Lecture No. 8

SPECIAL Continuous DISTRIBUTIONS

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Uniform Distribution

Uniform Distribution

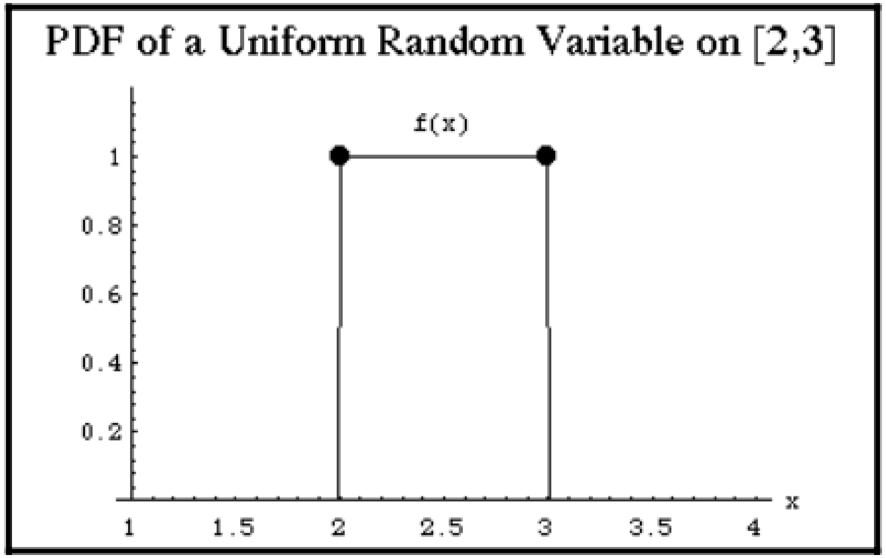
- **Definition**: A random variable X is said to be uniform on the interval [a, b] if its probability density function is of the form

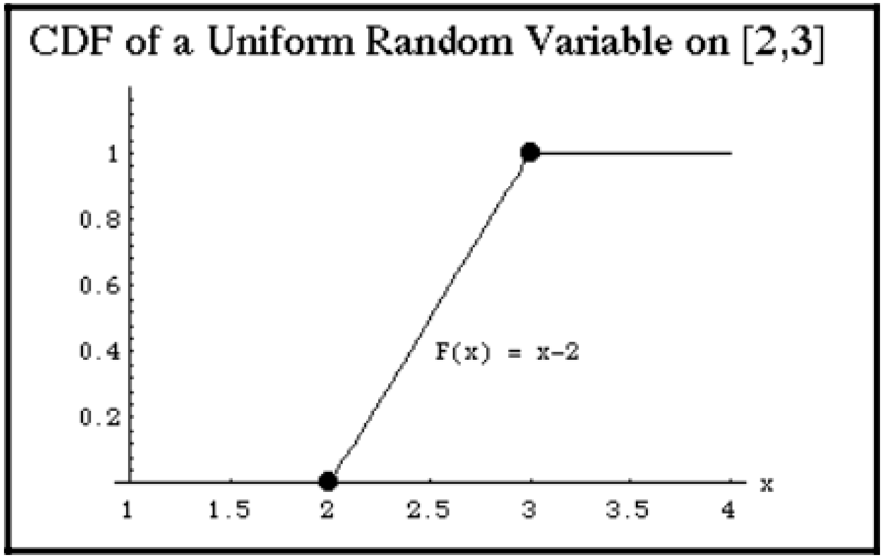
$$f(x) = \frac{1}{b-a} \quad , \qquad a \le x \le b$$

- where a and b are constants.

Uniform Distribution

- We denote a random variable X with the uniform distribution on the interval $[a, b]$ as $\boldsymbol{X \sim UNIF(a, b)}$.





CDF of Uniform Distribution

$$F_X(x) = \begin{cases} 0 & x \le a, \\ (x-a)/(b-a) & a < x \le b, \\ 1 & x > b. \end{cases}$$

CDF of Uniform Distribution

- The uniform distribution provides a probability model for selecting points at random from an interval $[a, b]$.

- An important application of uniform distribution lies in **random number generation**.

Uniform Distribution

- Expected Value:

$$E[X] = \frac{b+a}{2}$$

$$\begin{aligned} E(X) &= \int_a^b x\, f(x)\, dx \\ &= \int_a^b x \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \left[\frac{x^2}{2} \right]_a^b \\ &= \frac{1}{2}(b+a). \end{aligned}$$

Uniform Distribution

- Variance:

$$Var[X] = \frac{(b-a)^2}{12}$$

Uniform Distribution

- Variance:

$$
\begin{aligned}
E(X^2) &= \int_a^b x^2 \, f(x)\, dx \\
&= \int_a^b x^2 \, \frac{1}{b-a}\, dx \\
&= \frac{1}{b-a} \left[\frac{x^3}{3} \right]_a^b \\
&= \frac{1}{b-a} \, \frac{b^3 - a^3}{3} \\
&= \frac{1}{(b-a)} \, \frac{(b-a)\,(b^2 + ba + a^2)}{3} \\
&= \frac{1}{3}\,(b^2 + ba + a^2).
\end{aligned}
$$

Uniform Distribution

- Variance:

$$\boxed{Var(X) = E(X^2) - (E(X))^2}$$

$$= \frac{1}{3}(b^2 + ba + a^2) - \frac{(b+a)^2}{4}$$

$$= \frac{1}{12}\left[4b^2 + 4ba + 4a^2 - 3a^2 - 3b^2 - 6ba\right]$$

$$= \frac{1}{12}\left[b^2 - 2ba + a^2\right]$$

$$= \frac{1}{12}(b-a)^2.$$

Practice Problem

Buses arrive at a specified stop at 15-minute intervals starting at 7 A.M. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus;
(b) at least 12 minutes for a bus.

Practice Problem

(a) Probability that he waits less than 5 minutes for a bus

Let X denote the time in minutes past 7 A.M. that the passenger arrives at the stop.

Since X is a uniform random variable over the interval (0, 30), it follows that the passenger will have to wait less than 5 minutes if he arrives between 7:10 and 7:15 or between 7:25 and 7:30. Hence, the desired probability for (a) is

$$P\{10 < X < 15\} + P\{25 < X < 30\} = \frac{5}{30} + \frac{5}{30} = \frac{1}{3}$$

Practice Problem

(b) Probability that he waits at least 12 minutes for a bus

He would have to wait at least 12 minutes if he arrives between 7 and 7:03 or between 7:15 and 7:18, and so the probability for (b) is

$$\boldsymbol{P\{0 < X < 3\} + P\{15 < X < 18\} = \frac{3}{30} + \frac{3}{30} = \frac{1}{5}}$$

Exponential Distribution

Exponential Distribution

- **Definition**: A continuous random variable has an exponential distribution with parameter λ if its probability density function f is given by $\boldsymbol{f(x) = 0}$ $\boldsymbol{if\ x < 0}$ and

$$f(x) = \lambda e^{-\lambda x}, \qquad for\ \ x \geq 0$$

- We denote this distribution by **EXP(λ)** .

Exponential Distribution

- This is a legitimate PDF because

$$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^{\infty} = 1.$$

$$\mathbf{P}(X \geq a) = \int_a^{\infty} \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_a^{\infty} = e^{-\lambda a}.$$

- Note that the probability that X exceeds a certain value falls exponentially. Indeed, for any $a \geq 0$

Exponential Distribution

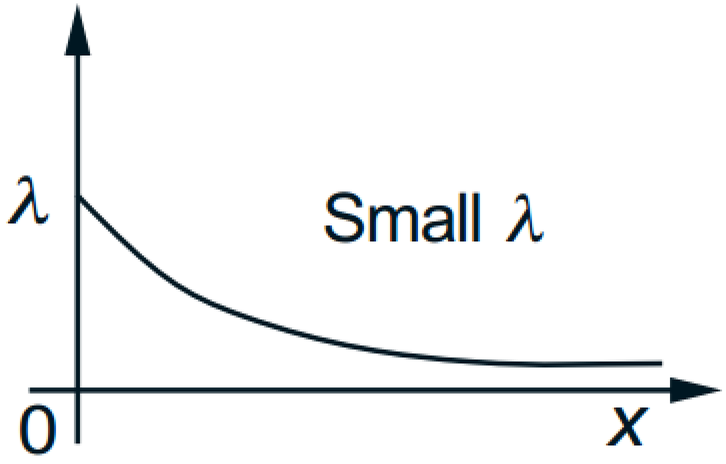
- The CDF of an EXP(λ) distribution is given by

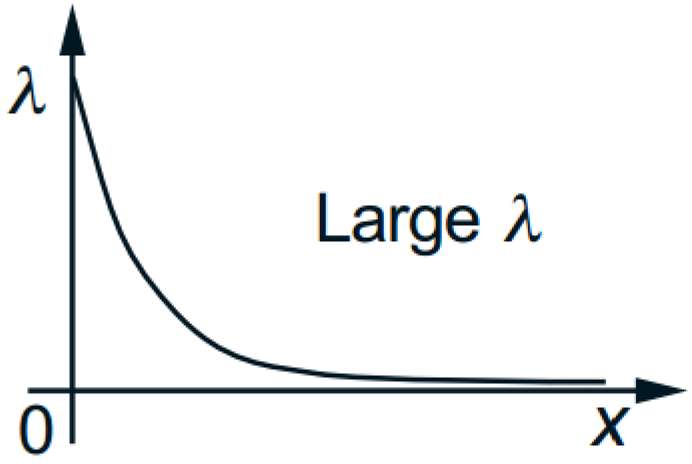
$$
\begin{aligned}
F(x) &= P\{X \le x\} \\
&= \int_0^x \lambda e^{-\lambda y}\, dy \\
&= 1 - e^{-\lambda x}, \qquad x \ge 0
\end{aligned}
$$

Exponential Distribution

An exponential random variable can be a very good model for the amount of time until a piece of equipment breaks down, until a light bulb burns out, or until an accident occurs.

Exponential Distribution





Exponential Distribution

- The mean and the variance can be calculated to be

$$\mathbf{E}[X] = \frac{1}{\lambda}, \qquad \operatorname{var}(X) = \frac{1}{\lambda^2}.$$

Exponential Distribution

$$\begin{aligned}\mathbf{E}[X] &= \int_0^\infty x\lambda e^{-\lambda x}\,dx \\ &= \left(-xe^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty e^{-\lambda x}\,dx \\ &= 0 - \frac{e^{-\lambda x}}{\lambda}\Bigg|_0^\infty \\ &= \frac{1}{\lambda}.\end{aligned}$$

Exponential Distribution

$$
\begin{aligned}
\mathbf{E}[X^2] &= \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx \\
&= \left(-x^2 e^{-\lambda x}\right)\Big|_0^\infty + \int_0^\infty 2x e^{-\lambda x}\, dx \\
&= 0 + \frac{2}{\lambda}\mathbf{E}[X] \\
&= \frac{2}{\lambda^2}.
\end{aligned}
$$

Exponential Distribution

Finally, using the formula $\text{var}(X) = \mathbf{E}[X^2] - \big(\mathbf{E}[X]\big)^2$, we obtain

$$\text{var}(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Practice Problem

Commonly, car cooling systems are controlled by electrically driven fans. Assuming that the lifetime T (in hours) of a fan manufactured by a company "Kenlowe" can be modelled by an exponential distribution with $\lambda = 0.0003$.

Find the proportion of fans which will give at least 10000 hours service.

If the fan is redesigned so that its lifetime may be modelled by an exponential distribution with $\lambda = 0.00035$, would you expect more fans or fewer to give at least 10000 hours service?

Practice Problem

We know that $f(t) = 0.0003e^{-0.0003t}$ so that the probability that a fan will give at least 10000 hours service is given by the expression

$$\mathrm{P}(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.0003e^{-0.0003t}\,dt = -\left[e^{-0.0003t}\right]_{10000}^{\infty} = e^{-3} \approx 0.0498$$

Hence about 5% of the fans may be expected to give at least 10000 hours service. After the redesign, the calculation becomes

$$\mathrm{P}(T > 10000) = \int_{10000}^{\infty} f(t)\,dt = \int_{10000}^{\infty} 0.00035e^{-0.00035t}\,dt = -\left[e^{-0.00035t}\right]_{10000}^{\infty} = e^{-3.5} \approx 0.0302$$

and so only about 3% of the fans may be expected to give at least 10000 hours service.

Hence, after the redesign we expect *fewer* fans to give 10000 hours service.

Homework Problem

The phase angle, Θ, of the signal at the input to a modem is uniformly distributed between 0 and 2π radians. Find the CDF, the expected value, and the variance of Θ.

Homework Problem(Solution)

The PDF is

$$f_\Theta(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta < 2\pi, \\ 0 & \text{otherwise.} \end{cases}$$

The CDF is

$$F_\Theta(\theta) = \begin{cases} 0 & \theta \leq 0, \\ \theta/(2\pi) & 0 < x \leq 2\pi, \\ 1 & x > 2\pi. \end{cases}$$

The expected value is $E[\Theta] = b/2 = \pi$ radians, and the variance is $\mathrm{Var}[\Theta] = (2\pi)^2/12 = \pi^2/3$ rad^2.

Normal/Guassian Distribution

Guassian Distribution

- **Definition**: A continuous random variable X is said to be **Normal** or **Gaussian** if it has a PDF of the form

$$f(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

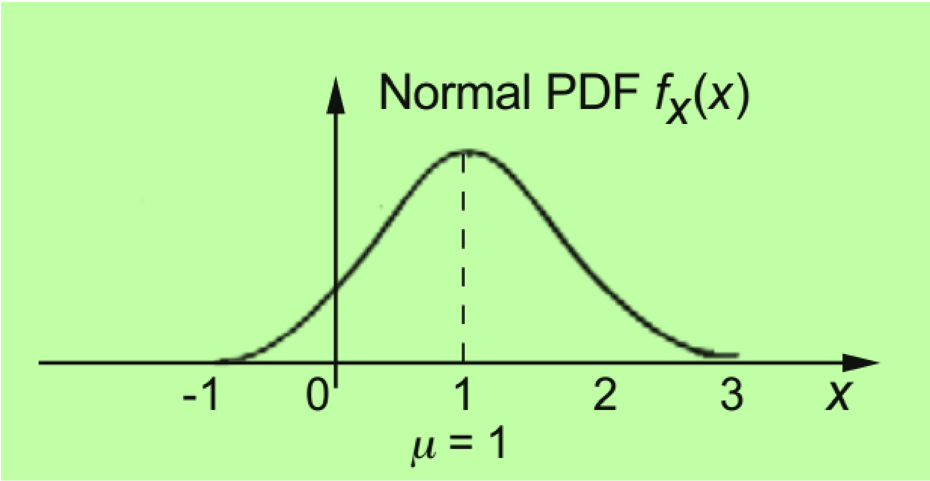
- where $\boldsymbol{\mu}$ and $\boldsymbol{\sigma}$ are two scalar parameters characterizing the PDF, with $\boldsymbol{\sigma}$ assumed nonnegative.
-

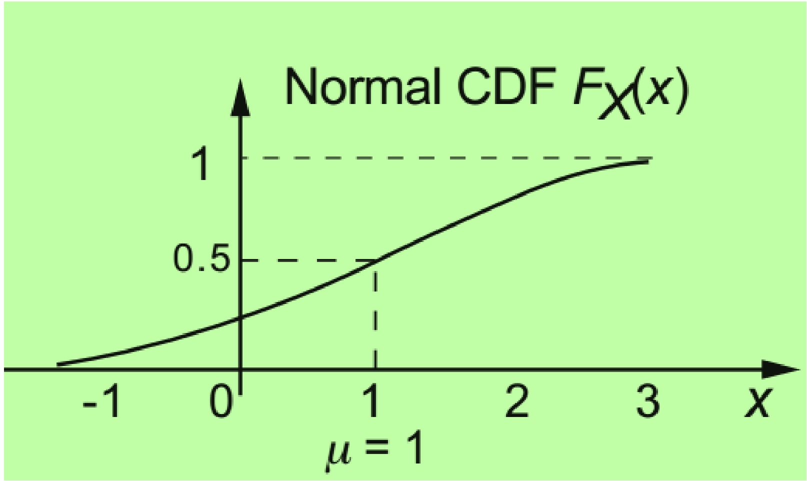
Normal Distribution

- It can be verified that the normalization property holds.

$$\frac{1}{\sqrt{2\pi\sigma}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = 1$$

Normal Distribution





- A normal PDF and CDF, with $\boldsymbol{\mu = 1}$ and $\boldsymbol{\sigma^2 = 1}$.
- We observe that the PDF is symmetric around its mean $\boldsymbol{\mu}$, and has a characteristic bell-shape.
- As $\boldsymbol{x}$ gets further from $\boldsymbol{\mu}$, the term $\mathbf{e}^{-\frac{(\mathbf{x}-\boldsymbol{\mu})^2}{2\boldsymbol{\sigma}^2}}$ decreases very rapidly.
- In this figure, the PDF is very close to zero outside the interval $[-1,\ 3]$

Normal Distribution

- The mean and the variance can be calculated to be

$$E[X] = \mu, \qquad Var[X] = \sigma^2$$

- To see this, note that the PDF is symmetric around **μ**, so its mean must be **μ**.

Normal Distribution

- The variance is given by

$$\operatorname{var}(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (x-\mu)^2 e^{-(x-\mu)^2/2\sigma^2}\, dx.$$

- Using the change of variables $\mathbf{y = (x - \mu)/\sigma}$ and integration by parts, we have

$$\begin{aligned}
\operatorname{var}(X) &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-y^2/2}\, dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \left(-y e^{-y^2/2}\right)\Big|_{-\infty}^{\infty} + \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \\
&= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy \\
&= \sigma^2.
\end{aligned}$$

Normal Distribution

- The last equality above is obtained by using the fact

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2}\, dy = 1,$$

- Which is just the normalization property of the normal PDF for the case where **$\mu = 0$** and **$\sigma = 1$**.

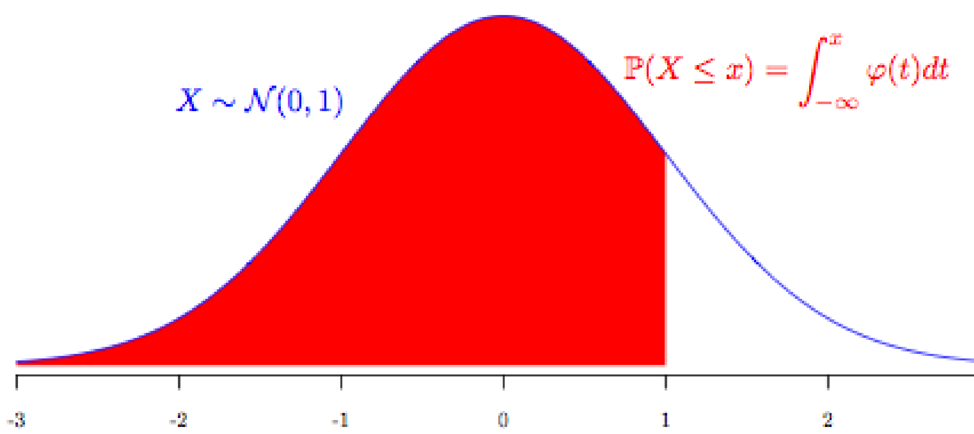
Normal Distribution

Standard Normal Random Variable

- A normal random variable Y with zero mean and unit variance is said to be a standard normal. Its CDF is denoted by $\mathbf{\Phi}$.
- The standard normal random variable Z is the **Gaussian (0, 1)** random variable.

$$\Phi(y) = \mathbf{P}(Y \le y) = \mathbf{P}(Y < y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt.$$

- It is recorded in a table (given in the next slide), and is a very useful tool for calculating various probabilities involving normal random variables.



	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706

Normal Distribution

- **Note:** Table only provides the values of $\boldsymbol{\Phi}(\boldsymbol{y})$ for y $\geq$ 0, because the omitted values can be found using the symmetry of the PDF.

- For example, if Y is a standard normal random variable, we have

$$\Phi(-0.5) = \mathbf{P}(Y \leq -0.5) = \mathbf{P}(Y \geq 0.5) = 1 - \mathbf{P}(Y < 0.5)$$
$$= 1 - \Phi(0.5) = 1 - .6915 = 0.3085.$$

Normal Distribution

- Let $\boldsymbol{X}$ be a normal random variable with mean μ and variance $\boldsymbol{\sigma^2}$. We "standardize" $\boldsymbol{X}$ by defining a new random variable $\boldsymbol{Y}$ given by

$$Y = \frac{X - \mu}{\sigma}.$$

- Thus, Y is a standard normal random variable. This fact allows us to calculate the probability of any event defined in terms of X: we redefine the event in terms of Y , and then use the standard normal table.

Normal Distribution

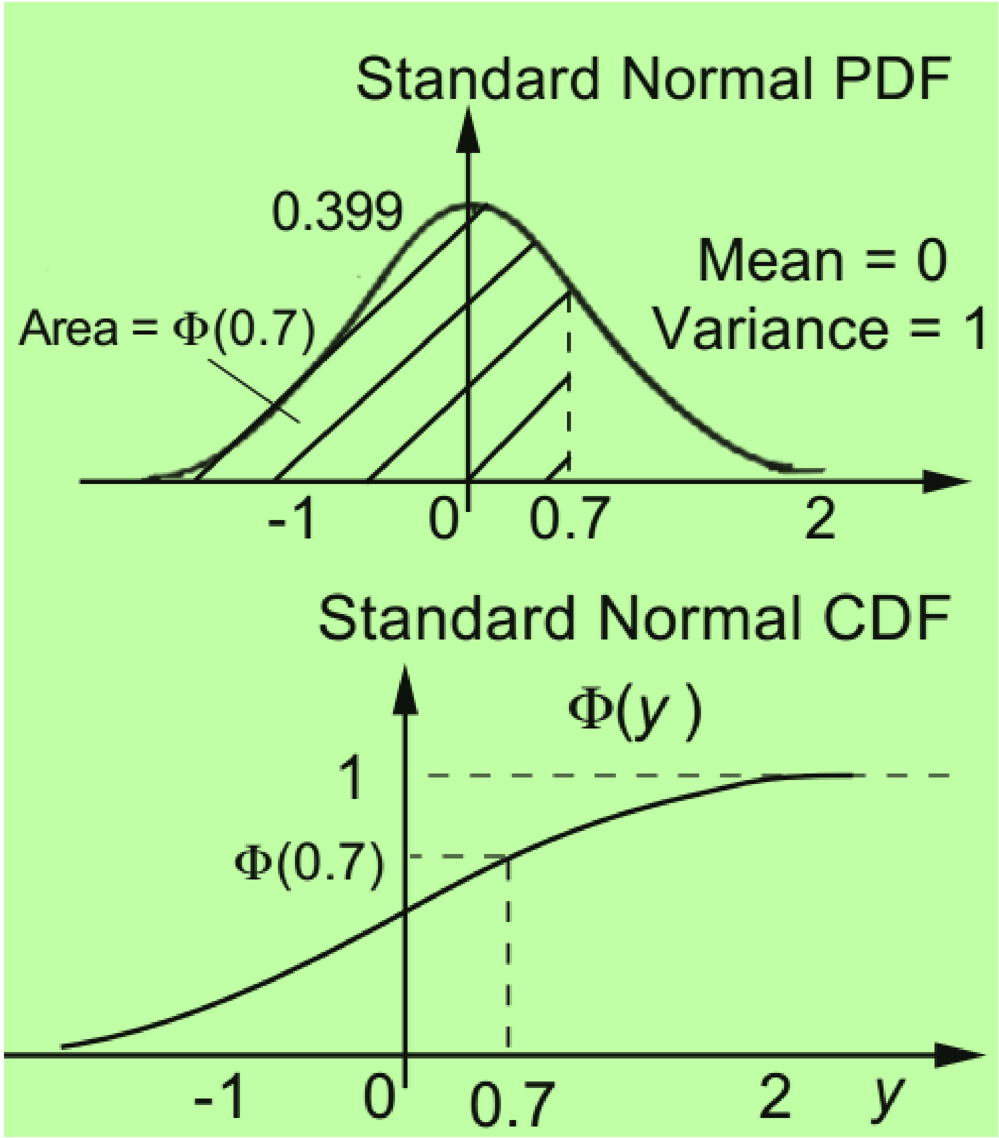
- If X is a **Gaussian(μ,σ)** random variable, the CDF of X is

$$F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

- The probability that X is in the interval $(a, b]$ is

$$P(a < X \leq b) = \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Normal Distribution



The PDF

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$$ of

the standard normal random variable.

Its corresponding CDF, which is denoted by $\Phi(y)$, is recorded in a table.

Normal Distribution

Example 3.8 (Bertsekas). Using the Normal Table.

The annual snowfall at a particular geographic location is modeled as a Normal random variable with a mean of $\mu = 60$ inches, and a standard deviation of $\sigma = 20$.

What is the probability that this year's snowfall will be at least 80 inches?

Normal Distribution

- Let X be the snow accumulation, viewed as a normal random variable, and let

$$Y = \frac{X - \mu}{\sigma} = \frac{X - 60}{20},$$

be the corresponding standard normal random variable. We want to find

$$\mathbf{P}(X \geq 80) = \mathbf{P}\left(\frac{X - 60}{20} \geq \frac{80 - 60}{20}\right) = \mathbf{P}\left(Y \geq \frac{80 - 60}{20}\right) = \mathbf{P}(Y \geq 1) = 1 - \Phi(1),$$

where $\mathbf{\Phi}$ is the CDF of the standard normal. We read the value $\mathbf{\Phi(1)}$ from the table:

$$\Phi(1) = 0.8413,$$

$$\mathbf{P}(X \geq 80) = 1 - \Phi(1) = 0.1587.$$

	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
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0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
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1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
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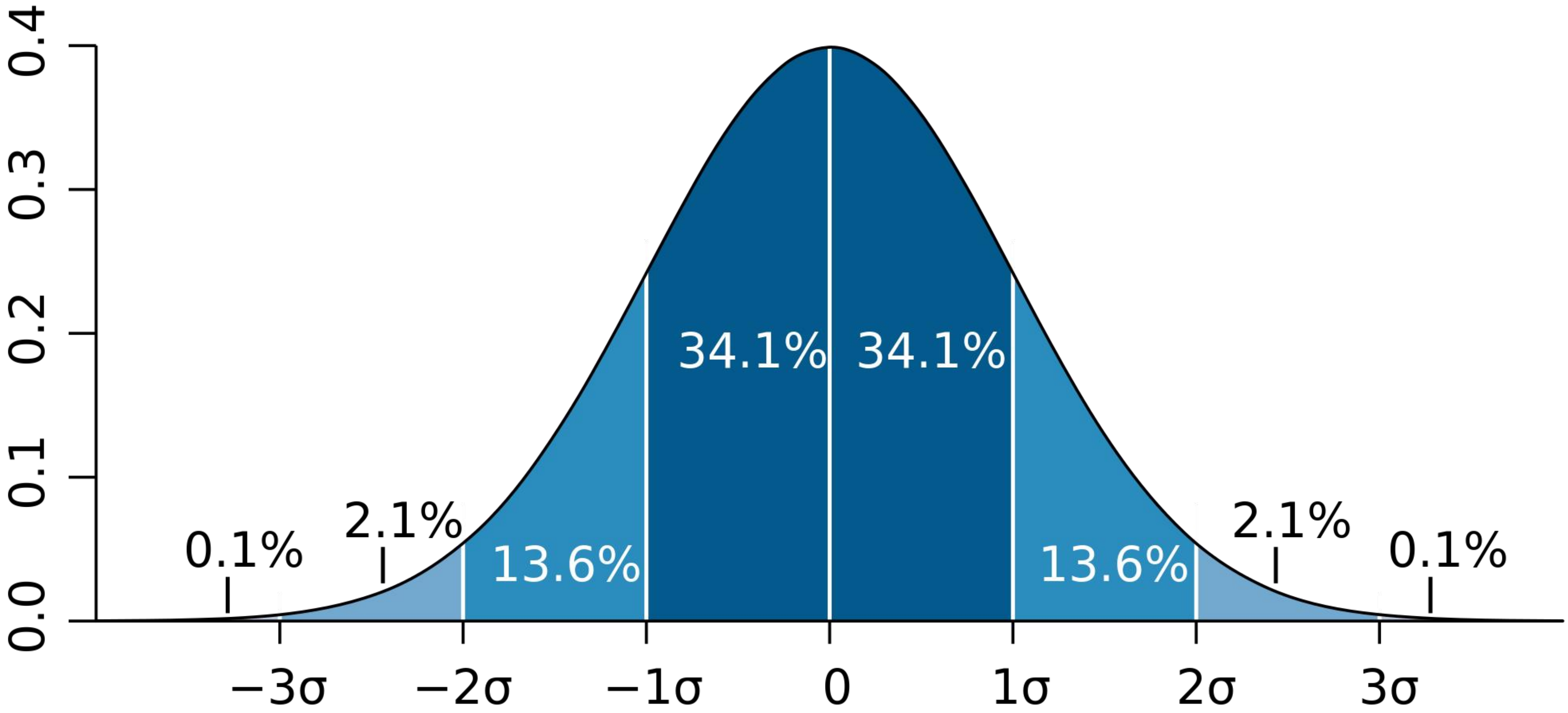
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767
2.0	0.9772	0.9778	0.9783	0.9788	0.9793	0.9798	0.9803	0.9808	0.9812	0.9817
2.1	0.9821	0.9826	0.9830	0.9834	0.9838	0.9842	0.9846	0.9850	0.9854	0.9857
2.2	0.9861	0.9864	0.9868	0.9871	0.9875	0.9878	0.9881	0.9884	0.9887	0.9890
2.3	0.9893	0.9896	0.9898	0.9901	0.9904	0.9906	0.9909	0.9911	0.9913	0.9916
2.4	0.9918	0.9920	0.9922	0.9925	0.9927	0.9929	0.9931	0.9932	0.9934	0.9936
2.5	0.9938	0.9940	0.9941	0.9943	0.9945	0.9946	0.9948	0.9949	0.9951	0.9952
2.6	0.9953	0.9955	0.9956	0.9957	0.9959	0.9960	0.9961	0.9962	0.9963	0.9964
2.7	0.9965	0.9966	0.9967	0.9968	0.9969	0.9970	0.9971	0.9972	0.9973	0.9974
2.8	0.9974	0.9975	0.9976	0.9977	0.9977	0.9978	0.9979	0.9979	0.9980	0.9981
2.9	0.9981	0.9982	0.9982	0.9983	0.9984	0.9984	0.9985	0.9985	0.9986	0.9986
3.0	0.9987	0.9987	0.9987	0.9988	0.9988	0.9989	0.9989	0.9989	0.9990	0.9990

Normal Distribution



Practice Problem

If X is a Gaussian random variable with $\mu = 61$ and $\sigma = 10$.
What is $P[51 < X \leq 71]$?

Practice Problem

$$P[51 < X \le 71] = \Phi\left(\frac{71-\mu}{\sigma}\right) - \Phi\left(\frac{51-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{71-61}{10}\right) - \Phi\left(\frac{51-61}{10}\right)$$

$$= \Phi(1) - \Phi(-1)$$

$$= \Phi(1) - [1 - \Phi(1)]$$

$$= 2\Phi(1) - 1$$

$$= 2\Phi(1) - 1 = 0.6826$$

Homework Problem

The peak temperature T , as measured in degrees Fahrenheit, on a July day in Sydney is the Gaussian (85, 10) random variable.

What is $P[T > 100]$, $P[T < 60]$ and $P[70 \leq T \leq 100]$?

$$
\begin{aligned}
P[T > 100] &= 1 - P[T \le 100] = 1 - F_T(100) = 1 - \Phi\left(\frac{100 - 85}{10}\right) \\
&= 1 - \Phi(1.5) = 1 - 0.933 = 0.066 \\
P[T < 60] &= \Phi\left(\frac{60 - 85}{10}\right) = \Phi(-2.5) \\
&= 1 - \Phi(2.5) = 1 - .993 = 0.007 \\
P[70 \le T \le 100] &= F_T(100) - F_T(70) \\
&= \Phi(1.5) - \Phi(-1.5) = 2\Phi(1.5) - 1 = .866
\end{aligned}
$$