{+\infty} = \frac{k}{2} = 1.$$

Hence, $k = 2$. Integrating the density, we get the cdf,

$$F(x) = \int_{-\infty}^{x} f(y)dy = \int_{1}^{x} \frac{2}{y^3}dy = -\frac{1}{y^2}\Big|_{y=1}^{x} = 1 - \frac{1}{x^2}$$

Next, compute the probability for the lifetime to exceed 5 years,

$$\boldsymbol{P}\{X > 5\} = 1 - F(5) = 1 - \left(1 - \frac{1}{5^2}\right) = 0.04.$$

Problem - 6

Jobs are sent to a printer at an average rate of 3 jobs per hour.

(a) What is the expected time between jobs?
(b) What is the probability that the next job is sent within 5 minutes?

Solution

Solution. Job arrivals represent rare events, thus the time T between them is Exponential with the given parameter $\lambda = 3$ hrs^{-1} (jobs per hour).

(a) $\mathbf{E}(T) = 1/\lambda = 1/3$ hours or 20 minutes between jobs;

(b) Convert to the same measurement unit: 5 min = (1/12) hrs. Then,

$$\boldsymbol{P}\{T < 1/12 \text{ hrs}\} = F(1/12) = 1 - e^{-\lambda(1/12)} = 1 - e^{-1/4} = \underline{0.2212}.$$

$\Diamond$

Problem - 7

Suppose that the average household income in some country is 900 coins, and the standard deviation is 200 coins.

Assuming the Normal distribution of incomes,

compute the proportion of "the middle class," whose income is between 600 and 1200 coins.

Solution

$$
\begin{aligned}
\boldsymbol{P}\{600 < X < 1200\} &= \boldsymbol{P}\left\{\frac{600-\mu}{\sigma} < \frac{X-\mu}{\sigma} < \frac{1200-\mu}{\sigma}\right\} \\
&= \boldsymbol{P}\left\{\frac{600-900}{200} < Z < \frac{1200-900}{200}\right\} = \boldsymbol{P}\{-1.5 < Z < 1.5\} \\
&= \Phi(1.5) - \Phi(-1.5) = 0.9332 - 0.0668 = \underline{0.8664}.
\end{aligned}
$$

Problem - 8

Suppose it has been observed that, on average, 180 cars per hour pass a specified point on a particular road in the morning rush hour.

Due to impending roadworks it is estimated that congestion will occur closer to the city centre if more than 5 cars pass the point in any one minute. What is the probability of congestion occurring?

Solution

Let X be the random variable X = number of cars arriving in any minute. We need to calculate the probability that more than 5 cars arrive in any one minute. Note that in order to do this we need to convert the information given on the average rate (cars arriving per hour) into a value for λ (cars arriving per minute). This gives the value $\lambda = 3$.

Using $\lambda = 3$ to calculate the required probabilities gives:

r	0	1	2	3	4	5	Sum
$P(X = r)$	0.04979	0.149361	0.22404	0.22404	0.168031	0.10082	0.91608

To calculate the required probability we note that

P(more than 5 cars arrive in one minute) $= 1 -$ P(5 cars or less arrive in one minute)

Thus

$$\begin{aligned} P(X > 5) &= 1 - P(X \leq 5) \\ &= 1 - P(X = 0) - P(X = 1) - P(X = 2) - P(X = 3) - P(X = 4) - P(X = 5) \end{aligned}$$

Then P(more than 5) $= 1 - 0.91608 = 0.08392$

Problem - 9

A Council is considering whether to base a recovery vehicle on a stretch of road to help clear incidents as quickly as possible. The road concerned carries over 5000 vehicles during the peak rush hour period. Records show that, on average, the number of incidents during the morning rush hour is 5.

The Council won't base a vehicle on the road if the probability of having more than 5 incidents in any one morning is less than 30%. Based on this information should the Council provide a vehicle?

Solution

Answer

We need to calculate the probability that more than 5 incidents occur i.e. $\mathsf{P}(X > 5)$. To find this we use the fact that $\mathsf{P}(X > 5) = 1 - \mathsf{P}(X \leq 5)$. Now, for this problem:

$$\mathsf{P}(X = r) = e^{-5}\frac{5^r}{r!}$$

Writing answers to 5 d.p. gives:

$$\begin{aligned}
\mathsf{P}(X=0) &= e^{-5}\frac{5^0}{0!} = 0.00674 \\
\mathsf{P}(X=1) &= 5 \times \mathsf{P}(X=0) = 0.03369 \\
\mathsf{P}(X=2) &= \frac{5}{2} \times \mathsf{P}(X=1) = 0.08422 \\
\mathsf{P}(X=3) &= \frac{5}{3} \times \mathsf{P}(X=2) = 0.14037 \\
\mathsf{P}(X=4) &= \frac{5}{4} \times \mathsf{P}(X=3) = 0.17547 \\
\mathsf{P}(X=5) &= \frac{5}{5} \times \mathsf{P}(X=4) = 0.17547
\end{aligned}$$

$$\begin{aligned}
\mathsf{P}(X \leq 5) &= \mathsf{P}(X=0) + \mathsf{P}(X=1) + \mathsf{P}(X=2) + \mathsf{P}(X=3) + \mathsf{P}(X=4) + \mathsf{P}(X=5) \\
&= 0.61596
\end{aligned}$$

The probability of more than 5 incidents is $\mathsf{P}(X > 5) = 1 - \mathsf{P}(X \leq 5) = 0.38403$, which is 38.4%

So the Council should provide a vehicle.