Lecture No.6
Continuous Random Variables

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Continuous Random Variables

Continuous Random Variable has an uncountable image.

- $\boldsymbol{S_X = (0, 1]}$

- $\boldsymbol{S_Y = R}$

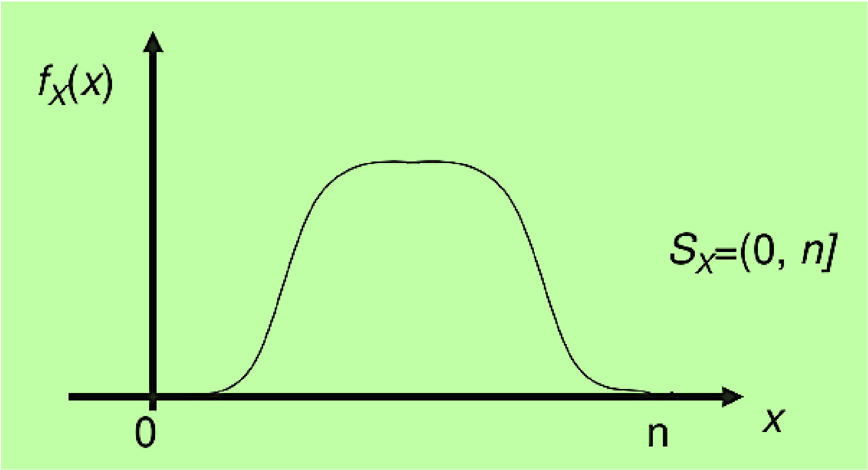
Probability Density Function (PDF) of a Continuous Random Variable

- The Probability Density Function (pdf) of a continuous random variable, $f_X(x)$ is a continuous function of the $x \in S_X$.

Properties of PDF:

$$f_X(x) \geq 0$$

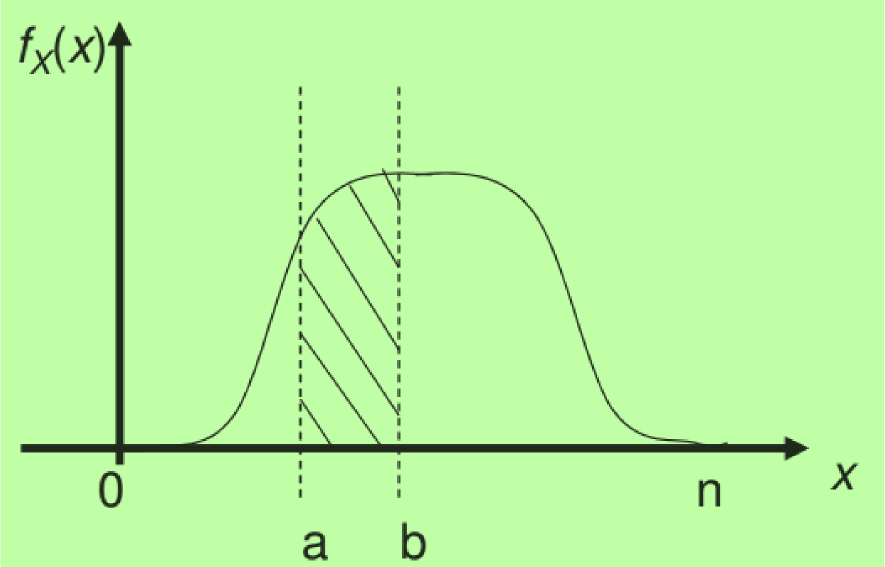
$$\int_{-\infty}^{\infty} f_X(x) = 1$$



PDF of a Continuous Random Variable

Properties of pdf:

$$P(a \leq x \leq b)$$
$$= P(a \leq x < b)$$
$$= P(a < x \leq b)$$
$$= P(a < x < b)$$
$$= \int_a^b f_X(x)\, dx$$



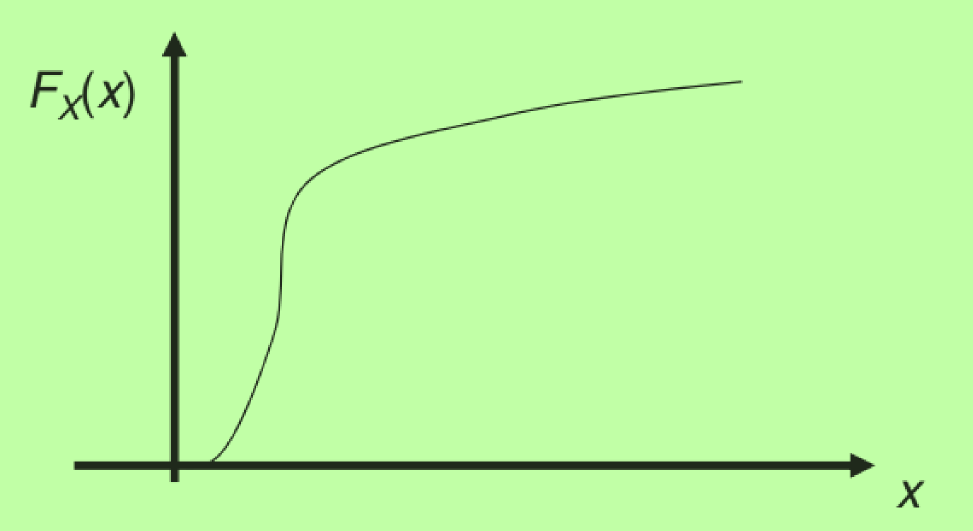
Cumulative Distribution Function (CDF) of a Continuous Random Variable

- CDF of a continuous RV is computed by integrating the pdf

$$F_X = P(X \leq t) = \int_{x=-\infty}^{t} f_X(x)\, dx$$

- Conversely, we also have:

$$f_X(x) = \frac{d}{dx} F_X(x)$$



Properties of the CDF

Properties of CDF:

- $$0 \leq F_X(x) \leq 1, -\infty < x < \infty$$
- $$a \leq b \quad \Rightarrow \quad F_X(a) \leq F_X(b)$$
- $$\lim_{x \to -\infty} F_X(x) = 0 \Rightarrow F(-\infty) = 0$$
- $$\lim_{x \to \infty} F_X(x) = 1 \Rightarrow F(\infty) = 1$$

Properties of the CDF

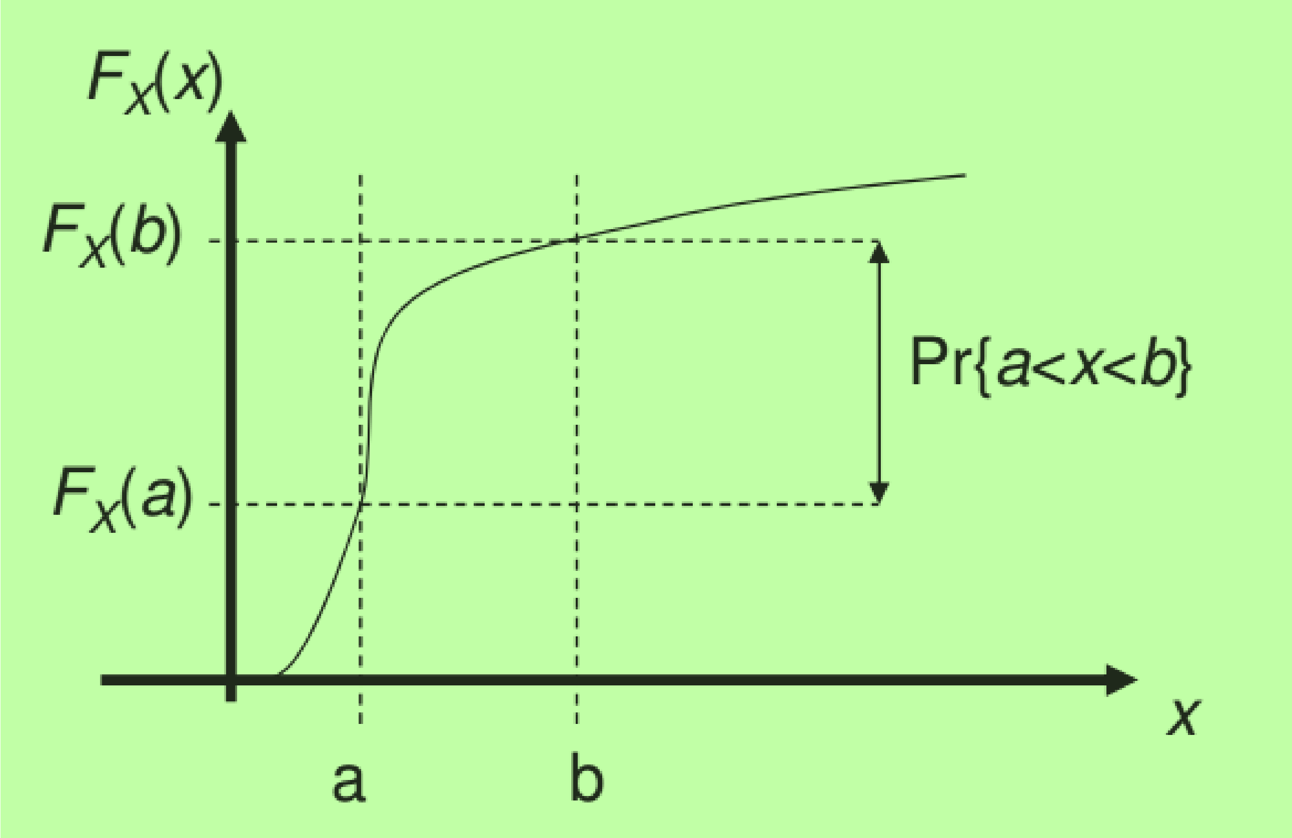
Properties of CDF:

- $$P(X = a) = P(a \le x \le a) = \int_a^a f_X(x)\, dx = 0$$

- $$P(a \le x \le b) = P(a \le x < b) = P(a < x \le b) = P(a < x \le b)$$

$$= F_X(b) - F_X(a)$$

Properties of the CDF



Practice Problem

Example

Let X denote the width in mm of metal pipes from an automated production line. If X has the probability density function $f(x) = 10e^{-10(x-5.5)}$ for $x \geq 5.5$, $f(x) = 0$ for $x < 5.5$.

Determine:

(i) $P(X < 5.7)$;

(ii) $P(X > 6)$;

(iii) $P(5.6 < X \leq 6)$.

Example

(i)

$$\begin{aligned} P(X < 5.7) &= \int_{5.5}^{5.7} 10e^{-10(x-5.5)}\,dx \\ &= -e^{-10(x-5.5)}\Big|_{5.5}^{5.7} \\ &= 1 - e^{-2} = 0.865. \end{aligned}$$

(ii)

$$\begin{aligned} P(X > 6) &= \int_6^{\infty} 10e^{-10(x-5.5)}\,dx \\ &= -e^{-10(x-5.5)}\Big|_6^{\infty} \\ &= e^{-5} = 0.007. \end{aligned}$$

(iii)

$$
\begin{aligned}
P(5.6 < X \le 6) &= \int_{5.6}^{6} 10e^{-10(x-5.5)}dx \\
&= -e^{-10(x-5.5)}\Big|_{5.6}^{6} \\
&= e^{-1} - e^{-5} = 0.361.
\end{aligned}
$$

Example

The random variable X measures the width in mm of metal pipes from an automated production line X has the probability density function $f(x) = 10e^{-10(x-5.5)}$ for $x \geq 5.5$, $f(x) = 0$ for $x < 5.5$. What is the cumulative distribution function of X?

For $x < 5.5$, $F(x) = 0$. For $x \geq 5.5$,

$$\begin{aligned} F(x) &= \int_{5.5}^{x} 10e^{-10(t-5.5)}\,dt \\ &= -e^{-10(t-5.5)}\Big|_{5.5}^{x} \\ &= 1 - e^{-10(x-5.5)}. \end{aligned}$$

Example

Let X denote the time in milliseconds for a chemical reaction to complete. The cumulative distribution function of X is

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-0.05x} & x \geq 0. \end{cases}$$

What is the probability density function of X? What is the probability that a reaction completes within 40 milliseconds?

The probability density function will be given by $\frac{dF(x)}{dx}$.

$$f(x) = \begin{cases} 0 & x < 0 \\ 0.05e^{-0.05x} & x \geq 0 \end{cases}$$

The probability that the reaction completes within 40 milliseconds is

$$P(X \leq 40) = F(40) = 1 - e^{-2} = 0.865.$$

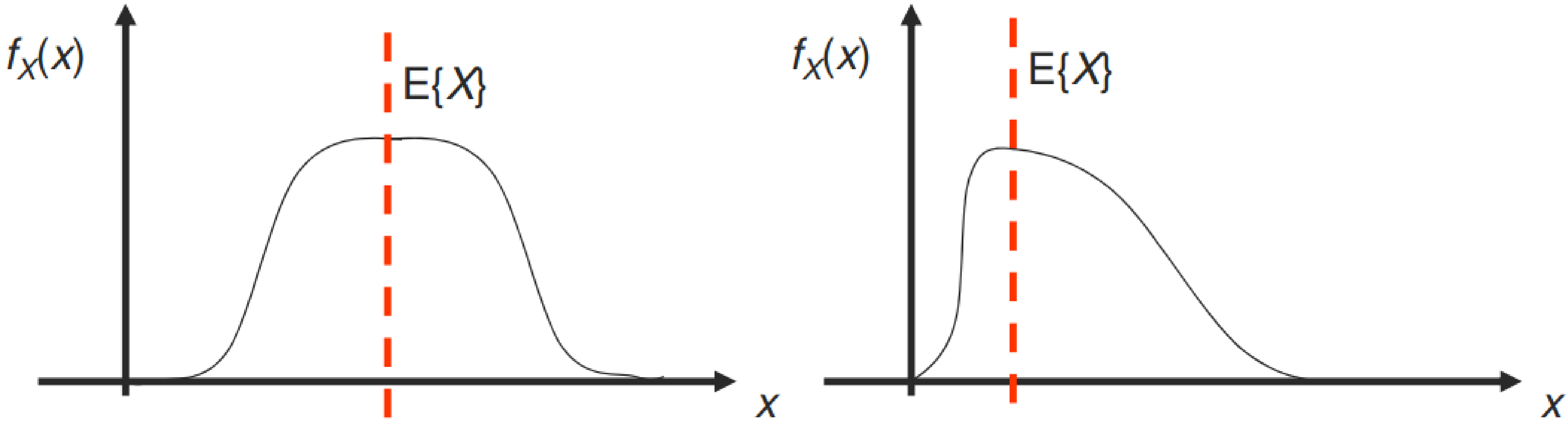
Expected Value of Continuous R.V.

- Expected value for Discrete RV

$$\boldsymbol{E[X] = \mu_X = \sum_{x \in S_X} x \cdot p_X(x)}$$

- Expected value for Continuous RV

$$\boldsymbol{E[X] = \mu_X = \int_{-\infty}^{\infty} x \cdot f_X(x) dx}$$



$f_X(x)$
E{X}
x
$f_X(x)$
E{X}
x

Variance of a Random Variable

- Variance for Discrete RV

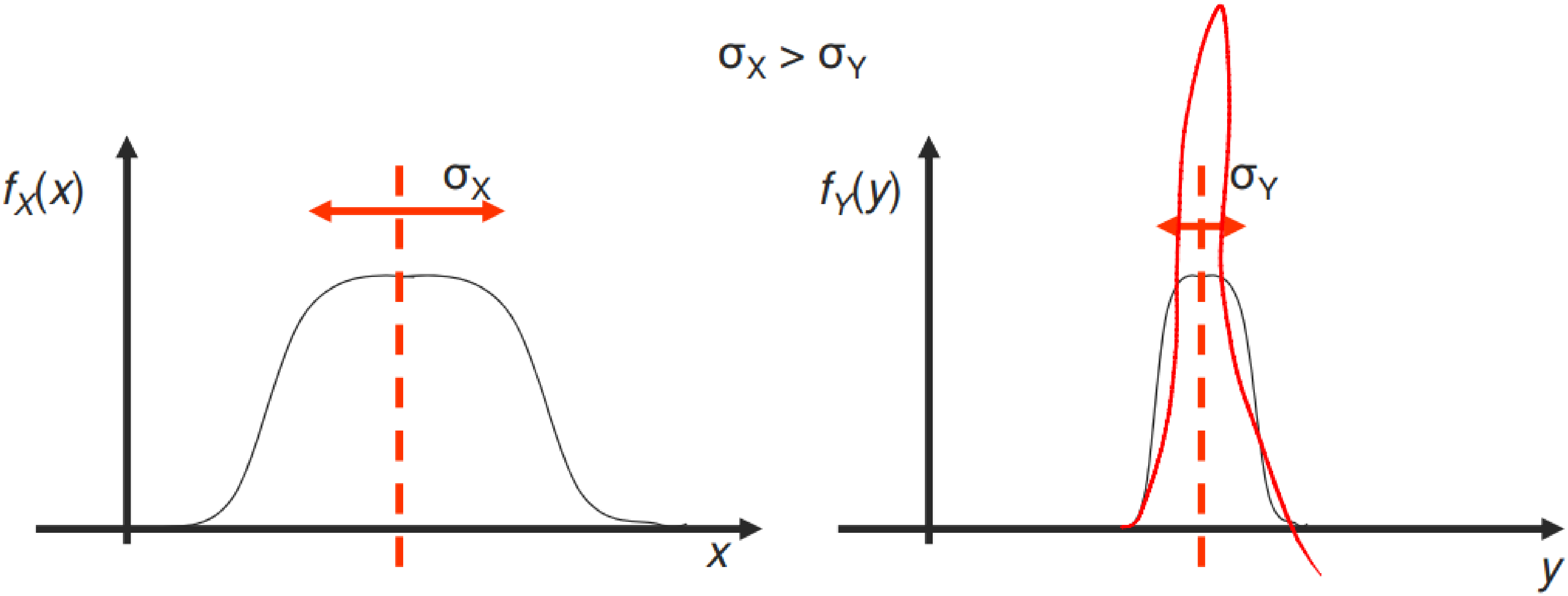
$$\boldsymbol{Var[X]} = \sigma_X^2 = \sum_{x \in S_X} (\boldsymbol{x} - \mu_X)^2 \cdot \boldsymbol{p_X(x)}$$

- Variance for Continuous RV

$$\boldsymbol{Var[X]} = \sigma_X^2 = \int_{-\infty}^{\infty} (\boldsymbol{x} - \mu_X)^2 \cdot \boldsymbol{f_X(x)dx}$$

Variance of a Random Variable

$\sigma_X > \sigma_Y$



Practice Problem

Is the real valued function defined by

$$f(x) = \begin{cases} 2\,x^{-2} & \text{if } 1 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

a probability density function for some random variable X?

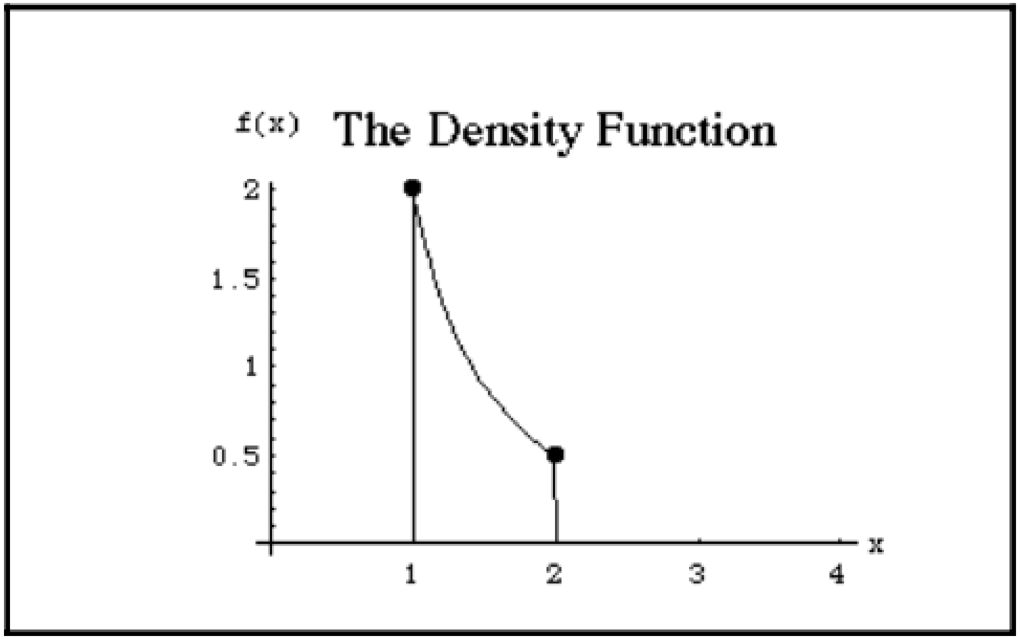
Practice Problem

For a valid pdf

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$f(x) = \begin{cases} 2x^{-2} & \text{if } 1 < x < 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\int_{1}^{2} 2x^{-2}dx = 1$$



Homework Problem

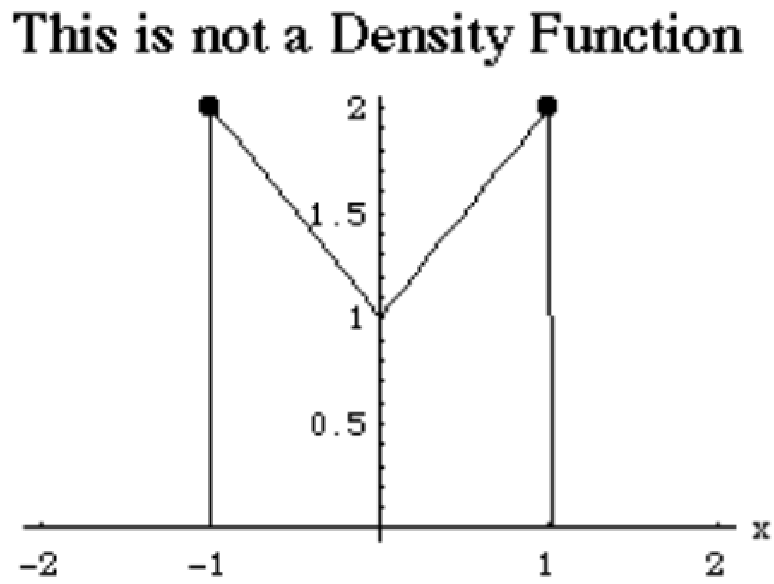
Is the real valued function defined by

$$f(x) = \begin{cases} 1 + |x| & \text{if } -1 < x < 1 \\ 0 & \text{otherwise,} \end{cases}$$

a probability density function for some random variable X?

Homework Problem

$$\int_{-1}^{1} (1 + |x|)dx$$



$$= \int_{-1}^{0} \big(1 + (-x)\big)dx + \int_{0}^{1} \big(1 + (x)\big)dx = 3$$

Homework Problem

What is the probability density function of the random variable whose cdf is

$$F(x) = \frac{1}{1 + e^{-x}}, \qquad -\infty < x < \infty$$

The pdf of the random variable is given by

$$
\begin{aligned}
f(x) &= \frac{d}{dx} F(x) \\
&= \frac{d}{dx}\left(\frac{1}{1+e^{-x}}\right) \\
&= \frac{d}{dx}\left(1+e^{-x}\right)^{-1} \\
&= (-1)\left(1+e^{-x}\right)^{-2} \frac{d}{dx}\left(1+e^{-x}\right) \\
&= \frac{e^{-x}}{(1+e^{-x})^2}.
\end{aligned}
$$