Lecture : 3

Instructor

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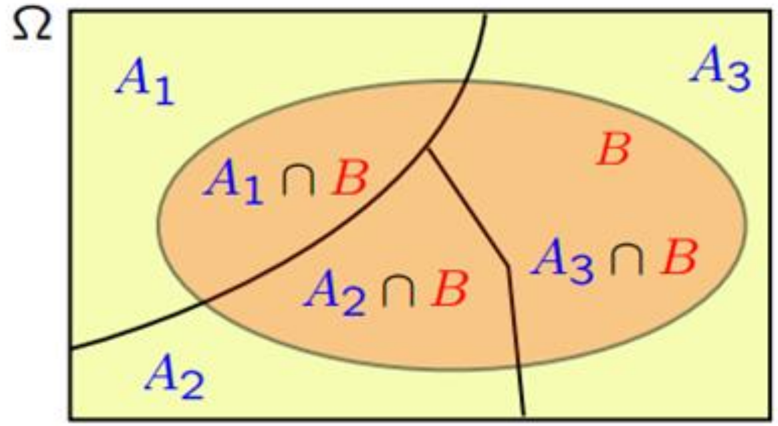
Total Probability Theorem

- Partition of sample space into A_1, A_2, A_3
- Have $P(A_i)$ for every i
- Have $P(B|A_i)$ for every i



$$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$$

$$= P(A_1)P(B|A_1) + P(A_2)P(B|A_2) + P(A_3)P(B|A_3)$$

Example

A company has three machines B1, B2, and B3 for making 1 kΩ resistors.
It has been observed that 80% of resistors produced by B1 are within 50 Ω of the nominal value.
Machine B2 produces 90% of resistors within 50 Ω of the nominal value.
The percentage for machine B3 is 60%.
Each hour, machine B1 produces 3000 resistors, B2 produces 4000 resistors, and B3 produces 3000 resistors.

All the resistors are mixed together at random in one bin and packed for shipment.
What is the probability that the company ships a resistor that is within 50 Ω of the nominal value?

Solution

A company has three machines B1, B2, and B3 for making 1 kΩ resistors. It has been observed that 80% of resistors produced by B1 are within 50 Ω of the nominal value. Machine B2 produces 90% of resistors within 50 Ω of the nominal value. The percentage for machine B3 is 60%. Each hour, machine B1 produces 3000 resistors, B2 produces 4000 resistors, and B3 produces 3000 resistors.

Let A = {resistor is within 50 Ω of the nominal value} = $\{950 \leq R \leq 1050\}$

Using the resistor accuracy information to formulate a probability model, we write

$$P[A|B_1] = 0.8, \quad P[A|B_2] = 0.9, \quad P[A|B_3] = 0.6$$

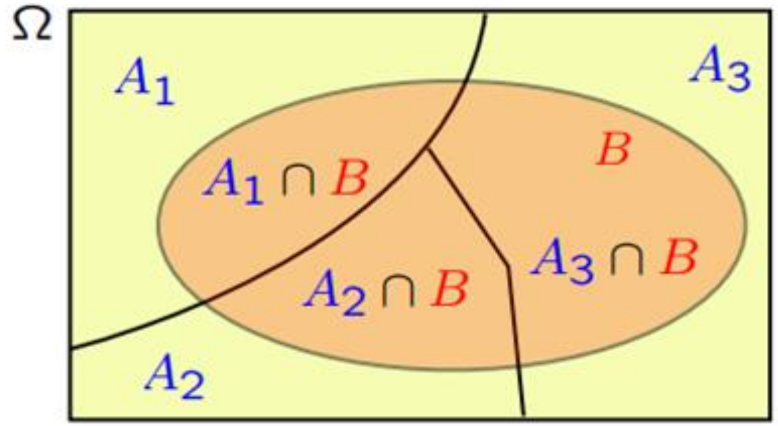
Apply the law of total probability to find the accuracy probability for all resistors shipped by the company:

$$P[A] = P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + P[A|B_3]P[B_3]$$
$$= (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = 0.78.$$

For the whole factory, **78%** of resistors are within 50 Ω of the nominal value.

Bayes' Rule

- Partition of sample space into A_1, A_2, A_3
- Have $P(A_i), P(B|A_i)$ for every i
- Wish to compute $P(A_i|B)$



$$P(A_i|B) = \frac{P(A_i \cap B)}{P(B)}$$

$$P(A_i|B) = \frac{P(A_i)P(B|A_i)}{\sum_j P(A_j)P(B|A_j)}$$

By using Bayes' Rule, we can find $P(A|B)$ if we know $P(B|A)$ and vice versa

Bayes' Rule

$$A_i \xrightarrow{cause\ effect} B$$

$$P(B|A_i)$$

$$A_i \xleftarrow{Inference} B$$

$$P(A_i|B)$$

$\Rightarrow$ Draw conclusion about causes.

Example

A laboratory blood test is 99 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply he or she has the disease.)

If 0.5 percent of the population actually has the disease, **what is the probability a person has the disease given that his test result is positive**?

Solution

A laboratory blood test is 99 percent effective in detecting a certain disease when it is, in fact, present. However, the test also yields a "false positive" result for 1 percent of the healthy persons tested. (That is, if a healthy person is tested, then, with probability .01, the test result will imply he or she has the disease.)
If 0.5 percent of the population actually has the disease, what is the probability a person has the disease given that his test result is positive?

Let **D** be the event that the tested person has the disease and **T** the event that his test result is positive.

Given: $P(T|D) = .99$, $P(T|D^C) = 0.01$, $P(D) = 0.005$

The desired probability P(D|T) is obtained by

$$P(D|T) = \frac{P(D \cap T)}{P(T)}$$

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$$P(D|T) = \frac{P(T \mid D)P(D)}{P(T|D)P(D) + P(T|D^C)P(D^C)}$$

$$P(D|T) = \frac{(0.99)(0.005)}{(0.99)(0.005) + (0.01)(0.995)} = 0.33$$

Practice Problem:

One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.

(a). What is the probability that Joe (a random person) tests positive?

(b). Joe just got the bad news that the test came back positive; what is the probability that Joe has the disease?

Solution

One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.

Let D be the event that Joe has the disease.

Let T be the event that Joe's test comes back positive.

We are told that $\boldsymbol{P(D) = 0.005}$, since 1/2% of the population has the disease.

We are also told that $\boldsymbol{P(T|D) = .98}$, since 2% of the time a person having the disease is missed ("false negative").

$$\boldsymbol{P(T^c|D) = .02 \Rightarrow P(T|D) = 1 - P(T^c|\mathrm{D}) = 0.98}$$

We are told that $\boldsymbol{P(T|D^c) = .03}$, since there are 3% false positives.

(a). We want to compute P(T). We do so by conditioning on whether or not Joe has the disease:

$$P(T) = P(T|D)P(D) + P(T|D^c)P(D^c) = (.98)(.005) + (.03)(.995)$$

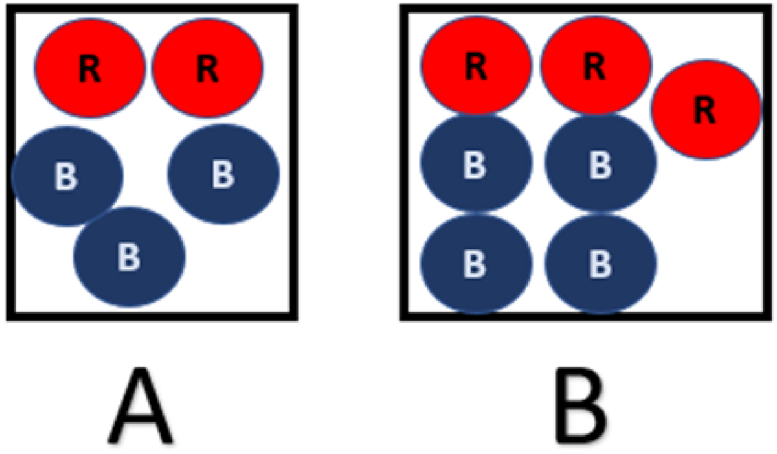
$\boldsymbol{P(A) = 1 - P(A^c), P(A^c) = 1 - P(A)}$

(b). Joe just got the bad news that the test came back positive; what is the probability that Joe has the disease?

$$P(D|T) = \frac{P(D \cap T)}{P(T)} = \frac{P(T|D)P(D)}{P(T|D)P(D) + P(T|D^c)P(D^c)} = \frac{(.98)(.005)}{(.98)(.005) + (.03)(.995)} \approx .14$$

11

Practice Problem:



$\Rightarrow$ P(R) if "Bag A is chosen" $P(R|A) = 2/5$

$\Rightarrow$ P ("Red ball is drawn from A")

$$P(A \cap R) = P(A)P(R|A) = \frac{1}{2} * \frac{2}{5} = \frac{1}{5}$$

$\Rightarrow$ What is the probability of red ball drawn:

$P(R) = P(A \cap R) + P(B \cap R)$ – Total Probability Theorem

$\Rightarrow$ Given that red ball is drawn. What is the probability that ball is drawn from A.

$$P(A|R) = \frac{P(A \cap R)}{P(R)} = \frac{P(A)P(R|A)}{P(A \cap R) + P(B \cap R)}$$