Lecture : 2

Instructor

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Practice Problem:

A file contains **1 Kbytes**. The probability that there exists at least one corrupted byte is **0.01**. The probability that at least two bytes are corrupted is **0.005**. Let the outcome of the experiment be the number of bytes in error.

(a) Define the sample space.

$\Omega = \{0,1,2,3 \ldots\ldots, 1000\}$

(b) Find P (no errors).

$$P(\text{“no error”}) = P(\{0\})$$

$$= P(\{1,2,3, \ldots, 1000\}^c)$$

$$= 1 - P(\{1,2,3,4,\ldots,1000\})$$

$$= 1 - P(\text{“at least one error”}) \quad = 1 - 0.01 = \mathbf{0.99}$$

Practice Problem:

A file contains **1 Kbytes**. The probability that there exists at least one corrupted byte is **0.01**. The probability that at least two bytes are corrupted is **0.005**. Let the outcome of the experiment be the number of bytes in error.

(c) P (exactly one byte in error) =?

A = {'atleast 1 error'}   B = {"atleast 2 error"}   , $B \subset A$

P({"atleast 1 error"}) = P({"exactly one error"}) + P({"atleast 2 error"})

P({1,2,3,4.....,1000}) = P({1}) + P({2,3,4,....,1000})

(d) P(at most one byte is in error)=?

P({0,1}) = 1 - P({2,3,4,...,1000}) = 1 – P({atleast 2 error})

= 1 - 0.005 = 0.995

Continuous Probability Models

The sample space is an uncountable set in continuous models. We compute the probability by measuring the probability "weight" of the desired event relative to the sample space.

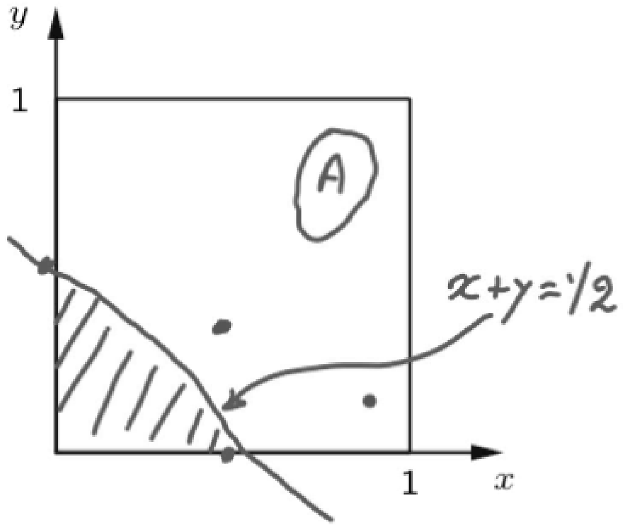
Example: I start driving to work in the morning at some time uniformly chosen in the interval [7:10, 8:00].

What is the probability that I start driving before 7: 35 ? **25/50**

Continuous Probability Models

Sample space: continuous example

- (x, y) such that $0 \le x, y \le 1$
- Uniform Probability law: Probability = Area



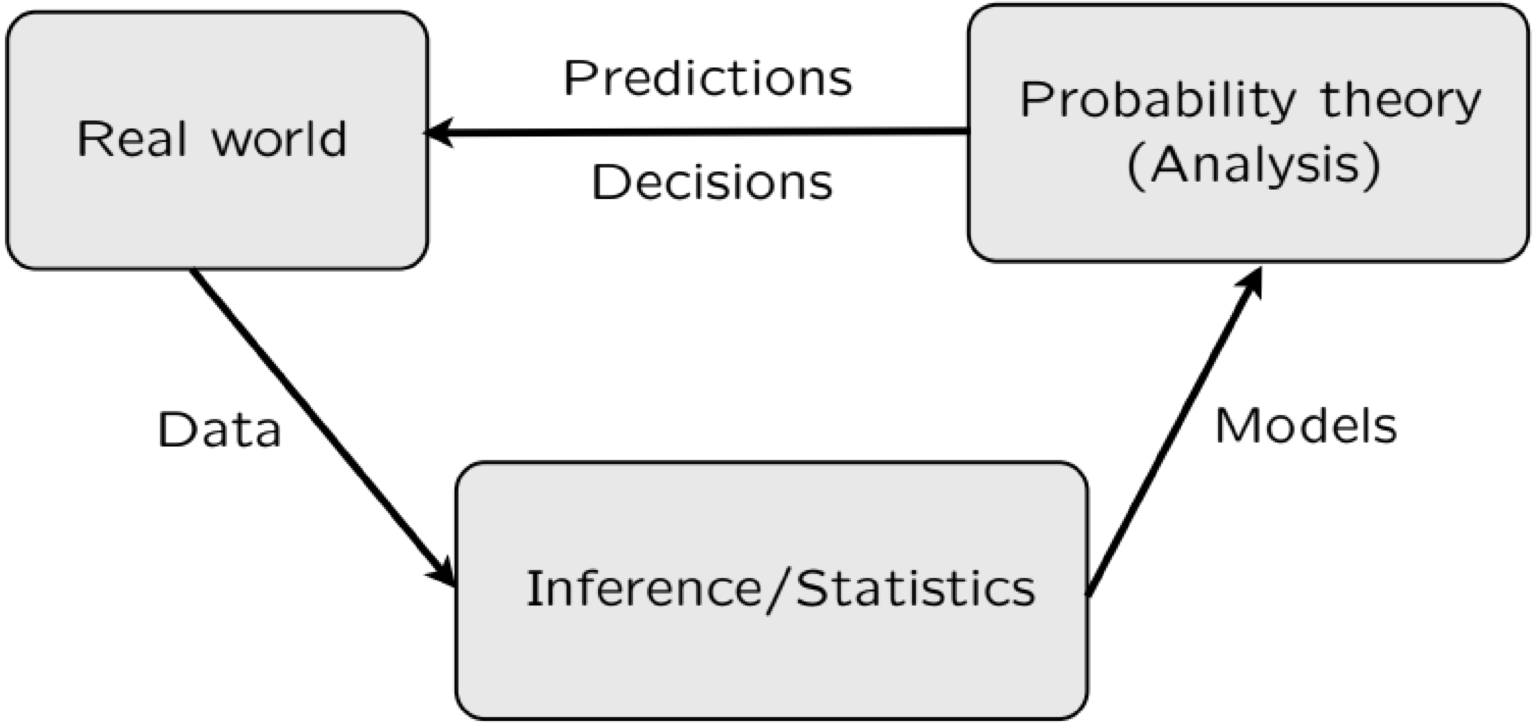
$$P\big(\{(x, y) \mid x + y \le 1/2\}\big) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$$

$$P\big(\{(0.5, 0.3)\}\big) = 0$$

Continuous Probability Models

The role of probability theory

A framework for analyzing phenomena with uncertain outcomes
– Rules for consistent reasoning
– Used for predictions and decisions

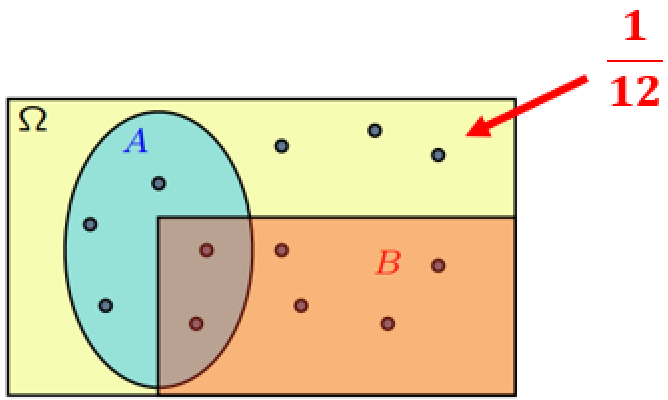


Conditional Probability

Conditional probability provides us with a way to reason about the outcome of an experiment, based on **partial information**.

The idea of conditioning

1. Assume 12 equally likely outcomes

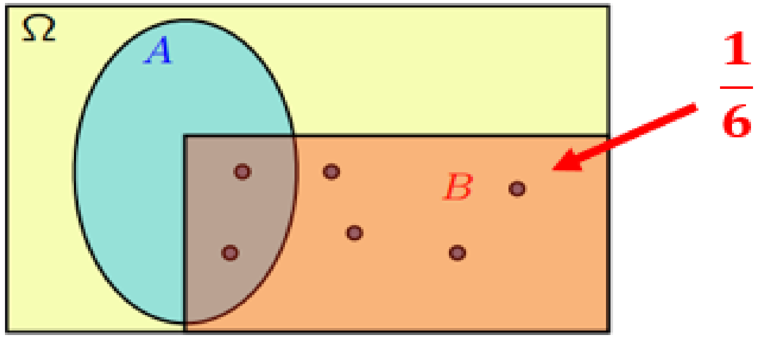


$$P(A) = \frac{5}{12} \quad , \qquad P(B) = \frac{6}{12}$$

Conditional Probability

Use new information to revise a model:

If told B occurred:

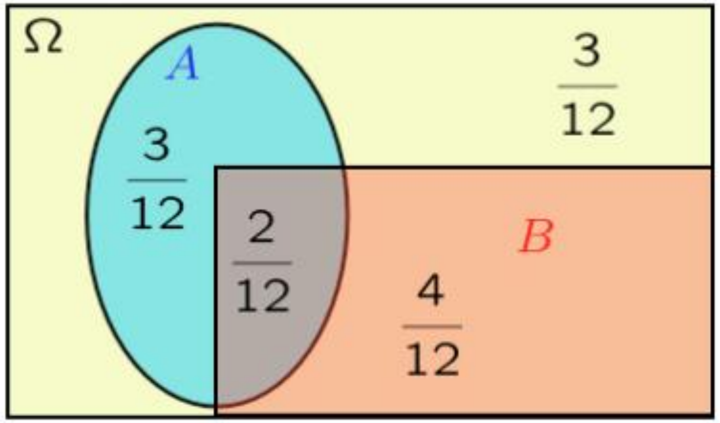


$$P(A|B) = \frac{2}{6} = \frac{1}{3} \quad , \quad P(B|B) = 1$$

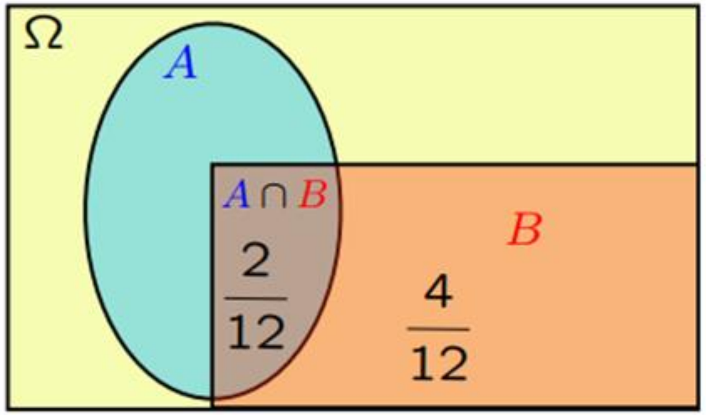
Conditional Probability

How about now, if the sample space does not consist of equally likely outcomes, but instead we're given the probabilities of different pieces of the sample space, as in this example

P (A | B) = "probability of A, given that B occurred"



If told B occurred:



$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

defined only when $P(B) > 0$

Example: two rolls of a 4-sided die

Y = Second roll

Y \ X	1	2	3	4
4	(1,4)	(2,4)	(3,4)	(4,4)
3	(1,3)	(2,3)	(3,3)	(4,3)
2	(1,2)	(2,2)	(3,2)	(4,2)
1	(1,1)	(2,1)	(3,1)	(4,1)

X = First roll

Events are equality likely: 1/16

Let B be the event: $\min(X, Y) = 2$

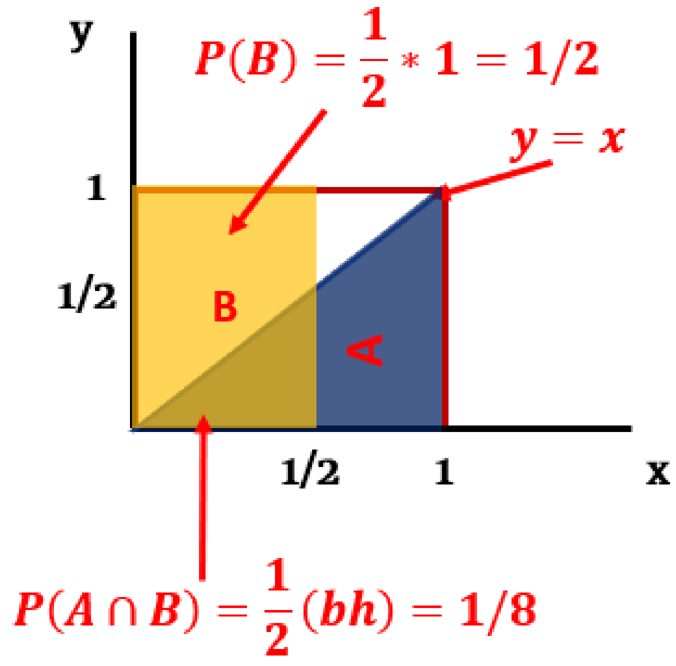
Let $M = \max(X, Y)$

$$\mathbf{P}(M = 1 \mid B) = \frac{P(M=1 \cap B)}{P(B)} = \frac{0}{5/16} = 0$$

$$\mathbf{P}(M = 3 \mid B) = \frac{P(M=3 \cap B)}{P(B)} = \frac{2/16}{5/16}$$

Exercise: Conditional probabilities in a continuous model

Let the sample space be the unit square, $\Omega = [0, 1]^2$, and let the probability of a set be the area of the set. Let A be the set of points $(x, y) \in [0, 1]^2$ for which $y \leq x$. Let B be the set of points for which $x \leq 1/2$. Find $P(A \mid B)$.



$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1}{4}$$

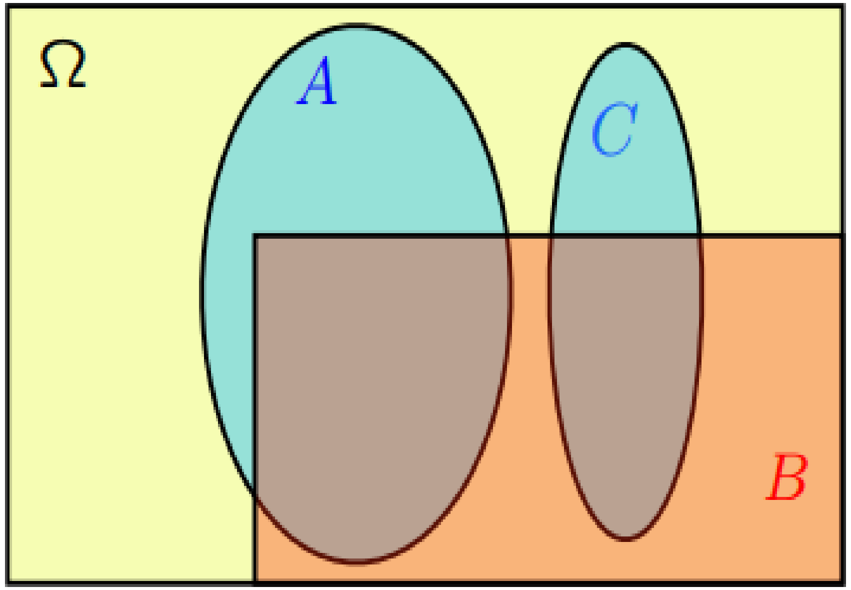
If $A \cap C = \emptyset$, then $P(A \cup C \mid B) = P(A|B) + P(C|B)$

$$P(A \cup C \mid B) = \frac{P\big((A \cup C) \cap B\big)}{P(B)}$$

$$= \frac{P\big((A \cap B) \cup (C \cap B)\big)}{P(B)}$$

$$= \frac{P(A \cap B) + P(C \cap B)}{P(B)}$$

$$= P(A|B) + P(C|B)$$



Practice Problem:

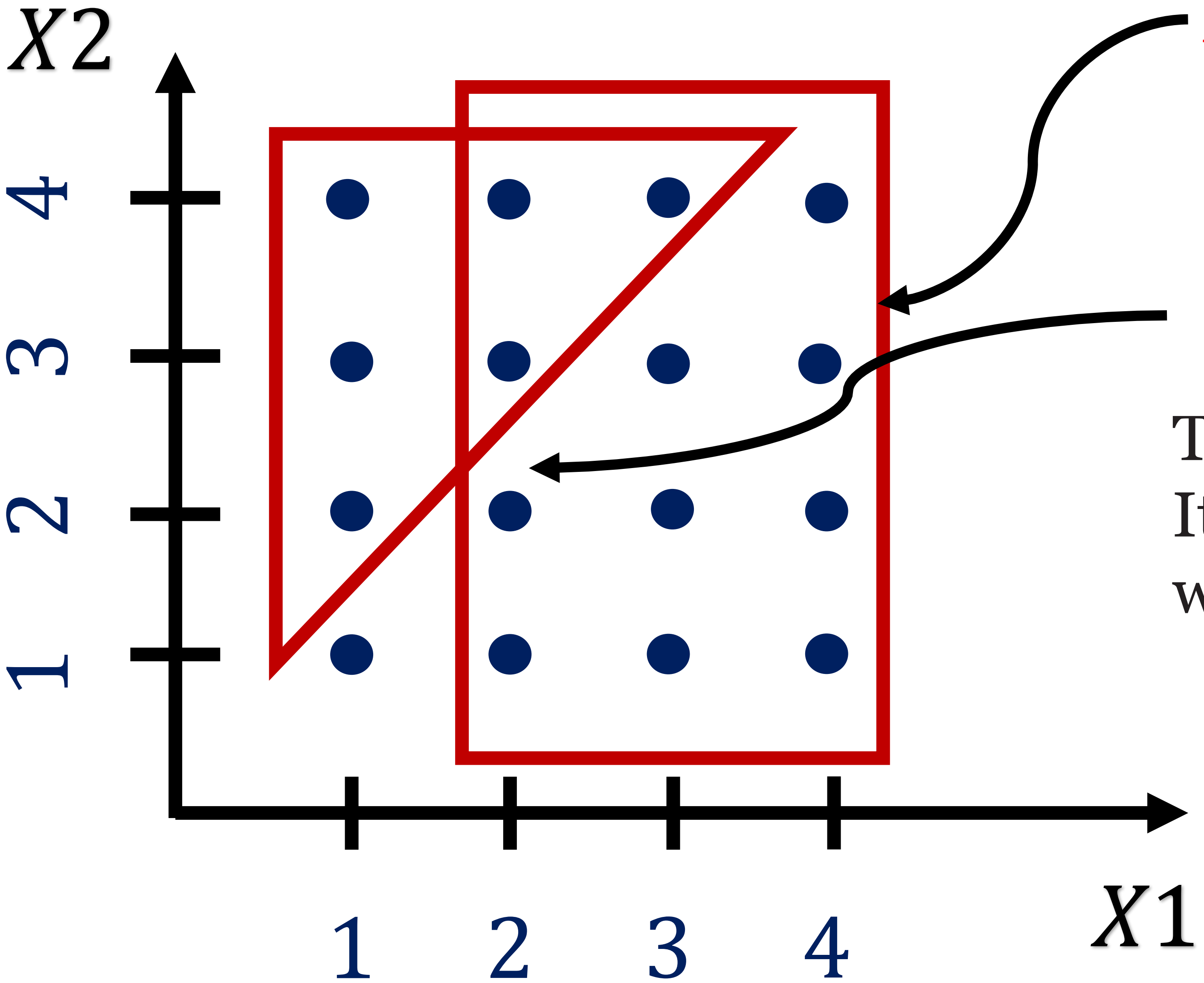
Roll two fair four-sided dice. Let X_1 and X_2 denote the number of dots that appear on die 1 and die 2, respectively. Let A be the event $X_1 \geq 2$.

- What is $P[A]$

Let B denote the event $X_2 > X_1$.

- What is $P[B]$

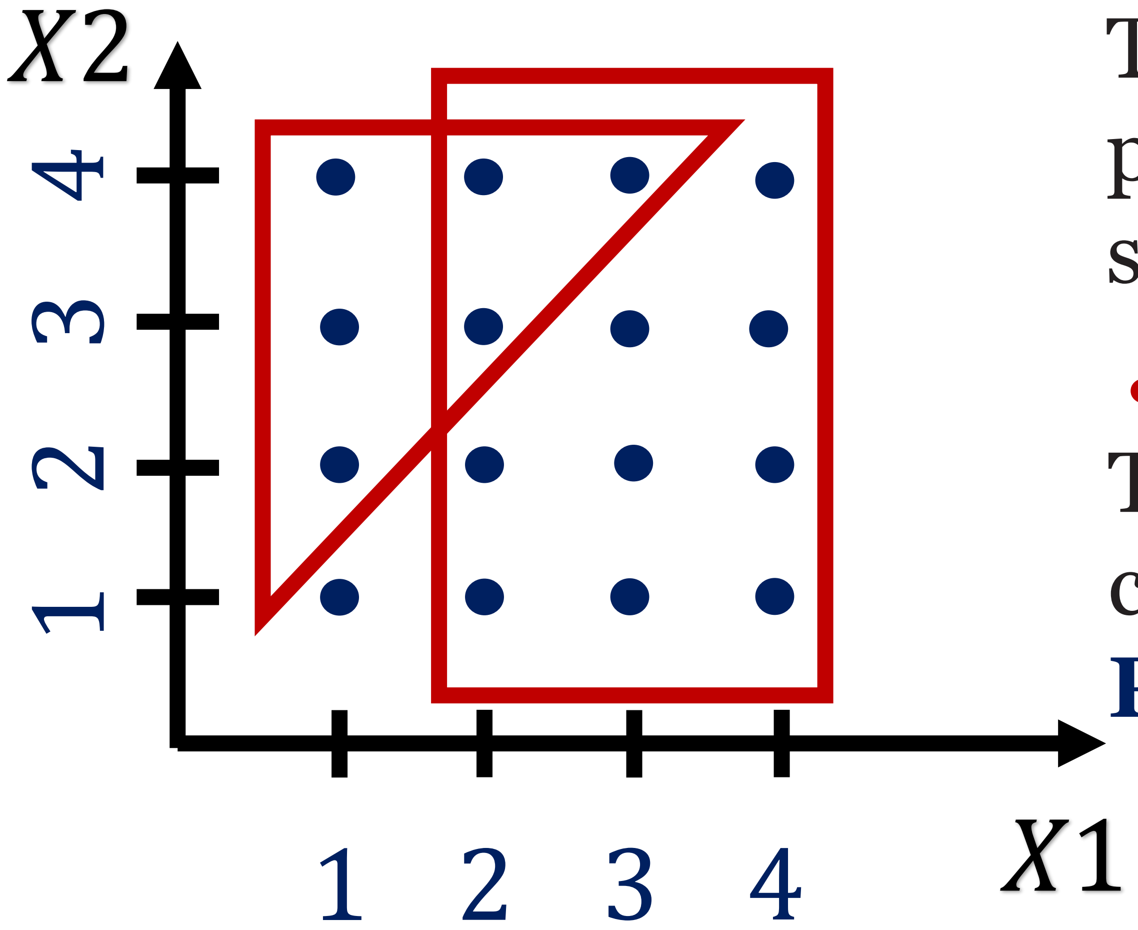
- What is $P[A|B]$



A be the event **$X1 \geq 2$**

B denote the event **$X2 > X1$**

The rectangle represents **A**.
It contains 12 outcomes, each with probability 1/16



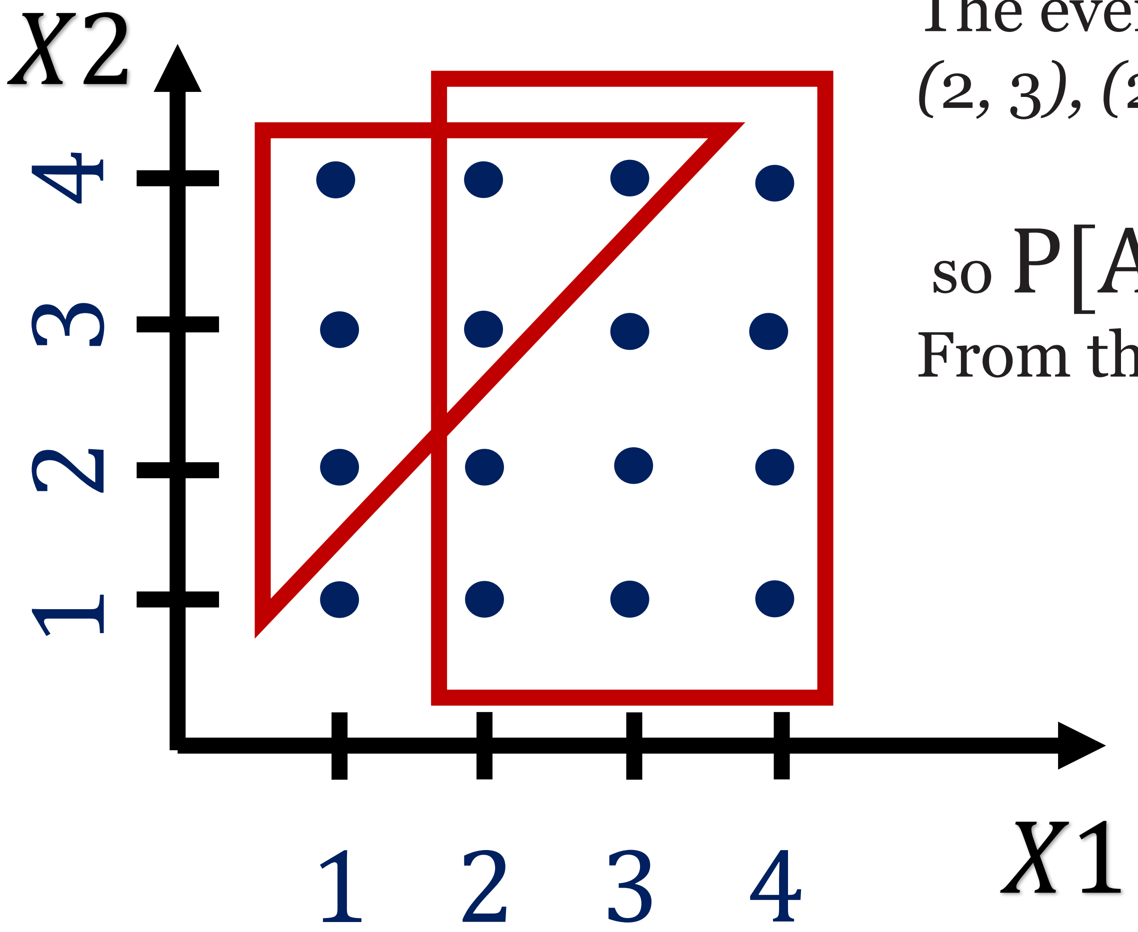
- **What is P[A]**

To find $P[A]$, we add up the probabilities of outcomes in A, so $\boldsymbol{P[A]}$ **= 12/16 = 3/4**

- **What is P[B]**

The triangle represents B. It contains six outcomes. Therefore **P[B] = 6/16 = 3/8**

- What is **P[A|B]**



The event $A \cap B$ has three outcomes,
(2, 3), (2, 4), (3, 4),

so $P[A \cap B] = 3/16.$
From the definition of conditional probability, we write

$$\mathbf{P}[A|B] = \frac{\mathbf{P}[A \cap B]}{\mathbf{P}[B]} = 1/2$$

Models based on conditional probabilities

Example 1.9. Radar detection. If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05.

1) What is the probability of false alarm (a false indication of aircraft presence)
2) What is the probability of missed detection (nothing registers, even though an aircraft is present)?
3) Find $\boldsymbol{P(A \cap B)}$
4) Find $\boldsymbol{P(B)}$
5) Find $\boldsymbol{P(A|B)}$

Solution

If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05.

Event A: Airplane is flying

Event B: Something registers on radar screen

Given information:

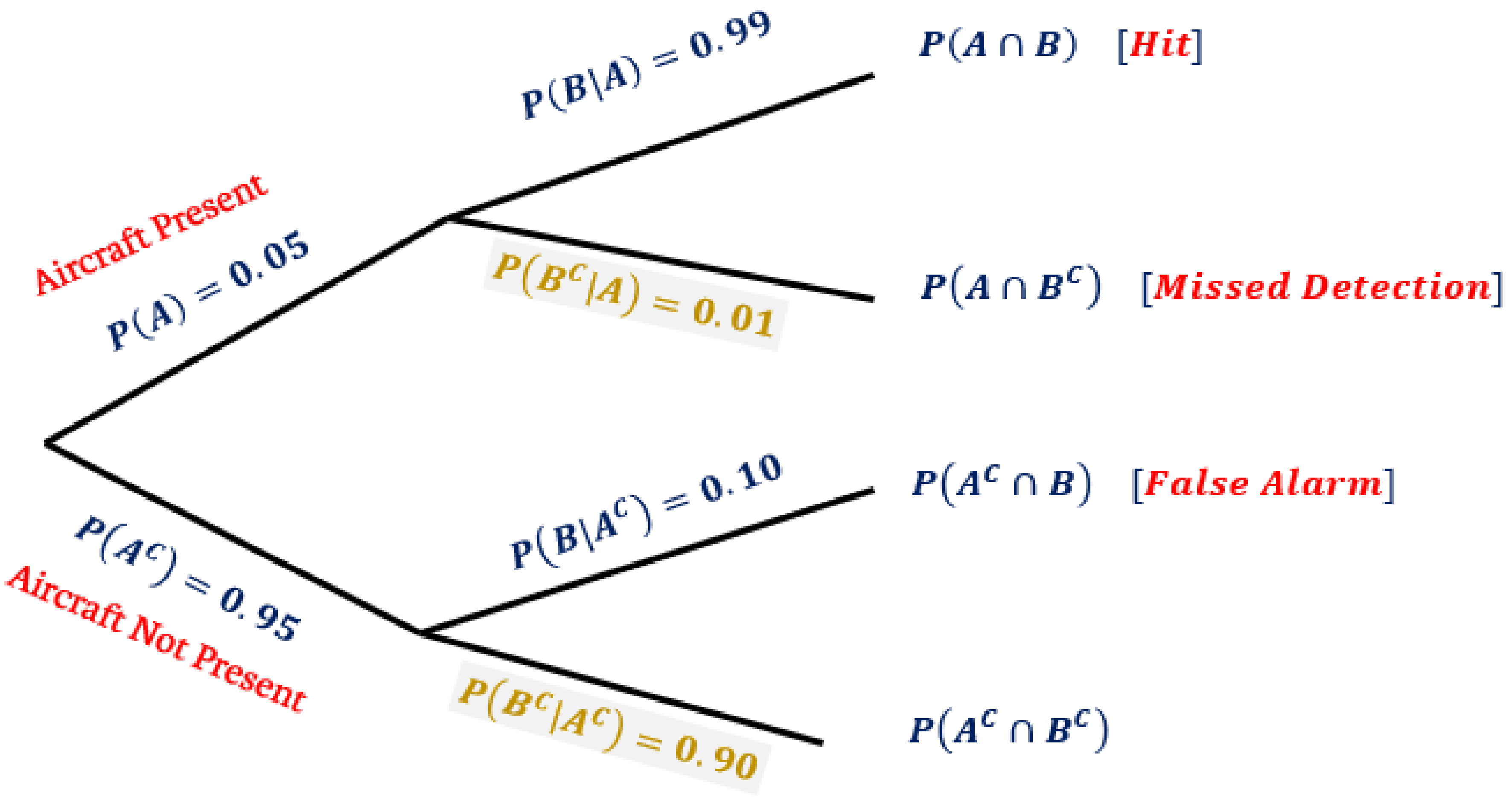
$$P(A) = 0.05, P(A^C) = 1 - P(A) = 0.95$$

$$P(B|A) = 0.99,$$

$$P(B|A^C) = 0.10$$

Solution

If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05.

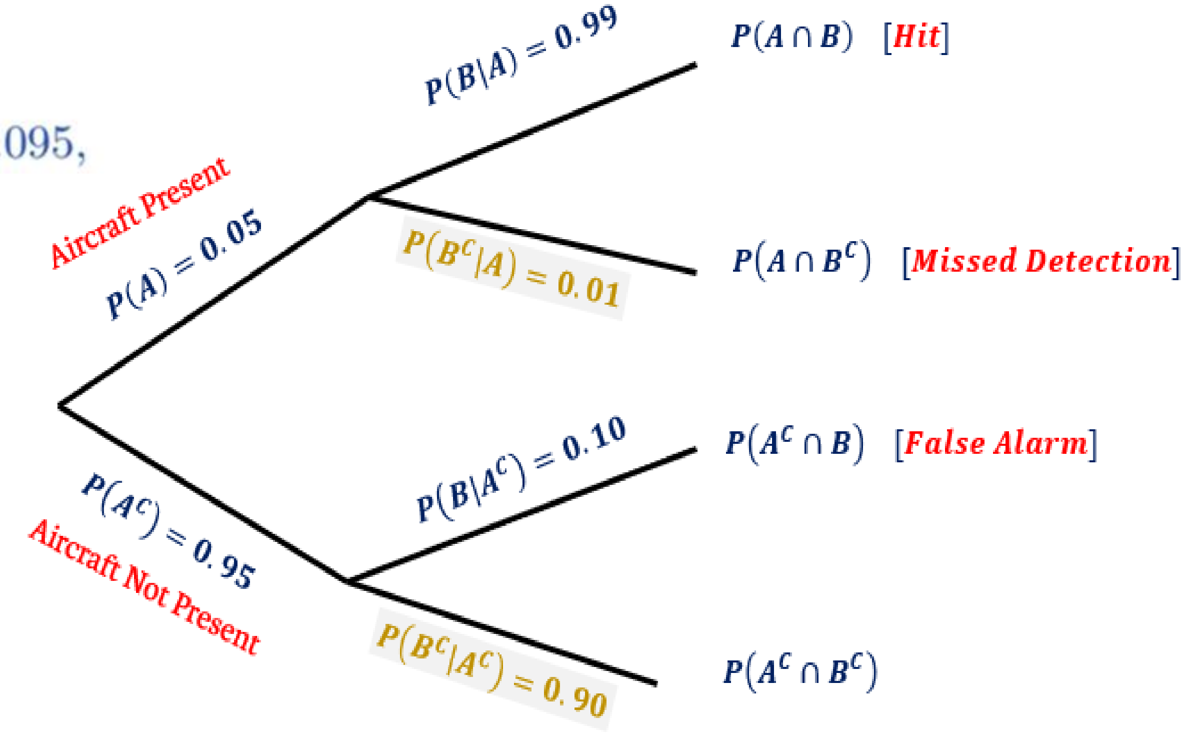


Solution

If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05.

1) **What is the probability of false alarm (a false indication of aircraft presence)**

$$\mathbf{P}(\text{false alarm}) = \mathbf{P}(A^c \cap B) = \mathbf{P}(A^c)\mathbf{P}(B \mid A^c) = 0.95 \cdot 0.10 = 0.095,$$



2) **What is the probability of missed detection (nothing registers, even though an aircraft is present)?**

$$\mathbf{P}(\text{missed detection}) = \mathbf{P}(A \cap B^c) = \mathbf{P}(A)\mathbf{P}(B^c \mid A) = 0.05 \cdot 0.01 = 0.0005.$$

Solution

If an aircraft is present in a certain area, a radar correctly registers its presence with probability 0.99. If it is not present, the radar falsely registers an aircraft presence with probability 0.10. We assume that an aircraft is present with probability 0.05.

3) Find $P(A \cap B)$

$$P(A \cap B) = P(A) * P(B|A) = 0.05 * 0.99$$
$$= 0.0495$$

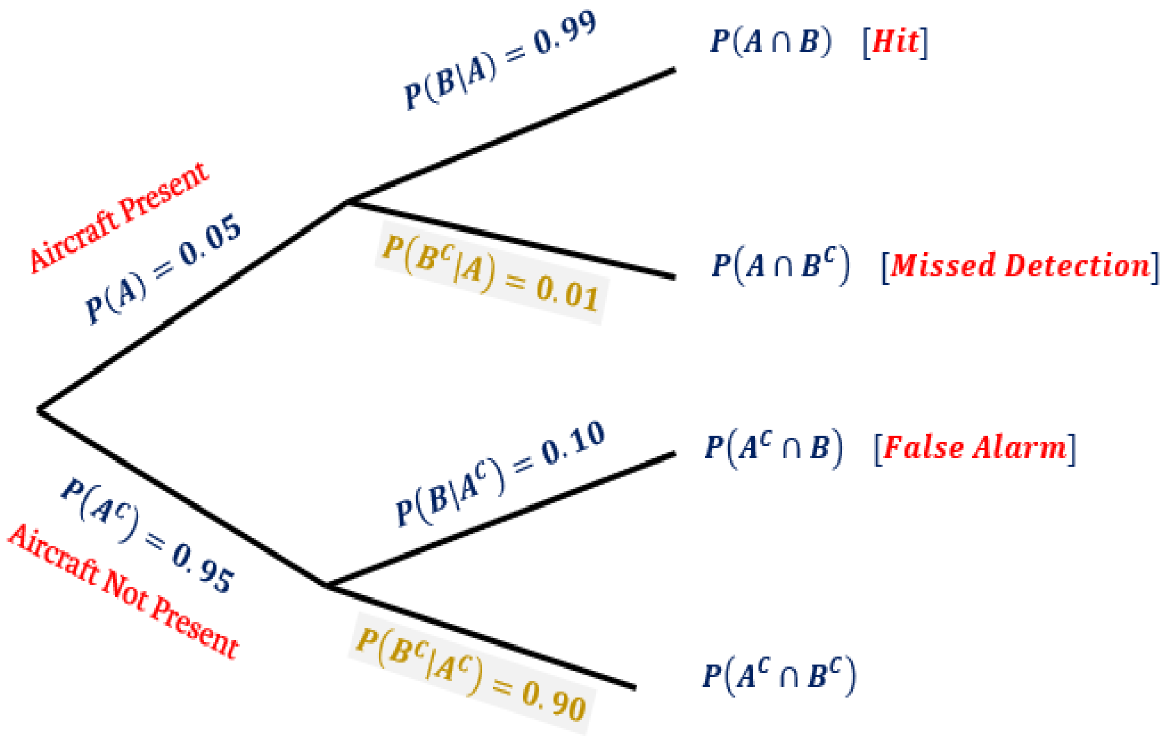
4) Find P(B)

$$P((B) = P(A \cap B) + P(A^C \cap B)$$

$$= P(A) * P(A|B) + P(A^C) * P(B|A^C) = 0.1445$$

5) Find P(A|B)

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = 0.34$$



“SOME PEOPLE DREAM OF SUCCESS WHILE OTHERS WAKE UP AND WORK HARD FOR IT”