Lecture No. 10

Functions of a Random Variable

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- In many practical situations, we observe sample values of a random variable and use these sample values to compute other quantities.

- One example that occurs frequently is an experiment in which the procedure is to measure the power level of the received signal in a cellular telephone.

- An observation is x, the power level in units of milliwatts. Frequently engineers convert the measurements to decibels by calculating $\boldsymbol{y = 10\, log_{10} x\ dBm}$ (decibels with respect to one milliwatt).

- If x is a sample value of a random variable X, implies that y is a sample value of a random variable Y.

- Because we obtain Y from another random variable, we refer to Y as a derived random variable.

Definition ***Derived Random Variable***

Each sample value y of a **derived random variable** Y is a mathematical function $g(x)$ of a sample value x of another random variable X. We adopt the notation $Y = g(X)$ to describe the relationship of the two random variables.

- If $Y = g(X)$ is a function of a random variable X, then Y is also a random variable, since it provides a numerical value for each possible outcome.
- This is because every outcome in the sample space defines a numerical value x for X and hence also the numerical value $y = g(x)$ for Y .
- If X is discrete with PMF $\boldsymbol{P_X}$, then Y is also discrete, and its PMF $\boldsymbol{P_Y}$ can be calculated using the PMF of X.
- In particular, to obtain $P_Y(y)$ for any y, we add the probabilities of all values of x such that $g(x) = y$:

$$p_Y(y) = \sum_{\{x \mid g(x)=y\}} p_X(x).$$

Example

- Let $Y = |X|$ and let us apply the preceding formula for the PMF P_Y to the case where

$$p_X(x) = \begin{cases} 1/9 & \text{if } x \text{ is an integer in the range } [-4, 4], \\ 0 & \text{otherwise.} \end{cases}$$

- The possible values of Y are $y = 0, 1, 2, 3, 4$. To compute $P_Y(y)$ for some given value y from this range, we must add $P_X(x)$ over all values x such that $|x| = y$. In particular, there is only one value of X that corresponds to $y = 0$, namely $x = 0$.

Example

$$p_Y(0) = p_X(0) = \frac{1}{9}.$$

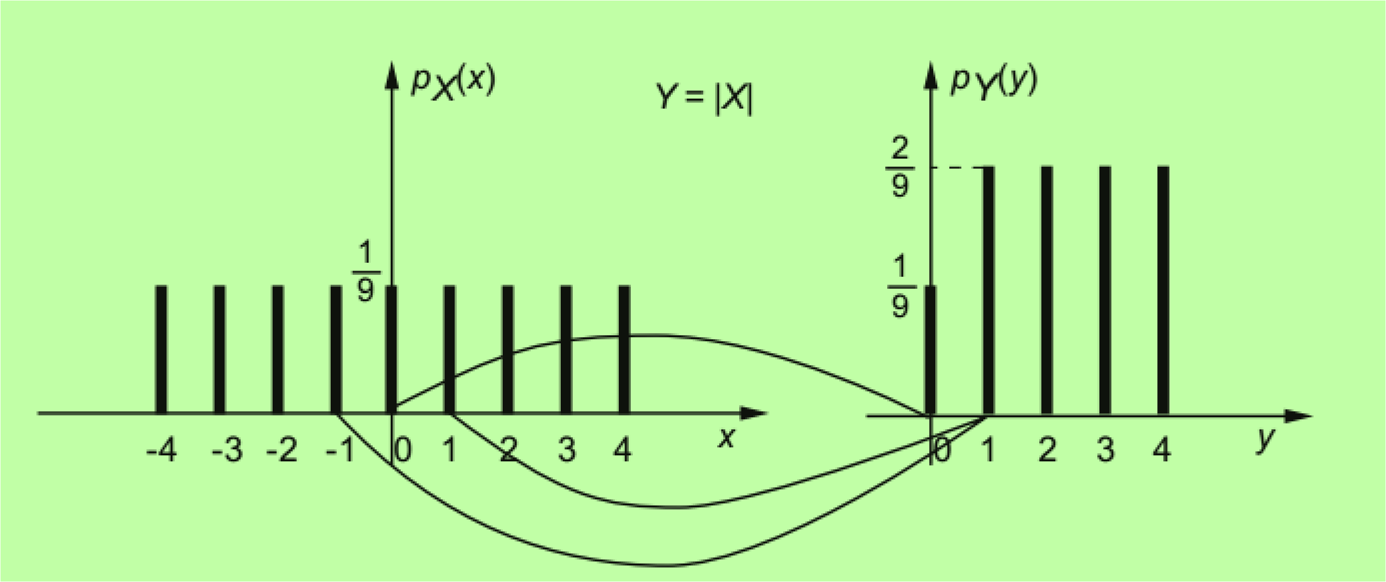
- Also, there are two values of X that correspond to each y = 1, 2, 3, 4 so for example,

$$p_Y(1) = p_X(-1) + p_X(1) = \frac{2}{9}.$$

Thus, the PMF of Y is

$$p_Y(y) = \begin{cases} 2/9 & \text{if } y = 1, 2, 3, 4, \\ 1/9 & \text{if } y = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Example



PMF of X and $Y = |X|$

Expected Value for Functions of Random Variables

Expected Value Rule for Functions of Random Variables

- Let X be a random variable with PMF $P_X(x)$, and let $g(X)$ be a real valued function of X. Then, the expected value of the random variable $g(X)$ is given by

$$\mathbf{E}[g(X)] = \sum_x g(x) p_X(x).$$

Practice Problem

Monitor three phone calls and observe whether each one is a voice call or a data call.
The random variable N is the number of voice calls. Assume N has PMF

$$P_N(n) = \begin{cases} 0.1 & n = 0, \\ 0.3 & n = 1, 2, 3, \\ 0 & \textit{otherwise}. \end{cases}$$

Voice calls cost 25 cents each and data calls cost 40 cents each. T cents is the cost of the three telephone calls monitored in the experiment.

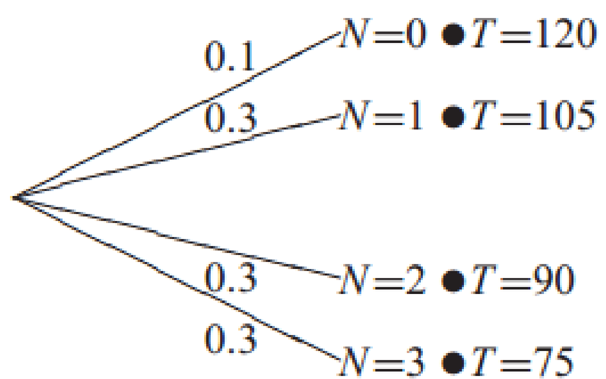
(1) Express T as a function of N. ***(2) Find $P_T(t)$ and $E[T]$.***

Practice Problem

(1) As a function of N, the cost T is

$$T = 25N + 40(3 - N) = 120 - 15N$$

(2) To find the PMF of T, we can draw the following tree:



Practice Problem

From the tree, we can write down the PMF of T:

$$P_T(t) = \begin{cases} 0.3 & t = 75, 90, 105 \\ 0.1 & t = 120 \\ 0 & \text{otherwise} \end{cases}$$

From the PMF $P_T(t)$, the expected value of T is

$$\begin{aligned} E[T] &= 75 P_T(75) + 90 P_T(90) + 105 P_T(105) + 120 P_T(120) \\ &= (75 + 90 + 105)(0.3) + 120(0.1) = 62 \end{aligned}$$

Theorem 1 *For any random variable X*

$$E[X - \mu_X] = 0$$

Proof Defining $g(X) = X - \mu_X$

$$E[g(X)] = \sum_{x \in S_X} (x - \mu_X) P_X(x) = \sum_{x \in S_X} x P_X(x) - \mu_X \sum_{x \in S_X} P_X(x).$$

The first term on the right side is μ_X by definition. In the second term, $\sum_{x \in S_X} P_X(x) = 1$, so both terms on the right side are μ_X and the difference is zero.

Theorem 2 For any random variable X

$$\boldsymbol{E[aX + b] = aE[X] + b}$$

Theorem 3 The variance of random variable X

$$\boldsymbol{Var[X] = E\left[(X - \mu_X)^2\right]}$$

$$E[(X-\mu_X)^2] = \sum_x (x-\mu_X)^2\, p_X(x)$$

$$= \sum_x (x^2 + \mu_X^2 - 2x\mu_X)\, p_X(x)$$

$$= \sum_x x^2 p_X(x) + \mu_X^2 \sum_x p_X(x) - 2\mu_X \sum_x x p_X(x)$$

$$= E[X^2] + \mu_X^2(1) - 2\mu_X E[X]$$

$$= E[X^2] + \mu_X^2 - 2\mu_X^2$$

$$= E[X^2] - (E[X])^2$$

Moments For any random variable X

(a) The nth **moment** is $E[X^n]$.

(b) The nth **central moment** is $E[(X - \mu_X)^n]$

Theorem 4 *For any random variable X*

$$Var[aX + b] = a^2 Var[X]$$

$Proof: Var[aX + b] = a^2 \, Var[X]$

$Let \ Y = aX + b$

$Var[Y] = E[Y^2] - (E[Y])^2$

$E[Y^2] = E[a^2X^2 + 2abX + b^2] = a^2E[X^2] + 2ab\mu_X + b^2$

$E[Y] = aE[X] + b$

$$(E[Y])^2 = (aE[X] + b)^2$$

$$\mu_Y^2 = a^2\mu_X^2 + 2ab\mu_X + b^2$$

$$Var[Y] = a^2E[X^2] + 2ab\mu_X + b^2 - (a^2\mu_X^2 + 2ab\mu_X + b^2)$$

$$Var[Y] = a^2E[X^2] - a^2\mu_X^2$$

$$= a^2(E[X^2] - \mu_X^2) = a^2Var[X]$$

Practice Problem

The amplitude V (volts) of a sinusoidal signal is a random variable with PMF

$$P_V(v) = \begin{cases} 1/7 & v = -3, -2, \ldots, 3, \\ 0 & \text{otherwise.} \end{cases}$$

A new voltmeter records the amplitude U in millivolts.

(1) Express U as a function of V.

(2) What is the variance of U?

Practice Problem

(1) Express U as a function of V.

$$U = 1000V$$

(2) What is the variance of U?

We first find the variance of V. The expected value of the amplitude is

$$\mu_V = 1/7[-3 + (-2) + (-1) + 0 + 1 + 2 + 3] = 0 \text{ volts.}$$

The second moment is

$$E\left[V^2\right] = 1/7[(-3)^2 + (-2)^2 + (-1)^2 + 0^2 + 1^2 + 2^2 + 3^2] = 4 \text{ volts}^2$$

Practice Problem

Therefore the variance is

$$Var[V] = E[V^2] - \mu_V^2 = 4\,volts^2$$

$$\boxed{Var\,[U] = 1000^2\,Var[V] = 4{,}000{,}000\,millivolts^2}$$

Homework Problem

X is a Gaussian random variable with $E[X] = 0$ and $P[|X| \leq 10] = 0.1$.

What is the standard deviation σ_X ?

Homework Problem

X is a Gaussian random variable with zero mean but unknown variance. We do know, however, that

$$P[|X| \leq 10] = 0.1 \tag{1}$$

We can find the variance Var[X] by expanding the above probability in terms of the $\Phi(\cdot)$ function.

$$P[-10 \leq X \leq 10] = F_X(10) - F_X(-10) = 2\Phi\left(\frac{10}{\sigma_X}\right) - 1 \tag{2}$$

This implies $\Phi(10/\sigma_X) = 0.55$. Using Table for the Gaussian CDF, we find that $10/\sigma_X = 0.15$ or $\sigma_X = 66.6$.