Lecture : 4

Instructor

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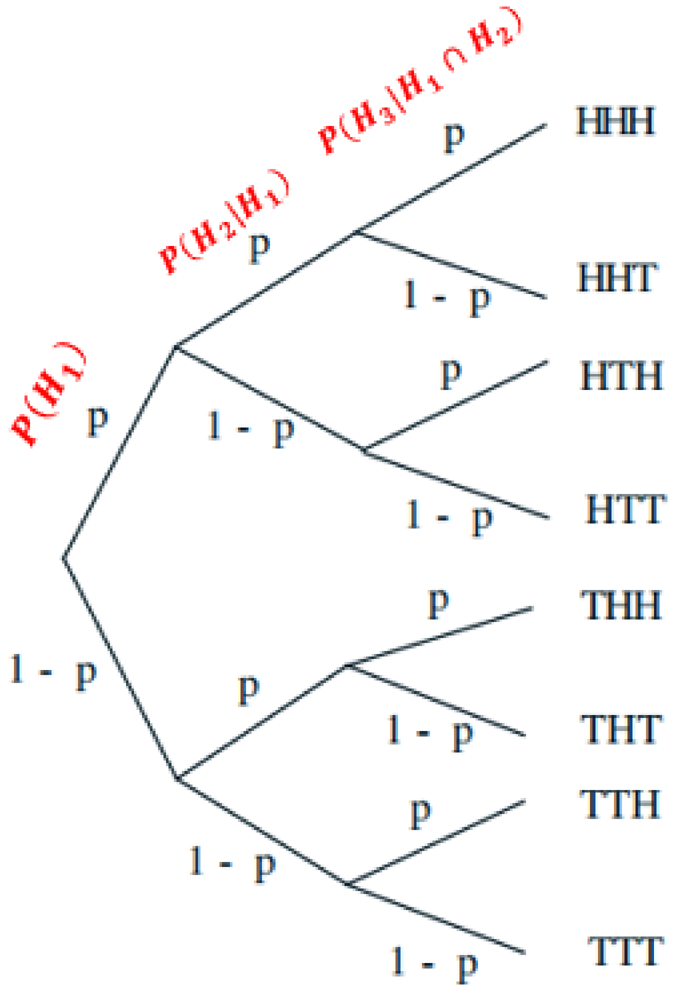
Independence

- 3 tosses of a biased coin: $\boldsymbol{P(H) = p,\ P(T) = 1 - p}$
- Multiplication rule: $\boldsymbol{P(THT)} = (1 - p)p(1 - p)$
- $\boldsymbol{P(1\ head)} = 3p(1-p)^2$ = {HTT,TTH,THT} = p(1-p)(1-p)+(1-p)(1-p)p+(1-p)p(1-p)
- $\boldsymbol{P(first\ toss\ is\ head\ |1\ head\ )}$

$$= \frac{P(H_1 \cap 1\ head)}{P(1\ head)} = \frac{p(1-p)^2}{3p(1-p)^2} = 1/3$$

❖ $P(H_2|H_1) = p = P(H_2|T_1)$



Independence

Independence of two events

- Intuitive “definition”: $P(B|A) = P(B)$
  - occurrence of A provides no new information about B

$$P(A \cap B) = P(A)P(B|A)$$

If independent: $P(B|A) = P(B)$

Definition of independence: $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$

$\Rightarrow$ Symmetric with respect to A and B

$\Rightarrow$ Implies $P(A|B) = P(A)$

Example

Two successive rolls of a 4-sided die

(a)Are the events independent?

$$A = \{First\ roll\ is\ 1\}$$

$$B = \{Sum\ of\ two\ rolls\ is\ 5\}$$

Solution:

$$A = \{(1,1), (1,2), (1,3), \mathbf{(1,4)}\}, \qquad P(A) = \frac{4}{16}$$

$$B = \{\mathbf{(1,4)}, (2,3), (3,2), (4,1)\}, \qquad P(B) = \frac{4}{16}$$

$$A \cap B = \{(1,4)\}, \qquad P(A \cap B) = \frac{1}{16}$$

$Since\ P(A \cap B) = P(A)P(B) \rightarrow \boldsymbol{A\ \&\ B\ are\ independent}.$

(a)Are the events independent?

$$A = \{\max of\ two\ rolls\ is\ 2\}$$

$$B = \{min\ of\ two\ rolls\ is\ 2\}$$

Solution:

$$A = \{(1,2), (2,1), \mathbf{(2,2)}\}, \qquad P(A) = \frac{3}{16}$$

$$B = \{\mathbf{(2,2)}, (2,3), (2,4), (3,2), (4,2)\}, \qquad P(B) = \frac{5}{16}$$

$$A \cap B = \{(2,2)\}, \qquad P(A \cap B) = \frac{1}{16}$$

$Since\ P(A)P(B) \neq P(A \cap B) \ \rightarrow$ ***A & B are not independent***.

Conditional Independence:

Events A and B are called conditionally independent if

$$\boldsymbol{P(A \cap B \mid C) = P(A|C)P(B|C)}$$

$\Rightarrow$ Unconditional Independence of two events A and B does not imply conditional independence and vice versa.

Example

Consider two independent fair coin tosses, in which all four possible outcomes are equally likely.

$$\boldsymbol{H_1 = \{First\ toss\ is\ a\ head\}}$$

$$\boldsymbol{H_2 = \{Second\ toss\ is\ a\ head\}}$$

$$\boldsymbol{D = \{Two\ tosses\ have\ different\ results\}}$$

$$\Omega = \{HH, HT, TH, TT\}$$

$$H_1 = \{HH, HT\}, \qquad P(H_1) = \frac{2}{4} = \frac{1}{2}$$

$$H_2 = \{HH, TH\}, \qquad P(H_2) = \frac{2}{4} = \frac{1}{2}$$

$$H_1 \cap H_2 = \{HH\}, \qquad P(H_1 \cap H_2) = \frac{1}{4}$$

$Since\ \boldsymbol{P(H_1 \cap H_2) = P(H_1)P(H_2)}$

$\rightarrow \boldsymbol{H_1\ \&\ H_2\ are\ unconditionally\ independent}.$

Example

Consider two independent fair coin tosses, in which all four possible outcomes are equally likely.

$$\boldsymbol{H_1 = \{First\ toss\ is\ a\ head\}}$$
$$\boldsymbol{H_2 = \{Second\ toss\ is\ a\ head\}}$$
$$\boldsymbol{D = \{Two\ tosses\ have\ different\ results\}}$$

$D = \{HT, TH\}$, $H_1 \cap D = \{HT\}$, $H_2 = \{TH\}$

$$(H_1 \cap H_2) \cap D = \emptyset$$

$$P(H_1|D) = \frac{P(H_1 \cap D)}{P(D)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(H_2|D) = \frac{P(H_2 \cap D)}{P(D)} = \frac{1/4}{2/4} = \frac{1}{2}$$

$$P(H_1 \cap H_2|D) = \frac{P\big((H_1 \cap H_2) \cap D\big)}{P(D)} = \frac{0}{2/4} = 0$$

$Since\ \boldsymbol{P(H_1 \cap H_2|D) \neq P(H_1|D)P(H_2|D)}$

$\rightarrow$ $\boldsymbol{H_1\ \&\ H_2\ are\ not\ conditionally\ independent}$.

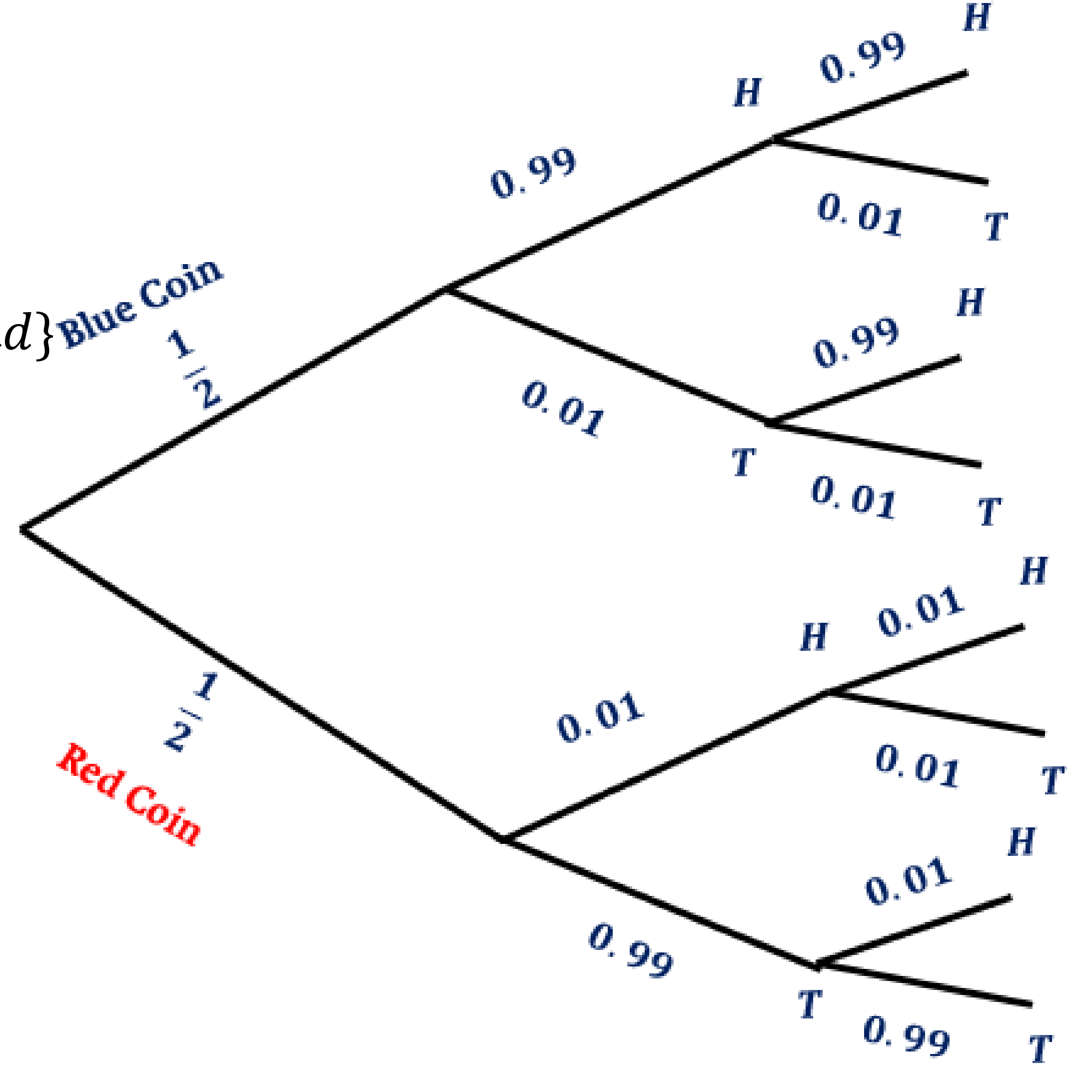
Example-1.19[Textbook]:

Two coins : Red + Blue (both are baised)

Blue: $P(H) = 0.99\,, P(T) = 0.01$

Red: $P(H) = 0.01\,, P(T) = 0.99$

$B = \{Blue\ coin\ is\ selected\}$ $H_i = \{ith\ toss\ resulted\ in\ head\}$



Given the choice of the coin, events H_1 *and* H_2 are independent (i.e impendent tosses), then

$$\boldsymbol{P(H_1 \cap H_2|B) = P(H_1|B)P(H_2|B)}$$
$$\boldsymbol{= 0.99 * 0.99}$$

Example-1.19[Textbook]:

On the other hand, events H_1 and H_2 are not independent

$$P(H_1) = P(B)P(H_1|B) + P(R)P(H_1|R)$$

$$P(H_1) = \frac{1}{2} * 0.99 + \frac{1}{2} * 0.01 = \frac{1}{2}$$

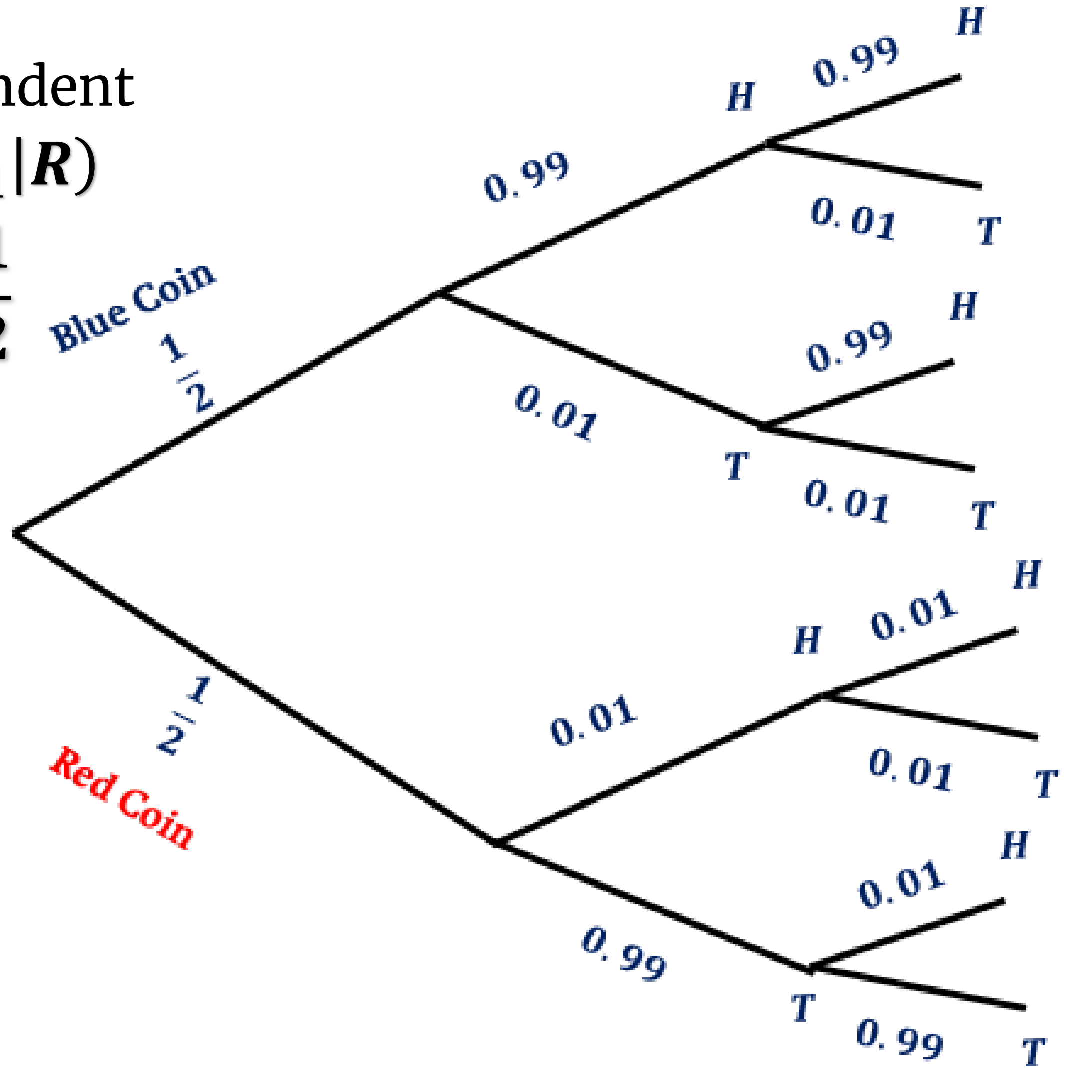
Similarly

$$P(H_2) = \frac{1}{2}$$

$$P(H_1 \cap H_2) = P(B)P(H_1 \cap H_2|B) + P(R)P(H_1 \cap H_2|R)$$

$$P(H_1 \cap H_2) = \frac{1}{2}(0.99 * 0.99) + \frac{1}{2}(0.01 * 0.01) \approx \frac{1}{2}$$

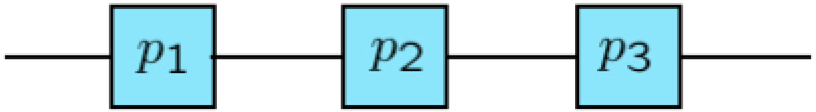
$$P(H_1 \cap H_2) \neq P(H_1)P(H_2)$$



Reliability

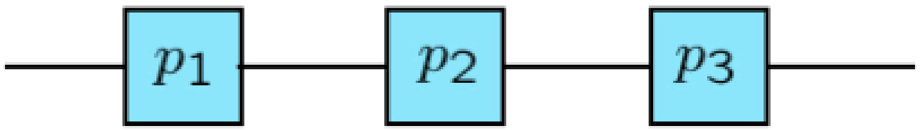
pi: probability that unit i is "up"
Independent units

- $Event\, \boldsymbol{U_i}$:ith unit is up
- $\boldsymbol{U_1}, \boldsymbol{U_2}, \ldots\ldots, \boldsymbol{U_n}$ *are independent.*
- $Event\, \boldsymbol{F_i}: i^{th}$ *unit is down*
- $\boldsymbol{F_i}$ *is independent.*



Probability that system is up?

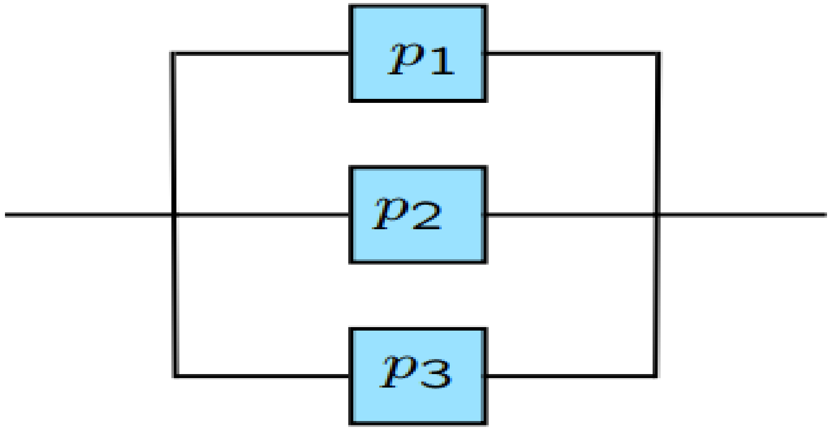
Reliability



$$P(sytem\ is\ up) = P(U_1 \cap U_2 \cap U_3)$$
$$= P(U_1)P(U_2)P(U_3)$$
$$= p_1 p_2 p_3$$

For parallel combination:

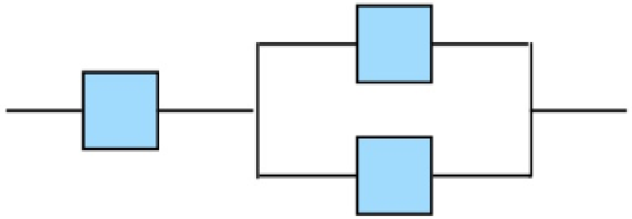
$$P(sytem\ is\ up) = P(U_1 \cup U_2 \cup U_3)$$
$$= 1 - P(F_1 \cap F_2 \cap F_3)$$
$$= 1 - P(F_1)P(F_2)P(F_3)$$
$$= 1 - (1 - p_1)(1 - p_2)(1 - p_3)$$



Example

Suppose that each unit of a system is up with probability 2/3 and down with probability 1/3. Different units are independent. For each one of the systems shown below, calculate the probability that the whole system is up (that is, that there exists a path from the left end to the right end, consisting entirely of units that are up).

What is the probability that the following system is up?

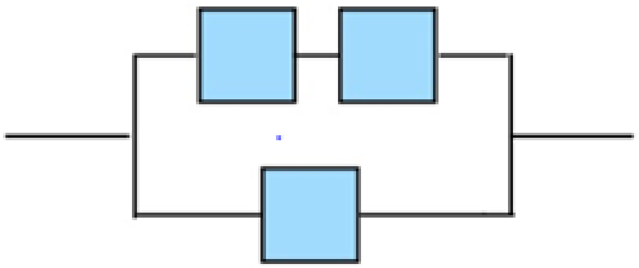


- Parallel connection on the right is down when both units fails $\frac{1}{3} * \frac{1}{3}$
- Parallel connection is up with probability $1 - \frac{1}{9} = \frac{8}{9}$
- The overall system is up if the first unit is up (probability 2/3) and the parallel connection is also up (probability 8/9), which happens with probability $\frac{2}{3} * \frac{8}{9} = \frac{16}{27}$

Example

Suppose that each unit of a system is up with probability 2/3 and down with probability 1/3. Different units are independent. For each one of the systems shown below, calculate the probability that the whole system is up (that is, that there exists a path from the left end to the right end, consisting entirely of units that are up).

What is the probability that the following system is up?



- Top path is up with probability $\frac{2}{3} * \frac{2}{3} = \frac{4}{9}$. Thus it fails with probability $1 - \frac{4}{9} = \frac{5}{9}$
- Overall system fails when top path fails (probability 5/9) and the bottom path also fails (probability 1/3)
- Probability of failure is $\frac{5}{9} * \frac{1}{3} = \frac{5}{27}$
- Probability that system is up: $1 - \frac{5}{27} = \frac{22}{27}$

Solution

A company has three machines B1, B2, and B3 for making 1 kΩ resistors. It has been observed that 80% of resistors produced by B1 are within 50 Ω of the nominal value. Machine B2 produces 90% of resistors within 50 Ω of the nominal value. The percentage for machine B3 is 60%. Each hour, machine B1 produces 3000 resistors, B2 produces 4000 resistors, and B3 produces 3000 resistors.

Let A = {resistor is within 50 Ω of the nominal value} = $\{950 \le R \le 1050\}$

Using the resistor accuracy information to formulate a probability model, we write

$$P[A|B_1] = 0.8, \quad P[A|B_2] = 0.9, \quad P[A|B_3] = 0.6$$

Apply the law of total probability to find the accuracy probability for all resistors shipped by the company:

$$\begin{aligned} P[A] &= P[A|B_1]P[B_1] + P[A|B_2]P[B_2] + P[A|B_3]P[B_3] \\ &= (0.8)(0.3) + (0.9)(0.4) + (0.6)(0.3) = 0.78. \end{aligned}$$

For the whole factory, **78%** of resistors are within 50 Ω of the nominal value.

Practice Problem:

One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.

(a). What is the probability that Joe (a random person) tests positive?

(b). Joe just got the bad news that the test came back positive; what is the probability that Joe has the disease?

Solution One half percent of the population has a particular disease. A test is developed for the disease. The test gives a false positive 3% of the time and a false negative 2% of the time.

Let D be the event that Joe has the disease.

Let T be the event that Joe's test comes back positive.

We are told that $\boldsymbol{P(D) = 0.005}$, since 1/2% of the population has the disease.

We are also told that $\boldsymbol{P(T|D) = .98}$, since 2% of the time a person having the disease is missed ("false negative").

$$\boldsymbol{P(T^c|D) = .02 \Rightarrow P(T|D) = 1 - P(T^c|\mathrm{D}) = 0.98}$$

We are told that $\boldsymbol{P(T|D^c) = .03}$, since there are 3% false positives.

(a). We want to compute P(T). We do so by conditioning on whether or not Joe has the disease:

$$P(T) = P(T|D)P(D) + P(T|D^c)P(D^c) = (.98)(.005) + (.03)(.995)$$

$\boldsymbol{P(A) = 1 - P(A^c), P(A^c) = 1 - P(A)}$